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Essays on Liquidity-Constrained Portfolio Optimization

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Abstract

This dissertation focuses on the study of the portfolios liquidity and how liquidity can be inserted into portfolio optimization models. The work is composed by the following articles: “Liquidity constraints for portfolio selection based on financial volume”; "Liquidity-constrained index tracking optimization models"; “Portfolio Optimization and Liquidity: a Comparison Between Individual Constraints and Portfolio Constraints”.

The first article aims to discuss the implementation of liquidity constraints in a portfolio selection model. It presents the main liquidity metrics and the main ways to insert liquidity as a constraint in the optimization model. A liquidity constraint is designed using the financial volume as the liquidity metric. The proposed constraint is empirically tested in various scenarios. Tests are carried out in three emerging markets: Brazil, Mexico and Turkey. The second article aims to expand the studies on the proposed liquidity constraint. The methodology is implemented in a Index Tracking model. Several empirical tests are carried out with the objective of analyzing the portfolios behavior with the imposed liquidity constraint. Two Brazilian market indices are considered: Ibovespa and SMLL, the main Brazilian market index and an index composed of less liquid assets, respectively. In addition, the portfolios diversification levels are studied, through two indexes: Herfindahl-Hirschman index and Gini index. In the third article, additional empirical tests are performed, comparing the proposed constraint with a traditional way of inserting liquidity constraints: individual constraints on each participating asset. Differently from the previous articles, the two compared constraints have the same liquidity definition, thus making it possible to directly compare the results. The proposed liquidity constraint results showed good consistency in all articles. Comparisons with other approaches showed good performance, taking into account factors such as diversification and risk of portfolios constrained by liquidity.

Keywords: Liquidity. Portfolio Optimization. Liquidity Constraints

Resumo

Esta tese tem como foco o estudo da liquidez de portfolios de investimentos e como a liquidez pode ser inserida em modelos de otimização de portfolios. O trabalho é composto pelos seguintes artigos: “Liquidity constraints for portfolio selection based on financial volume”; “Liquidity-constrained index tracking optimization models”; “Portfolio Optimization and Liquidity: a Comparison Between Individual Constraints and Portfolio Constraints”.

O primeiro artigo tem como objetivo discutir a implementação de restrições de liquidez em um modelo de seleção de portfolios. São apresentadas as principais métricas de liquidez e principais formas de inserir a liquidez como uma restrição no modelo de otimização. É realizado o desenvolvimento de uma restrição de liquidez baseada no volume financeiro como métrica de liquidez. A restrição proposta é testada empiricamente nos mais variados cenários. Os testes são realizados em três mercados emergentes: Brasil, México e Turquia. O segundo artigo visa expandir os estudos a respeito da restrição de liquidez proposta. A metodologia é implementada em um modelo de *Index Tracking*, sendo realizados diversos testes empíricos com o objetivo de analisar o comportamento dos portfolios formados com a restrição de liquidez imposta. Dois índices de mercado brasileiros são considerados: Ibovespa e SMLL, principal índice de mercado brasileiro e índice composto por ativos com menor liquidez, respectivamente. Além disso, são estudados os níveis de diversificação dos portfolios formados, através de dois índices: índice de Herfindahl-Hirschman e índice Gini. No terceiro artigo testes empíricos adicionais são performados, comparando a restrição proposta com uma outra forma tradicional de se inserir restrições de liquidez: restrições individuais em cada ativo participante. Diferentemente dos artigos anteriores, a definição de liquidez das duas restrições comparadas é a mesma, tornando possível assim a comparação direta dos resultados. Os resultados dos testes da restrição de liquidez proposta demonstraram boa consistência em todos os artigos. Comparações realizadas com outros approaches mostraram boa performance, levando em consideração fatores como diversificação e risco dos portfolios restritos pela liquidez.

Palavras-chave: Liquidez. Otimização de Portfolio. Restrições de liquidez

Contents

Introduction

The objective of this dissertation is to discuss liquidity and its insertion in a portfolio optimization model through the presentation of three articles. Liquidity represents the ability of an asset to be transacted in large quantities, in a short time and with low transaction costs. A liquid asset is an asset that can be bought or sold at stable prices in large quantities. Therefore, if an asset has low trading activity, that asset is considered illiquid. Hence, liquidity proves to be an essential factor in portfolio management.

The importance of liquidity is greater in emerging markets. In consolidated markets such as The New York Stock Exchange and London Stock Exchange Group, the difficulty of generating and liquidating a portfolio of significant monetary value is relatively low, as in these markets there are many assets with high liquidity available. However, in markets with lower trading levels such as B3 in Brazil and JSE Limited in South Africa, liquidity control becomes more important. The construction of a portfolio composed by assets with less liquidity can generate higher transaction costs.

Despite its importance, the study of liquidity has not been much explored by researchers, especially regarding the endogenous insertion of liquidity into portfolio optimization models. Traditional portfolio selection models usually rely on the assumption that assets can be transacted continuously in any quantity. These models do not take into account that liquidity can provide adverse effects for the investors. The main example of a study on this topic is presented in Lo, Petrov e Wierzbicki (2006).

The first article of this dissertation focuses on the design and application of liquidity constraints into portfolio optimization models. The proposed liquidity constraints are imposed to different models in different scenarios. The linearity of the proposed constraints facilitates the implementation, in addition to requiring only data that is easy to obtain and manipulate.

The article begins with a discussion regarding one of the main difficulties in studying liquidity: its multidimensionality. Financial volume, number of transactions, volatility, size of the company, bid-ask spread and share price are components of liquidity (DEMSETZ, 1968). The multidimensionality makes not possible a direct measurement of liquidity. There is no consensus among researchers on the best definition of liquidity. Some approximations for liquidity are proposed by several authors, trying to capture as precisely as possible the liquidity behavior of financial assets. Based on Lo, Petrov e Wierzbicki (2006) and on a brief study of liquidity metrics correlation, the one used in the proposed constraint is the financial volume negotiated.

The proposed approach is extensively tested, with several scenarios including different portfolio values and required liquidity levels, with the objective of analyzing the liquidity behavior in varied situations. Tests are performed in three emerging markets: Brazil, Turkey and Mexico. In addition, liquidity is compared with the liquidity of Risk Parity and Equally weighted portfolios. The proposed constraint is inserted both in the minimum variance model and in the mean-variance model.

The results showed good liquidity levels in the different analyzed scenarios, comparing with the liquidity of portfolios generated without any liquidity constraints. An increase in the risk of constrained portfolios was observed, as expected. The good results, with liquidity levels close to the required, in different emerging markets and different portfolio selection models demonstrate their robustness.

The second article aims to expand the studies on the liquidity constraint, proposed in the first article. The focus of the article is to study the behavior of an Index Tracking model with the application of liquidity constraints. Index Tracking is a passive investment strategy, which aims to build portfolios to mimic the performance of a market index.The base model takes the form of a convex quadratic programming optimization problem in which the objective is to minimize the gap betwenn portfolio returns and the index returns.

The empirical tests performed considering two market indices: Brazilian stock market (Ibovespa index) and Brazilian Small Cap index (SMLL). The Ibovespa Index is the main market index in Brazil while the SMLL is composed by less liquid assets. In addition, two approaches to liquidity constraints are considered. The first of them, proposed in Lo, Petrov e Wierzbicki (2006), referred here as Weighted Average Liquidity (WAL), defines the liquidity portfolio as the weighted average of participant assets liquidities. This approach allows the use of different liquidity metrics. Therefore, in order to expand the analysis, in addition to financial volume, turnover and the Amihud metric, proposed in Amihud (2002), were considered. The second considered approach, proposed in the first article, referred here to as Financial Value Liquidation (FVL), considers the portfolio liquidity as the possible liquidation percentage.

The impacts of the insertion of liquidity in the index tracking model are observed through the tracking error, compared to portfolios generated without the presence of liquidity constraints. The diversification level of the generated portfolios is analyzed through two indices: the Herfindahl-Hirschman index and the Gini index.

Overall, the results showed a significant increase in liquidity for portfolios constrained by both approaches. An increase in the tracking error of the constrained portfolios was observed. The biggest difference between the two approaches is related to diversification levels. Considering the WAL approach, portfolios with more intense liquidity constraints concentrated capital only on assets with high individual liquidity. In the FVL approach, more intense liquidity constraints generate higher portfolio diversification, thus increasing the number of participating assets.

Finally, the third article presents a comparison between the approach proposed in the first article and an approach in which constraints are individually applied to each of the participating assets, also proposed in Lo, Petrov e Wierzbicki (2006). The differential of this study lies in the fact that the two approaches considered here have the same definition of liquidity. This makes possible a direct comparison between the liquidity levels of the generated portfolios on both approaches, which was not possible in the previous studies.

Considering the minimum-variance portfolio selection model, liquidity results in the two approaches are compared, as well as the risk of the formed portfolios and the number of participating assets. In addition, it is mathematically demonstrated that in the specific case in which the requirement that the percentage liquidated is 100% of the generated portfolio value, the two approaches are equivalent. Equivalence is confirmed through empirical testing. The results were consistent with the results of previous studies. Both approaches considered in this article were able to provide the minimum required liquidity. The biggest difference between the two approaches is related to the risk of the generated portfolios. The individual constraints approach presented higher risk in all scenarios in which the required liquidation is below 100% of the portfolio value.

This dissertation is divided as follows. Part I presents the first article entitled "Liquidity constraints for portfolio selection based on financial volume", that has been accepted for publication in the Computational Economics. Part II presents the second article entitled "Liquidity-Constrained index tracking optimization models", that has been accepted for publication in the Annals of Operations Research. Finally, Part III presents the third article entitled "Portfolio Optimization and Liquidity: a Comparison Between Individual Constraints and Portfolio Constraints", that has not been submitted yet.

Part I

Liquidity constraints for portfolio selection based on financial volume

Abstract

This paper proposes liquidity constraints for portfolio selection models based on financial volume. The constraints consider different parameters such as the value of the portfolio and the acceptable liquidation level. Different portfolio selection models were tested in different scenarios.The liquidation level and its impact on the risk level are verified in the portfolio. The results are robust for all of the performed tests, with reasonable levels of portfolio liquidation. As expected, there is an increase in the level of risk of liquidity-restricted portfolios.

keywords: Liquidity. Portfolio selection. Liquidity constraint.

Note: this article has been accepted for publication in the Computational Economics.

1 Introduction

Liquidity control is an essential issue for portfolio management and it increases its relevance in smaller markets which present lower trading volumes. In some of the world's largest financial markets such as the USA, UK and Japan, the liquidity conditions are much more favourable to portfolio managers. The New York Stock Exchange, the Japan Exchange Group and the London Stock Exchange Group are among some of the most significant world's exchanges. In April 2018, these exchanges presented a market capitalization of USD 23,139 billion, USD 6,288 billion and USD 4,596 billion and monthly trading volumes of USD 1,452 billion, USD 481 billion and USD 219 billion, respectively [1]. The difficulty to generate and liquidate the portfolio is considerably reduced by the high levels of volume traded.

However, in markets with lower trading volumes, such as the B3 in Brazil, the Australian Securities Exchange in Australia and the JSE Limited in South Africa, the portfolio liquidity control becomes more important. In April 2018, these markets exhibited a market capitalization of USD 1,073 billion, USD 1,442 billion and USD 1,165 billion and monthly trading volumes of USD 62 billion, 56 USD billion and USD 29 billion, respectively. Compared to the most significant financial markets in the world, it is evident the greater portfolio liquidation difficulty in these markets. Therefore, the study of liquidity control can be justified, especially, for those who are in markets where the trading volume is smaller.

Liquidity represents the ability of an economic agent to quickly transact an asset with low cost and limited effects on market prices. It is a critical factor for market efficiency and financial stability. Liquidity risk can be divided into two types: market liquidity risk, i.e., the possibility of the investment fund not liquidating its positions in a given market breakdown; and cash flow liquidity risk, i.e., a fund's investment not having the capacity to honour its obligations or outflows (ACHARYA; PEDERSEN, 2005). These risks are intensified when portfolio managers apply a high proportion of the portfolio to fewer liquid assets while providing immediate or daily liquidity to the investor.

Traditional portfolio selection models such as the mean-variance (MARKOWITZ, 1952a) or the minimum variance models (BEST; GRAUER, 1992; DEMIGUEL; NOGALES, 2009), just to name a few, usually only consider the risk and return relationships in investment analysis. Moreover, assuming that the assets can be traded continuously in any quantity. Generally, these models do not consider the assets' liquidity, which can cause adverse effects for the investor. During the decision-making process, not only risk and

[1] Source: World Federation of Exchanges: www.world-exchanges.org/home/index.php/ monthly-reports-tool on 25 July 2018. This source was used in the second paragraph of this paper

return but also the period that the investor is willing to remain in the position and the possibility of quickly selling the asset are considered relevant.

The position of an investor in a particular asset is related to the number of assets owned and their price. However, depending on the quantity of assets owned, closing the position becomes extremely difficult at a given price. A small investor can probably sell all of his/her assets at the desired price, but a large investor will hardly have the same opportunity. A price reduction would be necessary to make it possible. Therefore, a large supply of assets leads to a drop in the asset price, reducing the return from its sale (AMIHUD; MENDELSON; PEDERSEN, 2012).

One of the main characteristics of liquidity is its multidimensionality, which makes it extremely hard to be measured directly. Thus, several metrics and approaches are proposed to capture more precisely the behaviour of the various determinants of asset liquidity. Generally, liquidity approximation metrics can be divided into two groups: price-based indexes such as the difference between the bid and ask prices, and volume-based indexes, such as turnover rate. Amihud's illiquidity metric (AMIHUD; MENDELSON, 1986) is considered one of the main liquidity measures. The financial volume and the turnover rate are also worth mentioning among the main liquidity metrics (GABRIELSEN; MARZO; ZAGAGLIA, 2011).

Several articles have studied the definition of liquidity and have proposed ways to capture its behaviour, for instance, Brennan, Chordia e Subrahmanyam (1998), Chordia, Subrahmanyam e Anshuman (2001) and Gabrielsen, Marzo e Zagaglia (2011). However, there are few examples of papers addressing the more practical problem of integrating liquidity directly on the portfolio selection process as Lo, Petrov e Wierzbicki (2006). Our paper's primary aim is to insert liquidity directly in the portfolio selection process. The model's construction follows some directions proposed by Lo, Petrov e Wierzbicki (2006), regarding the insertion of liquidity in the portfolio selection problem and choice of liquidity approximation metrics. However, our study differs from Lo, Petrov e Wierzbicki (2006) regarding the use financial volume as the liquidity measure which is (i) readily observable and obtained and (ii) more intuitive, especially, for portfolio managers or practitioners. Another difference in the present study is that it considers the portfolio monetary value directly in the imposed constraint. In Lo, Petrov e Wierzbicki (2006), portfolio liquidity is not a function of its monetary value. The present study considers the monetary value as an input parameter of the model, generating effects on the portfolio liquidity. The higher the value attributed to the portfolio, the harder the task to obtain the required liquidity.

The objective of this work is to design liquidity constraints for portfolio selection models. The implemented constraints consider some parameters of liquidity control. Furthermore, it proposes liquidity constraints which can be applied in any standard portfolio investment models; it analyzes the liquidity of the portfolio as a whole, and not the individual liquidity of the participating assets; it considers the portfolio monetary value as a liquidity variable.

In this article, three liquidity control parameters are proposed: the limit percentage of the total volume traded (ρ), the liquidation period (γ), and the acceptable liquidation level (ϕ) . The percentage limit of the total volume traded is the maximum percentage to be liquidated without an important impact on the asset price. The liquidation period is the maximum period for the value allocated on the asset to be liquidated. The level of acceptable liquidation represents a relaxation of the requirement of total liquidation of the position in such a way that complete liquidation might not be required within the considered term.

Mathematical optimization models can be of various types. The simplest and easiest to implement are the linear ones. There are also models with non-linear, integer, stochastic, among other characteristics. The constraints proposed in this work have the advantage of being linear, which facilitates the implementation, based on easily collected and manipulated parameters and data.

The insertion of the constraints proposed in the present study shows excellent results for the portfolios liquidation percentages. The results are consistent when compared to different portfolio selection models. The three tested emerging markets showed the model robustness. Comparing with portfolios formed in the absence of the constraint, a considerable increase in the liquidation is observed. When considering the out-of-sample performance, the percentage liquidated is very close to the designed in-sample acceptable level. As expected, the risk of liquidity-constrained portfolios is positively correlated with a higher demand for liquidity.

In the next section, a review of the literature on liquidity is presented; in Section 3, the proposed constraints are discussed; Section 4 presents several applications of the proposed constraints.

2 Literature Review

In this chapter, it is presented the theoretical basis to be used in the proposed constraints. First, the most used liquidity proxies are introduced. Next, the forms to implement liquidity constraints in the portfolio selection model are discussed.

2.1 Asset Liquidity Metrics

The study of liquidity has one central issue: the liquidity is multidimensional. It is impossible to observe liquidity directly. Using approximations inevitably leads to measurement errors, and there is no consensus among researchers about which measure is the best approximation. According to Demsetz (1968) and Lespagnol e Rouchier (2018), it is a determinant of the asset liquidity: the transacted financial volume, the number of transactions, the volatility, the size of the company, bid-ask spread, and the share price. Artificial indexes can be created to capture as accurately as possible liquidity's behaviour. The two main measure groups are the volume-based and price-based indexes.

2.1.1 Volume-based Indexes

Financial volume is widely used as an approximation for liquidity. According to Gabrielsen, Marzo e Zagaglia (2011), indexes based on information provided by the traded volume are related to the impact on the transactions price. This can be captured merely by measuring the total monetary value of shares traded in a defined period. It is intuitive to think that a stock that trades a lot has high liquidity. Despite the simplicity, the volume traded can be considered a reliable measure of liquidity, although it is not unanimous among the researchers.

In Brennan, Chordia e Subrahmanyam (1998), it is investigated whether the expected returns are explained by several characteristics, among them liquidity. They use the monetary volume transacted as a proxy for this variable, since they considered it a more appropriate measure for their study due to the broad availability of long series of data, allowing more robust hypothesis tests. In Zagst e Kalin (2007), a liquidity approach is proposed where the volume is considered in its formulation. In Darolles, Fol e Mero (2015), a study regarding the volume component in liquidity is presented.

Another volume-based index is the turnover rate. It relates the total volume transacted and total assets issued. Several papers which use this index as an approximation of liquidity are Datar, Naik e Radcliffe (1998), Marshall e Young (2003), Chan e Faff (2003) and Jun, Marathe e Shawky (2003). In the study of Chordia, Subrahmanyam e

Anshuman (2001), the relationship between expected returns and two proxies for liquidity were analyzed: the transacted volume and the turnover index. The liquidity rate of Hui e Heubel (1984) is also an alternative. This rate is a relationship between the volume traded during the five-day period and its impact on asset prices (GABRIELSEN; MARZO; ZAGAGLIA, 2011).

2.1.2 Price-based Indexes

A liquidity measure widely used by researchers is the difference between the offered price of an asset and its demanded price: the bid-ask spread. In Amihud e Mendelson (1986), liquidity is approximated by the bid-ask spread to examine the relationship between return and liquidity. Chung e Chuwonganant (2014) studied the effect of market uncertainty on liquidity using the bid-ask spread as a proxy. The bid-ask spread is also used in Moshirian et al. (2017) to evaluate the determinants and pricing of liquidity commonality in 39 markets. Several other authors such as for Brennan e Subrahmanyam (1996), Atkins e Dyl (1997), Jacoby, Fowler e Gottesman (2000) have also used this measure as an approximation for liquidity.

The direct analysis of the variation in prices is used to infer the liquidity of an asset or a market. According to Gabrielsen, Marzo e Zagaglia (2011), other price based indexes are the March and Rock liquidity rate and the variance rate. The Marsh and Rock liquidity ratio relates the price change to the total traded monetary value. However, the number of transactions is the focus and not the volume traded. Therefore, the assumption is that price changes are not influenced by the transacted volume. On the other hand, the variance ratio relates long and short term changes in asset prices. Further details can be found at Gabrielsen, Marzo e Zagaglia (2011).

In Amihud (2002), a measure of illiquidity is proposed as the ratio between the absolute return and the financial volume traded. This measure uses daily data and can be interpreted as the price response associated with the traded volume and, it is, therefore, a measure of price impact. Results show that returns are an increasing function of illiquidity. This proxy is used in Amihud et al. (2015) to study the illiquidity premium in stock markets across 45 countries. In Barardehi, Bernhardt e Davies (2018), it is proposed a measurement of high frequency illiquidity, based on Amihud's measure. In Ben-Rephael, Kadan e Wohl (2015), the liquidity premium in the US market is estimated using the turnover rate as metric, comparing also the results with Amihud's metric. Chacko, Das e Fan (2016) demonstrates a new measure of high-frequency liquidity is proposed, which is studied through correlation analysis with Amihud's metric. The Eq. 2.2 shows Amihud's measure of illiquidity.

$$A_i = \frac{1}{D_i} . \sum_{t=1}^{D_i} \frac{|r_{it}|}{vol_{it}} \tag{2.1}$$

Where r_{it} is the return of asset i on day t, vol_{it} is the financial volume of the asset i on day t and D_i is the number of days in the analysis.

2.2 Liquidity Constraints

We discuss various ways of including liquidity as constraints on portfolio selection problems. There are essentially three ways to have liquidity as a constraint: pre-filtering, the individual constraint on assets, and weighted average liquidity between assets.

2.2.1 Pre-filtering

The objective is to filter the sample previously, defining a cut for liquidity. Only assets with defined minimum liquidity may be part of the portfolio. It is not a constraint on the model itself but excluding assets without the minimum required liquidity. This works as a pre-processing which is carried out before the portfolio optimization procedure.

2.2.2 Individual Constraint on Assets

In this type of constraint, the financial value allocated in each asset, which is part of the portfolio is considered. The maximum amount allocated in a given asset depends on its liquidity. In this way, an asset will have less allocated value if its liquidity is low. For instance, the constraint for individual assets based on the financial volume can be written as:

$$x_i.TPV \leq l_i$$

Where x_i is the weight allocated in each asset i, TPV is the total portfolio value and l_i the financial volume traded on each asset in a specific period.

2.2.3 Weighted Average Liquidity

It is possible, rather than pre-filtering assets or considering the liquidity of assets individually, to use a constraint that considers the portfolio liquidity as a whole, defined by Lo, Petrov e Wierzbicki (2006). Using this liquidity metric in the portfolio, a constraint that ensures that the weighted average liquidity of the assets participating in the portfolio is higher than a certain level is proposed. In such a way, assets that could be excluded in the previous filtering can be used, increasing the universe of portfolio selection solutions. The liquidity constraint by the portfolio's weighted average liquidity can be written as:

$$\sum_{i=1}^{N} l_i.x_i \geq l_0$$

Where l_i is the liquidity of asset i, x_i is the weight allocated in asset i, l_0 is the minimum liquidity level required and N is the number of assets present in the portfolio.

Some limitations of this liquidity metric are addressed by Lo, Petrov e Wierzbicki (2006). If the weighted average is used as portfolio liquidity, no interaction between the assets is being considered. The liquidity of one asset is not affected by the liquidity of another. Also, imposing this constraint disregards the portfolio monetary value, it considers the weight of each of the asset only. Furthermore, it is not being considered portfolios with high financial values, as they are more difficult to liquidate.

2.3 Other Types of Constraints in Portfolio Selection

Constraints are imposed in different ways with different model's variables. A few examples of different constraints used in the classic portfolio selection model are discussed.

Lobo, Fazel e Boyd (2007) considered the portfolio optimization problem with transaction costs and risk exposure constraints. The expected return maximization is analyzed with different types of constraints. Transaction costs are inserted in two ways: fixed and variable transaction costs. The fixed cost would be the transaction rate, regardless of the transaction value. The variable cost depends on the transacted value. Diversification constraints are also studied by limiting the maximum amount invested in each asset. Other examples of transaction costs studies are: Magill e Constantinides (1976), Patel e Subrahmanyam (1982), Davis e Norman (1990), Kellerer, Mansini e Speranza (2000), Konno e Wijayanayake (2001) and Baixauli-Soler, Alfaro-Cid e Fernandez-Blanco (2011).

Chang et al. (2000) studied the insertion of constraints that limit not only the number of assets present in the portfolio but also the maximum percentage allocated in each asset. It is shown that the solution procedure becomes more complex and that efficient boundaries are discontinuous in the presence of the constraints analyzed. Other examples of studies that discuss carnality in portfolio selection problems are: Maringer e Kellerer (2003), Shaw, Liu e Kopman (2008), Bertsimas e Shioda (2009) and Anagnostopoulos e Mamanis (2011).

Another example of constraints imposed on the classic minimum variance model is in DeMiguel et al. (2009). The constraints are imposed on the weight vector norm of the formed portfolio. The weight vector norm is limited by a stipulated value. The results obtained are compared with other strategies, which have been found in the literature, such as in Jagannathan e Ma (2003), which presents a restrictive strategy in the short sale of the portfolio, and Ledoit e Wolf (2003), in which the mean of two estimators construct the covariance matrix.

In Bonami e Lejeune (2009) the portfolio selection model is considered with the minimization of variance, where the expected return on assets is deemed to be stochastic. An inserted probabilistic constraint imposes that the portfolio expected return is higher than the desired level with a specified confidence level. Other examples of stochastic constraint studies in investment portfolios are: Dentcheva e Ruszczynski (2003), Dentcheva e Ruszczyński (2006), Huang (2007) and Hasuike, Katagiri e Ishii (2009).

Several investment strategies, using portfolio optimization, are compared in Behr, Guettler e Miebs (2013). Constraints are proposed on the maximum and minimum weights allowed for each asset in the portfolio. The results obtained are compared to the benchmark portfolio, which is the equally distributed portfolio $(1/N)$ and the market portfolio. Besides the comparison with benchmark portfolios, the results are compared with different strategies such as constraints on short selling, constraints on weights vector norm and different forms of covariance matrix construction.

3 Methodology

The choice of the liquidity metric used in this work is based on the conclusions obtained in Lo, Petrov e Wierzbicki (2006) in which the authors compare three measures of liquidity: turnover rate, traded volume and bid-ask spread. They have found a strong correlation between the three measures. Furthermore, a correlation analysis between the Amihud's measure and the traded volume is performed in our study (refer to Appendix A). Amihud's measure of illiquidity is widely used as an approximation and presented a high correlation with the volume traded. Further details of the results can be found in Appendix A. Therefore, in the present study, the financial volume traded as a liquidity measure was chosen. Some advantages of the financial volume traded are: (i) easily observable and obtained; (ii) very intuitive, especially, for portfolio managers or practitioners. For the present study, the 30-day daily volume moving average was chosen to be used as the liquidity estimation.

3.1 The Generic Optimization Model

We consider a generic portfolio optimization model consisting of an objective function and its constraints. The variables of choice with portfolio optimization are the weights assigned in each asset. The general portfolio selection model is presented in Eq. 3.1 and Eq. 3.2

$$min f(x) \tag{3.1}$$

subject to

$$h_j(x) \geq d_j \tag{3.2}$$

where x are the weights allocated on each asset, h_j are constraint functions that depends on x and d_j are constants.

3.2 Liquidity Constraints

The present work proposes the use of three liquidity control parameters, which are inserted directly into the liquidity constraints. The percentage limit of the total volume traded (ρ) is the maximum percentage of the average total volume transacted that is believed not to have a significant impact on the asset price. The liquidation period (γ) is the maximum period in which the asset position is totally liquidated. The acceptable liquidation level (ϕ) represents a relaxation in the requirement of total liquidation of the position, not being required a total liquidation within the period.

The first constraint defines that the value possibly liquidated in each asset cannot be higher than the total value allocated to it. The amount allocated for each asset is the multiplication between the portfolio value and the weight allocated on each asset. The constraint is presented in Eq. 2.7.

$$\theta_i \leq TPV.x_i \tag{3.3}$$

where θ_i is the value that is possible to be liquidated on asset i and TPV is the total portfolio value.

The second constraint defines that the possible liquidation value on each asset cannot exceed the maximum value defined by the parameters ρ and γ. The maximum allowed liquidation value of asset i is given by $\rho.\gamma.vol_i$, where vol_i is the average volume of the last 30 days of the asset i. The constraint is presented in Eq. 2.8.

$$\theta_i \leq \rho.\gamma.vol_i \tag{3.4}$$

The third constraint indicates that the portfolio's liquidation must be equal to or greater than the acceptable liquidation value, given by $TPV.\phi$. The constraint is presented in Eq. 2.6.

$$TPV.\phi \leq \sum_i \theta_i \tag{3.5}$$

4 Empirical Tests

In this chapter, we apply the proposed liquidity constraints in several empirical tests. First, various cases are studied for the Brazilian market with the minimum variance model (Section 4.1). Then, in Section 4, additional tests are discussed: risk parity and equally weighted portfolios, mean-variance and other two emerging countries (Mexico and Turkey).

4.1 Liquidity Constraints in an Emerging Market

In this first empirical test, we focus the analysis for the Brazilian stock market. The minimum variance model is adopted, but any other model could have been chosen. The model aims to minimize the risk of the portfolio with the risk being measured by the variance of the portfolio's returns. The portfolio's variance is calculated considering the interaction between the assets, measured by the covariance. Aiming to improve the computational performance, an adaptation of the variance calculation discussed in Filomena e Lejeune (2012) is implemented. The objective function to be minimized can be written as:

$$min \sum_{i=1}^{N} \sum_{j=1}^{N} x_i . x_j . \sigma_{ij}$$

where x_i is the weight allocated on asset i, σ_{ij} covariance of the returns between asset i and asset j, in a portfolio consisting of N possible assets.

Moreover, we also included some additional constraints on the empirical tests. All the available capital will be used, in other words, the sum of the weights of all the assets that form the portfolio is equal to one. As a premise, for simplicity, short selling is not allowed. Therefore, the weights of all assets should be greater than or equal to zero.

$$\sum_{i=1}^{N} x_i = 1$$

$$x_i \geq 0, \quad \forall i$$

4.1.1 Data and Simulation - Brazilian Market

Daily closing prices and daily financial traded volume were used to structure the database. This data has been obtained for stocks listed on B3 (Brazilian Stock Exchange) through Economatica, one of the leading providers of financial data in Brazil . The initial sample had 610 assets, between the period between 1 January 2007 and 8 August 2016. The assets, which had less than 80% available data in the period were taken out of the sample. The lack of data on certain days was handled through linear interpolation of neighbours data. Besides, the duplicated shares of the same company (preferred and common shares) were removed, leaving only one asset per company (the asset with the highest trading volume). After the filtering, 252 assets remained in the sample. The holidays and dates with unavailable data were removed, and the resulting sample had 2370 days.

The tests considered four different intervals for portfolio rebalancing using a rolling horizon basis: 1 day, 20 days, 40 days and 60 days. The portfolio liquidation level obtained is measured, for each interval completion. Moreover, a new portfolio is generated, and the process is repeated until the end of the data sample. In each interval three monetary values were used for the portfolios. Portfolios of 1 million, 10 million and 100 million Brazilian Reais (approximately 0.25, 2.5 and 25 million dollars, respectively) were formed. Compared to the USA stock market, these numbers can be small but they are not small for the Brazilian stock market. Just as a comparison measure, in a typical trading day in 2017, the volume negotiated of Apple stock easily surpassed all the Brazilian stock market. Therefore, due to this lack of liquidity, it makes our study even more important for countries like Brazil. In the tests with liquidity constraints, the values of 30%, 50%, 70% and 100% were adopted for the portfolio's acceptable liquidation level. Considering all the possible combinations of parameters mentioned above, a total of 45 different scenarios are generated.

The first 250 days of the sample were used as the training period. Then, portfolios were generated in the remaining 2120 days of the sample once the training period had been removed. Then, for the interval of 20 days, portfolios are formed in 106 days of the analysis (sample containing 2120 days). Similarly, for the 40 and 60-day intervals, portfolios are formed in 53 and 35 days, respectively. In each test it is analyzed: the average percentage liquidated among all the portfolios formed, the average number of participant assets and the number of days in which portfolio formation was possible. The portfolio standard deviation measured portfolio risk.

When calculating the percentage of how much of the portfolio formed can be liquidated, the actual volume on the liquidation day is used rather than using the average volume of the last 30 days estimation calculated for the liquidity constraints. In addition, the portfolio value is corrected to the value on the liquidation day. The portfolio changes its value on the liquidation day due to changes in the stock prices during the period between its formation and liquidation.

Table 1 – Results for the 12 scenarios analyzed, with a one-day interval.

ϕ	TPV	Av. Liq	Av. St Dev	Av. Number of Assets
30%	BRL 1 million	49.17%	10.35%	44.32
30%	BRL 10 million	30.75%	10.97%	54.64
30%	BRL 100 million	29.81%	12.86%	69.85
50%	BRL 1 million	53.93%	10.74%	49.35
50%	BRL 10 million	49.71%	12.70%	68.66
50%	BRL 100 million	49.64%	17.71%	80.32
70%	BRL 1 million	69.67%	11.80%	57.32
70%	BRL 10 million	69.51%	15.56%	73.10
70%	BRL 100 million	69.47%	24.54%	87.33
100%	BRL 1 million	99.23%	15.82%	57.61
100%	BRL 10 million	99.21%	21.64%	63.11
100%	BRL 100 million	99.19%	36.90%	69.25

4.1.2 Results

The average liquidation percentage results, along with its average standard deviation, for the one-day interval case, is presented in Table 1. The results for 20, 40 and 60-day intervals are presented in Appendix B. The portfolios without the liquidity constraints generated average liquidity results of 30.58%, 15.07% and 5.33%, in the cases of portfolios of BRL 1 million, BRL 10 million and BRL 100 million, respectively. In the presence of liquidity constraints, it is possible to observe a greater portfolio average liquidation percentage. The average liquidation out-of-sample was close to the level of acceptable liquidation assigned in-sample. In two cases, the average liquidation was above the stipulated minimum level. The reason for this was because in cases which the portfolio value is worth BRL 1 million with acceptable liquidation levels of 30% and 50%, the liquidity constraint is not binding for a considerable number of days. Thus, on average, the resulting liquidation was above the acceptable level. This result is consistent compared to the average liquidation level in portfolios without liquidity constraints.

The assignment of the acceptable liquidation in the liquidity constraint proved useful in generating a higher average liquidation. In all the scenarios studied, the out-of-sample liquidation was very close to the level assigned in the constraint (in-sample). The average liquidation value slightly below ϕ can be explained by fluctuations in the traded volume resulting from the time between the portfolio formation and its liquidation.

When analyzing the percentage liquidated in different portfolio values, no significant differences were observed, in cases in which the portfolio is totally restricted by liquidity. In exception of cases where the liquidity constraint is not bidding, the percentage liquidated has not shown to depend on the stipulated portfolio value. Considering portfolios, which had different monetary values, once the liquidity constraint is active, the liquidity of the portfolio will not depend on the value of the portfolios. For the case of $\phi = 70\%$, the percentage liquidated values were 69.67%, 69.51% and 69.47% for the portfolio values of BRL 1 million, BRL 10 million and BRL 100 million, respectively. It is clear that the difficulty of forming high-value portfolios will be more significant, but once formed, its percentage liquidated will not depend on its monetary value.

The portfolio standard deviation is related to the difficulty of selecting the portfolio, that is, to the size of the universe of solutions. For the same level of acceptable liquidation level, the portfolios with the high monetary value presented the highest standard deviation. In addition, for the same monetary values, portfolios with the highest acceptable liquidation level produced the highest standard deviation.

Figure 1 presents an illustration of the variation between the average liquidation percentage and the average portfolios standard deviation, for the four different assigned acceptable liquidation levels. The financial portfolio value assigned is indicated in each graph.

It is possible, in all cases, to observe that reduced portfolio monetary values are related to higher percentage liquidated and smaller standard deviation. This difference was more evident in the cases of $\phi = 30\%$ and 50%, comparing the values of BRL 1 million or BRL 10 million. In cases of acceptable liquidation levels of 70% and 100%, no considerable differences were identified in the percentage liquidated. However, portfolios with higher monetary value had a higher average standard deviation in all cases.

The average portfolios standard deviation considering $\phi = 30\%$ and 50% were not far from the standard deviation of portfolios formed in the absence of liquidity constraints, especially considering the financial portfolio value of BRL 1 million. Assigning $\phi = 30\%$, for example, in BRL 1 million portfolios showed a standard deviation of 10.35%, close to the value of 10.17% obtained in the absence of liquidity constraints. However, the percentage liquidated increased considerably from 30.58% to 49.17%. Thus, it was discovered that imposing the proposed liquidity constraints with lower values of acceptable liquidation, together with lower financial value portfolios, can generate a relative increase in the percentage liquidated without a large change in the standard deviation as a result.

The results show that the number of assets has a high dependence on the acceptable liquidation level. The higher the liquidity desired, the greater the variety of assets required to meet the imposed level. However, it can be observed that the average number of participating assets when $\phi = 100\%$ is less than the number when $\phi = 70\%$. This can be explained by the required liquidity being so high, that there are not so many assets with such a high level of liquidity. Thus, there is a reduction in the average number of assets in this case. In all cases, portfolios with high monetary value required a greater quantity of participating assets to attend the desired liquidity.

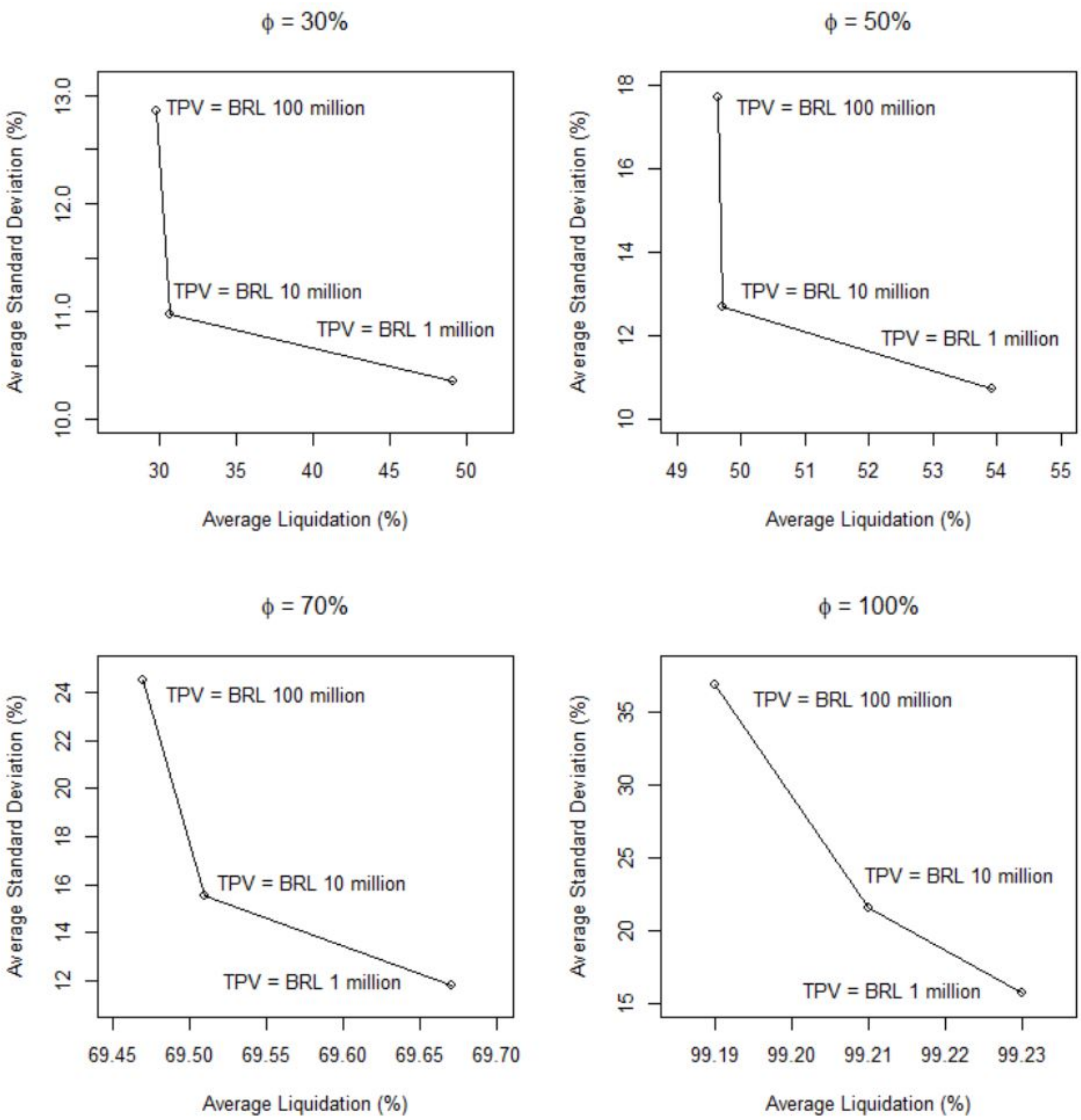

Figure 1 – Average Liquidation versus Average Standard Deviation, for $\phi = 30\%$, $\phi = 50\%$, $\phi = 70\%$ and $\phi = 100\%$

4.2 Additional Empirical Tests

Three additional tests were performed for the proposed liquidity constraints. The first is a comparison between the minimal variance model with the presence of liquidity constraints and the risk parity and equally weighted portfolios. The following is the implementation of the constraints in the mean-variance model; besides the liquidity constraints, the required returns are imposed. Finally, implementing the minimum variance model with liquidity constraints in the Mexican and Turkish markets is presented.

4.2.1 Comparing with Risk Parity and Equally Weighted Portfolios

In this section, the minimum variance with liquidity constraints is compared with the risk parity and equally weighted portfolios. The risk parity portfolio, also called equally weighted contributions portfolio, imposes asset's risk to be equally distributed on the portfolio. In other words, using the estimation of the covariance matrix, the risk parity seeks maximum risk diversification between the assets. It differs greatly from the minimum variance which focuses on minimizing the overall risk of the portfolio. The equally weighted portfolio also known as $1/N$ uniformly allocates capital between the assets.

We omit from the paper additional formalizations for risky parity and equally weighted portfolios to avoid a new set of mathematical definitions. Definitions, algorithms, properties and other considerations on risk parity can be found on Maillard, Roncalli e Teïletche (2010), Clarke, Silva e Thorley (2013), Bai, Scheinberg e Tutuncu (2016) and Roncalli e Weisang (2016). Further material on equally weighted portfolios is discussed by Benartzi e Thaler (2001), Windcliff e Boyle (2004) and DeMiguel, Garlappi e Uppal (2007).

We conduct the empirical tests using the same Brazilian dataset and training period from Section 4.1. However, due to the difficulty in estimating high dimensional covariance matrices, necessary to obtain the risk parity portfolios, the number of available assets was reduced to 60. The group of 60 assets was generated by dividing the initial sample of 252 assets into 6 groups based on their liquidity level. From each of the 6 groups, 10 assets were randomly selected. The new set of assets, to keep the comparison fair, is used for the three types of portfolios. So, we compute the results for the minimum variance with liquidity constraints again.

The minimum variance model with liquidity constraints test is performed for the same four acceptable liquidation levels used in the previous test. Portfolios of 10 million and 100 million Brazilian Reais were formed. The risk, measured by the standard deviation, and the liquidation levels are the main metrics analyzed. We considered the interval of 1 day between portfolio formation and liquidation. Please, notice that we did not apply the liquidity constraint on the risk parity and equally weighted portfolios because, conceptually, both present by definition the portfolio allocated already constrained. The risk parity distributes the risk uniformly between the assets and equally weighted assigns the capital homogeneously between the assets.

The results for the liquidity constrained portfolios are exhibited in Table 2. Portfolios without the liquidity constraints presented an average standard deviation of 12.78% and an average percentage liquidation results of 55.92% and 25.54%, in the cases of portfolios worth BRL 10 million and BRL 100 million, respectively. It is observed that in the case of TPV BRL 10 million and acceptable liquidation level 30% and 50%, the liquidity constraints are inactive on most days, since in those days the portfolio percentage liquidated without constraints is already above the minimum required.

The risk parity portfolios results are presented in Table 3. The average standard deviation found was 15,76%. The average standard deviation of risk parity portfolios was

Table 2 – Results for the 8 scenarios analyzed, with a one-day interval, using 60 assets.

ϕ	TPV	Av. Liq	Av. St Dev	Av. Number of Assets
30%	BRL 10 million	56.01%	12.81%	23.06
30%	BRL 100 million	30.12%	13.13%	23.64
50%	BRL 10 million	56.90%	12.88%	23.22
50%	BRL 100 million	49.89%	19.43%	28.08
70%	BRL 10 million	69.37%	16.82%	23.2
70%	BRL 100 million	69.62%	28.69%	33.52
100%	BRL 10 million	98.90%	26.18%	20.4
100%	BRL 100 million	98.48%	42.65%	32.16

Table 3 – Results for the 2 risk parity scenarios analyzed, with a one-day interval, using 60 assets.

TPV	Av. Liq	Av. St Dev
BRL 10 million	51.87%	15.76%
BRL 100 million	34.17%	15.76%

higher than the cases, in which, the minimum variance model had liquidity constraints with lower acceptable liquidation level. However, in the cases where liquidity is tightly constrained, its standard deviation exceeds the standard deviation of the risk parity portfolio.

Moreover, by purely comparing the liquidation level of the risk parity with the minimum variance without liquidity constraints, on one hand, demonstrates the minimum variance performs better for the 10 million (55.92% versus 51.87% of risk parity). Whereas, on the other hand, risk parity operates better for the 100 million (34.17% versus 25.54% of minimum variance). Except for case of 100 million and $\phi = 30\%$ (high portfolio value and relatively low ϕ), we observe that the percentage liquidated was lower compared with the model of minimum variance restricted by liquidity. Thus, as designed, the liquidity constraints actually improves the liquidation percentage. In some cases, it not only shows the average percentage liquidated but also the average standard deviation demonstrates better results with the liquidity restricted minimum variance model compared to risk parity being a considerable gain regarding performance.

The results for the equally weighted portfolios are shown in Table 4. As expected, the average standard deviation was higher than what was found in the risk parity portfolio. However, the average percentage of liquidated was also slightly higher than on risk parity. Comparing to the minimum variance model, the equally weighted portfolios show lower average percentage liquidated in almost all studied cases, especially when liquidity requirement is tighter.

Table 4 – Results for the 2 equally weighted scenarios analyzed, with a one-day interval, using 60 assets.

TPV	Av. Liq	Av. St Dev
BRL 10 million	52.82%	18.93%
BRL 100 million	38.74%	18.93%

4.2.2 Mean-Variance Portfolio

In this section, we apply the liquidity constraints to the classical mean-variance portfolio optimization (MARKOWITZ, 1952a; RUBINSTEIN, 2002). Thus, we have the addition of the required return to the minimum variance portfolio presented in previous sections.The mean-variance without liquidity constraints is considered with the same four acceptable liquidation levels adopted in Section 4.1 for the 252 available assets of the Brazilian market. We cover just the last two years of the Brazilian dataset to decrease the backtest's time effort. Hence, this test focuses on the period between 8 August 2014 and 8 August 2016 with 250 days as the training period. We implement six different annual required returns (0%, 3%, 5%, 10%, 15% and 30%) with the portfolio value of 10 million Brazilian Reais with daily rebalance as in Section 4.1.

The average standard deviation results for different required returns and liquidation levels are shown in Table 5. The value of $\phi = 0\%$ is related to the case without liquidity restrictions. The similar standard deviation values of 0%, 3% and 5% required returns can be explained by the fact that, in the analyzed period, the minimum variance portfolio showed positive returns over a large number of days. In all cases, as expected, for the same required return, the standard deviation rises as the acceptable liquidation level rises. The average number of assets results are presented in Table 6. It can be observed that by increasing the required return, there is a slight decrease in the number of participating assets, for a fixed liquidation level. This is because it increases the difficulty of finding assets, which can provide the necessary required return. The behaviour of the number of participating assets for the different acceptable liquidation levels were similar to the behaviour found in the previous test with the minimum variance.

The average percentage liquidated results can be observed in Table 7. The liquidation level of portfolios generated without liquidity restriction showed liquidation percentage above 45%. However, the liquidation level cannot be directly compared with the results of the minimum variance from Section 4.1, given that the testing period is much smaller. The liquidity constraint is rarely active for $\phi = 30\%$. The liquidation level is always very close to the required liquidation level, especially, for the more demanding requirements ($\phi = 50\%, 70\%, 100\%$). The liquidity constraints show that they are effective for all levels of the efficient frontier with similar patterns of liquidation observed on the minimum variance tests from Section 4.1. Hence, the constraints show their flexibility to be applied

Table 5 – Average standard deviation for the five different acceptable liquidation levels and six different required returns

	Required return					
ϕ	0%	3%	5%	10%	15%	30%
0%	10.76%	10.78%	10.79%	10.81%	10.86%	11.19%
30%	10.96%	10.98%	10.99%	11.05%	11.12%	11.60%
50%	11.79%	11.81%	11.82%	11.86%	11.96%	12.10%
70%	13.88%	13.91%	13.94%	13.99%	14.22%	14.66%
100%	17.70%	17.77%	17.87%	18.34%	18.84%	19.29%

Table 6 – Average number of participant assets for the five different acceptable liquidation levels and six different required returns

	Required return					
ϕ	0%	3%	5%	10%	15%	30%
0%	42.34	42.33	42.21	41.98	40.25	38.98
30%	42.3	42.24	42.23	41.87	40.21	38.89
50%	46.06	45.99	45.87	44.55	41.36	40.09
70%	55.70	55.72	55.67	53.09	50.02	47.83
100%	51.93	52.03	51.79	50.87	48.45	47.67

Table 7 – Average percentage liquidated for the five different acceptable liquidation levels and six different required returns

	Required return					
ϕ	0%	3%	5%	10%	15%	30%
0%	45.86%	45.85%	45.88%	45.86%	46.16%	47.41%
30%	45.87%	45.87%	45.91%	45.90%	46.21%	49.50%
50%	50.56%	50.54%	50.55%	50.51%	52.54%	53.45%
70%	68.92%	68.91%	68.89%	68.88%	68.93%	68.91%
100%	97.04%	97.19%	97.05%	98.02%	97.94%	97.58%

to many variants of portfolio optimization models.

4.2.3 Tests in Other Emerging Markets

We performed empirical tests of the proposed model in two other emerging markets: The Borsa Istanbul (Turkey) and The Mexican Stock Exchange (Mexico). These countries were selected because among the possible emergent markets these were the countries in which our database provided considerable available data. Please, notice that these are known as emerging countries sustaining reasonable liquidity level in their stock exchanges. For both cases, the period between 29 January 2012 and 20 February 2014 is considered; as this was the most reliable period we could access for these countries. After filtering the data, by using the previously explained methodology, the Mexican and the Turkish datasets consisted of 183 and 163 assets, respectively. The minimum variance model with liquidity constraints test is performed for the same four acceptable liquidation levels used in Section 4.1. For each market, it is considered two different portfolio's financial values: 10 million and 100 million Mexican pesos (Turkish lira) for the Mexican (Turkish) market.

Although the results cannot be compared directly, we can still infer that the outcomes from the Mexican and Turkish markets are very similar to the ones from the Brazilian market. The effect of the liquidity constraints in the minimum variance model was analogous to the Brazilan market. The percentage liquidated was also close to the acceptable liquidation level discussed in Section 4.1 when liquidity constraints are applied.

In the Mexican market, the portfolios generated without the liquidity constraints presented an average standard deviation of 11.28% being composed on average by 54.56 assets. The average liquidation percentage of these cases was 41.61% for TPV of MXN 10 million and 12.75% for MXN 100 million. Moreover, due to a large number of assets selected in the minimum variance portfolio, it was observed that the liquidity constraint is almost always inactive for the cases with the lowest TPV since the acceptable liquidation level is reached without the constraints functioning on most of the days. In the case of higher TPV, the liquidity constraints were almost always active, resulting in a percentage liquidated always close to the acceptable liquidation level.

In the Turkish market, the portfolios generated without the liquidity constraints presented an average standard deviation of 14.14%, which comprised of on average by 21.76 assets. The average liquidation percentage of these cases were 15.05% for TPV of TRY 10 million and 10.46% for TRY 100 million. The constraints were strongly active in this market, both for TPV of TRY 10 million and TRY 100 million, because the portfolios without liquidity constraints showed relatively low liquidated percentages. We found five days in which it was impossible to generate portfolios for the cases of $\phi = 70\%$ and 100%, considering the TPV of TRY 100 million. In other words, the combination of a high portfolio value with the requirement of high liquidated percentages did not have a feasible solution on these days.

Table 8 – Results for the 8 scenarios analyzed, with a one-day interval, using 183 assets in the Mexican market.

ϕ	TPV	Av. Liq	Av. St Dev	Av. Number of Assets
30%	MXN 10 million	41.69%	11.32%	54.61
30%	MXN 100 million	29.50%	12.29%	68.68
50%	MXN 10 million	49.79%	11.38%	57.80
50%	MXN 100 million	49.22%	16.06%	69.20
70%	MXN 10 million	69.09%	12.30%	64.88
70%	MXN 100 million	68.90%	22.03%	66.12
100%	MXN 10 million	97.94%	16.65%	65.01
100%	MXN 100 million	98.60%	34.66%	55.00

Table 9 – Results for the 8 scenarios analyzed, with a one-day interval, using 163 assets in the Turkish market.

ϕ	TPV	Av. Liq	Av. St Dev	Av. Number of Assets
30%	TRY 10 million	29.75%	14.56%	27.44
30%	TRY 100 million	29.67%	18.57%	36.55
50%	TRY 10 million	49.57%	15.90%	33.6
50%	TRY 100 million	49.41%	27.07%	43.98
70%	TRY 10 million	69.36%	18.00%	37.68
70%	TRY 100 million	69.43%	34.07%	51.92
100%	TRY 10 million	99.02%	22.01%	41.32
100%	TRY 100 million	99.06%	39.38%	50.86

5 Conclusions

Applying liquidity constraints in a portfolio selection model was performed in an empirical test conducted in some emerging markets. The measurement that was used in this study was the financial volume traded. This measurement was selected based on the comparison presented in Lo, Petrov e Wierzbicki (2006), in which a substantial correlation between the measures of volume traded, turnover rate and bid-ask spread are found. Portfolios liquidity control parameters were inserted into the proposed constraints. Different scenarios were analyzed in the Brazilian market as well as in the Turkish and Mexican markets. A comparison between the model with proposed liquidity constraints and the equally weighted portfolio and risk parity portfolio was performed. Further, we have also applied the constraints to the mean-variance model. The constraints designed in this work were considered adequate and the results obtained in the empirical test were coherent.

As a result, high liquidation levels were found, in the presence of the imposed liquidity constraint, when compared to the levels in cases without the imposed constraints. The liquidation levels obtained were also close to the acceptable levels imposed by the liquidity constraints. It was observed that the risk of the portfolio increases as the portfolio formation becomes more restricted due to the liquidity limitations of the available assets. The results were not only robust for different emerging markets, but also for distinct portfolio selection models tested.

The present research leaves room for a wide variety of extensions. For instance, the inclusion of the information related to the limit order book or large market makers capacity to execute the order are possible developments. In the same direction, the investigation of market impact on prices can be investigated.

Part II

Liquidity-Constrained index tracking optimization models

Abstract

This paper examines optimization models that use liquidity constraints to track an index. Liquidity is relevant from a risk management perspective but has hardly been explored in the portfolio optimization literature. The liquidity aspect is especially critical in emerging markets. We present two modeling approaches to instill liquidity in index tracking portfolio optimization. The first one defines the portfolio liquidity as the weighted average of assets' liquidities and accounts for financial volume, turnover, and Amihud's metric. The second one models liquidity with the introduction of financial practice parameters related to the liquidation level and the monetary value of the constructed portfolio. An extensive empirical analysis is conducted to replicate two Brazilian stock market indices: Ibovespa and SMLL. As expected, liquidity-constrained portfolios show higher liquidity and higher tracking errors. A counter-intuitive result is observed for the first liquidity-constrained approach in which the number of assets included in the portfolio decreases as the liquidity requirement gets tighter. In the second approach, as liquidity becomes tighter, the number of assets in the portfolio increases. This observation is confirmed by investigating the diversification of the constructed portfolios using the Gini index.

keywords: Liquidity. Index Tracking. Liquidity Constraint. Diversification.

Note: this article has been accepted for publication in the Annals of Operations Research.

1 Introduction

A liquid asset is one that can be quickly bought and sold at stable prices. Liquidity measures how easily financial assets can be converted into cash (WEGENER et al., 2019). In this sense, liquidity is a measure of how many buyers and sellers are present and whether transactions can easily occur without resulting in sudden asset price movements (DEMSETZ, 1968; LANGEDIJK; MONOKROUSSOS; PAPANAGIOTOU, 2018). High levels of liquidity happen when there is a substantial intensity in commercial activity and considerable supply and demand for an asset, making it easier for buyers/sellers to find a counterpart for their transactions. If there are few market participants and sparse trading activity, the market or asset is considered illiquid. Since higher liquidity is often associated with lower risk, this pose challenges to asset managers from a risk perspective. Moreover, as mark-to-market accounting is also influenced by liquidity, an investor must be willing to take the other side of the trade to allow a seller or buyer to quickly complete a trade without the price of the asset significantly deviating from its current market value.

Liquidity is especially important in emerging stock markets. A comparison between some of the S&P 500 stocks and those in the Ibovespa index (i.e., the leading benchmark for the Brazilian stock market) illustrates this. On August 30th, 2020, the two biggest companies in terms of weight in each index were, respectively, Apple and Microsoft (S&P), and Vale and Itau (Ibovespa). In 2019, Apple and Microsoft traded on average USD 5.82 and 3.16 billions daily, respectively, while, on the other hand, Vale and Itau's trades averaged USD 0.23 and 0.17 billion[1]. Notice that the Brazilian market is far from negligible. Based on the 2018 World Federation of Exchange (2019) market capitalization ranking, B3 (i.e., the main Brazilian stock exchange) was by far the biggest Latin American stock exchange and ranked 18th over 79 stock exchanges around the world.

There is no widely accepted definition of liquidity risk. However, two dimensions of the liquidity risk can be identified: (i) the risk associated with market participants not being able to liquidate easily their positions due to abrupt market movements; and (ii) the risk related to cash flow magnitude, which is related to the market participants' capacity to honor their obligations (SCANNELLA, 2016). In fact, these two dimensions are interrelated and have positive co-movements.

Furthermore, while most relevant for financial risk management, modeling liquidity is challenging due to its multidimensional nature and the difficulty to measure it directly. Several approaches have been used in previous studies to capture the liquidity components from different angles. Even though there is no overall consensus about what is the best

[1] Data collected on Yahoo Finance – August 30th, 2020.

proxy for asset liquidity, Amihud's illiquidity metric (AMIHUD; MENDELSON, 1986), volume, and turnover are considered to be the main liquidity measures (LO; PETROV; WIERZBICKI, 2006; GABRIELSEN; MARZO; ZAGAGLIA, 2011).

Despite its importance, asset liquidity has been vastly overlooked in traditional portfolio optimization models which rely on the assumption that assets trade continuously over time in any quantity (MARKOWITZ, 1952b; FAMA; FRENCH, 1993; FAMA; FRENCH, 2015; ROCKAFELLAR; URYASEV, 2000). We refer the interested reader to Kolm, Tütüncü e Fabozzi (2014) for a review of portfolio optimization developments over the last 60 years, including discussions on portfolio constraints (e.g., regulatory, exposure, trading, risk). Interestingly, liquidity constraints are not mentioned by the authors. To the best of our knowledge, only Lo, Petrov e Wierzbicki (2006) and Vieira e Filomena (2019) discuss explicitly the inclusion of liquidity constraints in portfolio optimization models. Particularly, the literature on index tracking investment encompasses a variety of optimization models (BEASLEY; MEADE; CHANG, 2003; CORIELLI; MARCELLINO, 2006), but has yet to pay attention to the liquidity problematic.

Index tracking is an investment strategy that aims to mimic a market benchmark or its segment. Tracking funds, also known as index funds, are designed to offer investors access to an entire index at a low cost via passive investment, thus reducing their exposure to unsystematic risks (CHEN; HUANG, 2010). These funds aim to replicate the performance of a market benchmark, and are usually assembled as mutual or exchange-traded funds to meet the fund's objective of following its target. Using a passive investment approach to track a market is based on the efficient market hypothesis (FAMA, 1970), which argues about the difficulty for investors to obtain long-term returns superior to those offered by the market (FAMA; FRENCH, 2010). Therefore, choosing a passive investment vehicle such as an index fund should allow the investor to match the market performance at a reduced cost, since this type of investment targets lower expense rates due to its passive characteristic and reduced trading activities (FRINO; GALLAGHER, 2001). Examples of indexes that could be replicated are the S&P 500 in the U.S. market, the FTSE 100 in the British market, or the Ibovespa in the Brazilian stock market[2].

To bridge the gap between liquidity and index tracking, this paper analyzes the liquidity impact in an index tracking optimization model. The objective of this article is to investigate the behavior of an index tracking model with the inclusion of liquidity constraints. Two different approaches are considered, each one relying on specific portfolio liquidity metrics. The first one, referred to as Weighted Average Liquidity (WAL), defines the portfolio liquidity as the weighted average of assets' liquidity in the same fashion

[2] For instance, we could mention the SPDR S&P 500 ETF Trust (SPY) and iShares Core S&P 500 ETF (IVV) as trackers for the S&P 500 in the U.S. market, the iShares S&P 500 FIC FI IE (IVVB11) as a tracker for the S&P 500 in the Brazilian market, or the iShares Ibovespa Fundo de Índice (BOVA11) as a tracker for the Ibovespa – the main Brazilian benchmark.

as Lo, Petrov e Wierzbicki (2006) and relies, separately, on financial volume, turnover, and Amihud's metric. The second one, referred to as Financial Value Liquidation (FVL), considers the monetary value of the constructed portfolio, and models liquidity with parameters used by portfolio managers in their practice simultaneously, namely the percentage limit of the total financial volume traded, the liquidation period, and the acceptable level of liquidation for portfolio as in Vieira e Filomena (2019). The overarching aim of this paper is threefold, as we seek (i) to understand the impact of a liquidity constraint on the index tracking problem, (ii) to compare the two approaches in terms of performance and liquidity of the tracking portfolios, and (iii) to analyze the diversification characteristics of the constructed portfolios.

We carry out an extensive empirical analysis based on the main benchmark for the Brazilian stock market (Ibovespa index) and Brazilian Small Cap index (SMLL) whose constituents are less liquid than the stocks included in the Ibovespa index. A rolling horizon framework with in-sample training and out-of-sample cross-validation period is adopted. Considering the Ibovespa and the SMLL, a total of 1209 and 294 portfolios are generated, respectively. Thus, including the two indexes and the benchmark portfolio, a comprehensive empirical investigation was conducted with a total of 1505 portfolios.

The overall results are consistent with the findings from previous studies (LO; PETROV; WIERZBICKI, 2006; VIEIRA; FILOMENA, 2019) in which constructed portfolios subjected to liquidity requirements turned out to be more liquid than the portfolios free of liquidity conditions. In fact, the results show that the proposed models that include liquidity constraints within the tracking optimization model have a higher liquidity, which is on average about 60% (WAL) and 16% (FVL) higher than the liquidity of the traditional tracking portfolios that do not include liquidity constraints. In scenarios that impose the highest liquidity requirement, liquidity increases amounting to 108% and 53% were observed with WAL and FVL respectively when compared to the traditional tracking portfolios. Yet, as one would expect intuitively, including an additional constraint in the optimization model to regulate portfolio liquidity leads to a larger tracking error, i.e., worse performance in terms of replicating the targeted market benchmark returns over time. As a result, one can argue on the practical suitability of the proposed liquidity constraints as they were able to produce portfolios with larger liquidity, thus demonstrating a trade-off between liquidity and tracking performance that could be addressed by portfolio managers especially when dealing with liquidity requirements imposed by market regulators.

Another insight provided by this study regards the number of assets included in the tracking portfolios and how it varies with the required liquidity level. On one hand, when considering the WAL liquidity-constrained portfolios (LO; PETROV; WIERZBICKI, 2006), it turns out that imposing stricter liquidity requirements considerably reduces the number of positions in the constructed portfolio since very few stocks can provide the prescribed liquidity level. To the limit, if the required portfolio liquidity is above the liquidity of every asset but one, the constructed portfolio will include this single asset only. On the other hand, when modeling the portfolio liquidity based on the FVL approach as in Vieira e Filomena (2019), stricter liquidity requirements result into tracking portfolios including a larger number of securities since this approach considers the sum of the monetary value traded of all portfolio components. Under this approach, the number of assets in the portfolios increase monotonically with the tightness of the liquidity requirements. The results related to the number of assets included in the constructed portfolios are in line with the conclusions obtained by investigating the diversification features of the WAL and FVL approaches. We use the Gini index to conduct an empirical test that shows that, as liquidity requirements get tighter, asset diversification decreases with the WAL approach, while, on the other hand, the strictness of the liquidity conditions has no impact on diversification with the FVL approach.

This paper contributes to the literature on both index tracking and risk by assessing the impact of including liquidity constraints on index tracking optimization models and on the composition of the constructed portfolios. Liquidity is a critical factor to mitigate financial risk. We show that the inclusion of liquidity requirements can considerably raise the liquidity level of the tracking portfolios and by corollary reduce financial risk. As liquidity is a most relevant factor to mitigate financial risk, we show that the inclusion of liquidity requirements can considerably raise the liquidity level of the tracking portfolios. Besides the liquidity metrics used in previous studies (see Section 3.2 and the comments on the WAL approach), this research explores and extends a new liquidity approach (see FVL approach presented in 3.3) introduced most recently by Vieira e Filomena (2019). We also provide an asset diversification analysis which shows quite contrasted outcomes for the WAL and FVL approaches. Furthermore, to the best of our knowledge, this study is the first one introducing and testing Amihud's illiquidity metric in a portfolio optimization problem.

This paper is structured as follows. In Section 2, we discuss the theoretical aspects regarding liquidity and index tracking. In Section 3, we present index tracking models representative of the two liquidity approaches investigated in this study. In Section 4, we describe the results of the numerical experiments. Finally, Section 5 concludes the paper. C, D, and E provide the results of additional tests that complement our main findings.

2 Liquidity Background and Overview

In this section, we describe the liquidity theoretical background. Subsection 2.1 presents the main liquidity metrics. Subsection 2.2 discusses the inclusion of liquidity requirements within portfolio optimization modelled as constraints. Finally, we provide a brief review of the index tracking literature (Subsection 2.3).

2.1 Asset Liquidity Metrics

Liquidity is a multi-dimensional concept encompassing mainly financial volume (i.e., volume in dollar amount, corresponding to the total value of all shares traded during a day, for example), number of transactions, volatility, size of the company, bid-ask spread, and share price (DEMSETZ, 1968). There is no general consensus over a precise liquidity definition, and the metrics mentioned above are viewed as proxied liquidity measures.

Financial volume is widely used as an approximation for liquidity. According to Gabrielsen, Marzo e Zagaglia (2011), indexes based on the volume traded are correlated with the impact on transaction costs, and such impact could be captured merely through the use of the total monetary value of shares traded. There are multiple studies that use volume as a liquidity proxy (BRENNAN; CHORDIA; SUBRAHMANYAM, 1998; ZAGST; KALIN, 2007; DAROLLES; FOL; MERO, 2015). The turnover rate relates the total volume transacted and the total assets outstanding and is another volume-based metric used for liquidity. Equation (2.1) presents the definition of turnover used in several studies (for instance Datar, Naik e Radcliffe (1998), Chordia, Subrahmanyam e Anshuman (2001), Chan e Faff (2003), Jun, Marathe e Shawky (2003), Marshall e Young (2003)) to approximate liquidity.

$$\text{Turnover}_i = \frac{\text{Volume}_i}{\text{Total Assets Outstanding}_i} \tag{2.1}$$

where Volume_i is the volume of asset i during a specific time window.

Another extensively used liquidity measure is the bid-ask spread, i.e., the difference between the price offered by the buyer and the price demanded by the seller. Amihud e Mendelson (1986) use the bid-ask spread to investigate the relationship between return and liquidity. Other studies that also use the bid-ask spread are Brennan e Subrahmanyam (1996), Atkins e Dyl (1997), Jacoby, Fowler e Gottesman (2000), and Chung e Chuwonganant (2014).

In Amihud (2002), illiquidity is defined as the ratio between the absolute return and the financial volume traded during some period of time. This measure can be interpreted as the price response associated with the volume, and can therefore be viewed as a measure of price impact. Equation (2.2) defines Amihud's illiquidity metric, which has been used in recent studies (see, for instance Amihud et al. (2015), Ben-Rephael, Kadan e Wohl (2015), Ho e Chang (2015), Chacko, Das e Fan (2016), Barardehi, Bernhardt e Davies (2018)):

$$A_i = \frac{1}{D_i} \sum_{t=1}^{D_i} \frac{|r_{it}|}{\lambda_{it}} \tag{2.2}$$

where r_{it} is the return of asset i at time t, λ_{it} is the financial volume for the i-th asset at t, and D_i is the number of time intervals (i.e., the number of days).

2.2 Liquidity Constraints

Liquidity can be factored in investment decisions in three ways: pre-filtering of the data, inclusion of individual liquidity constraints on each asset in the optimization problem, and inclusion of a global liquidity constraint on the weighted average liquidity of all stocks in the portfolio (LO; PETROV; WIERZBICKI, 2006).

By using pre-filtering, the goal is to treat and refine the considered asset universe based on a minimal liquidity level condition. As a result, only assets with a certain minimum liquidity level would remain under consideration. In this sense, pre-filtering is not a constraint inside an optimization problem but is based on the ex-ante pre-processing of the asset universe resulting in the exclusion of assets which do not satisfy a prescribed liquidity threshold.

Another way to enforce liquidity conditions is to incorporate individual asset constraints, which consider the financial value allocated in each asset included in the portfolio and the liquidity requirement. The maximum amount that can be invested in a given asset will depend on its liquidity. The individual liquidity asset constraints based on the financial volume can be written (VIEIRA; FILOMENA, 2019) as:

$$x_i \delta \leq l_i \tag{2.3}$$

where x_i is the weight of the i-th asset in the portfolio, δ is the total portfolio value, and l_i is the liquidity of the i-th asset in the portfolio.

Rather than pre-filtering assets or considering the liquidity of assets individually, a constraint that considers the portfolio liquidity as a whole can be formulated (see Lo, Petrov e Wierzbicki (2006)). Such a global liquidity constraint ensures that the weighted average liquidity of the portfolio components is higher than a certain minimum liquidity threshold. Hence, assets that would potentially be excluded if one would use pre-filtering can remain under consideration as the lower liquidity of an asset could be compensated by holding a position in an asset with above-average liquidity. The global liquidity constraint on the portfolio's weighted average liquidity (LO; PETROV; WIERZBICKI, 2006) reads:

$$\sum_{i=1}^{N} l_i x_i \geq l_p \tag{2.4}$$

where l_i is the liquidity of the i-th asset in the portfolio, x_i is the weight of the i-th asset in the portfolio, l_p is the minimum level required for the portfolio liquidity, and N is the number of assets.

Lo, Petrov e Wierzbicki (2006) also point out limitations associated with the approach based on the portfolio weighted average liquidity. Imposing this constraint has for consequence to disregard the portfolio monetary value, because it considers the weight of each asset individually. Thus, the use of portfolio's weighted average liquidity as defined above does not consider the impact of the capitalization of the portfolio on its liquidity. Indeed, liquidating portfolios with large financial value can be problematic. In order to address this shortcoming, Vieira e Filomena (2019) proposed an alternative way to define portfolio liquidity, in which the portfolio liquidity denoted l_p is defined as:

$$l_p = \sum_{i=1}^{N} \theta_i \tag{2.5}$$

where θ_i is the value of the i-th stock that can be liquidated. Note that θ_i is the minimum between the monetary volume allocated on the i-th asset in the portfolio and the maximum monetary value allowed to be liquidated on this asset, and can be modelled with the following system of equalities (2.2)-(2.4):

$$\theta_i = \min(\text{Allocated Value}, \text{Maximum Allowed Liquidation}) \tag{2.6}$$

$$\text{Allocated Value} = x_i \delta \tag{2.7}$$

$$\text{Maximum Allowed Liquidation} = \lambda_i \rho \gamma \tag{2.8}$$

Equations (2.3) and (2.4) respectively define the monetary volume allocated and the maximum monetary volume allowed to be liquidated for the i-th. The portfolio monetary value is denoted by δ, ρ is the percentage limit of the total volume traded, γ is the liquidation period, and λ_i is the monetary volume for i.

2.3 Index tracking

Index tracking is a form of passive investment approach, in which the objective is to compose a portfolio that reproduces the returns of a market index. The easiest way to build a tracking portfolio is to hold all assets of the tracked market index in the same proportion. However, this form of allocation can lead to several problems, such as high transaction costs and excessively low allocations for certain assets (BARRO; CANESTRELLI, 2009; CANAKGOZ; BEASLEY, 2009). For this reason, index tracking optimization often involves the addition of a cardinality constraint to restrict the size of the portfolio, as it has been discussed by, for instance, Murray e Shek (2012) and Scozzari et al. (2013). The challenge is to locate a portfolio that behaves as closely as possible to the tracked index and, at the same time, incur reduced costs (LEJEUNE, 2012; LEJEUNE; SAMATLI-PAÇ, 2013). We do not implicitly constraint the number of assets in our study, but we show the implications of different liquidity approaches on the number of assets.

Gaivoronski, Krylov e Wijst (2005) propose alternative approaches for index tracking. Heuristic methods are widely used by researchers, especially in cases where combinatorial and integrality restrictions require an intensive computational effort (KRINK; MITTNIK; PATERLINI, 2009; SCOZZARI et al., 2013). In Yu, Zhang e Zhou (2006), the downside risk is applied as a risk measure. Cointegration is another approach followed by several authors (DUNIS; HO, 2005; SANT'ANNA et al., 2019; SANT'ANNA; CALDEIRA; FILOMENA, 2020). Our index tracking model is aligned with the approach proposed by Sant'Anna et al. (2017). In Section 3, we describe the models used for the empirical analysis and present the formulation of the index tracking optimization model.

3 Index Tracking with Liquidity Requirements

In this section, we first present in Subsection 3.1 the base index tracking optimization model. Subsections 3.2 and 3.3 present two approaches (i.e., WAL and FVL) to enforce liquidity requirements via the incorporation of linear constraints.

3.1 Base Index Tracking Formulation

The proposed optimization model is an index tracking model that minimizes the square of errors between the portfolio and the index returns. The base model (BM) takes the form of a convex quadratic programming optimization problem in which the objective is to minimize the variance of the difference between portfolio and index returns (3.1) and has a linear feasible set defined by the budget (3.2) and no-shortselling (3.3) constraints

$$\min \quad \frac{1}{T}\sum_{t=1}^{T}\left[\left(\sum_{i=1}^{N} x_i r_{i,t}\right) - R_t\right]^2 \tag{3.1}$$

$$\text{s.to} \quad \sum_{i=1}^{N} x_i = 1 \tag{3.2}$$

$$x_i \geq 0 \qquad \forall i \in N \tag{3.3}$$

where x_i is the weight of the i-th stock in the portfolio, $r_{i,t}$ is the return of stock i at t, and R_t is the index return at t. The notation T represents the number of in-sample (training) time periods (i.e., days) while N is the number of securities in the asset universe.

Constraints (3.2) and (3.3), respectively, set that the weights of all stocks in the portfolio should sum up to 100%, and that short-selling is not allowed (i.e, $x_i < 0$). We use thereafter the acronym BM to refer to the index tacking model defined by (3.1)-(3.3).

We recall that the objective of this study is not to come up with the "optimal" way to model index tracking. Our goal is to understand the impact of liquidity constraints on index tracking. Even though the index tracking problem can be formulated differently than BM, Vieira e Filomena (2019) demonstrated that the set of liquidity constraints are not dependent on the chosen model; the authors have shown that for minimum-variance, mean-variance and risk parity. Furthermore, with regards to the number of assets in the tracking portfolios, we have not included a cardinality constraint to limit the number of components for the portfolios, as the inclusion of a cardinality constraint results in the formulation of a quadratic mixed-integer optimization problem which poses computational challenges and the algorithmic aspects are not central to this study.

The three liquidity metrics considered in this study are the financial volume (i.e., total value of all shares traded in a specific period), the turnover (see (2.1)), and Amihud's illiquidity metric (see (2.2)). To compare the obtained results, we use the normalization approach proposed by Lo, Petrov e Wierzbicki (2006) and described below:

$$l_{i,t} = \frac{\tilde{l}_{i,t} - \min_{k,\tau} \tilde{l}_{k,\tau}}{\max_{k,\tau} \tilde{l}_{k,\tau} - \min_{k,\tau} \tilde{l}_{k,\tau}} \tag{3.4}$$

Lo, Petrov e Wierzbicki (2006) et al. considered the monthly liquidity. In this paper, we use the daily liquidity and normalize the daily liquidity considering each stock liquidity during all the sample period. The notation $\tilde{l}_{i,t}$ represents the liquidity for security i on day t, and the maximum and minimum are estimated over all stocks k in every day τ. As a result of the normalization, we have $0 \leq l_{i,t} \leq 1$, $\forall i, t$. We use the reciprocal of Amihud's monthly illiquidity metric so that larger numerical values imply more higher liquidity, similarly to the other two metrics. Next, we apply these liquidity metrics on two different liquidity constraint approaches: Weighted Average Liquidity and Financial Value Liquidation.

3.2 Weighted Average Liquidity - WAL

This approach defines the portfolio liquidity as the weighted average liquidity of tracking portfolio components as discussed by Lo, Petrov e Wierzbicki (2006). We introduce the constraint (3.5) in BM

$$\sum_{i=1}^{N} x_i l_i \geq l_p \tag{3.5}$$

where l_i is the liquidity of stock i and l_p is the smallest admissible liquidity level for the portfolio. In Section 4.1.2, we will conduct a sensitivity analysis with respect to the liquidity threshold l_p and observe how the structure of the constructed portfolios varies with l_p.

3.3 Financial Value Liquidation - FVL

In the Financial Value Liquidation approach (FVL), the portfolio liquidity is defined by three parameters or criteria proposed by Vieira e Filomena (2019): the percentage limit of the total financial volume traded ρ, the liquidation period γ, and the acceptable level of liquidation for portfolio ϕ. Accordingly, we had the three following linear inequalities (2.6)-(2.8) in the base formulation BM:

$$\begin{cases} \phi\delta \leq \sum_{i=1}^{N} \theta_i & \text{(3.6a)} \\ \theta_i \leq x_i\delta \qquad \forall i \in N & \text{(3.6b)} \\ \theta_i \leq \lambda_i\rho\gamma \qquad \forall i \in N & \text{(3.6c)} \end{cases}$$

The parameters ρ, γ, and ϕ are such that: (i) constraint (2.6) stipulates that the sum of the value that can be liquidated across all portfolio components (right side of the inequality) has to be larger than (or equal to) the minimum level of liquidation required for the entire portfolio, where ϕ defines the portion of the total portfolio value required as the minimum accepted liquidation level; (ii) constraint (2.7) requires that the value θ_i of the i-th stock cannot exceed the portion of the total portfolio value corresponding to the weight of the i-th stock in the tracking portfolio (x_i) (i.e., position in stock i as defined by (2.3)); and (iii) constraint (2.8) determines that θ_i has to be no larger than the maximum allowed liquidation level for stock i (right hand side of the inequality) as defined by (2.4).

4 Empirical Tests

In this section, we conduct numerical tests to investigate the impact of liquidity requirements on the portfolios constructed with index tracking optimization models. The empirical analysis investigates how index tracking portfolios respond to the incorporation of liquidity constraints. As explained in Section 3, we consider liquidity based on WAL and FVL approaches, since liquidity is a financial metric that lacks a broadly accepted definition in the literature.

As discussed in the introduction, liquidity is even more important in emerging markets than in developed markets. In terms of liquidity risks, managing USD 200 million in the United Stated is probably much simpler than in an emerging market. It might take a few hours to sell USD 200 million allocated in S&P 500 stocks, but it would be much more challenging if the investment is in stocks of the Brazilian Ibovespa index. We have chosen the Brazilian stock market to carry out the empirical analysis. In terms of market capitalization, B3 (the main Brazilian exchange) is a fraction of the premier exchanges from developed markets, but it is bigger than the main exchanges from other important emerging economies such as South Africa, Singapore, Russia, Indonesia, Mexico, Chile and Turkey (World Federation of Exchange, 2019).

Furthermore, to corroborate and reinforce the results of the empirical analysis based on the Ibovespa index – the main benchmark for the Brazilian stock market that includes the main public companies in the country–, we also examine the Small Cap Index[1], which includes smaller companies that have considerably smaller liquidity relatively to the stocks in the Ibovespa index. Therefore, the empirical tests explore distinct liquidity metrics with different indices composed by both large and small companies from an emerging market, and assess the impact of explicitly accounting for liquidity requirements when building an index tracking portfolio.

In Subsection 4.1, we present the results for the Ibovespa empirical tests. Subsection 4.2 discusses the results for the SMLL index. Finally, Subsection 4.3 is devoted to diversification.

4.1 Empirical Tests with the Ibovespa

The empirical tests start with the Ibovespa index and its components, as it is the main benchmark for the Brazilian financial market and is traded on B3 (formerly

[1] For more on this index, see http://www.b3.com.br/en_us/market-data-and-indices/indices/indices-de-segmentos-e-setoriais/smallcap-index-smll.htm – acessed on 28 September 2020.

BM&FBovespa), i.e., the leading Brazilian stock exchange. We describe the data, as well as the numerical experiments and results for both the WAL and FVL models.

4.1.1 Data and Simulation

We have selected 113 stocks that were part of the index at some point in time during the study period. All data have been extracted from Economatica (one of the leading providers of financial data in Brazil), and cover the period ranging from January 1st 2010 to September 1st 2018.

The empirical tests are based on a rolling horizon framework using 150 in-sample daily data points. For Ibovespa, we have solved a series of 1209 optimization problems to recursively construct the tracking portfolios. Each optimization problem uses as inputs the data of the 150 previous trading days to determine the composition of the incumbent tracking portfolio. After constructing a portfolio, its performance is evaluated during the next 20, 40, and 60 business days that constitute the out-of-sample or testing dataset, which means that the portfolios are updated on a monthly, bimonthly, and quarterly basis. As a result, for Ibovespa, we have a total number of 93 (resp., 46 and 31) portfolios with monthly (resp., bimonthly and quarterly) rebalancing frequency.

Additionally, in order to benchmark the liquidity-constrained portfolios, we have also constructed tracking portfolios without any liquidity constraints using the BM model (3.1)-(3.3) presented in Section 3 and have followed the same procedure: 150 data points in-sample, portfolio updating every 20, 40, and 60 business days, and no limitation on the number of securities in the portfolio. Hereafter, we refer to these portfolios as *benchmark portfolios*. Table 10 displays the number of stocks included in the benchmark portfolios. Considering the 20-day rebalancing period (93 portfolios out-of-sample), the average number of stocks included in the portfolio is 58.59. Table 10 also provides the distribution of the positions in the constructed portfolios. For example, 80% of the capital is on average allocated to 26.74 assets.

Table 10 – Number of stock components in the benchmark portfolios for 20 days rebalancing period.

Top 40%	Top 60%	Top 80%	Average	Standard deviation
6.65	13.84	26.74	58.59	5.57

Table 11 compares the Ibovespa index and the benchmark portfolios in terms of return, volatility, and turnover. It can be seen that the out-of-sample performance of the benchmark portfolios is very close to that of the index fund. As expected, the average monthly turnover gets smaller as the time interval between portfolio updates moves from 20 to 60 business days. The tracking error (TE) is defined as:

$$TE_t = \Big(\sum_{i=1}^{N} x_i r_{i,t}\Big) - R_t \tag{4.1}$$

Notice that the TE should not be confused with the squared error from (3.1). The TE can be negative given that it is difference between the return of the portfolio and the benchmark in a given period and indicates that the constructed portfolio was outperformed by the benchmark at that time t. We observe that the average TE is 0.001% for all three rebalancing period portfolios, with low TE volatility.

Table 11 – Comparison of return, volatility, and turnover for Ibovespa and the benchmark portfolios[1]

	Ibovespa	20 days	40 days	60 days
Annual average return	4.77%	4.81%	4.34%	4.47%
Cumulative return	42.92%	43.28%	39.03%	40.19%
Annual volatility	22.60%	22.75%	22.78%	22.78%
Daily TE average	-	0.00%	0.00%	0.00%
Daily TE volatility	-	0.14%	0.15%	0.15%
Monthly turnover	-	11.00%	7.65%	6.00%

[1] 1.1 Average Annual Return refers to the average of the cumulative returns for each year from 2010 to 2018. Cumulative Return refers to the return calculated cumulatively during the entire out-of-sample period. Daily Volatility accounts for the standard deviation (σ) of daily returns from 2010 to 2018, whereas annual volatility refers to $\sigma\sqrt{252}$. Daily TE Average and Daily TE volatility account for the average and standard deviations of the daily tracking errors from 2010 to 2018. Monthly turnover refers to the average portfolio rebalancing monthly turnover, for instance, the yearly turnover dived by 12 months period.

The annual TEs for the benchmark portfolios is shown in Table 12. The TE corresponds to the difference between the cumulative return of each portfolio and the cumulative return of the index during each year. We can see that the TEs are especially high in 2013 and 2014. In 2014, the three benchmark portfolios have an annual tracking error below -8%, which is certainly a consequence of the 2014 huge financial instability in Brazil. On the other hand, we also have small TEs in some years, such as 2010, 2016, and 2017.

From a computational point of view, the insertion of the liquidity constraints does not pose any particular challenge as they take the form of linear inequalities in the two proposed approaches (WAL and FVL). The additional time needed to solve the optimization problems after their incorporation is close to zero. As an illustration, we have solved multiple times two versions of the portfolio optimization problem, one with and one without liquidity constraints. The average solution times of the problems were basically identical (i.e., 0.1935 and 0.1824 seconds respectively).

Table 12 – Annual tracking error for the benchmark portfolios

Interval	20 days	40 days	60 days
2010	0.58%	0.24%	0.30%
2011	1.53%	0.83%	1.57%
2012	-0.50%	-0.50%	-0.07%
2013	5.31%	5.20%	4.63%
2014	-8.17%	-8.16%	-8.99%
2015	1.89%	0.47%	0.69%
2016	0.74%	-0.87%	0.23%
2017	0.33%	0.45%	-0.09%
2018	-1.79%	-1.78%	-1.00%
Average	-0.01%	-0.46%	-0.30%

4.1.2 Empirical Tests with WAL

We have performed several empirical tests using WAL to generate liquidity-constrained portfolios. We have set threshold levels for the three different liquidity criteria, i.e., volume, turnover, and Amihud's metric to specify the corresponding constraints. The values assigned to the liquidity thresholds are above the average liquidity calculated for the benchmark portfolios as shown later in this subsection. We have considered three values - 0.1, 0.15, and 0.2 - for the threshold of the volume metric. In this subsection, we describe the results when using volume as the metric to set the liquidity requirements for the tracking portfolios. Due to the similarity among the results for volume, turnover, and Amihud's metric, we discuss the WAL results for turnover and Amihud's metric in C.

Table 13 reports the average number of stocks included in the liquidity-constrained tracking portfolios, as well as the distribution of the position weights in the portfolios. The number of stocks included in the constructed portfolios decreases as the level of required liquidity increases. On average, portfolios with a minimum liquidity level of 0.2 include less than 24 stocks. This result can be explained from the characteristics of the WAL portfolio liquidity perspective. Considering the liquidity of each portfolio as the weighted average of the liquidity of the participating assets, the larger the required portfolio liquidity level is, the lower is the number of assets that can be included in the portfolio.

Using this definition of portfolio liquidity, highly liquid portfolios only include assets with high individual liquidity level, regardless of the portfolio financial value. In particular, the portfolio with the highest possible liquidity level is fully invested in the asset which has the highest individual liquidity. The standard deviation of the number of positions is higher when a higher liquidity is required. The concentration in the liquidity-constrained portfolios is much more pronounced than in the benchmark portfolios. While 80% of the capital is committed to slightly than half of the assets included in the benchmark portfolios, this percentage drops by one third in the case of liquidity-constrained portfolios with a 0.2 liquidity threshold. Table 13 shows that, on average, 80% of the capital in liquidity-constrained portfolios is invested in 8.11 assets, as compared to 23.59 assets in the benchmark portfolios.

Table 13 – Number of assets of Ibovespa portfolios liquidity constrained based on WAL

Liquidity	Top 40%	Top 60%	Top 80%	Average	Standard deviation
0.1	5.67	12.20	23.83	54.78	5.27
0.15	3.59	7.28	14.97	39.00	11.96
0.2	2.34	4.00	8.11	23.59	12.32

Furthermore, we observe in Table 14 that as the requirement imposed by the liquidity constraint gets stricter, for instance with the threshold value moving from 0.1 to 0.2, the difference between the average annual return of the constructed portfolio and of the index increases; on average, the annual return of the index is 4.77% (Table 11). Among the three considered rebalancing frequencies and for each minimum liquidity level, there was no marked difference in the returns of the constructed portfolios. It can be seen that in the portfolios subjected to the most stringent liquidity requirements, the average annual return is always above the average annual index return of 4.77%. This occurs both because the liquidity constraint forces to have large positions in assets with high individual liquidity level the most liquid assets overperformed the index during the study period. The annual volatility of the returns is very close to that of the index (22.6% – Table 11), with a slight increase as the liquidity requirement increases.

Comparing the daily tracking error of the liquidity-constrained portfolios with that of the benchmark portfolios, we observe an increase in the TE as the required liquidity level increases. In particular, when high liquidity requirements are set, the average daily tracking error tends to be always positive due to the fact that portfolios have returns above the index returns. The daily tracking error volatility increases when the rebalancing period becomes larger and with the increase in the liquidity requirements. Finally, the monthly turnover of the liquidity-constrained portfolios is larger as compared to that of the benchmark portfolios. For the 60-day rebalancing period, the turnover goes from 6% for the benchmark portfolios to more than 12% for the liquidity-constrained portfolios portfolios with the 20% liquidity requirement. For smaller rebalancing periods, a higher monthly turnover is observed.

Table 14 – Descriptive results of Ibovespa portfolios generated based on WAL[1]

Liquidity	0.1			0.15		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	5.39%	4.47%	5.43%	6.48%	6.01%	7.05%
Cumulative return	48.53%	40.25%	48.90%	58.34%	54.12%	63.41%
Annual volatility	22.65%	22.77%	22.84%	23.04%	23.23%	23.24%
Daily TE average	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%
Daily TE volatility	0.08%	0.13%	0.14%	0.18%	0.23%	0.24%
Monthly turnover	14.59%	8.77%	6.97%	20.36%	11.50%	10.10%

Liquidity	0.2		
Interval	20 days	40 days	60 days
Annual average return	8.33%	7.94%	10.70%
Cumulative return	75.01%	71.50%	96.27%
Annual volatility	23.96%	23.70%	24.36%
Daily TE average	0.02%	0.02%	0.03%
Daily TE volatility	0.37%	0.41%	0.53%
Monthly turnover	26.11%	14.43%	12.23%

[1] 1.1 Average Annual Return refers to the average of the cumulative returns for each year from 2010 to 2018. Cumulative Return refers to the return calculated cumulatively during the entire out-of-sample period. Daily Volatility accounts for the standard deviation (σ) of daily returns from 2010 to 2018, whereas Annual Volatility refers to $\sigma \times \sqrt{252}$. Daily TE Average and Daily TE volatility account for the average and standard deviations of the daily tracking errors from 2010 to 2018. Monthly Turnover refers to the average portfolio rebalancing monthly turnover.

The annual tracking error for liquidity-constrained portfolios is presented in Table 15. Their average tracking error is larger than that of the benchmark portfolios (Table 12), which was expected since imposing an extra constraint on the minimum liquidity level requires the solution of a more constrained optimization problem. Considering the 20-day rebalancing period, the annual TE equals 0.76%, 1.85% and 3.70% when the liquidity threshold is set to 0.1, 0.15, and 0.2, respectively. In contrast, the benchmark portfolios have an average annual TE equal to -0.01% (Table 12). This shows that, as expected, the benchmark portfolio performs better in terms of TE than WAL, in other words, the TE is closer to zero.

Table 15 – Annual tracking error for Ibovespa portfolios constructed with WAL

Liquidity metric	0.1			0.15		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
2010	1.72%	0.96%	1.02%	3.92%	1.41%	2.04%
2011	0.98%	1.84%	1.55%	1.17%	2.63%	3.52%
2012	1.40%	0.06%	1.47%	4.30%	1.52%	4.10%
2013	4.62%	5.24%	7.17%	7.37%	9.98%	10.08%
2014	-4.42%	-6.92%	-7.74%	-4.76%	-5.64%	-5.65%
2015	1.63%	1.28%	1.31%	0.77%	0.48%	2.23%
2016	0.71%	0.41%	1.24%	0.66%	-0.37%	1.89%
2017	0.54%	-0.41%	0.07%	1.70%	1.85%	2.01%
2018	-0.39%	-0.70%	-0.10%	1.49%	3.77%	0.27%
Average	0.76%	0.19%	0.66%	1.85%	1.74%	2.28%

Liquidity metric	0.2		
Interval	20 days	40 days	60 days
2010	6.15%	0.52%	1.95%
2011	1.55%	1.38%	16.08%
2012	4.59%	0.59%	3.95%
2013	12.92%	18.08%	16.43%
2014	-4.98%	-2.69%	-3.33%
2015	1.00%	1.75%	1.35%
2016	0.56%	-1.60%	1.81%
2017	3.11%	4.09%	0.99%
2018	8.39%	10.89%	4.11%
Average	3.70%	3.67%	4.82%

To compare the benchmark portfolios with the liquidity-constrained tracking portfolios derived using WAL, we have calculated the liquidity level of the benchmark portfolios using the WAL liquidity approach (see Table 16). Also, the liquidity of the WAL liquidity-constrained portfolios is given in Table 17. On average, the liquidity of the WAL liquidity-constrained portfolios is very close, if not equal, to the minimum acceptable liquidity level, which indicates that the liquidity constraints are binding and impact the construction of the portfolios. Considering a 10% liquidity threshold, the average liquidity level is close to that of the benchmark portfolios (Table 16). However, in the benchmark portfolio, there are days when the liquidity is significantly below – close to 5% – the prescribed liquidity level. The liquidity standard deviation is low in the constrained portfolio when the portfolio is rebalanced every 20 days. Even though the average liquidity of the benchmark portfolio and the WAL portfolio is similar, there will be many days when the liquidity constraint will be active for the WAL liquidity-constrained portfolios. For example, when the required liquidity threshold is 20%, the constraint is active almost every day. This generates an even greater reduction in the standard deviation for the 20-day rebalancing period. The increase in the duration of rebalancing period generates a higher standard deviation of the liquidity. Considering the out-of-sample results obtained for the 20% liquidity requirement and 20-day rebalancing period, the minimal daily liquidity level is 18.92%. However, as the rebalancing periodicity gets larger, the deviation from the required liquidity level increases accompanied by increments in the standard deviation. The overall average liquidity of the portfolios increases from 0.097 (benchmark portfolios) to 0.156 for the liquidity-constrained portfolios constructed with WAL, i.e., a 60% increment. For the scenarios with the highest liquidity requirements, the average liquidity of the portfolios increases from 0.097 (benchmark portfolios) to 0.202 for the WAL portfolios, i.e., a 108% gain.

Table 16 – Liquidity of benchmark portfolios based on WAL considering volume as liquidity metric

Interval	20 days	40 days	60 days
Average	0.0970	0.0981	0.0963
Max	0.1913	0.1716	0.1654
Min	0.0510	0.0517	0.0554
Standard deviation	2.86%	2.80%	2.60%

Table 17 – Liquidity for Ibovespa portfolios constructed with WAL

Liquidity metric	0.1			0.15		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.1104	0.1140	0.1136	0.1504	0.1560	0.1548
Max	0.1868	0.1690	0.1814	0.1868	0.2541	0.2726
Min	0.0951	0.0834	0.0817	0.1419	0.1197	0.1028
Standard deviation	1.75%	2.19%	2.25%	0.50%	2.79%	3.57%

Liquidity metric	0.2		
Interval	20 days	40 days	60 days
Average	0.2010	0.2070	0.1993
Max	0.2121	0.3392	0.3013
Min	0.1892	0.1605	0.1303
Standard deviation	0.46%	3.90%	4.05%

4.1.3 Empirical Tests with FVL

We have considered twenty-seven scenarios to conduct the test for the liquidity-constrained portfolios constructed with FVL. The scenarios differ in the values assigned to the liquidity parameters of the model, namely the total portfolio value δ, the liquidation period γ, the acceptable liquidation level ϕ, and the rebalancing period.

Table 18 displays the out-of-sample liquidity results of the benchmark portfolios (BM) based on the FVL metrics. Keeping γ and the duration of the rebalancing period constant, we notice that the portfolios with higher monetary value δ have less liquidity. For instance, the portfolios constructed when δ is set to 30%, the rebalancing period is 20 days, and the liquidity required in 1 day (γ=1) have an average liquidity of 61.98%. When we set δ to 40% (instead of 30%) and keep all the other parameters unchanged, the liquidity drops to 53.10%.

To ease the presentation of the results (and keeping the tables of reasonable size) and because the FVL approach to liquidity endogenously considers the ratio of δ/γ to

Table 18 – Liquidity of Ibovespa benchmark portfolios constructed with FVL

γ-δ-Interval	1-0.2-20	1-0.2-40	1-0.2-60	1-0.3-20	1-0.3-40	1-0.3-60
Average	72.09%	72.50%	71.51%	61.98%	62.38%	61.50%
Max	90.43%	93.37%	86.16%	78.18%	78.66%	75.95%
Min	56.26%	57.11%	57.25%	45.74%	45.98%	49.15%
Standard deviation	9.16%	9.63%	8.94%	7.68%	8.04%	7.20%
γ-δ-Interval	1-0.4-20	1-0.4-40	1-0.4-60	4-0.8-20	4-0.8-40	4-0.8-60
Average	53.10%	53.58%	52.89%	72.10%	72.52%	71.31%
Max	70.83%	69.13%	65.64%	90.81%	93.24%	86.75%
Min	38.17%	39.07%	41.08%	56.30%	57.03%	56.94%
Standard deviation	7.13%	7.37%	6.78%	9.18%	9.63%	8.89%
γ-δ-Interval	4-1.2-20	4-1.2-40	4-1.2-60	4-1.6-20	4-1.6-40	4-1.6-60
Average	62.00%	62.37%	61.22%	53.14%	53.50%	52.64%
Max	79.14%	78.21%	75.47%	72.45%	68.58%	65.31%
Min	45.70%	45.94%	48.77%	38.08%	39.04%	40.46%
Standard deviation	7.68%	7.93%	7.18%	7.11%	7.25%	6.80%
γ-δ-Interval	8-1.6-20	8-1.6-40	8-1.6-60	8-2.4-20	8-2.4-40	8-2.4-60
Average	72.14%	72.49%	71.28%	62.04%	62.30%	61.08%
Max	91.77%	92.20%	87.52%	79.42%	76.95%	74.85%
Min	56.18%	56.66%	57.04%	45.45%	45.68%	48.65%
Standard deviation	9.20%	9.62%	8.94%	7.68%	7.79%	7.16%
γ-δ-Interval	8-3.2-20	8-3.2-40	8-3.2-60			
Average	53.19%	53.39%	52.46%			
Max	72.93%	67.17%	64.78%			
Min	37.74%	38.85%	39.56%			
Standard deviation	7.12%	7.11%	6.81%			

build the in-sample portfolios, we provide a simplification on the scenarios in which we relate the δ/γ ratio to specific combinations of δ and γ. Table 19 provides a summary of these combinations. Notice that in order to compare the out-of-sample liquidity, we use the specific values of δ and γ and not the ratio δ/γ;the out-of-sample liquidity might be impacted by different combinations of δ and γ even if the ratio δ/γ is constant.

Table 19 – Relationship between δ/γ ratio and specific combinations of δ and γ.

δ/γ	δ	γ
0.2	0.2	1
	0.8	4
	1.6	8
0.3	0.3	1
	1.2	4
	2.4	8
0.4	0.4	1
	1.6	4
	3.2	8

The number of portfolio positions for the scenarios of δ/γ and ϕ is given in Table 20. We observe that the number of positions is almost similar to the number of assets in the benchmark portfolio. Under the scenario with $\delta/\gamma = 30\%$ and $\phi = 50\%$, the average number of positions is 52.09 (as well as the distribution of the assets' weights) is fairly similar to that of the benchmark portfolios which is 58.59 (Table 10). However, the standard deviation of the number of assets in the portfolio was lower in all analyzed scenarios. This occurs because, on one hand, in the benchmark portfolio, there are days when very few assets are included in the portfolio. On the other hand, in the WAL liquidity-constrained portfolios, several assets are almost systematically included.

Table 20 – Number of participant assets of FVL Ibovespa portfolios

δ/γ; ϕ	Top 40%	Top 60%	Top 80%	Average	Standard deviation
0.3; 0.5	6.04	12.27	23.40	52.09	3.42
0.4; 0.5	6.12	12.40	23.57	52.47	3.22
0.3; 0.7	6.25	12.80	24.35	54.03	2.95
0.2; 1	6.17	12.49	23.94	55.05	2.95

Using the FVL liquidity approach, it appears that high-liquidity requirements leads to investments in a larger number of assets when compared to WAL. As the FVL liquidity concept depends on the traded volume of each asset, more stocks are needed to reach the liquidity requirement. The portfolio with the highest liquidity level is the one easier to turn into cash in a specified period. Therefore, there is a limitation regarding the portfolio value. Considering the extreme case ($\rho = 100\%$, $\gamma = 1$ day and $\phi = 100\%$), the largest admissible financial value is equal to the daily traded financial volume of all assets. In this case, the portfolio would consist of all available assets.

The characteristics of the portfolios constructed with the FVL approach are presented in Table 21. The table shows that in each scenario the constructed portfolios have a larger annual average return than this of the the benchmark portfolios (see Table 11). In the liquidity-constrained portfolios, capital is invested in assets exhibiting a performance superior to that of the index. In the most liquidity-constrained scenario with $\phi = 1$, the average annual return is twice large than the average annual return of the benchmark under the 40-day rebalancing period. Additionally, the annual volatility turns out to be similar to that of the benchmark. In some cases, the volatility of the liquidity-constrained portfolios is even lower than the volatility of the benchmark portfolios. In the more liquidity-constrained scenarios, TE volatility increases and becomes superior to that of the benchmark portfolio (Table 11). Finally, we notice that the monthly turnover of the liquidity-constrained portfolios is close to that of the benchmark portfolios in all scenarios.

Table 21 – Descriptive results of Ibovespa portfolios generated based on FVL[1]

δ/γ; ϕ	0.3; 0.5			0.4; 0.5		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	6.69%	7.11%	7.56%	6.71%	7.06%	7.55%
Cumulative return	60.20%	63.98%	68.03%	60.38%	63.54%	67.93%
Annual volatility	22.62%	22.64%	22.77%	22.62%	22.63%	22.76%
Daily TE average	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%
Daily TE volatility	0.13%	0.17%	0.18%	0.13%	0.17%	0.18%
Monthly turnover	11.18%	7.37%	5.91%	11.21%	7.33%	5.85%
δ/γ; ϕ	0.3; 0.7			0.2; 1.0		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	7.03%	6.95%	10.30%	7.92%	8.71%	8.65%
Cumulative return	63.29%	62.57%	92.73%	71.30%	78.37%	77.89%
Annual volatility	22.42%	22.31%	21.90%	22.78%	22.80%	22.92%
Daily TE average	0.01%	0.01%	0.03%	0.02%	0.02%	0.02%
Daily TE volatility	0.23%	0.28%	0.43%	0.20%	0.22%	0.22%
Monthly turnover	13.25%	8.18%	6.65%	11.86%	6.86%	5.40%

[1] 1.1 Average Annual Return refers to the average of the cumulative returns for each year from 2010 to 2018. Cumulative Return refers to the return calculated cumulatively during the entire out-of-sample period. Daily Volatility accounts for the standard deviation (σ) of daily returns from 2010 to 2018, whereas Annual Volatility refers to $\sigma \times \sqrt{252}$. Daily TE Average and Daily TE volatility account for the average and standard deviations of the daily tracking errors from 2010 to 2018. Monthly Turnover refers to the average portfolio rebalancing monthly turnover.

The annual tracking error for the portfolios constructed under the FVL approach is shown in Table 22. Compared to the benchmark portfolios (see Table 12), the annual tracking error is considerably higher. The tracking error for the liquidity-constrained portfolios is especially high in 2012 and 2013 (in each scenario), while, for the benchmark portfolios, the largest tracking error occurs in 2014. We also observe that the average annual tracking error grows with the strictness of the liquidity requirement.

Table 22 – Annual tracking error for Ibovespa portfolios constructed with WAL

δ/γ; ϕ	0.3; 0.5			0.4; 0.5		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
2010	0.61%	0.61%	0.84%	0.79%	0.55%	0.94%
2011	0.31%	3.01%	3.24%	0.32%	2.72%	3.00%
2012	8.15%	9.49%	10.57%	8.21%	9.48%	10.63%
2013	10.02%	10.97%	12.26%	10.03%	10.99%	12.26%
2014	-5.21%	-5.59%	-6.21%	-5.25%	-5.60%	-6.21%
2015	0.90%	0.44%	0.06%	0.90%	0.44%	0.06%
2016	0.81%	0.61%	2.22%	0.81%	0.61%	2.22%
2017	1.43%	1.29%	2.03%	1.41%	1.27%	2.00%
2018	0.27%	0.22%	0.10%	0.24%	0.17%	0.10%
Average	1.92%	2.34%	2.79%	1.94%	2.29%	2.78%
δ/γ; ϕ	0.3; 0.7			0.2; 1		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
2010	-0.78%	-2.86%	4.88%	1.55%	1.49%	1.52%
2011	2.95%	3.46%	23.59%	0.36%	3.46%	4.09%
2012	9.17%	10.24%	11.27%	11.86%	13.15%	12.31%
2013	10.68%	11.73%	12.61%	14.50%	15.52%	16.94%
2014	-5.11%	-5.44%	-6.32%	-4.52%	-3.72%	-4.73%
2015	0.73%	0.30%	-0.23%	-1.61%	-0.74%	-2.39%
2016	0.79%	0.47%	2.02%	2.74%	2.40%	4.50%
2017	1.81%	1.69%	1.94%	3.31%	3.29%	2.52%
2018	0.13%	0.06%	0.06%	0.19%	0.60%	0.21%
Average	2.26%	2.18%	5.53%	3.15%	3.94%	3.89%

Table 23 shows the liquidity level with the FVL approach for ϕ equal to 0.5, 0.7, and 1. The FVL portfolios are more liquid than benchmark portfolios (see Table 18), even in the case where the liquidity constraint is relatively soft ($\phi = 0.5$). Considering $\phi = 0.5$, the average liquidity level is above the level prescribed. Indeed, in most days, the constraint is not active, that is, the benchmark portfolio has a liquidity higher than the level required. However, the minimum liquidity approaches the requirement in all scenarios. Considering $\phi = 0.7$, there are some scenarios in which the average liquidity is above the level required. For instance, when $\phi = 1$, the liquidation of the entire portfolio is necessary in a specified period, which means that the liquidity constraint is active at every period (i.e., day) of the training period.

Table 23 – Liquidity for Ibovespa portfolios constructed with WAL

$\phi = 0.5$						
γ-δ-Interval	1-0.3-20	1-0.3-40	1-0.3-60	1-0.4-20	1-0.4-40	1-0.4-60
Average	66.13%	67.03%	66.90%	55.76%	56.64%	56.19%
Max	81.96%	79.94%	80.58%	73.78%	70.74%	69.16%
Min	51.59%	54.33%	54.61%	47.78%	45.38%	45.45%
Standard deviation	6.14%	6.44%	6.53%	5.71%	6.41%	6.44%
γ-δ-Interval	4-1.2-20	4-1.2-40	4-1.2-60	4-1.6-20	4-1.6-40	4-1.6-60
Average	66.19%	66.93%	66.55%	55.79%	56.50%	55.87%
Max	83.32%	79.26%	80.13%	75.48%	68.45%	68.82%
Min	50.88%	53.68%	54.33%	46.89%	45.28%	45.17%
Standard deviation	6.15%	6.30%	6.60%	5.74%	6.23%	6.44%
γ-δ-Interval	8-2.4-20	8-2.4-40	8-2.4-60	8-3.2-20	8-3.2-40	8-3.2-60
Average	66.26%	66.77%	66.33%	55.84%	56.31%	55.68%
Max	83.60%	78.52%	79.41%	75.87%	67.78%	68.09%
Min	49.93%	52.82%	54.37%	46.10%	44.72%	44.82%
Standard	6.11%	6.17%	6.58%	5.79%	6.09%	6.37%
$\phi = 0.7$						
γ-δ-Interval	1-0.3-20	1-0.3-40	1-0.3-60	4-1.2-20	4-1.2-40	4-1.2-60
Average	71.06%	70.98%	70.04%	70.98%	70.90%	69.77%
Max	81.96%	84.22%	82.18%	83.32%	84.45%	81.75%
Min	67.40%	61.50%	58.57%	65.48%	61.59%	57.64%
Standard deviation	2.76%	5.43%	6.15%	2.99%	5.35%	6.22%
γ-δ-Interval	8-2.4-20	8-2.4-40	8-2.4-60			
Average	70.92%	70.73%	69.62%			
Max	83.60%	84.69%	81.00%			
Min	64.51%	61.68%	56.83%			
Standard deviation	3.28%	5.24%	6.26%			
$\phi = 1$						
γ-δ-Interval	1-0.2-20	1-0.2-40	1-0.2-60	4-0.8-20	4-0.8-40	4-0.8-60
Average	99.21%	95.09%	93.36%	98.59%	94.97%	93.11%
Max	100.00%	99.82%	99.97%	99.97%	99.84%	99.99%
Min	96.67%	85.89%	82.40%	92.90%	85.64%	81.57%
Standard deviation	0.77%	3.45%	4.78%	1.23%	3.38%	4.97%
γ-δ-Interval	8-1.6-20	8-1.6-40	8-1.6-60			
Average	97.96%	94.84%	93.00%			
Max	99.99%	99.82%	99.96%			
Min	91.22%	85.62%	80.30%			
Standard deviation	1.60%	3.27%	5.00%			

The fact that the liquidity constraint is constantly active implies that liquidity has less variability (volatility) since it is always close to the required level, which in turn explains the sharp decrease in the liquidity standard deviation. The overall average liquidity of the portfolios raises from 62.32% (benchmark portfolios) to 72.19% for the liquidity-constrained portfolios constructed with FVL, i.e., approximately a 16% increment. For the scenarios with the highest liquidity requirements, the average liquidity of the portfolios increases from 62.32% (benchmark portfolios) to 95.57% for the the most liquidity-constrained scenarios of FVL portfolio, i.e., a 53% gain.

To sum up, the above results show the consistency of the results obtained with both

WAL and FVL that allow for the construction of more liquid index tracking portfolios. However, a direct liquidity comparison between the two models in terms of liquidity is not possible. We can not say which approach generates portfolios with higher liquidity since both models enforce a different liquidity proxy. Thus, although liquidity increases on average, there is no guarantee that a portfolio generated in WAL will have an acceptable FVL liquidity level.

In D, we present the formal statistical tests we conducted and whose results show that the liquidity level of the proposed liquidity-constrained portfolios is statistically different from that of the benchmark portfolio which is liquidity-unrestricted, as postulated in this subsection.

4.2 Empirical tests with the SMLL index

Additionally to the empirical analysis based on the Ibovespa index, we have also constructed and analyzed liquidity-constrained portfolios for the Small Cap Index market benchmark (SMLL) which is composed by a set of small cap companies traded on the Brazilian stock market and can be viewed - to some extent - as the Brazilian equivalent to the Russell 2000 index. Since SMLL includes small cap firms, its constituents have considerably lower liquidity in comparison to the assets in the Ibovespa index. Hence, this analysis will allow us to assess the impact of liquidity constraints in index tracking models when only securities with (very) low liquidity are considered.

Our dataset contains the daily closing prices for the SMLL index and a set of 101 stocks included in it for the period ranging from January 1st 2016 to August 8th 2020. As this index has a high turnover with regular changes in its constituents and several small companies are relatively new in the stock market, data availability, in particular before 2016, is a challenge. That is why we have decided to restrict our attention to the 2016-2020 period (different from the Ibovespa analysis carried out on data starting in 2010). The total of 101 stocks were part of SMLL at some point during the study period. The empirical tests on the SMLL index are executed in the same way they were for the Ibovespa index. For this analysis, we have only used volume as liquidity metric in the WAL approach. We have not used the turnover and Amihud's metric since the results for the Ibovespa index (see C) showed that these two metrics led to similar findings as those obtained with the volume metric.

Table 24 shows the results for the number of portfolio positions in the benchmark portfolios. On average, the benchmark portfolios have 65.69 components. Similarly to what was observed for the Ibovespa index, approximately 80% of the capital is allocated to 50% of the assets included in the portfolios. The descriptive results and the annual tracking error of the benchmark portfolios are shown in Tables 25 and 26. The average annual return of the benchmark portfolios is close to the index return index. The same observation prevails for the volatility. We also note that increasing the portfolio rebalancing frequency tends to raise the annual tracking error.

Table 24 – Number of participant assets of SMLL benchmark portfolios

Top 40%	Top 60%	Top 80%	Average	Standard deviation
11.86	20.81	33.95	65.69	4.92

Table 25 – Descriptive results of SMLL and the benchmark portfolios[1]

	SMLL	20 days	40 days	60 days
Annual average return	11.93%	12.05%	12.31%	13.04%
Cumulative return	59.67%	60.24%	61.55%	65.20%
Annual volatility	24.81%	24.89%	24.83%	25.01%
Daily TE average	-	0.00%	0.00%	0.01%
Daily TE volatility	-	0.21%	0.21%	0.23%
Monthly turnover	-	17.21%	11.59%	9.34%

[1] 1.1 Average Annual Return refers to the average of the cumulative returns for each year from 2016 to 2020. Cumulative Return refers to the return calculated cumulatively during the entire out-of-sample period. Daily Volatility accounts for the standard deviation (σ) of daily returns from 2016 to 2020, whereas Annual Volatility refers to $\sigma \times \sqrt{252}$. Daily TE Average and Daily TE volatility account for the average and standard deviations of the daily tracking errors from 2016 to 2020. Monthly Turnover refers to the average portfolio rebalancing monthly turnover.

Table 26 – Annual tracking error for the SMLL benchmark portfolios

Interval	20 days	40 days	60 days
2017	1.06%	1.35%	1.44%
2018	-0.35%	0.18%	1.82%
2019	1.39%	1.79%	3.59%
2020	-1.53%	-1.44%	-1.32%
Average	0.15%	0.47%	1.38%

Tables 27 and 28 show the liquidity of the benchmark portfolios estimated using the WAL and FVL approaches, respectively. For the WAL approach, three different minimum levels of liquidity are needed. As explained above, the tests are reported when using volume to proxy liquidity. For the FVL approach, we consider four cases with distinct required liquidity levels (in a similar manner to the tests for Ibovespa described in Section 4.1.3).

Table 27 – Liquidity of SMLL benchmark portfolios, based on WAL

Metric – interval	Volume-20	Volume-40	Volume-60
Average	0.0132	0.0143	0.0156
Max	0.0646	0.0621	0.0636
Min	0.0088	0.0089	0.0087
Standard deviation	0.83%	1.09%	1.34%

Table 28 – Liquidity of SMLL benchmark portfolios, based on FVL

γ-δ-Interval	1-0.03-20	1-0.05-20	1-0.07-20	1-0.09-20
Average	0.8280	0.6720	0.5629	0.4830
Max	0.9751	0.9568	0.9160	0.8662
Min	0.7192	0.5196	0.4180	0.3514
Standard deviation	6.38%	8.19%	8.78%	8.76%
γ-δ-Interval	2-0.06-20	2-0.10-20	2-0.14-20	2-0.18-20
Average	0.8289	0.6733	0.5644	0.4844
Max	0.9745	0.9562	0.9126	0.8639
Min	0.7221	0.5274	0.4248	0.3572
Standard deviation	6.32%	8.13%	8.73%	8.71%
γ-δ-Interval	1-0.03-40	1-1.05-40	1-0.07-40	1-0.09-40
Average	0.8190	0.6762	0.5722	0.4944
Max	0.9800	0.9466	0.9190	0.8856
Min	0.7203	0.5461	0.4336	0.3602
Standard deviation	7.25%	9.32%	10.44%	10.91%
γ-δ-Interval	2-0.06-40	2-0.10-40	2-0.14-40	2-0.18-40
Average	0.8194	0.6771	0.5730	0.4952
Max	0.9800	0.9458	0.9173	0.8832
Min	0.7190	0.5461	0.4336	0.3602
Standard deviation	7.20%	9.28%	10.42%	10.87%
γ-δ-Interval	1-0.03-60	1-0.05-60	1-0.07-60	1-0.09-60
Average	0.8188	0.6812	0.5772	0.4905
Max	0.9853	0.9538	0.9388	0.8970
Min	0.6913	0.5523	0.4426	0.3728
Standard deviation	6.77%	9.08%	11.16%	11.97%
γ-δ-Interval	2-0.06-60	2-0.10-60	2-0.14-60	2-0.18-60
Average	0.8201	0.6822	0.5778	0.4909
Max	0.9846	0.9535	0.9381	0.8960
Min	0.7008	0.5557	0.4460	0.3760
Standard deviation	6.74%	9.07%	11.14%	11.94%

We consider three liquidity requirement levels (0.03, 0.04 and 0.05) to estimate the liquidity-constrained tracking portfolios constructed with the WAL approach. Table 29 reports the number of assets included in the liquidity-constrained portfolios with the WAL approach. We notice that the number of positions decreases once the required liquidity level gets larger. This is due to the fact that the liquidity constraint eliminates the assets with relatively low individual liquidity and hence contributes to concentrate the positions hold in a small subset of securities. Table 30 also displays some descriptive statistics for the three considered scenarios. We notice that the tracking error is dependent on the required liquidity level. For instance, if we consider portfolios with a 20-day rebalancing period, the daily average TE equals 0.02%, 0.03% and 0.04% for portfolios using a liquidity requirement equal to 0.03, 0.04 and 0.05, respectively. Table 31 shows that this result applies to each analyzed year. The TE volatility also tends to increase with the required liquidity level, and the same conclusions can be made regarding the volatility of the returns.

Table 29 – Number of positions in portfolios tracking SMLL – WAL

Liquidity level	Top 40%	Top 60%	Top 80%	Average	Standard deviation
0.03	5.71	10.38	17.62	38.31	11.69
0.04	3.69	6.74	11.71	27.14	12.53
0.05	2.88	4.83	8.00	18.93	10.77

Table 30 – Descriptive results of SMLL portfolios generated constructed with WAL[1]

Liquidity	0.03			0.04		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	15.46%	14.98%	13.01%	17.73%	16.66%	13.49%
Cumulative return	77.29%	74.90%	65.04%	88.65%	83.28%	67.45%
Annual volatility	26.47%	26.24%	26.15%	28.00%	27.90%	27.48%
Daily TE average	0.02%	0.02%	0.01%	0.03%	0.03%	0.01%
Daily TE volatility	0.39%	0.39%	0.38%	0.57%	0.56%	0.54%
Monthly turnover	25.68%	15.99%	12.48%	29.48%	18.19%	14.02%

Liquidity	0.05		
Interval	20 days	40 days	60 days
Annual average return	18.42%	20.05%	14.54%
Cumulative return	92.10%	100.24%	72.71%
Annual volatility	30.11%	31.10%	33.42%
Daily TE average	0.04%	0.05%	0.06%
Daily TE volatility	0.78%	0.79%	0.82%
Monthly turnover	30.37%	18.32%	14.10%

[1] 1.1 Average Annual Return refers to the average of the cumulative returns for each year from 2016 to 2020. Cumulative Return refers to the return calculated cumulatively during the entire out-of-sample period. Daily Volatility accounts for the standard deviation (σ) of daily returns from 2016 to 2020, whereas Annual Volatility refers to $\sigma \times \sqrt{252}$. Daily TE Average and Daily TE volatility account for the average and standard deviations of the daily tracking errors from 2016 to 2020. Monthly Turnover refers to the average portfolio rebalancing monthly turnover.

Table 31 – Annual tracking error for SMLL portfolios constructed with WAL

Liquidity metric	0.03			0.04		
Year	20 days	40 days	60 days	20 days	40 days	60 days
2017	3.58%	-1.12%	-0.23%	6.81%	-0.48%	1.22%
2018	7.94%	8.07%	3.03%	15.92%	13.28%	4.67%
2019	7.10%	6.86%	3.28%	6.42%	7.91%	1.92%
2020	-0.89%	1.53%	-0.59%	0.24%	3.31%	0.39%
Average	4.43%	3.84%	1.37%	7.35%	6.01%	2.05%

Liquidity metric	0.05		
Year	20 days	40 days	60 days
2017	10.02%	3.89%	4.65%
2018	14.43%	15.89%	16.47%
2019	5.56%	8.36%	7.30%
2020	1.43%	5.16%	3.48%
Average	7.86%	8.32%	7.98%

The liquidity of the tracking portfolios constructed with the WAL approach is presented in Table 32. We can see that the liquidity of these portfolios is close to the required liquidity level in each case. Often, since the liquidity constraint is not active on several days, the average liquidity is above the specified requirement. Also, the standard deviation of the liquidity level in the liquidity-constrained portfolios is slightly larger than the one of the benchmark portfolios.

Table 32 – Liquidity for SMLL portfolios constructed with WAL

Liquidity metric	0.03			0.04		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.0314	0.0329	0.0343	0.0417	0.0437	0.0454
Max	0.0962	0.1141	0.1123	0.1246	0.1558	0.1508
Min	0.0184	0.0181	0.0200	0.0229	0.0220	0.0241
Standard deviation	1.17%	1.91%	2.26%	1.56%	2.66%	3.08%
Liquidity metric	0.05					
Interval	20 days	40 days	60 days			
Average	0.0522	0.0549	0.0568			
Max	0.1519	0.1969	0.1890			
Min	0.0275	0.0259	0.0282			
Standard deviation	1.95%	3.42%	3.88%			

The tests for the FVL liquidity approach (see Tables 33-36) have been carried out on four scenarios, which differ in liquidity level (threshold) imposed by the liquidity constraint. Table 33 shows that the number of positions is fairly similar in each scenario. Compared with the benchmark portfolio, there is a slight increase in the number of positions in the liquidity-constrained portfolios. Tables 34 and 35 present descriptive statistics and the annual tracking errors, respectively, for the four considered scenarios. The observations for the SMLL index are the same as those reported for Ibovespa: increasing the required liquidity level makes it more difficult to follow the index leading to a larger tracking error.

Table 33 – Number of positions in portfolios tracking SMLL – FVL

δ/γ; ϕ	Top 40%	Top 60%	Top 80%	Average	Standard deviation
0.03; 0.5	11.88	20.79	33.93	68.52	4.89
0.07; 0.7	13.12	23.40	39.00	69.62	6.16
0.09; 0.7	12.49	23.22	39.88	69.78	4.95
0.05; 1	13.48	24.98	42.38	69.88	4.83

Table 34 – Descriptive results of SMLL portfolios generated constructed with FVL[1]

δ/γ; ϕ	0.03; 0.5			0.07; 0.7		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	11.96%	12.30%	12.99%	13.15%	13.12%	13.04%
Cumulative return	59.78%	61.51%	64.97%	65.77%	65.60%	65.19%
Annual volatility	24.91%	24.84%	25.01%	25.10%	25.06%	25.19%
Daily TE average	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%
Daily TE volatility	0.21%	0.21%	0.23%	0.21%	0.21%	0.21%
Monthly turnover	17.19%	11.55%	9.33%	15.52%	10.35%	7.45%
δ/γ; ϕ	0.09; 0.7			0.05; 1		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	12.44%	14.02%	12.81%	14.60%	14.52%	13.17%
Cumulative return	62.22%	70.09%	64.05%	73.01%	72.59%	65.86%
Annual volatility	25.42%	25.38%	25.53%	25.76%	25.78%	25.97%
Daily TE average	0.00%	0.01%	0.01%	0.02%	0.02%	0.01%
Daily TE volatility	0.28%	0.26%	0.25%	0.25%	0.26%	0.25%
Monthly turnover	18.41%	11.56%	8.19%	14.86%	9.65%	6.62%

[1] 1.1 Average Annual Return refers to the average of the cumulative returns for each year from 2016 to 2020. Cumulative Return refers to the return calculated cumulatively during the entire out-of-sample period. Daily Volatility accounts for the standard deviation (σ) of daily returns from 2016 to 2020, whereas Annual Volatility refers to $\sigma \times \sqrt{252}$. Daily TE Average and Daily TE volatility account for the average and standard deviations of the daily tracking errors from 2016 to 2020. Monthly Turnover refers to the average portfolio rebalancing monthly turnover.

Table 35 – Annual tracking error for SMLL portfolios constructed with FVL

δ/γ; ϕ	0.03; 0.5			0.07; 0.7		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
2017	1.05%	1.38%	1.47%	5.07%	3.31%	1.14%
2018	-0.33%	0.18%	1.82%	0.92%	2.08%	2.41%
2019	0.92%	1.72%	3.34%	1.77%	1.98%	3.42%
2020	-1.53%	-1.44%	-1.32%	-1.53%	-1.30%	-1.32%
Average	0.03%	0.46%	1.33%	1.56%	1.52%	1.41%
δ/γ; ϕ	0.09; 0.7			0.05; 1		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
2017	-3.05%	4.05%	0.82%	8.45%	5.17%	0.76%
2018	4.06%	4.17%	1.38%	1.95%	3.45%	3.75%
2019	3.82%	3.57%	4.47%	5.22%	5.75%	4.20%
2020	-1.78%	-0.87%	-1.80%	-1.88%	-1.05%	-2.13%
Average	0.76%	2.73%	1.22%	3.44%	3.33%	1.65%

Table 36 shows the results for the liquidity levels of the FVL portfolios. These results can be compared to the findings for the benchmark portfolios presented in Table 28. As in Subsection 4.1.3, we have considered several values for δ and γ. For instance, $\delta = 0.18$ and $\gamma = 2$ is a possible combination for $\delta/\gamma = 0.09$. The results show an increase in the liquidity level of the constrained portfolios, with the liquidation percentage close to the required level. In addition, there was a slight reduction in the liquidity standard deviation in comparison to the benchmark portfolios. To summarize, the liquidity constraints introduced in the index tracking optimization model have to a large extent the same effect for both Ibovespa and SMLL (see also D).

Table 36 – Liquidity for SMLL portfolios constructed with FVL

δ; γ; ϕ	0.03; 1; 0.5			0.07; 1; 0.7		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.8197	0.8192	0.8282	0.6816	0.6786	0.6907
Max	0.9853	0.9800	0.9751	0.9160	0.9190	0.9388
Min	0.6913	0.7203	0.7192	0.5539	0.5455	0.5644
Standard deviation	6.78%	7.22%	6.36%	6.01%	7.81%	8.90%
δ; γ; ϕ	0.09; 1; 0.7			0.05; 1; 1		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.6639	0.6587	0.6669	0.9174	0.8984	0.9045
Max	0.9018	0.9173	0.9157	0.9999	0.9999	0.9990
Min	0.5113	0.5156	0.5496	0.7804	0.7558	0.7971
Standard deviation	6.14%	8.12%	9.05%	5.06%	6.10%	6.05%
δ; γ; ϕ	0.06; 2; 0.5			0.14; 2; 0.7		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.8210	0.8195	0.8291	0.6825	0.6795	0.6911
Max	0.9846	0.9800	0.9745	0.9126	0.9173	0.9381
Min	0.7008	0.7190	0.7221	0.5516	0.5485	0.5683
Standard deviation	6.75%	7.18%	6.30%	6.06%	7.80%	8.88%
δ; γ; ϕ	0.18; 2; 0.7			0.1; 2; 1		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.6644	0.6589	0.6672	0.9178	0.8985	0.9043
Max	0.9001	0.9145	0.9153	0.9999	0.9999	0.9990
Min	0.5088	0.5156	0.5543	0.7818	0.7586	0.7971
Standard deviation	6.20%	8.08%	8.97%	5.06%	6.11%	5.96%

4.3 Diversification

In this subsection, we discuss the diversification implications of our study using the Gini index (GINI, 1912). The Gini index is defined as the relative mean absolute difference of all pairs of portfolio weights and takes value between 0 and 1, where 0 refers to the evenly distributed portfolio ($1/N$). The larger the absolute difference between weights in the portfolio, the larger the Gini index (more concentration). The Gini index was previously used (see Chaves et al. (2012), Bellalah et al. (2015)) to assess the diversification of a portfolio. We have calculated the Gini index for the benchmark portfolio and for the liquidity-constrained portfolios based on the WAL and FVL approaches. Table 37 reports the Gini coefficient for the two liquidity approaches and the benchmark portfolio. For the WAL approach, we only present the results for the financial volume. The results for turnover and Amihud's metric are relegated to E. The results are discussed in terms of the Gini coefficient. We note that that the conclusions obtained with the Gini are the same if we use the Herfindahl-Hirschman index (HIRSHMAN, 1964; RHOADES, 1993); see Table 37.

As shown in Table 37, the benchmark portfolios have smaller values for the Gini

Table 37 – Diversification coefficients results with Gini and Herfindahl-Hirschman (HH) for Ibovespa portfolios.

Benchmark		
-	HH	Gini
-	0.026	0.567
WAL		
Liquidity metric	HH	Gini
0.1	0.034	0.614
0.15	0.073	0.755
0.2	0.153	0.855
FVL		
δ/γ; ϕ	HH	Gini
0.3; 0.5	0.031	0.619
0.4; 0.5	0.031	0.616
0.3; 0.7	0.027	0.583
0.2; 1	0.029	0.604

index, indicating that these portfolios are more diversified than the liquidity-constrained portfolios. Nonetheless, the diversification levels of the benchmark portfolio and FVL are close. Contrasting and counter-intuitive results can be observed by comparing the the diversification level and Gini index of the portfolios built with the WAL and FVL approaches. On one hand, as liquidity requirements get tighter, the Gini index of the WAL liquidity-constrained portfolios quickly raises. For instance, the Gini index goes from 0.614 to 0.855 once the liquidity requirement moves from 10% to 20%. In other words, the WAL liquidity-constrained portfolios become more concentrated (less diversified) as the liquidity requirement becomes more stringent. On the other hand, the FVL liquidity-constrained portfolios are more diversified and have a lower Gini index value than the WAL liquidity-constrained portfolios. In addition, the Gini value of the FVL liquidity-constrained portfolios is somewhat invariant with and shows minimal sensitivity to the strictness of the liquidity requirement.

The diversification results related to the WAL and FVL liquidity-constrained portfolios are in line with the conclusions related to the number of positions included in the portfolios (see Subsections 4.1.2 and 4.1.3). A smaller number of assets in a portfolio tends to generate a higher concentration level, as confirmed by the value of the Gini index for the WAL liquidity-constrained portfolios. The WAL liquidity-constrained portfolios do not have any limit on the capital allocated to each asset, independently of the portfolio monetary value. Pushed to the extreme, the WAL liquidity-constrained portfolio could be reduced to a single asset. On the other hand, the FVL liquidity approach prevents large capital concentration in specific assets because a maximum allocation per asset is imposed by the liquidity constraints which account for the portfolio total capital (monetary value).

Thus, especially for portfolios with higher monetary value (i.e. hundreds of millions or billions of USD), a high concentration in a few assets is not possible in the FVL liquidity-constrained model. The WAL liquidity-constrained approach might be effective for small- or medium-size portfolio value in a developed marked. If one has an USD 100 million portfolio and invest everything in the Apple stock, one might be able to sell everything in a couple of hours. However, in an emerging market, it might take days or weeks to unwind the same portfolio. The diversification advantage of the FVL liquidity-constrained portfolios stems from the endogenously imposed maximum that can be invested in every asset, thereby preventing a large concentration in specific assets. There is no such limitation for the WAL liquidity-constrained portfolios, making the construction of highly concentrated portfolios possible.

5 Conclusions

This study investigates the inclusion of portfolio liquidity constraints for the construction of index tracking portfolios. We propose two liquidity modeling approaches for index tracking. In the first approach (WAL), portfolio liquidity is viewed as the weighted average of the liquidity of the assets included in the portfolio and relies on financial volume, turnover, and Amihud's metric. In the second approach (FVL), portfolio liquidity is based on the liquidity perspective recently introduced by Vieira e Filomena (2019) that takes into account several financial-practice parameters related to the liquidation of an asset and the monetary value of the constructed portfolio.

An extensive empirical analysis is conducted on the basis of Brazilian stock market data and, in particular, on the two Brazilian indices Ibovespa and SMLL (Small Caps index). Besides analyzing the key features of the liquidity-constrained index tracking portfolios constructed with these two liquidity approaches, we have also built tracking portfolios free of any liquidity restriction. These serve as benchmarks and permit to better comprehend the impact of the inclusion of liquidity requirements on the composition of index tracking portfolios. Compared to the benchmark portfolios free of liquidity requirements, the portfolios obtained with the two proposed liquidity approaches permit a significant increase in the liquidity level of the constructed portfolios. The overall average liquidity of the benchmark portfolios increases from 0.097 to 0.156 compared to the liquidity-constrained portfolios constructed with the WAL approach, i.e., a 60% jump. In the same vein, the overall average liquidity of the benchmark portfolios raises from 62.32% to 72.19% compared to the liquidity-constrained portfolios constructed with the FVL approach, a 16% jump. For the scenarios with the highest liquidity requirements, the average liquidity level of the WAL liquidity-constrained portfolios is 108% larger than the one for the portfolios free of liquidity restrictions (benchmark portfolios). Similarly, the average liquidity level of the FVL liquidity-constrained portfolios increase is 53% larger than that of the benchmark portfolios.

The number of assets included in the tracking portfolios and the diversification analysis based in the Gini index point to similar conclusions in terms of portfolio diversification. The number of assets in the most tightly liquidity-constrained portfolios differs significantly in the two WAL and FVL liquidity approaches. Under the WAL approach, a portfolio with high liquidity is generated by concentrating the capital in the few assets that have the largest individual liquidity level, regardless of the portfolio's value. Under the FVL approach, the number of assets in the portfolio raises with the strictness of the liquidity requirements. Additionally, the WAL liquidity-constrained portfolios have a limited number of positions but also a high monthly turnover. The FVL liquidity-constrained portfolios have a low monthly turnover, a low variability in their liquidity, and include a larger number of securities. The conclusions related to the number of assets in the portfolios are corroborated by the diversification analysis based on the Gini index. The FVL liquidity-constrained portfolios are more diversified and have a lower Gini index value than the WAL liquidity-constrained portfolios. Furthermore, the Gini value of the FVL liquidity-constrained portfolios is somewhat invariant with and shows minimal sensitivity to the strictness of the liquidity requirement.

Part III

Portfolio Optimization and Liquidity: a Comparison Between Individual Constraints and Portfolio Constraints

Abstract

This paper compares two liquidity constraints approaches into a portfolio optimization model. The first individually restricts the amount allocated to participating assets. The second, called Financial Value Liquidation (FVL), proposed in Vieira e Filomena (2019), acts on the portfolio as a whole. The results can be directly compared as the two approaches use the same definition of portfolio liquidity. Empirical tests are performed on stocks listed in the Brazilian market using the individual constraints approach formulated along with the minimum variance model, while results for the FVL are directly extracted from Vieira e Filomena (2019). Even though both approaches yield reasonable liquidity levels as these requirements are closely reached, the FVL demonstrate superiority under risk perspective as its portfolios are less risky in comparison to individual constraints. A mathematical equivalence between both methods for a special case is also demonstrated.

keywords: Liquidity. Portfolio Optimization. Liquidity Constraint.

1 Introduction

One of the main aspects of portfolio management is the liquidity control, especially for portfolios with high monetary value in markets where the traded volume is relatively low. Environments like that are commonly found in emerging countries where liquidity control becomes a relevant topic. Liquidity measures the easiness of converting financial assets in a portfolio into cash. A portfolio, or asset, with high liquidity provides the ability to trade in big amounts, with great speed and low transaction costs. Liquidity control involves the manager allocating adequate amounts in certain assets with less liquidity. The large supply of less liquid assets leads to a drop in the asset price, reducing the return from its sale (AMIHUD; MENDELSON; PEDERSEN, 2012).

Despite its importance, liquidity is a scant subject in portfolio optimization studies. Many classic portfolio optimization models only consider the relationship between return and risk in investment analysis, supposing that assets would be traded continuously in any quantity. Some examples of classical models are: mean-variance model (MARKOWITZ, 1952a) and minimum-variance models (BEST; GRAUER, 1992; DEMIGUEL; NOGALES, 2009). The insertion of a liquidity constraint in the optimization model is more recently observed in Lo, Petrov e Wierzbicki (2006), Vieira e Filomena (2019) and Vieira et al. (2021).

The study of liquidity impose obstacles such as a proper definition for measurement justified by its multidimensionality nature. Such impossibility of direct measurement makes necessary the use of approximations. Among the current strategies, the most popular tailors characteristics thereby depending on the application aspects. However, the best way for approximating liquidity still remains as an open topic. Another challenge is to determine the most appropriated mathematical formulation for representing liquidity constraint in the optimization model. It is usually enforced by the definition of a minimum value of the average liquidity of participating assets which is weighted by the amount allocated to each asset, either by constraining on the amount allocated to each asset individually (more details in Lo, Petrov e Wierzbicki (2006)). Vieira e Filomena (2019) propose a novel methodology for adding liquidity constraints which is based on control coefficients and uses the financial volume as liquidity metric.

The present work aims to compare two liquidity constraints applied to a portfolio optimization model: individual constraint on assets (LO; PETROV; WIERZBICKI, 2006) and the Financial Value Liquidation (FVL), proposed in Vieira e Filomena (2019). The first restricts the amount allocated to each asset, requiring liquidity specifications from each participant in the portfolio. The second is concerned only with the total portfolio value, not demanding specific liquidity levels in each individual participant. A direct comparison of results is possible due to the fact that the same definition of liquidity and the same metric are used to approximate it.

In all analyzed scenarios, the liquidation obtained by both methods was close to the required liquidity level with slightly advantage for the portfolios with individual constraints. However, the FVL approach offers portfolios with lower risk, still holding acceptable liquidation prerequisites, than portfolios generated with individual constraints.

This article is divided as follows. Section 2 describes the Liquidity constraints and the optimization model. Then, Section 3 presents the empirical tests. Finally, Section 4 concludes the study.

2 Liquidity Constraints and the Optimization Model

In this section, a brief review of the main liquidity metrics used in the literature, and the different ways to insert liquidity constraints in an optimization model are provided. Additionally, the optimization model and the liquidity constraints used in this study are formalized.

Liquidity metrics are used in an attempt to capture the complex behavior of the liquidity of an asset. According to Lespagnol e Rouchier (2018), the transacted financial volume, the number of transactions, the volatility, the size of the company, bid-ask spread, and the share price are all correlated to asset liquidity. Liquidity metrics can be divided in volume-based and price-based indexes. Volume-based indexes can be represented by the financial volume and the turnover rate. Price-based indexes, in turn, are known as the bid-ask spread and the Amihud metric (AMIHUD, 2002; VIEIRA; FILOMENA, 2019).

The simplest approach to include liquidity requirements in optimization model is preprocessing data so that assets which have liquidity below the assigned limit are eliminated before the portfolio optimization procedure. Another way is to restrict the amount allocated to each participating asset by its liquidity capacity. Alternatively, it is possible to use a constraint that takes in account the portfolio liquidity as a whole, defined by Lo, Petrov e Wierzbicki (2006). This last strategy makes possible to ensure that weighted average liquidity of the assets surpasses certain threshold value, but overlooks the monetary value of the portfolio. In order to tackle this problem, a new liquidity constraint was proposed in Vieira e Filomena (2019). The constraint, which requires the use of financial volume as a liquidity metric, defines a minimum monetary percentage required to be liquidated from the portfolio. The proposed constraint can be written as:

$$l_p = \sum_{i=1}^{N} \theta_i \tag{2.1}$$

where the portfolio liquidity is denoted as l_p and θ_i is the value of the i-th stock that can be liquidated. Note that θ_i is the minimum value between the monetary volume allocated on the i-th asset in the portfolio and the maximum monetary value allowed to be liquidated on this asset. θ_i definition can be written as:

$$\theta_i = \min(\text{Allocated Value}, \text{Maximum Allowed Liquidation}) \tag{2.2}$$

$$\text{Allocated Value} = x_i \delta \tag{2.3}$$

$$\text{Maximum Allowed Liquidation} = \lambda_i \rho \gamma \tag{2.4}$$

where δ is the portfolio monetary value, ρ is the percentage limit of the total volume traded, γ is the liquidation period, and λ_i is the monetary volume for i.

Although the minimum variance model to test the liquidity constraints was selected, as discussed by Vieira e Filomena (2019), the liquidity constraints could be applied to any portfolio optimization model without loss of generality.

2.1 Optimization Model

The minimum variance model is adopted in the present work as in Vieira e Filomena (2019). The model aims to minimize the risk of the portfolio with the risk being measured by the variance of the portfolio's returns. The objective function to be minimized can be written as:

$$min \sum_{i=1}^{N} \sum_{j=1}^{N} x_i . x_j . \sigma_{ij}$$

where x_i is the weight allocated on asset i, σ_{ij} covariance of the returns between asset i and asset j, in a portfolio consisting of N possible assets. The added constraints define that all available capital is invested and that short selling is not allowed.

$$\sum_{i=1}^{N} x_i = 1$$

$$x_i \geq 0, \quad \forall i$$

2.2 Liquidity Constraint Approaches

The use of the methodology proposed in Vieira e Filomena (2019) requires a specific definition of portfolio liquidity, and use of the financial volume metric. Therefore, the present work considers the same definition of portfolio liquidity in the two studied approaches. This supposition makes possible a direct comparison between liquidity values in the two approaches. As in Vieira e Filomena (2019), portfolio liquidity can be written as:

$$liquidity = (\sum_{i=1}^{N} min(\rho.\gamma.\lambda_i \ , \delta.x_i))/\delta$$

where δ is the total portfolio value and λ_i is the average volume of the last 30 days of the asset i. In essence, liquidity is determined as the portfolio liquidation percentage after the interval between formations. Using such definition of portfolio liquidity, it is possible to assign minimum required liquidity values in the imposed constraints.

Next, both techniques are used for conducting tests. Initially it is required that the assets individually achieve certain threshold of liquidation. This experiment results are compared to the restriction of portfolio as a whole so that a predefined liquidation percentage is settled regardless the contribution of each asset. These two analysis represent respectively Individual constraint on assets and the Financial Value Liquidation - FVL.

An unique definition of liquidity is used for method proposed in Vieira e Filomena (2019) and the individual asset constraint approach by virtue of correspondence. This metric applies individual constraint on assets, limiting the allocation of each participating asset by its individual liquidity. The liquidation percentage of each asset is restricted to be equal or greater than the required portfolio liquidation threshold. The individual constraint is formalized in Eq. (2.5).

$$\delta.\phi.x_i \ \leq \rho.\gamma.\lambda_i \tag{2.5}$$

where ϕ is the minimum acceptable liquidation percentage. The FVL approach is defined by three linear inequalities presented in Eqs. (2.6)-(2.8).

$$\phi\delta \leq \sum_{i=1}^{N} \theta_i \tag{2.6}$$

$$\theta_i \leq x_i\delta \qquad \forall i \in N \tag{2.7}$$

$$\theta_i \leq \lambda_i\rho\gamma \qquad \forall i \in N \tag{2.8}$$

In the empirical tests, the optimized portfolios are evaluated by the liquidity, the risk and the number of participant assets. Additionally, both approaches are tested in a equalized and particular scenario in which the complete portfolio liquidation is required (i.e. $\phi = 100\%$). The mathematical proof for such equivalence is presented below.

The total portfolio liquidation is only possible if:

$$\theta_i = \delta.x_i \ \ \forall i$$

From the FVL model we have:

$$\theta_i \leq \rho.\gamma.\lambda_i$$

So:

$$\delta.x_i \leq \rho.\gamma.\lambda_i \tag{2.9}$$

Thus, Eq. (2.9) is equivalent to the individual constraint (2.5) for $\phi = 100\%$.

3 Empirical Tests

In this section, empirical tests are carried out to investigate the impact of distinct liquidity requirements on the generated portfolios driven by two liquidity constraint approaches. As explained in Section 2, the addition of liquidity into optimization model relies on individual constraint and FVL approaches. For the FVL approach, the results are taken directly out from Vieira e Filomena (2019). Thus, the performed tests are conduct for the individual constraint approach. In order to obtain an adequate comparison, the tests conducted in this work use the exact same database used in Vieira e Filomena (2019). The tests are equivalent in all aspects, apart from the applied liquidity constraint approach. The data consists in stocks listed on B3 (Brazilian Stock Exchange) and were obtained through Economatica. There are 252 stocks present in the sample, for the period between 1 January 2007 and 8 August 2016.

The scenarios were defined considering the same required liquidation levels and the same portfolio values. Tests are performed with portfolio values of 1 million, 10 million and 100 million Brazilian Reais, with minimum liquidity required of 30%, 50%, 70% and 100%, resulting in 12 different scenarios. Three additional scenarios were extracted from Vieira e Filomena (2019), for comparison purposes: portfolios with values of 1 million, 10 million and 100 million Brazilian Reais but without any imposed liquidity constraints. The results for the FVL approach are presented in Table 38. The average liquidity levels along with the average standard deviation of the portfolio returns and the average number of participants are shown for the 12 liquidity constrained scenarios. The portfolios without the liquidity constraints generated average standard deviation of 10.17% and average liquidity results of 30.58%, 15.07% and 5.33%, in the cases of portfolios of BRL 1 million, BRL 10 million and BRL 100 million, respectively. Table 39 shows the results for the individual constraints approach, for the same 12 scenarios. Setting $\phi = 100\%$ generated the same results on both approaches, as expected.

Both approaches demonstrate good results for portfolio liquidity in all scenarios. In all cases, the liquidation percentage was very close to the minimum required. The individual constraints approach generated slightly higher liquidity than the FVL, and in some cases the results were even above the required liquidity. However, the approach of individual constraints showed higher levels of risk in the generated portfolios. This excess liquidity can be interpreted as unnecessary since the objective is to construct the portfolio with the lowest risk which has acceptable level of liquidity. That said, the FVL approach proved to be superior and more efficient than individual constraint approach as it yields lower values of risk and still holds feasible levels of liquidity. This advantage is observed even in cases whose liquidity is similar for both strategies.

Table 38 – Results for the 12 scenarios analyzed, FVL approach.

ϕ	δ	Av. Liq	Av. St Dev	Av. Number of Assets
30%	BRL 1 million	49.17%	10.35%	44.32
30%	BRL 10 million	30.75%	10.97%	54.64
30%	BRL 100 million	29.81%	12.86%	69.85
50%	BRL 1 million	53.93%	10.74%	49.35
50%	BRL 10 million	49.71%	12.70%	68.66
50%	BRL 100 million	49.64%	17.71%	80.32
70%	BRL 1 million	69.67%	11.80%	57.32
70%	BRL 10 million	69.51%	15.56%	73.10
70%	BRL 100 million	69.47%	24.54%	87.33
100%	BRL 1 million	99.23%	15.82%	57.61
100%	BRL 10 million	99.21%	21.64%	63.11
100%	BRL 100 million	99.19%	36.90%	69.25

Table 39 – Results for the 12 scenarios analyzed, individual constraints approach.

ϕ	δ	Av. Liq	Av. St Dev	Av. Number of Assets
30%	BRL 1 million	53.14%	11.45%	49.49
30%	BRL 10 million	40.35%	13.86%	56.26
30%	BRL 100 million	30.09%	17.93%	66.83
50%	BRL 1 million	57.44%	12.35%	55.83
50%	BRL 10 million	51.42%	16.26%	65.57
50%	BRL 100 million	49.97%	24.33%	76.89
70%	BRL 1 million	70.11%	13.70%	59.60
70%	BRL 10 million	69.78%	19.10%	78.78
70%	BRL 100 million	69.71%	30.84%	85.45
100%	BRL 1 million	99.23%	15.82%	57.61
100%	BRL 10 million	99.21%	21.64%	63.11
100%	BRL 100 million	99.19%	36.90%	69.25

4 Conclusions

This study investigates the application of two different approaches of liquidity constraints and the comparison of the generated portfolios behavior. Direct comparison between results is possible because both approaches use the same definition of portfolio liquidity, which is related to the possible monetary value to be liquidated in the portfolio. The first approach considered is based on individual constraints on each participating asset, limiting the monetary value allocated to each one of them. In the second approach (FVL) there is no individual asset limitation, only some portfolio liquidation level is required.

Twelve scenarios that are constrained by different levels of liquidation and with different portfolio values are simulated. The results of the FVL approach are directly pulled from the Vieira e Filomena (2019). Those same scenarios and the exact same database are also inputted in the tests executed for the individual constraints approach allowing a straight correspondence between both methods. It was demonstrated, both mathematically and empirically, the equivalence of the two liquidity approaches in the particular case whose 100% of portfolio liquidation is required.

The results showed good liquidity levels in the two approaches studied. In all analyzed scenarios, the liquidation obtained was close to the required. The risk levels showed better results in the FVL approach, being possible to generate portfolios with lower standard deviation, compared to portfolios generated through individual constraints. Although in some cases liquidity is lower than the liquidity obtained using individual constraints, the FVL approach presents better quality results, as it is capable of providing acceptable liquidation levels and considerably less risk.

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Appendix

APPENDIX A – Correlation between Amihud's measure of illiquidity and volume traded

The methodology to obtain the correlations used in the present study relates Amihud's liquidity measure and its financial trading volume component. According to Lou e Shu (2017), it is possible to obtain the volume-traded component by assigning constant returns in the Eq. 2.2. Next, the volume component of Amihud's measure is shown.

$$AC_i = \frac{1}{D_i} . \sum_{t=1}^{D_i} \frac{1}{vol_{it}}$$

The correlations between the volume component and the Amihud's measure were obtained for monthly, quarterly, semiannual, and annual periods. Besides the correlation of the variables in level, an analysis of the correlation of the differences was also performed. The difference vector gives the percentage variation of a given variable between the instant t and the instant $t-1$.

The data, which has been used in the correlation analysis, excludes assets that do not have data throughout all the sample period, between January 2007 and August 2016. After this filtering, 100 assets remained in the sample. The rolling window technique was used throughout the sample so that the number of data was not reduced, especially in the semiannual and annual measurements. The results are presented in Table 40.

Table 40 – Correlations between Amihud's measure of illiquidity and volume traded

	Monthly	Quarterly	Semiannual	Annual
On level	0.7585	0.8188	0.8510	0.8709
On the differences	0.3926	0.4531	0.4931	0.5124

APPENDIX B – Results for the 20, 40 and 60-day intervals

The average liquidation percentage results and its average standard deviation, for the 20-day, 40-day and 60-day interval cases, are presented in Tables 41, 42 and 43 respectively. The results were similar to the one-day results. Therefore, the interval between the formation and liquidation of the portfolio showed no significant influence on portfolio liquidity. Only a small reduction in liquidity was observed in most cases, due to the increase in the interval. However, in these cases, liquidation levels close to acceptable levels were also observed. Regarding the mean standard deviation and number of participants, no interval dependence was observed.

Table 41 – Results for the 12 scenarios analyzed, with a 20-day interval.

ϕ	TPV	Av. Liq	Av. St Dev	Av. Number of Assets
30%	BRL 1 million	48.80%	10.56%	47.6
30%	BRL 10 million	28.56%	11.15%	52.7
30%	BRL 100 million	26.91%	12.94%	71.3
50%	BRL 1 million	52.06%	11.17%	50.9
50%	BRL 10 million	44.71%	12.94%	66.5
50%	BRL 100 million	44.69%	18.16%	80.6
70%	BRL 1 million	63.54%	11.87%	58.3
70%	BRL 10 million	61.96%	15.88%	73.5
70%	BRL 100 million	62.35%	24.88%	90.1
100%	BRL 1 million	85.90%	15.86%	60.4
100%	BRL 10 million	87.28%	21.68%	64.6
100%	BRL 100 million	88.26%	37.30%	70.0

Table 42 – Results for the 12 scenarios analyzed, with a 40-day interval.

ϕ	TPV	Av. Liq	Av. St Dev	Av. Number of Assets
30%	BRL 1 million	49.98%	10.38%	48.68
30%	BRL 10 million	28.95%	11.11%	51.77
30%	BRL 100 million	27.39%	13.19%	70.94
50%	BRL 1 million	54.06%	10.88%	50.77
50%	BRL 10 million	45.51%	12.97%	66.23
50%	BRL 100 million	45.09%	18.21%	84.47
70%	BRL 1 million	64.21%	11.87%	57.81
70%	BRL 10 million	62.68%	15.93%	73.21
70%	BRL 100 million	62.97%	24.70%	90.58
100%	BRL 1 million	87.25%	15.99%	60.29
100%	BRL 10 million	89.19%	21.86%	65.63
100%	BRL 100 million	89.14%	37.37%	70.64

Table 43 – Results for the 12 scenarios analyzed, with a 60-day interval.

ϕ	TPV	Av. Liq	Av. St Dev	Av. Number of Assets
30%	BRL 1 million	48.80%	10.56%	47.6
30%	BRL 10 million	28.56%	11.15%	52.7
30%	BRL 100 million	26.91%	12.94%	71.3
50%	BRL 1 million	52.06%	11.17%	50.9
50%	BRL 10 million	44.71%	12.94%	66.5
50%	BRL 100 million	44.69%	18.16%	80.6
70%	BRL 1 million	63.54%	11.87%	58.3
70%	BRL 10 million	61.96%	15.88%	73.5
70%	BRL 100 million	62.35%	24.88%	90.1
100%	BRL 1 million	85.90%	15.86%	60.4
100%	BRL 10 million	87.28%	21.68%	64.6
100%	BRL 100 million	88.26%	37.30%	70.0

APPENDIX C – WAL results on Turnover and Amihud as liquidity metrics

In Subsection 4.1.2, we have described the results for the liquidity-constrained portfolios based on the WAL approach and the volume liquidity metric. However, as above-mentioned, two other indicators have been considered (within the WAL approach) to represent liquidity, i.e., turnover and Amihud's metric. We now report and analyze the results based on turnover and Amihud's metric for liquidity. The turnover is defined in (2.1), whereas Amihud's metric is modelled by (2.2). For these tests, we follow the same guidelines described in Subsections 4.1.1 and 4.1.2.

Three values of the required liquidity level are considered for the two metrics.

Tables 44 to 49 present the corresponding results and show the consistency of the results obtained with the two liquidity metrics (turnover and Amihud's metric). The conclusions are very similar to those described for the volume metric. The portfolios obtained when imposing liquidity constraints based on turnover and Amihud's metric have higher liquidity than the benchmark portfolios. The liquidity-constrained portfolios have also a larger tracking error. The number of assets include in the liquidity-constrained portfolios goes down when the required liquidity level increases, similarly to the results presented for the volume metric in Subsection 4.1.2.

Table 44 – Number of assets of Ibovespa portfolios liquidity constrained based on WAL considering Turnover and Amihud as liquidity metrics

Turnover					
Liquidity	Top 40%	Top 60%	Top 80%	Average	Standard deviation
0.02	6.43	12.78	23.41	51.27	9.68
0.025	5.91	11.25	20.33	45.27	10.64
0.03	5.14	9.75	17.48	39.74	10.77
Amihud					
Liquidity	Top 40%	Top 60%	Top 80%	Average	Standard deviation
0.1	6.07	12.99	25.34	57.01	6.03
0.15	3.79	8.38	17.61	44.44	12.40
0.2	2.54	4.92	10.79	31.06	16.08

Table 45 – Descriptive results of Ibovespa portfolios liquidity constrained based on WAL considering Turnover and Amihud as liquidity metrics[1]

Turnover						
Liquidity metric	0.02			0.025		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	5.55%	5.17%	6.41%	6.02%	6.11%	7.57%
Cumulative return	49.93%	46.54%	57.70%	54.21%	55.01%	68.09%
Annual volatility	22.70%	22.84%	22.88%	22.82%	22.94%	23.00%
Daily TE average	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%
Daily TE volatility	0.12%	0.16%	0.16%	0.17%	0.21%	0.21%
Monthly turnover	16.59%	10.92%	8.14%	19.95%	12.92%	9.38%
Liquidity metric	0.03					
Interval	20 days	40 days	60 days			
Annual average return	6.62%	7.28%	8.67%			
Cumulative return	59.57%	65.54%	78.03%			
Annual volatility	23.02%	23.11%	23.20%			
Daily TE average	0.01%	0.01%	0.02%			
Daily TE volatility	0.24%	0.27%	0.27%			
Monthly turnover	22.41%	14.32%	10.29%			
Amihud						
Liquidity metric	0.1			0.15		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Annual average return	5.06%	4.18%	5.37%	5.06%	3.89%	5.28%
Cumulative return	45.49%	37.59%	48.29%	45.50%	34.99%	47.56%
Annual volatility	22.61%	22.73%	22.82%	22.77%	22.92%	23.17%
Daily TE average	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Daily TE volatility	0.08%	0.12%	0.13%	0.16%	0.19%	0.22%
Monthly turnover	13.45%	9.01%	7.43%	17.21%	10.98%	8.26%
Liquidity metric	0.2					
Interval	20 days	40 days	60 days			
Annual average return	5.65%	3.95%	5.47%			
Cumulative return	50.85%	35.55%	49.25%			
Annual volatility	23.50%	23.75%	24.01%			
Daily TE average	0.00%	0.00%	0.00%			
Daily TE volatility	0.33%	0.34%	0.40%			
Monthly turnover	20.79%	12.78%	9.56%			

[1] 1.1 Average Annual Return refers to the average of the cumulative returns for each year from 2010 to 2018. Cumulative Return refers to the return calculated cumulatively during the entire out-of-sample period. Daily Volatility accounts for the standard deviation (σ) of daily returns from 2010 to 2018, whereas Annual Volatility refers to $\sigma \times \sqrt{252}$. Daily TE Average and Daily TE volatility account for the average and standard deviations of the daily tracking errors from 2010 to 2018. Monthly Turnover refers to the average portfolio rebalancing monthly turnover.

Table 46 – Annual tracking error of Ibovespa portfolios liquidity constrained based on WAL considering Turnover metric

Liquidity metric	0.02			0.025		
Year	20 days	40 days	60 days	20 days	40 days	60 days
2010	0.67%	0.52%	0.40%	1.00%	0.87%	0.70%
2011	0.81%	1.59%	1.96%	1.00%	2.15%	2.92%
2012	1.77%	1.54%	3.18%	3.10%	3.35%	5.80%
2013	3.82%	3.25%	6.37%	4.13%	3.67%	6.66%
2014	-4.22%	-6.53%	-7.94%	-2.85%	-5.04%	-6.52%
2015	-0.89%	-0.49%	0.22%	-2.85%	-1.31%	-0.57%
2016	2.75%	2.77%	4.30%	4.77%	5.49%	7.17%
2017	7.50%	9.49%	7.22%	11.13%	14.27%	11.08%
2018	-4.02%	-4.08%	-0.94%	-6.95%	-6.92%	-2.06%
Average	0.91%	0.89%	1.64%	1.39%	1.84%	2.80%

Liquidity metric	0.03		
Year	20 days	40 days	60 days
2010	1.31%	1.11%	0.98%
2011	1.25%	2.89%	3.47%
2012	4.51%	5.31%	8.15%
2013	4.14%	3.99%	6.25%
2014	-1.52%	-3.66%	-5.17%
2015	-4.10%	-1.40%	-1.75%
2016	7.09%	8.79%	10.53%
2017	15.26%	19.29%	15.65%
2018	-10.11%	-9.28%	-3.00%
Average	1.98%	3.00%	3.90%

Table 47 – Annual tracking error of Ibovespa portfolios liquidity constrained based on WAL considering Amihud metric

Liquidity metric	0.1			0.15		
Year	20 days	40 days	60 days	20 days	40 days	60 days
2010	0.34%	0.04%	-0.12%	-0.13%	-0.23%	-0.44%
2011	0.54%	0.83%	0.98%	-0.14%	-0.13%	1.09%
2012	0.90%	0.39%	2.42%	3.48%	1.15%	4.12%
2013	3.86%	4.32%	6.20%	3.76%	4.79%	5.74%
2014	-4.60%	-6.98%	-7.67%	-5.57%	-8.76%	-8.32%
2015	1.61%	1.13%	1.89%	3.34%	2.56%	2.45%
2016	0.67%	0.39%	1.61%	-1.81%	-2.39%	0.02%
2017	0.87%	-0.29%	0.16%	1.26%	0.25%	0.07%
2018	-0.42%	-0.73%	-0.10%	-0.42%	-0.73%	-0.10%
Average	0.42%	-0.10%	0.60%	0.42%	-0.39%	0.52%

Liquidity metric	0.2		
Year	20 days	40 days	60 days
2010	-0.89%	-1.47%	-2.10%
2011	-0.62%	1.51%	3.34%
2012	1.62%	-0.90%	-0.31%
2013	5.98%	7.12%	7.65%
2014	-3.31%	-9.48%	-11.35%
2015	6.64%	4.45%	6.41%
2016	-1.48%	-4.86%	2.83%
2017	1.49%	1.28%	-0.04%
2018	-0.32%	-0.59%	-0.10%
Average	1.01%	-0.33%	0.70%

Table 48 – Benchmark liquidity based on WAL considering Turnover and Amihud as liquidity metrics

Turnover			
Interval	20 days	40 days	60 days
Average	0.0137	0.0138	0.0135
Max	0.0203	0.0195	0.0190
Mn	0.0091	0.0090	0.0092
Standard deviation	0.24%	0.23%	0.25%
Amihud			
Interval	20 days	40 days	60 days
Average	0.1174	0.1134	0.1181
Max	0.2753	0.2303	0.2367
Min	0.0548	0.0555	0.0568
Standard deviation	5.09%	4.74%	4.85%

Table 49 – Liquidity of constrained portfolios based on WAL considering Turnover and Amihud as liquidity metrics

Turnover						
Liquidity metric	0.02			0.025		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.0200	0.0202	0.0197	0.0250	0.0251	0.0244
Max	0.0212	0.0320	0.0321	0.0266	0.0402	0.0403
Min	0.0190	0.0157	0.0134	0.0237	0.0184	0.0156
Standard deviation	0.04%	0.31%	0.39%	0.05%	0.42%	0.51%
Liquidity metric	0.03					
Interval	20 days	40 days	60 days			
Average	0.0299	0.0301	0.0291			
Max	0.0320	0.0485	0.0487			
Min	0.0285	0.0208	0.0178			
Standard deviation	0.06%	0.53%	0.64%			
Amihud						
Liquidity metric	0.1			0.15		
Interval	20 days	40 days	60 days	20 days	40 days	60 days
Average	0.1274	0.1287	0.1263	0.1618	0.1655	0.1643
Max	0.2823	0.2449	0.2327	0.2823	0.2757	0.2725
Min	0.0940	0.0639	0.0639	0.1391	0.0892	0.0932
Standard deviation	4.57%	4.58%	4.47%	2.97%	3.95%	4.87%
Liquidity metric	0.2					
Interval	20 days	40 days	60 days			
Average	0.2046	0.2102	0.2098			
Max	0.2823	0.3672	0.4004			
Min	0.1828	0.1153	0.1155			
Standard deviation	1.63%	4.38%	6.47%			

APPENDIX D – Difference Between Out-of-Sample Average Daily Liquidity

In Sections 4.1 and 4.2, it is argued that the inclusion of a liquidity constraint has a significant impact on the tracking portfolios in terms of their liquidity level over time. We present in this Appendix a statistical analysis to support the aforementioned arguments in favor of using the liquidity-constrained approach for index tracking. To this end, we test the difference between the average daily liquidity of the liquidity-constrained portfolios against that of the benchmark portfolio. The goal is to verify if the out-of-sample average daily liquidity of the liquidity-constrained portfolios differs from that of the benchmark portfolios which are liquidity-unrestricted.

To do so, we first compute the out-of-sample daily liquidity according to Equation 3.4 for every period t (i.e., every business day) for four liquidity-constrained tracking portfolios related to the Ibovespa and for four liquidity-constrained portfolios tracking the SMLL index. Additionally, we compute the daily liquidity of the benchmark portfolios for both indices. Second, we test the difference between the out-of-sample average daily liquidity of each of the four liquidity-constrained portfolios for the Ibovespa against the Ibovespa benchmark portfolio. We proceed in the exact same manner for the SMLL index.

The test for the difference between average daily liquidity is carried out as follows. Given the time-series of daily liquidity $l_{c,t}$ and $l_{b,t}$, for every $t = 1, 2, \ldots T$, where respectively $l_{c,t}$ refers to the liquidity for a liquidity-constrained portfolio and $l_{b,t}$ refers to the liquidity at every period t for the benchmark portfolio, we apply a t-test to verify the null hypothesis $H_0 : \mu_{l_c} = \mu_{l_b}$. If the test fails to reject the null hypothesis, then we can consider that the daily liquidity of the liquidity-constrained portfolio and that of the benchmark portfolio are statistically identical, therefore concluding that the liquidity constraint does not have any statistically meaningful impact. Contrariwise, rejecting the null hypothesis confirms that the liquidity constraint results in a significant change in the portfolio liquidity over time. Hence, our expectation is to reject the null hypothesis to confirm the relevance of the liquidity constraint from a statistical viewpoint.

We use the bootstrapping technique and carry out a series of t-test to conduct the statistical analysis. We present the approach followed below. Sant'Anna et al. (2019) provide a more detailed description of this methodology.

For two time-series $l_{c,t}$ and $l_{b,t}$ for $t = 1, 2, \ldots T$, we randomly select V values in each of the two time-series, $V \leq T$, therefore forming two subsets $l^s_{c,v}$ and $l^s_{b,v}$, where $v = 1, 2, \ldots V$. This random sampling approach is repeated S times, $s = 1, 2, \ldots S$. For each

s, we test the null hypothesis $H_0 : \mu_{l_c^s} = \mu_{l_b^s}$ and calculate a statistical value $z_s = \mu_{l_c^s} - \mu_{l_b^s}$. As a result, we form a set z_s, $s = 1, 2, \ldots S$, which is used to compute the lower and upper limits CI- and CI+ of the confidence interval with confidence level $1 - \alpha$. By doing so, we fail to reject the null hypothesis for the difference between means of $l_{c,t}$ and $l_{b,t}$ if 0 (zero) falls inside the interval [CI-,CI+].

For the tests, we chose parameters $V = 50$, $S = 1000$ and $\alpha = 0.05$ (i.e. 95% confidence interval). To carry out the analysis of difference between means, we chose Ibovespa tracking portfolios estimated using parameters $[\phi; \delta]$ equal to $[1; 0.2]$, $[0.5; 0.4]$, $[0.5; 0.3]$ and $[0.7; 0.3]$; and SMLL tracking portfolios using $[\phi; \delta]$ equal to $[1; 0.05]$, $[0.7; 0.07]$, $[0.5; 0.03]$ and $[0.7; 0.09]$. All portfolios use 40 days as rebalancing window. Starting with the Ibovespa index, we analyze the difference between the liquidity average for each of the four portfolios mentioned above and that for the benchmark portfolio. We then do the same for the SMLL index.

For the Ibovespa index, since our data sample goes from January 2010 to September 2018, we test the difference between liquidity averages year-by-year from 2010 to 2017. For the SMLL index, as the available data cover the period January 2016-August 2020, we test the difference between liquidity averages from 2016 to 2019.

Tables 50 (Ibovespa) and 51 (SMLL) present the results. In Table 50, we show the results for each of the four liquidity-constrained tracking portfolios for distinct values of ϕ and δ against the benchmark portfolio. The results present only four cases (out of 32) in which the null hypothesis could not be rejected, hence confirming the statistical significance in daily liquidity when adopting a liquidity-constrained optimization model. Likewise, Table 51 shows a somewhat similar result for the SMLL index, in which case, for portfolios with $[\phi; \delta]$ set to $[1; 0.05]$, $[0.7; 0.07]$, and $[0.7; 0.09]$, the conclusion was to reject the null hypothesis in all years. This underscores effectiveness of the constraints to create more liquid portfolios.

The exception were portfolios with $[\phi; \delta]$ equal to $[0.5; 0.03]$ for which we could not reject the null hypothesis in all four years. Nonetheless, such finding cannot be considered as a surprise, nor setback for the use of the liquidity constrained model, since setting $\phi = 0.5$ and $\delta = 0.03$ comes up to relaxing the liquidity constraint (i.e., close to solve an unconstrained portfolio), thereby producing liquidity levels similar to those obtained with the benchmark portfolio.

Table 50 – Difference between means - tracking portfolios for the Ibovespa

Ibovespa, $\phi = 1$, $\delta = 0.2$				Ibovespa, $\phi = 0.5$, $\delta = 0.3$			
Year	CI-	CI+	Decision	Year	CI-	CI+	Decision
2010	-0.3387	-0.3253	Reject H0	2010	-0.1110	-0.0964	Reject H0
2011	-0.3046	-0.2888	Reject H0	2011	-0.0946	-0.0657	Reject H0
2012	-0.2496	-0.2312	Reject H0	2012	-0.0353	0.0018	Fail to reject H0
2013	-0.2685	-0.2466	Reject H0	2013	-0.0869	-0.0725	Reject H0
2014	-0.1600	-0.1472	Reject H0	2014	-0.0507	-0.0332	Reject H0
2015	-0.1109	-0.0978	Reject H0	2015	-0.0350	-0.0232	Reject H0
2016	-0.1547	-0.1425	Reject H0	2016	-0.0479	-0.0223	Reject H0
2017	-0.1351	-0.1105	Reject H0	2017	-0.0074	0.0139	Fail to reject H0
Ibovespa, $\phi = 0.5$, $\delta = 0.4$				Ibovespa, $\phi = 0.7$, $\delta = 0.3$			
Year	CI-	CI+	Decision	Year	CI-	CI+	Decision
2010	-0.1029	-0.0876	Reject H0	2010	-0.1358	-0.1249	Reject H0
2011	-0.0763	-0.0523	Reject H0	2011	-0.1590	-0.1397	Reject H0
2012	-0.0319	0.0060	Fail to reject H0	2012	-0.1056	-0.0728	Reject H0
2013	-0.0540	-0.0376	Reject H0	2013	-0.1221	-0.1037	Reject H0
2014	-0.0179	-0.0008	Reject H0	2014	-0.0691	-0.0503	Reject H0
2015	-0.0294	-0.0179	Reject H0	2015	-0.0371	-0.0246	Reject H0
2016	-0.0434	-0.0115	Reject H0	2016	-0.0710	-0.0445	Reject H0
2017	-0.0091	0.0078	Fail to reject H0	2017	-0.0671	-0.0480	Reject H0

Table 51 – Difference between means - tracking portfolios for the SMLL

SMLL, $\phi = 1$, $\delta = 0.05$				SMLL, $\phi = 0.5$, $\delta = 0.03$			
Year	CI-	CI+	Decision	Year	CI-	CI+	Decision
2016	-0.2089	-0.1924	Reject H0	2016	-0.0053	0.0057	Fail to reject H0
2017	-0.2251	-0.1843	Reject H0	2017	-0.0126	0.0119	Fail to reject H0
2018	-0.2858	-0.2673	Reject H0	2018	-0.0099	0.0105	Fail to reject H0
2019	-0.1922	-0.1751	Reject H0	2019	-0.0096	0.0028	Fail to reject H0
SMLL, $\phi = 0.7$, $\delta = 0.07$				SMLL, $\phi = 0.7$, $\delta = 0.09$			
Year	CI-	CI+	Decision	Year	CI-	CI+	Decision
2016	-0.1422	-0.1245	Reject H0	2016	-0.1825	-0.1619	Reject H0
2017	-0.1067	-0.0705	Reject H0	2017	-0.1703	-0.1331	Reject H0
2018	-0.1632	-0.1443	Reject H0	2018	-0.2060	-0.1919	Reject H0
2019	-0.0833	-0.0637	Reject H0	2019	-0.1501	-0.1309	Reject H0

The statistical analysis corroborates the results exposed in Sections 4.1 and 4.2. As the liquidity constraints gets tighter, we show with statistical significance that liquidity-constrained portfolios permit to construct more liquid portfolios than models which are liquidity-unrestricted.

APPENDIX E – Other diversification results

In Section 4.3, we have presented diversification results for the FVL approach and for the WAL approach based on financial volume. We report here additional diversification tests. First, as in C, we provide diversification results for the WAL approach based on turnover and Amihud's metric. The results for the SMLL index are also given.

Table 52 displays the diversification results for the turnover and the Amihud liquidity metrics, which were analyzed with the Ibovespa index. The same observation can be made for three metrics (i.e., volume, trunover Amihud) used in the WAL approach: the higher the required liquidity level, the lower the value of the Gini index.

Table 52 – WAL Diversification coefficients results with Gini and Herfindahl-Hirschman (HH) for Ibovespa portfolios, using Turnover and Amihud metrics.

WAL Turnover		
Liquidity metric	HH	Gini
0.02	0.029	0.617
0.025	0.035	0.665
0.03	0.043	0.711
WAL Amihud		
Liquidity metric	HH	Gini
0.1	0.030	0.597
0.15	0.069	0.715
0.2	0.154	0.821

Table 53 shows the results for the diversification indices obtained for SMLL. It can be seen that the diversification of benchmark portfolio and that of the liquidity-constrained portfolios based on the both the FVL and WAL approaches. The scenarios presented in Section 4.2 are also considered here for SMLL. The benchmark portfolio is the most diversified and is followed by the liquidity-constrained portfolio constructed using the FVL approach. The insights about the effect of the required liquidity level on the diversification of the constructed portfolios are similar for both Ibovespa and SMLL.

Table 53 – Diversification coefficients results with Gini and Herfindahl-Hirschman (HH) for SMLL portfolios.

Benchmark		
-	HH	Gini
-	0.014	0.614
WAL		
Liquidity metric	HH	Gini
0.03	0.049	0.806
0.04	0.088	0.842
0.05	0.198	0.874
FVL		
δ/γ; ϕ	HH	Gini
0.03; 0.5	0.018	0.684
0.07; 0.7	0.015	0.657
0.09; 0.7	0.015	0.637
0.05; 1	0.015	0.627