AVERAGES ALONG UNIFORMLY DISTRIBUTED DIRECTIONS ON A CURVE

JOSE BARRIONUEVO

(Communicated by J. Marshall Ash)

ABSTRACT. We obtain a sharp $L^2$ estimate for the maximal operator associated with uniformly distributed directions on a curve of finite type in $\mathbb{R}^n$.

INTRODUCTION

Let $\gamma : [0, 1] \to S^{n-1}$ be a smooth curve crossing each hyperplane of $\mathbb{R}^n$ a finite number of times. If $\mathcal{B}_N$ denotes the family of all cylinders in $\mathbb{R}^n$ having eccentricity $N$ and direction in $\gamma$, it is proved in [C] that the maximal operator

$$M_N f(x) = \sup_{R \in \mathcal{B}_N} \frac{1}{|R|} \int_R |f(y)| \, dy$$

satisfies the estimate

$$\|M_N f\|_{L^2} \leq C_\gamma (\log N)^2 \|f\|_{L^2}$$

where $C_\gamma$ is independent of $N$.

The purpose of this note is to show that by imposing an additional condition on $\gamma$ one can prove a stronger result.

Let $\gamma$ be a smooth curve satisfying

(*) For all $t \in [0, 1]$, the set $\{\gamma(j/2^m) : 0 \leq j < 2^m\}$ spans $\mathbb{R}^n$.

For a positive integer $m$ let $\mathcal{B}_m$ denote the family of all cylinders in $\mathbb{R}^n$ pointing in the direction of $\gamma(j/2^m)$ for some $0 \leq j < 2^m$. Let $\mathcal{M}_m f(x) = \sup_{R \in \mathcal{B}_m} (1/|R|) \int_R |f(y)| \, dy$. Then we will prove the following:

Theorem. If $\gamma$ satisfies (*) then

$$\|\mathcal{M}_m f\|_{L^2} \leq C_\gamma m \|f\|_{L^2}$$

where $C_\gamma$ is independent of $m$.

If $n = 2$ or if $\gamma$ is contained in a 2-dimensional subspace, (2) is known to be true (see [S] or [B]). Also, since $\mathcal{M}_m$ dominates $M_{2^m}$, (2) implies an improved version of (1).

In what follows all the constants are independent of $m$. 

Received by the editors March 4, 1991 and, in revised form, March 9, 1992.
1991 Mathematics Subject Classification. Primary 42B10.
We will now prove some consequences of (*) that will be used to prove the theorem.

A simple compactness argument shows that if \( y \) satisfies (*), then there exist an integer \( L \) and \( c > 0 \) such that for all \( \xi \in S^{n-1} \) and \( t \in [0, 1] \)

\[
\sum_{i=0}^{L} |\xi \cdot y^{(i)}(t)| \geq c.
\]

For \( j = 0, 1, 2 \) let \( \mathcal{U}_j = \{ (\xi, t) \in S^{n-1} \times [0, 1]: |\xi \cdot y^{(j)}(t)| \leq c2^{-(l+1)} \} \) for \( l \leq j \). Then we have

**Lemma 1.** There exist \( \delta_j > 0 \) and \( c_j > 0 \) such that for all \( (\xi, t) \in \mathcal{U}_j \)

\[
|s - t| < \delta_j \Rightarrow |\xi \cdot (y^{(j)}(s) - y^{(j)}(t))| \geq c_j |s - t|^{L-j}.
\]

**Proof.** If the lemma is false, we can find sequences \( \varepsilon_k \rightarrow 0 \), \( \delta_k \rightarrow 0 \), \( (\xi_k, t_k) \in S^{n-1} \times [0, 1] \), and \( s_k \) such that \( |s_k - t_k| < \delta_k \) and

\[
|\xi_k \cdot (y^{(j)}(s_k) - y^{(j)}(t_k))| < \varepsilon_k |s_k - t_k|^{L-j}.
\]

Since \( \mathcal{U}_j \) is compact, by passing to a subsequence, we can assume that \( (\xi_k, t_k) \) converges to \( (\xi, t) \in \mathcal{U}_j \). By Taylor's theorem (5) implies that \( \xi \cdot y^{(l)}(t) = 0 \) for \( l = j + 1, \ldots, L \). This contradicts (3).

**Lemma 1** implies that there exist integers \( N_j \) (\( \sim \delta_j^{-1} \)) such that for all \( \xi \in S^{n-1} \) the function \( \xi \cdot y^{(j)}(t) \) has at most \( N_j \) zeros on \( \{ t \in [0, 1]: |\xi \cdot y^{(j)}(t)| < c2^{-(l+1)} \} \) for \( 0 \leq l \leq j - 1 \).

For \( \xi \in S^{n-1} \) let \( v_\xi(t) = \xi \cdot y(t) \), \( \mathcal{V}_\xi^1 = \{ t \in [0, 1]: |v_\xi(t)| > c/2 \} \), and \( \mathcal{V}_\xi^2 = \{ t \in [0, 1]: |v_\xi(t)| < c/2 \text{ and } |v_\xi'(t)| > c/4 \} \). Since \( \mathcal{V}_\xi^1 \) and \( \mathcal{V}_\xi^2 \) are open (in [0, 1]) and disjoint, we can write each \( \mathcal{V}_\xi^j \) as a countable union of disjoint intervals. Since between each two intervals of \( \mathcal{V}_\xi^1 \) there exists a \( t \) for which either \( v_\xi(t) = 0 \) or \( v'_\xi(t) = 0 \), Lemma 1 implies that \( \mathcal{V}_\xi^1 \) is the union of at most \( N_0 + N_1 \) (independent of \( \xi \)) intervals. A similar argument applied to \( \mathcal{V}_\xi^2 \) in the complement of \( \mathcal{V}_\xi^1 \) together with the fact that, on the complement of \( \mathcal{V}_\xi^1 \cup \mathcal{V}_\xi^2 \), \( v'_\xi(t) \) has at most \( N_2 \) zeros shows that the complement of \( \mathcal{V}_\xi^1 \cup \mathcal{V}_\xi^2 \) can be written as a union of no more than \( 2(N_0 + N_1 + N_2) \) closed intervals where, on each of these, \( v'_\xi(t) \) is monotonic. Let \( I = [a, b] \) be one such interval, and let \( t_0 \in [a, b] \) be such that \( |v'_\xi(t_0)| = \min_I |v'_\xi(t)| \). Then we have

\[
v'_\xi(t) = \frac{L-1}{j!} v^{(j+1)}(t_0) (t - t_0)^j + R_{t_0}(t)
\]

\[
= p'_\xi(t) + R_{t_0}(t) \text{ where } |R_{t_0}(t)| \leq C|t - t_0|^L.
\]

Thus if \( \delta_\gamma = 1/2 \min \{ \min_j \delta_j, c_1 C^{-1} \} \) we have for \( |t - t_0| < \delta_\gamma \) and \( t \neq t_0 \)

\[
\left| \frac{p'_\xi(t)}{v'_\xi(t)} - 1 \right| \leq \frac{C|t - t_0|^L}{|v'_\xi(t)|} \leq \frac{1}{2},
\]
which implies
\[ |p'_\xi(t)| \leq |v'_\xi(t)| \leq 2|p'_\xi(t)|. \]
If we let \( p'_\xi(t) - v'_\xi(t_0) = q_\xi(t) \), we have by Lemma 1 that there exist \( c_\xi > 0 \) such that \( |q_\xi(t)| \approx c_\xi |t - t_0|^k \) for \( |t - t_0| < \delta_\xi \) and for some \( k \) with \( 1 \leq k \leq L - 1 \). If \( |t - t_0| > \delta_\xi \) and \( t \in I \), Lemma 1 implies that \( |v'_\xi(t)| \geq c_1 \delta_\xi^{L-1} \). Since \( v_\xi(t) \) has at most two zeros on \( I \), we can divide \{ \( t \in I: |t - t_0| < \delta_\xi \) \} in no more than four intervals where \( v_\xi(t) \) is monotonic and of constant sign satisfying estimates like the above. Thus, if we let \( N_\gamma = 10(N_0 + N_1 + N_2) \) and \( c_\gamma = \min\{c/4, c_1 \delta_\xi^{L-1}\} \), we obtain

**Lemma 2.** There exist an integer \( N_\gamma \) and \( c_\gamma > 0 \) such that for all \( \xi \) in \( S^{n-1} \) we have
\[ [0, 1] = U_1^1 \cup \cdots \cup U_{N_\xi}^1 \cup V_1^1 \cup \cdots \cup V_{N_\xi}^1 \cup W_1^1 \cup \cdots \cup W_{N_\xi}^1, \]
where \( N_\xi + M_\xi + K_\xi \leq N_\gamma \) and where the \( U_i^1 \)'s, \( V_i^1 \)'s, and \( W_i^1 \)'s are closed intervals with disjoint interiors for which
(i) \( |v_\xi(t)| \geq c_\gamma \) on \( \bigcup_j U_j^1 \),
(ii) \( |v_\xi(t)| \geq c_\gamma \) on \( \bigcup_j V_j^1 \), and
(iii) for each \( i \leq K_\xi \) there exist \( c_\xi^1 > 0 \), \( t_0 \in W_j^1 \), and \( k = k_{\xi,i,t_0} \) with \( 1 \leq k \leq L \) such that
\[ |v_\xi(t)| \approx |v_\xi(t_0)| + c_\xi^1 |t - t_0|^k \quad \text{and} \quad |v_\xi(t)| \approx c_\xi^k |t - t_0|^{k-1}. \]

**Proof of Theorem.** The proof is based in a square function argument following the ideas in [W, NSW].

Let \( \varphi \in C_0^\infty(\mathbb{R}) \) be nonnegative, with \( \varphi \equiv 1 \) on \([-\frac{1}{4}, \frac{1}{4}] \) and such that \( \int_{-\infty}^{\infty} \varphi(t) \, dt = 1 \). For \( h > 0 \) let \( \varphi_h(t) = h^{-1} \varphi(h^{-1}t) \) and let \( \omega_j = \gamma(j2^{-m}) \).

For \( 0 \leq j \leq 2^m \) let
\[ T_{h,j}^m f(x) = \int_{-\infty}^{\infty} f(x - t\omega_j) \varphi_h(t) \, dt, \quad T^m_f(x) = \sup_{h,j} |T_{h,j}^m f(x)|. \]
Then a simple geometric argument shows that it suffices to prove that
\[ \|T^m f\|_{L^2} \leq C_\gamma m \|f\|_{L^2} \quad \text{for} \quad f \geq 0. \]

For \( m = 1 \), (6) follows from the boundedness of the one-dimensional Hardy-Littlewood maximal operator. Suppose (6) is true for \( m = 1 \). Then for \( f \geq 0 \)
\[ T^m f(x) \leq \sup_{h,j} |T_{h,j}^1 f(x) - T_{h,j-1}^1 f(x)| \]
and (6) will follow if we can show that
\[ \|\sup_{h,j} H_{h,j}^m f\|_{L^2} \leq C_\gamma \|f\|_{L^2}. \]

For \( j = 1, \ldots, 2^m \), let \( \Gamma_j \) be the cone \( \{ \xi \in \mathbb{R}^n: |\xi \cdot \omega_j| > c_1 2^{-jL} |\xi| \} \) and let \( K_j \) be the complement of \( \Gamma_j \), and for \( j = 1, \ldots, 2^{m-1} \), let \( f(x) = f_j(x) + r_j(x) \) where \( f_j(\xi) = \chi_{K_{2j} \cup K_{2j-1}}(\xi) \hat{f}(\xi) \). \( \hat{f} \) denotes the Fourier transform.
An argument similar to the one in [W, p. 88] shows that
\[
\sup_{h,j} H^m_{h,j} f(x) \leq C (g_1(f)(x) + g_2(f)(x))
\]
where
\[
g_1(f)(x) = \left( \sum_{j=1}^{2^{m-1}} \left( \sup_h T_{h,2j}^m |r_j(x)| + \sup_h T_{h,2j-1}^m |r_j(x)| \right)^2 \right)^{1/2},
\]
\[
g_2(f)(x) = \left( \int_0^\infty \sum_{j=1}^{2^{m-1}} \left| T_{h,2j}^m f_j(x) - T_{h,2j-1}^m f_j(x) \right|^2 \frac{dh}{h} \right)^{1/2}.
\]
Thus (8) and hence the theorem will be a consequence of the following two estimates:
\[
(9) \quad \|g_1(f)\|_{L^2} \leq C_\gamma \|f\|_{L^2},
\]
\[
(10) \quad \|g_2(f)\|_{L^2} \leq C_\gamma \|f\|_{L^2}.
\]

**Proof of (9).** By the boundedness of the one-dimensional Hardy-Littlewood maximal operator and Plancherel's theorem, one has
\[
(11) \quad \|g_1(f)\|_{L^2}^2 \leq C \int_{\mathbb{R}^n} \left( \sum_{j=1}^{2^{m-1}} |r_j(x)|^2 \right) dx = C \int_{\mathbb{R}^n} \sum_{j=1}^{2^{m-1}} \chi_{K_{2j} \cup K_{2j-1}}(\xi) |\hat{f}(\xi)|^2 d\xi.
\]
Since the $K_j$'s are conic, it is enough to prove that no $\xi \in S^{n-1}$ belongs to more than $C_\gamma$ of the $K_j$'s. Given $\xi$ in $K_{j_0}$ we have $|\nu(\xi)| \leq c_1 2^{-mL}$. By (4), if $k > c_1 1^L$, then $\xi$ does not belong to $K_{j_0 \pm k}$. Thus $\xi$ does not belong to more than $N_\gamma c_1^{1/2}$ of the $K_j$'s.

**Proof of (10).** Plancherel's theorem implies that
\[
(12) \quad \|g_2(f)\|_{L^2}^2 = \int_{\mathbb{R}^n} \sum_{j=1}^{2^{m-1}} \int_0^\infty |\phi(h \xi \cdot \omega_{2j}) - \phi(h \xi \cdot \omega_{2j-1})|^2 |\hat{f}_j(\xi)|^2 \frac{dh}{h} d\xi
\]
\[
= \int_{\mathbb{R}^n} m(\xi) |\hat{f}(\xi)|^2 d\xi,
\]
where
\[
(13) \quad m(\xi) = \sum_{j=1}^{2^{m-1}} \int_0^\infty |\phi(h \xi \cdot \omega_{2j}) - \phi(h \xi \cdot \omega_{2j-1})|^2 \chi_{\Gamma_{2j} \cap \Gamma_{2j-1}}(\xi) \frac{dh}{h},
\]
and we are left to prove that $m(\xi) \leq C_\gamma$. This is accomplished by dividing the curve $\gamma$ in pieces where one has control over the decay of $\phi(h \xi \cdot \omega_j)$ in $\xi$ and $j$ in estimating (13). The details are below.

By Lemma 2 we can, for each $\xi$, split the sum in (13) in no more than $N_\gamma$ sums of the form $\sum_{j_2-m \in U_\xi^1}$, $\sum_{j_2-m \in V_\xi}$, and $\sum_{j_2-m \in W_\xi}$. Thus the theorem will follow if we can show that each of these sums is bounded with bound independent of $m$ and $\xi$. By homogeneity we only need to consider $\xi \in S^{n-1}$. 

Since \(|\xi \cdot \omega_j| \geq c_\gamma\) for \(j^{2^m-1} \in U_\xi^j\), and since \(\hat{\phi}\) is a Schwartz function, we have that \(|\hat{\phi}(h\xi \cdot \omega_{j+1}) - \hat{\phi}(h\xi \cdot \omega_j)|^2 \leq Ch^2|\omega_{j+1} - \omega_j|^2|\hat{\phi}'(h\xi \cdot u_j)|^2\) with \(|\xi \cdot u_j| \geq c_\gamma\). This implies

\[
\sum_{j^{2^m-1} \in U_\xi^j} \int_0^\infty |\hat{\phi}(h\xi \cdot \omega_{j+1}) - \hat{\phi}(h\xi \cdot \omega_j)|^2 \frac{dh}{h} \leq C_\gamma.
\]

We now prove a similar estimate for \(W_\xi^j\). There is no lack of generality in assuming that \(W_\xi^j = [0, \epsilon]\) and that \(v_\xi(0) = 0\). Lemma 2 implies that for \(j^{2^m-1} \in W_\xi^j\)

\[
|\xi \cdot (\omega_{2j} - \omega_{2j-1})| \leq Cc_\xi^j - 1 \cdot 2^{-m_k},
\]

\[
|\xi \cdot \omega_j| \geq Cc_\xi^j 2^{-m_k}.
\]

Since \(\hat{\phi}\) is smooth and rapidly decreasing, by (15) and (16) we obtain

\[
|\hat{\phi}(h\xi \cdot \omega_{2j}) - \hat{\phi}(h\xi \cdot \omega_{2j-1})|^2 \leq Cc_\xi^j 2^{2(k-1) - 2m_k} h^2,
\]

\[
|\hat{\phi}(h\xi \cdot \omega_j)|^2 \leq Cc_\xi^{-2\alpha} j^{-2\alpha} 2^{-2m_k} h^{-2\alpha}.
\]

Splitting each integral in \(\int_0^{a_j} + \int_{a_j}^\infty\) where the \(a_j\)'s are to be determined later and using (17) and (18) on each integral respectively we obtain that \(\sum_{j^{2^m-1} \in W_\xi^j}\) is dominated by

\[
\sum_{j^{2^m-1} \in W_\xi^j} Cc_\xi^j 2^{2(k-1) - 2m_k} a_j^2 + Cc_\xi^{-2\alpha} j^{-2\alpha} 2^{-2m_k} a_j^{-2\alpha}.
\]

To finish, put \(\beta = k - \frac{1}{4}\), \(\alpha = 3\), and let \(a_j = c_\xi^{-1} j^{-\beta} 2^{m_k}\) in (19) obtaining

\[
\sum_{j^{2^m-1} \in W_\xi^j} \leq C \sum_{1}^{2^{m+1}} j^{2(k-\beta-1)} + j^{2\alpha(\beta-k)} \leq C \sum_{1}^{\infty} j^{-3/2}.
\]

The terms \(\sum_{j^{2^m-1} \in W_\xi^j}\) can be handled similarly with \(k = 1\).

Since (20) is independent of \(\xi\) and \(m\), the proof is complete.

References


Department of Mathematics and Statistics, University of South Alabama, Mobile, Alabama 36688

E-mail address: FJAB@USOUTHAL.BITNET