Ergod. Th. & Dynam. Sys. (2005), **25**, 605–628
DOI: 10.1017/S0143385704000732

Sub-actions for Anosov flows

A. O. LOPES† and PH. THIEULLEN‡

† *Instituto de Matemática, UFRGS, Porto Alegre 91501-970, Brazil*
(e-mail: alopes@mat.ufrgs.br)
‡ *Département de Mathématiques, Université Paris-Sud, 91405 Orsay cedex, France*
(e-mail: philippe.thieullen@math.u-bordeaux.fr)

(*Received* 14 *April* 2003 *and accepted in revised form* 23 *July* 2003)

Abstract. Let $(M, \{\phi^t\})$ be a smooth (not necessarily transitive) Anosov flow without fixed points generated by a vector field $X(x) = (d/dt)|_{t=0}\phi^t(x)$ on a compact manifold M. Let $A : M \to \mathbb{R}$ be a globally Hölder function defined on M. Assume that $\int_0^T A \circ \phi^t(x)\,dt \geq 0$ for any periodic orbit $\{\phi^t(x)\}_{t=0}^{t=T}$ of period T. Then there exists a Hölder function $V : M \to \mathbb{R}$, called a sub-action, smooth in the flow direction, such that

$$A(x) \geq L_X V(x), \quad \text{for all } x \in M$$

(where $L_X V = (d/dt)|_{t=0} V \circ \phi^t(x)$ denotes the Lie derivative of V). If A is $\mathcal{C}^r$ then $L_X V$ is $\mathcal{C}^r$ on any local center-stable manifold.

1. *Notations and main results*

We consider a compact Riemannian manifold M and a smooth Anosov flow $\{\phi^t\}$ on M without fixed points ($\mathcal{C}^2$ would be enough but then V could not be more than $\mathcal{C}^2$ in the center-stable direction). By hypothesis, the tangent space TM splits continuously into three sub-bundles:

$$T_x M = E_x^u \oplus E_x^0 \oplus E_x^s,$$

where E_x^0 has dimension one and is generated by the direction of the flow $X(x)$, which is non-zero by hypothesis, and E_x^u and E_x^s are respectively the unstable and stable directions and their dimensions are respectively u and s. There exist constants K, $\Lambda^s < \lambda^s < 0 < \lambda^u < \Lambda^u$ such that

$$K^{-1}\exp(t\lambda^u)\|v\| \leq \|T_x\phi^t(v)\| \leq K\exp(t\Lambda^u)\|v\|, \quad \text{for all } v \in E_x^u,$$
$$K^{-1}\exp(t\Lambda^s)\|v\| \leq \|T_x\phi^t(v)\| \leq K\exp(t\lambda^s)\|v\|, \quad \text{for all } v \in E_x^s.$$

We also consider an observable $A : M \to \mathbb{R}$ and define its minimal average $m(A)$:

$$m(A) = \inf\left\{ \int A\,d\mu \,\middle|\, \mu \in \mathcal{M}_1(M, \phi^t) \right\}$$

where $\mathcal{M}_1(M, \phi^t)$ denotes the set of all $\{\phi^t\}$-invariant probability measures. A measure μ which attains the infimum is called a minimizing measure (or ground-state measure for A, as in [**8**]). Such measures always exist by compactness of $\mathcal{M}_1(M, \phi^t)$. In the sequel A is assumed to be (at least) Hölder, that is, A satisfies

$$|A(x) - A(y)| \le Cd(x, y)^\alpha$$

for some constants $C > 0$ and $\alpha \in]0, 1]$. The main result we prove is the following.

THEOREM 1. *Let $A : M \to \mathbb{R}$ be a Hölder observable on M. Then there exists $V : M \to \mathbb{R}$ Hölder, smooth along the flow direction, such that for every $x \in M$ and every $T > 0$*

$$\int_0^T A \circ \phi^t(x)\, dt \ge V \circ \phi^T(x) - V(x) + m(A)T$$

or, equivalently,

$$\text{for all } x \in M, \quad A(x) = m + L_X V(x) + H(x), \quad H(x) \ge 0 \quad \text{and} \quad m(H) = 0,$$

for some constant m and some function H smooth along the flow direction, globally $\alpha\beta$-Hölder, of Hölder norm depending uniformly with respect to A,

$$\text{Höld}_{\alpha\beta}(H) \le K\, \text{Höld}_\alpha(A), \quad 0 < \beta < 1,$$

for some constants K, β depending only on the flow and not on A. Furthermore, if A is $\mathcal{C}^r$ on M, V and H are also $\mathcal{C}^r$ along the center-stable direction.

The function V which appears in Theorem 1 is certainly not unique; we call such functions 'sub-actions'. They are analogous to weak KAM (see [**5**]) sub-solutions or 'dominated functions' of the Hamilton–Jacobi equation in Fathi and Siconolfi's theory [**5–7**]. Notice also that, for a non-negative and continuous H, $m(H) = 0$ if and only if $H = 0$ on the support of some invariant measure. On the support of such measures, A is cohomologous to a constant. We notice that, by changing the direction of the flow, our construction gives another sub-action which is, for instance, $\mathcal{C}^r$ in the center-unstable direction if A is $\mathcal{C}^r$ on M. We do not know how to construct sub-actions using thermodynamic formalism as we did in [**4**].

Our proof gives a global regularity for V smaller than that of A. In some cases, V may have more regularity. Suppose N is a compact Riemannian manifold with negative curvature and $M = T_1N$ is the unitary tangent bundle. Let us denote by $\{\phi^t\}$ the geodesic flow on M and by $\pi : M \to N$ the canonical projection. As the referee pointed out to us, the following proposition can be deduced from [**6**] using the notion of 'dominated function'. We will reproduce both the referee's proof implying a $\mathcal{C}^1$ regularity for V and our initial proof which just implies a Lipschitz regularity and can be seen as a simple example to help readers understand the proof of our main Theorem 1.

PROPOSITION 2. *For every closed 1-form $\omega \in \Omega(N)$ (seen as a function on T_1N), there exists a $\mathcal{C}^1$ super-action $V : N \to \mathbb{R}$ such that, for all $(x, v) \in T_1N$,*

$$\int_0^T \omega \circ \phi^t(x, v)\, dt \le T\|\omega\|_s + V \circ \pi \circ \phi^T(x, v) - V \circ \pi(x, v),$$

where $\|\omega\|_s$ denotes the stable norm of ω defined by

$$\|\omega\|_s = \sup\left\{ \int \omega(x, v)\, d\mu(x, v) \,\middle|\, \mu \in \mathcal{M}_1(M, \phi^t) \right\}.$$

We refer the reader to [**12**] for definitions and properties about the stable norm. In [**10**, section 2], we established the existence of sub-actions for Anosov diffeomorphisms. The proof is simpler and it may help the reader to understand it before reading the proof of Proposition 4.

2. *Plan of the proof*

We explain in this section the main steps of the proof of Theorem 1. The proof is divided essentially into two parts. In the first part we construct a special Poincaré section of the flow and show the existence of a sub-action for a discretized observable. In the second part we extend the previous sub-action to the whole manifold so that it stays globally Hölder and smooth along any local center-stable manifold.

Although the notion of a Poincaré section is simple, our construction depends on the choice of a particular section and we spend some time giving details of the precise definitions.

Definition 3. A *Poincaré section of uniform size* is a family (Σ, γ) of transverse charts where Σ is a disjoint union of open sets $\{\Sigma_i\}_{i\in I}$ of $\mathbb{R}^{u+s}$, each containing a ball $B(0, \epsilon^*)$, and γ_i is a smooth diffeomorphism defined on a neighborhood of $\overline{\Sigma_i} \times [0, \tau^*]$ onto an open set of M, for some $\tau^* > 0$. We assume the following.

(i) M is covered by the union of sets $U_i = \gamma_i(\Sigma_i \times \,]0, \tau^*[)$.

(ii) For each $x \in \Sigma_i$, $0 \le s < t \le \tau^*$, $\phi^{t-s} \circ \gamma_i(x, s) = \gamma_i(x, t)$.

(iii) $\gamma_i(\Sigma_i)$ is tangent to $E^u_{\gamma_i(0)} \oplus E^s_{\gamma_i(0)}$ at $\gamma_i(0)$ (in order to simplify the notations we use $\gamma_i(x)$ instead of $\gamma_i(x, t)$ when $t = 0$).

(iv) The $\mathcal{C}^2$-size of γ_i and $(\gamma_i)^{-1}$ is bounded from above by a constant K^*, the angle of the direction of the flow and the tangent space of $\gamma_i(\Sigma_i)$ are bounded from below by a constant $(K^*)^{-1}$.

(v) The sections $\gamma_i(\overline{\Sigma_i})$ are pairwise disjoint. The return time between two successive sections is bounded from below by a uniform constant τ_*.

If we want to be more precise, we say that (Σ, γ) is of, uniform size ϵ^*, τ_*, τ^*, K^*. We will also use the term 'weak Poincaré section' to mean a Poincaré section satisfying assumptions (i)–(iv) but not (v).

To any Poincaré section (Σ, γ) we associate a return map ψ defined in the following way (see Figure 1). Let Σ be the disjoint union of each $\gamma_i(\Sigma_i)$, (we intentionally use the same letter). If x belongs to $\gamma_i(\Sigma_i)$, let $\tau(x)$ be the smallest $t \ge 0$ such that $\phi^t \circ \gamma_i(x)$ belongs to $\gamma_j(\Sigma_j)$ for some $j \ne i$. Such a τ exists since $\phi^{\tau^*} \circ \gamma_i(x)$ belongs to U_j for some $j \ne i$. Moreover, τ satisfies $\tau(x) < \tau^*$. The return map is then defined by $\psi(x) = \gamma_j^{-1} \circ \phi^{\tau(x)} \circ \gamma_i(x)$.

The reader will have noticed that our Poincaré map is not defined on the manifold itself but rather through the charts $\{\gamma_i\}_{i\in I}$. In the first part of the proof, we show the existence

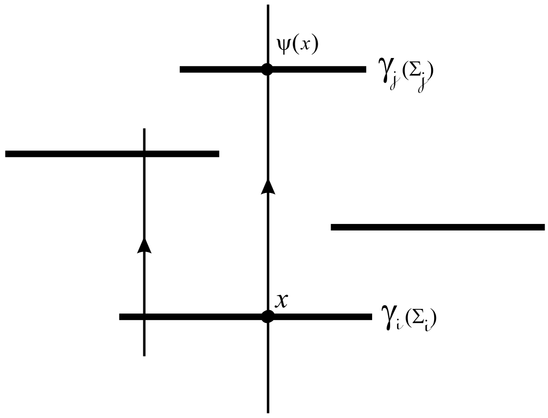

FIGURE 1. A Poincaré section.

of a discretized sub-action $\mathcal{V}$ for the associated discretized observable

$$\mathcal{A}(x) = \int_0^{\tau(x)} (A - m(A)) \circ \phi^t \circ \gamma(x)\, dt.$$

In the sequel, W^{cs}_{loc} denotes any local center-stable manifold; similarly W^{s}_{loc} denotes any local strong-stable manifold. They are both embedded sub-manifolds of M. We will later define similar sub-manifolds $W^{s}_{\mathrm{loc}}(\omega)$, $W^{u}_{\mathrm{loc}}(\omega)$; but they will be sub-manifolds of $\mathbb{R}^{u+s}$.

PROPOSITION 4. *There exists a Poincaré section* (Σ, γ) *and a globally Hölder function* $\mathcal{V} : \Sigma \to \mathbb{R}$ *satisfying*

$$\mathcal{A}(x) \geq \mathcal{V} \circ \psi(x) - \mathcal{V}(x) \quad \text{for all } x \in \Sigma.$$

There exist constants K, β *depending only on the flow such that*

$$\text{Höld}_{\alpha\beta}(\mathcal{V}) \leq K\, \text{Höld}_{\alpha}(\mathcal{A}), \quad 0 < \beta < 1.$$

Moreover, $\mathcal{V}$ *is* C^r *on* $\gamma^{-1}(W^{cs}_{\mathrm{loc}}) \cap \Sigma$ *if* A *is* C^r *on* M.

Notice that $\mathcal{V}$ is Hölder although $\mathcal{A}$ is not even continuous. The proof of the existence of $\mathcal{V}$ is similar to the case of Anosov diffeomorphisms [**10**]. The exponent β is related to the Hölder regularity of the stable foliation. In the proof of Proposition 14(iii), an explicit formula for β is given:

$$\beta = \frac{-\lambda^s_*}{\Lambda^u_* - \lambda^s_*}$$

where λ^s_* and Λ^u_* can be chosen as close as we want to $\min(\tau)\lambda^s$ and $\max(\tau)\Lambda^u$ by letting the diameter of the section go to 0. In the second part of the proof, we extend $\mathcal{V}$ to the whole space M.

PROPOSITION 5. *We use the notation of Proposition 4. There exists a Poincaré sub-section* (Σ', γ'), *(*$\Sigma' \subset \Sigma$ *and* γ' *denotes the restriction of* γ *to* Σ'*) and a non-negative*

Hölder function $H' : M \to \mathbb{R}^+$, smooth along the flow direction, null in a neighborhood of $\gamma'(\overline{\Sigma'})$, such that for any $x \in \Sigma'$

$$\mathcal{A}'(x) - (\mathcal{V} \circ \psi'(x) - \mathcal{V}(x)) = \int_0^{\tau'(x)} H' \circ \phi^t \circ \gamma(x)\, dt$$

(where τ', ψ' and $\mathcal{A}'$ have the same definitions as τ, ψ and $\mathcal{A}$ using (Σ', γ') instead of (Σ, γ)). Moreover, H' is $\mathcal{C}^r$ on any W^{cs}_{loc} if A is $\mathcal{C}^r$ on M.

We are now able to give an explicit formula for the sub-action V in Theorem 1. We first define the backward return time

$$T'(x) = \inf\{T \geq 0 \mid \phi^{-T}(x) \in \gamma'(\Sigma')\} \quad \text{for all } x \in M,$$

we then define V first on $\gamma'(\Sigma')$ by $V = \mathcal{V} \circ (\gamma')^{-1}$ and second on M by

$$V(x) = V \circ \phi^{-T'(x)}(x) + \int_{-T'(x)}^{0} (A - m(A) - H') \circ \phi^t(x)\, dt$$

(or equivalently $A = m(A) + L_X V + H'$). Although T' is again highly discontinuous, we claim that the function V just defined possesses all the required properties. Let $x_0 \in M$ and Σ'_i be a section disjoint from x_0 that meets the backward orbit $\{\phi^t(x) \mid -\tau^* \leq t \leq 0\}$. Let T'_i be a smooth backward return time to Σ'_i defined locally about x_0. For any x close to x_0, though the piece of orbit $\{\phi^t(x) \mid -T'_i(x) \leq t \leq 0\}$ may encounter other sections Σ'_k, the property which characterizes H' implies that

$$V(x) = V \circ \phi^{-T'_i(x)}(x) + \int_{-T'_i(x)}^{0} (A - m(A) - H') \circ \phi^t(x)\, dt.$$

This explicit formula proves the claim.

Before going into the proof of Theorem 1 we show how to construct a super-action in a simpler case that may help the reader to understand the general proof. We assume here that $M = T_1N$ is the unitary tangent bundle of a compact Riemannian manifold N of negative curvature and that our observable A is actually a closed 1-form $\omega : T_1N \to \mathbb{R}$ restricted to M. Our proof of Proposition 2 gives only a Lipschitz super-action whereas the use of Fathi and Siconolfi's dominated functions, suggested by the referee, gives a $\mathcal{C}^1$ super-action.

Proof of Proposition 2. Part I. We construct an explicit Lipschitz super-action V:

$$V(x) = \sup\left\{ \int_0^T \omega(c(t), \dot{c}(t))\, dt - T\|\omega\|_s \;\middle|\; c : [0, T] \to N \text{ and } c(T) = x \right\},$$

where the supremum is taken over all piecewise $\mathcal{C}^1$ paths $c : [0, T] \to N$ with constant speed equal to 1 and finishing at x. If x, y are two points in N and $\gamma : [0, d] \to N$ is a minimizing path between these two points, then $d = d(x, y)$ and, by definition of V,

$$V(x) + \int_0^d \omega(\gamma(t), \dot{\gamma}(t))\, dt - d\|\omega\|_s \leq V(y),$$
$$|V(x) - V(y)| \leq 2\|\omega\|_\infty\, d(x, y).$$

V is then Lipschitz. Let us prove that $V \circ \pi$ is a super-action. Let $(x, v) \in M$ and $\gamma : [0, T] \to N$ be a geodesic beginning at (x, v). Then $\phi^t(x, v) = (\gamma(t), \dot\gamma(t))$ and, again by definition of V,

$$V(\gamma(0)) + \int_0^T \omega \circ \phi^t(x, v)\, dt - T\|\omega\|_s \leq V(\gamma(T)).$$

The only non-trivial fact we are left to prove is that V is actually a finite function. Let $c : [0, L] \to N$ be a piecewise $\mathcal{C}^1$ path ending to x. We close this path by joining the two extremities and obtain a new path c_0 of length at most $L + \mathrm{diam}(N)$. The path c_0 is homotopic (as a closed curve) to a path c_1 of minimum length $l(c_1)$. Necessarily c_1 is a closed geodesic and $(1/l(c_1)) \int_0^{l(c_1)} \delta_{(c_1(t),\dot c_1(t))}\, dt$ defines an invariant measure. Then

$$\int_{c_0} \omega = \int_{c_1} \omega \leq l(c_1)\|\omega\|_s \leq l(c_0)\|\omega\|_s,$$

$$\int_0^L \omega(c(t), \dot c(t))\, dt - L\|\omega\|_s \leq \mathrm{diam}(N)\|\omega\|_s,$$

$$V(x) \leq \mathrm{diam}(N)\|\omega\|_s.$$

We have proved that V is a well-defined function.

Part II. We use Fathi and Siconolfi's work to construct a $\mathcal{C}^1$ super-action. Let $L(x, v) = \|v\|_x/2$ be the standard Lagrangian in Riemannian geometry. Mañé's critical value is given by

$$-\alpha(\omega) = \inf \int (L - \omega)\, d\mu$$

where the infimum is taken over all invariant probability measures with compact support in TM. One can show that the support of any minimizing measure is included in the energy level set $\|\omega\|_S$, that $\alpha(\omega) = \|\omega\|_S/2$ and that after renormalizing μ to a probability measure in T^1N, μ is maximizing for ω. Let $U : N \to \mathbb{R}$ be a dominated function associated to $L - \omega$ and $\gamma : [0, T] \to N$ be a curve of constant speed 1. We reparametrize γ so that it has constant speed $\|\omega\|_S = \sqrt{2\alpha}$

$$c(t) = \gamma(t\sqrt{2\alpha}), \quad \text{for all } t \in [0, T/\sqrt{2\alpha}].$$

By definition of U,

$$U \circ c(T/\sqrt{2\alpha}) - U \circ c(0) \leq \alpha T/\sqrt{2\alpha} + \int_0^{T/\sqrt{2\alpha}} (L - \omega)(c(t), \dot c(t))\, dt,$$

or equivalently

$$U \circ \gamma(T) - U \circ \gamma(0) \leq T\|\omega\|_S - \int_0^T \omega(\gamma, \dot\gamma)\, dt.$$

$V = U \circ \pi$ is the super-action we are looking for. □

2.1. *Discretized sub-actions.* We begin the construction of $\mathcal{V}$ by first refining the notion of Poincaré section. The next lemma shows that we can choose $\mathrm{diam}(\Sigma_i)$ as small as we want independently of the length of the return time τ and the uniform size of the section.

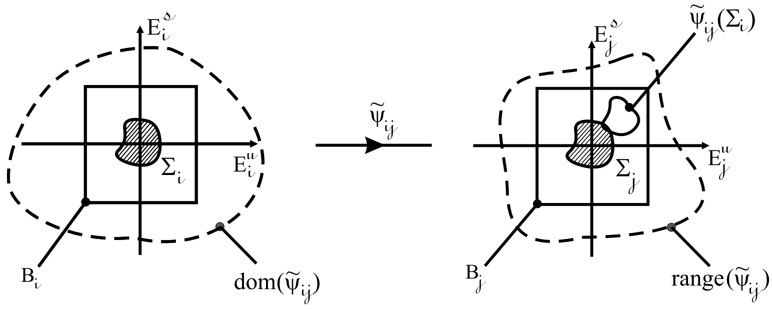

FIGURE 2. Lyapunov charts.

LEMMA 6. *There exist constants* $(\epsilon^*, \tau_*, \tau^*, K^*)$ *such that for any* $\epsilon < \epsilon^*$, *one can construct a weak Poincaré section* $(\tilde{\Sigma}, \tilde{\gamma})$ *of size* $(\epsilon^*, \tau^*, K^*)$ *and a Poincaré sub-section* (Σ, γ) *of small base* ϵ *with the additional properties:*

(i) *for all* $i \in I$, $\Sigma_i = B(0, \epsilon) \subset \tilde{\Sigma}_i$ *and* γ_i *is the restriction of* $\tilde{\gamma}_i$ *to* Σ_i*;*

(ii) *the sets* $\{\gamma_i(\Sigma_i \times]0, \tau^*[)\}_{i \in I}$ *cover* M *and the sections* $\gamma_i(\overline{\Sigma_i})$ *are pairwise disjoint;*

(iii) *the return time* τ *associated to the base* (Σ, γ) *is at least* τ_*.

From now on we choose a weak Poincaré section $(\tilde{\Sigma}, \tilde{\gamma})$ with a sub-section (Σ, γ) of size ϵ; the size ϵ will be determined later and will be small. Let ψ, τ be the return map and return time associated to (Σ, γ).

Definition 7. Let $i, j \in I$. We say that $i \to j$ is a *simple transition* if there exists $x \in \Sigma_i$ such that $\psi(x) \in \Sigma_j$. Let $\tilde{\tau}_{ij}$, $\tilde{\psi}_{ij}$ be the extended return time and return map from $\tilde{\Sigma}_i$ to $\tilde{\Sigma}_j$:

$$\tilde{\tau}_{ij} = \inf\{t \in]0, \tau^*[\mid \phi^t(x) \in \tilde{\Sigma}_j\}, \quad \tilde{\psi}_{ij} = \tilde{\gamma}_j^{-1} \circ \phi^{\tilde{\tau}_{ij}} \circ \tilde{\gamma}_i,$$
$$\mathrm{dom}(\tilde{\psi}_{ij}) = \mathrm{dom}(\tilde{\tau}_{ij}) = \{x \in \tilde{\Sigma}_i \mid \exists t \in]0, \tau^*[\quad \phi^t(x) \in \tilde{\Sigma}_j\}.$$

If ϵ is small enough, $\Sigma_i \subset \mathrm{dom}(\tilde{\psi}_{ij})$ and $\Sigma_j \subset \mathrm{range}(\tilde{\psi}_{ij})$ for any simple transition $i \to j$ (see Figure 2). Since $\tilde{\tau}_{ij}$ is uniformly bounded from below for all transitions independently of ϵ, by choosing ϵ small enough, we can construct a family of norms $\{\|\cdot\|_i\}_i$, called Lyapunov norms, adapted to the hyperbolicity of $\tilde{\psi}_{ij}$.

LEMMA 8. *There exist constants* $\Lambda_*^s < \lambda_*^s < 0 < \lambda_*^u < \Lambda_*^u$ *such that for any* $\delta > 0$, *one can construct a family of norms* $\{\|\cdot\|_i\}_{i \in I}$ *and a family of splittings* $\mathbb{R}^{u+s} = E_i^u \oplus E_i^s$ *such that:*

(i) *for every* $v \in E_i^u$, $w \in E_i^s$, $\|v + w\|_i = \max(\|v\|_i, \|w\|_i)$*;*

(ii) *if* B_i *denotes the unit ball of the norm* $\|\cdot\|_i$, *for any transition* $i \to j$, $\Sigma_i \subset B_i \subset \mathrm{dom}(\tilde{\psi}_{ij})$ *and* $B_j \subset \mathrm{range}(\tilde{\psi}_{ij})$*;*

(iii) *if* $D\tilde{\psi}_{ij}(x) = \begin{bmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{bmatrix}$, $x \in B_i$, *with respect to the splitting* $E_i^u \oplus E_i^s$ *then*

$$\|A_{ij} \cdot v\|_j \geq \exp(\lambda_*^u)\|v\|_i, \quad \|B_{ij} \cdot w\|_j \leq \delta\|w\|_i,$$
$$\|D_{ij} \cdot w\|_j \leq \exp(\lambda_*^s)\|w\|_i, \quad \|C_{ij} \cdot v\|_j \leq \delta\|v\|_i;$$

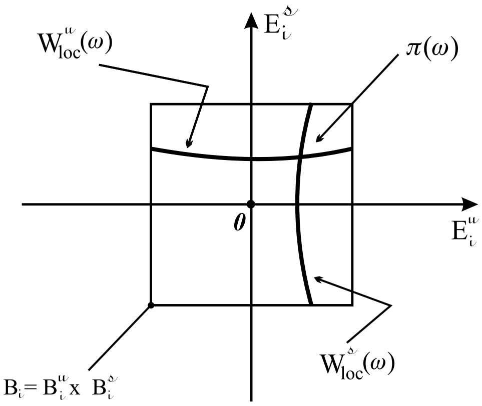

FIGURE 3. Bowen's shadowing lemma.

(iv) *for any $x \in B_i$ and any chain of simple transitions $i_0 \to \cdots \to i_n$ such that $\tilde{\psi}^n = \tilde{\psi}_{i_{n-1}i_n} \circ \cdots \circ \tilde{\psi}_{i_0 i_1}$ exists locally about x, for any $v \in \mathbb{R}^{u+s}$,*

$$(K^*)^{-1} \exp(n\Lambda^s_*)\|v\|_{i_0} \le \|D\tilde{\psi}^n(x) \cdot v\|_{i_n} \le K^* \exp(n\Lambda^u_*)\|v\|_{i_0}.$$

We actually show that $\Lambda^u_* = \tau^*\Lambda^u$, $\Lambda^s_* = \tau^*\Lambda^s$ and that λ^u_* and λ^s_* can be any real numbers satisfying $\tau_*\lambda^s < \lambda^s_* < 0 < \lambda^u_* < \tau_*\lambda^u$. The existence of Lyapunov charts enables us to use Bowen's shadowing lemma along pseudo-orbits. A pseudo-orbit is a doubly sided sequence of simple transitions. Let Ω be the set

$$\Omega = \{(\ldots, \omega_{-1}|\omega_0, \omega_1, \ldots) \mid \omega_i \to \omega_{i+1} \text{ is a simple transition } \forall\, i \in \mathbb{Z}\}.$$

We notice that Ω is a sub-shift of finite type and we denote by $\sigma : \Omega \to \Omega$ the associated left shift. For each $\omega \in \Omega$, define

$$\tilde{\psi}_\omega = \tilde{\psi}_{\omega_0\omega_1} : B_{\omega_0} \to \tilde{\Sigma}_{\omega_1}, \quad \tilde{\psi}^{-1}_\omega = (\tilde{\psi}_{\omega_{-1}\omega_0})^{-1}, \quad \tilde{\tau}_\omega = \tilde{\tau}_{\omega_0\omega_1},$$

and, more generally, $\tilde{\psi}^n_\omega = \tilde{\psi}_{\sigma^{n-1}(\omega)} \circ \cdots \circ \tilde{\psi}_\omega$. The following proposition is standard; it uses the theory of graph transform and will not be proved (see Figure 3).

PROPOSITION 9. (Bowen's shadowing lemma) *For each $\omega \in \Omega$, define*

$$W^s_{\mathrm{loc}}(\omega) = \{x \in B_{\omega_0} \mid \forall n \ge 0,\ \tilde{\psi}^n_\omega(x) \in B_{\omega_n}\},$$
$$W^u_{\mathrm{loc}}(\omega) = \{x \in B_{\omega_0} \mid \forall n \ge 0,\ \tilde{\psi}^{-n}_\omega(x) \in B_{\omega_{-n}}\}.$$

Then the local stable $W^s_{\mathrm{loc}}(\omega)$ and unstable $W^u_{\mathrm{loc}}(\omega)$ manifolds satisfy:
(i) *$W^s_{\mathrm{loc}}(\omega)$ and $W^u_{\mathrm{loc}}(\omega)$ are $\mathcal{C}^2$ graphs above $E^s_{\omega_0}$ and $E^u_{\omega_0}$;*
(ii) *$W^s_{\mathrm{loc}}(\omega)$ depends only on $(\omega_0, \omega_1, \ldots)$ and $W^u_{\mathrm{loc}}(\omega)$ on $(\ldots, \omega_{-1}, \omega_0)$;*
(iii) *for any $\omega, \zeta \in \Omega$, $W^s_{\mathrm{loc}}(\omega)$ intersects $W^u_{\mathrm{loc}}(\zeta)$ at a unique point* $[\omega, \zeta]$.

We call $B = \cup_{i \in I} B_i$ the *disjoint union* of all balls B_i and we introduce the more condensed notation:

$$\tilde{\psi} = \begin{cases} \Omega \times B \to \Omega \times \tilde{\Sigma}, \\ (\omega, x) \mapsto (\sigma(\omega), \tilde{\psi}_\omega(x)). \end{cases}$$

We also extend our previous discretized $\mathcal{A}$ to the space $\Omega \times B$ by:

$$\tilde{\mathcal{A}}(\omega, x) = \int_0^{\tilde{\tau}(\omega,x)} \big(A - m(A)\big) \circ \phi^t \gamma(x)\, dt.$$

For each $\omega \in \Omega$ we define $\pi(\omega)$, the unique point at the intersection of the local stable and unstable manifolds,

$$\pi(\omega) = [\omega, \omega] = W^s_{\mathrm{loc}}(\omega) \cap W^u_{\mathrm{loc}}(\omega).$$

For each i let R_i be the projection $\pi\{\omega \in \Omega \mid \omega_0 = i\}$ of the ith cylinder of Ω and R the disjoint union of all these $\{R_i\}_{i\in I}$. The space $\Omega \times R$ is again not invariant under $\tilde{\psi}$ but possesses an additional property called the *Markov property*. Bowen also uses a notion of Markov rectangles in [**1–3**], but our approach is different.

LEMMA 10. *Let* $\omega \in \Omega$.

(i) *The local stable manifold restricted to the rectangle R can be written as*

$$\begin{aligned} W^s_{\mathrm{loc}}(\omega) \cap R_\omega &= \{[\omega, \zeta] \mid \zeta \in \Omega,\ \zeta_0 = \omega_0\} \\ &= \{\pi(\zeta) \mid \eta \in \Omega,\ \zeta_n = \omega_n,\ \forall n \geq 0\}. \end{aligned}$$

(ii) $\tilde{\psi}_\omega$ *stabilizes the local stable manifold*

$$\tilde{\psi}_\omega(W^s_{\mathrm{loc}}(\omega) \cap R_\omega) \subset W^s_{\mathrm{loc}}(\sigma(\omega)) \cap R_{\sigma(\omega)}.$$

(iii) $\tilde{\psi}_\omega^{-1}$ *stabilizes the local unstable manifold*

$$\tilde{\psi}_\omega^{-1}(W^u_{\mathrm{loc}}(\omega) \cap R_\omega) \subset W^u_{\mathrm{loc}}(\sigma^{-1}(\omega)) \cap R_{\sigma^{-1}(\omega)}.$$

We can see $(\Omega \times R, \tilde{\psi})$ as an extension of (Σ, ψ) in the following way: each $x \in \Sigma$ admits a canonical pseudo-orbit $\theta(x) \in \Omega$ by taking the successive sections which $\psi^n(x)$ crosses

$$\theta(x) = (\ldots, \theta_{-1}|\theta_0, \theta_1, \ldots) \quad \text{where } \psi^n(x) \in \Sigma_{\theta_n}, \text{ for all } n \in \mathbb{Z}.$$

The pseudo-orbit $\theta(x)$ satisfies $\pi \circ \theta(x) = x$ for all $x \in \Sigma$ and defines a discontinuous injection $\tilde{\theta}$ into $\tilde{\Omega} = \mathrm{graph}(\pi)$. We then obtain a topological dynamical system $(\tilde{\Omega}, \tilde{\psi})$, in the usual sense, which commutes with (Σ, ψ):

$$\tilde{\theta} \begin{cases} \Sigma \to \tilde{\Omega}, \\ x \mapsto (\theta(x), x), \end{cases} \qquad \begin{array}{ccc} \Sigma & \xrightarrow{\psi} & \Sigma \\ \downarrow{\scriptstyle\tilde{\theta}} & & \downarrow{\scriptstyle\tilde{\theta}} \\ \tilde{\Omega} & \xrightarrow{\tilde{\psi}} & \tilde{\Omega} \end{array}$$

Moreover, $\tilde{\mathcal{A}}$ can be seen as an extension of $\mathcal{A}$: $\mathcal{A} = \tilde{\mathcal{A}} \circ \tilde{\theta}$.

Our main goal is, first, to find a sub-action $\tilde{\mathcal{V}} : \tilde{\Omega} \to \mathbb{R}$ satisfying $\tilde{\mathcal{A}} \geq \tilde{\mathcal{V}} \circ \tilde{\psi} - \tilde{\mathcal{V}}$ on $\tilde{\Omega}$, second to show that $\tilde{\mathcal{V}}$ can be of the form $\tilde{\mathcal{V}} = \mathcal{V} \circ \pi$ for some $\mathcal{V} : R \to \mathbb{R}$ and third to show that $\mathcal{V}$ is Hölder on Σ. We first define two cocycles along the stable leaves.

Definition 11. For any $(\omega, x) \in \Omega \times B$ such that $x \in W^s_{\mathrm{loc}}(\omega)$ we define

$$\begin{aligned} b^s(\omega, x) &= \sum_{n\geq 0} \tilde{\tau} \circ \tilde{\psi}^n(\omega, x) - \tilde{\tau} \circ \tilde{\psi}^n(\omega, \pi(\omega)), \\ \Delta^s(\omega, x) &= \sum_{n\geq 0} \tilde{\mathcal{A}} \circ \tilde{\psi}^n(\omega, x) - \tilde{\mathcal{A}} \circ \tilde{\psi}^n(\omega, \pi(\omega)), \\ w^s(\omega, x) &= \phi^{b^s(\omega,x)} \circ \gamma(x). \end{aligned}$$

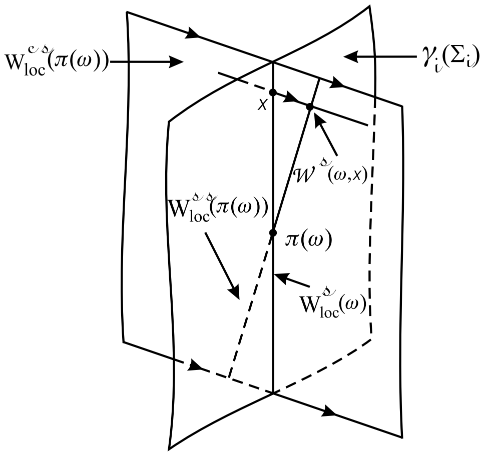

FIGURE 4. Strong-stable manifold.

The two series converge because both x and $\pi(\omega)$ are on the same stable manifold.

PROPOSITION 12. *For any $\omega \in \Omega$:*

(i) *the map $x \in W^s_{\mathrm{loc}}(\omega) \mapsto w^s(\omega, x)$ is a parametrization of the local strong stable manifold of the flow passing through $\gamma \circ \pi(\omega)$ (see Figure 4);*

(ii) *$\gamma\big(W^s_{\mathrm{loc}}(\omega)\big)$ is equal to the intersection of $\gamma(B_\omega)$ and the local center-stable manifold $W^{cs}_{\mathrm{loc}}\big(\gamma \circ \pi(\omega)\big)$ passing through $\gamma \circ \pi(\omega)$;*

(iii) *the stable cocycle Δ^s admits the equivalent form;*

$$\Delta^s(\omega, x) = \int_0^\infty (A \circ \phi^t \circ w^s(\omega, x) - A \circ \phi^t \circ \gamma(\pi(\omega)))\, dt$$
$$+ \int_0^{b^s(\omega,x)} (A - m(A)) \circ \phi^t \circ \gamma(x)\, dt.$$

The proposition says in particular that, if $\pi(\omega) = \pi(\omega')$ and $x \in W^s_{\mathrm{loc}}(\omega)$, then

$$W^s_{\mathrm{loc}}(\omega) = W^s_{\mathrm{loc}}(\omega'), \quad b^s(\omega, x) = b^s(\omega', x),$$
$$\Delta^s(\omega, x) = \Delta^s(\omega', x), \quad w^s(\omega, x) = w^s(\omega', x).$$

We are now able to define a discretized sub-action $\tilde{\mathcal{V}}$. Let $S_n\tilde{\mathcal{A}} = \sum_{k=0}^{n-1} \tilde{\mathcal{A}} \circ \tilde{\psi}^k$ be the Birkhoff sum of $\tilde{\mathcal{A}}$. The cocycle Δ^s is similar to what Bowen uses to prove that any Hölder function on the two sided shift is cohomologous to one which depends only on positive coordinates.

Definition 13. For any $\omega \in \Omega$ let

$$\tilde{\mathcal{V}}(\omega) = \inf\{S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\omega, \zeta]) + \Delta^s(\omega, [\omega, \zeta]) \mid n \geq 0,\ \zeta \in \Omega,\ \zeta_0 = \omega_0\}$$

where the infimum is taken over all $n \geq 0$ and all $\zeta \in \Omega$ satisfying $\zeta_0 = \omega_0$.

We are using for the first time the particular choice of the normalizing constant $m(A)$ to guarantee that the infimum is finite. Proposition 4 is then a consequence of the following.

PROPOSITION 14. *The sub-action $\tilde{\mathcal{V}}$ defined previously satisfies:*

(i) *when $\tilde{\mathcal{V}}$ is restricted to $\tilde{\Omega} = \mathrm{graph}(\pi)$ we have $\tilde{\mathcal{A}} \geq \tilde{\mathcal{V}} \circ \tilde{\psi} - \tilde{\mathcal{V}}$ on $\tilde{\Omega}$;*

(ii) *there exists $\mathcal{V} : R \to \mathbb{R}$ such that $\tilde{\mathcal{V}}(\omega) = \mathcal{V} \circ \pi(\omega)$ for all $\omega \in \Omega$;*

(iii) *$\mathcal{V}$ is globally Hölder on Σ and has the same regularity as A on any $W^s_{\mathrm{loc}} \cap \Sigma$.*

The proof of Proposition 4 follows easily: we know that $\tilde{\mathcal{A}} \geq \tilde{\mathcal{V}} \circ \tilde{\psi} - \tilde{\mathcal{V}}$; since $\mathcal{A} = \tilde{\mathcal{A}} \circ \tilde{\theta}$, $\mathcal{V} = \tilde{\mathcal{V}} \circ \tilde{\theta}$ and $\tilde{\psi} \circ \tilde{\theta} = \tilde{\theta} \circ \psi$, we then obtain $\mathcal{A} \geq \mathcal{V} \circ \psi - \mathcal{V}$ on Σ.

2.2. *Extension of discretized sub-actions.* We constructed in the first part a weak Poincaré section $(\tilde{\Sigma}, \tilde{\gamma})$ of uniform size $(\epsilon_*, \tau_*, K_*)$ and a Poincaré sub-section (Σ, γ), where each $\Sigma_i \subset \tilde{\Sigma}_i$ has a diameter at most ϵ and ϵ can be as small as we want compared with the size of $(\tilde{\Sigma}, \tilde{\gamma})$ and the minimal return time τ_* on Σ. We also constructed a sub-action $\mathcal{V} : \Sigma \to \mathbb{R}$ satisfying

$$\int_0^{\tau(x)} (A - m(A)) \circ \phi^t \circ \gamma(x)\, dt \geq \mathcal{V} \circ \psi(x) - \mathcal{V}(x)$$

where (τ, ψ) is the return time and the return map to the section Σ. We now choose a Poincaré sub-section (Σ', γ') of (Σ, γ) as given in the following lemma.

LEMMA 15. *There exists for each i an open subset Σ'_i such that, denoting by γ'_i the restriction of γ_i to Σ'_i:*

(i) $\overline{\Sigma'_i} \subset \Sigma_i$;

(ii) *the sets $U'_i = \gamma'_i(\Sigma'_i \times \,]0, \tau^*[)$ cover M (as do $U_i = \gamma_i(\Sigma_i \times \,]0, \tau^*[)$);*

(iii) *for any i, j, U_i intersects $\gamma_j(\Sigma_j)$ if and only if U'_i intersects $\gamma'_j(\Sigma'_j)$.*

We now introduce the return time and return map (τ', ψ') associated to the Poincaré sub-section (Σ', γ'), and for any $x \in \Sigma'$ we define

$$\mathcal{H}'(x) = \int_0^{\tau'(x)} (A - m(A)) \circ \phi^t \circ \gamma'(x)\, dt - (\mathcal{V} \circ \psi'(x) - \mathcal{V}(x)) \geq 0.$$

Our final goal is to find $H' : M \to \mathbb{R}^+$ smooth in the flow direction, globally Hölder, $\mathcal{C}^r$ along any W^{cs}_{loc} if A is $\mathcal{C}^r$ and such that for any $x \in \Sigma'$,

$$\mathcal{H}'(x) = \int_0^{\tau'(x)} H' \circ \phi^t \circ \gamma'(x)\, dt.$$

This construction uses a refinement of the notion of transitions (see Figure 5).

Definitions 16.

(i) Let i, j be given, we say that $i \Rightarrow j$ is a *multiple transition* if there exist $x \in \Sigma_i$ and $n \geq 1$ such that $\psi^n(x) \in \Sigma_j$ and $\tau_n(x) = \sum_{k=0}^{n-1} \tau \circ \psi^k(x) < \tau^*$.

(ii) We say that $i_0 \to i_1 \to \cdots \to i_n$ is a *chain of simple transitions* of length n if each $i_k \to i_{k+1}$ is a simple transition.

(iii) We define the *rank of a multiple transition* $i \Rightarrow j$ as the largest $n \geq 1$ such that there exists a chain of simple transitions of length n starting at i and ending at j.

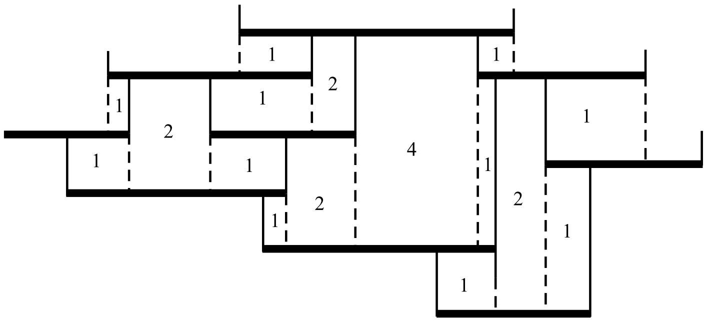

FIGURE 5. Rank of a multiple transition.

We notice that the notion of multiple transitions has been defined with respect to Σ. Thanks to Lemma 15, we could have obtained the same notion using Σ' instead of Σ. The following lemma shows that the notion of rank is meaningful.

LEMMA 17. *The rank of any multiple transition is bounded by* $2\tau^*/\tau_*$.

This enables us to define a maximal rank N:

$$N = \max\{\text{rank}(i, j) \mid i \Rightarrow j \text{ is a multiple transition}\}.$$

We choose now for the rest of the proof, for each i, a sequence of open sets $\{\Sigma_i^k\}_{k=0}^N$ satisfying $\overline{\Sigma}_i^{k+1} \subset \Sigma_i^k$, $\Sigma_i^0 = \Sigma_i$ and $\Sigma_i^N = \Sigma_i'$.

Definitions 18. Let $i \Rightarrow j$ be a multiple transition of rank n and $N \geq k \geq n$.

(i) We define τ_{ij} and ψ_{ij} to be the return time and return map from Σ_i to Σ_j and define Σ_{ij}^k to be the domain of ψ_{ij} considered as a map from Σ_i^k to Σ_j^k

$$\Sigma_{ij}^k = \{x \in \Sigma_i^k \mid \psi_{ij}(x) \in \Sigma_j^k\}.$$

(ii) We define the flow box of rank n and size k associated to the transition $i \Rightarrow j$ the subset in M as

$$B_{ij}^k = \gamma_i(\{(x, t) \mid x \in \Sigma_{ij}^k \text{ and } 0 \leq t \leq \tau_{ij}(x)\}).$$

(iii) We define a partial flow box of B_{ij}^k as any set B of the form

$$B = \gamma_i(\{(x, t) \mid x \in C \text{ and } 0 \leq t \leq \tau_{ij}(x)\}),$$

where C is any open set in Σ_{ij}^k.

(iv) For each $x \in \Sigma_{ij}^0$, $\tau_{ij}(x)$ is a sum of return times $\tau \circ \psi^k$. We extend $\mathcal{H}$ by

$$\mathcal{H}_{ij}(x) = \int_0^{\tau_{ij}(x)} (A - m(A)) \circ \phi^t \circ \gamma(x)\, dt - (\mathcal{V} \circ \psi_{ij}(x) - \mathcal{V}(x)).$$

We thus obtain a family of flow boxes which will enable us to construct by induction on the rank of the function H'. Each $\mathcal{H}_{ij}(x)$ is equal to a sum of $\mathcal{H} \circ \psi^k(x)$ and is therefore non-negative. Moreover, for any $x \in \Sigma'$, each $\mathcal{H}'(x) = \mathcal{H}_{ij}(x)$ for some simple transition $i \to j$. Their main properties are given by the following lemma.

LEMMA 19.
(i) *For any $1 \leq n \leq N$, the union of flow boxes of any rank and fixed size n is equal to M.*
(ii) *If B^n_{ij} is a flow box of rank $n+1$ and size n and $\gamma_k(\Sigma^n_k)$ intersects the interior of B^n_{ij}, the partial flow box (of rank $n+1$) defined by*

$$\{x \in B^n_{ij} \mid \exists t \in]-\tau^*, \tau^*[\textit{ such that } \phi^t(x) \in \gamma_k(\Sigma_k)\}$$

is equal to the union of two partial flow boxes of size n and rank $\leq n$, $B^n_{ik} \cap B^n_{ij}$ and $B^n_{kj} \cap B^n_{ij}$.
(iii) *If B and B' are two flow boxes of rank $n+1$ and size $n+1$, such that* $\mathrm{int}(B) \cap \mathrm{int}(B') \neq \emptyset$*, then $B \cap B'$ is a partial flow box of rank $\leq n$ (and size $n+1$).*

The first step of the construction of H' is to find functions H^0_{ij} on each flow box B^0_{ij}, where $i \Rightarrow j$ is a multiple transition, independently of each other. The following lemma explains the construction.

LEMMA 20. *Let $i \Rightarrow j$ be a multiple transition of any rank.*
(i) *There exists a smooth non-negative function $h : M \to \mathbb{R}^+$, null in a neighborhood of $\gamma(\overline{\Sigma})$ such that*

$$\int_0^{\tau(x)} h \circ \phi^t \circ \gamma(x)\, dt \geq 1 \quad \textit{for all } x \in M.$$

(ii) *Let H^0_{ij} be defined on the flow box B^0_{ij} by*

$$H^0_{ij} \circ \phi^t \circ \gamma(x) = \mathcal{H}_{ij}(x) \frac{h \circ \phi^t \circ \gamma(x)}{\int_0^{\tau_{ij}(x)} h \circ \phi^t \circ \gamma(x)\, dt} \quad \textit{for all } x \in \Sigma^0_{ij}.$$

Then H^0_{ij} is globally Hölder, has the same regularity as A on $W^{cs}_{\mathrm{loc}} \cap B^0_{ij}$, is null in a neighborhood of $\gamma(\overline{\Sigma})$ and satisfies

$$\mathcal{H}_{ij}(x) = \int_0^{\tau_{ij}(x)} H^0_{ij} \circ \phi^t \circ \gamma(x)\, dt \quad \textit{for all } x \in \Sigma^0_{ij}.$$

We now explain in the following lemma the global strategy. Let $\mathcal{U}^n$ be the union of all flow boxes of rank $\leq n$ and size n:

$$\mathcal{U}^n = \bigcup \{B^n_{ij} \mid i \Rightarrow j \text{ is a transition of rank } \leq n\}.$$

By the previous lemma, we notice that $\mathcal{U}^N = M$. In the sequel 'regular' means globally Hölder, smooth along the flow and as regular as A on any local center-stable manifold.

LEMMA 21. *There exist, for each $1 \leq n \leq N$, regular non-negative functions, $H^n : \mathcal{U}^n \to \mathbb{R}^+$ null in a neighborhood of $\gamma(\overline{\Sigma})$ satisfying the two properties:*
(i) *for any multiple transition $i \Rightarrow j$ of rank n and any $x \in \Sigma^n_{ij}$,*

$$\mathcal{H}_{ij}(x) = \int_0^{\tau_{ij}(x)} H^n \circ \phi^t \circ \gamma(x)\, dt;$$

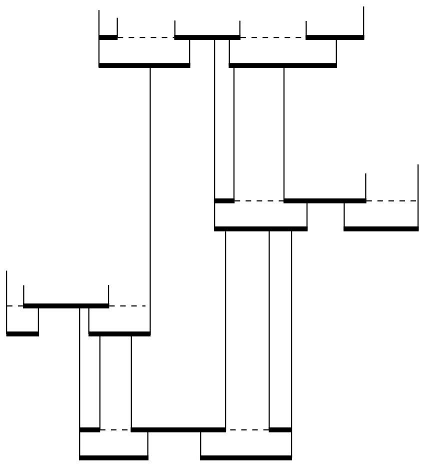

FIGURE 6. Poincaré section with small base.

(ii) *for any transition $i \Rightarrow j$ of rank n and any $x \in B_{ij}^{n+1}$,*

$$H^{n+1}(x) = H^n(x).$$

The function H' we are looking for is given by the last one: $H' = H^N$. For any $1 \le n \le N$ and any transition $i \Rightarrow j$ of rank n, H' coincides with H^n on B_{ij}^N.

3. *Technical proofs*

The proof of our main result is essentially divided into two parts: in the first part we construct a discretized sub-action $\mathcal{V}$ and in the second part we extend $\mathcal{V}$ to M. In both parts, one of the main steps is to construct a Poincaré section (Σ, γ) with small base compared to the minimum of τ (see Figure 6). It is needed in the first part in order to define Lyapunov charts and in the second part to be able to define a notion of rank. The precise statement is given in Lemma 6 but before proving it we need the following simple covering lemma.

LEMMA 22. *For any dimension D ($D = u + s$ in the sequel), there exists a covering number $C(D)$ depending only on D such that for any $\epsilon > 0$, one can cover any open set $\Sigma \subset \mathbb{R}^D$ by balls $\{B_{jk}(\epsilon)\}_{j \in J, 1 \le k \le C(D)}$ of size ϵ such that, for each $1 \le k \le C(D)$, the balls $\{B_{jk}\}_{j \in J}$ are pairwise disjoint.*

Proof. Let Σ be an open set of $\mathbb{R}^D$. We first cover Σ by a maximal net $\{x_j\}_{j \in J}$ of size 2ϵ: that is, $d(x_j, x_{j'}) \ge 2\epsilon$ and $\bigcup_{j \in J} B(x_j, 2\epsilon)$ covers Σ. The balls $\{B(x_j, \epsilon)\}_{j \in J}$ are pairwise disjoint. We then translate simultaneously each center x_j by a vector

$$\vec{v}_k = \frac{\epsilon}{\sqrt{D}} \vec{k}, \quad \vec{k} = (k_1, k_2, \ldots, k_D),$$

where each k_i is an integer and satisfies $|k_i| < 2\sqrt{D}$. We obtain $(2\sqrt{D} + 1)^D = C(D)$ vectors. Each point in $B(x_j, 2\epsilon)$ belongs to one of the balls $B(x_j + \vec{v}_k, \epsilon)$ and, for each k, the balls $\{B(x_j + \vec{v}_k, \epsilon)\}_{j \in J}$ are pairwise disjoint. $\square$

Proof of Lemma 6. We first choose a fixed Poincaré section $(\tilde{\Sigma}, \tilde{\gamma})$. By definition the sets $\{\tilde{\gamma}_i(\Sigma_i \times]0, \tau^*[)\}_{i\in I}$ cover M and the sets $\{\tilde{\gamma}_i(\overline{\tilde{\Sigma}_i})\}_{i\in I}$ are disjoint. We call τ_* the minimum of $\tilde{\tau}$ on $\tilde{\Sigma}_i$ and choose ϵ^* small enough so that $2\epsilon^* < \tau_*$ and the sets $\{\tilde{\gamma}_i(\tilde{\Sigma}'_i \times]0, \tau^* - 2\epsilon^*[)\}_{i\in I}$ still cover M, where

$$\tilde{\Sigma}'_i = \{x \in \tilde{\Sigma}_i \mid d(x, \mathbb{R}^{u+s} \setminus \tilde{\Sigma}_i) > 2\epsilon^*\}.$$

Let $C(D)$ be the covering number given by Lemma 22. We can cover each $\tilde{\Sigma}'_i$ by balls $\{B_{ijk}\}_{j\in J, 1\le k\le C(D)}$ of size $\epsilon < \epsilon^*$ and center x_{ijk} so that, for i, k fixed, the balls $\{B_{ijk}\}_{j\in J}$ are pairwise disjoint. Our choice of ϵ^* implies that $B(x_{ijk}, \epsilon^*) \subset \tilde{\Sigma}_i$. We then stack $C(D)$ copies of $\tilde{\Sigma}_i$ one above the over along the flow, or formally, we just define new charts:

$$\tilde{\gamma}_{ijk} : \begin{cases} (\tilde{\Sigma}_i - x_{ijk}) \times]0, \tau^* - \epsilon^*[\to M, \\ (x, t) \mapsto \tilde{\gamma}_i(x + x_{ijk}, t + (k/C(D))\epsilon^*). \end{cases}$$

Let $\tilde{\Sigma}_{ijk} = \tilde{\Sigma}_i - x_{ijk}$ and $\Sigma_{ijk} = B(0, \epsilon)$. We notice that the maximum height of the stack is $\epsilon^* < \tau_*/2$ and the minimum return time between two Σ_{ijk} is at least $\epsilon^*/C(D)$. The sets $\{\tilde{\gamma}_{ijk}(\overline{\Sigma_{ijk}})\}_{ijk}$ are therefore disjoint and the sets $\{\tilde{\gamma}_{ijk}(\Sigma_{ijk} \times]0, \tau^* - \epsilon^*[)\}_{ijk}$ cover M since $\tilde{\tau} < \tau^* - 2\epsilon^*$. □

We give two consequences of the existence of a Poincaré section of small base. We first show how to define on each Σ_i a new norm $\|\cdot\|_i$ so that the new unit ball B_i for this norm is still small, contains Σ_i and is such that the very first iterate $\tilde{\psi}_{ij}$ is uniformly hyperbolic for all transitions $i \to j$.

Proof of Lemma 8. By definition of a Poincaré section, $\gamma_i(\Sigma_i)$ is tangent to $E^u_{\gamma_i(0)} \oplus E^s_{\gamma_i(0)}$ at the point $\gamma_i(0)$ and $\mathbb{R}^{u+s}$ therefore admits a decomposition $\mathbb{R}^{u+s} = E^u_i \oplus E^s_i$ such that

$$D_0\gamma_i \cdot E^u_i = E^u_{\gamma_i(0)}, \quad D_0\gamma_i \cdot E^s_i = E^s_{\gamma_i(0)}.$$

Let K^* denote the $\mathcal{C}^2$-norm of γ_i and $(\gamma_i)^{-1}$,

$$\frac{1}{K^*}\|v\| \le \|D_x\gamma_i \cdot v\| \le K^*\|v\| \quad \text{for all } v \in \mathbb{R}^{u+s}, \text{ for all } x \in \Sigma_i$$

(where $D_x\gamma_i$ denotes the tangent map of γ_i at x). We now define a (Finsler) norm on each $E^{u,s}_i$ in the following way. We first fix $\eta^* > 0$ small, $\rho^* > 0$, T^* large, to be determined later, and we also choose once and for all some constants μ^s, μ^s_*, μ^u, μ^u_* satisfying:

$$\lambda^s < \mu^s < \mu^s_* < 0 < \mu^u_* < \mu^u < \lambda^u,$$

(where λ^u and λ^s are constants given by the flow). We then define for any $v^s \in E^s_i$ and $v^u \in E^u_i$:

$$\|v^s\|_i = \rho^* \int_0^{T^*} \|D_0(\phi^t \circ \gamma_i) \cdot v^s\| \exp(-t\mu^s)\, dt,$$
$$\|v^u\|_i = \rho^* \int_0^{T^*} \|D_0(\phi^{-t} \circ \gamma_i) \cdot v^u\| \exp(t\mu^u)\, dt.$$

The two norms $\|\cdot\|$ and $\|\cdot\|_i$ are related as follows

$$\begin{aligned}\|v^s\|_i &\geq \|v^s\| \frac{\rho^*}{KK^*} \int_0^{T^*} \exp(t(\Lambda^s - \mu^s))\,dt \\ &\geq \|v^s\| \frac{\rho^*}{KK^*(\mu^s - \Lambda^s)}(1 - \exp(-T^*(\mu^s - \Lambda^s))) \\ &\geq \frac{1}{2\epsilon^*}\|v^s\|,\end{aligned}$$

where T^* is chosen so that $\exp(T^*(\Lambda^s - \mu^s)) < 1/2$ and ρ^* is large compared to $1/\epsilon^*$. In particular, we obtain that the unit ball of $\|\cdot\|_i$ is included in $\tilde{\Sigma}_i$.

Let $i \to j$ be a simple transition and $v^s \in E_i^s$. We want to compare now the two norms $\|D_0\tilde{\psi}_{ij} \cdot v^s\|_j$ and $\|v^s\|_i$. For any $w \in \mathbb{R}^{u+s}$, we denote by $w = [w]^u + [w]^s$ its decomposition in $E_i^u \oplus E_i^s$. Then

$$\begin{aligned}\|[D_0\tilde{\psi}_{ij} \cdot v^s]^s\|_j &= \rho^* \int_0^{T^*} \|D_0(\phi^t \circ \gamma_j)[D_0\tilde{\psi}_{ij} \cdot v^s]^s\| \exp(-t\mu^s)\,dt, \\ \|[D_0\tilde{\psi}_{ij} \cdot v^s]^u\|_j &= \rho^* \int_0^{T^*} \|D_0(\phi^{-t} \circ \gamma_j)[D_0\tilde{\psi}_{ij} \cdot v^s]^u\| \exp(t\mu^u)\,dt.\end{aligned}$$

If ϵ is small enough, since $\tilde{\psi}_{ij}(\Sigma_i)$ intersects Σ_j, $\tilde{\psi}_{ij}(0)$ is close to 0 and $D_0\tilde{\psi}_{ij}$ is close to $D_0\phi^{\tilde{\tau}_{ij}(0)}$. To simplify the notation let $\tau = \tilde{\tau}_{ij}(0)$. We fix η^* small and T^* large that will be determined later. Then, for ϵ small enough,

$$\begin{aligned}\|D_0(\phi^t \circ \gamma_j)[D_0\tilde{\psi}_{ij} \cdot v^s]^s\| &\leq (1 + \eta^*)\|D_0(\phi^{t+\tau} \circ \gamma_i) \cdot v^s\|, \\ \|[D_0\tilde{\psi}_{ij} \cdot v^s]^u\| &\leq \eta^*\|v^s\|.\end{aligned}$$

On the one hand,

$$\begin{aligned}&\|[D_0\tilde{\psi}_{ij} \cdot v^s]^s\|_j \\ &\quad\leq (1 + \eta^*)\exp(\tau\mu^s)\rho^* \int_\tau^{T^*+\tau} \|D_0(\phi^t \circ \gamma_i) \cdot v^s\| \exp(-t\mu^s)\,dt \\ &\quad\leq (1 + \eta^*)\exp(\tau\mu_*^s)\left[\|v^s\|_i + \rho^* \int_{T^*}^{+\infty} \|D_0(\phi^t \circ \gamma_i) \cdot v^s\| \exp(-t\mu^s)\,dt\right].\end{aligned}$$

The integral $\int_{T^*}^{+\infty}$ can be bounded from above using that $\|v^s\| \leq 2\epsilon^*\|v^s\|_i$, that the flow contracts the stable manifold and that T^* is chosen large enough compared to $\ln(1/\eta_*)$ to get

$$\frac{2\epsilon^* KK^*\rho^*}{\mu^s - \lambda^s} \exp(T^*(\lambda^s - \mu^s))\|v^s\|_i \leq \eta^* \exp(\tau\mu^s)\|v^s\|_i.$$

We obtain finally, provided that η^* satisfies $(1 + \eta^*)^2 \leq \exp(\tau_*(\mu_*^s - \mu^s))$:

$$\|[D_0\tilde{\psi}_{ij} \cdot v^s]^s\|_j \leq (1 + \eta^*)^2 \exp(\tau\mu^s)\|v^s\|_i \leq \exp(\tau_*\mu_*^s)\|v^s\|_i.$$

On the other hand, for η^* small enough:

$$\|[D_0\tilde{\psi}_{ij} \cdot v^s]^u\|_j \leq \frac{2\rho^* KK^*\epsilon^*\eta^*}{\mu^u - \lambda^u}\|v^s\|_i \leq \delta\|v^s\|_i.$$

The other estimates for $v^u \in E_i^u$ are obtained similarly. □

We give now a second application of the existence of a Poincaré section.

Proof of Lemma 17. Let $i \Rightarrow j$ be a multiple transition and $i = i_0 \to \cdots \to i_n = j$ a chain of simple transitions joining i and j. We want to prove that n is uniformly bounded. We first extend some notations we have already introduced. Let $\tilde{\tau}_{ij}$, $\tilde{\psi}_{ij}$ be the first return time and return map from $\tilde{\Sigma}_i$ to $\tilde{\Sigma}_j$ (the flow may cross other $\tilde{\Sigma}_k$ in between). Let $\tilde{\Sigma}_{ij}$ be the domain of definition of $\tilde{\tau}_{ij}$ or $\tilde{\psi}_{ij}$ and $\tilde{U}_{ij}$ the open set $\tilde{\gamma}_i(\tilde{\Sigma}_{ij} \times]0, \tau^*[) \subset M$. Each $\gamma_k(\Sigma_k)$ that intersects $\tilde{U}_{ij}$ can be seen as a graph in $\tilde{\Sigma}_{ij} \times]0, \tau^*[$ of a function σ_k defined on a domain $\Delta_k \subset \tilde{\Sigma}_{ij}$:

$$\tilde{\gamma}_i(\{(x, \sigma_k(x)) \mid x \in \Delta_k\}) = \gamma_k(\Sigma_k) \cap \tilde{U}_{ij}.$$

Let η^* small enough be defined later. If ϵ is sufficiently small, each Δ_k has a diameter less than η^* and the oscillation of σ_k is less than $\tau_*/2$ where τ_* is the minimum value of all return times to Σ:

$$|\sigma_k(x) - \sigma_k(y)| < \tfrac{1}{2}\tau_* \quad \text{for all } x, y \in \Delta_k.$$

Moreover, since $\tilde{\psi}_{ij}(0)$ is close to 0 within ϵ, we can assume that $\tilde{\Sigma}_{ij}$ contains the ball $B(0, \epsilon^*/2)$.

We want to prove that actually each $\gamma_{i_k}(\Sigma_{i_k})$ is entirely contained in $\tilde{U}_{ij}$. Suppose that this is true for $k = 1, 2, \dots, m$, $m \le n$, then each Δ_{i_k} intersects $\Delta_{i_{k-1}}$ at some point $x_{i_k} \in \tilde{\Sigma}_{ij}$. By definition of τ_*, $\sigma_{i_k}(x_{i_k}) - \sigma_{i_{k-1}}(x_{i_k}) \ge \tau_*$ and thanks to the distortion estimate of $\sigma_{i_{k-1}}$, $\sigma_{i_{k-1}}(x_{i_k}) - \sigma_{i_{k-1}}(x_{i_{k-1}}) \ge \tau_*/2$. We obtain in particular that m is uniformly bounded:

$$\tau^* \ge \sigma_{i_k}(x_{i_k}) \ge \frac{m}{2}\tau_* \quad \Rightarrow \quad m \le 2\frac{\tau^*}{\tau_*} = N^*.$$

Furthermore, if we choose η^* so that $(N^* + 1)\eta^* < \epsilon^*/2$, since the distance of Δ_{i_m} from 0 is bounded by $m\eta^* \le N^*\eta^*$, we have just proved that $\gamma_{i_{m+1}}(\Sigma_{i_{m+1}})$ is again totally included in $\tilde{\Sigma}_{ij}$. $\square$

Existence of Lyapunov charts enables us to use the theory of graph transform to construct local stable and unstable manifolds. We do not prove this fact (Lemma 8), which can be viewed as an improvement of the standard Bowen's shadowing lemma: points for instance in the local stable manifold $W^s_{\text{loc}}(\omega)$ positively shadow a given pseudo-orbit $(\omega_0, \omega_1, \dots)$. We prove, nonetheless, the Markov property these manifolds possess.

Proof of Lemma 10. Let $\zeta, \omega \in \Omega$ with $\zeta_0 = \omega_0$. Then $[\omega, \zeta]$ denotes the unique intersection point of $W^s_{\text{loc}}(\omega)$ and $W^u_{\text{loc}}(\omega)$. This point can also be obtained as:

$$[\omega, \zeta] = \pi(\dots, \zeta_{-2}, \zeta_{-1} | \omega_0, \omega_1, \dots).$$

If z belongs to $W^s_{\text{loc}}(\omega) \cap R_\omega$, $z = \pi(\zeta)$ for some $\zeta \in \Omega$, $\zeta_0 = \omega_0$. Let $z' = [\omega, \zeta]$. Then z and z' belong to the same unstable manifold $W^u_{\text{loc}}(\zeta)$ and simultaneously to the same stable manifold $W^s_{\text{loc}}(\omega)$: they have to coincide by transversality of these manifolds, $z = z'$. We have just proved assertion (i). Assertions (ii) and (iii) come from the following remark:

$$\psi_\omega \circ \pi(\dots, \zeta_{-2}, \zeta_{-1} | \omega_0, \omega_1, \dots) = \pi(\dots, \zeta_{-2}, \eta_{-1}, \omega_0 | \omega_1, \omega_2, \dots),$$

$$\psi_\omega^{-1} \circ \pi(\dots, \omega_{-2}, \omega_{-1} | \zeta_0, \zeta_1, \dots) = \pi(\dots, \omega_{-3}, \omega_{-2} | \omega_{-1}, \zeta_0, \zeta_1, \dots),$$

and the other form of the unstable manifold restricted to R:

$$W^u_{\mathrm{loc}}(\omega) \cap R = \{[\zeta, \omega] \mid \zeta \in \Omega \quad \text{and} \quad \zeta_0 = \omega_0\}. \qquad \square$$

The local stable manifold apparently depends on the choice of the pseudo-orbit ω projecting to the same true orbit $\pi(\omega) = x$. The content of Proposition 12 says that $\gamma(W^s_{\mathrm{loc}}(\omega))$ is a geometric object in the manifold M depending only on $\gamma(\Sigma)$ and not on the way we code the orbits.

Proof of Proposition 12. Let $x \in W^s_{\mathrm{loc}}(\omega)$. We want to show that the distance between $\phi^{t+b^s(\omega,x)} \circ \gamma(x)$ and $\phi^t \circ \gamma(\pi(\omega))$ converges exponentially to 0 when $t \to +\infty$. It is enough to prove this convergence along a sub-sequence of times $\{\tau_n\}$ satisfying $\tau_* \leq \tau_{n+1} - \tau_n \leq \tau^*$. Let

$$\tau_n = \tau_n(\omega, \pi(\omega)) = \sum_{k=0}^{n-1} \tau \circ \tilde{\psi}(\omega, \pi(\omega)).$$

On the one hand, the cocycle $b^s(\omega, x)$ satisfies the cocycle equation

$$b^s(\omega, x) + \tau_n = b^s \circ \tilde{\psi}^n(\omega, x) + \tau_n(\omega, x),$$

which implies

$$\phi^{\tau_n + b^s(\omega,x)} \circ \gamma(x) = \phi^{b^s \circ \tilde{\psi}^n(\omega,x)} \circ \gamma \circ \tilde{\psi}^n_\omega(x).$$

On the other hand,

$$\phi^{\tau_n} \circ \gamma(\pi(\omega)) = \gamma \circ \tilde{\psi}^n_\omega(\pi(\omega)).$$

Since $\tilde{\psi}^n_\omega(x)$ converges exponentially to $\tilde{\psi}^n_\omega(\pi(\omega))$, and b^s equals 0 restricted to the graph of π, we obtain the exponential convergence of $\phi^{\tau_n + b^s(\omega,x)} \circ \gamma(x)$ to $\phi^{\tau_n} \circ \gamma(\pi(\omega))$:

$$\phi^{b^s(\omega,x)} \circ \gamma(x) \in W^{ss}_{\mathrm{loc}} \circ \gamma(\pi(\omega)).$$

This proves (i) and (ii). To prove (iii) we compute

$$\begin{aligned}
(*) &= \sum_{k=0}^{n-1} \tilde{\mathcal{A}} \circ \tilde{\psi}^n_\omega(x) - \sum_{k=0}^{n-1} \tilde{\mathcal{A}} \circ \tilde{\psi}^n_\omega(\pi(\omega)) \\
&= \int_0^{\tau_n(\omega,x)} (A - m(A)) \circ \phi^t \circ \gamma(x)\, dt \\
&\quad - \int_0^{\tau_n(\omega,\pi(\omega))} (A - m(A)) \circ \phi^t \circ \gamma(\pi(\omega))\, dt.
\end{aligned}$$

We split the first integral into two parts $\int_0^{b^s(\omega,x)} + \int_{b^s(\omega,x)}^{\tau_n(\omega,x)}$, we use the cocycle equation $\tau_n(\omega, x) - b^s(\omega, x) = \tau_n - b_n$ where $b_n = b^s \circ \tilde{\psi}^n(\omega, x)$, and we obtain

$$\begin{aligned}
(*) &= \int_0^{b^s(\omega,x)} \big(A - m(A)\big) \circ \phi^t \circ \gamma(x)\, dt \\
&\quad + \int_0^{\tau_n} \{A \circ \phi^t \circ w^s(\omega, x) - A \circ \phi^t \circ \gamma(\pi(\omega))\}\, dt \\
&\quad - \int_{\tau_n - b_n}^{\tau_n} (A - m(A)) \circ \phi^t \circ w^s(\omega, x)\, dt.
\end{aligned}$$

We use again the fact that $b_n \to 0$ to eliminate the last integral and we finally obtain (iii). $\square$

The existence of a Markov section R and of a stable cocycle Δ^s are the main tools we use to construct a sub-action as we did in [**10**] and [**4**]. We divide the proof of Proposition 14 into three parts: in the first part we show that our definition of $\tilde{\mathcal{V}}$ immediately implies the sub-cocycle equation $\tilde{\mathcal{A}} \geq \tilde{\mathcal{V}} \circ \tilde{\psi} - \tilde{\mathcal{V}}$ on $\tilde{\Omega} = \mathrm{graph}(\pi)$; in the second part we show that $\tilde{\mathcal{V}}$ depends actually on $\pi(\omega)$ and not on ω; we finally show in the last part that $\mathcal{V}$ is Hölder on Σ.

Proof of Proposition 14, parts (i) and (ii). For any $\omega, \zeta \in \Omega$, $\omega_0 = \zeta_0$, let $\xi = (\ldots, \zeta_{-2}, \zeta_{-1} | \omega_0, \omega_1, \ldots)$. Then $\pi(\xi) = [\omega, \zeta]$ and $\tilde{\mathcal{A}}(\xi, \pi(\xi)) = \tilde{\mathcal{A}}(\omega, \pi(\xi))$. The definition of $\tilde{\mathcal{V}}$ implies

$$\begin{aligned} S_n \tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\omega, \zeta]) &+ \Delta^s(\omega, [\omega, \zeta]) + \tilde{\mathcal{A}}(\omega, \pi(\omega)) \\ &= S_n \tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\xi, \pi(\xi)) + \tilde{\mathcal{A}}(\xi, \pi(\xi)) + \Delta^s(\sigma(\omega), \pi \circ \sigma(\xi)) \\ &= S_{n+1} \tilde{\mathcal{A}} \circ \psi^{-(n+1)}(\sigma(\xi), \pi \circ \sigma(\xi)) + \Delta^s(\sigma(\omega), \pi \circ \sigma(\xi)). \end{aligned}$$

The Markov property implies that $\pi \circ \sigma(\xi) \in W^s_{\mathrm{loc}}(\sigma(\omega))$ and we obtain:

$$\tilde{\mathcal{A}}(\omega, \pi(\omega)) + \tilde{\mathcal{V}}(\omega) \geq \tilde{\mathcal{V}} \circ \sigma(\omega).$$

Now to prove the second part. For any $\omega, \omega', \zeta \in \Omega$, $\omega_0 = \omega'_0 = \zeta_0$ and $\pi(\omega) = \pi(\omega')$,

$$x = [\omega, \zeta] = [\omega', \zeta],$$

since both points $[\omega, \zeta]$ and $[\omega', \zeta]$ belong to $W^u_{\mathrm{loc}}(\zeta)$ and $W^s_{\mathrm{loc}}(\omega) = W^s_{\mathrm{loc}}(\omega')$. Moreover $b^s(\omega, x) = b^s(\omega', x)$, $w^s(\omega, x) = w^s(\omega', x)$ and (iii) of Proposition 12 implies that $\Delta^s(\omega, x) = \Delta^s(\omega', x)$. We just have proved that $\tilde{\mathcal{V}}(\omega) = \tilde{\mathcal{V}}(\omega')$. □

Before proving part (iii) of Proposition 14 we need two lemmas.

LEMMA 23. *There exists a constant $\eta > 0$ depending on Σ such that for any $i \in I$, for any $x, y \in \Sigma_i$, if $d(x, y) < \eta$ then there exists a simple transition $i \to j$ and $m, n \geq 1$ such that*

$$\psi^m(x) \in \Sigma_j, \quad \psi^n(y) \in \Sigma_j, \quad \tau_m(x) \leq \tau^*, \quad \tau_n(y) \leq \tau^*.$$

(Or in other words, ψ^m (respectively ψ^n) coincides with $\tilde{\psi}_{ij}$ about x (respectively y)).

Proof. Let $U_i = \gamma_i(\Sigma_i \times]0, \tau^*[)$. For any simple transition $i \to j$, $\gamma_j(\Sigma_j) \cap U_i$ can be seen as a graph in $\Sigma_i \times]0, \tau^*[$ over an open domain $\Delta_{ij} \in \Sigma_i$. As the sets $\{\Delta_{ij}\}_j$ cover Σ_i, we can find a Lebesgue number $\eta > 0$, that is, a positive real number small enough that any ball of radius η is included in one of the sets Δ_{ij}. □

In the following lemma, Λ^u_* is a Lyapunov exponent given by Lemma 8.

LEMMA 24. *For any $k \geq 1$, any $i \in I$, any $x, y \in \Sigma_i$, if $d(x, y)$ is less than $(\eta/K^*)\exp(-k\Lambda^u_*)$ then there exist $\omega, \zeta \in \Omega$ such that $\pi(\omega) = x$, $\pi(\zeta) = y$ and their symbols coincide during k times, $\omega_0 = \zeta_0, \ldots, \omega_k = \zeta_k$.*

Proof. Let $x_0 = x$, $y_0 = y$ on the same section Σ_i. By the previous lemma there exist m_1, n_1 and $i_1 \in I$ such that

$$x_1 = \psi^{m_1}(x_0) \in \Sigma_{i_1}, \quad y_1 = \psi^{n_1}(y_0) \in \Sigma_{i_1},$$
$$\tau_{m_1}(x_0) \leq \tau^*, \quad \tau_{n_1}(y_1) \leq \tau^*.$$

Since $d(x_1, y_1) < \eta$, we repeat again the construction and find two sequences $(m_1, \dots, m_k)$ and $(n_1, \dots, n_k)$ such that if

$$x_l = \psi^{m_1+\cdots+m_l}(x_0), \quad y_l = \psi^{n_1+\cdots+n_l}(y_0),$$

x_l and y_l belong to the same section Σ_{i_l} and $d(x_l, y_l) < \eta$. Let ω and ξ be the canonical orbits associated to x_0 and y_0, $m = m_1 + \cdots + m_k$ and $n = n_1 + \cdots + n_k$. We define a new pseudo-orbit:

$$\zeta = (\dots, \xi_{-2}, \xi_{-1} | \omega_0, \omega_1, \dots, \omega_m, \xi_{n+1}, \xi_{n+2}, \dots).$$

By construction, for all $1 \le l \le k$, $\tilde{\psi}_\zeta^{m_1+\cdots+m_l}(y) = \psi^{n_1+\cdots+n_l}(y) \in \Sigma$; since $\sigma^m(\zeta)$ coincide with $\sigma^n(\xi)$, for all $k \ge 0$, we have $\tilde{\psi}_\zeta^{m+k}(y) = \psi^{n+k}(y) \in \Sigma$ and $\tilde{\psi}_\zeta^{-k}(y) = \psi^{-k}(y) \in \Sigma$.

Since m_l is bounded by τ^* and ϵ (or η) can be as small as we want independently from τ^*, we obtain that $\tilde{\psi}_\zeta^k(y) \in B_{\zeta_k}$ for all $k \in \mathbb{Z}$ and therefore $y = \pi(\zeta)$. □

Proof of Proposition 14, part (iii). We first prove that $\mathcal{V}$ is globally Hölder on Σ. Let x, x' be two points of Σ; we assume that $d(x, x')$ is smaller than η/K^*, where η is given by Lemma 23. Let $N = N(x, x')$ be the unique integer satisfying

$$\frac{\eta}{K^*} \exp(-(N+1)(\Lambda_*^u - \lambda_*^s)) \le d(x, x') \le \frac{\eta}{K^*} \exp(-N(\Lambda_*^u - \lambda_*^s)).$$

By Lemma 24, one can find ω, ω' such that

$$\pi(\omega) = x, \quad \pi(\omega') = x' \quad \text{and} \quad \omega_0 = \omega'_0, \dots, \omega_N = \omega'_N.$$

Let $n \ge 0$, $\zeta \in \Omega$, $\zeta_0 = \omega_0$, $y = [\omega, \zeta]$ and $y' = [\omega', \zeta]$. We may assume that $\xi_k = \omega_k$ for all $0 \le k \le N$. We want to find an upper bound of the expression

$$(**) = \{S_n \tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, y) + \Delta^s(\omega, y)\} - \{S_n \tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, y') + \Delta^s(\omega', y')\}$$

in terms of $d(x, x')$. In the sequel we use the notations

$$x_n = \tilde{\psi}_\omega^n(x), \quad x'_n = \tilde{\psi}_{\omega'}^n(x'), \quad y_n = \tilde{\psi}_\omega^n(y), \quad y'_n = \tilde{\psi}_{\omega'}^n(y').$$

We first estimate $\exp(N\lambda_*^s)$ in terms of $d(x, x')$. The definition of N gives

$$N + 1 \ge \frac{1}{\Lambda_*^u - \lambda_*^s} \ln\left(\frac{K^*}{\eta d(x, x')}\right), \quad \exp(N\lambda_*^s) \le K_0(\eta) d(x, x')^\beta,$$

for some constants $K_0(\eta) = \exp(-\lambda_*^s)(K^*/\eta)^\beta$ and $\beta = -\lambda_*^s/(\Lambda_*^u - \lambda_*^s)$.

Since y and y' belong to the same unstable manifold, since x and y, on the one hand, x' and y', on the other hand, belong to the same stable manifolds we have

$$\begin{aligned}
d(y, y') &\le \exp(-N\lambda_*^u) d(y_N, y'_N), \\
d(y_N, y'_N) &\le d(y_N, x_N) + d(x_N, x'_N) + d(x'_N, y'_N), \\
d(y_N, x_N) &\le \exp(N\lambda^s), \\
d(x'_N, y'_N) &\le \exp(N\lambda^s), \\
d(x_N, x'_N) &\le K^* \exp(N\Lambda_*^u) d(x, x') \le \eta \exp(N\lambda_*^s).
\end{aligned}$$

We first obtain that the stable holonomy is β_0-Hölder:

$$d(y, y') \le 3\exp(-N(\lambda^u - \lambda^s)) \le K_1(\eta) d(x, x')^{\beta_0}$$

for some exponent $\beta_0 = (\lambda^u_* - \lambda^s_*)/(\Lambda^u_* - \lambda^s_*)$. We also obtain

$$d(x_N, x'_N), d(y_N, y'_N), d(x_N, y_N), d(x'_N, y'_N) \le K_0(\eta) d(x, x')^{\beta}.$$

We now split the expression $(**)$ into five parts:

$$\begin{aligned}
(**)_1 &= S_n \tilde{\mathcal{A}}^{-n}(\zeta, y) - S_n \tilde{\mathcal{A}}^{-n}(\zeta, y'),\\
(**)_2 &= \sum_{k=0}^{N-1} \{\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, y) - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', y')\},\\
(**)_3 &= \sum_{k=0}^{N-1} \{\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', x') - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, x)\},\\
(**)_4 &= \sum_{k\ge N} \{\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, y) - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega, x)\},\\
(**)_5 &= \sum_{k\ge N} \{\tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', y') - \tilde{\mathcal{A}} \circ \tilde{\psi}^k(\omega', x')\}.
\end{aligned}$$

The first two terms are estimated as follows, using the fact that the orbits of ω and ω' coincide during the first n steps.

$$\begin{aligned}
|(**)_1 + (**)_2| &= |S_{n+N} \tilde{\mathcal{A}} \circ \tilde{\psi}^{-n-N}(\sigma^N(\zeta), y_N) - S_{n+N} \tilde{\mathcal{A}} \circ \tilde{\psi}^{-n-N}(\sigma^N(\zeta), y'_N)|\\
&\le \mathrm{H\ddot{o}ld}_\alpha(\tilde{\mathcal{A}}) \sum_{k=1}^{n+N} \exp(-k\alpha\lambda^u_*) d(y_N, y'_N)^\alpha\\
&\le K_2(\eta) \mathrm{H\ddot{o}ld}_\alpha(\tilde{\mathcal{A}}) d(x, x')^{\alpha\beta}.
\end{aligned}$$

To estimate the third term, we again use $\omega_k = \omega'_k$ for $0 \le k \le N$,

$$\begin{aligned}
d(x_k, x'_k) &\le K^* \exp(k\Lambda^u_*) d(x, x') \le \eta \exp(-(N-k)\Lambda^u_*) \exp(N\lambda^s_*)\\
&\le \eta K_0(\eta) \exp(-(N-k)\Lambda^u_*) d(x, x')^\beta,\\
|(**)_3| &\le (\eta K_0(\eta))^\alpha \sum_1^N \exp(k\alpha\Lambda^u_*) \mathrm{H\ddot{o}ld}_\alpha(\tilde{\mathcal{A}}) d(x, x')^{\alpha\beta}\\
&\le K_3(\eta) \mathrm{H\ddot{o}ld}_\alpha(\tilde{\mathcal{A}}) d(x, x')^{\alpha\beta}.
\end{aligned}$$

To estimate the last two terms we use that x and y, on the one hand, x' and y', on the other hand, are on the same stable manifold:

$$\begin{aligned}
|(**)_4|,\ |(**)_5| &\le K(\eta)^\alpha \sum_0^{+\infty} \exp(k\alpha\lambda^s_*) \mathrm{H\ddot{o}ld}_\alpha(\tilde{\mathcal{A}}) d(x, x')^{\alpha\beta}\\
&\le K_4(\eta) \mathrm{H\ddot{o}ld}_\alpha(\tilde{\mathcal{A}}) d(x, x')^{\alpha\beta}.
\end{aligned}$$

We finally prove that $\mathcal{V}$ is as smooth as $\tilde{\mathcal{A}}$ on any $W^s_{\text{loc}} \cap \Sigma$. Indeed, if $\omega \in \Omega$ and x, x' are two points on $W^s_{\text{loc}}(\omega) \cap \Sigma$, there exist $\xi, \xi' \in \Omega$ such that

$$\pi(\xi) = x, \quad \pi(\xi') = x', \quad \xi_n = \xi'_n = \omega_n \quad \text{for all } n \geq 0.$$

For any $\zeta \in \Omega$, the following brackets are equal and correspond to a unique point:

$$y = [\xi, \zeta] = [\xi', \zeta] = [\omega, \zeta].$$

To compute $\mathcal{V}(x) - \mathcal{V}(x')$ we estimate as follows:

$$\begin{aligned}
(***) &= \{S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\xi, \zeta]) + \Delta^s(\xi, [\xi, \zeta])\} \\
&\quad - \{S_n\tilde{\mathcal{A}} \circ \tilde{\psi}^{-n}(\zeta, [\xi', \zeta]) + \Delta^s(\xi', [\xi', \zeta])\} \\
&= \Delta^s(\xi, y) - \Delta^s(\xi', y) \\
&= \sum_{n\geq 0}\{\tilde{\mathcal{A}} \circ \psi^n(\omega, x') - \tilde{\mathcal{A}} \circ \psi^n(\omega, x)\} \\
&= \mathcal{V}(x) - \mathcal{V}(x').
\end{aligned}$$

We have obtained the last equality because the last but one term is independent of ζ. Since x and x' belong to the same stable manifold $d(\tilde{\psi}^n_\omega(x), \tilde{\psi}^n_\omega(x')) \to 0$ exponentially fast and $\mathcal{V}$ possesses the required regularity. $\square$

We now extend $\mathcal{V}$ to the whole manifold. Since we are going to use a partition of unity, we actually extend the restriction $\mathcal{V}'$ of $\mathcal{V}$ to a Poincaré sub-section (Σ', γ'). We also want to preserve the notion of multiple transitions; namely, we want to keep

$$U_i \cap \gamma_j(\Sigma_j) \neq \emptyset \quad \Leftrightarrow \quad U'_i \cap \gamma'_i(\Sigma'_i) \neq \emptyset$$

(where U_i denotes the open set $\gamma_i(\Sigma_i \times]0, \tau^*[)$ and similarly for U'_i).

Proof of Lemma 15. Since $\{U_i\}_{i\in I}$ is a covering of M, for any sufficiently small $\epsilon > 0$ (depending on a Lebesgue number of this covering and on the $\mathcal{C}^1$-size of $\{\gamma_i\}$), the sets $\{U'_i\}$ associated to $\Sigma'_i = \{x \in \Sigma_i \mid d(x, \mathbb{R}^{u+s} \setminus \Sigma_i) > \epsilon\}$ again cover M. Moreover, for each i, j such that $U_i \cap \gamma_j(\Sigma_j) \neq \emptyset$ we choose a point $x_{ij} \in U_i \cap \gamma_j(\Sigma_j)$ and, for ϵ small, x_{ij} remains in $U'_i \cap \gamma'_j(\Sigma'_j)$. $\square$

The notion of rank we have introduced enables us to cover M by sets, called 'flow boxes', having the property that whenever two such flow boxes of same rank meet, their intersection is included into a flow box of rank strictly inferior.

Proof of Lemma 19. Part (i). Since (Σ^n, γ^n) is a Poincaré section, the sets $U^n_i = \gamma_i\{(x, t) \mid x \in \Sigma^n_i,\ 0 < t < \tau^*\}$ cover M and any orbit $\{\phi^t(x)\}_{t=0}^{t=\tau^*}$, $x \in \Sigma_i$, has to intersect another $\gamma_j(\Sigma_j)$ (the point $\phi^{\tau^*}(x)$ has to belong to a distinct set U^n_j).

Part (ii). If $\gamma_k(\Sigma^n_k)$ meets some B^n_{ij}, where $i \Rightarrow j$ is a transition of rank $n+1$, then $i \Rightarrow k$ and $k \Rightarrow j$ are multiple transitions,

$$\text{rank}(i, j) \geq \text{rank}(i, k) + \text{rank}(k, j),$$

and each sub-transition has therefore a rank strictly smaller.

Part (iii). Let B^n_{ij} and $B^n_{i'j'}$ be two flow boxes of rank n and size n and suppose that their interiors meet at a common point x. We order the indices i, j, i', j' in the following way: let k, l be two indices among i, j, i', j', we say that $k < l$ if there exist $s < t$, in $]-\tau^*, \tau^*[$ such that $\phi^s(x) \in \gamma_k(\Sigma_k)$ and $\phi^t(x) \in \gamma_l(\Sigma_l)$. The orders

$$i < i' < j' < j \quad \text{and} \quad i' < i < j < j'$$

are impossible by the sub-additivity of the rank function. We are thus left with two cases:

$$i < i' < j < j' \quad \text{or} \quad i' < i < j' < j.$$

The rank of $i' \Rightarrow j$ or $i \Rightarrow j'$ is strictly smaller than n and $B^n_{ij} \cap B^n_{i'j'}$ is thus a partial flow box included either in $B^n_{i'j}$ or in $B^n_{ij'}$. □

Proof of Lemma 21. By induction, we are going to construct a family $\{H^n_{ij}\}$ of regular non-negative functions, defined on each B^n_{ij} for all multiple transitions $i \Rightarrow j$ of rank $\le n$, which coincide on the intersection of two such flow boxes.

If $i \Rightarrow j$ has rank 1, we define $H^1 = H^0_{ij}$ as in Lemma 20 on B^1_{ij}. Since two such flow boxes can only meet on $\gamma(\overline{\Sigma})$ and since H^0_{ij} is null on a neighborhood of $\gamma(\overline{\Sigma})$, this construction defines a global function H^1 on $\mathcal{U}^1$ which satisfies

$$\mathcal{H}_{ij}(x) = \int_0^{\tau_{ij}(x)} H^1 \circ \phi^t \circ \gamma_i(x)\, dt \quad \text{for all } x \in \Sigma^1_{ij} \text{ and } \mathrm{rank}(i, j) = 1.$$

Suppose by induction that we have defined a function $H^n : \mathcal{U}^n \to \mathbb{R}^+$ which satisfies the following 'integrability condition' on any flow box of rank at most n, that is, for any multiple transition $i \Rightarrow j$ of rank $\le n$ and for any $x \in \Sigma^n_{ij}$,

$$\mathcal{H}_{ij}(x) = \int_0^{\tau_{ij}(x)} H^n \circ \phi^t \circ \gamma_i(x)\, dt.$$

Let $i \Rightarrow j$ be a multiple transition of rank $n + 1$. We want to construct a function H^{n+1} on B^{n+1}_{ij}. For any k such that $\gamma_k(\Sigma^n_k)$ meets B^n_{ij}, $\gamma_k(\Sigma^n_k) \cap B^n_{ij}$ can be seen as a graph in $\Sigma^n_{ij} \times]0, \tau^*[$ over a domain we denote by Σ^n_{ijk}. Lemma 19 implies that the partial flow box

$$U^n_{ijk} = \gamma_i\{(x, t) \mid x \in \Sigma^n_{ijk},\ 0 \le t \le \tau_{ij}(x)\}$$

is equal to the union of the two partial flow boxes

$$\gamma_i\{(x, t) \mid x \in \Sigma^n_{ijk},\ 0 \le t \le \tau_{ik}(x)\},$$
$$\gamma_i\{(x, t) \mid x \in \Sigma^n_{ijk},\ \tau_{ik}(x) \le t \le \tau_{ij}(x)\}$$

and they are both of rank $\le n$. Since H^n is already defined on $\bigcup_k U^n_{ijk}$, H^n, restricted to this set, can be viewed as a function $H^n_{ij}(x, t)$ on

$$\left\{(x, t) \,\middle|\, x \in \bigcup_k \Sigma^n_{ijk},\ 0 \le t \le \tau_{ij}(x)\right\}$$

satisfying the integrability condition on $\bigcup_k \Sigma^n_{ijk}$,

$$\mathcal{H}_{ij}(x) = \int_0^{\tau_{ij}(x)} H^n_{ij}(x, t)\, dt \quad \text{for all } x \in \bigcup_k \Sigma^n_{ijk}.$$

On the other hand H^0_{ij}, defined in Lemma 20, can also be seen as a function $H^0_{ij}(x,t)$ satisfying the same integrability condition but on a larger set Σ^0_{ij}. We now choose two smooth functions $p, q : \Sigma^n_{ij} \to [0,1]$ such that $p+q=1$ on $\overline{\Sigma}^{n+1}_{ij}$, $\text{supp}(p) \subset \bigcup_k \Sigma^n_{ijk}$ and $\text{supp}(q) \subset \Sigma^n_{ij} \setminus \bigcup_k \overline{\Sigma}^{n+1}_{ijk}$. Then the function

$$H^{n+1}_{ij}(x,t) = p(x)H^n_{ij}(x,t) + q(x)H^0_{ij}(x,t)$$

satisfies again the integrability condition on a smaller set Σ^{n+1}_{ij} and extends $H^n_{ij}(x,t)$ on $\bigcup_k \Sigma^{n+1}_{ijk}$.

We summarize the previous construction: we constructed for any transition $i \Rightarrow j$ of rank $n+1$ a regular non-negative function H^{n+1}_{ij} defined on B^{n+1}_{ij} which extends H^n on $\tilde{\mathcal{U}}^n \cap B^{n+1}_{ij}$, where $\tilde{\mathcal{U}}^n$ represents the union of all flow boxes of rank $\le n$ but of size a little bit smaller, namely of size $n+1$. Since the intersection of two such boxes B^{n+1}_{ij} and $B^{n+1}_{i'j'}$ is included into $\tilde{\mathcal{U}}^n$, the two definition H^{n+1}_{ij} and $H^{n+1}_{i'j'}$ coincide on $B^{n+1}_{ij} \cap B^{n+1}_{i'j'}$ and the collection of $\{H^{n+1}_{ij}\}_{i\Rightarrow j}$ defines a global function H^{n+1} on $\mathcal{U}^{n+1}$. □

Acknowledgements. We thank the referee for their careful reading of the paper, especially for having shown us that the Anosov flow need not be transitive and for having improved Proposition 2. The first author was partially supported by CNPq-Pronex Sistemas Dinamicos and Instituto do Milenio. The second author was partially supported by Brazil–France agreement.

REFERENCES

[1] R. Bowen. Periodic orbits for hyperbolic flows. *Amer. J. Math.* **94** (1972), 1–30.

[2] R. Bowen. Symbolic dynamics for hyperbolic flows. *Amer. J. Math.* **95** (1973), 429–459.

[3] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.* **29** (1975), 181–202.

[4] G. Contreras, A. O. Lopes and Ph. Thieullen. Lyapunov minimizing measures for expanding maps of the circle. *Ergod. Th. & Dynam. Sys.* **21** (2001), 1379–1409.

[5] A. Fathi. Weak KAM theorem in Lagrangian dynamics. *Preliminary version*, 2001.

[6] A. Fathi and A. Siconolfi. Existence of $\mathcal{C}^1$ critical sub-solutions of the Hamilton–Jacobi equation. *Invent. Math.* **155**(2) (2004), 363–388.

[7] A. Fathi and E. Maderna. Weak KAM theorem on non compact manifolds. *Preprint*, 2002.

[8] H. O. Georgii. *Gibbs Measures and Phase Transitions*. W. de Gruyter, New York, 1988.

[9] A. N. Livsic. Cohomology properties of dynamical systems. *Math. USSR-Izv.* **6** (1972), 1278–1301.

[10] A. O. Lopes and Ph. Thieullen. Sub-actions for Anosov diffeomorphisms. *Astérisque, Geometric Methods in Dynamics* **287** (2003), 135–146.

[11] A. O. Lopes and Ph. Thieullen. Sub-actions for the geodesic flow in constant negative curvature. *Annales de l'Institut Henry Poincaré, Analyze Non Lineaire* to appear.

[12] D. Massart. Norme stable des surfaces. *C. R. Acad. Sci. Paris, Ser. I, Math.* **324** (1997), 221–224.

[13] R. Souza. Hölder sub-actions for weakly hyperbolic one-dimensional systems. *Ergod. Th. & Dynam. Sys.* to appear.