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Weak magnetohydrodynamic turbulence theory revisited

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ABSTRACT

Two recent papers, P. H. Yoon and G. Choe, Phys. Plasmas **28**, 082306 (2021) and Yoon *et al.*, Phys. Plasmas **29**, 112303 (2022), utilized in the derivation of the kinetic equation for the intensity of turbulent fluctuations the assumption that the wave spectra are isotropic, that is, the ensemble-averaged magnetic field tensorial fluctuation intensity is given by the isotropic diagonal form, $\langle \delta B_i \delta B_j \rangle_k = \langle \delta B^2 \rangle_k \delta_{ij}$. However, it is more appropriate to describe the incompressible magnetohydrodynamic turbulence involving shear Alfvénic waves by modeling the turbulence spectrum as being anisotropic. That is, the tensorial fluctuation intensity should be different in diagonal elements across and along the direction of the wave vector, $\langle \delta B_i \delta B_j \rangle_k = \frac{1}{2} \langle \delta B_\perp^2 \rangle_k (\delta_{ij} - k_i k_j / k^2) + \langle \delta B_\parallel^2 \rangle_k (k_i k_j / k^2)$. In the present paper, we thus reformulate the weak magnetohydrodynamic turbulence theory under the assumption of anisotropy and work out the form of nonlinear wave kinetic equation.

I. INTRODUCTION

In a paper recently published, methods of the weak turbulence theory developed in the context of kinetic theory have been employed to develop a weak turbulence approach to incompressible magnetohydrodynamics (MHD).[1] The formulation employed MHD equations developed with the use of Elsasser variables, incorporating iterative solutions that took into account contributions up to second order. In addition to the derivation of an equation for the time evolution of the amplitudes of magnetic fluctuations, Ref. 1 obtained an equation for the residual energy, which is the difference between the particle and field energies.[1]

In another and more recent work,[2] the authors addressed the problem of weak MHD turbulence theory with the use of the original equations of MHD theory, without employing the Elsasser variables. The iterative solution for the nonlinear momentum equation is kept up to third-order terms, leading to an additional contribution to the term describing the so-called nonlinear frequency shift. Despite this additional effect, the wave kinetic equation obtained is the same as that obtained in Ref. 1. However, the third-order nonlinear correction turns out to be essential for calculation of the total and residual energies.[2]

The works developed in Refs. 1 and 2 had in common the use of an assumption regarding the nature of turbulence, that is, the turbulence fluctuation intensities are isotropic. Here, by isotropic, we mean that the ensemble-averaged fluctuating magnetic field tensor intensity is given by an isotropic diagonal form, $\langle \delta B_i \delta B_j \rangle_{\mathbf{k}} = \langle \delta B^2 \rangle_{\mathbf{k}} \delta_{ij}$, where $\delta \mathbf{B}$ is the fluctuating magnetic field vector, and the bracket denotes the ensemble average. That is, the form of diagonal tensor implicitly assumed in Refs. 1 and 2 is in a scalar matrix form, with all the diagonal elements being equal. Thus, the notion of isotropy in this sense implies that the turbulent fluctuation intensity has no preferred spatial direction with respect to the wave vector. The isotropic model is not strictly valid in a physical sense since the magnetic field perturbation cannot have a component along the $\mathbf{k}$ vector, as the divergence-free conditions (Gauss' law) dictate. That is, upon writing

$$\delta \mathbf{B}_{\mathbf{k}} = \delta \mathbf{B}_{\mathbf{k}}^{\perp} + \delta \mathbf{B}_{\mathbf{k}}^{\parallel},$$

$$\delta \mathbf{B}_{\mathbf{k}}^{\perp} = \frac{(\mathbf{k} \times \delta \mathbf{B}_{\mathbf{k}}) \times \mathbf{k}}{k^2}, \quad \delta \mathbf{B}_{\mathbf{k}}^{\parallel} = \frac{\mathbf{k}(\mathbf{k} \cdot \delta \mathbf{B}_{\mathbf{k}})}{k^2},$$

it is evident that $\delta \mathbf{B}_{\mathbf{k}}^{\parallel}$ should be zero by virtue of the fact that $\mathbf{k} \cdot \delta \mathbf{B}_{\mathbf{k}} = 0$. As a result, the proper relationship that dictates the ensemble average of tensorial fluctuations associated with a stationary and homogeneous turbulence should be given by an anisotropic (or non-scalar) diagonal form,

$$\langle \delta B_i \delta B_j \rangle_{\mathbf{k}} = \frac{1}{2}\langle \delta B_\perp^2 \rangle_{\mathbf{k}} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) + \langle \delta B_\parallel^2 \rangle_{\mathbf{k}} \frac{k_i k_j}{k^2} = \frac{1}{2}\langle \delta B_\perp^2 \rangle_{\mathbf{k}} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$$

with $\langle \delta B_\parallel^2 \rangle_{\mathbf{k}} = 0$, leaving only the transverse fluctuations to remain nonvanishing. In short, the isotropic diagonal form of the magnetic field spectrum, $\langle \delta B_i \delta B_j \rangle_{\mathbf{k}} = \langle \delta B^2 \rangle_{\mathbf{k}} \delta_{ij}$, adopted in Refs. 1 and 2 is simply incorrect.

In the literature, the notion of "isotropic" turbulence often has a different meaning. That is, the more widely accepted definition of isotropic turbulence is that the magnetic field spectrum only depends on the modulus of wave vector, $\langle \delta B_\perp^2 \rangle_{\mathbf{k}} = \mathscr{E}(k)$.[3–8] It is important to note that even for such an isotropic spectrum, the diagonal form of the spectral tensor should still be anisotropic (or non-scalar), $\langle \delta B_i \delta B_j \rangle_{\mathbf{k}} = \frac{1}{2}\mathscr{E}(k)(\delta_{ij} - k_i k_j / k^2)$. It is commonly believed that if the mean magnetic field is weak (or even absent), then the isotropic model of the turbulence represents an effective theory. On the other hand, in the presence of a finite background magnetic field the incompressible MHD turbulence is generally anisotropic in $\mathbf{k}$ space. Turbulence theories that emphasize the dependence of the spectrum on $k_\perp$ and $k_\parallel$, where $k_\perp$ and $k_\parallel$ are wave vector components perpendicular and parallel to the averaged (ambient) magnetic field vector $\mathbf{B}_0$, are known as the anisotropic turbulence models.[9,10] In short, according to this categorization, the isotropy vs anisotropy means the wave vector dependence of the spectrum. References 1 and 2 belong to the latter category, that is, these references deal with the anisotropic turbulence in the sense of the turbulence spectrum having distinct dependence on $k_\perp$ and $k_\parallel$, but these works made an unjustifiable assumption of isotropic diagonal spectral function, $\langle \delta B_i \delta B_j \rangle_{\mathbf{k}} = \langle \delta B^2 \rangle_{\mathbf{k}} \delta_{ij}$.

In the present work, we revisit the discussion of weak MHD turbulence theory, starting from the equation of incompressible MHD theory and proceeding with the use of an iterative approach, but avoiding the use of assumption about the isotropic diagonal form of turbulent fluctuation spectrum tensor. In the present paper, we will re-derive the nonlinear wave kinetic equation of the weak incompressible MHD turbulence theory under the correct anisotropic formulation.

The present paper is organized as follows: In Sec. II, the equations of weak MHD turbulence are obtained from the basic equations of incompressible MHD theory by means of an iterative solution keeping contributions up to third order. The correct anisotropic equation for the time evolution of the amplitudes of the normal modes is obtained and compared with the equation obtained in Ref. 2 under the incorrect assumption of isotropic form of diagonal fluctuating spectral tensor. Section III summarizes and concludes the present paper.

II. ALTERNATIVE DERIVATION OF EQUATIONS FOR WEAK MHD TURBULENCE

We start from the equations of incompressible MHD,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} &= -\nabla P_* + (\mathbf{b}\cdot\nabla)\mathbf{b} + (\mathbf{c}_A\cdot\nabla)\mathbf{b} + \nu\nabla^2\mathbf{u}, \\ \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{b} &= (\mathbf{b}\cdot\nabla)\mathbf{u} + (\mathbf{c}_A\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{b}, \\ \nabla\cdot\mathbf{u} &= 0 = \nabla\cdot\mathbf{b}, \end{aligned} \tag{1}$$

where

$$\mathbf{c}_A = \frac{\mathbf{B}_0}{(4\pi\rho)^{1/2}}, \quad \mathbf{b} = \frac{\mathbf{B}}{(4\pi\rho)^{1/2}}, \quad P_* = \frac{P}{\rho} + \frac{b^2}{2}, \tag{2}$$

are Alfvén velocity, perturbed magnetic field, and total normalized pressure, respectively. Here, $\mathbf{u}$ represents the fluid momentum vector, and we have made an assumption that the viscosity and magnetic resistivity are the same, both being represented by ν. Finally, the fluid mass density, ρ, is assumed constant, from which follows the divergence-free condition for $\mathbf{u}$. Here, we note that Eq. (1) presupposes the presence of a finite (and constant) ambient magnetic field vector, $\mathbf{B}_0$. If the background field is extremely weak in comparison with the fluctuating magnetic field $\mathbf{b}$ (that is, $|\mathbf{B}_0| \ll |b|$), or even entirely absent, then we may take $\mathbf{c}_A \to 0$ in Eq. (1). Such a limit is appropriate for the discussion of isotropic (in wave vector space) MHD turbulence.[3–8] We are not concerned with such a limit. After some manipulations and performing the spectral transformation, we arrive at the following equations for the spectral components of velocity and magnetic field,

$$(\omega + ik^2\nu)u^i_{\mathbf{k},\omega} + k_\parallel c_A b^i_{\mathbf{k},\omega} = \left(\delta_{il} - \frac{k_i k_l}{k^2}\right) k_j \sum_{\mathbf{k}',\omega'} (u^j_{\mathbf{k}',\omega'} u^l_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - b^j_{\mathbf{k}',\omega'} b^l_{\mathbf{k}-\mathbf{k}',\omega-\omega'}), \tag{3}$$

$$(\omega + ik^2\nu)b^i_{\mathbf{k},\omega} + k_\parallel c_A u^i_{\mathbf{k},\omega} = \sum_{\mathbf{k}',\omega'} k_j (u^j_{\mathbf{k}',\omega'} b^i_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - u^i_{\mathbf{k}-\mathbf{k}',\omega-\omega'} b^j_{\mathbf{k}',\omega'}). \tag{4}$$

Suppose that we solve the velocity equation iteratively by expanding $u^i_{\mathbf{k},\omega}$ as

$$u^i_{\mathbf{k},\omega} = u^{i(1)}_{\mathbf{k},\omega} + u^{i(2)}_{\mathbf{k},\omega} + \cdots, \tag{5}$$

where the velocity field is expanded in a series with each term proportional to the power of b-field,

$$u^{i(n)}_{\mathbf{k},\omega} \propto (b^i_{\mathbf{k},\omega})^n. \tag{6}$$

Then, from

$$\begin{aligned} &(\omega + ik^2\nu)(u^{i(1)}_{\mathbf{k},\omega} + u^{i(2)}_{\mathbf{k},\omega} + \cdots) + k_\parallel c_A b^i_{\mathbf{k},\omega} \\ &= \left(\delta_{il} - \frac{k_i k_l}{k^2}\right) k_j \sum_{\mathbf{k}',\omega'} \Big[(u^{j(1)}_{\mathbf{k}',\omega'} + u^{j(2)}_{\mathbf{k}',\omega'} + \cdots) \\ &\quad \times (u^{l(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} + u^{l(2)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} + \cdots) - b^j_{\mathbf{k}',\omega'} b^l_{\mathbf{k}-\mathbf{k}',\omega-\omega'} \Big], \end{aligned} \tag{7}$$

we have order-by-order equations,

$$(\omega + ik^2\nu)u^{i(1)}_{\mathbf{k},\omega} + k_\parallel c_A b^i_{\mathbf{k},\omega} = 0, \tag{8a}$$

$$(\omega + ik^2\nu)u^{i(2)}_{\mathbf{k},\omega} = \left(\delta_{il} - \frac{k_i k_l}{k^2}\right) k_j \sum_{\mathbf{k}',\omega'} (u^{j(1)}_{\mathbf{k}',\omega'} u^{l(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - b^j_{\mathbf{k}',\omega'} b^l_{\mathbf{k}-\mathbf{k}',\omega-\omega'}), \tag{8b}$$

$$\begin{aligned} (\omega + ik^2\nu)u^{i(3)}_{\mathbf{k},\omega} &= \left(\delta_{il} - \frac{k_i k_l}{k^2}\right) k_j \\ &\quad \times \sum_{\mathbf{k}',\omega'} (u^{j(2)}_{\mathbf{k}',\omega'} u^{l(1)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'} + u^{j(1)}_{\mathbf{k}',\omega'} u^{l(2)}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}), \\ &\cdots \end{aligned} \tag{8c}$$

Iterative solution leads to

$$u_q^{i(1)} = -\frac{k_\| c_A}{\omega + ik^2\nu}\, b_q^i, \tag{9a}$$

$$u_q^{i(2)} = -\frac{P_{il}(\mathbf{k})k_j}{\omega}\sum_{q'}(1 - f_{q'}f_{q-q'})\, b_{q'}^j b_{q-q'}^l, \tag{9b}$$

$$u_q^{i(3)} = \frac{P_{il}(\mathbf{k})k_j}{\omega}\sum_{q'}\sum_{q''}\left[\frac{P_{jm}(\mathbf{k}')k_n'}{\omega'}\,(1 - f_{q''}f_{q'-q''})f_{q-q'}\, b_{q-q'}^l b_{q''}^n b_{q'-q''}^m + \frac{P_{lm}(\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')_n}{\omega - \omega'}\,(1 - f_{q''}f_{q-q'-q''})f_{q'}\, b_{q'}^j b_{q''}^n b_{q-q'-q''}^m\right], \dots \tag{9c}$$

where

$$q = (\mathbf{k}, \omega), \quad P_{il}(\mathbf{k}) = \delta_{il} - \frac{k_i k_l}{k^2}, \quad f_q = f_{\mathbf{k},\omega} = \frac{k_\| c_A}{\omega} \tag{10}$$

and where in the nonlinear term the small dissipative term associated with ω is ignored.

Writing the velocity up to the second order of nonlinear corrections, and using it in the equation for magnetic fluctuations, we obtain

$$\left(\omega + ik^2\nu - \frac{k_\|^2 c_A^2}{\omega + ik^2\nu}\right) b_q^i = \sum_{q'}\left[P_{il}(\mathbf{k})f_q(1 - f_{q'}f_{q-q'}) - \delta_{il}(f_{q'} - f_{q-q'})\right]k_j b_{q'}^j b_{q-q'}^l. \tag{11}$$

Considering that the dissipative term in the denominator is a small contribution, and taking into account that for the waves of interest we expect the solution of the form $\omega \simeq k_\| c_A$, we can write in approximate form the following expression:

$$D(q)b_q^i = \sum_{q'}\chi_{ijl}(q'|q - q')b_{q'}^j b_{q-q'}^l, \tag{12}$$

where

$$D(q) = \omega - k_\| c_A f_q + 2ik^2\nu, \tag{13a}$$

$$\chi_{ijl}(q'|q - q') = \frac{1}{2}\left[P_{il}(\mathbf{k})k_j + P_{ij}(\mathbf{k})k_l\right]f_q(1 - f_{q'}f_{q-q'}) - \frac{1}{2}\,(\delta_{il}k_j - \delta_{ij}k_l)(f_{q'} - f_{q-q'}). \tag{13b}$$

The nonlinear coefficients satisfy the following symmetry properties:

$$\chi_{ijl}(q'|q - q') = \chi_{ilj}(q - q'|q'), \quad \chi_{ijl}(-q'| - q + q') = -\chi_{ijl}(q'|q - q'). \tag{14}$$

Up to this point, the derivation follows the same steps employed in the derivation of the formulation presented in Ref. 2. For the sequence of the development, in our previous work we have introduced at this point an assumption, $\langle b_i(\mathbf{k},\omega)b_j(\mathbf{k},\omega)\rangle = \delta_{ij}\langle b^2\rangle_{\mathbf{k},\omega}$, which is not valid. That is, we have assumed that the tensorial turbulent spectrum is given in the form of a scalar matrix, with the diagonal elements being identical. According to this assumption, the spectral tensor is isotropic in any spatial orientation. The general property of homogeneous and stationary turbulent spectrum tensor dictates that the diagonal elements should be different along and across the local magnetic field direction. In short, the following expression should hold:

$$\langle b_i(\mathbf{k},\omega)b_j(\mathbf{k},\omega)\rangle = \frac{k_i k_j}{k^2}\langle b_\|^2\rangle_{\mathbf{k},\omega} + \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\langle b_\perp^2\rangle_{\mathbf{k},\omega}. \tag{15}$$

The factor $\frac{1}{2}$ relates to the fact that there are two degrees of freedom associated with the transverse direction. Since magnetic field fluctuations must satisfy the Gauss law of magnetism, we must have $b_\|(\mathbf{k},\omega) = (\mathbf{b}\cdot\mathbf{k})/k = 0$. This implies that $\langle b_\perp^2\rangle = \langle b^2\rangle$. Therefore, magnetic field fluctuations should be given by an anisotropic diagonal (that is, non-scalar) matrix form

$$\langle b_i(\mathbf{k},\omega)b_j(\mathbf{k},\omega)\rangle = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\langle b^2\rangle_{\mathbf{k},\omega} = \frac{1}{2}\,P_{ij}(\mathbf{k})\langle b^2\rangle_{\mathbf{k},\omega}. \tag{16}$$

In the case of incompressible MHD, the velocity fluctuations satisfy the divergence-free condition, $\nabla\cdot\mathbf{u}$, so the velocity fluctuations also satisfy a relationship similar to Eq. (16), with u^2 instead of b^2.

Taking Eq. (12), multiplying by b_{-q}^j, and applying the averaging procedure, we obtain

$$D(q)\langle b_q^i b_{-q}^l\rangle = \sum_{q'}\chi_{ijm}(q'|q - q')\langle b_{q'}^j b_{q-q'}^m b_{-q}^l\rangle. \tag{17}$$

To this, we substitute Eq. (16), and take the projection operator, $P_{il}(\mathbf{k}) = \delta_{il} - k_i k_l/k^2$. We then make use of the property $P_{il}(\mathbf{k})P_{il}(\mathbf{k}) = 2$ to obtain

$$D(q)\langle b^2\rangle_q = \sum_{q'}P_{il}(\mathbf{k})\chi_{ijm}(q'|q - q')\langle b_{q'}^j b_{q-q'}^m b_{-q}^l\rangle. \tag{18}$$

Making use of the property $P_{il}(\mathbf{k})\chi_{ijm}(q'|q - q') = \chi_{ljm}(q'|q - q')$, we find that the nonlinear wave equation is given by

$$D(q)\langle b^2\rangle_q = \sum_{q'}\chi_{ijm}(q'|q - q')\langle b_{q'}^j b_{q-q'}^m b_{-q}^i\rangle. \tag{19}$$

This happens to be formally identical to Eq. (19) of Ref. 2, although that equation was derived under an implicit assumption of isotropic tensorial turbulence spectral intensity, whereas Eq. (19) was re-derived with the anisotropic wave spectrum (16).

In order to obtain the fluctuating quantities that appear at the triple correlation on the right-hand side of Eq. (19), we utilize Eq. (12). We consider that the magnetic fluctuations are constituted by a linear part and a nonlinear correction, and we take into account that the linear part must satisfy the linear dispersion relation, $D(q)b_q^{(0)i} = 0$. From the nonlinear wave equation (12), we therefore obtain

$$D(q)b_q^{(1)i} \approx \sum_{q'}\chi_{ijm}(q'|q - q')b_{q'}^{(0)j}b_{q-q'}^{(0)m} + \cdots, \tag{20}$$

where we have ignored higher-order corrections in the nonlinear term. From this expression, we can obtain

$$b_{q'}^{(1)j} = \frac{1}{D(q')}\sum_{q''}\chi_{jln}(q''|q' - q'')b_{q''}^{(0)l}b_{q'-q''}^{(0)n},$$
$$b_{q-q'}^{(1)m} = \frac{1}{D(q - q')}\sum_{q''}\chi_{mln}(q''|q - q' - q'')b_{q''}^{(0)l}b_{q-q'-q''}^{(0)n}, \tag{21}$$
$$b_{-q}^{(1)i} = -\frac{1}{D^*(q)}\sum_{q''}\chi_{iln}(q''| - q - q'')b_{q''}^{(0)l}b_{-q-q''}^{(0)n}.$$

The triple correlation in Eq. (19) can therefore be written as follows:

$$\begin{aligned}\langle b^j_{q'} b^m_{q-q'} b^i_{-q}\rangle &= \langle b^{(1)j}_{q'} b^{(0)m}_{q-q'} b^{(0)i}_{-q}\rangle + \langle b^{(0)j}_{q'} b^{(1)m}_{q-q'} b^{(0)i}_{-q}\rangle + \langle b^{(0)j}_{q'} b^{(0)m}_{q-q'} b^{(1)i}_{-q}\rangle \\ &= \frac{1}{D(q')}\sum_{q''}\chi_{jln}(q''|q'-q'')\langle b^l_{q''} b^n_{q'-q''} b^m_{q-q'} b^i_{-q}\rangle \\ &+ \frac{1}{D(q-q')}\sum_{q''}\chi_{mln}(q''|q-q'-q'')\langle b^l_{q''} b^n_{q-q'-q''} b^j_{q'} b^i_{-q}\rangle \\ &- \frac{1}{D^*(q)}\sum_{q''}\chi_{iln}(q''|-q-q'')\langle b^l_{q''} b^n_{-q-q''} b^j_{q'} b^m_{q-q'}\rangle.\end{aligned} \quad (22)$$

Here, for simplicity, we have dropped the superscript (0) in going from the first to the second equality. The fourth-order correlations appearing on the right-hand side of Eq. (22) can be written in terms of products of second-order correlations. Omitting the details of the calculation, we arrive at the following:

$$\begin{aligned}\langle b^j_{q'} b^m_{q-q'} b^i_{-q}\rangle &= \frac{1}{D(q')}\Big[\chi_{jln}(-q+q'|q)\langle b_m b_l\rangle_{q-q'}\langle b_i b_n\rangle_q \\ &+\chi_{jnl}(-q+q'|q)\langle b_m b_n\rangle_{q-q'}\langle b_i b_l\rangle_q\Big] \\ &+\frac{1}{D(q-q')}\Big[\chi_{mln}(-q'|q)\langle b_j b_l\rangle_{q'}\langle b_i b_n\rangle_q \\ &+\chi_{mnl}(-q'|q)\langle b_j b_n\rangle_{q'}\langle b_i b_l\rangle_q\Big] \\ &+\frac{1}{D^*(q)}\Big[\chi_{iln}(q'|q-q')\langle b_j b_l\rangle_{q'}\langle b_m b_n\rangle_{q-q'} \\ &+\chi_{inl}(q'|q-q')\langle b_j b_n\rangle_{q'}\langle b_m b_l\rangle_{q-q'}\Big].\end{aligned} \quad (23)$$

We again make use of anisotropic turbulence spectra, Eq. (16), to express the triple correlation as follows:

$$\begin{aligned}\langle b^j_{q'} b^m_{q-q'} b^i_{-q}\rangle &= \frac{1}{4}\frac{1}{D(q')}\Big[P_{ml}(\mathbf{k}-\mathbf{k}')P_{in}(\mathbf{k})\chi_{jln}(-q+q'|q) \\ &+P_{mn}(\mathbf{k}-\mathbf{k}')P_{il}(\mathbf{k})\chi_{jnl}(-q+q'|q)\Big]\langle b^2\rangle_{q-q'}\langle b^2\rangle_q \\ &+\frac{1}{4}\frac{1}{D(q-q')}\Big[P_{jl}(\mathbf{k}')P_{in}(\mathbf{k})\chi_{mln}(-q'|q) \\ &+P_{jn}(\mathbf{k}')P_{il}(\mathbf{k})\chi_{mnl}(-q'|q)\Big]\langle b^2\rangle_{q'}\langle b^2\rangle_q \\ &+\frac{1}{4}\frac{1}{D^*(q)}\Big[P_{jl}(\mathbf{k}')P_{mn}(\mathbf{k}-\mathbf{k}')\chi_{iln}(q'|q-q') \\ &+P_{jn}(\mathbf{k}')P_{ml}(\mathbf{k}-\mathbf{k}')\chi_{inl}(q'|q-q')\Big]\langle b^2\rangle_{q'}\langle b^2\rangle_{q-q'}.\end{aligned} \quad (24)$$

Inserting this result in the wave equation (19), we arrive at the following result:

$$\begin{aligned}D(q)\langle b^2\rangle_q &= \frac{1}{4}\sum_{q'}\Bigg(\frac{1}{D(q')}\Big[P_{ml}(\mathbf{k}-\mathbf{k}')P_{in}(\mathbf{k})\chi_{ijm}(q'|q-q') \\ &\times\chi_{jln}(-q+q'|q)+P_{mn}(\mathbf{k}-\mathbf{k}')P_{il}(\mathbf{k}) \\ &\times\chi_{ijm}(q'|q-q')\chi_{jnl}(-q+q'|q)\Big]\langle b^2\rangle_{q-q'}\langle b^2\rangle_q \\ &+\frac{1}{D(q-q')}\Big[P_{jl}(\mathbf{k}')P_{in}(\mathbf{k})\chi_{ijm}(q'|q-q') \\ &\times\chi_{mln}(-q'|q)+P_{jn}(\mathbf{k}')P_{il}(\mathbf{k})\chi_{ijm}(q'|q-q') \\ &\times\chi_{mnl}(-q'|q)\Big]\langle b^2\rangle_{q'}\langle b^2\rangle_q \\ &+\frac{1}{D^*(q)}\Big[P_{jl}(\mathbf{k}')P_{mn}(\mathbf{k}-\mathbf{k}')\chi_{ijm}(q'|q-q') \\ &\times\chi_{iln}(q'|q-q')+P_{jn}(\mathbf{k}')P_{ml}(\mathbf{k}-\mathbf{k}') \\ &\times\chi_{ijm}(q'|q-q')\chi_{inl}(q'|q-q')\Big]\langle b^2\rangle_{q'}\langle b^2\rangle_{q-q'}\Bigg).\end{aligned} \quad (25)$$

After some simplification and reshuffling of dummy indexes, it is verified that some of the terms appearing in the expression can be combined, and the wave equation becomes of the form given as follows:

$$\begin{aligned}D(q)\langle b^2\rangle_q &= \frac{1}{2}\sum_{q'}\Bigg(\frac{1}{D(q')}P_{ml}(\mathbf{k}-\mathbf{k}')\chi_{ijm}(q'|q-q') \\ &\times\chi_{jli}(-q+q'|q)\langle b^2\rangle_{q-q'}\langle b^2\rangle_q \\ &+\frac{1}{D(q-q')}P_{jl}(\mathbf{k}')\chi_{ijm}(q'|q-q')\chi_{mli}(-q'|q)\langle b^2\rangle_{q'}\langle b^2\rangle_q \\ &+\frac{1}{D^*(q)}P_{jl}(\mathbf{k}')P_{mn}(\mathbf{k}-\mathbf{k}')\chi_{ijm}(q'|q-q') \\ &\times\chi_{iln}(q'|q-q')\langle b^2\rangle_{q'}\langle b^2\rangle_{q-q'}\Bigg).\end{aligned} \quad (26)$$

Making use of the general definition for the second order-susceptibilities—see Eq. (13b),

$$\begin{aligned}\chi_{ijm}(q_1|q_2) &= \frac{1}{2}\Bigg\{\Bigg[\left(\delta_{im}-\frac{(\mathbf{k}_1+\mathbf{k}_2)_i(\mathbf{k}_1+\mathbf{k}_2)_m}{(\mathbf{k}_1+\mathbf{k}_2)^2}\right)(\mathbf{k}_1+\mathbf{k}_2)_j \\ &+\left(\delta_{ij}-\frac{(\mathbf{k}_1+\mathbf{k}_2)_i(\mathbf{k}_1+\mathbf{k}_2)_j}{(\mathbf{k}_1+\mathbf{k}_2)^2}\right)(\mathbf{k}_1+\mathbf{k}_2)_m\Bigg] \\ &\times f_{q_1+q_2}(1-f_{q_1}f_{q_2}) \\ &-\Big[\delta_{im}(\mathbf{k}_1+\mathbf{k}_2)_j-\delta_{ij}(\mathbf{k}_1+\mathbf{k}_2)_m\Big](f_{q_1}-f_{q_2})\Bigg\},\end{aligned} \quad (27)$$

and after some straightforward but relatively lengthy calculations, we arrive at a more explicit expression for the terms on the right-hand side of Eq. (26). For instance, for the first term within the large parenthesis on the right-hand side, we obtain

$$\begin{aligned}&P_{ml}(\mathbf{k}-\mathbf{k}')\chi_{ijm}(q'|q-q')\chi_{jli}(-q+q'|q) \\ &= P_{ml}(\mathbf{k}-\mathbf{k}')k_m k'_l\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)(1+f_qf_{q'}-f_qf_{q-q'}-f_{q'}f_{q-q'}) \\ &= \frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)(1+f_qf_{q'}-f_qf_{q-q'}-f_{q'}f_{q-q'}).\end{aligned} \quad (28)$$

Proceeding in the same way with the other two terms, we arrive at the following form of wave equation:

$$
\begin{aligned}
D(q)\langle b^2\rangle_q = \frac{1}{2}\sum_{q'}\Bigg\{ & \frac{1}{D(q')}\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right) \\
& \times (1+f_qf_{q'}-f_qf_{q-q'}-f_{q'}f_{q-q'})\langle b^2\rangle_{q-q'}\langle b^2\rangle_q \\
& +\frac{1}{D(q-q')}\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right) \\
& \times (1+f_qf_{q-q'}-f_qf_{q'}-f_{q'}f_{q-q'})\langle b^2\rangle_{q'}\langle b^2\rangle_q \\
& +\frac{2}{D^*(q)}\left[\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)(1-f_{q'}f_{q-q'})\right. \\
& \left.-\left(\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}-\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\right)f_q(f_{q'}-f_{q-q'})\right] \\
& \times \langle b^2\rangle_{q'}\langle b^2\rangle_{q-q'}\Bigg\}.
\end{aligned} \tag{29}
$$

We note that the term associated with the inverse of $D(q')$ and the same associated with the inverse of $D(q-q')$ are identical if one interchanges the dummy integral variables, $(q' \leftrightarrow q-q')$. The quantities that appear within the square bracket associated with the term $2/D^*(q)$ can be written in symmetrical form by permutating the dummy integral variables, $(q' \leftrightarrow q-q')$. This leads to

$$
\begin{aligned}
D(q)\langle b^2\rangle_q = \frac{1}{2}\sum_{q'}(\mathbf{k}\times\mathbf{k}')^2\Bigg\{ & \frac{1}{D(q')}\frac{1}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right) \\
& \times (1+f_qf_{q'}-f_qf_{q-q'}-f_{q'}f_{q-q'})\langle b^2\rangle_{q-q'}\langle b^2\rangle_q \\
& +\frac{1}{D(q-q')}\frac{1}{k'^2}\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right) \\
& \times (1+f_qf_{q-q'}-f_qf_{q'}-f_{q'}f_{q-q'})\langle b^2\rangle_{q'}\langle b^2\rangle_q \\
& +\frac{1}{D^*(q)}\left[\frac{1}{k'^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)(1-f_{q'}f_{q-q'})\right. \\
& +\frac{1}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2k'^2}\right)(1-f_{q'}f_{q-q'}) \\
& \left.-2\left(\frac{1}{k'^2}-\frac{1}{(\mathbf{k}-\mathbf{k}')^2}\right)f_q(f_{q'}-f_{q-q'})\right]\langle b^2\rangle_{q'}\langle b^2\rangle_{q-q'}\Bigg\}.
\end{aligned} \tag{30}
$$

At this stage, we introduce the slow-time derivative to the linear response,

$$
D(q)\langle b^2\rangle_q \to D\left(\mathbf{k},\omega+i\frac{\partial}{\partial t}\right)\langle b^2\rangle_q \simeq D(q)\langle b^2\rangle_q+\frac{i}{2}\frac{\partial D(q)}{\partial\omega}\frac{\partial\langle b^2\rangle_q}{\partial t}. \tag{31}
$$

Taking the real part of the dispersion relation given by Eq. (30), while ignoring nonlinear terms, we obtain the angular frequency of the normal modes,

$$
\mathrm{Re}D(q)\langle b^2\rangle_q = 0, \to \omega - k_\parallel c_A f_q = 0 \to \omega^2 - k_\parallel^2 c_A^2 = 0. \tag{32}
$$

Therefore, we write

$$
\begin{aligned}
&\omega = \sigma\omega_{\mathbf{k}}, \quad \omega_{\mathbf{k}} = k_\parallel c_A, \quad \sigma = \pm 1, \\
&\langle b^2\rangle_q = \sum_{\sigma=\pm1} I_{\mathbf{k}}^\sigma \delta(\omega-\sigma\omega_{\mathbf{k}}),
\end{aligned} \tag{33}
$$

assuming that the magnetic fluctuations are those associated with normal modes propagating in forward ($\sigma = 1$) and backward ($\sigma = -1$) directions along the direction of ambient magnetic field vector. Moreover, we can write

$$
f_q \to \sigma, \quad f_{q'} \to \sigma', \quad f_{q-q'} \to \sigma''. \tag{34}
$$

We consider the following:

$$
\mathrm{Re}D(q) = \omega - \frac{k_\parallel^2 c_A^2}{\omega}, \quad \frac{\partial \mathrm{Re}D(q)}{\partial\omega} = 1+\frac{k_\parallel^2 c_A^2}{\omega^2}. \tag{35}
$$

Proceeding, we evaluate $D(q)$ in the proximity of the frequency of a normal mode (neglecting the small imaginary part),

$$
D(q) \simeq D(q)|_{\omega=\sigma\omega_{\mathbf{k}}} + (\omega-\sigma\omega_{\mathbf{k}})\frac{\partial D(q)}{\partial\omega}\bigg|_{\omega=\sigma\omega_{\mathbf{k}}} = 0 + 2(\omega-\sigma\omega_{\mathbf{k}}). \tag{36}
$$

We, therefore, obtain

$$
\begin{aligned}
\frac{1}{D(q)} &\simeq \sum_\sigma \frac{1}{\omega-\sigma\omega_{\mathbf{k}}}\frac{1}{2} = \frac{1}{2}\sum_\sigma \lim_{\Delta\to0^+}\frac{\overbrace{\omega-\sigma\omega_{\mathbf{k}}}^{\simeq 0}-i\Delta}{(\omega-\sigma\omega_{\mathbf{k}})^2+\Delta^2} \\
&= -\frac{i\pi}{2}\sum_\sigma \delta(\omega-\sigma\omega_{\mathbf{k}}),
\end{aligned} \tag{37}
$$

which leads to

$$
\begin{aligned}
\frac{1}{D(q')} &= -\frac{i\pi}{2}\sum_{\sigma'}\delta(\omega'-\sigma'\omega_{\mathbf{k}'}), \\
\frac{1}{D(q-q')} &= -\frac{i\pi}{2}\sum_{\sigma''}\delta(\omega-\omega'-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}), \\
\frac{1}{D(q)^*} &= \frac{i\pi}{2}\sum_\sigma \delta(\omega-\sigma\omega_{\mathbf{k}}).
\end{aligned} \tag{38}
$$

Equation (30) can therefore be written as follows:

$$
\begin{aligned}
&\left(\omega - \sigma k_\parallel c_A + 2ik^2\nu + i\frac{\partial}{\partial t}\right)\sum_{\sigma=\pm1} I_{\mathbf{k}}^\sigma\delta(\omega-\sigma\omega_{\mathbf{k}}) \\
&= -\frac{i\pi}{4}\sum_{q'}\sum_{\sigma\sigma'\sigma''=\pm1}\Bigg\{\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right) \\
&\quad \times (1+\sigma\sigma'-\sigma\sigma''-\sigma'\sigma'')I_{\mathbf{k}}^\sigma I_{\mathbf{k}-\mathbf{k}'}^{\sigma''} + \frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2} \\
&\quad \times \left(1+\frac{|\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')|^2}{k^2|\mathbf{k}-\mathbf{k}'|^2}\right)(1+\sigma\sigma''-\sigma\sigma'-\sigma'\sigma'')I_{\mathbf{k}}^\sigma I_{\mathbf{k}'}^{\sigma'} \\
&\quad - \left[\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)(1-\sigma'\sigma'')\right. \\
&\quad + \frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}'))^2}{k^2k'^2}\right)(1-\sigma'\sigma'') \\
&\quad \left. -2\left(\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}-\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\right)\sigma(\sigma'-\sigma'')\right]I_{\mathbf{k}'}^{\sigma'}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''}\Bigg\} \\
&\quad \times \delta(\omega-\sigma\omega_{\mathbf{k}})\delta(\omega'-\sigma'\omega_{\mathbf{k}'})\delta(\omega-\omega'-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}).
\end{aligned} \tag{39}
$$

Isolating the terms $\sum_{\sigma=\pm 1}\delta(\omega-\sigma\omega_{\mathbf{k}})$, integrating over ω, and replacing the useful notation $\sum_{q'}$ by the original integrals over $\mathbf{k}'$ and ω', and integrating over ω', we obtain an equation for the time evolution of the spectral intensities of the normal modes,

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma}}{\partial t} &= -2k^2\nu I_{\mathbf{k}}^{\sigma} - \frac{\pi}{4}\int d\mathbf{k}' \sum_{\sigma'\sigma''=\pm 1}\left\{\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)\right.\\ &\times(1+\sigma\sigma'-\sigma\sigma''-\sigma'\sigma'')I_{\mathbf{k}}^{\sigma}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''}\\ &+\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}\left(1+\frac{|\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')|^2}{k^2|\mathbf{k}-\mathbf{k}'|^2}\right)\\ &\times(1+\sigma\sigma''-\sigma\sigma'-\sigma'\sigma'')I_{\mathbf{k}}^{\sigma}I_{\mathbf{k}'}^{\sigma'}\\ &-\left[\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)(1-\sigma'\sigma'')\right.\\ &+\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}'))^2}{k^2k'^2}\right)(1-\sigma'\sigma'')\\ &\left.\left.-2\left(\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}-\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\right)\sigma(\sigma'-\sigma'')\right]I_{\mathbf{k}'}^{\sigma'}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''}\right\}\\ &\times\delta(\sigma\omega_{\mathbf{k}}-\sigma'\omega_{\mathbf{k}'}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}).\end{aligned} \tag{40}$$

We proceed by taking into account all combinations of σ' and σ'', developing and rearranging the terms. This results in the following relatively compact form:

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma}}{\partial t} &= -2k^2\nu I_{\mathbf{k}}^{\sigma}\\ &-\pi\int d\mathbf{k}'\frac{(\mathbf{k}\times\mathbf{k}')^2}{(\mathbf{k}-\mathbf{k}')^2}\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)\\ &\times(I_{\mathbf{k}}^{\sigma}-I_{\mathbf{k}'}^{\sigma})I_{\mathbf{k}-\mathbf{k}'}^{-\sigma}\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'})\\ &-\pi\int d\mathbf{k}'\frac{(\mathbf{k}\times\mathbf{k}')^2}{k'^2}\left(1+\frac{|\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')|^2}{k^2|\mathbf{k}-\mathbf{k}'|^2}\right)\\ &\times(I_{\mathbf{k}}^{\sigma}-I_{\mathbf{k}-\mathbf{k}'}^{\sigma})I_{\mathbf{k}'}^{-\sigma}\delta(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}).\end{aligned} \tag{41}$$

Equation (41) describes the time evolution of the anisotropic intensities of magnetic fluctuations associated with the normal modes, i.e., shear Alfvénic turbulence, which can be contrasted to the following form, which was derived based on the incorrect notion of implicit tensorial turbulent fluctuations given by a scalar matrix form (with isotropic diagonal elements), and published in Ref. 2:

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma}}{\partial t} &= -2k^2\nu I_{\mathbf{k}}^{\sigma}\\ &-4\pi\int d\mathbf{k}'\left[(\mathbf{k}\cdot\mathbf{k}')\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)I_{\mathbf{k}}^{\sigma}-2k^2I_{\mathbf{k}'}^{\sigma}\right]\\ &\times I_{\mathbf{k}-\mathbf{k}'}^{-\sigma}\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'})\\ &-4\pi\int d\mathbf{k}'\left[[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]\left(1+\frac{[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2(\mathbf{k}-\mathbf{k}')^2}\right)I_{\mathbf{k}}^{\sigma}-2k^2I_{\mathbf{k}-\mathbf{k}'}^{\sigma}\right]\\ &\times I_{\mathbf{k}'}^{-\sigma}\,\delta(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'}).\end{aligned} \tag{42}$$

Note that we may rewrite Eq. (41) in a slightly different form. In the last line of Eq. (41), we may define $\mathbf{k}''=\mathbf{k}-\mathbf{k}'$, so $d\mathbf{k}''=-d\mathbf{k}'$, and then denote $\mathbf{k}''$ as $\mathbf{k}'$ again, since it is a dummy integral variable. With such a procedure, Eq. (41) can be written more succinctly as follows:

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma}}{\partial t} &= -2k^2\nu I_{\mathbf{k}}^{\sigma}\\ &-2\pi\int d\mathbf{k}'\frac{|\mathbf{k}\times\mathbf{k}'|^2}{|\mathbf{k}-\mathbf{k}'|^2}\left(1+\frac{|\mathbf{k}\cdot\mathbf{k}'|^2}{k^2k'^2}\right)\\ &\times(I_{\mathbf{k}}^{\sigma}-I_{\mathbf{k}'}^{\sigma})I_{\mathbf{k}-\mathbf{k}'}^{-\sigma}\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}).\end{aligned} \tag{43}$$

The same can be done to the previous (and erroneous) form,

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma}}{\partial t} &= -2k^2\nu I_{\mathbf{k}}^{\sigma}\\ &-8\pi\int d\mathbf{k}'\left[(\mathbf{k}\cdot\mathbf{k}')\left(1+\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\right)I_{\mathbf{k}}^{\sigma}-2k^2I_{\mathbf{k}'}^{\sigma}\right]\\ &\times I_{\mathbf{k}-\mathbf{k}'}^{-\sigma}\,\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'}).\end{aligned} \tag{44}$$

The difference between the time evolution ruled by the (correct) equation of anisotropic turbulence, either in the form of Eq. (41) or in the form of Eq. (43), vs the incorrect isotropic version (42) or (44), has to be evaluated by numerical methods, which is the subject of a follow-up paper. However, some basic differences can be already seen by a simple examination of both expressions. For instance, it can be seen from Eq. (43) that the waves with $\mathbf{k}'$ nearly parallel or anti-parallel to $\mathbf{k}$ play negligible role in the nonlinear dynamics, while the same cannot be concluded from Eq. (44). The implication is that according to the correct anisotropic formalism, the nonlinear cascade along the strictly perpendicular direction, that is, along $k_\perp$ (as defined with respect to the ambient magnetic field) will be ineffective, whereas the incorrect previous formalism does indeed allow for such a process. As a matter of fact, the earlier paper, i.e., Ref. 1, made use of such a property to discuss the perpendicular cascade of incompressible MHD turbulence, but we now know that such a result may not be valid.

III. SUMMARY AND DISCUSSION

In the present paper, we have reformulated the theory of MHD weak turbulence theory, by revisiting the earlier theory in Refs. 1 and 2. That is, in the previous two papers, an implicit assumption of isotropic turbulence spectra was made, that is, the turbulent fluctuation spectral tensor was assumed to be given by a scalar matrix form, which was not immediately evident at the time. At this point, it is useful to reminisce upon the underlying cause of how the implicit assumption of isotropy crept in during the process of theoretical development, although such a hypothesis was not explicitly made at the outset. The reason was as follows: During the process of taking the ensemble average of the nonlinear wave equation (12),

$$D(q)b_q^i=\sum_{q'}\chi_{ijk}(q'|q-q')b_{q'}^jb_{q-q'}^k,$$

the proper procedure should have been to take the product of this equation with the wave amplitude $b_{q'}^j$ and take the ensemble average, as in Eq. (17). Instead, in Refs. 1 and 2, this equation was simply multiplied with b_{-q}^i and the definition of wave spectral intensity given by

$$\langle b_q^ib_{-q}^i\rangle=\langle b^2\rangle_q$$

was invoked. This straightforward definition hides the fact that not all scalar product components of the vector b_q^i are nonzero. In fact, only the transverse components of the scalar product should be nonzero, while the longitudinal component should vanish, $\langle b_q^i b_{-q}^i \rangle = \langle b_\perp^2 \rangle_q + \langle b_\parallel^2 \rangle_q = \langle b_\perp^2 \rangle_q = \langle b^2 \rangle_q$, where $\langle b_\parallel^2 \rangle_q = 0$. In the present paper, we have corrected this shortcoming by means of more rigorous definitions, Eqs. (15) and (16). The result is the correct form of nonlinear wave kinetic equation for weak anisotropic incompressible MHD turbulence, Eq. (41) or Eq. (43), which contrasts with the incorrect form, Eq. (42) or Eq. (44).

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Luiz F. Ziebell: Conceptualization (supporting); Formal analysis (equal); Funding acquisition (equal); Investigation (supporting); Writing – original draft (lead); Writing – review & editing (equal). **Peter H. Yoon:** Conceptualization (lead); Formal analysis (equal); Funding acquisition (equal); Investigation (lead); Writing – original draft (supporting); Writing – review & editing (equal). **Gwangson Choe:** Conceptualization (supporting); Formal analysis (equal); Funding acquisition (equal); Investigation (supporting); Writing – original draft (supporting); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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