Objective and subjective product differentiation in a Cournot duopoly

Diferenciação objetiva e subjetiva de produtos em um duopólio de Cournot

Diferenciación objetiva y subjetiva de productos en un duopolio de Cournot

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Abstract: Cournot duopoly is a mathematical model of an imperfectly competitive market, where two profit maximizing firms simultaneously choose the quantities of product to produce, considering each other's behavior. The main objective of this paper is to analytically obtain the equilibrium quantities, prices, and profits of a Cournot duopoly in a market with two objectively differentiated products composed of two properties, and also considering that the consumers subjectively differentiate these properties. We begin by introducing subjective properties differentiation into a Lancaster consumer's choice problem, where the consumer maximizes a symmetrical CES utility function depending on the properties consumed, subject to a budget constraint and to the Lancaster linear transformation of these properties in the final goods. Then we solve this optimization problem through the method of Lagrange multipliers and obtain the final products direct and inverse demand functions, and finally find the equilibrium of the proposed Cournot duopoly. The results show that if the properties are perceived by the consumers as highly differentiated, then a higher subjective properties differentiation implies that the Cournot duopoly has a well-defined equilibrium only for a smaller range of objective product differentiation. Furthermore, an increase in the consumers' subjective differentiation of properties and/or in the objective product differentiation increases the monopoly power of the firms, i.e. with a higher differentiation, the firms produce smaller quantities of products, charge higher prices and earn higher profits. The model pro-

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posed here may serve as a basis for the study of imperfectly competitive markets with product differentiation and advertising.

Keywords: mathematical economics; microeconomics; Cournot duopoly; product differentiation; mathematical modeling.

Resumo: Duopólio de Cournot é um modelo matemático de mercado de concorrência imperfeita, em que duas firmas maximizadoras de lucros decidem simultaneamente as quantidades de produtos a produzir, considerando o comportamento uma da outra. O objetivo principal deste trabalho é obter, analiticamente, as quantidades, preços e lucros de equilíbrio do duopólio de Cournot em um mercado com dois produtos objetivamente diferenciados compostos por duas propriedades, também considerando que os consumidores diferenciam subjetivamente estas propriedades. Inicialmente introduzimos diferenciação subjetiva de propriedades no problema de escolha do consumidor de Lancaster, em que se maximiza uma função de utilidade CES simétrica dependente das propriedades consumidas, sujeito à restrição orçamentária e à transformação linear de Lancaster das propriedades em produtos. Então resolvemos este problema de otimização através do método dos multiplicadores de Lagrange, obtendo as funções de demanda inversa e direta dos produtos finais. Encontramos o equilíbrio do duopólio de Cournot proposto. Os resultados mostram que se os consumidores percebem as propriedades como altamente diferenciadas, então uma maior diferenciação subjetiva implica que o duopólio de Cournot possui um equilíbrio bem definido apenas para uma faixa menor de diferenciação objetiva de produtos. Ainda, um aumento na diferenciação subjetiva das propriedades do consumidor e/ou da diferenciação objetiva de produtos aumenta o poder de monopólio das firmas, i.e. com uma maior diferenciação, as firmas produzem menores quantidades de produtos, cobram preços maiores e auferem lucros maiores. O modelo proposto aqui pode servir de base para o estudo de mercados de concorrência imperfeita com diferenciação de produtos e propaganda.

Palavras-chave: economia matemática; microeconomia; Duopólio de Cournot; diferenciação de produtos; modelagem matemática.

Resumen: El duopolio de Cournot es un modelo matemático de mercado de competencia imperfecta, en el que dos empresas maximizadoras de beneficios deciden simultáneamente las cantidades de productos a producir, considerando el comportamiento de la otra. El objetivo de este trabajo es obtener analíticamente las cantidades, precios y las ganancias de equilibrio del duopolio de Cournot en un mercado con dos productos objetivamente diferenciados compuestos por dos propiedades, considerando también que los consumidores diferencian subjetivamente dichas propiedades. Inicialmente, introducimos esa diferenciación subjetiva en el problema de elección del consumidor de Lancaster, donde se maximiza una función de utilidad CES simétrica dependiente de las propiedades, sujeta a la restricción presupuestaria y a la transformación lineal de Lancaster de esas propiedades en los productos. Luego resolvemos ese problema utilizando el método de los multiplicador de Lagrange, obteniendo las funciones de demanda inversa y directa de los productos, y encontramos el equilibrio del duopolio de Cournot propuesto. Los resultados muestran que si los consumidores perciben las propiedades como altamente diferenciadas, entonces una mayor diferenciación subjetiva de propiedades implica que el duopolio de Cournot tiene un equilibrio bien definido para un rango menor de diferenciación objetiva. Además, un aumento en la diferenciación subjetiva de las propiedades del consumidor y/o la diferenciación objetiva incrementa el poder de monopolio de las empresas, es decir, las empresas producen menores cantidades de productos, cobran precios más altos y obtienen mayores beneficios. El modelo aquí propuesto puede servir como base para el estudio de mercados de competencia imperfecta con diferenciación de productos y publicidad.

Palabras clave: economía matemática; microeconomía; Duopolio de Cournot; diferenciación del producto; modelización matemática.

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1 Introduction

Real imperfectly competitive markets usually are formed by few rival firms competing by the consumers, and in this way each firm does not have the demand for their products guaranteed, since this depends on the production and pricing decisions taken by each competitor in the market. In such situation, the firms have two courses of action to soften competition: differentiate their products and/or use informative or persuasive advertising[1] aiming to get more consumers for their products (Lauga; Ofek; Katona, 2022; Shy, 1995). These features can be empirically observed in real state, automotive, clothing, cultural, and smartphones markets, among others (Belleflamme; Peitz, 2015; Lancaster, 1966, 1971; Nicholson; Snyder, 2012; Puu, 2018; Rosen, 1974; Shy, 1995). Focusing on product differentiation, on one hand it can be objective, in the physical sense that products can have distinct construction designs (or properties), as considered by Lancaster (1966, 1971) in his consumer's theory. Objective product differentiation is under the direct control of the firms during the productive process. On the other hand, product differentiation can also be subjective, in the sense that consumers can psychologically perceive products more or less differentiated, depending on their cognitive capabilities or on the availability of public information about the products in the markets. This subjective perception of differentiation by the consumers can be indirectly influenced by the

[1] According to Shy (1995, p. 283), "[...] persuasive advertising intends to enhance consumer tastes for a certain product, whereas informative advertising carries basic product information such as characteristics, prices, and where to buy it".

firms through investment in advertising (Belleflamme; Peitz, 2015; Dixit; Norman, 1978; Dorfman; Steiner, 1954; Shy, 1995).

Intrinsic or objective product differentiation was first introduced into the consumer's choice problem by Lancaster (1966, 1971), as a critique to its complete absence in the traditional microeconomic consumer theory (Jehle; Reny, 2011). In his approach, Lancaster considered each product as being described by a vector of exogenously given intrinsic properties. Consider as an example a smartphone, whose properties' vector has as elements its size, battery life, amount of memory, screen resolution, and processor speed, among other possible characteristics. In this formulation, the consumer derives his utility from the quantity of each property consumed, and not directly from the total quantity of each product that consumes, as occurs in the traditional consumer theory. The great advantage of this methodology is to enable the objective differentiation of products through realistic properties. This can be important in the study of imperfectly competitive markets, such as oligopolies, where objective product differentiation is common and consumers seek specific qualities in the products they consume rather than to simply maximize the amount of goods consumed (Belleflamme; Peitz, 2015; Lancaster, 1966; Puu, 2017; 2018; Rosen, 1974; Shy, 1995). In particular, the Cournot duopoly with objective product differentiation – considering a symmetrical Cobb-Douglas utility function (Cobb; Douglas, 1928) – was already studied in detail by Puu (2017, 2018) and Juchem Neto (2023). In these works it is shown – among other results – that when the marginal costs of the firms are equal, more differentiated products imply in lower quantities, higher prices, and higher profits in the duopoly equilibrium for both firms, what means that they have their monopoly power increased with a higher objective product differentiation.

The main objective of this theoretical work is to introduce subjective properties differentiation in a Cournot duopoly model with objective product differentiation, generalizing in this way the model considered in Puu (2017, 2018) and Juchem Neto (2023), and to study the impact of such subjective differentiation in the duopoly's equilibrium quantities, prices, and profits. This subjective differentiation is mathematically introduced in the Lancaster consumer's choice model through the use of a symmetrical constant elasticity of substitution (CES) utility function (Nicholson; Snyder, 2012), function that is commonly used to model monopolistic competition market structures in the core-periphery types of models in the New Economic Geography literature (Brakman; Garretsen; Van Marrewijk, 2009; Krugman, 1991). In this way, we first obtain the Lancaster demand functions with objective and subjective differentiation, and then use the inverse Lancaster demand functions in computing the equilibrium of the proposed Cournot duopoly. The model proposed here may serve as a basis for future theoretical and empirical studies of imperfectly competitive markets with product differentiation and advertising, constituting an alternative approach to the model proposed by Lauga, Ofek and Katona (2022), for example. In their approach, these authors consider that firms differentiate products through quality, which is described by a single parameter, and use informative advertising aiming to increase their market share.

This paper is structured as follows. In Section 2 we introduce subjective properties differentiation into the Lancaster consumer's choice problem through a symmetrical constant elasticity of substitution (CES) utility function, where the consumer maximizes his utility subject to her budget constrain and to the Lancaster linear transformation of properties in final products. For simplicity, we consider a market with two intrinsically differentiated products composed of two properties, as in Puu (2017, 2018) and Juchem Neto (2023). In Section 3 we solve this consumer's constrained maximization problem by applying the classical method of Lagrange multipliers (Araújo, 2022; Chiang; Wainwright, 2005; Simon; Blume, 2008; Sundaran, 2011; Sydsaeter *et al.*, 2016), and obtain the final products direct and inverse demand functions (the so called Lancaster demand functions), the properties demand functions and their shadow (or imputed) prices, finally presenting some limiting cases for the inverse demand functions at the end of this section. With these basic results at hand, in Section 4 we analytically obtain the equilibrium quantities, prices, and profits of the Cournot duopoly in a market with two objectively differentiated products composed of two subjectively differentiated properties, analyze the impact caused by small changes in the differentiation parameters in this equilibrium, and present a numerical example. Finally, in Section 5 we present our final remarks.

2 Consumer's choice problem

In this section we propose the modified Lancaster consumer's choice problem, whose solution will give rise to the Lancaster demand functions that will be used to obtain the equilibrium of the Cournot duopoly proposed in this work. In this problem we consider, as in Puu (2017, 2018), a market with two products, 1 and 2, composed of two independent properties, 1 and 2, following the one parameter properties parametrization proposed by Juchem Neto (2023). This base model already shows objective product differentiation. Defining $q_1, q_2 \geq 0$ and $p_1, p_2 \geq 0$ as the quantities and prices of the final products 1 e 2, respectively, $x_1, x_2 \geq 0$ as the total quantities of properties 1 e 2, $a \in [0, 1]$ as the proportion of the property 1 (2) in the final product 1 (2), and $1 - a$ as the proportion of the property 1 (2) in the final product 2 (1), the Lancaster transformation of properties in final products is given by the following linear transformation:

$$\begin{aligned} x_1 &= aq_1 + (1-a)q_2, \\ x_2 &= (1-a)q_1 + aq_2. \end{aligned} \tag{1}$$

Here we propose the introduction of subjective properties differentiation in the consumer's problem through a symmetrical CES utility function dependent on the quantities of properties consumed. In this way, if its income is given by $I > 0$, the consumer aims to maximize this CES utility function subject to its budget constraint, $p_1q_1 + p_2q_2 = I$, and to the Lancaster linear transformation (1), as follows:

$$\begin{aligned} \max\ & U(x_1,x_2) = \left(\frac{1}{2}x_1^\delta + \frac{1}{2}x_2^\delta\right)^{\frac{1}{\delta}},\ \delta \le 1 \\ \text{s.t.}\ & x_1 = aq_1 + (1-a)q_2,\ 0 \le a \le 1 \\ & x_2 = (1-a)q_1 + aq_2 \\ & p_1q_1 + p_2q_2 = I,\ p_1,\ p_2,\ I > 0. \end{aligned} \tag{2}$$

This model is a generalization of the model proposed by Juchem Neto (2023), since the symmetrical CES utility function considered here includes the symmetrical Cobb-Douglas utility function considered in that work as a particular case when $\delta \to 0$, in which case the elasticity of substitution between properties 1 and 2 is perceived as unitary[2]. Indeed, the utility function considered here presents the following convenient limiting cases (Nicholson; Snyder, 2012):

$$U(x_1,x_2) = \left(\frac{1}{2}x_1^\delta + \frac{1}{2}x_2^\delta\right)^{\frac{1}{\delta}} \to \begin{cases} \frac{1}{2}x_1 + \frac{1}{2}x_2, \text{ if } \delta = 1 \\ \sqrt{x_1x_2}, \text{ as } \delta \to 0 \text{ (symmetrical Cobb-Douglas).} \\ \min\{x_1,x_2\}, \text{ as } \delta \to -\infty \end{cases} \tag{3}$$

Since this CES utility function is modeling the consumer's preferences for the properties, in this work we interpret its exogenous parameter $\delta \le 1$ as a measure of how the consumer subjectively perceives the distinction between properties 1 and 2: if $\delta = 1$, he perceives both properties as perfect substitutes, or identical; if $\delta \to -\infty$, he perceives both properties as perfect complements, or completely differentiated or independent. Besides, the lower the value of the parameter δ is, the more differentiated the consumer perceives properties 1 and 2. This parameter could be influenced by advertising, for example.

[2]The elasticity of substitution between properties 1 and 2 for a CES utility function is given by $\rho = \frac{1}{1-\delta}$ (Nicholson; Snyder, 2012, p. 105).

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Moreover, the exogenous parameter $a \in [0,1]$, present in the Lancaster transformation (1), is associated with the objective (construction) differentiation of the products 1 and 2, since it informs the physical proportion of each property used in the production of each of them. The value of this parameter, that gives the design of both products, is defined a priori by the firms. Distinct values for a could possibly imply distinct costs of production, although we do not consider such subtleties in this work. If $a = \frac{1}{2}$, the final products 1 and 2 are objectively identical, while if $a = 0$ or $a = 1$, they are objectively completely differentiated. Besides, the further away from $1/2$ the value of a is, the more objectively differentiated the products 1 and 2 are.

3 Lancaster demand functions

Following Nicholson and Snyder (2012), it is algebraically easier to solve the consumer's problem (2) considering the following monotonic transformation of the CES utility function:

$$\tilde{U}(x_1, x_2) = \frac{2}{\delta} U(x_1, x_2)^\delta = \frac{x_1^\delta}{\delta} + \frac{x_2^\delta}{\delta}. \tag{4}$$

In this way, the Lagrangian of this problem can be written as:

$$\mathcal{L} = \frac{x_1^\delta}{\delta} + \frac{x_2^\delta}{\delta} - \lambda_1 \left(x_1 - aq_1 - (1-a)q_2\right) - \lambda_2 \left(x_2 - (1-a)q_1 - aq_2\right) - \lambda_3 \left(p_1 q_1 + p_2 q_2 - I\right).$$

The objective function (4) is concave, and the constraints $x_1 \le aq_1 + (1-a)q_2$, $x_2 \le (1-a)q_1 - aq_2$, and $p_1q_1 + p_2q_2 \le I$ are linear in the variables (x_1, x_2, q_1, q_2). If the Lagrange multipliers λ_1, λ_2, and λ_3 are positive, the points that satisfy the first order conditions of the Lagrangian are such that $x_1 = aq_1 + (1-a)q_2$, $x_2 = (1-a)q_1 - aq_2$, $p_1q_1 + p_2q_2 = I$, and so they are maxima for the problem (2). Then, the following first order conditions for this problem are sufficient for a maximum, provided $\lambda_1,\ \lambda_2,\ \lambda_3 > 0$ (Araújo, 2022; Sundaran, 2011):

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_1^{\delta-1} - \lambda_1 = 0, \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_2^{\delta-1} - \lambda_2 = 0, \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = aq_1 + (1-a)q_2 - x_1 = 0, \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = aq_2 + (1-a)q_1 - x_2 = 0, \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_3} = I - p_1q_1 - p_2q_2 = 0, \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial q_1} = a\lambda_1 + (1-a)\lambda_2 - p_1\lambda_3 = 0, \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = (1-a)\lambda_1 + a\lambda_2 - p_2\lambda_3 = 0. \tag{11}$$

In the following we solve the system (5)-(11) in order to obtain the consumer's inverse and direct product demand functions, the properties demand functions, and their shadow prices.

3.1 Inverse Lancaster demand functions

To obtain the consumer's inverse demand functions (products' prices as a function of their quantities), which in fact we will use in solving the Cournot duopoly in the Section 4 below, first note that from equations (10)-(11) we get:

$$p_1 q_1 = \frac{\lambda_1}{\lambda_3} a q_1 + \frac{\lambda_2}{\lambda_3}(1-a) q_1, \tag{12}$$

$$p_2 q_2 = \frac{\lambda_1}{\lambda_3}(1-a) q_2 + \frac{\lambda_2}{\lambda_3} a q_2. \tag{13}$$

Adding equations (12) and (13), and using (9), we obtain:

$$I = p_1 q_1 + p_2 q_2 = \frac{\lambda_1}{\lambda_3}\left(a q_1 + (1-a) q_2\right) + \frac{\lambda_2}{\lambda_3}\left((1-a) q_1 + a q_2\right). \tag{14}$$

Plugging (7)-(8) into (5)-(6) yields the shadow (or imputed) prices of properties 1 and 2 as functions of the quantities q_1 and q_2, which are given by the Lagrange multipliers λ_1 and λ_2, respectively:

$$\lambda_1(q_1, q_2) = \left(a q_1 + (1-a) q_2\right)^{\delta - 1} > 0, \tag{15}$$

$$\lambda_2(q_1, q_2) = \left((1-a) q_1 + a q_2\right)^{\delta - 1} > 0. \tag{16}$$

Substituting (15)-(16) into (14), we get:

$$\lambda_3(q_1, q_2) = \frac{\delta U}{I} = \frac{x_1^\delta + x_2^\delta}{I} = \frac{\left(a q_1 + (1-a) q_2\right)^\delta + \left((1-a) q_1 + a q_2\right)^\delta}{I} > 0. \tag{17}$$

Finally, applying (15), (16) and (17) in equations (10) and (11), and isolating p_1 and p_2, we get the final products Lancaster inverse demand functions:

$$p_1(q_1, q_2) = I\left[\frac{a\left(a q_1 + (1-a) q_2\right)^{\delta-1} + (1-a)\left((1-a) q_1 + a q_2\right)^{\delta-1}}{\left(a q_1 + (1-a) q_2\right)^\delta + \left((1-a) q_1 + a q_2\right)^\delta}\right], \tag{18}$$

$$p_2(q_1, q_2) = I\left[\frac{(1-a)\left(a q_1 + (1-a) q_2\right)^{\delta-1} + a\left((1-a) q_1 + a q_2\right)^{\delta-1}}{\left(a q_1 + (1-a) q_2\right)^\delta + \left((1-a) q_1 + a q_2\right)^\delta}\right]. \tag{19}$$

In addition, note that the properties demand functions $x_1(q_1, q_2)$ and $x_2(q_1, q_2)$ are given by the Lancaster linear transformation itself (1).

3.2 **Direct Lancaster demand functions**

In order to obtain the consumer's direct demand functions for products 1 and 2 (product's quantities as a function of their prices), we start solving the linear system (7)-(8) for the quantities q_1 and q_2:

$$q_1 = \frac{a}{2a-1}x_1 - \frac{(1-a)}{2a-1}x_2, \tag{20}$$

$$q_2 = \frac{a}{2a-1}x_2 - \frac{(1-a)}{2a-1}x_1. \tag{21}$$

Now, solving the linear system (10)-(11) for λ_1 and λ_2, we obtain:

$$\lambda_1 = \frac{\lambda_3}{2a-1}\left(ap_1 - (1-a)p_2\right), \tag{22}$$

$$\lambda_2 = \frac{\lambda_3}{2a-1}\left(ap_2 - (1-a)p_1\right). \tag{23}$$

Applying (22)-(23) and (5)-(6) into (20)-(21), we get:

$$q_1 = \frac{(2a-1)^{\frac{\delta}{1-\delta}}}{\lambda_3^{\frac{1}{1-\delta}}}\left[a\left(ap_1 - (1-a)p_2\right)^{\frac{1}{\delta-1}} - (1-a)\left(ap_2 - (1-a)p_1\right)^{\frac{1}{\delta-1}}\right], \tag{24}$$

$$q_2 = \frac{(2a-1)^{\frac{\delta}{1-\delta}}}{\lambda_3^{\frac{1}{1-\delta}}}\left[a\left(ap_2 - (1-a)p_1\right)^{\frac{1}{\delta-1}} - (1-a)\left(ap_1 - (1-a)p_2\right)^{\frac{1}{\delta-1}}\right]. \tag{25}$$

Substituting (5)-(6) in (17), and considering (22)-(23) we get that:

$$\frac{(2a-1)^{\frac{\delta}{1-\delta}}}{\lambda_3^{\frac{1}{1-\delta}}} = \frac{I}{\left(ap_1 - (1-a)p_2\right)^{\frac{\delta}{1-\delta}} + \left(ap_2 - (1-a)p_1\right)^{\frac{\delta}{1-\delta}}}, \tag{26}$$

what implies that:

$$\lambda_3(p_1, p_2) = \frac{(2a-1)^{\delta}}{I^{1-\delta}}\left[\left(ap_1 - (1-a)p_2\right)^{\frac{\delta}{1-\delta}} + \left(ap_2 - (1-a)p_1\right)^{\frac{\delta}{1-\delta}}\right]^{1-\delta}. \tag{27}$$

Therefore, plugging (26) in (24) and (25), we get the Lancaster direct demand functions for the final products[3]:

$$q_1 = I\left[\frac{a\left(ap_1 - (1-a)p_2\right)^{\frac{1}{\delta-1}} - (1-a)\left(ap_2 - (1-a)p_1\right)^{\frac{1}{\delta-1}}}{\left(ap_1 - (1-a)p_2\right)^{\frac{\delta}{1-\delta}} + \left(ap_2 - (1-a)p_1\right)^{\frac{\delta}{1-\delta}}}\right], \tag{28}$$

$$q_2 = I\left[\frac{a\left(ap_2 - (1-a)p_1\right)^{\frac{1}{\delta-1}} - (1-a)\left(ap_1 - (1-a)p_2\right)^{\frac{1}{\delta-1}}}{\left(ap_1 - (1-a)p_2\right)^{\frac{\delta}{1-\delta}} + \left(ap_2 - (1-a)p_1\right)^{\frac{\delta}{1-\delta}}}\right]. \tag{29}$$

[3]As noted by Puu (2018) for the case considering the symmetrical Cobb-Doublas utility function, here the Lancaster direct demands are also less well-behaved than the inverse demands (18)-(19). For example, the demands (28)-(29) will not result in non-negative quantities for all values of the ratio p_1/p_2. Since we will not use these direct demands in the proposed Cournot duopoly in the following, we will not pursue this case in further details here.

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Finally, the shadow prices of properties 1 and 2 as a function of the final product's prices, $\lambda_1(p_1, p_2)$ and $\lambda_2(p_1, p_2)$, can be obtained substituting (27) in (22)-(23); then plugging this shadow prices in (5)-(6) we can get the properties demand functions $x_1(p_1, p_2)$ and $x_2(p_1, p_2)$.

3.3 Limiting cases for the inverse Lancaster demand functions

Some limiting cases of interest are when the products are objectively identical ($a = 1/2$) or completely differentiated ($a = 0$ or $a = 1$), and when the properties are perceived as identical ($\delta = 1$), when their elasticity of substitution is perceived as unitary ($\delta \to 0$), or when they are perceived as completely differentiated ($\delta \to -\infty$). Below we illustrate these limiting cases for the final products inverse demand functions (18)-(19), since these are the demand functions that we will use in solving the Cournot duopoly in Section 4. Similar limiting results can easily be obtained for the direct demand functions (28)-(29).

On one hand, if both properties are subjectively perceived as identical ($\delta = 1$), *ceteris paribus*, then (18) and (19) reduce to:

$$p_1(q_1, q_2) = p_2(q_1, q_2) = \frac{I}{q_1 + q_2}, \tag{30}$$

regardless of the objective product differentiation given by the parameter a. On the other hand, if both products are objectively identical ($a = 1/2$), *ceteris paribus*, then (18) and (19) reduce to the same expression (30):

$$p_1(q_1, q_2) = p_2(q_1, q_2) = \frac{I}{q_1 + q_2}, \tag{31}$$

regardless of the subjective product differentiation given by the parameter δ. In both cases, (30)-(31), the prices depend only on the total quantity consumed ($q_1 + q_2$), or vice-versa. This result is consistent with the results of the classical Cournot duopoly considering homogeneous products (Nicholson; Snyder, 2012; Puu, 2018).

In the particular case where the elasticity of substitution between properties are perceived as unitary, $\delta \to 0$, *ceteris paribus*, (18) and (19) reduces to the inverse demand functions:

$$p_1(q_1, q_2) = \frac{I}{2}\left[\frac{a}{aq_1 + (1-a)q_2} + \frac{1-a}{(1-a)q_1 + aq_2}\right], \tag{32}$$

$$p_2(q_1, q_2) = \frac{I}{2}\left[\frac{1-a}{aq_1 + (1-a)q_2} + \frac{a}{(1-a)q_1 + aq_2}\right], \tag{33}$$

which are the ones obtained in the particular model considering the symmetrical Cobb-Douglas utility function (Juchem Neto, 2023).

When both properties are subjectively perceived as completely differentiated ($\delta \to -\infty$), *ceteris paribus*, then (18) and (19) reduces to following inverse demand functions defined by parts:

$$p_1(q_1,q_2) = \begin{cases} I\left[\dfrac{a}{aq_1+(1-a)q_2}\right], \text{ if } q_1 < q_2 \\ \dfrac{I}{2}\left[\dfrac{a}{aq_1+(1-a)q_2} + \dfrac{1-a}{(1-a)q_1+aq_2}\right], \text{ if } q_1 = q_2, \\ I\left[\dfrac{1-a}{(1-a)q_1+aq_2}\right], \text{ if } q_1 > q_2 \end{cases} \tag{34}$$

$$p_2(q_1,q_2) = \begin{cases} I\left[\dfrac{1-a}{aq_1+(1-a)q_2}\right], \text{ if } q_1 < q_2 \\ \dfrac{I}{2}\left[\dfrac{a}{aq_1+(1-a)q_2} + \dfrac{1-a}{(1-a)q_1+aq_2}\right], \text{ if } q_1 = q_2, \\ I\left[\dfrac{a}{(1-a)q_1+aq_2}\right], \text{ if } q_1 > q_2 \end{cases} \tag{35}$$

which depend only on the objective product differentiation parameter, a. On the other hand, when both products are completely objectivelly differentiated ($a = 0$ or $a = 1$), *ceteris paribus*, then (18) and (19) reduce to the following:

$$p_1(q_1,q_2) = \frac{I}{q_1}\left[\frac{1}{1+\left(\frac{q_2}{q_1}\right)^{\delta}}\right], \quad p_2(q_1,q_2) = \frac{I}{q_2}\left[\frac{1}{1+\left(\frac{q_1}{q_2}\right)^{\delta}}\right], \tag{36}$$

and the inverse demand functions depend only on the subjective product differentiation parameter δ.

Finally, in the case where we have complete subjective and objective differentiation, i.e $a = 0$ (or $a = 1$) and $\delta \to -\infty$, from (34)-(35) and (36) we have that:

$$p_1(q_1,q_2) = p_1(q_1) = \begin{cases} \dfrac{I}{q_1}, \text{ if } q_1 < q_2 \\ \dfrac{I}{2q_1}, \text{ if } q_1 = q_2 \\ 0, \text{ if } q_1 > q_2 \end{cases}, \quad p_2(q_1,q_2) = p_2(q_2) = \begin{cases} 0, \text{ if } q_1 < q_2 \\ \dfrac{I}{2q_2}, \text{ if } q_1 = q_2. \\ \dfrac{I}{q_2}, \text{ if } q_1 > q_2 \end{cases} \tag{37}$$

and so, in the regions where the prices are positive, the final product inverse demand functions are independent of each other: p_1 depends only on q_1 and vice-versa; and p_2 depends only on q_2, and vice-versa.

4 Cournot duopoly with objective and subjective product differentiation

A Cournot duopoly is an imperfectly competitive market structure, where two profit maximizing firms (the duopolists) simultaneously choose the quantities of product to be produced, considering each other's best response (Nicholson; Snyder, 2012). Following Juchem Neto (2023) and Puu (2017, 2018), we will consider a Cournot duopoly where the firm 1 (2) produces only the final product 1 (2), both using some proportion of properties 1 and 2 given by the parameter a. Moreover, we consider that both firms have the same linear cost function, with constant marginal cost $c > 0$, and no fixed costs: $C_i(q_i) = cq_i,\ i = 1, 2$. In this way, the profit functions of the firms, $\pi_i(q_1, q_2),\ i = 1, 2$, can be written as:

$$\begin{aligned} \pi_1(q_1, q_2) &= (p_1(q_1, q_2) - c)q_1, \\ \pi_2(q_1, q_2) &= (p_2(q_1, q_2) - c)q_2, \end{aligned} \tag{38}$$

where $p_i(q_1, q_2),\ i = 1, 2$, are the inverse demand functions (18)-(19). The first order conditions for the maximization of profits (38), with each firm considering the output decision of its competitor as given, are the following:

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Leftrightarrow p_1 + q_1 \frac{\partial p_1}{\partial q_1} = c, \tag{39}$$

$$\frac{\partial \pi_2}{\partial q_2} = 0 \Leftrightarrow p_2 + q_2 \frac{\partial p_2}{\partial q_2} = c. \tag{40}$$

Equation (39) gives, implicitly, the best response curve for firm 1, $q_1^*(q_2)$, while (40) gives, also implicitly, the best response curve for firm 2, $q_2^*(q_1)$. Unfortunately, given the nature of the inverse demand functions (18)-(19), it is not possible to obtain, in general, these best response curves explicitly. However, since in this paper we are considering a symmetric CES utility function in the consumer problem (2), and equal cost functions for both firms, the equilibrium of the Cournot duopoly – which is obtained finding the quantities (q_1^*, q_2^*) that solve (39)-(40) simultaneously – will be symmetric, i.e. $q_1^* = q_2^* = q^* = x^*$, what implies that $p_1^* = p_2^* = p^*$, and $\pi_1^* = \pi_2^* = \pi^*$. Using this facts in (18)-(19) and (39)-(40), after some algebra we can analytically obtain the Cournot equilibrium of the duopoly, which is given by:

$$q^* = \frac{I}{4c}\left[2 + 2(\delta - 1)(a^2 + (1 - a)^2) - \delta\right] = x^*, \tag{41}$$

$$p^* = \frac{2c}{2 + 2(\delta - 1)(a^2 + (1 - a)^2) - \delta}, \tag{42}$$

$$\pi^* = \frac{I}{4}\left[\delta - 2(\delta - 1)(a^2 + (1 - a)^2)\right]. \tag{43}$$

Besides, we also get the equilibrium value for the shadow price of the properties:

$$\lambda^* = \left\{\frac{4c}{I[2 + 2(\delta - 1)(a^2 + (1 - a)^2) - \delta]}\right\}^{1-\delta}, \tag{44}$$

and the optimal value of λ_3:

$$\lambda_3^* = \frac{2}{I^{1-\delta}}\left[\frac{2 + 2(\delta - 1)(a^2 + (1 - a)^2) - \delta}{4c}\right]^{\delta}. \tag{45}$$

Now note that the Cournot equilibrium (41)-(43) is well defined only when $q^* \geq 0$, since q^* is the quantity of the final product produced by each firm. Then, given a degree of subjective differentiation $\delta \leq 1$, from (41) this will be the case when:

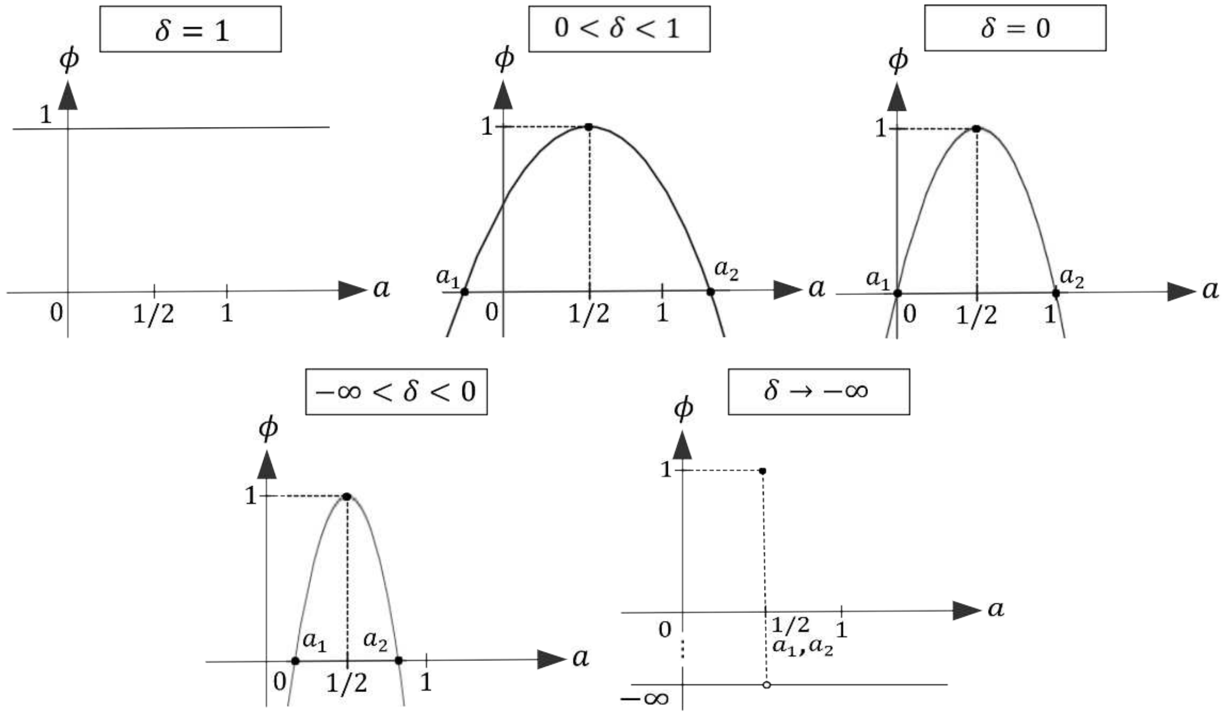
$$\phi(a) = 4(\delta - 1)a^2 - 4(\delta - 1)a + \delta \geq 0, \tag{46}$$

what happens when the objective differentiation parameter a is in the closed interval $[\max\{0, a_1\}, \min\{1, a_2\}]$, where:

$$a_1 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1-\delta}}\right) \text{ and } a_2 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\delta}}\right). \tag{47}$$

In Figure 1 we show the interval of the objective product differentiation parameter, a, for which the Cournot equilibrium is well defined, considering various cases for the values of subjective properties differentiation, δ. As we can see, if the consumer sees both properties as homogeneous ($\delta = 1$) or slightly differentiated ($0 \leq \delta < 1$), then the equilibrium exists for all possible values of objective differentiation, i.e. for $0 \leq a \leq 1$. Otherwise, if the consumer perceives the properties as highly differentiated, $\delta < 0$, then the equilibrium is well defined only for a narrower range of the objective differentiation parameter values, $0 < a_1 \leq a \leq a_2 < 1$, where the products are objectively less differentiated.

Figura 1: Given a subjective differentiation parameter $\delta \leq 1$, the Cournot equilibrium is well defined only for values of objective differentiation a where the parabola $\phi(a)$ assumes non-negative values.



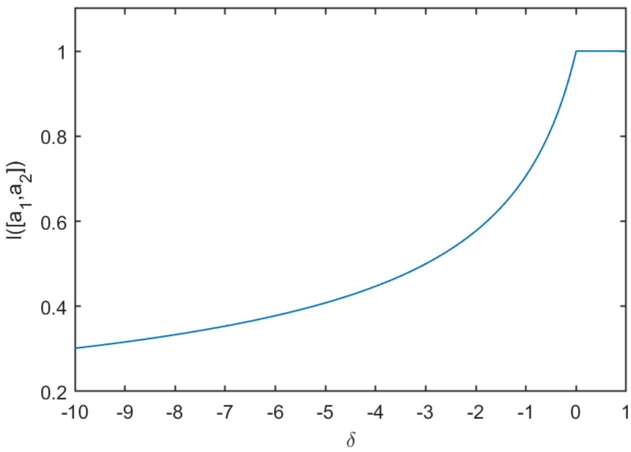
Source: Elaborated by the authors.

In fact, the length of the interval $[a_1, a_2]$, given by:

$$\ell([a_1, a_2]) = \min\{a_2 - a_1, 1\} = \min\left\{\frac{1}{\sqrt{1-\delta}}, 1\right\}, \tag{48}$$

which is illustrated graphically in Figure 2, can be thought as a measure of the possibilities of objective product differentiation in the Cournot duopoly, as a function of the subjective differentiation parameter, δ: if $\ell([a_1, a_2]) = 0$, the products can only be objectively identical; if $\ell([a_1, a_2]) = 1$, the products can be objectively differentiated within the maximum range of possibilities; finally, if $0 < \ell < 1$, then the products can be partially, but not completely, objectively differentiated. Economically, this means that, in the presence of highly subjective differentiation by the consumer, the duopoly has a well defined equilibrium only when the products are objectively less differentiated, i.e. it does not make sense to objectively differentiate the products too much, since the consumer already perceives their properties as very differentiated.

Figura 2: Possibilities of objective product differentiation in a Cournot duopoly, as a function of the subjective differentiation parameter, $\ell([a_1, a_2]) = \min\left\{\frac{1}{\sqrt{1-\delta}}, 1\right\}$.



Source: Elaborated by the authors.

In the limit when $\delta \to -\infty$, $\ell([a_1, a_2])$ shrinks to zero, and the Cournot equilibrium is defined only for objectively identical products ($a = \frac{1}{2}$), which is given by $q^* = \frac{I}{4c}$, $p^* = 2c$, and $\pi^* = \frac{I}{4}$, limiting case that is also illustrated in Figure 1. The analysis above can be summarized as the following Proposition 1. Below we use the up arrow ($\uparrow$) to denote a marginal (or small) increase, and the down arrow ($\downarrow$) to denote a marginal decrease in the corresponding variable.

Proposition 1: If the properties are perceived by the consumer as highly differentiated, $\delta < 0$, then a higher subjective properties differentiation ($\downarrow \delta$) implies that the Cournot duopoly has a well defined equilibrium only for a smaller range of objective product differentiation ($\downarrow \ell([a_1, a_2])$).

4.1 Impact of a marginal increase in the subjective and/or objective differentiation on the Cournot equilibrium

In the following we obtain some results about the impact of a marginal increase in the subjective and/or objective differentiation of properties and products on the Cournot equilibrium of the duopoly.

Proposition 2 (impact of a subjective differentiation increase): If the final products are objectively differentiated ($a \neq 1/2$), then an increase in the subjective differentiation of the properties ($\downarrow \delta$), *ceteris paribus*, causes: (i) a decrease in the equilibrium quantities of the final products ($q^* \downarrow$), and of the properties ($x^* \downarrow$); (ii) an increase in the equilibrium prices ($p^* \uparrow$); and (iii) an increase in the equilibrium profits of the duopolists ($\pi^* \uparrow$). If the final products are objectively identical ($a = 1/2$), then changes in the properties' subjective differentiation (δ) have no effect in the equilibrium values of q^*, x^*, p^*, and π^*; in this case, $q^* = x^* = \frac{I}{4c}$, $p^* = 2c$, and $\pi^* = \frac{I}{4}$.
Proof: It is enough to note that: (i) $\frac{\partial q^*}{\partial \delta} = \frac{\partial x^*}{\partial \delta} = \frac{I}{4c}[2(a^2+(1-a)^2)-1]$ is positive for $a \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$; (ii) $\frac{\partial p^*}{\partial \delta} = \frac{2c[1-2(a^2+(1-a)^2)]}{[2+2(\delta-1)(a^2+(1-a)^2)-\delta]^2}$ is negative for $a \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$; and (iii) $\frac{\partial \pi^*}{\partial \delta} = \frac{I}{4}[1-2(a^2+(1-a)^2)]$ is also negative for $a \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Besides, note that the partial derivatives above are zero for $a = 1/2$. Finally, letting $a = 1/2$ in (41), (42), and (43), the result follows. □

Proposition 3 (impact of an objective differentiation increase): If the final products are objective and subjectively differentiated ($a \neq 1/2$ and $\delta < 1$), then an increase in the objective differentiation, *ceteris paribus*, causes: (i) a decrease in the equilibrium quantities of the final products ($q^* \downarrow$) and of the properties ($x^* \downarrow$); (ii) an increase in the equilibrium prices ($p^* \uparrow$); and (iii) an increase in the equilibrium profits of the duopolists ($\pi^* \uparrow$). If the final products are objective or subjectively identical ($a = 1/2$ or $\delta = 1$), then changes in the objective differentiation parameter, a, have no effect in q^*, x^*, p^*, and π^*; in this case, $q^* = x^* = \frac{I}{4c}$, $p^* = 2c$, and $\pi^* = \frac{I}{4}$.
Proof: The result follows from the fact that: (i) $\frac{\partial q^*}{\partial a} = \frac{\partial x^*}{\partial \delta} = \frac{I}{c}(\delta-1)(2a-1)$ is negative for $a \in (\frac{1}{2}, 1]$ and $\delta < 1$, and positive for $a \in \left[0, \frac{1}{2}\right)$ and $\delta < 1$; (ii) $\frac{\partial p^*}{\partial a} = \frac{-8c(\delta-1)(2a-1)}{[2+2(\delta-1)(a^2+(1-a)^2)-\delta]^2}$ is positive for $a \in (\frac{1}{2}, 1]$ and $\delta < 1$, and negative for $a \in \left[0, \frac{1}{2}\right)$ and $\delta < 1$; and (iii) $\frac{\partial \pi^*}{\partial a} = -I(\delta-1)(2a-1)$ is also

JUCHEM NETO, João Plínio; ARAÚJO, Jorge Paulo de; WIEHE, Martin. Objective and subjective product differentiation in a Cournot duopoly. **REMAT**: Revista Eletrônica da Matemática, Bento Gonçalves, RS, v. 10, n. 2, p. e3010, December 17, 2024. https://doi.org/10.35819/remat2024v10i2id7367.

positive for $a \in (\frac{1}{2}, 1]$ and $\delta < 1$, and negative for $a \in \left[0, \frac{1}{2}\right)$ and $\delta < 1$. Finally, note that the partial derivatives above are zero if $a = \frac{1}{2}$ or $\delta = 1$. Then, letting $a = 1/2$ in (41), (42), and (43), the result follows. □

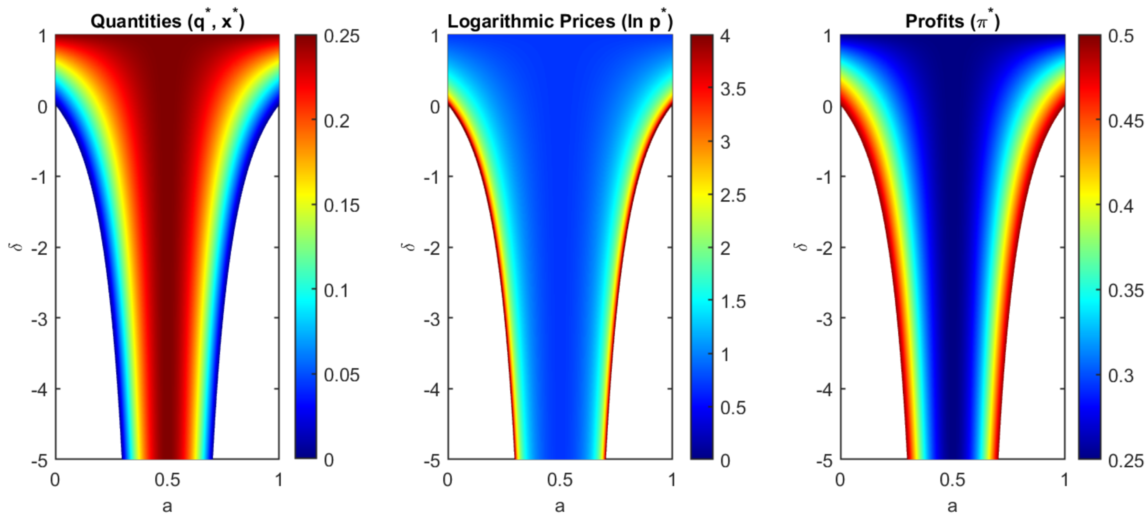
Economically, the results obtained in propositions 2 and 3 imply that an increase in the consumers' subjective differentiation of properties and/or an increase in the objective differentiation of the final products end up increasing the monopoly power of the firms, what means that, with a higher differentiation, the firms can produce smaller quantities of the final products, charge higher prices, and earn higher profits. In this way, from the point of view of the duopolists, it makes sense to increase their monopoly power, since this increases their profits. On the contrary, from the consumers' point of view, a higher monopoly power of the firms implies a lower consumer well-being, since they have to pay higher prices and thus can buy lower quantities of the products. Nevertheless, since the present work focus on the firms, these results show that the firms have two instruments in order to increase their monopoly power: either they intensify the objective differentiation of their products, changing the parameter a, which would require changes in the design of the products, and consequently in the production process, or they intensify the subjective differentiation of the consumers, decreasing the parameter δ, which would require investments in persuasive/informative advertising. It is up to the duopolists to do a cost-benefit analysis of what would be the best strategy to follow in order to increase their monopoly power. These results go in line with the ones found in the literature: using a different approach, Lauga, Ofek and Katona (2022) found that in a scenario where the informative advertising is very cost effective, the duopolists end up advertising and differentiating the quality of the products quite intensely.

4.2 A numerical example

In Figure 3 we present color maps of the Cournot equilibrium, (41)-(43), considering the particular case when $I = c = 1$, $0 \leq a \leq 1$, and $-5 \leq \delta \leq 1$. The white regions in these color maps indicate the set of parameter for which the duopoly has no optimal solution. As we can see, all the results presented in propositions 1, 2 and 3 can be verified in this figure.

In Figure 4 we present the equilibrium quantities, prices, and profits graphs of the duopoly, as a function of the objective differentiation parameter a, considering $\delta = 1, 0.5, 0, -1, -5$, what constitutes another way of presenting the results. In particular, let us focus in the impact of increasing subjective differentiation ($\downarrow \delta$) in the duopolist's profits, as presented in the third column of the Figure 4. For a given degree of objective differentiation $a \neq 1/2$, note that a higher degree of subjective differentiation ($\downarrow \delta$) implies higher profits. When compared with the results considering a symmetrical Cobb-Douglas utility function (Juchem Neto, 2023), as presented in the third line of the figure, we see that, for a given $a \neq 1/2$ the profit levels are lower when the subjective differentiation is less intense ($\delta > 0$), and higher when the subjective differentiation is more intense ($\delta < 0$). For this last case, the admissible range of objective product differentiation is narrower, what suggests that, if we include objective differentiation costs in the model, this last scenario would be even more advantageous for the duopolists, since a higher profit level is attainable without so much objective product differentiation.

Figura 3: Cournot equilibrium color maps as function of the objective, $0 \leq a \leq 1$, and subjective differentiation parameters, $-5 \leq \delta \leq 1$, considering $I = 1$, $c = 1$.



Source: Elaborated by the authors.

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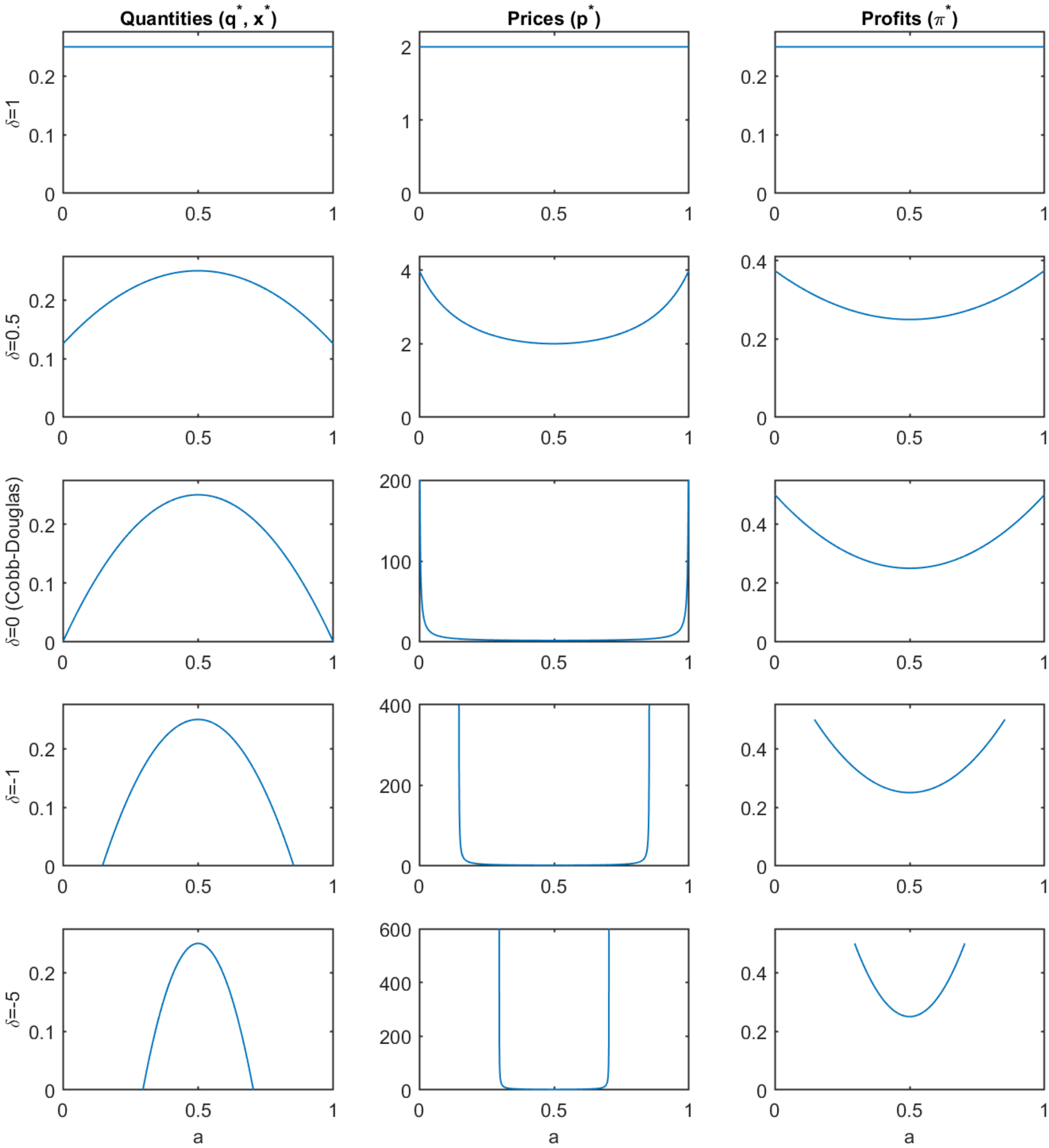
Figura 4: Cournot equilibrium graphs as function of the objective differentiation parameter, $0 \leq a \leq 1$, considering $\delta = 1, 0.5, 0, -1, -5$, $I = 1$, and $c = 1$.



Source: Elaborated by the authors.

5 Final remarks

In this paper we have obtained the equilibrium quantities, prices, and profits of a Cournot duopoly in a market with two objectively differentiated products composed of two properties, considering that the consumers subjectively differentiate these properties. We have shown that if the properties are perceived by the consumer as highly differentiated, then a higher subjective properties differentiation implies that the Cournot duopoly only has an equilibrium solution for a smaller range of objective product differentiation. In addition, we have shown that an increase in the consumers' subjective differentiation of properties (e.g. via advertising) and/or in the objective product differentiation (e.g. via changes in product design) increases the monopoly power of the duopolists, that is, with a higher objective and/or subjective differentiation, the firms produce smaller quantities of products, charge higher prices and earn higher profits. Besides, we have presented a numerical example of these results, which go in line with the literature, but considering a more realistic way to differentiate products, following the Lancaster's approach.

In order to analyse the proposed Courtot duopoly with product differentiation, first we have introduced subjective properties differentiation into the Lancaster consumer's choice problem, through a symmetrical constant elasticity of substitution (CES) utility function, considering a market with two products characterized by two intrinsic properties. We have solved this problem analytically through the Lagrange multipliers method, obtained the final products direct and inverse Lancaster demand functions, the properties demand functions, and their shadow (or imputed) prices.

Moreover, we have analyzed the following limiting cases for the product inverse demand functions: when the products are objectively or subjectively identical, the prices depend on the total quantity consumed, which is a result consistent with what is expected for identical products, as in the classical Cournot duopoly; in the particular case where the elasticity of substitution between properties is perceived as unitary, the inverse demand functions reduce to the ones implied by a symmetrical Cobb-Douglas utility function, as considered in previous works; when both properties are subjectively perceived as completely differentiated, then the limiting inverse demands depend only on the objective product differentiation parameter; when both products are completely objectively differentiated, the inverse demand functions depend only on the subjective product differentiation parameter; and finally, in the case where there are complete subjective and objective differentiation, the final product inverse demand functions are independent of each other in the regions where the prices are positive, i.e. p_1 depends only on q_1 and vice-versa, and p_2 depends only on q_2, and vice-versa.

Future research, still from a theoretical point of view, may focus in explicitly introducing advertising investments by the duopolists in the model, aiming to influence the perception of differentiation between properties by the consumers, and also considering objective differentiation costs. In addition, future works may also compare the theoretical results obtained here with empirical data.

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