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The signless Laplacian matrix of hypergraphs

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Abstract: In this article, we define signless Laplacian matrix of a hypergraph and obtain structural properties from its eigenvalues. We generalize several known results for graphs, relating the spectrum of this matrix to structural parameters of the hypergraph such as the maximum degree, diameter, and the chromatic number. In addition, we characterize the complete signless Laplacian spectrum for the class of power hypergraphs from the spectrum of its base hypergraph.

Keywords: hypergraph, signless Laplacian, spectral radius, power hypergraph

MSC 2020: 05C65, 05C50, 15A18

1 Introduction

The goal of spectral graph theory is to study structural properties of graphs by means of eigenvalues and eigenvectors of matrices associated with them. Researchers, motivated by the success of this theory, have studied many hypergraph matrices aiming to develop a spectral hypergraph theory. See for example [1–7]. In 2012, Cooper and Dutle presented a new approach, and in their article [8], they proposed the study of hypergraphs through tensors, causing a revolution in this area. Consequently, the study of hypergraph from its matrices has been put aside. Because determining the spectrum of a tensor has a high computational cost [9], the application of this theory has its toll. Perhaps for this reason, recently, some authors have renewed the interest to study matrix representations of hypergraphs, as for example in [10–16]. Therefore, we believe that the study of hypergraphs through matrices remains important.

Let $\mathcal{H}$ be a hypergraph whose incidence matrix is $\mathbf{B}(\mathcal{H})$. The *signless Laplacian matrix* of $\mathcal{H}$ is defined as $\mathbf{Q}(\mathcal{H}) = \mathbf{B}\mathbf{B}^T$. The aim of this article is the study of this matrix. We say that the eigenvalues of $\mathbf{Q}$ are the signless Laplacian eigenvalues of $\mathcal{H}$. The matrix $\mathbf{Q}$ has many interesting properties such as being symmetric, non-negative, positive semi-definite, and irreducible. Thus, important theorems such as the Perron-Frobenius theorem and Rayleigh's principle can be inherited directly from matrix theory. In this article, we prove generalizations of some results for this matrix in the context of graphs and show it is possible to determine structural properties of the hypergraph from $\mathbf{Q}$. For example, we show that the number of edges of the hypergraph can be determined from the sum of its signless Laplacian eigenvalues. We also show that the number of distinct eigenvalues of $\mathbf{Q}$ is larger than the diameter of the hypergraph. The spectral radius is bounded by the degrees of the hypergraph, and the chromatic number is bounded by the spectral radius.

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An important property of the signless Laplacian matrix in the context of spectral graph theory is the relation between the eigenvalue zero and the existence of bipartite components in the graph [17]. In an attempt to obtain similar results, we study when the signless Laplacian matrix has the eigenvalue zero. Here, we prove that if the hypergraph has eigenvalue zero, then it is partially bipartite, and if the hypergraph is balanced partially bipartite, then it has eigenvalue zero (see definitions in Section 5). We note that the properties we prove here generalize the results already known for graphs, since a bipartite graph is both partially bipartite and balanced partially bipartite.

Another interesting property for graphs is the fact that a regular graph is completely characterized by spectral properties. Here we generalize this result by showing how to determine whether a hypergraph is regular by analyzing its spectral radius or its principal eigenvector, according to the result below.

Theorem 1. *Let $\mathcal{H}$ be a connected k-graph with n vertices, and $\rho(\mathcal{H})$ its spectral radius. The following statements are equivalent*:
(*a*) *$\mathcal{H}$ is regular*;
(*b*) $\rho(\mathcal{H}) = kd(\mathcal{H})$;
(*c*) $\rho(\mathcal{H}) = k\Delta(\mathcal{H})$;
(*d*) *The principal eigenvector of* $\mathbf{Q}(\mathcal{H})$ *is* $\mathbf{x} = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$.

As an application of our developed theory, we also study the spectrum of the signless Laplacian matrix of the class of hypergraphs called power hypergraphs (see definition in Section 7). We show how to construct the whole spectrum of the power hypergraph from the signless Laplacian eigenvalues of its base hypergraph.

The remaining of the article is organized as follows. In Section 2, we present some basic definitions about hypergraphs and matrices. In Section 3, we study the incidence matrix and exploit some properties of line and clique multigraphs. In Section 4, we study the signless Laplacian matrix, extending many classical results of this matrix to the context of hypergraphs. In Section 5, we study structural characteristics of a hypergraph, such as being regular or partially bipartite, analyzing its signless Laplacian eigenvalues. In Section 6, we correlate classical and spectral parameters of a hypergraph, such as chromatic number and diameter, with spectral radius and number of distinct eigenvalues. In Section 7, we study the spectrum of the signless Laplacian matrix of a power hypergraph. Finally, in Section 8, we discuss our results and propose a few open problems.

2 Preliminaries

In this section, we shall present some basic definitions about hypergraphs and matrices, as well as terminology, notation, and concepts that will be useful in our proofs. More details about hypergraphs can be found in [18].

A *hypergraph* $\mathcal{H} = (V, E)$ is a pair composed of a set of vertices $V(\mathcal{H})$ and a set of (hyper)edges $E(\mathcal{H}) \subseteq 2^V$, where 2^V is the power set of V. $\mathcal{H}$ is said to be a *k-uniform* (or a k-graph) for $k \geq 2$ if all edges have cardinality k. For hypergraphs $\mathcal{H} = (V, E)$ and $\mathcal{H}' = (V', E')$, if $V' \subseteq V$ and $E' \subseteq E$, then $\mathcal{H}'$ is a *subgraph* of $\mathcal{H}$.

The *neighborhood* of a vertex $v \in V(\mathcal{H})$, denoted by $N(v)$, is the set of vertices distinct from v, which have some edge in common with v. The *edge neighborhood* of a vertex $v \in V$, denoted by $E_{[v]}$, is the set of all edges that contain v.

The *degree* of a vertex $v \in V$, denoted by $d(v)$, is the number of edges that contain v. More precisely, $d(v) = |E_{[v]}|$. A hypergraph is *r-regular* if $d(v) = r$ for all $v \in V$. We define the *maximum*, *minimum*, and *average* degrees, respectively, as

$$\Delta(\mathcal{H}) = \max_{v \in V}\{d(v)\}, \quad \delta(\mathcal{H}) = \min_{v \in V}\{d(v)\}, \quad d(\mathcal{H}) = \frac{1}{|V|}\sum_{v \in V} d(v).$$

When we are working with more than one hypergraph, we can use the notation $d_{\mathcal{H}}(v)$, to avoid ambiguity.

Let $\mathcal{H}$ be a hypergraph. A *walk* of length l is a sequence of vertices and edges $v_0e_1v_1e_2 \dots e_lv_l$, where v_{i-1} and v_i are distinct vertices contained in e_i for each $i = 1, \dots, l$. The *distance* between two vertices is the length of the shortest walk connecting these two vertices. The *diameter* of the hypergraph is the largest distance between two of its vertices. The hypergraph is *connected*, if for each pair of vertices u, w there is a walk $v_0e_1v_1e_2 \cdots e_lv_l$, where $u = v_0$ and $w = v_l$. Otherwise, the hypergraph is *disconnected*.

A *multigraph* is an ordered pair $\mathcal{G} = (V, E)$, where V is a set of vertices and E is a multiset of pairs of distinct, unordered vertices, called edges. Its *adjacency matrix* $\mathbf{A}(\mathcal{G})$ is the square matrix of order $|V|$, where $a_{ii} = 0$ and if $i \neq j$, then a_{ij} is the number of edges connecting the vertices i and j.

Let $\mathbf{M}$ be a symmetric, square matrix of order n. We denote its characteristic polynomial by $P_{\mathbf{M}}(\lambda) = \det(\lambda\mathbf{I}_n - \mathbf{M})$. Its eigenvalues will be denoted by $\lambda_1(\mathbf{M}) \geq \cdots \geq \lambda_n(\mathbf{M})$. If $\mathbf{x}$ is an eigenvector from eigenvalue λ, then the pair $(\lambda, \mathbf{x})$ will be called *eigenpair* of $\mathbf{M}$. The *spectral radius* $\rho(\mathbf{M})$ is the largest modulus of an eigenvalue.

3 Incidence matrix, clique, and line multigraphs

In this section, we will study the incidence matrix of a hypergraph. More specifically, we will analyze the relationship of this matrix with two multigraphs associated with it: the line and clique multigraphs. The results of this section are generalizations of well-known properties of the incidence matrix and line graphs [19,20].

Definition 3.1. Let $\mathcal{H} = (V, E)$ be a hypergraph. The *incidence matrix* $\mathbf{B}(\mathcal{H})$ is defined as the matrix of order $|V| \times |E|$, where $b(v, e) = 1$ if $v \in e$ and $b(v, e) = 0$ otherwise. Its *matrix of degrees* $\mathbf{D}(\mathcal{H})$ is a square matrix of order $|V|$, where $d_{ii} = d(i)$ and if $i \neq j$, then $d_{ij} = 0$.

The *clique multigraph* $\mathcal{C}(\mathcal{H})$ has the same vertices as $\mathcal{H}$. The number of edges between two vertices of this multigraph is equal to the number of hyperedges containing them in $\mathcal{H}$. The vertices of the *line multigraph* $\mathcal{L}(\mathcal{H})$ are the hyperedges of $\mathcal{H}$. The number of edges between two vertices of this multigraph is equal to the number of vertices in common in the two respective hyperedges.

Example 3.2. The clique and line multigraphs of $\mathcal{H} = (\{1, \dots, 5\}, \{123, 145, 345\})$ are illustrated in Figure 1.

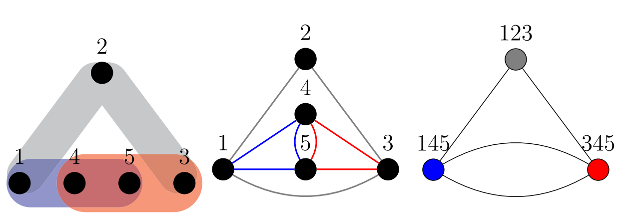

Figure 1: Clique $\mathcal{C}(\mathcal{H})$ and line $\mathcal{L}(\mathcal{H})$ multigraphs.

Our first result is the following observation. We believe it is worth mentioning because it opens the possibility of studying hypergraphs from the spectrum of multigraphs.

Theorem 2. *Let $\mathcal{H}$ be a k-graph, $\mathbf{B}$ its incidence matrix, $\mathbf{D}$ its degree matrix, $\mathbf{A}_{\mathcal{L}}$ and $\mathbf{A}_{\mathcal{C}}$ the adjacency matrices of its line and clique multigraphs, respectively. Then*

$$\mathbf{B}^T\mathbf{B} = k\mathbf{I} + \mathbf{A}_{\mathcal{L}} \quad \text{and} \quad \mathbf{B}\mathbf{B}^T = \mathbf{D} + \mathbf{A}_{\mathcal{C}}.$$

Proof. Let $\mathbf{C} = \mathbf{B}^T\mathbf{B}$. Note that c_{ij} is the number of vertices in common between the hyperedges e_i and e_j. So, if $i \neq j$, then c_{ij} is the number of edges between the vertices i and j in the line multigraph $\mathcal{L}(\mathcal{H})$, otherwise $c_{ii} = k$. Therefore, we conclude $\mathbf{C} = k\mathbf{I} + \mathbf{A}_{\mathcal{L}}$.

Now, let $\mathbf{M} = \mathbf{B}\mathbf{B}^T$. Note that m_{ij} is the number of hyperedges that contain at the same time the vertices i and j. So we have $m_{ii} = d(i)$ for all $i \in V$, and if $i \neq j$, then m_{ij} is the number of edges between the vertices i and j in the clique multigraph $\mathcal{C}(\mathcal{H})$. Therefore, we conclude $\mathbf{M} = \mathbf{D} + \mathbf{A}_{\mathcal{C}}$. □

Proposition 3. *If $\mathcal{H}$ is a k-graph, r-regular, with n vertices and m edges, then*

$$P_{\mathbf{A}_{\mathcal{L}}}(\lambda) = (\lambda + k)^{m-n} P_{\mathbf{A}_{\mathcal{C}}}(\lambda - r + k).$$

Proof. Let $\mathbf{B}$ be the incidence matrix of $\mathcal{H}$. Consider the following matrices:

$$U = \begin{bmatrix} \lambda\mathbf{I}_n & -\mathbf{B} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}, \quad V = \begin{bmatrix} \mathbf{I}_n & \mathbf{B} \\ \mathbf{B}^T & \lambda\mathbf{I}_m \end{bmatrix} \Rightarrow UV = \begin{bmatrix} \lambda\mathbf{I}_n - \mathbf{B}\mathbf{B}^T & \mathbf{0} \\ \mathbf{B}^T & \lambda\mathbf{I}_m \end{bmatrix}, \quad VU = \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{0} \\ \lambda\mathbf{B}^T & \lambda\mathbf{I}_m - \mathbf{B}^T\mathbf{B} \end{bmatrix}.$$

We know that $\det(VU) = \det(UV)$. So,

$$\lambda^n \det(\lambda\mathbf{I}_m - \mathbf{B}^T\mathbf{B}) = \lambda^m \det(\lambda\mathbf{I}_n - \mathbf{B}\mathbf{B}^T). \tag{1}$$

Thus,

$$\begin{aligned} P_{\mathbf{A}_{\mathcal{L}}}(\lambda) &= \det(\lambda\mathbf{I}_m - \mathbf{A}_{\mathcal{L}}) \\ &= \det((\lambda + k)\mathbf{I}_m - \mathbf{B}^T\mathbf{B}) \\ &= (\lambda + k)^{m-n} \det((\lambda + k)\mathbf{I}_n - \mathbf{B}\mathbf{B}^T) \\ &= (\lambda + k)^{m-n} \det((\lambda + k - r)\mathbf{I}_n - \mathbf{A}_{\mathcal{C}}) \\ &= (\lambda + k)^{m-n} P_{\mathbf{A}_{\mathcal{C}}}(\lambda - r + k). \end{aligned}$$

Therefore, the result follows. □

Lemma 4. *Let $\mathcal{H}$ be a k-graph and $\mathcal{L}(\mathcal{H})$ its line graph. If $u \in V(\mathcal{L}(\mathcal{H}))$ is a vertex corresponding to the edge $e \in E(\mathcal{H})$, then*

$$d_{\mathcal{L}}(u) = \left(\sum_{v \in e} d_{\mathcal{H}}(v) \right) - k.$$

Proof. Note that, for each $v \in e$, there exist other $d_{\mathcal{H}}(v) - 1$ hyperedges containing it. That is, this vertex will generate $d_{\mathcal{H}}(v) - 1$ edges containing u in $\mathcal{L}(\mathcal{H})$. Using the same argument for the other vertices of e, we conclude that the degree of the vertex u in the line multigraph must be $d_{\mathcal{L}}(u) = \sum_{v \in e}(d_{\mathcal{H}}(v) - 1)$. □

4 Signless Laplacian matrix

In this section, we study some properties of the signless Laplacian matrix of a hypergraph, generalizing important results of this matrix in the context of spectral graph theory. Those main results may be found in the series of papers by Cvetković et al. [17,21–23] and references therein.

Definition 4.1. Let $\mathcal{H}$ be a hypergraph and $\mathbf{B}$ its incidence matrix. The *signless Laplacian matrix* is defined as $\mathbf{Q}(\mathcal{H}) = \mathbf{B}\mathbf{B}^T$. We note that the eigenvalues of $\mathbf{Q}$ are the squares of the singular values of the incidence matrix $\mathbf{B}$, which is a fundamental object for hypergraph theory.

An *oriented hypergraph* $H = (\mathcal{H}, \sigma)$ is a hypergraph, where for each vertex-edge incidence (v, e), a label $\sigma(v, e) \in \{+1, -1\}$ is given. In [4], Reff and Rusnak defined the incidence matrix of an oriented hypergraph $\mathfrak{B}(H)$ by $\mathfrak{b}(v, e) = \sigma(v, e)$ if $v \in e$ and $\mathfrak{b}(v, e) = 0$ otherwise. The *Laplacian matrix* for oriented hypergraphs is defined as $\mathfrak{L}(H) = \mathfrak{B}\mathfrak{B}^T$. We observe that if $\sigma(v, e) = 1$ for all vertex-edge incidence (v, e), then this definition coincides with our definition of signless Laplacian matrix.

Remark 4.2. If $\mathcal{H}$ is a k-graph, its signless Laplacian matrix $\mathbf{Q}$ has some simple but useful properties, such as being symmetric, non-negative, and positive semi-definite. Furthermore, if $\mathcal{H}$ is connected, then $\mathbf{Q}$ is irreducible. Therefore, Rayleigh's principle and the Perron-Frobenius theorem are applicable to this matrix. The normalized positive vector obtained in the Perron-Frobenius theorem is referred to as *principal eigenvector* of $\mathbf{Q}$. Sometimes, we will denote the spectral radius $\rho(\mathbf{Q})$ as $\rho(\mathcal{H})$, and say it is the spectral radius of $\mathcal{H}$.

We finish this section proving some basic properties of the signless Laplacian matrix.

Lemma 5. *Let $\mathcal{H}$ be a k-graph and $\mathbf{Q} = (q_{ij})$ its signless Laplacian matrix. Then, for each $i \in V$, we have*

$$\sum_{j\in V} q_{ij} = kd(i).$$

Proof. By the characterization of signless Laplacian matrix of Theorem 2, we have

$$\sum_{j\in V} q_{ij} = d(i) + \sum_{j\in N(i)} a_{ij} = d(i) + (k-1)d(i) = kd(i).$$ □

Proposition 6. *If $\mathcal{H}$ is a k-graph with n vertices and m edges, then*

$$P_{\mathbf{A}_{\mathcal{L}}}(\lambda) = (\lambda + k)^{m-n} P_{\mathbf{Q}}(\lambda + k).$$

Proof. Note that, by equation (1), we have

$$\begin{aligned} P_{\mathbf{A}_{\mathcal{L}}}(\lambda) &= \det(\lambda \mathbf{I}_m - \mathbf{A}_{\mathcal{L}}) = \det((\lambda + k)\mathbf{I}_m - \mathbf{B}^T\mathbf{B}) \\ &= (\lambda + k)^{m-n} \det((\lambda + k)\mathbf{I}_n - \mathbf{B}\mathbf{B}^T) = (\lambda + k)^{m-n} P_{\mathbf{Q}}(\lambda + k). \end{aligned}$$

Therefore, the result follows. □

Remark 4.3. Here we highlight two interesting consequences of Proposition 6. First, if λ is an eigenvalue of $\mathbf{A}_{\mathcal{L}}$, then $\lambda \geq -k$. Second, we see that $\rho(\mathbf{A}_{\mathcal{L}}) = \rho(\mathbf{Q}) - k$.

We now introduce the following notation. Let $\mathcal{H} = (V, E)$ be a hypergraph. For each non-empty subset of vertices $\alpha = \{v_1, \dots, v_t\} \subset V$ and each vector $\mathbf{x} = (x_i)$ of dimension $n = |V|$, we denote $x(\alpha) = x_{v_1} + \cdots + x_{v_t}$. Under these conditions, we can write

$$(\mathbf{Q}\mathbf{x})_u = d(u)x_u + \sum_{w\in N(u)} a_{uw}x_w = \sum_{e\in E_{[u]}} x(e).$$

Let $\mathcal{G}$ and $\mathcal{H}$ be k-graphs. We define and denote its *Cartesian product* $\mathcal{G} \times \mathcal{H}$, as the k-graph, with the following sets of vertices $V(\mathcal{G} \times \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$ and edges $E(\mathcal{G} \times \mathcal{H}) = \{\{v\} \times e : v \in V(\mathcal{G}), e \in E(\mathcal{H})\} \cup \{a \times \{u\} : u \in V(\mathcal{H}), a \in E(\mathcal{G})\}$.

Proposition 7. *If $\mathcal{G}$ and $\mathcal{H}$ are two k-graphs with signless Laplacian eigenvalues μ of multiplicity m_1 and λ of multiplicity m_2, respectively, then $\mu + \lambda$ is an eigenvalue of $\mathbf{Q}(\mathcal{G} \times \mathcal{H})$ with multiplicity $m_1 \cdot m_2$.*

Proof. Suppose $\mathbf{x}$ is an eigenvector of λ for $\mathbf{Q}(\mathcal{H})$ and $\mathbf{y}$ is an eigenvector of μ for $\mathbf{Q}(\mathcal{G})$. Consider a vertex (v, u) of $\mathcal{G} \times \mathcal{H}$ and define a vector $\mathbf{z}$ by $z_{(v,u)} = y_v x_u$. Thus,

$$(\mathbf{Q}\mathbf{z})_{(v,u)} = \sum_{\alpha\in E_{[(v,u)]}} z(\alpha) = \sum_{e\in E_{[u]}} y_v x(e) + \sum_{a\in E_{[v]}} y(e)x_u = \lambda y_v x_u + \mu y_v x_u = (\mu + \lambda) z_{(v,u)}.$$

Therefore, the result is true. □

The result below may be seen as a corollary of Proposition 4.4 of [3].

Proposition 8. *Let $\mathcal{H}$ be a k-graph with n vertices. For each vector $\mathbf{x} \in \mathbb{R}^n$, we have*

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = \sum_{e \in E} [x(e)]^2.$$

Proof. Note that, for each edge $e \in E$, it is true that $(\mathbf{B}^T\mathbf{x})_e = x(e)$. Therefore,

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = \mathbf{x}^T(\mathbf{B}\mathbf{B}^T)\mathbf{x} = (\mathbf{B}^T\mathbf{x})^T(\mathbf{B}^T\mathbf{x}) = \sum_{e \in E} [x(e)]^2. \qquad \square$$

Proposition 9. *Let $\mathcal{H}$ be a k-graph. If $\mathcal{H}'$ is a subgraph of $\mathcal{H}$, then*

$$\rho(\mathcal{H}') \leq \rho(\mathcal{H}).$$

Proof. Let $\mathbf{x}$ be a unitary eigenvector of $\rho(\mathcal{H}')$. Define a new vector $\overline{\mathbf{x}}$ of dimension $n = |V(\mathcal{H})|$ by $\overline{x}_i = x_i$ if $i \in V(\mathcal{H}')$ and $\overline{x}_i = 0$ otherwise. By Rayleigh's principle, we have

$$\rho(\mathcal{H}) \geq \sum_{e \in E(\mathcal{H})} [\overline{x}(e)]^2 = \sum_{e \in E(\mathcal{H}')} [x(e)]^2 = \rho(\mathcal{H}'). \qquad \square$$

5 Structural and spectral properties

In this section, we will determine structural characteristics of a hypergraph from its signless Laplacian spectrum. More precisely, we will study regular and partially bipartite uniform hypergraphs through their signless Laplacian eigenvalues.

Theorem 1. *Let $\mathcal{H}$ be a connected k-graph with n vertices and $\rho(\mathcal{H})$ its spectral radius. The following statements are equivalent*:

(*a*) *$\mathcal{H}$ is regular*;
(*b*) $\rho(\mathcal{H}) = kd(\mathcal{H})$;
(*c*) $\rho(\mathcal{H}) = k\Delta(\mathcal{H})$;
(*d*) *the principal eigenvector of $\mathbf{Q}(\mathcal{H})$ is* $\mathbf{x} = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$.

Proof. We will prove the result through the following chain of implications:

$$(a) \Rightarrow (b) \Rightarrow (d) \Rightarrow (c) \Rightarrow (a).$$

If $\mathcal{H}$ is r-regular, then for each vertex u, we have $|E_{[u]}| = r$. Observing that the sum of the entries in the row v of this matrix is equal to k times the degree of v, we conclude that

$$(\mathbf{Q1})_u = \sum_{e \in E_{[u]}} x(e) = kr.$$

That is, $\mathbf{1} = (1, 1, \dots, 1)$ is an eigenvector associated with the eigenvalue kr, and since $\mathcal{H}$ is regular, then $r = d(\mathcal{H})$. Since the hypergraph is connected, we can apply the Perron-Frobenius theorem, so we have that $\rho(\mathcal{H}) = kd(\mathcal{H})$.

Now, suppose $\rho(\mathcal{H}) = kd(\mathcal{H})$. We note that the vector $\mathbf{x} = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$ solves the following optimization problem:

$$\rho(\mathcal{H}) = \max_{||\mathbf{y}||=1}\{\mathbf{y}^T\mathbf{Q}\mathbf{y}\} \geq \mathbf{x}^T\mathbf{Q}\mathbf{x} = \sum_{i \in V} \frac{kd(i)}{n} = kd(\mathcal{H}).$$

Since $\mathcal{H}$ is connected, by Rayleigh's principle, we conclude that $\mathbf{x}$ is the principal eigenvector of $\mathbf{Q}$.

Let $\mathbf{x} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$ be the principal eigenvector of $\mathbf{Q}$. If $u \in V$ is a vertex of maximum degree, since the sum of the entries in the row u of $\mathbf{Q}$ is equal to $kd(u)$, we have that

$$\rho(\mathcal{H})\left(\frac{1}{\sqrt{n}}\right) = (\mathbf{Q}\mathbf{x})_u = \sum_{e \in E_{[u]}} x(e) = \Delta k\left(\frac{1}{\sqrt{n}}\right) \Rightarrow \rho(\mathcal{H}) = k\Delta(\mathcal{H}).$$

If $\rho(\mathcal{H}) = k\Delta(\mathcal{H})$ is the spectral radius of the irreducible matrix $\mathbf{Q}$ and $\mathbf{x}$ its principal eigenvector, let $u \in V$ be a vertex, such that $x_u \geq x_v$ for all $v \in V$. Thus,

$$k\Delta x_u = \sum_{e \in E[u]} x(e).$$

We observe that this equality is only possible if $d(u) = \Delta$ and $x_v = x_u$ for all $v \in N(u)$. Hence, we conclude that every vertex that has maximum value in the eigenvector $\mathbf{x}$ must have maximum degree. Moreover, every vertex that is a neighbor of another vertex that has maximum value in the eigenvector must also have maximum value. By the connectivity of the hypergraph, we conclude that all the vertices have maximum value in the principal eigenvector and, therefore, maximum degree, i.e., $\mathcal{H}$ is regular. □

Lemma 10. *Let $\mathcal{H}$ be a k-graph. Thus, $(0, \mathbf{x})$ is an eigenpair of $\mathbf{Q}(\mathcal{H})$ if, and only if, for each edge $e \in E$ we have $x(e) = 0$.*

Proof. If $(0, \mathbf{x})$ is a signless Laplacian eigenpair of $\mathcal{H}$, then $\mathbf{Q}\mathbf{x} = \mathbf{0}$. So,

$$\mathbf{x}^T\mathbf{Q}\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0 \Rightarrow \sum_{e \in E}[x(e)]^2 = 0 \Rightarrow x(e) = 0, \quad \forall e \in E.$$

Conversely, let $\mathbf{x}$ be a vector of dimension $n = |V(\mathcal{H})|$ such that $x(e) = 0$ for each edge $e \in E$. So,

$$(\mathbf{Q}\mathbf{x})_u = \sum_{e \in E_{[u]}} x(e) = 0 \ \forall u \in V \Rightarrow (0, \mathbf{x}) \text{ is an eigenpair of } \mathbf{Q}(\mathcal{H}).$$ □

We note that for a graph, the condition $x_{v_i} + x_{v_j} = 0$ for all $e = \{v_i, v_j\} \in E$ implies a bipartition of vertices. Unfortunately, for $k \geq 3$, we do not have such a trivial characterization.

Definition 5.1. A hypergraph $\mathcal{H}$ is *partially bipartite*, if we can separate the set of vertices into three disjoint subsets $V = V_0 \cup V_1 \cup V_2$, where V_1 and V_2 are non-empty and each edge is fully contained in V_0 or has vertices in both V_1 and V_2.

Theorem 11. *Let $\mathcal{H} = (V, E)$ be a k-graph. If $\lambda = 0$ is an eigenvalue of $\mathbf{Q}(\mathcal{H})$, then $\mathcal{H}$ is partially bipartite.*

Proof. Let $\mathbf{x}$ be a eigenvector of $\lambda = 0$. Define

$$V_1 = \{v \in V : x_v > 0\}, \ V_2 = \{v \in V : x_v < 0\}, \ V_0 = \{v \in V : x_v = 0\}.$$

As $x(e) = 0$ for each edge $e \in E$, then the edge is contained in V_0, or it must have some vertices in V_1 and others in V_2, i.e., $\mathcal{H}$ is partially bipartite. □

The converse of Theorem 11 is not true. For example, $\mathcal{H} = (\{1, 2, 3, 4\}, \{123, 124, 134, 234\})$ has a partial bipartition $V_1 = \{1, 2\}$, $V_2 = \{3, 4\}$, and $V_0 = \varnothing$, but the eigenvalues of $\mathbf{Q}$ are $\rho = 9$ and $\lambda = 1$ with multiplicity 3. In view of this, we leave here the following question.

Question 5.2. How to characterize uniform hypergraphs with signless Laplacian eigenvalue zero?

Definition 5.3. A hypergraph $\mathcal{H}$ is *balanced partially bipartite*, if it is partially bipartite and there exists a constant $c > 0$, such that for each edge $e \not\subseteq V_0$, it happens $\frac{|e \cap V_1|}{|e \cap V_2|} = c$.

Theorem 12. *Let $\mathcal{H} = (V, E)$ be a k-graph. If it is balanced partially bipartite, then $\lambda = 0$ is an eigenvalue of $\mathbf{Q}(\mathcal{H})$.*

Proof. Since $\mathcal{H}$ is balanced partially bipartite, there is a constant $c = \frac{|e \cap V_1|}{|e \cap V_2|}$, where $e \in E(\mathcal{H})$. So we define a vector $\mathbf{x}$ of dimension $n = |V|$ by

$$x_v = \begin{cases} 1 & \text{if} \quad v \in V_1, \\ 0 & \text{if} \quad v \in V_0, \\ -c & \text{if} \quad v \in V_2. \end{cases}$$

Thus, $x(e) = 0$, for each edge $e \in E$. By Lemma 10, we conclude the result. □

Example 5.4. We illustrate here that balanced partially bipartite hypergraphs are abundant by presenting two examples that are easy to find. Let $\mathcal{H} = (V, E)$ be a k-graph in which
(1) $\mathcal{H}$ has a vertex v with the property that each edge containing it also contains another vertex of degree 1;
(2) $\mathcal{H}$ has a couple of vertices which are contained in exactly the same edges.

In both cases $\mathcal{H}$ is balanced partially bipartite.

6 Relating classical and spectral parameters

In this section, we will relate classic and spectral parameters of a hypergraph. More precisely, we will relate the spectral radius to the degrees and the chromatic number, the number of edges to the sum of the eigenvalues, and the diameter to the number of distinct eigenvalues of the signless Laplacian matrix.

Theorem 13. *If $\mathcal{H}$ is a k-graph and $\rho(\mathcal{H})$ is its spectral radius, then*

$$\min_{e \in E} \left\{ \sum_{v \in e} d(v) \right\} \leq \rho(\mathcal{H}) \leq \max_{e \in E} \left\{ \sum_{v \in e} d(v) \right\}.$$

Proof. For each $u \in V(\mathcal{L}(\mathcal{H}))$, let $e_u \in E(\mathcal{H})$ be the (hyper)edge associated with the vertex u. By Lemma 4, we have $d_{\mathcal{L}}(u) = \left(\sum_{v \in e_u} d_{\mathcal{H}}(v)\right) - k$. Now, by Theorem 6, we have $\rho(\mathbf{A}_{\mathcal{L}}) = \rho(\mathcal{H}) - k$. For graphs and multigraphs, we know that the spectral radius of the adjacency matrix is bounded by the maximum and minimum degrees, with the bounds being reached if, and only if, the multigraph is regular [24]. So, we have

$$\min_{u \in V(\mathcal{L}(\mathcal{H}))} d_{\mathcal{L}}(u) \leq \rho(\mathbf{A}_{\mathcal{L}}) \leq \max_{u \in V(\mathcal{L}(\mathcal{H}))} d_{\mathcal{L}}(u).$$

Therefore,

$$\min_{\{v_1, \ldots, v_k\} \in E} \left\{ \left(\sum_{i=1}^{k} d(v_i) \right) - k \right\} \leq \rho(\mathcal{H}) - k \leq \max_{\{v_1, \ldots, v_k\} \in E} \left\{ \left(\sum_{i=1}^{k} d(v_i) \right) - k \right\}.$$

Adding k in each of the three parts of the inequalities, we obtain the desired result. □

The result below may be seen as a corollary of Propositions 4.7 and 4.12 in [3].

Corollary 14. *If $\mathcal{H}$ is a k-graph and $\rho(\mathcal{H})$ is its spectral radius, then*

$$kd(\mathcal{H}) \leq \rho(\mathcal{H}) \leq k\Delta(\mathcal{H}).$$

Equalities hold if, and only if, the hypergraph is regular.

Proof. If $n = |V|$, define $\mathbf{x} = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$. By Rayleigh's principle and Theorem 13, we have

$$kd(\mathcal{H}) = \mathbf{x}^T\mathbf{Q}\mathbf{x} \leq \rho(\mathcal{H}) \leq \max_{e \in E}\left\{\sum_{v \in e} d(v)\right\} \leq k\Delta(\mathcal{H}).$$

By Theorem 1, we note that equalities are achieved only when the hypergraph is regular. □

Definition 6.1. For a k-graph $\mathcal{H}$, a function $f : V \to \{1, \dots, r\}$ is a (vertex) *r-coloring* of $\mathcal{H}$ if for every edge $e = \{v_1, \dots, v_k\}$, there exists $i \neq j$ such that $f(v_i) \neq f(v_j)$. The *chromatic number* $\chi(\mathcal{H})$ is the minimum integer r such that $\mathcal{H}$ has an r-coloring.

The proof of next result is a reprise of the classical proof for graphs by Wilf in [25]. Our result is similar to Theorem 3.10 in [8], where for a uniform hypergraph $\mathcal{H}$, it is proved that $\chi(\mathcal{H}) \leq \lambda + 1$. It is worth noting that λ is the spectral radius of the adjacency tensor of $\mathcal{H}$. The theorem proved here is based on matrices, so we believe it is computationally more efficient than the one demonstrated by Cooper and Dutle.

Theorem 15. *Let $\mathcal{H}$ be a connected k-graph. If $\chi(\mathcal{H})$ is its chromatic number, then*

$$\chi(\mathcal{H}) \leq \frac{1}{k}\rho(\mathcal{H}) + 1.$$

Proof. We will define an order for the vertices of $\mathcal{H}$ as follows. Let $\mathcal{H}(n) = \mathcal{H}$ and v_n be a vertex of minimum degree in $\mathcal{H}(n)$. For each $t = 2, \dots, n$, let $\mathcal{H}(t-1)$ be the subgraph obtained after removing a vertex v_t with minimum degree from $\mathcal{H}(t)$.

Let us use the ordering $v_1, v_2, \dots, v_n$ as input of a greedy coloring algorithm, which paints v_t with the smallest color that makes $\mathcal{H}(t)$ properly colored.

Note that $\chi(\mathcal{H}(1)) = 1 \leq \frac{1}{k}\rho(\mathcal{H}) + 1$. Inductively, suppose $\mathcal{H}(t-1)$ is properly colored with up to $\frac{1}{k}\rho(\mathcal{H}) + 1$ distinct colors. We see that v_t has a minimum degree in $\mathcal{H}(t)$. Thus, in the worst case, each edge containing v_t has all the other vertices painted with the same color, and each of these edges uses one of the colors $1, 2, \dots, d_{\mathcal{H}(t)}(v_t)$. So we should paint v_t with the color $d_{\mathcal{H}(t)}(v_t) + 1$. Thus,

$$d_{\mathcal{H}(t)}(v_t) + 1 = \delta(\mathcal{H}(t)) + 1 \leq \frac{1}{k}\rho(\mathcal{H}(t)) + 1 \leq \frac{1}{k}\rho(\mathcal{H}) + 1.$$

By the inductive hypothesis, we have $\chi(\mathcal{H}(t)) \leq \frac{1}{k}\rho(\mathcal{H}) + 1$. So, $\chi(\mathcal{H}) \leq \frac{1}{k}\rho(\mathcal{H}) + 1$. □

Proposition 16. *Let $\mathcal{H}$ be a k-graph with characteristic polynomial*

$$P_{\mathbf{Q}}(\lambda) = \lambda^n + q_1\lambda^{n-1} + \dots + q_{n-1}\lambda + q_n.$$

The number of edges of $\mathcal{H}$ is $m = -\frac{q_1}{k}$, or equivalently $km = \lambda_1 + \dots + \lambda_n$.

Proof. If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are all eigenvalues of the matrix $\mathbf{Q}$, then

$$q_1 = -(\lambda_1 + \dots + \lambda_n) = -\operatorname{Tr}(\mathbf{Q}) = -km.$$ □

Theorem 17. *Let $\mathcal{H}$ be a connected k-graph with diameter D. The number of distinct eigenvalues of the matrix $\mathbf{Q}$ is at least $D + 1$.*

Proof. First we will show the following claim.

Claim 6.2. If there is a walk with length l connecting two distinct vertices i and j, then $(\mathbf{Q}^l)_{ij} > 0$, otherwise $(\mathbf{Q}^l)_{ij} = 0$.

The proof is by induction on l. We first note that if $l = 1$, then the signless Laplacian matrix has the desired properties. Now suppose the statement is true for $l \geq 1$. Note that

$$(\mathbf{Q}^{l+1})_{ij} = \sum_{t=1}^{n} (\mathbf{Q}^{l})_{it}(\mathbf{Q})_{tj}.$$

Thus, if there is no walk with length $l + 1$ linking i and j, then there can be no walk linking i to a neighbor of j. This implies that if u is a neighbor of j, then $(\mathbf{Q}^{l})_{iu} = 0$ and, otherwise, $(\mathbf{Q})_{uj} = 0$. Therefore, $(\mathbf{Q}^{l+1})_{ij} = 0$. On the other hand, assuming there is a walk with length $l + 1$ linking i and j, then there must be a walk with length l linking i to a neighbor u of j. So, $(\mathbf{Q}^{l})_{iu} > 0$ and $(\mathbf{Q})_{uj} > 0$. Therefore, $(\mathbf{Q}^{l+1})_{ij} > 0$. The claim is proven.

Returning to the proof of the theorem, we let $\lambda_1, \lambda_2 \ldots, \lambda_t$ be all the distinct eigenvalues of $\mathbf{Q}$. So, $(\mathbf{Q} - \lambda_1\mathbf{I})(\mathbf{Q} - \lambda_2\mathbf{I})\cdots(\mathbf{Q} - \lambda_t\mathbf{I}) = 0$. Thus, $\mathbf{Q}^t + a_1\mathbf{Q}^{t-1} + \cdots + a_t\mathbf{I} = 0$. Suppose, by way of contradiction, that $D \geq t$. Hence, there must exist i and j such that its distance is t. Thus, $(\mathbf{Q}^t)_{ij} = -a_1(\mathbf{Q}^{t-1})_{ij} - \cdots - a_t(\mathbf{I})_{ij} = 0$, because there should be no walk shorter than t linking the vertices i and j. This contradicts the claim. Therefore, $t \geq D + 1$. □

Theorem 18. *Let $\mathcal{H}$ be a connected k-graph with more than one edge. If the diameter of $\mathcal{H}$ is D, then*

$$D \leq \left\lfloor 1 + \frac{\log((1 - x_{\min}^2)/x_{\min}^2)}{\log(\lambda_1/\lambda_2)} \right\rfloor,$$

where $\lambda_1 > \lambda_2$ are the greatest eigenvalues of $\mathbf{Q}$ and $x_{\min}$ is the smallest entry of the principal eigenvector.

Proof. As $\mathbf{Q}$ is real symmetric, we may consider the orthonormal eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ from the eigenvalues $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$, respectively. In this case, $\mathbf{x}_1$ is the principal eigenvector. Let i and j be vertices such that its distance is D. Using the spectral decomposition of $\mathbf{Q}$, for each integer t, we have

$$\begin{aligned}
(\mathbf{Q}^t)_{ij} &= \sum_{l=1}^{n} \lambda_l^t (\mathbf{x}_l \mathbf{x}_l^T)_{ij} \\
&\geq \lambda_1^t (\mathbf{x}_1)_i (\mathbf{x}_1)_j - \left| \sum_{l=2}^{n} \lambda_l^t (\mathbf{x}_l \mathbf{x}_l^T)_{ij} \right| \\
&\geq \lambda_1^t x_{\min}^2 - \lambda_2^t \left(\sum_{l=2}^{n} (\mathbf{x}_l)_i^2 \right)^{\frac{1}{2}} \left(\sum_{l=2}^{n} (\mathbf{x}_l)_j^2 \right)^{\frac{1}{2}} \\
&\geq \lambda_1^t x_{\min}^2 - \lambda_2^t (1 - (\mathbf{x}_1)_i^2)^{\frac{1}{2}} (1 - (\mathbf{x}_1)_j^2)^{\frac{1}{2}} \\
&\geq \lambda_1^t x_{\min}^2 - \lambda_2^t (1 - x_{\min}^2).
\end{aligned} \tag{2}$$

Note that if the expression (2) is positive, then $(\mathbf{Q}^t)_{ij}$ is positive and, therefore, $t \geq D$.

$$\lambda_1^t x_{\min}^2 - \lambda_2^t (1 - x_{\min}^2) > 0 \;\Rightarrow\; t > \frac{\log((1 - x_{\min}^2)/x_{\min}^2)}{\log(\lambda_1/\lambda_2)}.$$

Therefore, the result follows. □

7 Power hypergraph

In this section, we will study the spectrum of the class of power hypergraphs, relating its signless Laplacian eigenvalues to those of its base hypergraph. The spectrum of this class has already been studied in the context of tensors. See for example [26,27].

Definition 7.1. For a k-graph $\mathcal{H} = (V, E)$, let $s \geq 1$ and $r \geq ks$ be integers. The (generalized) *power hypergraph* $\mathcal{H}_s^r$ is defined as the r-graph with the following sets of vertices and edges:

$$V(\mathcal{H}_s^r) = \left(\bigcup_{v \in V} \varsigma_v\right) \cup \left(\bigcup_{e \in E} \varsigma_e\right) \quad \text{and} \quad E(\mathcal{H}_s^r) = \left\{\varsigma_e \cup \varsigma_{v_1} \cup \cdots \cup \varsigma_{v_k} : e = \{v_1, \ldots, v_k\} \in E\right\},$$

where $\varsigma_v = \{v^1, \ldots, v^s\}$ for each vertex $v \in V(\mathcal{H})$ and $\varsigma_e = \{v_e^1, \ldots, v_e^{r-ks}\}$ for each edge $e \in E(\mathcal{H})$.

Informally, we say that $\mathcal{H}_s^r$ is obtained from a *base hypergraph* $\mathcal{H} = (V, E)$ by replacing each vertex $v \in V$ by a set ς_v of cardinality s, and by adding a set ς_e with $r - ks$ new vertices for each edge $e \in \mathcal{H}$.

Example 7.2. The power hypergraph $(P_4)_2^5$ of the path P_4 is illustrated in Figure 2.

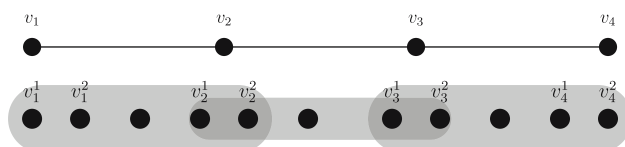

Figure 2: The power hypergraph $(P_4)_2^5$.

Let $\mathcal{H}_s^r$ be a power hypergraph. For each edge $e = \{i_1, \ldots, i_k\} \in E(\mathcal{H})$, we denote by $e_s^r = \varsigma_{i_1} \cup \cdots \cup \varsigma_{i_k} \cup \varsigma_e \in E(\mathcal{H}_s^r)$ the edge obtained from $e \in E(\mathcal{H})$. For simplicity, we will write $\mathcal{H}^r = \mathcal{H}_1^r$ and $\mathcal{H}_s = \mathcal{H}_s^{ks}$. We identify a vertex in each of the sets ς_v with the vertex v and say that it is a *main vertex* of $\mathcal{H}_s^k$, while the other vertices in ς_v are called *copies*. The vertices in some of the sets ς_e will be called *additional vertices*.

We start this section by proving some algebraic properties of this class.

Lemma 19. *Let $\mathcal{H}$ be a k-graph having two vertices u and v which are contained exactly in the same edges. If $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{Q}$ with $\lambda > 0$, then $x_u = x_v$.*

Proof. We just note that

$$\lambda x_u = \sum_{e \in E_{[u]}} x(e) = \sum_{e \in E_{[v]}} x(e) = \lambda x_v.$$

Since $\lambda \neq 0$, then the result is true. □

Lemma 20. *Let $\mathcal{H}$ be a k-graph and $s \geq 1$ an integer. If $(\lambda, \mathbf{x})$ is a signless Laplacian eigenpair of $\mathcal{H}_s$ with $\lambda > 0$, then for each edge $e_s \in E(\mathcal{H}_s)$, we have $x(e_s) = sx(e)$.*

Proof. By Lemma 19, we have $x(\varsigma_u) = x_u + x_{u^2} + \cdots + x_{u^s} = sx_u$. Hence,

$$x(e_s) = x\left(\varsigma_{u_1}\right) + \cdots + x\left(\varsigma_{u_k}\right) = s\left(x_{u_1} + \cdots + x_{u_k}\right) = sx(e). \qquad \square$$

Proposition 21. *Let $\mathcal{H}$ be a k-graph and $s \geq 1$ an integer. If $\mu \neq 0$ is a signless Laplacian eigenvalue of $\mathcal{H}$, then $\lambda = s\mu$ is an eigenvalue of $\mathbf{Q}(\mathcal{H}_s)$.*

Proof. Suppose $\mathbf{y}$ is an eigenvector of $\mathbf{Q}(\mathcal{H})$ associated with μ. Define a vector $\mathbf{x}$ of dimension $|V(\mathcal{H}_s)|$ by $x_u = y_v$ if $u \in \varsigma_v$. Thus,

$$(\mathbf{Q}(\mathcal{H}_s)\mathbf{x})_u = \sum_{e_s \in E(\mathcal{H}_s)_{[u]}} x(e_s) = \sum_{e \in E(\mathcal{H})_{[u]}} sx(e) = s\mu x_u. \qquad \square$$

Lemma 22. *Let $\mathcal{H}$ be a k-graph and $r \geq k$ be an integer. Then, for each eigenpair $(\lambda, \mathbf{x})$ of $\mathbf{Q}(\mathcal{H}^r)$ with $\lambda > r - k$, we have*

$$x_u = \frac{x(e)}{\lambda - r + k} \quad \text{if } u \text{ is an additional vertex of edge } e^r \in E(\mathcal{H}^r).$$

Proof. Suppose $\varsigma_e = \{i_e^1, \ldots, i_e^{r-k}\}$ and denote $u = i_e^1$. By Lemma 19, we know that $x_u = x_{i_e^2} \cdots = x_{i_e^{r-k}}$. So, $\lambda x_u = x(e^r) = (r-k)x_u + x(e)$. Thus,

$$(\lambda - r + k)x_u = x(e) \;\Rightarrow\; x_u = \frac{x(e)}{\lambda - r + k}. \qquad \square$$

Proposition 23. *Let $\mathcal{H}$ be a k-graph. If $\mu \neq 0$ is a signless Laplacian eigenvalue of $\mathcal{H}$, then $\lambda = \mu + r - k$ is an eigenvalue of $\mathbf{Q}(\mathcal{H}^r)$.*

Proof. Suppose $\mathbf{y}$ is an eigenvector of $\mathbf{Q}(\mathcal{H})$ associated with μ. Define a vector $\mathbf{x}$ of dimension $|V(\mathcal{H}^r)|$ by

$$x_i = \begin{cases} y_i & \text{if } i \text{ is a main vertex,} \\ \dfrac{y(e)}{\mu} & \text{if } i \text{ is an additional vertex of the edge } e^r. \end{cases}$$

If u is a main vertex, we have

$$\begin{aligned}(\mathbf{Q}(\mathcal{H}^r)\mathbf{x})_u &= \sum_{e^r \in E(\mathcal{H}^r)_{[u]}} x(e^r) \\ &= \sum_{e \in E(\mathcal{H})_{[u]}} \left(y(e) + (r-k)\frac{y(e)}{\mu} \right) \\ &= \sum_{e \in E(\mathcal{H})_{[u]}} y(e) + \left(\frac{r-k}{\mu}\right) \sum_{e \in E(\mathcal{H})_{[u]}} y(e) \\ &= (\mu + r - k)y_u \\ &= (\mu + r - k)x_u.\end{aligned}$$

Now, if u is an additional vertex, we have

$$(\mathbf{Q}(\mathcal{H}^r)\mathbf{x})_u = x(e^r) = y(e) + (r-k)\frac{y(e)}{\mu} = (\mu + r - k)\frac{y(e)}{\mu} = (\mu + r - k)x_u.$$

Therefore, the result follows. $\square$

Theorem 24. *Let $\mathcal{H}$ be k-graph, $s \geq 1$ and $r \geq ks$ be two integers. Then, $\lambda > r - ks$ is an eigenvalue of $\mathbf{Q}(\mathcal{H}_s^r)$ if, and only if, there is a signless Laplacian eigenvalue $\mu > 0$ of $\mathcal{H}$ such that $\lambda = s\mu + r - ks$.*

Proof. If μ is a signless Laplacian eigenvalue of $\mathcal{H}$, then $s\mu$ is an eigenvalue of $\mathbf{Q}(\mathcal{H}_s)$. So, $\lambda = s\mu + r - ks$ is a signless Laplacian eigenvalue of $(\mathcal{H}_s)^r = \mathcal{H}_s^r$.

Conversely, let $\mathbf{x}$ be an eigenvector associated with λ in $\mathbf{Q}(\mathcal{H}_s^r)$. Thus,

$$\begin{aligned}\lambda x_u &= \sum_{e_s^r \in E(\mathcal{H}_s^r)_{[u]}} x(e_s^r) \\ &= \sum_{e_s^r \in E(\mathcal{H}_s^r)_{[u]}} (r - ks)x_{i_e^1} + x(e_s) \\ &= \sum_{e_s \in E(\mathcal{H}_s)_{[u]}} (r-ks)\frac{x(e_s)}{\lambda - r + ks} + x(e_s) \\ &= \left(\frac{\lambda s}{\lambda + ks - r}\right) \sum_{e \in E(\mathcal{H})_{[u]}} x(e).\end{aligned}$$

Therefore,

$$\sum_{e\in E(\mathcal{H})_{[u]}} x(e) = \frac{\lambda + ks - r}{s} x_u.$$

That is, $\mathcal{H}$ has a signless Laplacian eigenvalue μ, such that

$$\mu = \frac{\lambda + ks - r}{s} \Rightarrow \lambda = s\mu + r - ks. \qquad \square$$

We note that Theorem 24 characterizes all signless Laplacian eigenvalues greater than $r - ks$ of a power hypergraph $\mathcal{H}^r_s$. Now we will study its other eigenvalues.

Proposition 25. *Let $\mathcal{H}$ be a k-graph. If $s \geq 1$ is an integer, then the multiplicity of $\lambda = 0$ as eigenvalue of $\mathbf{Q}(\mathcal{H}_s)$ is $s|V| - t$, where t is the rank of $\mathbf{Q}(\mathcal{H})$.*

Proof. If $\mathbf{z}$ is an eigenvector of $\lambda = 0$ in $\mathbf{Q}(\mathcal{H})$, define a new vector $\mathbf{x}$ of dimension $|V(\mathcal{H}_s)|$ by $x_v = z_u$ if $v \in \varsigma_u$. Note that

$$(\mathbf{Q}(\mathcal{H}_s)\mathbf{x})_v = \sum_{e_s\in E(\mathcal{H}_s)_{[v]}} x(e_s) = s \sum_{e\in E(\mathcal{H})_{[v]}} z(e) = 0.$$

Hence, for each eigenvector of $\lambda = 0$ in $\mathbf{Q}(\mathcal{H})$, we build one for $\mathcal{H}_s$, i.e., we construct a family of $|V| - t$ linearly independent eigenvectors.

Now, for each $v \in V(\mathcal{H})$, suppose $\varsigma_v = \{v, v_2 \dots, v_s\}$ and $2 \leq j \leq s$. We can construct the following family of $s - 1$ linearly independent vectors:

$$\mathbf{x}^j = \begin{cases} (x^j)_v = 1, \\ (x^j)_{v_j} = -1, \\ (x^j)_u = 0, \quad \text{for } u \in V(\mathcal{H}_s) - \{v_1, v_j\}. \end{cases}$$

Note that these vectors are eigenvectors of $\lambda = 0$ in $\mathbf{Q}(\mathcal{H}_s)$. Repeating this construction for the other main vertices of $\mathcal{H}_s$, we obtain $(s - 1)|V|$ linearly independent eigenvectors. Observe that these vectors are linearly independent from those constructed from the zero eigenvectors of the base hypergraph $\mathcal{H}$. To see this, we observe that the former vectors have constant sign in each ς_u, while these new vectors have more than one sign in these sets. Therefore, we have $s|V| - t$ linearly independent eigenvectors of $\lambda = 0$. $\square$

Remark 7.3. Let $\mathcal{H}$ be a k-graph with n vertices. For $s \geq 1$, the eigenvalues of $\mathbf{Q}(\mathcal{H}_s)$ are $s\lambda_1, \dots, s\lambda_t$ and 0 with multiplicity $sn - t$, where t is the rank of $\mathbf{Q}(\mathcal{H})$.

Proposition 26. *Let $\mathcal{H}$ be a k-graph. If $s \geq 1$ and $r > ks$ are two integers, then the multiplicity of $\lambda = 0$ as eigenvalue of $\mathbf{Q}(\mathcal{H}^r_s)$ is at least $(r - ks - 1)|E| + s|V|$.*

Proof. Let $e \in E(\mathcal{H})$ be an edge. Suppose $\varsigma_e = \{u_1, \dots, u_{r-ks}\}$ and $2 \leq j \leq r - ks$. Similar to the proof of Proposition 25, we can construct the following family of $r - ks - 1$ linearly independent vectors:

$$\mathbf{y}^j = \begin{cases} (y^j)_{u_1} = 1, \\ (y^j)_{u_j} = -1, \\ (y^j)_u = 0, \quad \text{for } u \in V(\mathcal{H}^r_s) - \{u_1, u_j\}. \end{cases}$$

Repeating this construction for the other edges of $\mathcal{H}$, we obtain $(r - ks - 1)|E|$ linearly independent eigenvectors associated with $\lambda = 0$.

Now, let $w \in V(\mathcal{H}_s)$ and consider $e_1, \dots, e_p$, all edges of $\mathcal{H}^r_s$ that contain the vertex w. For each edge, take $w_i \in e_i$ to be an additional vertex. So we can build the vector

$$\mathbf{z} = \begin{cases} z_w = 1, \\ z_{w_i} = -1, \quad \text{for } 1 \le i \le p, \\ z_u = 0, \quad \text{for } u \in V(\mathcal{H}_s^r) - \{w, w_1, \ldots, w_p\}. \end{cases}$$

Repeating this construction for the other vertices of $\mathcal{H}_s$, we obtain $s|V|$ linearly independent eigenvectors associated with $\lambda = 0$. Together with the previously created sets, we obtain a total of $(r - ks - 1)|E| + s|V|$ linearly independent eigenvectors. □

Theorem 27. *Let $\mathcal{H}$ be a k-graph. If $s \ge 1$ and $r > ks$ are integers, then the multiplicity of $\lambda = r - ks$ as eigenvalue of $\mathbf{Q}(\mathcal{H}_s^r)$ is $|E| - t$, where t is the rank of the signless Laplacian matrix $\mathbf{Q}(\mathcal{H})$.*

Proof. First, we observe that $\lambda = 0$ is an eigenvalue of multiplicity $|E| - t$ from $\mathbf{B}^T\mathbf{B}$. Let $\mathbf{z} = (z_{e_1}, \ldots, z_{e_m})$ be an eigenvector of $\lambda = 0$ in $\mathbf{B}^T\mathbf{B}$. We note that

$$\sum_{v \in V} \left(\sum_{e \in E_{[v]}} z_e \right)^2 = \mathbf{z}^T\mathbf{B}^T\mathbf{B}\mathbf{z} = 0 \quad \Rightarrow \quad \sum_{e \in E_{[v]}} z_e = 0 \quad \forall v \in V.$$

Now, define a vector $\mathbf{x}$ of dimension $|V(\mathcal{H}_s^r)|$ by

$$\mathbf{x} = \begin{cases} x_v = z_e \ \text{ if } v \text{ is an additional vertex of } e, \\ x_v = 0 \ \text{ if } v \text{ is not an additional vertex.} \end{cases}$$

If u is an additional vertex, then

$$(\mathbf{Q}(\mathcal{H}_s^r)\mathbf{x})_u = x(e) = (r - ks)(z_e) = (r - ks)x_u.$$

If u is a main or copy vertex, then

$$(\mathbf{Q}(\mathcal{H}_s^r)\mathbf{x})_u = \sum_{e \in E_{[u]}} x(e) = \sum_{e \in E_{[v]}} z_e = 0 = (r - ks)x_u.$$

Therefore, the result follows. □

Remark 7.4. If $\mathcal{H}$ is a k-graph with n vertices, m edges, and having signless Laplacian eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_t > \lambda_{t+1} = \cdots = \lambda_n = 0$, then the eigenvalues of $\mathbf{Q}(\mathcal{H}_s^r)$ are $(s\lambda_1 + r - ks), \ldots, (s\lambda_t + r - ks)$, and $(r - ks)$ with multiplicity $m - t$, as well as 0 with multiplicity $(r - ks - 1)m + sn$.

8 Conclusion

Here, we made contributions to spectral hypergraph theory. More precisely, we determined many properties and parameters associated with the signless Laplacian matrix of a hypergraph and generalized important results of spectral graph theory. In particular, we have proved that structural properties of a hypergraph such as being regular or partially bipartite can be studied from the eigenvalues of this matrix. We also have shown that spectral parameters such as the number of distinct eigenvalues and the spectral radius are related to structural parameters such as diameter, maximum degree, and chromatic number.

We note that many topics of spectral graph theory have not been explored in the context of hypergraphs, perhaps because the spectral hypergraph theory is a recent research area. In addition, hypergraphs are structures with several characteristics that are not observed in graphs. Hence, we believe that spectral hypergraph theory offers many research opportunities, and perhaps in a few years, this spectral theory will produce results that transcend what we know for graphs.

To conclude this article, we present some research possibilities about the signless Laplacian matrix of hypergraphs.

(1) Here, we proved that if zero is a signless Laplacian eigenvalue of a uniform hypergraph, then it is partially bipartite. With some additional assumptions, we prove that if a uniform hypergraph is balanced partially bipartite, then zero is a signless Laplacian eigenvalue. It is an open problem to determine an intermediate hypergraph class, which is characterized by the existence of zero as signless Laplacian eigenvalue.
(2) A very important and well-developed topic in spectral graph theory is the study of the line graph and its relation with the signless Laplacian matrix (see [19]). Here, we proved that the line multigraphs associated with k-graphs have eigenvalues greater than or equal to $-k$. An interesting topic of research is to study the spectrum of the line multigraphs and its relation with the signless Laplacian spectrum of the hypergraph.
(3) In [28], the author defined a new spectral parameter Γ from the principal eigenvector of the signless Laplacian matrix that can be thought of as a measure of the regularity of the edges of the hypergraph. We believe it may be possible to obtain bounds relating Γ to structural and spectral parameters. It would be interesting to determine which hypergraphs maximize Γ.
(4) The definition of the signless Laplacian matrix is made for general hypergraphs, so we believe that many results proven here can be generalized or adapted to non-uniform hypergraphs. In addition, some results obtained in the following articles could be generalized for the general case.
    (a) In [29], the authors defined and studied the energy of the signless Laplacian matrix for uniform hypergraphs.
    (b) In [28], the author defined and studied the principal eigenvector of the signless Laplacian matrix for uniform hypergraphs.
    (c) In [14], the authors defined and studied the signless Laplacian Estrada index of uniform hypergraphs.
    (d) The incidence energy of a hypergraph can be computed as the sum of the square roots of its signless Laplacian eigenvalues. In [16], the authors obtained some lower and upper bounds for this energy. At the same time, their corresponding extremal hypergraphs were characterized.

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