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An essay on mixed-frequency data, aggregated time series, and causality.

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Resumo

Entender as complexidades das relações econômicas é crucial para formuladores de políticas, pesquisadores e analistas. A agregação temporal, onde a frequência de geração de dados excede a frequência de coleta de dados, apresenta desafios significativos na análise econômica. Essa discrepância pode levar a realizações não observáveis do processo estocástico original, afetando as propriedades dos dados de séries temporais. Abordar esses desafios é vital para detectar e interpretar com precisão as relações causais entre variáveis econômicas. Nossa pesquisa visa identificar como a agregação temporal pode interferir na detecção de causalidade entre séries temporais. Também demonstramos como um teste de causalidade Sims modificado pode ser empregado para detectar causalidade em modelos de frequências mistas. Nossas simulações de Monte Carlo mostram boas propriedades de tamanho e poder para amostras finitas. Finalmente, testamos a causalidade entre o PIB dos EUA e indicadores macroeconômicos mensais.

Palavras-Chave: Séries temporais, Agregação temporal, Causalidade de Granger, Causalidade de Sims.

Abstract

Temporal aggregation, where the data generation frequency exceeds the data collection frequency, poses significant challenges in economic analysis. This discrepancy can lead to unobservable realizations of the original stochastic process, which in turn affects the properties of time series data. Consequently, addressing these challenges is crucial for accurately detecting and interpreting causal relationships between economic variables. In our research, we aim to identify how temporal aggregation can interfere with the detection of causality between time series. Furthermore, we demonstrate the application of a modified Sims causality test to detect causality in mixed-frequency models. Our Monte Carlo simulations indicate that this test exhibits good finite sample size and power properties. Finally, we apply our methodology to test the causality between U.S. GDP and macroeconomic monthly indicators.

Keywords: Time series, Temporal aggregation, Granger causality, Sims causality.

Contents

1 Introduction

Understanding the intricacies of economic relationships is crucial for policymakers, researchers, and analysts. Temporal aggregation, where the data generation frequency exceeds the data collection frequency, poses significant challenges in economic analysis, as described in Zellner e Montmarquette (1971). This discrepancy can lead to unobservable realizations of the original stochastic process, affecting the properties of time series data. Consequently, addressing these challenges is vital to accurately detecting and interpreting causal relationships between economic variables.

In this work, our first goal is to understand how temporal aggregation affects the detection of causality between two time series. Secondly, we aim to propose a method for correctly testing causality in situations where spurious causality exists. To achieve this, we demonstrate the existence or absence of spurious causality using four theoretical models considering a bivariate time series model. We then show that, with a slight modification, the Sims causality methodology is suitable for detecting causality when one series is aggregated and the other is not.

As stated in Marcellino (1999), temporal aggregation occurs when the frequency of data generation is higher than the frequency of data collection, resulting in some realizations of the original stochastic process being unobservable. The author also provides a detailed examination of which properties of the disaggregated time series remain invariant after aggregation and which do not, referring to this discrepancy as temporal aggregation bias. Further instances of common estimation problems associated with time aggregations can be found in Weiss (1984) and Swanson e Granger (1997).

Granger first introduced the causality concepts that became known as Granger causality in Granger (1963). Although Granger's concept relates to the ability of one time series to predict another, conditional on a given information set, Chalak e White (2012) demonstrates that Granger's concept is closely linked with the causal notions of the Pearl Causal Model discussed in Pearl (2009). Later, several empirical investigations, such as those in Weiss (1984), demonstrated that causality properties are not invariant to temporal aggregation. This can lead to spurious conclusions about the relationships between time series, a phenomenon referred to as spurious causality. To address this issue, McCrorie e Chambers (2006) proposed formulating models in continuous time to correct the effects of temporal aggregation in observed discrete data through a discrete-time analog. Similarly, Renault et al. (1998) utilized a continuous-time model to distinguish between true and spurious causality. Moreover, Breitung e Swanson (2002) observed that causality relationships seem to change when moving to a finer sampling interval. For more examples of the effects of temporal aggregation on causality testing, see Rajaguru e Abeysinghe (2010) and Xu (1996).

An alternative approach to identifying causality between non-aggregated observations was introduced in Sims (1972). Sims suggested that in a regression analysis if causality only moves from current and past values of exogenous variables to an endogenous variable, the coefficients for future values of the exogenous variables should be zero. This approach, known as the Sims causality test, has been widely applied. For example, in Macunovich e Easterlin (1988), Granger-Sims causality tests were applied to monthly age-specific data, demonstrating the technique's value in pinpointing the effective lag between business cycles and fertility in the United States. Similarly, Chow (1987) explored the causal link between export growth and industrial development in eight Newly Industrializing Countries using Sims' causality test to reveal a robust bidirectional causality between export growth and industrial development in most of them. For further applications of the Sims method, see Heckman (2000) and Holland (1986).

The paper is organized as follows: Section 2 explores the characteristics of Granger causality; Section 3 illustrates how aggregation leads to spurious causality; Section 4 introduces the proposed modified Sims causality test; Section 5 discuss the technical implementation details; Section 6 reports the finite sample behavior of the proposed Sims test; Section 7 presents an empirical application with quarterly GDP and monthly US indicators; Section 8 concludes the paper and the Appendix showcases the intermediate results and expands upon the results presented in Section 3 using the same aggregation pattern as presented in the GDP series.

2 Granger Causality

This section focuses on Granger's proposed measurement of causality between two variables, introduced in Granger (1969). Hamilton (1994) definition based on Vector Autoregression models (VARs) highlights the forecasting effectiveness of certain variables on others. Under this approach, the researcher aims to determine if lagged observations of the series $\{x_t\}_{t=0}^{\infty}$ can contribute to forecasting the series $\{y_t\}_{t=0}^{\infty}$. If not, it can be concluded that the past values of x_t do not Granger-cause present values of y_t. This concept is denoted as x_t not Granger-causing y_t if:

$$\text{MSE}[E(y_t|y_{t-1}, y_{t-2}, \ldots)] = \text{MSE}[E((y_t|y_{t-1}, y_{t-2}, \ldots, x_{t-1}, x_{t-2}, \ldots)],$$

which means, for the linear case, that the past of x_t lacks explanatory power for y_t whenever the mean squared error (MSE) of forecasting y_t, conditioned exclusively on its historical values, is statistically equivalent to the MSE of forecasting y_t using both its own and x_t's history.

According to Hamilton (1994), considering past values of y_t and x_t and assuming a lag length p, we can define the augmented autoregressive model as:

$$y_t = c + \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} + \beta_1 x_{t-1} + \cdots + \beta_p x_{t-p} + v_t \tag{2.1}$$

where the error term v_t is assumed to be independent and identically distributed. In other words, as stated in Kuersteiner (2010a), testing for Granger causality involves assessing whether the coefficients associated with lags of x_t in the y_t series are statistically equal to zero. If this condition is not rejected, it implies that $x_t \not\to y_t$.

As shown in Marcellino (1999), testing for Granger causality between aggregated time series variables can lead to spurious conclusions since not all time series properties are invariant to aggregation. The divergence from the actual causal relationship between two time series and the false relationship is known as temporal aggregation bias, commonly referred to as spurious causality. This effect has also been mentioned in Breitung e Swanson (2002) and Götz et al. (2016).

In a related study, Renault et al. (1998) investigates another reason for spurious causality, attributing it to the use of discrete data in causality analysis. The authors state that using discrete data overlooks events that occur within time intervals, essentially disregarding valuable information within a series of observations. They describe significant misleading causal effects in discrete time that do not emerge from the continuous-time generating process. For further discussions on using continuous models, see Florens e Fougere (1996) and Comte e Renault (1996).

Ghysels et al. (2016) addresses the problem of spurious causality in a Mixed Frequency Vector Autoregressive (MF-VAR) setting by assuming that the low-frequency sampled process is not temporally aggregated but instead sampled at its true frequency. This approach effectively rules out any spurious causality by construction. The finite sample properties presented in Ghysels et al. (2016) indicate that Granger testing within the MF-VAR approach exhibits higher asymptotic power for small differences in sampling frequencies, such as quarterly/monthly mixtures. Götz et al. (2016) also proposed a causality test using MF-VAR, demonstrating that a Bayesian methodology improves the sensitivity of the Granger test when examining causality from a very high frequency to low frequency. This improvement is shown through tests of causality from daily to quarterly observations, with adjustments made to handle parameter proliferation. Tank et al. (2019) conducts an investigation into Granger causality in mixed-frequency time series models, focusing specifically on the identifiability of the structural vector.

Following the methodology presented in Ghysels et al. (2016), the series are stacked, resulting in an MF vector, $\boldsymbol{X}(\tau)$, described by equation

$$\boldsymbol{X}(\tau) = \left[\boldsymbol{x}_H(\tau,1)', \ldots, \boldsymbol{x}_H(\tau,s)', \boldsymbol{x}_L(\tau)'\right]',$$

where $\tau \in \{1, \ldots, T\}$ is the low-frequency (LF) time index, $\boldsymbol{x}_L(\tau)$ represents the LF variable, and the set $\{\boldsymbol{x}_H(\tau,1)', \ldots, \boldsymbol{x}_H(\tau,s)'\}$ consists of the high-frequency (HF) observations, with s being the frequency ratio between the HF and LF variables. For example, for each quarterly GDP observation, there are three observations of a monthly indicator. Specifically, if $\boldsymbol{x}_L(\tau)$ denotes a quarterly variable, then $\boldsymbol{x}_H(\tau,1)$ represents the first month of each quarter, $\boldsymbol{x}_H(\tau,2)$ the second month, and so on.

Under some standard assumptions, as discussed in Ghysels et al. (2016), consider a bivariate case where the structural form for the MF-VAR includes the current low-frequency variable and three lags of the high-frequency variable, given by

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -d & 1 & 0 & 0 \\ 0 & -d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\equiv \boldsymbol{N}} \begin{bmatrix} x_H(\tau,1) \\ x_H(\tau,2) \\ x_H(\tau,3) \\ x_L(\tau) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & d & c_1 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_3 \\ b_3 & b_2 & b_1 & a \end{bmatrix}}_{\equiv \boldsymbol{M}} \begin{bmatrix} x_H(\tau-1,1) \\ x_H(\tau-1,2) \\ x_H(\tau-1,3) \\ x_L(\tau-1) \end{bmatrix} + \underbrace{\begin{bmatrix} \xi_H(\tau,1) \\ \xi_H(\tau,2) \\ \xi_H(\tau,3) \\ \xi_L(\tau) \end{bmatrix}}_{\equiv \xi(\tau)}$$

or $\boldsymbol{N}\boldsymbol{X}(\tau) = \boldsymbol{M}\boldsymbol{X}(\tau-1) + \boldsymbol{\xi}(\tau)$. It also assumed that x_H follows a AR(1) with coefficient d. The impact of lagged x_L on x_H is governed by c_1, c_2, and c_3. x_L follows a AR(1) with coefficient a. The impact of lagged x_H on x_L is governed by b_1, b_2, and b_3. Premultiply both sides of the structural form by

$$\boldsymbol{N}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ d^2 & d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

to get the reduced form $\boldsymbol{X}(\tau) = \boldsymbol{A}_1\boldsymbol{X}(\tau-1) + \boldsymbol{\epsilon}(\tau)$, where

$$\boldsymbol{A}_1 = \boldsymbol{N}^{-1}\boldsymbol{M} = \begin{bmatrix} 0 & 0 & d & \sum_{i=1}^{1} d^{1-i}c_i \\ 0 & 0 & d^2 & \sum_{i=1}^{2} d^{2-i}c_i \\ 0 & 0 & d^3 & \sum_{i=1}^{3} d^{3-i}c_i \\ b_3 & b_2 & b_1 & a \end{bmatrix}$$

and $\boldsymbol{\epsilon}(\tau) = \boldsymbol{N}^{-1}\boldsymbol{\xi}(\tau)$.

We can demonstrate that x_H does not cause x_L in the mixed frequency setting if and only if $b_1 = b_2 = b_3 = 0$. Non-causality from x_L to x_H involves c_1, c_2 and c_3, the AR(1) coefficient of x_H, as seen in the upper-right block of $\boldsymbol{A}_1$.

Unlike Ghysels et al. (2016) and Götz et al. (2016), we assume that aggregated series are generated at a higher frequency, resulting in missing information once aggregated. Thus, in the following section, we illustrate how this assumption influences Granger causality analysis.

3 Aggregation and Spurious Causality

In this section, we build upon Granger's concept of causality as introduced in Section 2 for two time series, y_t and x_t, and their aggregation. By illustrating the spurious causality effects using simple models, we can better understand its impact on more complex models. The series x_t and y_t have a causal relationship described as

$$\begin{array}{rcl} y_t & = & \theta_1 y_{t-1} + \theta_2 x_{t-1} + v_{1,t} \\ x_t & = & \lambda_1 y_{t-1} + \lambda_2 x_{t-1} + v_{2,t} \end{array}, \quad v_{j,t} \sim N(0, \sigma^2), \text{ for } j = \{1, 2\}. \tag{3.1}$$

with both y_t and x_t modeled after one lag of themselves and one of each other. The errors $v_{j,t}$, $j = \{1, 2\}$, are assumed to be independent and identically distributed. If the coefficients $\theta_2 \neq 0$ or $\lambda_1 \neq 0$, this indicates that we are working with a scenario where lagged values of x_t are influential in present values of y_t or lagged values of y_t are influential in present values of x_t, respectively. Therefore, we have four possible cases of Granger causality between x_t and y_t:

(i) x_t doesn't Granger cause y_t and y_t doesn't Granger cause x_t,

$$\theta_2 = 0 \text{ and } \lambda_1 = 0, \quad x_t \not\rightarrow y_t \text{ and } y_t \not\rightarrow x_t,$$

(ii) x_t Granger cause y_t and y_t doesn't Granger cause x_t,

$$\theta_2 \neq 0 \text{ and } \lambda_1 = 0, \quad x_t \rightarrow y_t \text{ and } y_t \not\rightarrow x_t$$

(iii) x_t doesn't Granger cause y_t and y_t Granger cause x_t,

$$\theta_2 = 0 \text{ and } \lambda_1 \neq 0, \quad x_t \not\rightarrow y_t \text{ and } y_t \rightarrow x_t$$

(iv) x_t Granger cause y_t and y_t Granger cause x_t,

$$\theta_2 \neq 0 \text{ and } \lambda_1 \neq 0, \quad x_t \rightarrow y_t \text{ and } y_t \rightarrow x_t$$

Now, let us focus on the effects of aggregation on the measurement of causality. For this, let Y_τ and X_τ be potentially aggregated time series composed of lagged values of y_t and x_t, respectively, where $\tau = \{1, 2, \dots\}$ represents periods corresponding to $t = \{s, 2s, \dots\}$, with s being the frequency ratio between the high-frequency and the low-frequency variables. We can then analyze the effects of $Y_{\tau-1}$ and $X_{\tau-1}$ on Y_τ, as expressed by

$$Y_\tau = \Theta_1 Y_{\tau-1} + \Theta_2 X_{\tau-1} + u_t$$

The coefficients $\hat{\Theta}_1$ and $\hat{\Theta}_2$ can be retrieved by least squares,

$$\begin{bmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \end{bmatrix} = det(\hat{M})^{-1} \begin{bmatrix} (X'_{\tau-1}X_{\tau-1})(Y'_{\tau-1}Y_\tau) - (X'_{\tau-1}Y_{\tau-1})(X'_{\tau-1}Y_\tau) \\ (Y'_{\tau-1}Y_{\tau-1})(X'_{\tau-1}Y_\tau) - (X'_{\tau-1}Y_{\tau-1})(Y'_{\tau-1}Y_\tau) \end{bmatrix} \quad (3.2)$$

where $det(\hat{M}) = (Y'_{\tau-1}Y_{\tau-1})(X'_{\tau-1}X_{\tau-1}) - (X'_{\tau-1}Y_{\tau-1})^2$. Furthermore, under usual time series linear regression assumptions and assuming the errors are i.i.d., we have

$$E\begin{bmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \end{bmatrix} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = det(M)^{-1} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}, \quad (3.3)$$

where π_1 and π_2 are related to the second part of the right-hand side of the equation (3.2).

Considering the relationship between the variables x_t and y_t as depicted in equation (3.1), our objective is to illustrate how spurious conclusions about the relationship between them can emerge in the aggregate setting, utilizing the series Y_τ and X_τ. The following four examples spotlight instances of what may be deemed spurious causality between these series. These examples help to describe scenarios in testing for Granger causality that might be prone to the spurious causality problem.

Notice that, for the next examples, the series are not subsampled but only aggregated, as subsampling does not cause spurious causality. To illustrate this, assume that model (3.1) is sampled every two observations. In this scenario, $\{y_{t-1}, x_{t-1}\}$ would not be observable, but $\{y_{t-2}, x_{t-2}\}$ would be. Consequently, even if $\theta_2 = 0$ and $\lambda_1 \neq 0$, the projection of y_t on $\{y_{t-2}, x_{t-2}\}$ would result in $E[\tilde{\theta}_1] = \theta_1^2$ and $E[\tilde{\Theta}_2] = 0$, thus not leading to spurious causality. This simplification allows us to derive the effects of spurious causality without loss of generality.

Example 1

In this initial example, we assume Case (i), $\theta_2 = 0$ and $\lambda_1 = 0$, i.e., $x_t \nrightarrow y_t$ and $y_t \nrightarrow x_t$. Thus,

$$\begin{array}{rcl} y_t &=& \theta_1 y_{t-1} + v_{1,t} \\ x_t &=& \lambda_2 x_{t-1} + v_{2,t} \end{array}, \quad v_{j,t} \sim N(0, \sigma^2), \text{ for } j = \{1, 2\} \quad (3.4)$$

with a simple aggregation pattern defined by

$$Y_\tau = y_t + y_{t-1}, \quad Y_{\tau-1} = y_{t-1} + y_{t-2}, \quad X_{\tau-1} = x_{t-1} + x_{t-2}. \quad (3.5)$$

Intuitively, we consider a present value of the y_t series and a further lagged value of the x_t series. Solving for Θ_1 and Θ_2 in equations (3.2) and (3.3), we obtain

$$\begin{array}{rcl} det(M) &=& 4\sigma_x^2\sigma_y^2 + 4\theta_1\sigma_x^2\sigma_y^2 + 4\lambda_2\sigma_x^2\sigma_y^2 + 4\theta_1\lambda_2\sigma_x^2\sigma_y^2, \\ \pi_1 &=& 4\theta_1\sigma_x^2\sigma_y^2 + 2\sigma_x^2\sigma_y^2 + 2\theta_1^2\sigma_x^2\sigma_y^2 + 4\theta_1\lambda_2\sigma_x^2\sigma_y^2 + 2\lambda_2\sigma_x^2\sigma_y^2 + 2\theta_1^2\lambda_2\sigma_x^2\sigma_y^2, \\ \pi_2 &=& 0. \end{array}$$

The result shows that $E[\hat{\Theta}_2]$ equals zero, indicating that past values of X_τ have no influence on current values of Y_τ. This is similar to the relationship between x_t and y_t. Hence, the non-causality among the non-aggregated series results in no spurious causality when using the aggregated setting.

Example 2

Now, we assume Case (iii), $\theta_2 = 0$ but $\lambda_1 \neq 0$, i.e., $y_t \to x_t$ but $x_t \not\to y_t$, or

$$\begin{array}{rcl} y_t & = & \theta_1 y_{t-1} + v_{1,t} \\ x_t & = & \lambda_1 y_{t-1} + \lambda_2 x_{t-1} + v_{2,t} \end{array}, \quad v_{j,t} \sim N(0, \sigma^2), \text{ for } j = \{1, 2\} \tag{3.6}$$

with the same aggregation patter defined in the first example, see equation (3.5), the relation between Y_τ given $Y_{\tau-1}$ and $X_{\tau-1}$ is given by

$$\begin{array}{rcl} det(M) & = & \sigma_x^2\sigma_y^2(4 + 4\theta_1 + 4\lambda_2 + 4\theta_1\lambda_2) - \lambda_1^2\sigma_y^4 \\ & & +\sigma_{xy}(-4^2 - 4\theta_1^2 - \theta_1^2 + 2\theta_1\lambda_1\sigma_y^2 - 4\lambda_2^2 - 2\lambda_2\lambda_1\sigma_y^2 - \lambda_2^2 - 2\theta_1\lambda_2^2), \\ \pi_1 & = & \sigma_x^2\sigma_y^2(4\theta_1 + 2 + 2\theta_1^2 + 4\theta_1\lambda_2 + 2\lambda_2 + 2\theta_1^2\lambda_2) + \\ & & \sigma_{xy}^2(-5\theta_1 - 4\theta_1^2 - 2 - \theta_1^3 - 2\theta_1\lambda_2 - \lambda_2 - \theta_1^2\lambda_2) + \\ & & \sigma_y^2\sigma_{xy}(+2\theta_1\lambda_1 + \lambda_1 + \theta_1^2\lambda_1) \\ & & +\sigma_{xy}^2(-5\theta_1 - 4\theta_1^2 - 2 - \theta_1^3 - 2\theta_1\lambda_2 - \lambda_2 - \theta_1^2\lambda_2) \\ & & +\sigma_{xy}(2\theta_1\lambda_1\sigma_y^2 + 1\lambda_1\sigma_y^2 + \theta_1^2\lambda_1\sigma_y^2), \\ \pi_2 & = & \sigma_y^2\sigma_{xy}(\theta_1 + 2\theta_1^2 + \theta_1^3 - 2\theta_1\lambda_2 - \lambda_2 - \theta_1^2\lambda_2). \\ & & +\sigma_y^4(-2\theta_1\lambda_1 - \lambda_1 - \theta_1^2\lambda_1). \end{array}$$

Despite the absence of θ_2 in the model, the results indicate that Θ_2 is likely different from zero, demonstrating a spurious causality caused by aggregation. The aggregated time series reveals an influence of X_τ on the Y_τ series, even though such an influence is non-existent when the time series is non-aggregated.

We calculate potential values for the Θ_2 coefficient, assuming all σ_{xy}, σ_y, and σ_x as functions of λ_1, λ_2, and θ_1, with $\sigma_{v,j} = 1$ for $j = 1, 2$. Setting $\theta_2 = 0$ and varying all other coefficients between -0.95 and 0.95 in increments of 0.01, Figure 3.1 illustrates the frequency of the expected estimator. The chart shows that values of Θ_2 are distinct from zero in several cases, making the spurious causality evident. It is important to note that although the estimations of Θ_2 are evenly distributed around zero, this does not imply that its true value is expected to be zero. For instance, with $\lambda_1 = \lambda_2 = \theta_1 = 0.3$, the expected value for Θ_2 equals -0.068.

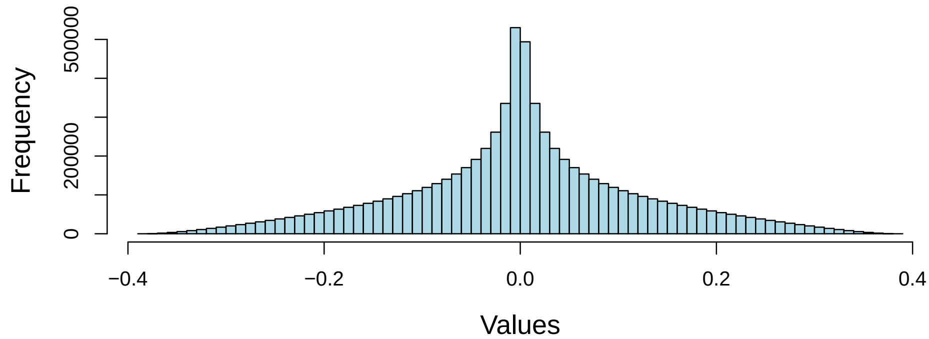

Figure 3.1: Histogram of possible values for Θ_2 accordingly to model (3.6), equation (3.3), and aggregation (3.5). With $\theta_2 = 0$ and λ_1, λ_2, and θ_1 between -0.95 and 0.95 in increments of 0.01.

Example 3

Similarly to previous example, we assume Case (iii), where $x_t \not\rightarrow y_t$ and $y_t \rightarrow x_t$, as presented in the model (3.6). However, we now assume that X_τ is non-aggregated, thus measuring Granger causality from a non-aggregated to an aggregated series, i.e.,

$$Y_\tau = y_t + y_{t-1}, \quad Y_{\tau-1} = y_{t-1} + y_{t-2}, \quad X_{\tau-1} = x_{t-2} \tag{3.7}$$

With this aggregation pattern, our objective is to assess whether there are any spurious effects when only one of the series is aggregated. In this setup, the furthest lag of the x_t series coincides with the furthest lag of the y_t series. The results are given by

$$\begin{aligned} det(M) &= 2\sigma_x^2\sigma_y^2 + 2\theta_1\sigma_x^2\sigma_y^2 - 3\theta_1\sigma_{xy}^2 - \sigma_{xy}^2, \\ \pi_1 &= 2\theta_1\sigma_x^2\sigma_y^2 + \sigma_x^2\sigma_y^2 + \theta_1^2\sigma_x^2\sigma_y^2 - \theta_1^3\sigma_{xy}^2 - 2\theta_1^2\sigma_{xy}^2 - \theta_1\sigma_{xy}^2, \\ \pi_2 &= \theta_1^2\sigma_y^2\sigma_{xy} + \theta_1^3\sigma_y^2\sigma_{xy} - \theta_1\sigma_y^2\sigma_{xy} - \sigma_y^2\sigma_{xy}. \end{aligned}$$

Notice that the expected value of $\hat{\Theta}_2$ still deviates from zero, indicating a persistent issue of spurious causality. As shown in Figure 3.2, the distribution of Θ_2 appears more concentrated around zero compared to Figure 3.1.

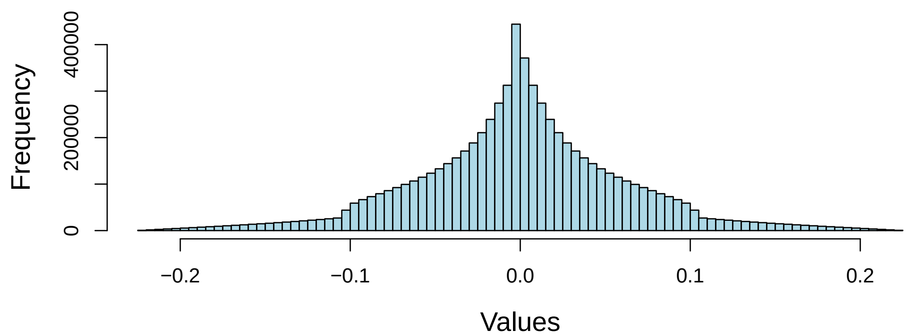

Figure 3.2: Histogram of possible values for Θ_2 accordingly to model (3), equation (3.3), and aggregation (3.7). With $\theta_2 = 0$ and λ_1, λ_2, and θ_1 between -0.95 and 0.95 in increments of 0.01.

One could argue that introducing a higher lag might reduce substantially the spurious effect. However, although this is true, it will not eliminate the spurious causality. To test if a higher lag value of x_t is sufficient to reduce spurious effects considerably, we replicate the calculations and simulations mentioned earlier, this time with $X_{\tau-1}$ being non-aggregated and equal to x_{t-5}.

$$Y_\tau = y_t + y_{t-1}, \quad Y_{\tau-1} = y_{t-1} + y_{t-2}, \quad X_{\tau-1} = x_{t-5} \tag{3.8}$$

with results given by

$$\begin{aligned} det(M) &= 2\sigma_x^2\sigma_y^2 + 2\theta_1\sigma_x^2\sigma_y^2 - 3\theta_1\sigma_{xy}^2 - \sigma_{xy}^2, \\ \pi_1 &= 2\theta_1\sigma_x^2\sigma_y^2 + \sigma_x^2\sigma_y^2 + \theta_1^2\sigma_x^2\sigma_y^2 - \theta_1^3\sigma_{xy}^2 - 2\theta_1^2\sigma_{xy}^2 - \theta_1\sigma_{xy}^2, \\ \pi_2 &= \theta_1^5\sigma_y^2\sigma_{xy} + \theta_1^6\sigma_y^2\sigma_{xy} - \theta_1^4\sigma_y^2\sigma_{xy} - \theta_1^3\sigma_y^2\sigma_{xy}. \end{aligned}$$

The distribution of Θ_2 seen in Figure 3.3 is now closer to zero than in the previous example, but it is not exactly zero. Since it differs from zero, the rejection rate of $H_0 : \hat{\Theta}_2 = 0$ will tend to 1 as $n \to \infty$.

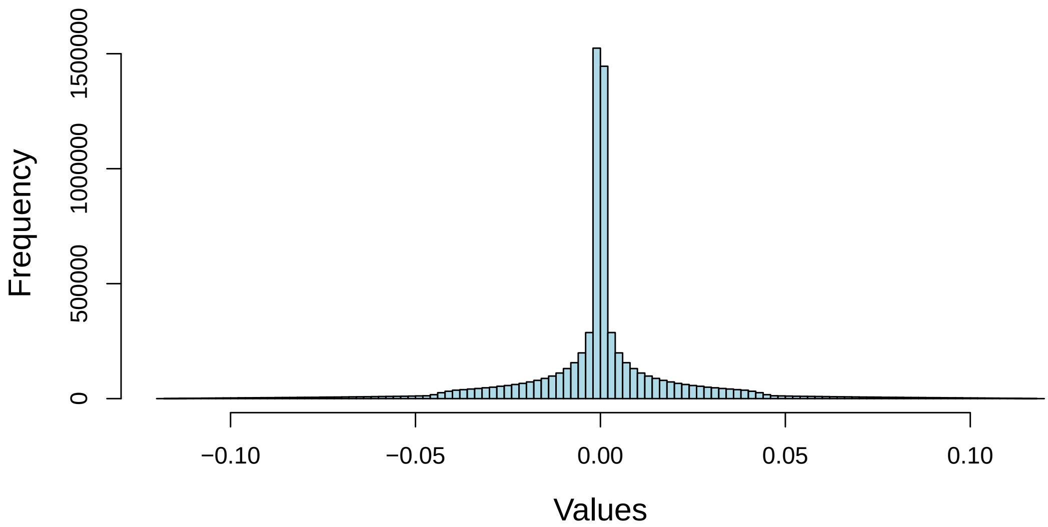

Figure 3.3: Histogram of possible values for Θ_2 accordingly to model (3), equation (3.3), and aggregation (3.8). With $\theta_2 = 0$ and λ_1, λ_2, and θ_1 between -0.95 and 0.95 in increments of 0.01.

Example 4

Finally, we examine the same scenario, Case (iii), $x_t \not\to y_t, y_t \to x_t$, model (3.6), but this time, we reverse the aggregation, with Y_τ and $Y_{\tau-1}$ being non-aggregated and X_τ being aggregated, i.e.,

$$Y_\tau = y_t, \quad Y_{\tau-1} = y_{t-1}, \quad X_{\tau-1} = x_{t-1} + x_{t-2}. \tag{3.9}$$

This allows us to investigate the causality direction from an aggregated series to a non-aggregated one. The results are given by

$$\begin{aligned} det(M) &= 2\sigma_x^2\sigma_y^2 - \sigma_{xy}^2 - 2\theta_1\sigma_{xy}^2 - \theta_1^2\sigma_{xy}^2 + 2\lambda_1\sigma_y^2\sigma_{xy} + 2\lambda_2\sigma_x^2\sigma_y^2, \\ \pi_1 &= 2\theta_1\sigma_x^2\sigma_y^2 - \theta_1\sigma_{xy}^2 - 2\theta_1^2\sigma_{xy}^2 - \theta_1^3\sigma_{xy}^2 + 2\theta_1\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1\lambda_2\sigma_x^2\sigma_y^2, \\ \pi_2 &= 0. \end{aligned}$$

Notice that the result for Θ_2 is now zero, indicating that there are no issues related to spurious causality when the x_t series is not aggregated, contrary to Example 3. In summary, testing Granger causality with an aggregated series, such as $Y_\tau = y_t + y_{t-1}$, and a non-aggregated series, such as x_t, allows for the correct measurement of causality from y_t to x_t.

REMARK: We derived the above results by implementing an algorithm tailored to this purpose. Written in R and adjusted to model (3.1), the algorithm efficiently solves the symbolic expressions. In the Appendix, we present the results for the GDP aggregation, which are explained in Section 6.

4 Sims-causality for Mixed-Frequencies

According to Chamberlain (1982), Granger causality is defined such that if x_t does not cause y_t then

$$\text{(G) } y_t \text{ is independent of } x_{t-1}, x_{t-2}, \ldots, \text{ conditional on } y_{t-1}, y_{t-2}, \ldots, \forall t \quad (4.1)$$

in contrast, Sims (1972) proposed a distinct measure of causality, or more precisely, the measurement of strict exogeneity, where if x_t does not cause y_t, then the regression of x_t against leads and lags of y_t will result in null coefficients for the lead variables. The Sims's causality definition by Chamberlain (1982) is

$$\text{(S) } x_t \text{ is independent of } y_{t+1}, y_{t+1}, \ldots, \text{ conditional on } y_t, y_{t-1}, \ldots, \forall t \quad (4.2)$$

Chamberlain (1982) argue that non-causality is a more stringent criterion than strict exogeneity, suggesting that equation (4.1) implies equation (4.2) while the inverse is not necessarily true. This result is similar to the one presented in Florens e Mouchart (1982), where is also pointed that both conditions, equations (4.1) and (4.2), became equivalent when x_t and y_t are Gaussian processes. Kuersteiner (2010b) shows that the equivalence between Sims and Granger causality no longer holds when additional covariates are included in the analysis. The author demonstrates that when considering three time series z_t, y_t, and x_t in a causality relation where $z_t \to x_t$, $z_t \to y_t$, and x_t is not directly causing y_t, using the Granger approach we will conclude that $x_t \not\to y_t$. However, using the Sims approach, we would wrongly conclude that $x_t \to y_t$, demonstrating that the Sims approach is not suited for multivariate systems.

We know from Example 4, Section 3, that temporal aggregation does not compromise the measurement of causality between the aggregated variable and the regular variable. The same holds for Sims causality, which can be measured using

$$Y_\tau = \sum_{j=-p}^{p} \gamma_{t-j} x_{t-j} + v_t \quad (4.3)$$

where $\tau = \{1, 2, \ldots\}$ represents periods corresponding to $t = \{s, 2s, \ldots\}$, with s been the frequency ratio between the high-frequency and the low-frequency variables. If any coefficients of $\{x_{t+1}, \ldots, x_{t+p}\}$, or the conjunction of all coefficients, are significant then we say that Y_τ Sims-cause x_t.

To test if x_t causes Y_τ, we propose a modification to the Sims test that avoids any possible overlap between future values of Y_τ and the present value of x_t. This modified Sims test can be described by

$$x_t = \sum_{j=0}^{\infty} \gamma_{t-j} Y_{\tau-j} + \sum_{j=1}^{p} \gamma_{t+j} Y_{\tau+j} I(j) + v_t \tag{4.4}$$

where $I(j)$ is a indicator function that is equal to zero when the aggregated variable $Y_{\tau+j} = y_{(\tau+j)\cdot s} + y_{(\tau+j)\cdot s-1} + \cdots + y_{(\tau+j)\cdot s-k}$ contains any element that occurred at same time or before x_t.

Theorem 1. *Given the conditions of Sims (1972) and equations (4.3)-(4.4), if $Y_\tau \to x_t$ or $Y_\tau \not\to x_t$, then $y_t \to x_t$ or $y_t \not\to x_t$, respectively. Also if $x_t \to Y_\tau$ or $x_t \not\to Y_\tau$ then $x_t \to y_t$ or $x_t \not\to y_t$, respectively.*

Sketch. The proof of equation (4.3) is straightforward and is therefore omitted. For equation (4.4), the proof is also straightforward when both $x_t \not\to y_t$ and $y_t \not\to x_t$. However, when $x_t \not\to y_t$ but $y_t \to x_t$, we have the system

$$\begin{aligned} y_t &= \theta_1 y_{t-1} + v_{1,t}, \\ x_t &= \lambda_1 y_{t-1} + \lambda_2 x_{t-1} + v_{2,t}. \end{aligned}$$

Note that x_t can be expressed as a function solely of past values of y_t, specifically as

$$x_t = \sum_{j=1}^{\infty} \lambda_1 \lambda_2^{j-1} y_{t-j} + v_{2,t} + \sum_{j=1}^{\infty} \lambda_2^j v_{2,t-j},$$

or, more simply, as

$$x_t = \sum_{j=1}^{\infty} \tilde{\lambda}_j y_{t-j} + \tilde{v}_{2,t}.$$

Substituting this into equation (4.4), we obtain

$$\left(\sum_{j=1}^{\infty} \tilde{\lambda}_1 y_{t-j} + \tilde{v}_{2,t} \right) = \sum_{j=0}^{\infty} \gamma_{t-j} Y_{\tau-j} + \sum_{j=1}^{p} \gamma_{t+j} Y_{\tau+j} I(j) + v_t,$$

which simplifies to

$$(1 + \tilde{\lambda}_j L + \cdots + \tilde{\lambda}_j L^k) y_t = \sum_{j=0}^{\infty} \gamma_{t-j} Y_{\tau-j} + \sum_{j=1}^{p} \gamma_{t+j} Y_{\tau+j} I(j) + \tilde{v}_t.$$

Given that $Y_\tau = (1 + a_1 L + \cdots + a_k L^k) y_t$, where the a_i, $i = 1, \ldots, k$, terms are the temporal aggregation weights and the fact that y_t can be also be written in terms of its past, the infinite summation on the right-hand side of (4.4) can fully explain the left-hand side. Thus leading to the γ_{t+j}, $j = 1, \ldots, p$, coefficients to be zero. □

The theorem above states that, assuming Y_τ to be a potentially aggregated time series composed of lagged values of y_t, any conclusion made regarding the causality relationship between Y_τ and x_t will also be drawn between the non-aggregated y_t and x_t. In other words, any causality conclusion made when considering an aggregated quarterly variable, such as GDP, and any other monthly indicator can be extended to the relationship between the same monthly variable and GDP in its non-aggregated state. In Section 6, we will present, by simulation, how the implementation of the proposed Sims test reflects the behavior presented by the theorem.

5 Technical Procedures

Considering two time series x_t and y_t and their aggregated versions, X_τ and Y_τ, composed of lagged values of x_t and y_t, respectively, we will assess the causality cases described as $x_t \to Y_\tau$, $Y_\tau \to x_t$, $X_\tau \to Y_\tau$, and $Y_\tau \to X_\tau$. In these cases, we will evaluate the performance of the tests mentioned in previous sections by implementing each test and comparing their sensitivity to spurious effects.

To analyze the Granger non-causality testing behavior, we follow the structure shown in equation (2.1). By defining the order as one, we choose to consider a single lag for each series. This corresponds to using the same equation structure presented in equation (3.1) for each x_t and y_t series. We refer to this implementation as $G_{yx}(1)$. Note that this test will be only used to test the aggregation explained in Example 4, equation (3.9).

Two tests were implemented considering the Sims causality testing methodology presented in Chapter 4. The first test is the Sims test with a fixed order, represented as $S_{xy}(1)$, where we consider the number of lags and leads to be equal to one. For the $x_t \to Y_\tau$ scenario, we follow equation (4.3), and for the $Y_\tau \to x_t$ scenario, we follow equation (4.4). In the second test, we determine the maximum number of lags for each series using the function $0.5 \cdot (n/s)^{(1/3)}$. We refer to this test as $S_{xy}(k)$, with k being the test order represented by the lag function. In both tests, the number of lead observations is determined by the direction of causality being tested. For the $x_t \to Y_\tau$ scenario, we consider the lead to be 1. For the $Y_\tau \to x_t$ scenario, we consider the lead to be 2, skipping the first lead of Y_τ.

Another causality test is conducted on both y_t and x_t in their original frequencies and without aggregation. This test uses the Granger methodology to serve as the benchmark scenario for causality testing between the two series. We refer to this implementation as the *Non-Agg* testing approach. In contrast, the *Agg* implementation involves aggregating the higher frequency series to match the lower frequency series before performing the Granger test in both directions: $X_\tau \to Y_\tau$ and $Y_\tau \to X_\tau$. However, as discussed in Section 3, this method introduces spurious causality effects. The objective of this implementation is to investigate these effects further and compare them with the results obtained using the proposed tests.

6 Finite sample properties

In this chapter, we aim to simulate a fundamental relationship in Economics: the interaction between GDP and monthly indicators. GDP observations are reported quarterly, without overlap months, and are obtained by summing the GDP values of the preceding three months, i.e., $Y_\tau = \bar{y}_t + \bar{y}_{t-1} + \bar{y}_{t-2}$ where m denotes monthly observations. As shown in Taufemback (2023), the difference between the current quarter's GDP, Y_τ, and the previous quarter, $Y_{\tau-1}$, reflects changes or innovations within that quarter. Equation (6.1) illustrates how monthly GDP innovations are structured following a quarterly difference, under the assumption that $\bar{y}_t = \sum_{j=0}^{\infty} y_{t-j}$, where y_{t-j} are the monthly innovations and $\bar{y}_t \sim I(1)$, with Δ_q denoting the difference between consecutive quarters.

$$\begin{aligned}
\Delta_q Y_\tau &= \bar{y}_t + \bar{y}_{t-1} + \bar{y}_{t-2} - (\bar{y}_{t-3} + \bar{y}_{t-4} + \bar{y}_{t-5}) \\
&= \sum_{j=0}^{\infty} y_{t-j} + \sum_{j=1}^{\infty} y_{t-j} + \sum_{j=2}^{\infty} y_{t-j} - \sum_{j=3}^{\infty} y_{t-j} - \sum_{j=4}^{\infty} y_{t-j} - \sum_{j=5}^{\infty} y_{t-j} \\
&= y_t + 2y_{t-1} + 3y_{t-2} + 2y_{t-3} + y_{t-4}.
\end{aligned} \tag{6.1}$$

Now, let the following models be defined by

$$\begin{aligned}
\text{Model 1: } y_t &= 0.6y_{t-1} + u_{1t}, \\
x_t &= 0.5x_{t-1} - 0.2x_{t-2} + u_{2t}, \\
\\
\text{Model 2: } y_t &= 0.25y_{t-1} + 0.5x_{t-1} + u_{2t}, \\
x_t &= 0.65x_{t-1} + u_{1t}, \\
\\
\text{Model 3: } y_t &= 0.65y_{t-1} + u_{1t}, \\
x_t &= 0.25y_{t-1} + 0.5x_{t-1} + u_{2t}, \\
\\
\text{Model 4: } y_t &= 0.65y_{t-1} + 0.15x_{t-1} + u_{1t}, \\
x_t &= -0.25x_{t-1} + 0.55y_{t-1} + u_{2t}, \\
\\
\text{Model 5: } y_t &= 0.5y_{t-1} - 0.2y_{t-2} + 0.4x_{t-1} - 0.1x_{t-2} + 0.15x_{t-3} - 0.05x_{t-4} + u_{1t}, \\
x_t &= 0.65x_{t-1} + u_{2t}, \\
\\
\text{Model 6: } y_t &= 0.65y_{t-1} + u_{1t}, \\
x_t &= 0.5x_{t-1} - 0.2x_{t-2} + 0.4y_{t-1} - 0.1y_{t-2} + 0.15y_{t-3} - 0.05y_{t-4} + u_{2t}.
\end{aligned}$$

The six models presented above were designed to illustrate the different causality effects discussed in Section 3. For each series, the errors u_1 and u_2 were generated by sampling from a i.i.d. normal distribution, and the aggregation of series y_t follows Equation (6.1). In Model 1, both x_t and y_t consist solely of their lags, indicating the absence of a causal relationship between the series. Model 2 introduces a scenario where y_t includes one lag of x_t, while x_t depends only on its past values, suggesting x_t causes y_t but not vice versa. Conversely, Model 3 demonstrates y_t causing x_t while x_t does not cause y_t. Model 4 incorporates lagged values from both series, indicating a bidirectional causality scenario. In Model 5, y_t includes lags from x_t lags up to $t-4$ and contains only one lag of itself. Finally, Model 6 mirrors the structure of Model 5 but with the roles of x_t and y_t reversed.

Our findings, as shown in Tables 6.1 and 6.2, indicate that both the $S_{xy}(k)$ and $S_{xy}(1)$ tests maintain rejection rates closer to the *Non-Agg* test, demonstrating correct sensitivity in detecting causal relationships between variables. Conversely, the *Agg* test results suggest spurious causality, as expected. The rejection rate of the $G_{yx}(1)$ test gradually increases to one when the alternative hypothesis is true, particularly for models 3, 4, and 6 in Table 6.2. However, this increase is consistently slower than that observed for the $S_{xy}(k)$ and $S_{xy}(1)$ tests. Notably, in cases where the *Agg* tests diverge, both the $S_{xy}(k)$ and $S_{xy}(1)$ tests maintain rejection rates around 0.05.

Analyzing each model individually, for Model 1 in Table 6.1, the *Non-Agg* test rejection rate presents values around 0.05. The *Agg* test rejection rate is slightly over-sized for $n \in \{80, 120\}$ but stabilizes around 0.05 for $n \in \{160, 400\}$. Both Sims tests, $S_{xy}(1)$ and $S_{xy}(k)$, are slightly under-sized. In Table 6.2, the *Non-Agg* test is under-sized for all sample sizes of n, while the *Agg* test maintains a rejection rate around 0.05. The $G_{yx}(1)$ test is also under-sized but remains close to 0.05. Both $S_{xy}(1)$ and $S_{xy}(k)$ tests consistently vary around 0.05, which is desirable.

For Model 2, as shown in Table 6.1, the *Non-Agg* test consistently exhibits a 100% rejection rate across all sample sizes of n. Similarly, the rejection rate for the *Agg* test approaches 100% as n increases. Both the $S_{xy}(1)$ and $S_{xy}(k)$ tests also converge to 100% rejection rates, with $S_{xy}(1)$ showing a higher rejection rate than $S_{xy}(k)$. Table 6.2 shows that the *Non-Agg* test consistently maintains rejection rates around 0.05 across different sample sizes. However, the rejection rate of the *Agg* test diverges. The $G_{yx}(1)$ test, along with the $S_{xy}(1)$ and $S_{xy}(k)$ tests, shows rejection rates close to 0.05.

For Model 3, as shown in Table 6.1, the *Non-Agg* test maintains a stable rejection rate around 0.05 across all sample sizes. In contrast, the *Agg* test exhibits rejection rates that diverge from 0.05 as n increases. The $S_{xy}(k)$ test also maintains rejection rates close to 0.05, although they are slightly elevated for $n \in \{120, 160\}$. Conversely, $S_{xy}(1)$ shows values closer to the expected 0.05. In Table 6.2, the *Non-Agg* test exhibits a 100% rejection rate. Similarly, the *Agg* test approaches a 100% rejection rate as n increases. The $G_{yx}(1)$ test shows a gradual increase in rejection rates with increasing n. Both Sims tests, $S_{xy}(k)$ and $S_{xy}(1)$, also demonstrate increasing rejection rates as n grows, converging towards 100

For Model 4, as shown in Table 6.1, both the *Non-Agg* and *Agg* tests exhibit rejection rates approaching 100% as n increases. Similarly, both Sims tests show increasing rejection rates with increasing n, approaching 100%. In Table 6.2, the *Non-Agg* test consistently shows a 100% rejection rate across all sample sizes. Likewise, the *Agg* test demonstrates a rejection rate that converges to 100% as n increases. The Granger $G_{yx}(1)$ test exhibits a very low rejection rate, with values increasing slowly as n grows. Both Sims tests, $S_{xy}(k)$ and $S_{xy}(1)$, also show increasing rejection rates as n increases, nearing 100% for $n = 400$.

For Model 5, as presented in Table 6.1, both the *Non-Agg* and *Agg* tests exhibit a 100% rejection rate. Similarly, both Sims tests show rejection rates that converge to 100% as n increases. In Table 6.2, the *Non-Agg* test maintains a rejection rate close to 0.05 across different sample sizes. Conversely, the *Agg* test demonstrates high rejection rates that increase significantly as n grows. The Granger $G_{yx}(1)$ test initially shows a slightly inflated rejection rate for $n = 80$, which converges towards 0.05 as n increases. Both Sims tests, $S_{xy}(k)$ and $S_{xy}(1)$, exhibit rejection rates close to 0.05 across sample sizes.

For Model 6, as shown in Table 6.1, the rejection rate of the *Non-Agg* test is slightly elevated for $n = 80$, but remains around 0.05 for other values of n. The *Agg* test exhibits consistently high rejection rates that increase with n. Both the $S_{xy}(k)$ and $S_{xy}(1)$ tests show rejection rates slightly below 0.05, but close to the expected level. In Table 6.2, the *Non-Agg* test demonstrates a 100% rejection rate across all sample sizes. Similarly, the *Agg* test shows rejection rates close to 100%. The Granger $G_{yx}(1)$ test exhibits a rejection rate that increases slowly as n increases. Both Sims tests, $S_{xy}(k)$ and $S_{xy}(1)$, initially show low rejection rates for $n = 80$, but these rates converge towards 100% as n increases.

Table 6.1: Average simulated rejections for $H_0 : x_t \nrightarrow y_t$ with $\alpha = 0.05$.

Model 1 - $\mathbf{x_t \nrightarrow y_t, y_t \nrightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$S_{xy}(k)$	$S_{xy}(1)$
80	0.055	0.060	0.037	0.043
120	0.041	0.063	0.036	0.035
160	0.050	0.059	0.046	0.044
400	0.060	0.049	0.038	0.035

Model 2 - $\mathbf{x_t \rightarrow y_t, y_t \nrightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$S_{xy}(k)$	$S_{xy}(1)$
80	0.998	0.825	0.685	0.694
120	1.000	0.955	0.861	0.868
160	1.000	0.982	0.925	0.937
400	1.000	1.000	1.000	1.000

Model 3 - $\mathbf{x_t \nrightarrow y_t, y_t \rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$S_{xy}(k)$	$S_{xy}(1)$
80	0.049	0.210	0.065	0.067
120	0.043	0.211	0.087	0.083
160	0.047	0.266	0.072	0.069
400	0.053	0.498	0.063	0.069

Model 4 - $\mathbf{x_t \rightarrow y_t, y_t \rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$S_{xy}(k)$	$S_{xy}(1)$
80	0.859	0.765	0.294	0.307
120	0.941	0.856	0.361	0.370
160	0.981	0.947	0.461	0.473
400	1.000	1.000	0.832	0.839

Model 5 - $\mathbf{x_t \rightarrow y_t, y_t \nrightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$S_{xy}(k)$	$S_{xy}(1)$
80	1.000	0.987	0.762	0.775
120	1.000	0.999	0.911	0.910
160	1.000	1.000	0.966	0.967
400	1.000	1.000	1.000	1.000

Model 6 - $\mathbf{x_t \nrightarrow y_t, y_t \rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$S_{xy}(k)$	$S_{xy}(1)$
80	0.065	0.264	0.048	0.052
120	0.052	0.356	0.042	0.044
160	0.058	0.407	0.036	0.037
400	0.039	0.800	0.038	0.039

Note: Results are obtained considering the aggregation presented in equation (6.1), models from Section 6, and the procedures presented in Section 5.

Table 6.2: Average simulated rejections for $H_0 : y_t \not\rightarrow x_t$ with $\alpha = 0.05$.

Model 1 - $\mathbf{x_t \not\rightarrow y_t, y_t \not\rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$G_{yx}(1)$	$S_{yx}(k)$	$S_{yx}(f)$
80	0.049	0.051	0.052	0.052	0.049
120	0.042	0.051	0.037	0.060	0.058
160	0.046	0.049	0.043	0.055	0.051
400	0.036	0.043	0.039	0.041	0.042

Model 2 - $\mathbf{x_t \rightarrow y_t, y_t \not\rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$G_{yx}(1)$	$S_{yx}(k)$	$S_{yx}(f)$
80	0.055	0.154	0.052	0.047	0.044
120	0.053	0.212	0.045	0.043	0.049
160	0.048	0.266	0.046	0.058	0.048
400	0.067	0.515	0.046	0.042	0.037

Model 3 - $\mathbf{x_t \not\rightarrow y_t, y_t \rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$G_{yx}(1)$	$S_{yx}(k)$	$S_{yx}(f)$
80	0.998	0.834	0.092	0.341	0.345
120	1.000	0.946	0.086	0.456	0.464
160	1.000	0.985	0.106	0.624	0.624
400	1.000	1.000	0.213	0.949	0.947

Model 4 - $\mathbf{x_t \rightarrow y_t, y_t \rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$G_{yx}(1)$	$S_{yx}(k)$	$S_{yx}(f)$
80	0.999	0.947	0.067	0.344	0.353
120	1.000	0.983	0.061	0.481	0.496
160	1.000	0.998	0.069	0.589	0.599
400	1.000	1.000	0.116	0.938	0.943

Model 5 - $\mathbf{x_t \rightarrow y_t, y_t \not\rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$G_{yx}(1)$	$S_{yx}(k)$	$S_{yx}(f)$
80	0.050	0.254	0.063	0.047	0.052
120	0.062	0.342	0.055	0.047	0.069
160	0.048	0.432	0.058	0.052	0.058
400	0.041	0.818	0.055	0.054	0.041

Model 6 - $\mathbf{x_t \not\rightarrow y_t, y_t \rightarrow x_t}$

n_{agg}	*Non-Agg*	*Agg*	$G_{yx}(1)$	$S_{yx}(k)$	$S_{yx}(f)$
80	1.000	0.982	0.045	0.636	0.621
120	1.000	0.999	0.057	0.831	0.809
160	1.000	1.000	0.066	0.910	0.899
400	1.000	1.000	0.098	1.000	1.000

Note: Results are obtained considering the aggregation presented in equation (6.1), models from Section 6, and the procedures presented in Section 5.

7 Empirical analysis

In this section, we apply the tests described in Section 5 to U.S. macroeconomic series data within a bivariate framework. We consider the same set of variables as in Ghysels et al. (2016), specifically U.S. inflation (CPI), monthly crude oil price fluctuations (OIL), and quarterly real GDP growth. Additionally, we include other U.S. indicators utilized in Stock e Watson (1989) and Mariano e Murasawa (2003), such as income per person (INC), industrial production (IP), industries sales (IS), and total employment (EMP). The dataset spans from January 1959 to December 2019, with all data publicly available and sourced from research.stlouisfed.org. Each series was made stationary following the guidelines outlined in McCracken e Ng (2016). Table 7.1 presents the p-values for each test, along with the tested causality direction.

Table 7.1: Empirical results

Test direction	*Agg*	$G_{yx}(1)$	$S_{xy}(k)$	$S_{xy}(1)$
OIL $\nrightarrow$ GDP	0.0586	-	0.2771	0.3455
GDP $\nrightarrow$ OIL	0.7002	0.3215	0.1483	0.3986
CPI $\nrightarrow$ GDP	**0.0000**	-	0.5358	0.4368
GDP $\nrightarrow$ CPI	**0.0163**	0.6380	0.1009	0.5459
INC $\nrightarrow$ GDP	**0.0000**	-	**0.0256**	**0.0132**
GDP $\nrightarrow$ INC	**0.0000**	**0.0002**	**0.0000**	**0.0000**
IP $\nrightarrow$ GDP	**0.0000**	-	0.4341	0.9833
GDP $\nrightarrow$ IP	0.1321	**0.0081**	**0.0013**	**0.0156**
EMP $\nrightarrow$ GDP	**0.0000**	-	0.3490	0.2431
GDP $\nrightarrow$ EMP	0.07609	**0.0000**	**0.0269**	**0.0001**
IS $\nrightarrow$ GDP	**0.0000**	-	**0.0077**	**0.0128**
GDP $\nrightarrow$ IS	**0.0136**	**0.0000**	0.0936	**0.0001**

Note: All p-values highlighted in bold are inferior to the 5% rejection level.

The $S_{xy}(k)$ test and the $S_{xy}(1)$ test both fail to detect causality from OIL, CPI, IP, and EMP towards GDP. However, they do identify causality from GDP to INC, IP, EMP, and IS. The only discrepancy between the tests occurs in detecting causality from GDP to IS, where the $S_{xy}(k)$ test only rejects at 10%. Neither test rejects the hypothesis of non-causality in either direction when examining GDP with OIL and CPI. The findings of Ghysels et al. (2016) indeed indicate no causality between GDP and OIL, but they detect causality to and from GDP with respect to CPI.

The $G_{yx}(1)$ test consistently detects causality from GDP to the monthly indicators in most scenarios despite its lower rejection power. However, similar to the $S_{xy}(k)$ and $S_{xy}(1)$ tests, it does not reject the hypothesis of non-causality when testing with CPI and OIL. This reinforces our findings and contrasts with those of Ghysels et al. (2016) regarding CPI.

As previously discussed, temporal aggregation can lead to spurious causality effects, which might explain why the *Agg* testing sometimes detects causality when other tests do not.

8 Conclusion

Aggregating high-frequency variables can result in missing information and potentially misleading conclusions when testing causality between time series. In this study, we propose a modification to the Sims causality test designed to address this issue. Specifically, the modification allows for detecting causality when one series is aggregated while the other remains not aggregated, thereby avoiding spurious conclusions.

In practice, it is challenging to draw conclusions about the causal relationship between economic variables solely based on a bivariate time series model. The structure of the relationship can only be accurately derived by including all relevant variables in the model. Consequently, since many economic variables interact and are important, high-dimensional time series model-building is necessary. As discussed in Lütkepohl (1982), a low-dimensional sub-process may not fully capture the dynamics of a higher-dimensional system. Thus, even if x_t does not cause Y_τ in the bivariate context, Y_τ could still respond to changes in x_t within a broader, multivariate framework.

Reference

Breitung, J. e Swanson, N. R. (2002). Temporal aggregation and spurious instantaneous causality in multiple time series models. *Journal of Time Series Analysis*, 23(6):651–665.

Chalak, K. e White, H. (2012). Causality, conditional independence, and graphical separation in settable systems. *Neural Computation*, 24(7):1611–1668.

Chamberlain, G. (1982). The general equivalence of granger and sims causality. *Econometrica*, 50(3):569–581.

Chow, P. C. (1987). Causality between export growth and industrial development: Empirial evidence from the nics. *Journal of development Economics*, 26(1):55–63.

Comte, F. e Renault, E. (1996). Noncausality in continuous time models. *Econometric theory*, 12(2):215–256.

Florens, J.-P. e Fougere, D. (1996). Noncausality in continuous time. *Econometrica: Journal of the Econometric Society*, pages 1195–1212.

Florens, J.-P. e Mouchart, M. (1982). A note on noncausality. *Econometrica: Journal of the Econometric Society*, pages 583–591.

Ghysels, E., Hill, J. B., e Motegi, K. (2016). Testing for granger causality with mixed frequency data. *Journal of Econometrics*, 192(1):207–230.

Götz, T. B., Hecq, A., e Smeekes, S. (2016). Testing for granger causality in large mixed-frequency vars. *Journal of Econometrics*, 193(2):418–432.

Granger, C. W. (1969). Investigating causal relations by econometric models and cross-spectral methods. *Econometrica: journal of the Econometric Society*, pages 424–438.

Granger, C. W. J. (1963). Economic processes involving feedback. *Information and control*, 6(1):28–48.

Hamilton, J. D. (1994). *Time Series Analysis*. Princeton University Press.

Heckman, J. J. (2000). Causal parameters and policy analysis in economics: A twentieth century retrospective. *The Quarterly Journal of Economics*, 115(1):45–97.

Holland, P. W. (1986). Statistics and causal inference. *Journal of the American statistical Association*, 81(396):945–960.

Kuersteiner, G. M. (2010a). Granger-sims causality. In *Macroeconometrics and time series analysis*, pages 119–134. Springer.

Kuersteiner, G. M. (2010b). Granger-sims causality. In Durlauf, S. N. e Blume, L. E., editors, *Macroeconometrics and Time Series Analysis*, pages 119–134. Palgrave Macmillan UK, London.

Lütkepohl, H. (1982). Non-causality due to omitted variables. *Journal of econometrics*, 19(2-3):367–378.

Macunovich, D. J. e Easterlin, R. A. (1988). Application of granger-sims causality tests to monthly fertility data, 1958–1984. *Journal of Population Economics*, 1(1):71–88.

Marcellino, M. (1999). Some consequences of temporal aggregation in empirical analysis. *Journal of Business & Economic Statistics*, 17(1):129–136.

Mariano, R. S. e Murasawa, Y. (2003). A new coincident index of business cycles based on monthly and quarterly series. *Journal of applied Econometrics*, 18(4):427–443.

McCracken, M. W. e Ng, S. (2016). Fred-md: A monthly database for macroeconomic research. *Journal of Business & Economic Statistics*, 34(4):574–589.

McCrorie, J. R. e Chambers, M. J. (2006). Granger causality and the sampling of economic processes. *Journal of econometrics*, 132(2):311–336.

Pearl, J. (2009). *Causality*. Cambridge university press.

Rajaguru, G. e Abeysinghe, T. (2010). The distortionary effects of temporal aggregation on granger causality. *Some recent developments in statistical theory and applications: Selected Proceedings of the International Conference on Recent Developments in Statistics, Econometrics and Forecasting, University of Allahabad, India.*

Renault, E., Sekkat, K., e Szafarz, A. (1998). Testing for spurious causality in exchange rates. *Journal of Empirical Finance*, 5(1):47–66.

Sims, C. A. (1972). Money, income, and causality. *The American economic review*, 62(4):540–552.

Stock, J. H. e Watson, M. W. (1989). New indexes of coincident and leading economic indicators. *NBER macroeconomics annual*, 4:351–394.

Swanson, N. R. e Granger, C. W. (1997). Impulse response functions based on a causal approach to residual orthogonalization in vector autoregressions. *Journal of the American Statistical Association*, 92(437):357–367.

Tank, A., Fox, E., e Shojaie, A. (2019). Identifiability and estimation of structural vector autoregressive models for subsampled and mixed-frequency time series. *Biometrika*, 106(2):433–452.

Taufemback, C. G. (2023). Asymptotic behavior of temporal aggregation in mixed-frequency datasets. *Oxford Bulletin of Economics and Statistics*, 85(4):894–909.

Weiss, A. A. (1984). Systematic sampling and temporal aggregation in time series models. *Journal of Econometrics*, 26(3):271–281.

Xu, Z. (1996). On the causality between export growth and gdp growth: an empirical reinvestigation. *Review of International Economics*.

Zellner, A. e Montmarquette, C. (1971). A study of some aspects of temporal aggregation problems in econometric analyses. *The Review of Economics and Statistics*, pages 335–342.

Appendix

The set of equations bellow represent the covariance between values of the y_t and x_t series considering a different lag h, for any $h \in [1, 10]$, for model (3.1). We use $\gamma_y(h)$ to represent a covariance between a present value y_t and y_{t-h}. We utilize $\psi_{yx}(h)$ to represent a covariance between a present value y_t and x_{t-h}. We use $\phi_{xy}(h)$ to represent a covariance between a present value x_t and y_{t-h}. Lastly, $\upsilon_x(h)$ represents a covariance between a present value x_t and x_{t-h}, for all cases we assume $h > 0$.

$$\begin{aligned}
\gamma_y(0) &= \sigma_y^2, & \gamma_y(1) &= \theta_1^1\sigma_y^2, & \gamma_y(2) &= \theta_1^2\sigma_y^2,\\
\gamma_y(3) &= \theta_1^3\sigma_y^2, & \gamma_y(4) &= \theta_1^4\sigma_y^2, & \gamma_y(5) &= \theta_1^5\sigma_y^2,\\
\gamma_y(6) &= \theta_1^6\sigma_y^2, & \gamma_y(7) &= \theta_1^7\sigma_y^2, & \gamma_y(8) &= \theta_1^8\sigma_y^2,\\
\gamma_y(9) &= \theta_1^9\sigma_y^2, & \gamma_y(10) &= \theta_1^{10}\sigma_y^2, & &\\
\psi_{yx}(0) &= \sigma_{xy}^2, & \psi_{yx}(1) &= \theta_1^1\sigma_{xy}^2, & \psi_{yx}(2) &= \theta_1^2\sigma_{xy}^2,\\
\psi_{yx}(3) &= \theta_1^3\sigma_{xy}^2, & \psi_{yx}(4) &= \theta_1^4\sigma_{xy}^2 & \psi_{yx}(5) &= \theta_1^5\sigma_{xy}^2,\\
\psi_{yx}(6) &= \theta_1^6\sigma_{xy}^2, & \psi_{yx}(7) &= \theta_1^7\sigma_{xy}^2 & \psi_{yx}(8) &= \theta_1^8\sigma_{xy}^2,\\
\psi_{yx}(9) &= \theta_1^9\sigma_{xy}^2, & \psi_{yx}(10) &= \theta_1^{10}\sigma_{xy}^2 & &
\end{aligned}$$

$$\begin{aligned}
\phi_{xy}(0) &= \sigma_{xy}^2,\\
\phi_{xy}(1) &= (\lambda_1\sigma_y^4 + \lambda_2\sigma_{xy}^2),\\
\phi_{xy}(2) &= (\theta_1^1\lambda_1\sigma_y^4 + \lambda_2\lambda_1\sigma_y^4 + \lambda_2^2\sigma_{xy}^2),\\
\phi_{xy}(3) &= (\theta_1^2\lambda_1\sigma_y^4 + \lambda_2\theta_1^1\lambda_1\sigma_y^4 + \lambda_2^2\lambda_1\sigma_y^4 + \lambda_2^3\sigma_{xy}^2),\\
\phi_{xy}(4) &= (\theta_1^3\lambda_1\sigma_y^4 + \lambda_2^4\sigma_{xy}^2 + \lambda_2^3\lambda_1\sigma_y^4 + \lambda_2^2\theta_1\lambda_1\sigma_y^4 + \lambda_2\theta_1^2\lambda_1\sigma_y^4),\\
\upsilon_x(0) &= \sigma_x^4,\\
\upsilon_x(1) &= (\lambda_1\sigma_{xy}^2 + \lambda_2\sigma_x^4),\\
\upsilon_x(2) &= (\lambda_1\theta_1\sigma_{xy}^2 + \lambda_1\lambda_2\sigma_{xy}^2 + \lambda_2^2\sigma_x^4),\\
\upsilon_x(3) &= (\theta_1^2\lambda_1\sigma_{xy}^2 + \lambda_2^3\sigma_x^4 + \lambda_2^2\lambda_1\sigma_{xy}^2 + \lambda_2\lambda_1\theta_1\sigma_{xy}^2),\\
\upsilon_x(4) &= (\theta_1^3\lambda_1\sigma_{xy}^2 + \lambda_2^4\sigma_x^4 + \lambda_2^3\lambda_1\sigma_{xy}^2 + \lambda_2^2\theta_1\lambda_1\sigma_{xy}^2 + \lambda_2\theta_1^2\lambda_1\sigma_{xy}^2)
\end{aligned}$$

Consider Y_τ, $Y_{\tau-1}$, and X_τ to be potentially aggregated time series composed of lagged values of the series y_t and x_t, both having a structure represented by Equation (3.1). Here, $Y_{\tau-1}$ represents an aggregation function of lagged values of y_t. The following results expand upon the estimations presented for simple models in Section 3, now utilizing the same aggregation pattern observed in the GDP series.

Example 1

In this initial example, we assume Case (i), $\theta_2 = 0$ and $\lambda_1 = 0$, i.e., $x_t \nrightarrow y_t$ and $y_t \nrightarrow x_t$. Thus,

$$\begin{array}{rcl} y_t & = & \theta_1 y_{t-1} + v_{1,t} \\ x_t & = & \lambda_2 x_{t-1} + v_{2,t} \end{array} , \quad v_{j,t} \sim N(0, \sigma^2), \text{ for } j = \{1, 2\} \tag{8.1}$$

which the aggregation pattern defined by

$$\begin{array}{rcl} Y_\tau & = & y_t + 2y_{t-1} + 3y_{t-2} + 2y_{t-3} + y_{t-4}, \\ Y_{\tau-1} & = & y_{t-3} + 2y_{t-4} + 3y_{t-5} + 2y_{t-6} + y_{t-7}, \\ X_\tau & = & x_{t-3} + 2x_{t-4} + 3x_{t-5} + 2x_{t-6} + x_{t-7}. \end{array} \tag{8.2}$$

Solving for Θ_1 and Θ_2, see equation (3.3), we obtain

$$\begin{array}{rcl} det(M) & = & 361\sigma_X^2\sigma_Y^2 + 608\alpha\sigma_X^2\sigma_Y^2 + 380\alpha^2\sigma_X^2\sigma_Y^2 + 152\alpha^3\sigma_X^2\sigma_Y^2 + 38\alpha^4\sigma_X^2\sigma_Y^2 \\ & & +608\beta\sigma_X^2\sigma_Y^2 + 380\beta^2\sigma_X^2\sigma_Y^2 + 152\beta^3\sigma_X^2\sigma_Y^2 + 38\beta^4\sigma_X^2\sigma_Y^2 + 1024\alpha\beta\sigma_X^2\sigma_Y^2 \\ & & +640\alpha\beta^2\sigma_X^2\sigma_Y^2 + 256\alpha\beta^3\sigma_X^2\sigma_Y^2 + 64\alpha\beta^4\sigma_X^2\sigma_Y^2 + 640\alpha^2\beta\sigma_X^2\sigma_Y^2 \\ & & +400\alpha^2\beta^2\sigma_X^2\sigma_Y^2 + 160\alpha^2\beta^3\sigma_X^2\sigma_Y^2 + 40\alpha^2\beta^4\sigma_X^2\sigma_Y^2 + 256\alpha^3\beta\sigma_X^2\sigma_Y^2 \\ & & +160\alpha^3\beta^2\sigma_X^2\sigma_Y^2 + 64\alpha^3\beta^3\sigma_X^2\sigma_Y^2 + 16\alpha^3\beta^4\sigma_X^2\sigma_Y^2 \\ & & +64\alpha^4\beta\sigma_X^2\sigma_Y^2 + 40\alpha^4\beta^2\sigma_X^2\sigma_Y^2 + 16\alpha^4\beta^3\sigma_X^2\sigma_Y^2 + 4\alpha^4\beta^4\sigma_X^2\sigma_Y^2 \\ \pi_1 & = & 361\alpha^3\sigma_X^2\sigma_Y^2 + 304\alpha^2\sigma_X^2\sigma_Y^2 + 209\alpha\sigma_X^2\sigma_Y^2 + 76\sigma_X^2\sigma_Y^2 + 304\alpha^4\sigma_X^2\sigma_Y^2 \\ & & +190\alpha^5\sigma_X^2\sigma_Y^2 + 76\alpha^6\sigma_X^2\sigma_Y^2 + 19\alpha^7\sigma_X^2\sigma_Y^2 + 608\alpha^3\beta\sigma_X^2\sigma_Y^2 \\ & & +380\alpha^3\beta^2\sigma_X^2\sigma_Y^2 + 152\alpha^3\beta^3\sigma_X^2\sigma_Y^2 + 38\alpha^3\beta^4\sigma_X^2\sigma_Y^2 + 512\alpha^2\beta\sigma_X^2\sigma_Y^2 \\ & & +320\alpha^2\beta^2\sigma_X^2\sigma_Y^2 + 128\alpha^2\beta^3\sigma_X^2\sigma_Y^2 + 32\alpha^2\beta^4\sigma_X^2\sigma_Y^2 + 352\alpha\beta\sigma_X^2\sigma_Y^2 \\ & & +220\alpha\beta^2\sigma_X^2\sigma_Y^2 + 88\alpha\beta^3\sigma_X^2\sigma_Y^2 + 22\alpha\beta^4\sigma_X^2\sigma_Y^2 + 128\beta\sigma_X^2\sigma_Y^2 \\ & & +80\beta^2\sigma_X^2\sigma_Y^2 + 32\beta^3\sigma_X^2\sigma_Y^2 + 8\beta^4\sigma_X^2\sigma_Y^2 + 512\alpha^4\beta\sigma_X^2\sigma_Y^2 + 320\alpha^4\beta^2\sigma_X^2\sigma_Y^2 \\ & & +128\alpha^4\beta^3\sigma_X^2\sigma_Y^2 + 32\alpha^4\beta^4\sigma_X^2\sigma_Y^2 + 320\alpha^5\beta\sigma_X^2\sigma_Y^2 + 200\alpha^5\beta^2\sigma_X^2\sigma_Y^2 \\ & & +80\alpha^5\beta^3\sigma_X^2\sigma_Y^2 + 20\alpha^5\beta^4\sigma_X^2\sigma_Y^2 + 128\alpha^6\beta\sigma_X^2\sigma_Y^2 + 80\alpha^6\beta^2\sigma_X^2\sigma_Y^2 \\ & & +32\alpha^6\beta^3\sigma_X^2\sigma_Y^2 + 8\alpha^6\beta^4\sigma_X^2\sigma_Y^2 + 32\alpha^7\beta\sigma_X^2\sigma_Y^2 + 20\alpha^7\beta^2\sigma_X^2\sigma_Y^2 \\ & & +8\alpha^7\beta^3\sigma_X^2\sigma_Y^2 + 2\alpha^7\beta^4\sigma_X^2\sigma_Y^2 \\ \pi_2 & = & 0 \end{array}$$

Example 2

Now, we assume Case (iii), $\theta_2 = 0$ but $\lambda_1 \neq 0$, i.e., $y_t \to x_t$ but $x_t \not\to y_t$, or

$$\begin{array}{rcl} y_t & = & \theta_1 y_{t-1} + v_{1,t} \\ x_t & = & \lambda_1 y_{t-1} + \lambda_2 x_{t-1} + v_{2,t} \end{array} , \quad v_{j,t} \sim N(0, \sigma^2), \text{ for } j = \{1, 2\} \tag{8.3}$$

with the same aggregation patter defined in the first example, see equation (3.5).

$$
\begin{aligned}
det(M) \;=\;& 361\sigma_x^2\sigma_y^2 + 608\theta_1\sigma_x^2\sigma_y^2 + 380\theta_1^2\sigma_x^2\sigma_y^2 + 152\theta_1^3\sigma_x^2\sigma_y^2 + 38\theta_1^4\sigma_x^2\sigma_y^2 - 361\sigma_{xy}^2 \\
& -608\theta_1\sigma_{xy}^2 - 636\theta_1^2\sigma_{xy}^2 - 472\theta_1^3\sigma_{xy}^2 - 266\theta_1^4\sigma_{xy}^2 - 112\theta_1^5\sigma_{xy}^2 - 36\theta_1^6\sigma_{xy}^2 \\
& -8\theta_1^7\sigma_{xy}^2 - \theta_1^8\sigma_{xy}^2 + 608\lambda_2\sigma_x^2\sigma_y^2 + 512\theta_1\lambda_1\sigma_y^2\sigma_{xy} + 380\lambda_2^2\sigma_x^2\sigma_y^2 \\
& +640\theta_1^2\lambda_1\sigma_y^2\sigma_{xy} + 152\lambda_2^3\sigma_x^2\sigma_y^2 + 456\theta_1^3\lambda_1\sigma_y^2\sigma_{xy} + 200\theta_1^2\lambda_2\lambda_1\sigma_y^2\sigma_{xy} \\
& +38\lambda_2^4\sigma_x^2\sigma_y^2 + 1024\theta_1\lambda_2\sigma_x^2\sigma_y^2 + 640\theta_1\lambda_2^2\sigma_x^2\sigma_y^2 + 256\theta_1\lambda_2^3\sigma_x^2\sigma_y^2 \\
& +224\theta_1^4\lambda_1\sigma_y^2\sigma_{xy} + 160\theta_1^3\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 64\theta_1\lambda_2^4\sigma_x^2\sigma_y^2 + 640\theta_1^2\lambda_2\sigma_x^2\sigma_y^2 \\
& +400\theta_1^2\lambda_2^2\sigma_x^2\sigma_y^2 + 160\theta_1^2\lambda_2^3\sigma_x^2\sigma_y^2 + 72\theta_1^5\lambda_1\sigma_y^2\sigma_{xy} + 72\theta_1^4\lambda_2\lambda_1\sigma_y^2\sigma_{xy} \\
& +32\theta_1^3\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 40\theta_1^2\lambda_2^4\sigma_x^2\sigma_y^2 + 256\theta_1^3\lambda_2\sigma_x^2\sigma_y^2 + 160\theta_1^3\lambda_2^2\sigma_x^2\sigma_y^2 \\
& +64\theta_1^3\lambda_2^3\sigma_x^2\sigma_y^2 + 16\theta_1^6\lambda_1\sigma_y^2\sigma_{xy} + 16\theta_1^5\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 16\theta_1^4\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} \\
& +16\theta_1^3\lambda_2^4\sigma_x^2\sigma_y^2 + 64\theta_1^4\lambda_2\sigma_x^2\sigma_y^2 + 40\theta_1^4\lambda_2^2\sigma_x^2\sigma_y^2 + 16\theta_1^4\lambda_2^3\sigma_x^2\sigma_y^2 \\
& +2\theta_1^7\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1^6\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1^5\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} \\
& +2\theta_1^4\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 4\theta_1^4\lambda_2^4\sigma_x^2\sigma_y^2 - 608\lambda_2\sigma_{xy}^2 - 636\lambda_2^2\sigma_{xy}^2 - 472\lambda_2^3\sigma_{xy}^2 \\
& -266\lambda_2^4\sigma_{xy}^2 - 256\lambda_1^2\sigma_y^4 - 2562\lambda_2\lambda_1\sigma_y^2\sigma_{xy} - 320\lambda_2\lambda_1^2\sigma_y^4 - 320\theta_1\lambda_1^2\sigma_y^4 \\
& -640\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} - 228\lambda_2^2\lambda_1^2\sigma_y^4 - 128\theta_1\lambda_2\lambda_1^2\sigma_y^4 - 228\theta_1^2\lambda_1^2\sigma_y^4 \\
& -256\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} - 112\lambda_2^3\lambda_1^2\sigma_y^4 - 32\theta_1\lambda_2^2\lambda_1^2\sigma_y^4 - 32\theta_1^2\lambda_2\lambda_1^2\sigma_y^4 \\
& -112\theta_1^3\lambda_1^2\sigma_y^4 - 1122\lambda_2^4\lambda_1\sigma_y^2\sigma_{xy} - 112\lambda_2^5\sigma_{xy}^2 - 512\theta_1\lambda_2\sigma_{xy}^2 - 320\theta_1^2\lambda_2\sigma_{xy}^2 \\
& -128\theta_1^3\lambda_2\sigma_{xy}^2 - 32\theta_1^4\lambda_2\sigma_{xy}^2 - 1002\theta_1\lambda_2\lambda_1^2\sigma_y^4 - 1002\theta_1\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} \\
& -160\theta_1^2\lambda_2\lambda_1^2\sigma_y^4 - 160\theta_1\lambda_2^2\lambda_1^2\sigma_y^4 - 160\theta_1\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} - 36\theta_1^4\lambda_1^2\sigma_y^4 \\
& -72\theta_1^3\lambda_2\lambda_1^2\sigma_y^4 - 40\theta_1^2\lambda_2^2\lambda_1^2\sigma_y^4 - 72\theta_1\lambda_2^3\lambda_1^2\sigma_y^4 - 72\theta_1\lambda_2^4\lambda_1\sigma_y^2\sigma_{xy} - 36\lambda_2^4\lambda_1^2\sigma_y^4 \\
& -72\lambda_2^5\lambda_1\sigma_y^2\sigma_{xy} - 36\lambda_2^6\sigma_{xy}^2 - 320\theta_1\lambda_2^2\sigma_{xy}^2 - 200\theta_1^2\lambda_2^2\sigma_{xy}^2 \\
& -80\theta_1^3\lambda_2^2\sigma_{xy}^2 - 20\theta_1^4\lambda_2^2\sigma_{xy}^2 - 42\theta_1^2\lambda_2^2\lambda_1^2\sigma_y^4 - 32\theta_1^2\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} \\
& -8\theta_1^5\lambda_1^2\sigma_y^4 - 16\theta_1^4\lambda_2\lambda_1^2\sigma_y^4 - 24\theta_1^3\lambda_2^2\lambda_1^2\sigma_y^4 - 24\theta_1^2\lambda_2^3\lambda_1^2\sigma_y^4 \\
& -16\theta_1^2\lambda_2^4\lambda_1\sigma_y^2\sigma_{xy} - 16\theta_1\lambda_2^4\lambda_1^2\sigma_y^4 - 16\theta_1\lambda_2^5\lambda_1\sigma_y^2\sigma_{xy} - 8\lambda_2^5\lambda_1^2\sigma_y^4 \\
& -16\lambda_2^6\lambda_1\sigma_y^2\sigma_{xy} - 8\lambda_2^7\sigma_{xy}^2 - 128\theta_1\lambda_2^3\sigma_{xy}^2 - 80\theta_1^2\lambda_2^3\sigma_{xy}^2 - 32\theta_1^3\lambda_2^3\sigma_{xy}^2 \\
& -8\theta_1^4\lambda_2^3\sigma_{xy}^2 - \theta_1^6\lambda_1^2\sigma_y^4 - 2\theta_1^5\lambda_2\lambda_1^2\sigma_y^4 - 3\theta_1^4\lambda_2^2\lambda_1^2\sigma_y^4 - 4\theta_1^3\lambda_2^3\lambda_1^2\sigma_y^4 \\
& -2\theta_1^3\lambda_2^4\lambda_1\sigma_y^2\sigma_{xy} - 3\theta_1^2\lambda_2^4\lambda_1^2\sigma_y^4 \\
& -2\theta_1^2\lambda_2^5\lambda_1\sigma_y^2\sigma_{xy} - 2\theta_1\lambda_2^5\lambda_1^2\sigma_y^4 - 2\theta_1\lambda_2^6\lambda_1\sigma_y^2\sigma_{xy} - \lambda_2^6\lambda_1^2\sigma_y^4 - 2\lambda_2^7\lambda_1\sigma_y^2\sigma_{xy} \\
& -\lambda_2^8\sigma_{xy}^2 - 32\theta_1\lambda_2^4\sigma_{xy}^2 - 20\theta_1^2\lambda_2^4\sigma_{xy}^2 - 8\theta_1^3\lambda_2^4\sigma_{xy}^2 - 2\theta_1^4\lambda_2^4\sigma_{xy}^2
\end{aligned}
$$

$$
\begin{aligned}
\pi_1 \quad = \quad & 361\theta_1^3\sigma_x^2\sigma_y^2 + 304\theta_1^2\sigma_x^2\sigma_y^2 + 209\theta_1\sigma_x^2\sigma_y^2 + 76\sigma_x^2\sigma_y^2 + 304\theta_1^4\sigma_x^2\sigma_y^2 \\
& +190\theta_1^5\sigma_x^2\sigma_y^2 + 76\theta_1^6\sigma_x^2\sigma_y^2 + 19\theta_1^7\sigma_x^2\sigma_y^2 - 733\theta_1^3\sigma_{xy}^{-}812\theta_1^4\sigma_{xy}^2 \\
& -710\theta_1^5\sigma_{xy}^2 - 488\theta_1^6\sigma_{xy}^2 - 266\theta_1^7\sigma_{xy}^2 - 504\theta_1^2\sigma_{xy}^2 - 254\theta_1\sigma_{xy}^2 \\
& -76\sigma_{xy}^2 - 112\theta_1^8\sigma_{xy}^2 - 36\theta_1^9\sigma_{xy}^2 - 8\theta_10\sigma_{xy}^2 - \theta_11\sigma_{xy}^2 + 512\theta_1^3\lambda_1\sigma_y^2\sigma_{xy} \\
& +608\theta_1^3\lambda_2\sigma_x^2\sigma_y^2 + 521\theta_1^4\lambda_1\sigma_y^2\sigma_{xy} + 265\theta_1^3\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 380\theta_1^3\lambda_2^2\sigma_x^2\sigma_y^2 \\
& +412\theta_1^5\lambda_1\sigma_y^2\sigma_{xy} + 252\theta_1^4\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 92\theta_1^3\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 152\theta_1^3\lambda_2^3\sigma_x^2\sigma_y^2 \\
& +247\theta_1^6\lambda_1\sigma_y^2\sigma_{xy} + 183\theta_1^5\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 83\theta_1^4\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 19\theta_1^3\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} \\
& +38\theta_1^3\lambda_2^4\sigma_x^2\sigma_y^2 + 382\theta_1^2\lambda_1\sigma_y^2\sigma_{xy} + 512\theta_1^2\lambda_2\sigma_x^2\sigma_y^2 + 208\theta_1^2\lambda_2\lambda_1\sigma_y^2\sigma_{xy} \\
& +320\theta_1^2\lambda_2^2\sigma_x^2\sigma_y^2 + 75\theta_1^2\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 128\theta_1^2\lambda_2^3\sigma_x^2\sigma_y^2 + 16\theta_1^2\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} \\
& +32\theta_1^2\lambda_2^4\sigma_x^2\sigma_y^2 + 216\theta_1\lambda_1\sigma_y^2\sigma_{xy} + 352\theta_1\lambda_2\sigma_x^2\sigma_y^2 + 126\theta_1\lambda_2\lambda_1\sigma_y^2\sigma_{xy} \\
& +220\theta_1\lambda_2^2\sigma_x^2\sigma_y^2 + 48\theta_1\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 88\theta_1\lambda_2^3\sigma_x^2\sigma_y^2 + 11\theta_1\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} \\
& +22\theta_1\lambda_2^4\sigma_x^2\sigma_y^2 + 45\lambda_1\sigma_y^2\sigma_{xy} + 128\lambda_2\sigma_x^2\sigma_y^2 + 40\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 80\lambda_2^2\sigma_x^2\sigma_y^2 \\
& +16\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 32\lambda_2^3\sigma_x^2\sigma_y^2 + 4\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 8\lambda_2^4\sigma_x^2\sigma_y^2 + 512\theta_1^4\lambda_2\sigma_x^2\sigma_y^2 \\
& +320\theta_1^4\lambda_2^2\sigma_x^2\sigma_y^2 + 128\theta_1^4\lambda_2^3\sigma_x^2\sigma_y^2 + 112\theta_1^7\lambda_1\sigma_y^2\sigma_{xy} + 96\theta_1^6\lambda_2\lambda_1\sigma_y^2\sigma_{xy} \\
& +56\theta_1^5\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 16\theta_1^4\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 32\theta_1^4\lambda_2^4\sigma_x^2\sigma_y^2 + 320\theta_1^5\lambda_2\sigma_x^2\sigma_y^2 \\
& +200\theta_1^5\lambda_2^2\sigma_x^2\sigma_y^2 + 80\theta_1^5\lambda_2^3\sigma_x^2\sigma_y^2 + 36\theta_1^8\lambda_1\sigma_y^2\sigma_{xy} + 36\theta_1^7\lambda_2\lambda_1\sigma_y^2\sigma_{xy} \\
& +26\theta_1^6\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 10\theta_1^5\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 20\theta_1^5\lambda_2^4\sigma_x^2\sigma_y^2 + 128\theta_1^6\lambda_2\sigma_x^2\sigma_y^2 \\
& +80\theta_1^6\lambda_2^2\sigma_x^2\sigma_y^2 + 32\theta_1^6\lambda_2^3\sigma_x^2\sigma_y^2 + 8\theta_1^9\lambda_1\sigma_y^2\sigma_{xy} + 8\theta_1^8\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 8\theta_1^7\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} \\
& +4\theta_1^6\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 8\theta_1^6\lambda_2^4\sigma_x^2\sigma_y^2 + 32\theta_1^7\lambda_2\sigma_x^2\sigma_y^2 + 20\theta_1^7\lambda_2^2\sigma_x^2\sigma_y^2 + 8\theta_1^7\lambda_2^3\sigma_x^2\sigma_y^2 \\
& +\theta_10\lambda_1\sigma_y^2\sigma_{xy} + \theta_1^9\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + \theta_1^8\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + \theta_1^7\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1^7\lambda_2^4\sigma_x^2\sigma_y^2 \\
& -308\theta_1^3\lambda_2\sigma_{xy}^2 - 190\theta_1^3\lambda_2^2\sigma_{xy}^2 - 76\theta_1^3\lambda_2^3\sigma_{xy}^2 - 19\theta_1^3\lambda_2^4\sigma_{xy}^2 - 266\theta_1^2\lambda_2\sigma_{xy}^2 \\
& -160\theta_1^2\lambda_2^2\sigma_{xy}^2 - 64\theta_1^2\lambda_2^3\sigma_{xy}^2 - 16\theta_1^2\lambda_2^4\sigma_{xy}^2 - 176\theta_1\lambda_2\sigma_{xy}^2 - 100\theta_1\lambda_2^2\sigma_{xy}^2 \\
& -40\theta_1\lambda_2^3\sigma_{xy}^2 - 10\theta_1\lambda_2^4\sigma_{xy}^2 - 83\lambda_2\sigma_{xy}^2 - 56\lambda_2^2\sigma_{xy}^2 - 26\lambda_2^3\sigma_{xy}^2 - 8\lambda_2^4\sigma_{xy}^2 \\
& -16\lambda_1^2\sigma_y^4 - 162\lambda_2\lambda_1\sigma_y^2\sigma_{xy} - 10\lambda_2\lambda_1^2\sigma_y^4 - 10\theta_1\lambda_1^2\sigma_y^4 - 20\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} \\
& -4\lambda_2^2\lambda_1^2\sigma_y^4 - 4\theta_1\lambda_2\lambda_1^2\sigma_y^4 - 4\theta_1^2\lambda_1^2\sigma_y^4 - 8\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} - \lambda_2^3\lambda_1^2\sigma_y^4 - \theta_1\lambda_2^2\lambda_1^2\sigma_y^4 \\
& -\theta_1^2\lambda_2\lambda_1^2\sigma_y^4 - \theta_1^3\lambda_1^2\sigma_y^4 - 2\lambda_2^4\lambda_1\sigma_y^2\sigma_{xy} - \lambda_2^5\sigma_{xy}^2 - 257\theta_1^4\lambda_2\sigma_{xy}^2 - 160\theta_1^4\lambda_2^2\sigma_{xy}^2 \\
& -64\theta_1^4\lambda_2^3\sigma_{xy}^2 - 16\theta_1^4\lambda_2^4\sigma_{xy}^2 - 160\theta_1^5\lambda_2\sigma_{xy}^2 - 100\theta_1^5\lambda_2^2\sigma_{xy}^2 - 40\theta_1^5\lambda_2^3\sigma_{xy}^2 \\
& -10\theta_1^5\lambda_2^4\sigma_{xy}^2 - 64\theta_1^6\lambda_2\sigma_{xy}^2 - 40\theta_1^6\lambda_2^2\sigma_{xy}^2 - 16\theta_1^6\lambda_2^3\sigma_{xy}^2 - 4\theta_1^6\lambda_2^4\sigma_{xy}^2 - 16\theta_1^7\lambda_2\sigma_{xy}^2 \\
& -10\theta_1^7\lambda_2^2\sigma_{xy}^2 - 4\theta_1^7\lambda_2^3\sigma_{xy}^2 - \theta_1^7\lambda_2^4\sigma_{xy}^2
\end{aligned}
$$

$$\begin{aligned}
\pi_2 \;=\;& 362\theta_1^3\sigma_y^2\sigma_{xy} + 504\theta_1^4\sigma_y^2\sigma_{xy} + 519\theta_1^5\sigma_y^2\sigma_{xy} + 412\theta_1^6\sigma_y^2\sigma_{xy} + 247\theta_1^7\sigma_y^2\sigma_{xy}\\
& +184\theta_1^2\sigma_y^2\sigma_{xy} + 45\theta_1\sigma_y^2\sigma_{xy} + 112\theta_1^8\sigma_y^2\sigma_{xy} + 36\theta_1^9\sigma_y^2\sigma_{xy}\\
& +8\theta_10\sigma_y^2\sigma_{xy} + \theta_11\sigma_y^2\sigma_{xy} - 45\lambda_1\sigma_y^4 - 45\lambda_2\sigma_y^2\sigma_{xy}\\
& -184\theta_1\lambda_1\sigma_y^4 - 144\theta_1\lambda_2\sigma_y^2\sigma_{xy} + -362\theta_1^2\lambda_1\sigma_y^4 - 236\theta_1^2\lambda_2\sigma_y^2\sigma_{xy}\\
& -504\theta_1^3\lambda_1\sigma_y^4 - 296\theta_1^3\lambda_2\sigma_y^2\sigma_{xy} - 519\theta_1^4\lambda_1\sigma_y^4 - 254\theta_1^4\lambda_2\sigma_y^2\sigma_{xy}\\
& -265\theta_1^3\lambda_2\lambda_1\sigma_y^4 - 190\theta_1^3\lambda_2^2\sigma_y^2\sigma_{xy} - 412\theta_1^5\lambda_1\sigma_y^4 - 252\theta_1^4\lambda_2\lambda_1\sigma_y^4\\
& -92\theta_1^3\lambda_2^2\lambda_1\sigma_y^4 - 76\theta_1^3\lambda_2^3\sigma_y^2\sigma_{xy} - 247\theta_1^6\lambda_1\sigma_y^4 - 183\theta_1^5\lambda_2\lambda_1\sigma_y^4\\
& -83\theta_1^4\lambda_2^2\lambda_1\sigma_y^4 - 19\theta_1^3\lambda_2^3\lambda_1\sigma_y^4 - 19\theta_1^3\lambda_2^4\sigma_y^2\sigma_{xy} - 208\theta_1^2\lambda_2\lambda_1\sigma_y^4\\
& -160\theta_1^2\lambda_2^2\sigma_y^2\sigma_{xy} - 75\theta_1^2\lambda_2^2\lambda_1\sigma_y^4 - 64\theta_1^2\lambda_2^3\sigma_y^2\sigma_{xy} - 16\theta_1^2\lambda_2^3\lambda_1\sigma_y^4\\
& -16\theta_1^2\lambda_2^4\sigma_y^2\sigma_{xy} - 126\theta_1\lambda_2\lambda_1\sigma_y^4 - 110\theta_1\lambda_2^2\sigma_y^2\sigma_{xy} - 48\theta_1\lambda_2^2\lambda_1\sigma_y^4\\
& -44\theta_1\lambda_2^3\sigma_y^2\sigma_{xy} - 11\theta_1\lambda_2^3\lambda_1\sigma_y^4 - 11\theta_1\lambda_2^4\sigma_y^2\sigma_{xy} - 40\lambda_2\lambda_1\sigma_y^4\\
& -40\lambda_2^2\sigma_y^2\sigma_{xy} - 16\lambda_2^2\lambda_1\sigma_y^4 - 16\lambda_2^3\sigma_y^2\sigma_{xy} - 4\lambda_2^3\lambda_1\sigma_y^4\\
& -4\lambda_2^4\sigma_y^2\sigma_{xy} - 160\theta_1^4\lambda_2^2\sigma_y^2\sigma_{xy} - 64\theta_1^4\lambda_2^3\sigma_y^2\sigma_{xy} - 112\theta_1^7\lambda_1\sigma_y^4 - 96\theta_1^6\lambda_2\lambda_1\sigma_y^4\\
& -56\theta_1^5\lambda_2^2\lambda_1\sigma_y^4 - 16\theta_1^4\lambda_2^3\lambda_1\sigma_y^4 - 16\theta_1^4\lambda_2^4\sigma_y^2\sigma_{xy} - 160\theta_1^5\lambda_2\sigma_y^2\sigma_{xy} - 100\theta_1^5\lambda_2^2\sigma_y^2\sigma_{xy}\\
& -40\theta_1^5\lambda_2^3\sigma_y^2\sigma_{xy} - 36\theta_1^8\lambda_1\sigma_y^4 - 36\theta_1^7\lambda_2\lambda_1\sigma_y^4 - 26\theta_1^6\lambda_2^2\lambda_1\sigma_y^4\\
& -10\theta_1^5\lambda_2^3\lambda_1\sigma_y^4 - 10\theta_1^5\lambda_2^4\sigma_y^2\sigma_{xy} - 64\theta_1^6\lambda_2\sigma_y^2\sigma_{xy} - 40\theta_1^6\lambda_2^2\sigma_y^2\sigma_{xy}\\
& -16\theta_1^6\lambda_2^3\sigma_y^2\sigma_{xy} - 8\theta_1^9\lambda_1\sigma_y^4 - 8\theta_1^8\lambda_2\lambda_1\sigma_y^4 - 8\theta_1^7\lambda_2^2\lambda_1\sigma_y^4 - 4\theta_1^6\lambda_2^3\lambda_1\sigma_y^4\\
& -4\theta_1^6\lambda_2^4\sigma_y^2\sigma_{xy} - 16\theta_1^7\lambda_2\sigma_y^2\sigma_{xy} - 10\theta_1^7\lambda_2^2\sigma_y^2\sigma_{xy} - 4\theta_1^7\lambda_2^3\sigma_y^2\sigma_{xy} - \theta_10\lambda_1\sigma_y^4\\
& -\theta_1^9\lambda_2\lambda_1\sigma_y^4 - \theta_1^8\lambda_2^2\lambda_1\sigma_y^4 - \theta_1^7\lambda_2^3\lambda_1\sigma_y^4 - \theta_1^7\lambda_2^4\sigma_y^2\sigma_{xy}
\end{aligned}$$

Example 3

Again, in Case (iii), $x_t \not\to y_t, y_t \to x_t$, model (3.6). However, we assume that X_τ is non-aggregated, so we are now measuring Granger causality from a non-aggregated to an aggregated series, i.e., which the aggregation pattern defined by

$$\begin{aligned}
Y_\tau &= y_t + 2y_{t-1} + 3y_{t-2} + 2y_{t-3} + y_{t-4},\\
Y_{\tau-1} &= y_{t-6} + 2y_{t-7} + 3y_{t-8} + 2y_{t-9} + y_{t-10},\\
X_\tau &= x_{t-2}.
\end{aligned} \tag{8.4}$$

The results are given by

$$\begin{aligned}
det(M) \;=\;& 19\sigma_x^2\sigma_y^2 + 32\theta_1\sigma_x^2\sigma_y^2 + 20\theta_1^2\sigma_x^2\sigma_y^2 + 8\theta_1^3\sigma_x^2\sigma_y^2 + 2\theta_1^4\sigma_x^2\sigma_y^2\\
& -\theta_1^8\sigma_{xy}^2 - 4\theta_1^7\sigma_{xy}^2 - 10\theta_1^6\sigma_{xy}^2 - 16\theta_1^5\sigma_{xy}^2 - 19\theta_1^4\sigma_{xy}^2 - 16\theta_1^3\sigma_{xy}^2\\
& -10\theta_1^2\sigma_{xy}^2 - 4\theta_1\sigma_{xy}^2 - \sigma_{xy}^2 h\\
\pi_1 \;=\;& 19\theta_1^3\sigma_x^2\sigma_y^2 + 16\theta_1^2\sigma_x^2\sigma_y^2 + 11\theta_1\sigma_x^2\sigma_y^2 + 4\sigma_x^2\sigma_y^2 + 16\theta_1^4\sigma_x^2\sigma_y^2 + 10\theta_1^5\sigma_x^2\sigma_y^2\\
& +4\theta_1^6\sigma_x^2\sigma_y^2 + \theta_1^7\sigma_x^2\sigma_y^2 - \theta_11\sigma_{xy}^2 - 4\theta_10\sigma_{xy}^2 - 10\theta_1^9\sigma_{xy}^2 - 16\theta_1^8\sigma_{xy}^2\\
& -19\theta_1^7\sigma_{xy}^2 - 16\theta_1^6\sigma_{xy}^2 - 10\theta_1^5\sigma_{xy}^2 - 4\theta_1^4\sigma_{xy}^2 - \theta_1^3\sigma_{xy}^2\\
\pi_2 \;=\;& 71\theta_1^7\sigma_y^2\sigma_{xy} + 50\theta_1^8\sigma_y^2\sigma_{xy} + 21\theta_1^9\sigma_y^2\sigma_{xy} + 6\theta_10\sigma_y^2\sigma_{xy} + \theta_11\sigma_y^2\sigma_{xy}\\
& +56\theta_1^6\sigma_y^2\sigma_{xy} - \theta_1^5\sigma_y^2\sigma_{xy} - 58\theta_1^4\sigma_y^2\sigma_{xy} - 73\theta_1^3\sigma_y^2\sigma_{xy}\\
& -50\theta_1^2\sigma_y^2\sigma_{xy} - 19\theta_1\sigma_y^2\sigma_{xy} - 4\sigma_y^2\sigma_{xy}
\end{aligned}$$

As example, testing if further lag values of x_t are sufficient to reduce considerable spurious effects, we replicate the calculations and simulations mentioned earlier, with X_τ being non-aggregated and equal to x_{t-3}, representing one lag further than the previous simulations.

which the aggregation pattern defined by

$$\begin{aligned} Y_\tau &= y_t + 2y_{t-1} + 3y_{t-2} + 2y_{t-3} + y_{t-4}, \\ Y_{\tau-1} &= y_{t-6} + 2y_{t-7} + 3y_{t-8} + 2y_{t-9} + y_{t-10}, \\ X_\tau &= x_{t-3}. \end{aligned} \quad (8.5)$$

with results given by

$$\begin{aligned} det(M) &= 19\sigma_x^2\sigma_y^2 + 32\theta_1\sigma_x^2\sigma_y^2 + 20\theta_1^2\sigma_x^2\sigma_y^2 + 8\theta_1^3\sigma_x^2\sigma_y^2 + 2\theta_1^4\sigma_x^2\sigma_y^2 - \theta_1 0\sigma_{xy}^2 \\ &\quad -4\theta_1^9\sigma_{xy}^2 - 10\theta_1^8\sigma_{xy}^2 - 16\theta_1^7\sigma_{xy}^2 - 19\theta_1^6\sigma_{xy}^2 - 16\theta_1^5\sigma_{xy}^2 - 10\theta_1^4\sigma_{xy}^2 \\ &\quad -4\theta_1^3\sigma_{xy}^2 - \theta_1^2\sigma_{xy}^2 \\ \pi_1 &= 19\theta_1^3\sigma_x^2\sigma_y^2 + 16\theta_1^2\sigma_x^2\sigma_y^2 + 11\theta_1\sigma_x^2\sigma_y^2 + 4\sigma_x^2\sigma_y^2 + 16\theta_1^4\sigma_x^2\sigma_y^2 + 10\theta_1^5\sigma_x^2\sigma_y^2 \\ &\quad +4\theta_1^6\sigma_x^2\sigma_y^2 + \theta_1^7\sigma_x^2\sigma_y^2 - \theta_1 3\sigma_{xy}^2 - 4\theta_1 2\sigma_{xy}^2 - 10\theta_1 1\sigma_{xy}^2 - 16\theta_1 0\sigma_{xy}^2 \\ &\quad -19\theta_1^9\sigma_{xy}^2 - 16\theta_1^8\sigma_{xy}^2 - 10\theta_1^7\sigma_{xy}^2 - 4\theta_1^6\sigma_{xy}^2 - \theta_1^5\sigma_{xy}^2 \\ \pi_2 &= 71\theta_1^8\sigma_y^2\sigma_{xy} + 50\theta_1^9\sigma_y^2\sigma_{xy} + 21\theta_1 0\sigma_y^2\sigma_{xy} + 6\theta_1 1\sigma_y^2\sigma_{xy} + \theta_1 2\sigma_y^2\sigma_{xy} \\ &\quad +56\theta_1^7\sigma_y^2\sigma_{xy} - \theta_1^6\sigma_y^2\sigma_{xy} - 58\theta_1^5\sigma_y^2\sigma_{xy} - 73\theta_1^4\sigma_y^2\sigma_{xy} - 50\theta_1^3\sigma_y^2\sigma_{xy} \\ &\quad -19\theta_1^2\sigma_y^2\sigma_{xy} - 4\theta_1\sigma_y^2\sigma_{xy} \end{aligned}$$

Example 4

Finally, we examine the same causality, Case (iii), $x_t \not\to y_t, y_t \to x_t$, model (3.6), but this time, we reverse the aggregation, with Y_τ and $Y_{\tau-1}$ being non-aggregated and X_τ being aggregated, i.e.,

$$\begin{aligned} Y_\tau &= y_t \\ Y_{\tau-1} &= y_{t-1}, \\ X_\tau &= x_{t-6} + 2x_{t-7} + 3x_{t-8} + 2x_{t-9} + x_{t-10}. \end{aligned} \quad (8.6)$$

This allows us to investigate the causality direction from an aggregated series to a non-aggregated one. The results are given by

$$\begin{aligned} det(M) &= 19\sigma_x^2\sigma_y^2 - \theta_1^4\sigma_{xy}^2 - 4\theta_1^5\sigma_{xy}^2 - 10\theta_1^6\sigma_{xy}^2 - 16\theta_1^7\sigma_{xy}^2 - 19\theta_1^8\sigma_{xy}^2 - 16\theta_1^9\sigma_{xy}^2 \\ &\quad -10\theta_1 0\sigma_{xy}^2 - 4\theta_1 1\sigma_{xy}^2 - \theta_1 2\sigma_{xy}^2 + 32\lambda_1\sigma_y^2\sigma_{xy} + 32\lambda_2\sigma_x^2\sigma_y^2 \\ &\quad +20\theta_1\lambda_1\sigma_y^2\sigma_{xy} + 20\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 20\lambda_2^2\sigma_x^2\sigma_y^2 + 8\theta_1^2\lambda_1\sigma_y^2\sigma_{xy} \\ &\quad +8\theta_1\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 8\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 8\lambda_2^3\sigma_x^2\sigma_y^2 \\ &\quad +2\theta_1^3\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1^2\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 2\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 2\lambda_2^4\sigma_x^2\sigma_y^2 \\ \pi_1 &= 19\theta_1\sigma_x^2\sigma_y^2 - \theta_1^5\sigma_{xy}^2 - 4\theta_1^6\sigma_{xy}^2 - 10\theta_1^7\sigma_{xy}^2 - 16\theta_1^8\sigma_{xy}^2 - 19\theta_1^9\sigma_{xy}^2 - 16\theta_1 0\sigma_{xy}^2 \\ &\quad -10\theta_1 1\sigma_{xy}^2 - 4\theta_1 2\sigma_{xy}^2 - \theta_1 3\sigma_{xy}^2 + 32\theta_1\lambda_1\sigma_y^2\sigma_{xy} + 32\theta_1\lambda_2\sigma_x^2\sigma_y^2 \\ &\quad +20\theta_1^2\lambda_1\sigma_y^2\sigma_{xy} + 20\theta_1\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 20\theta_1\lambda_2^2\sigma_x^2\sigma_y^2 + 8\theta_1^3\lambda_1\sigma_y^2\sigma_{xy} \\ &\quad +8\theta_1^2\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 8\theta_1\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 8\theta_1\lambda_2^3\sigma_x^2\sigma_y^2 + 2\theta_1^4\lambda_1\sigma_y^2\sigma_{xy} \\ &\quad +2\theta_1^3\lambda_2\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1^2\lambda_2^2\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1\lambda_2^3\lambda_1\sigma_y^2\sigma_{xy} + 2\theta_1\lambda_2^4\sigma_x^2\sigma_y^2 \\ \pi_2 &= 0 \end{aligned}$$