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Peter H. Yoon[1,a] and Luiz F. Ziebell[2]

AFFILIATIONS

[1]Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742-2431, USA
[2]Instituto de Física, Universidade Federal do Rio Grande do Sul—UFRGS, 91501-970 Porto Alegre, RS, Brazil

[a]Author to whom correspondence should be addressed: yoonp@umd.edu

ABSTRACT

Electrostatic weak turbulence theory for plasmas immersed in an ambient magnetic field is developed by employing a hybrid two-fluid and kinetic theories. The nonlinear susceptibility response function is calculated with the use of warm two-fluid equations. The linear dispersion relations for longitudinal electrostatic waves in magnetized plasmas are also obtained within the warm two-fluid theoretical scheme. However, dissipations that arise from linear and nonlinear wave–particle interactions cannot be discussed with the macroscopic two-fluid theory. To compute such collisionless dissipation effects, linearized kinetic theory is utilized. Moreover, a particle kinetic equation, which is necessary for a self-consistent description of the problem, is derived from the quasilinear kinetic theory. The final set of equations directly generalizes the electrostatic weak turbulence theory in unmagnetized plasmas, which could be applied for a variety of problems including the electron beam–plasma interactions in magnetized plasma environments.

I. INTRODUCTION

Among the methods in nonlinear plasma theory, the weak turbulence theory occupies a special place. It was developed by early pioneers of modern plasma physics—see, e.g., Refs. 1–11. The recent monograph by one of the present authors (P.H.Y.) expounds on such a theory from a modern perspective.[12] Its usefulness has been demonstrated by numerous examples including the non-thermal electrons measured in the solar wind[13–17] and the solar type II and type III radio bursts.[18–21] The kappa distribution[22] was introduced in order to empirically fit the observed non-thermal solar wind electron distribution, but the weak turbulence theory provided the first-principle based explanation. Specifically, Refs. 12, 23, and 24 demonstrated that the formation of electron kappa distribution is intimately related to the long-time evolution of Langmuir turbulence. The weak turbulence theory is also successfully employed to explain the solar radio bursts.[25–33] The validity of weak turbulence theory was recently confirmed against the particle-in-cell (PIC) simulation.[34] Several PIC simulations of electron beam-generated Langmuir turbulence and the ensuing electromagnetic (EM) radiation emission have been carried out in the literature.[35–48] However, Ref. 34 stands out in that the PIC simulation and weak turbulence theory was compared quantitatively.

Despite its successes, the standard weak turbulence theory found in the literature is mostly limited to unmagnetized plasmas. A fully general version of such a theory for magnetized plasmas does not yet exist. Some early efforts[49–51] attempted to formulate such a theory from a fully general kinetic plasma theory, but the usefulness of such efforts is obscured by the inherent complexity. The more concrete weak turbulence theories for magnetized plasmas that readily lend themselves to theoretical and/or numerical analyses, instead, have been developed by making certain simplifying assumptions at the outset. For instance, by assuming a low-frequency and long-wavelength regime, the magnetohydrodynamic weak turbulence theory was formulated and solved.[52–60] The mode-coupling process among electrostatic cyclotron-harmonic waves was discussed within the framework of weak turbulence ordering.[61–64] Recently, a weak turbulence theory that involves the whistler-mode and lower-hybrid waves was formulated and applied to a number of space plasma situations.[65–68] The weak turbulence theory was extended to interpret the polarization of solar coronal type III radio bursts.[51,69–75] A weak turbulence theory for general magnetized plasmas was formulated and solved but under the strict assumption of either parallel or perpendicular propagation.[76–80]

A major obstacle for extending the weak turbulence theory to fully magnetized plasmas is the computation of nonlinear susceptibility, as evidenced by the above-referenced early attempts.[49–51] A method was recently proposed to overcome such a difficulty. In a recent work,[81] one of the present authors (P.H.Y.) noted that one can partially reformulate the kinetic weak turbulence theory by resorting to the warm two-fluid theory. Specifically, it was noted that the nonlinear decay interactions among Langmuir and ion-sound waves could be fully discussed by resorting to the warm two-fluid approach. The cold two-fluid theory was pointed out as being inadequate since certain decay interaction coefficients are inversely proportional to the electron temperature. Obviously, the cold plasma theory is inapplicable for such processes. While Ref. 81 pointed out the usefulness of the warm two-fluid approach under a general situation, including the magnetized plasmas, for actual demonstration, Ref. 81 only considered the unmagnetized plasma problem as an example that proves the basic concept.

The weakly turbulent processes in unmagnetized plasmas that involve the interaction of Langmuir wave, ion-acoustic waves, transverse radiation, and the particles form the basic building blocks for the so-called plasma emission, which is the fundamental radiation emission mechanism responsible for the solar coronal and interplanetary type II and type III radio bursts. However, for type III radio bursts close to the solar active regions, the effects of finite background magnetic field can be an important factor in the interpretation of data. A recent particle-in-cell simulation of the plasma emission process in magnetized plasmas shows that the assumption of unmagnetized plasmas may be valid under certain conditions, particularly, when the medium is characterized by a high ratio of electron plasma frequency to electron–cyclotron frequency; but as the same ratio is reduced, say to order ten or less, the underlying wave–particle interaction as well as the mode-coupling (that is, the weakly turbulent plasma) processes undergo some dramatic shifts in their characteristics, which call for further theoretical development that reflects plasma magnetization.[82]

The purpose of this paper is to consider the first example of utilizing the warm two-fluid formalism to derive the basic equations of weak turbulence theory in magnetized plasmas, with a long-term focus of extending the existing unmagnetized plasma theory of plasma emission to that of magnetized plasma theory of plasma emission. To simplify the analysis, however, we first consider the electrostatic problem. Obviously, a fully electromagnetic version should follow, but it is a subject of future tasks. Without the electromagnetic effects, the radiation emission cannot be discussed, but the wave–particle interaction between the type III-emitting electron beams and electrostatic turbulence can be discussed with the electrostatic weak turbulence theory. For an unmagnetized plasma, the type III electron beams interact with the Langmuir waves, which undergo nonlinear interaction with the ion-sound waves and the background protons. The magnetized plasma analog of such a process will involve the electron beam interacting with the upper-hybrid waves, which undergo nonlinear interaction with the low-frequency sonic type of modes as well as the protons. The electrostatic weak turbulence theory in warm magnetized plasmas to be discussed herein is meant to provide a quantitative description of these processes.

As the remainder of this paper will illustrate, we begin the discourse based on the warm two-fluid equations (Sec. II). Section II is subdivided into subsections that deal with the first- and second-order iterative solutions, which are then combined into a nonlinear wave equation. Section III discusses the generic form of an electrostatic wave kinetic equation under weak turbulence ordering. A detailed derivation of the equations for electrostatic weak turbulence theory in magnetized plasmas is given in Sec. IV. This section is also subdivided into subsections, where each subsection deals with various subtopics, which includes adding kinetic effects for the complete descriptions of wave–particle interaction. Finally, the findings of the present paper are summarized in Sec. V, and some discussions related to future directions of the research are presented therein.

II. NONLINEAR WARM TWO-FLUID EQUATIONS

In the present analysis, n_a denotes the fluid density for species a; $\mathbf{v}_a$ denotes the fluid velocity; m_a, e_a, and c denote the mass, unit electric charge, and speed of light in vacuum, respectively; $\mathbf{E}$ is the electrostatic field vector; and $\mathbf{B}_0$ is the ambient magnetic field. We start from the electrostatic two-fluid equation in magnetized plasmas as follows:

$$\begin{aligned}
&\frac{\partial n_a}{\partial t} + \nabla\cdot(n_a\mathbf{v}_a) = 0,\\
&m_a n_a\frac{d\mathbf{v}_a}{dt} + \nabla P_a - e_a n_a\left(\mathbf{E} + \frac{1}{c}\,\mathbf{v}_a\times\mathbf{B}_0\right) = 0,\\
&\nabla\cdot\mathbf{E} = \sum_{a=e,i} 4\pi e_a n_a,
\end{aligned} \tag{1}$$

where $d/dt = \partial/\partial + \mathbf{v}_a\cdot\nabla$, and $a = e, i$ denotes electrons and ions, respectively. Assuming that the pressure is given by the product of density and temperature, $P_a = n_a T_a$, and separating density into an average and fluctuation term, $n_a = n_0 + \delta n_a$, while also denoting the velocity and electric field with δ preceding them to indicate that they are fluctuating quantities, we have

$$\begin{aligned}
&\frac{\partial \delta n_a}{\partial t} + n_0\nabla\cdot\delta\mathbf{v}_a + \nabla\cdot(\delta n_a\,\delta\mathbf{v}_a) = 0,\\
&\frac{\partial \delta\mathbf{v}_a}{\partial t} - \Omega_a\delta\mathbf{v}_a\times\mathbf{b} - \frac{e_a}{m_a}\,\delta\mathbf{E} + \frac{\nabla\delta n_a}{n_0}\,v_{Ta}^2\\
&\quad + \delta\mathbf{v}_a\cdot\nabla\delta\mathbf{v}_a - \frac{\delta n_a}{n_0}\frac{\nabla\delta n_a}{n_0}\,v_{Ta}^2 = 0,\\
&\nabla\cdot\delta\mathbf{E} = \sum_a 4\pi e_a\,\delta n_a,
\end{aligned} \tag{2}$$

where $v_{Ta}^2 = T_a/m_a$ represents the square of fluid thermal speed and $\Omega_a = e_a B_0/m_a c$ is the cyclotron frequency for species a. Here, we assume that the temperature is defined in units of energy; hence, the Boltzmann constant is set equal to unity, $k_B = 1$. Note that v_{Ta}^2 differs from the kinetic counterpart where it is defined by $v_{Ta}^2 = 2T_a/m_a$. We employ an iterative method to obtain the solution, $\delta n_a = n_a^{(1)} + n_a^{(2)} + \cdots$ and $\delta\mathbf{v}_a = \mathbf{v}_a^{(1)} + \mathbf{v}_a^{(2)} + \cdots$, where $n_a^{(1)}$ and $\mathbf{v}_a^{(1)}$ are proportional to $\mathcal{O}(\delta E)$, $n_a^{(2)}$ and $\mathbf{v}_a^{(2)}$ are proportional to $\mathcal{O}(\delta E^2)$, etc. That is, we follow the standard weak turbulence ordering where particle quantities are expanded in power series with each term proportional to the power of the field intensity. We then organize the resulting equations for each order. We write down the result in spectral form where the spectral transformation is defined by $f_{\mathbf{k},\omega} = (2\pi)^{-4}\int d\mathbf{r}\int dt f(\mathbf{r},t)e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}$, together with the inverse transformation, $f(\mathbf{r},t) = \int d\mathbf{k}\int d\omega f_{\mathbf{k},\omega}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} = \sum_{\mathbf{k},\omega} f_{\mathbf{k},\omega}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}$. Let us adopt the following shorthand notations:

$$\begin{aligned}
&q = (\mathbf{k},\omega), \quad r_q = \frac{n_{\mathbf{k},\omega}^a}{n_0},\\
&\mathbf{u}_q = \mathbf{v}_{\mathbf{k},\omega}^a, \quad \eta = \frac{\Omega_a}{\omega}.
\end{aligned} \tag{3}$$

We then have the following for each order:

$$
\begin{aligned}
r_q^{(1)} &= \frac{\mathbf{k}\cdot\mathbf{u}_q^{(1)}}{\omega},\\
\mathbf{u}_q^{(1)} - i\eta\,\mathbf{u}_q^{(1)}\times\mathbf{b} - \frac{\mathbf{k}v_{Ta}^2}{\omega}\,r_q^{(1)} - \frac{ie_a}{m_a\omega}\,\mathbf{E}_q &= 0,\\
r_q^{(2)} &= \frac{\mathbf{k}\cdot\mathbf{u}_q^{(2)}}{\omega} + \frac{\mathbf{k}}{\omega}\cdot\sum_{q'} r_{q'}^{(1)}\mathbf{u}_{q-q'}^{(1)},\\
\mathbf{u}_q^{(2)} - i\eta\,\mathbf{u}_q^{(2)}\times\mathbf{b} - \frac{\mathbf{k}v_{Ta}^2}{\omega}\,r_q^{(2)}&\\
-\frac{1}{2\omega}\sum_{q'}\Big[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{u}_{q'}^{(1)}\mathbf{u}_{q-q'}^{(1)} + \mathbf{k}'\cdot\mathbf{u}_{q-q'}^{(1)}\mathbf{u}_{q'}^{(1)}\Big]&\\
+\frac{\mathbf{k}v_{Ta}^2}{2\omega}\sum_{q'} r_{q'}^{(1)}r_{q-q'}^{(1)} &= 0,\\
\mathbf{k}\cdot\mathbf{E}_q &= -i\textstyle\sum_a 4\pi e_a n_0\,\big(r_q^{(1)} + r_q^{(2)}\big).
\end{aligned}
\tag{4}
$$

Let us pay attention to the momentum equations. One may rearrange these equations to obtain

$$
\begin{aligned}
(1-\eta^2)\mathbf{u}_q^{(1)} &= \frac{v_{Ta}^2}{\omega}\,(\mathbf{k} - i\eta\mathbf{b}\times\mathbf{k} - \eta^2\mathbf{b}\mathbf{k}\cdot\mathbf{b})r_q^{(1)}\\
&\quad + \frac{ie_a}{m_a\omega}\,(\mathbf{E}_q - i\eta\mathbf{b}\times\mathbf{E}_q - \eta^2\mathbf{b}\mathbf{b}\cdot\mathbf{E}_q),\\
(1-\eta^2)\mathbf{u}_q^{(2)} &= \frac{v_{Ta}^2}{\omega}\,(\mathbf{k} - i\eta\mathbf{b}\times\mathbf{k} - \eta^2\mathbf{b}\mathbf{k}\cdot\mathbf{b})r_q^{(2)}\\
&\quad + \frac{1}{2\omega}\sum_{q'}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{u}_{q'}^{(1)}\\
&\quad \times(\mathbf{u}_{q-q'}^{(1)} - i\eta\mathbf{b}\times\mathbf{u}_{q-q'}^{(1)} - \eta^2\mathbf{b}\mathbf{b}\cdot\mathbf{u}_{q-q'}^{(1)})\\
&\quad + \frac{1}{2\omega}\sum_{q'}\mathbf{k}'\cdot\mathbf{u}_{q-q'}^{(1)}\\
&\quad \times(\mathbf{u}_{q'}^{(1)} - i\eta\mathbf{b}\times\mathbf{u}_{q'}^{(1)} - \eta^2\mathbf{b}\mathbf{b}\cdot\mathbf{u}_{q'}^{(1)})\\
&\quad - \frac{v_{Ta}^2}{2\omega}\sum_{q'}(\mathbf{k} - i\eta\mathbf{b}\times\mathbf{k} - \eta^2\mathbf{b}\mathbf{k}\cdot\mathbf{b})r_{q'}^{(1)}r_{q-q'}^{(1)}.
\end{aligned}
\tag{5}
$$

Let us define

$$
q_\omega^{ij} = \delta_{ij} + \frac{i\Omega_a}{\omega}\,\varepsilon_{ijk}b_k - \frac{\Omega_a^2}{\omega^2}\,b_ib_j. \tag{6}
$$

Then, we obtain a compact notation

$$
\begin{aligned}
u_q^{(1)i} &= \frac{q_\omega^{ij}}{\omega(1-\eta^2)}\left(k_jv_{Ta}^2r_q^{(1)} + \frac{ie_a}{m_a}\,E_q^j\right),\\
u_q^{(2)i} &= \frac{1}{\omega(1-\eta^2)}\Big(q_\omega^{ij}k_jv_{Ta}^2r_q^{(2)}\\
&\quad + \frac{1}{2}\sum_{q'}\Big\{\Big[q_\omega^{ij}k'_k + q_\omega^{ik}(\mathbf{k}-\mathbf{k}')_j\Big]u_{q'}^{(1)j}u_{q-q'}^{(1)k}\\
&\quad - q_\omega^{ij}k_jv_{Ta}^2r_{q'}^{(1)}r_{q-q'}^{(1)}\Big\}\Big).
\end{aligned}
\tag{7}
$$

Making use of the velocity fluctuations given by Eq. (7), we may construct the density fluctuations as follows:

$$
\begin{aligned}
r_q^{(1)} &= \frac{k_iq_\omega^{ij}}{\omega^2(1-\eta^2)}\left(k_jv_{Ta}^2r_q^{(1)} + \frac{ie_a}{m_a}\,E_q^j\right),\\
r_q^{(2)} &= \frac{1}{\omega^2(1-\eta^2)}\Big(q_\omega^{ij}k_ik_jv_{Ta}^2r_q^{(2)}\\
&\quad + \frac{1}{2}\sum_{q'}\Big\{k_i\Big[q_\omega^{ij}k'_k + q_\omega^{ik}(\mathbf{k}-\mathbf{k}')_j\Big]u_{q'}^{(1)j}u_{q-q'}^{(1)k}\\
&\quad - q_\omega^{ij}k_ik_jv_{Ta}^2r_{q'}^{(1)}r_{q-q'}^{(1)}\Big\}\Big) + \frac{1}{\omega}\sum_{q'} r_{q'}^{(1)}\mathbf{k}\cdot\mathbf{u}_{q-q'}^{(1)}.
\end{aligned}
\tag{8}
$$

A. First-order solution

The first-order solution can be obtained from coupled equations given by Eqs. (7) and (8) by ignoring the nonlinear terms and second-order terms. This results in the following:

$$
\begin{aligned}
r_q^{(1)} &= \frac{ie_a}{m_a}\frac{k_iq_\omega^{ij}E_q^j}{\omega^2(1-\eta^2) - \big[k^2 - \eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]v_{Ta}^2},\\
u_q^{(1)i} &= \frac{ie_a}{m_a\omega(1-\eta^2)}\left(\frac{\Big[\omega^2(1-\eta^2) - \big[k^2-\eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]v_{Ta}^2\Big]q_\omega^{ij}E_q^j}{\omega^2(1-\eta^2) - \big[k^2 - \eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]v_{Ta}^2}\right.\\
&\quad \left. + \frac{q_\omega^{ij}k_jk_kq_\omega^{kl}v_{Ta}^2E_q^k}{\omega^2(1-\eta^2) - \big[k^2 - \eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]v_{Ta}^2}\right).
\end{aligned}
\tag{9}
$$

We make note of the fact that the fluid correction becomes important near resonances. Thus, the various terms associated with v_{Ta}^2 affect the denominators but can be ignored in the numerator. Adopting such an approximation scheme, we have the first-order solution as follows:

$$
\begin{aligned}
u_q^{(1)i} &= \frac{ie_a}{m_a|k|}\frac{\omega q_\omega^{ij}k_j\,E_q}{\omega^2(1-\eta^2) - \big[k^2 - \eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]v_{Ta}^2},\\
r_q^{(1)} &= \frac{ie_a}{m_a|k|}\frac{\big[k^2 - \eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]\,E_q}{\omega^2(1-\eta^2) - \big[k^2 - \eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]v_{Ta}^2},\\
q_\omega^{ij} &= \delta_{ij} + i\eta\,\varepsilon_{ijk}b_k - \eta^2\,b_ib_j,
\end{aligned}
\tag{10}
$$

where we have expressed the electric field vector as

$$
E_q^j = \frac{k_j}{|k|}\,E_q \tag{11}
$$

since we are dealing with an electrostatic problem.

B. Second-order solution

The second-order solution can be discussed with the density $r_q^{(2)}$ only since the Poisson equation involves only $r_q^{(1)}$ and $r_q^{(2)}$. Upon making use of the first-order solution given by Eq. (10), it is possible to show, after some tedious but otherwise straightforward algebraic manipulations, that $r_q^{(2)}$ is given by

$$
\begin{aligned}
r_q^{(2)} = &-\sum_{q'}\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega'(\omega-\omega')}{k'|\mathbf{k}-\mathbf{k}'|R_qR_{q'}R_{q-q'}}\\
&\times\Big[k_i q_\omega^{il}k'_m k'_j q_{\omega'}^{lj}(\mathbf{k}-\mathbf{k}')_k q_{\omega-\omega'}^{mk}\\
&+k_i q_\omega^{im}k'_j q_{\omega'}^{lj}(\mathbf{k}-\mathbf{k}')_l(\mathbf{k}-\mathbf{k}')_k q_{\omega-\omega'}^{mk}\Big]E_{q'}E_{q-q'}\\
&-\sum_{q'}\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega^2-\Omega_a^2}{\omega k'|\mathbf{k}-\mathbf{k}'|R_qR_{q'}R_{q-q'}}\\
&\times\Big[(\omega-\omega')\big[k'^2-\eta'^2(\mathbf{k}'\cdot\mathbf{b})^2\big]k_l(\mathbf{k}-\mathbf{k}')_k q_{\omega-\omega'}^{lk}\\
&+\omega'\{(\mathbf{k}-\mathbf{k}')^2-\eta''^2\big[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}\big]^2\}k_l k'_j q_{\omega'}^{lj}\Big]E_{q'}E_{q-q'},
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
R_q &= \omega^2-\Omega_a^2-\big[k^2-\eta^2(\mathbf{k}\cdot\mathbf{b})^2\big]v_{Ta}^2,\\
R_{q'} &= \omega'^2-\Omega_a^2-\big[k'^2-\eta'^2(\mathbf{k}'\cdot\mathbf{b})^2\big]v_{Ta}^2,\\
R_{q-q'} &= (\omega-\omega')^2-\Omega_a^2\\
&\quad-\{(\mathbf{k}-\mathbf{k}')^2-\eta''^2\big[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}\big]^2\}v_{Ta}^2,\\
\eta &= \frac{\Omega_a}{\omega},\quad \eta'=\frac{\Omega_a}{\omega'},\quad \eta''=\frac{\Omega_a}{\omega-\omega'}.
\end{aligned}
\tag{13}
$$

It is instructive to carry out the vector multiplications associated with the following quantities: $k_i q_\omega^{il}k'_m k'_j q_{\omega'}^{lj}(\mathbf{k}-\mathbf{k}')_k q_{\omega-\omega'}^{mk}$, $k_i q_\omega^{im}k'_j q_{\omega'}^{lj}(\mathbf{k}-\mathbf{k}')_l(\mathbf{k}-\mathbf{k}')_k q_{\omega-\omega'}^{mk}$, $k_l(\mathbf{k}-\mathbf{k}')_k q_{\omega-\omega'}^{lk}$, $k_l k'_j q_{\omega'}^{lj}$. After carrying out explicit manipulations of these quantities, we have

$$
\begin{aligned}
r_q^{(2)} = &-\sum_{q'}\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega'(\omega-\omega')}{k'|\mathbf{k}-\mathbf{k}'|R_qR_{q'}R_{q-q'}}\\
&\times(\{(1+\eta\eta')(\mathbf{k}\cdot\mathbf{k}')-(\eta^2+\eta'^2+\eta\eta'-\eta^2\eta'^2)(\mathbf{k}\cdot\mathbf{b})(\mathbf{k}'\cdot\mathbf{b})\\
&+i(\eta+\eta')(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{b}\}\times\{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')-i\eta''(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{b}\\
&-\eta''^2(\mathbf{k}'\cdot\mathbf{b})[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]\}+\{(1+\eta\eta'')[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]\\
&-(\eta^2+\eta''^2+\eta\eta''-\eta^2\eta''^2)(\mathbf{k}\cdot\mathbf{b})[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]\\
&-i(\eta+\eta'')(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{b}\}\{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')+i\eta'(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{b}\\
&-\eta'^2(\mathbf{k}'\cdot\mathbf{b})[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]\})E_{q'}E_{q-q'}\\
&-\sum_{q'}\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega^2-\Omega_a^2}{\omega k'|\mathbf{k}-\mathbf{k}'|R_qR_{q'}R_{q-q'}}\\
&\times((\omega-\omega')[k'^2-\eta'^2(\mathbf{k}'\cdot\mathbf{b})^2]\{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\\
&-i\eta''(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{b}-\eta''^2(\mathbf{k}\cdot\mathbf{b})[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]\}\\
&+\omega'\{(\mathbf{k}-\mathbf{k}')^2-\eta''^2[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]^2\}\{\mathbf{k}\cdot\mathbf{k}'+i\eta'(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{b}\\
&-\eta'^2(\mathbf{k}\cdot\mathbf{b})(\mathbf{k}'\cdot\mathbf{b})\})E_{q'}E_{q-q'}.
\end{aligned}
\tag{14}
$$

In what follows, we restrict $\mathbf{k}$ and $\mathbf{k}'$ to lie in the xz plane, $\mathbf{k}=\hat{\mathbf{x}}k_\perp+\hat{\mathbf{z}}k_\parallel$ and $\mathbf{k}'=\hat{\mathbf{x}}k'_\perp+\hat{\mathbf{z}}k'_\parallel$, while assuming $\mathbf{b}=\hat{\mathbf{z}}$. Then $(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{b}=0$. This is a reasonable assumption since the nonlinear term in the wave equation is associated with the integral $\sum_{q'}=\int d\mathbf{k}'$, which includes the average over azimuthal angle associated with the $\mathbf{k}'$ vector. Therefore, physical quantities orthogonal to both $\mathbf{k}$ and $\mathbf{k}'$ will be averaged over the azimuthal angle. The terms associated with $\mathbf{k}\times\mathbf{k}'$ are such quantities. As such, it is reasonable to assume that both $\mathbf{k}$ and $\mathbf{k}'$ lie in the xz plane and the resultant nonlinear terms can be averaged over the azimuthal angle in the end. This simplifies the expression as follows:

$$
\begin{aligned}
r_q^{(2)} = &-\sum_{q'}\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega'(\omega-\omega')}{k'|\mathbf{k}-\mathbf{k}'|R_qR_{q'}R_{q-q'}}\\
&\times([(1+\eta\eta')(\mathbf{k}\cdot\mathbf{k}')-(\eta^2+\eta'^2+\eta\eta'-\eta^2\eta'^2)k_\parallel k'_\parallel]\\
&\times[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')-\eta''^2k'_\parallel(k_\parallel-k'_\parallel)]+\{(1+\eta\eta'')[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')]\\
&-(\eta^2+\eta''^2+\eta\eta''-\eta^2\eta''^2)k_\parallel(k_\parallel-k'_\parallel)\}\\
&\times[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')-\eta'^2k'_\parallel(k_\parallel-k'_\parallel)])E_{q'}E_{q-q'}\\
&-\sum_{q'}\frac{1}{2}\frac{e_a^2}{m_a^2}\frac{\omega^2-\Omega_a^2}{\omega k'|\mathbf{k}-\mathbf{k}'|R_qR_{q'}R_{q-q'}}((\omega-\omega')(k'^2-\eta'^2k_\parallel'^2)\\
&\times[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')-\eta''^2k_\parallel(k_\parallel-k'_\parallel)]\\
&+\omega'[(\mathbf{k}-\mathbf{k}')^2-\eta''^2(k_\parallel-k'_\parallel)^2]\\
&\times(\mathbf{k}\cdot\mathbf{k}'-\eta'^2k_\parallel k'_\parallel))E_{q'}E_{q-q'}.
\end{aligned}
\tag{15}
$$

C. Nonlinear wave equation

We substitute the densities, $r_q^{(1)}$ and $r_q^{(2)}$ to Poisson equation in Eq. (4). The result can be written in long-hand notation as

$$
\varepsilon(\mathbf{k},\omega)E_{\mathbf{k},\omega}=\sum_{\mathbf{k}',\omega'}\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')E_{\mathbf{k}',\omega'}E_{\mathbf{k}-\mathbf{k}',\omega-\omega'},\tag{16}
$$

where $\varepsilon(\mathbf{k},\omega)$ is the fluid version of the linear dielectric constant and $\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')$ is the (second-order) nonlinear susceptibility, which are defined, respectively, by

$$
\varepsilon(\mathbf{k},\omega)=1-\sum_a\frac{\omega_{pa}^2}{k^2}\frac{k^2-\eta^2k_\parallel^2}{R_{\mathbf{k},\omega}},\tag{17}
$$

$$
\begin{aligned}
&\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\
&=\sum_a\frac{i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{kk'|\mathbf{k}-\mathbf{k}'|R_{\mathbf{k},\omega}R_{\mathbf{k}',\omega'}R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}\\
&\quad\times[\omega'(\omega-\omega')\Big([(1+\eta\eta')(\mathbf{k}\cdot\mathbf{k}')\\
&\quad-(\eta^2+\eta'^2+\eta\eta'-\eta^2\eta'^2)k_\parallel k'_\parallel]\\
&\quad\times\Big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')-\eta''^2k'_\parallel(k_\parallel-k'_\parallel)\Big]\\
&\quad+\{(1+\eta\eta'')\big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\big]\\
&\quad-(\eta^2+\eta''^2+\eta\eta''-\eta^2\eta''^2)k_\parallel(k_\parallel-k'_\parallel)\}\\
&\quad\times\Big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')-\eta'^2k'_\parallel(k_\parallel-k'_\parallel)\Big]\Big)\\
&\quad+\frac{\omega^2-\Omega_a^2}{\omega}((\omega-\omega')(k'^2-\eta'^2k_\parallel'^2)\big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\\
&\quad-\eta''^2k_\parallel(k_\parallel-k'_\parallel)]+\omega'\big[(\mathbf{k}-\mathbf{k}')^2\\
&\quad-\eta''^2(k_\parallel-k'_\parallel)^2\big](\mathbf{k}\cdot\mathbf{k}'-\eta'^2k_\parallel k'_\parallel))\Big].
\end{aligned}
\tag{18}
$$

In the above, the resonance denominators are now expressed in long-hand notation as $R_{\mathbf{k},\omega}=\omega^2-\Omega_a^2-(k^2-\eta^2k_\parallel^2)v_{Ta}^2$, $R_{\mathbf{k}',\omega'}=\omega'^2-\Omega_a^2-(k'^2-\eta'^2k_\parallel'^2)v_{Ta}^2$, $R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}=(\omega-\omega')^2-\Omega_a^2-[(\mathbf{k}-\mathbf{k}')^2-\eta''^2(k_\parallel-k'_\parallel)^2]v_{Ta}^2$.

III. GENERAL WAVE KINETIC EQUATION FOR ELECTROSTATIC WEAK TURBULENCE

Note that Eq. (16) is given by the standard form of electrostatic weak turbulence theory. We start from the general form of nonlinear wave equation given by Eq. (16), regardless of the specific form of $\varepsilon(\mathbf{k},\omega)$ or $\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')$, which is compactly rewritten as

$$0=\varepsilon(K)E_j(K)+\sum_{1+2=K}\chi(1|2)E_j(1)E_k(2), \tag{19}$$

where $K=(\mathbf{k},\omega)$, $1=K_1=(\mathbf{k}_1,\omega_1)$, and $2=K_2=(\mathbf{k}_2,\omega_2)$. We multiply this equation with $E_i(K')$ and take the ensemble average

$$\begin{aligned}0&=\varepsilon(K)\langle E^2\rangle_K\delta(K+K')\\&\quad+\sum_1\chi(1|K-1)\langle E(K')E(1)E(K-1)\rangle,\end{aligned} \tag{20}$$

where we have made use of the property of homogeneous and stationary turbulence, $\langle E(K)E(K')\rangle=\langle E^2\rangle_K\delta(K+K')$. The third-body correlation $\langle E(K')E(1)E(K-1)\rangle$ can be obtained in the customary way. That is, we iteratively solve the wave equation by writing $E(K)=E^{(0)}(K)+E^{(1)}(K)$, where $E^{(0)}(K)$ satisfies the linear dispersion relation, $\varepsilon(K)E^{(0)}(K)=0$. Then, the next-order correction is obtained via the nonlinear term. Specifically, we have $E^{(1)}(K')=-\varepsilon^{-1}(K)\sum_2\chi(2|K'-2)E^{(0)}(2)E^{(0)}(K'-2)$, $E^{(1)}(1)=-\varepsilon^{-1}(1)\sum_2\chi(2|1-2)E^{(0)}(2)E^{(0)}(1-2)$, and $E^{(1)}(K-1)=-\varepsilon^{-1}(K-1)\sum_2\chi(2|K-1-2)E^{(0)}(2)E^{(0)}(K-1-2)$. Since odd cumulants of $E^{(0)}$ vanish, the desired third-body cumulant $\langle E(K')E(1)E(K-1)\rangle$ can be obtained by adding contributions from $\langle E^{(1)}(K')E^{(0)}(1)E^{(0)}(K-1)\rangle$, $\langle E^{(0)}(K')E^{(1)}(1)E^{(0)}(K-1)\rangle$, and $\langle E^{(0)}(K')E^{(0)}(1)E^{(1)}(K-1)\rangle$. This process leads to the four-body cumulants, but we close this hierarchy by writing the four-body cumulants as products of two-body cumulants, while ignoring the irreducible four-body correlation function. In manipulating the products of two-body cumulants, we ignore the correlations when the argument becomes zero, since such quantities represent spatial correlations separated by an infinite distance, and temporal correlations separated by an infinitely long interval. Such terms are clearly unphysical. These procedures are well-described by the recent monograph,[12] but also discussed in standard literature, including Ref. 83. This method of closure is known as the quasi-normal closure in the literature. The result is the following:

$$\begin{aligned}&\langle E(K')E(1)E(K-1)\rangle\\&\quad=-2\delta(K+K')\left(\frac{\chi(-1|-K+1)\langle E^2\rangle_1\langle E^2\rangle_{K-1}}{\varepsilon(-K)}\right.\\&\qquad\left.+\frac{\chi(K|1-K)\langle E^2\rangle_K\langle E^2\rangle_{K-1}}{\varepsilon(1)}+\frac{\chi(K|-1)\langle E^2\rangle_K\langle E^2\rangle_1}{\varepsilon(K-1)}\right).\end{aligned} \tag{21}$$

From this, we obtain the formal nonlinear spectral balance equation, but in doing so, we make note of the symmetry property associated with the linear and nonlinear response functions, $\varepsilon(-K)=\varepsilon^*(K)$, $\chi(1|2)=\chi(2|1)$, and $\chi(1|2)=\chi(1+2|-2)$. We also take the customary approach of replacing the leading linear dielectric constant by a term that retains the slow time derivative, $\varepsilon(\mathbf{k},\omega)\langle E^2\rangle_{\mathbf{k},\omega}\to\varepsilon(\mathbf{k},\omega)\langle E^2\rangle_{\mathbf{k},\omega}+(i/2)[\partial\varepsilon(\mathbf{k},\omega)/\partial\omega](\partial\langle E^2\rangle_{\mathbf{k},\omega}/\partial t)$. The result is the following:

$$\begin{aligned}0&=\frac{i}{2}\frac{\partial\varepsilon(\mathbf{k},\omega)}{\partial\omega}\frac{\partial\langle E^2\rangle_{\mathbf{k},\omega}}{\partial t}+\varepsilon(\mathbf{k},\omega)\langle E^2\rangle_{\mathbf{k},\omega}\\&\quad+2\int d\mathbf{k}'\int d\omega'\Big\{\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\&\quad\times\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\&\quad\times\left(\frac{\langle E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\varepsilon(\mathbf{k}',\omega')}+\frac{\langle E^2\rangle_{\mathbf{k}',\omega'}}{\varepsilon(\mathbf{k}-\mathbf{k}',\omega-\omega')}\right)\langle E^2\rangle_{\mathbf{k},\omega}\\&\quad-\chi(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\chi^*(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\&\quad\times\frac{\langle E^2\rangle_{\mathbf{k}',\omega'}\langle E^2\rangle_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}{\varepsilon^*(\mathbf{k},\omega)}\Big\}.\end{aligned} \tag{22}$$

By taking the real part of Eq. (22) while ignoring the nonlinear part, we have $\mathrm{Re}\,\varepsilon(\mathbf{k},\omega)=0$, from which we obtain the wave dispersion relation, $\omega=\omega_{\mathbf{k}}^{\alpha}$, where α denotes the possibility of multiple roots. This also leads to $\langle E^2\rangle_{\mathbf{k},\omega}=\sum_{\sigma=\pm1}\sum_\alpha I_{\mathbf{k}}^{\sigma\alpha}\delta(\omega-\sigma\omega_{\mathbf{k}}^{\alpha})$. Upon substituting this back to Eq. (22), we obtain the final form of weak turbulence wave kinetic equation under electrostatic approximation as follows:

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma\alpha}}{\partial t}&=-\frac{2\,\mathrm{Im}\,\varepsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}{\varepsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}I_{\mathbf{k}}^{\sigma\alpha}-\frac{4\pi}{\varepsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}\sum_{\beta,\gamma}\sum_{\sigma',\sigma''=\pm1}\\&\quad\times\int d\mathbf{k}'\,|\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})|^2\\&\quad\times\left(\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}I_{\mathbf{k}}^{\sigma\alpha}}{\varepsilon'(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta})}+\frac{I_{\mathbf{k}'}^{\sigma'\beta}I_{\mathbf{k}}^{\sigma\alpha}}{\varepsilon'(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})}-\frac{I_{\mathbf{k}'}^{\sigma'\beta}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''\gamma}}{\varepsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}\right)\\&\quad\times\delta(\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{\gamma})\\&\quad-\frac{4}{\varepsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}\,\mathrm{Im}\sum_\beta\sum_{\sigma'=\pm1}\int d\mathbf{k}'\\&\quad\times\mathscr{P}\frac{2\{\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})\}^2}{\varepsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})}I_{\mathbf{k}'}^{\sigma'\beta}I_{\mathbf{k}}^{\sigma\alpha},\end{aligned} \tag{23}$$

where $\mathscr{P}$ denotes the principal value and

$$\varepsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})=\frac{\partial\,\mathrm{Re}\,\varepsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{\alpha})}{\partial(\sigma\omega_{\mathbf{k}}^{\alpha})}. \tag{24}$$

We remind the readers that Eq. (23) is a generic wave equation for electrostatic weak turbulence. In the context of the present warm two-fluid theoretical result, Eqs. (16)–(18), however, the imaginary part of $\varepsilon(\mathbf{k},\omega)$ is absent. However, as we will discuss later, we complement the formulation of electrostatic weak turbulence theory for magnetized plasmas by computing the imaginary part of $\varepsilon(\mathbf{k},\omega)$ from the kinetic theoretical calculation. For this reason, we leave $\mathrm{Im}\,\varepsilon(\mathbf{k},\omega)$ intact in the subsequent formulation.

IV. ELECTROSTATIC WEAK TURBULENCE THEORY IN MAGNETIZED PLASMAS

We are now ready for a concrete formulation of electrostatic weak turbulence theory in magnetized plasmas. The first step in the discussion is the dispersion relation. We then make use of the linear dispersive properties of the normal mode to simplify the nonlinear susceptibility, which leads to the desired wave kinetic equation. The warm two-fluid theoretical approach is thus adequate for formulating the nonlinear wave kinetic equation in magnetized plasmas. However, to complete the analysis, one must supplement the formalism by computing the imaginary part of the dielectric function as well as to provide a self-consistent description of the dynamical evolution of the particle distribution function. The kinetic equation for the particles will thus be provided by invoking the quasilinear kinetic theory.

A. Dispersion relation

We start from a discussion of the wave dispersion relation and the related properties of the normal modes. If we ignore nonlinear terms in Eq. (16), the dispersion relation is given by

$$0 = \varepsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2 - k^2 v_{Te}^2 + (\Omega_e^2/\omega^2)k_\parallel^2 v_{Te}^2}\left(1 - \frac{\Omega_e^2}{\omega^2}\frac{k_\parallel^2}{k^2}\right) - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2 - k^2 v_{Ti}^2 + (\Omega_i^2/\omega^2)k_\parallel^2 v_{Ti}^2}\left(1 - \frac{\Omega_i^2}{\omega^2}\frac{k_\parallel^2}{k^2}\right). \quad (25)$$

Figure 1 plots the numerical solution of the dispersion relation given by Eq. (25) for $\omega_{pe}/|\Omega_e| = 5$ and $T_i/T_e = 0.1$. The numerical solution was obtained for low values of T_i/T_e since ion-sound waves damp for high T_i. Although the fluid dispersion relation given by Eq. (25) does not have an imaginary part, the ion-sound waves damp for high values of T_i when compared with the electron temperature when we include the collisional damping effects by adding the kinetic effects. Figure 1 (top-left) corresponds to the Langmuir mode solution for quasi-parallel angle of propagation, which gradually turns into the upper-hybrid mode for quasi-perpendicular angle of propagation. In Fig. 1 (top-left), the cases of $\theta = 5°$ and $\theta = 80°$ correspond to quasi-parallel and quasi-perpendicular angles of propagation, with the intermediate value $\theta = 45°$ also plotted. Figure 1 (top-right) shows the electron–cyclotron mode, whose frequency is close to the electron–cyclotron frequency for quasi-parallel propagation, which gradually decreases as θ increases. The bottom-left panel of Fig. 1 plots the ion-acoustic branch of the solution, which shows that the curves for all three angles of propagation almost overlap. Finally, Fig. 1 (bottom-right) plots the ion–cyclotron mode. As with the electron–cyclotron mode, the ion–cyclotron mode frequency ω is close to Ω_i when θ is low, but as θ increases, the frequency decreases. For $\theta = 90°$, both cyclotron modes reduce to zero frequency. The numerical solution

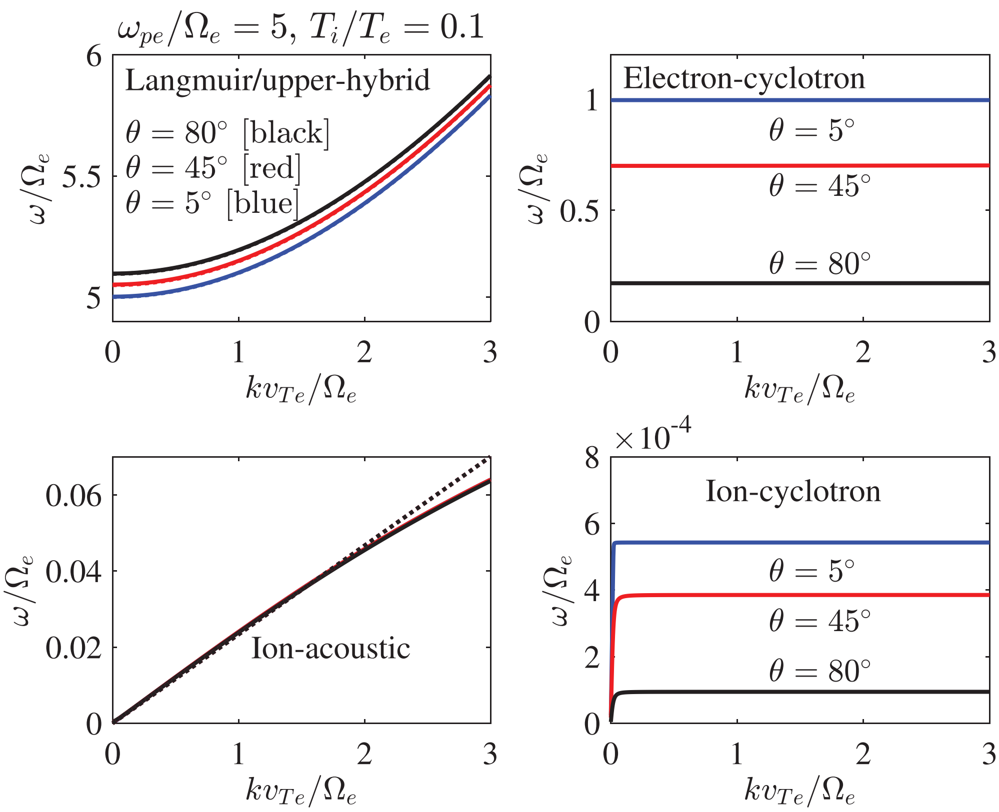

FIG. 1. Numerical solution, $\omega/|\Omega_e|$ vs $kv_{Te}/|\Omega_e|$, to the dispersion relation given by Eq. (25) for $\omega_{pe}/|\Omega_e| = 5$ and $T_i/T_e = 0.1$. Top-left panel corresponds to the Langmuir/upper-hybrid mode, top-right plots the electron–cyclotron mode, bottom-left shows the ion-acoustic mode, and bottom-right displays the ion–cyclotron mode. For the Langmuir/upper-hybrid and ion-acoustic modes, we superpose the analytical dispersion curves. For the Langmuir/upper-hybrid mode, the analytic solution is so close to the numerical solution that the curves almost completely overlap. For the ion-acoustic mode, the analytical solution is shown with a dotted line, which overlaps almost perfectly for most ranges of $kv_{Te}/|\Omega_e|$ until $kv_{Te}/|\Omega_e|$ becomes quite high.

shows that the Langmuir/upper-hybrid mode and the ion-acoustic mode are the propagating thermal modes that directly generalize the unmagnetized plasma modes. The cyclotron modes are not modified by thermal effects. We are interested in the extension of the unmagnetized plasma turbulence—where the Langmuir and ion-sound waves participate in the wave–particle and wave–wave interactions—to the magnetized plasma turbulence. For this purpose, we focus on the Langmuir/upper-hybrid mode and the ion-acoustic mode participating in the wave–wave and wave–particle resonances. As such, we will consider these two thermal modes as the fundamental normal modes of the magnetized plasma. For Langmuir/upper-hybrid and ion-acoustic modes, we superpose the analytic solutions of these modes (to be discussed later) on top of numerical solutions. The analytic Langmuir/upper-hybrid mode is so close to the numerical solution that the curves almost completely overlap. For the ion-acoustic mode, the analytical solution is shown with a dotted line, which overlaps almost perfectly for most ranges of $kv_{Te}/|\Omega_e|$ until $kv_{Te}/|\Omega_e|$ becomes quite high. For high $kv_{Te}/|\Omega_e|$, the ion-acoustic mode will be heavily damped.

In the present analysis, we generally consider ω_{pe}, which is sufficiently higher than $|\Omega_e|$. In the example shown in Fig. 1, we chose $\omega_{pe}/|\Omega_e| = 5$. Generally, we assume $\omega_{pe}^2/\Omega_e^2 \gg 1$. This means that $\omega_{pe}/|\Omega_e|$ can be as low as ~ 2 or ~ 3, but generally values higher than these are to be considered. For $\omega_{pe}/|\Omega_e| = 2$, the square of the frequency ratio is $\omega_{pe}^2/\Omega_e^2 = 4$, which can be marginally satisfying the requirement $\omega_{pe}^2/\Omega_e^2 \gg 1$, but for $\omega_{pe}/|\Omega_e| = 3$, the square of the frequency is $\omega_{pe}^2/\Omega_e^2 = 9$, which is certainly significantly higher than unity. Approximate, analytical solution to dispersion relation given by Eq. (25) is of relevance. For the high-frequency, Langmuir/upper-hybrid mode, we assume

$$\omega^2 \gg \Omega_e^2, \quad \omega^2 \gg k^2 v_{Te}^2. \tag{26}$$

We also ignore the ion response. If we ignore thermal effects altogether, then we have

$$0 = \omega^2(\omega^2 - \omega_{pe}^2 - \Omega_e^2) + \omega_{pe}^2\Omega_e^2 - \omega_{pe}^2\Omega_e^2 \frac{k_\perp^2}{k^2} \tag{27}$$

whose solution is

$$\omega^2 = \frac{1}{2}\left[\omega_{uh}^2 + \left((\omega_{pe}^2 - \Omega_e^2)^2 + 4\omega_{pe}^2\Omega_e^2 \frac{k_\perp^2}{k^2}\right)^{1/2}\right], \tag{28}$$

where $\omega_{uh}^2 = \omega_{pe}^2 + \Omega_e^2$ is the square of the upper-hybrid frequency. To simplify further, we replace the above by an approximate form

$$\omega = \left(\omega_{pe}^2 + \Omega_e^2 \frac{k_\perp^2}{k^2}\right)^{1/2} = (\omega_{pe}^2 + \Omega_e^2 \sin^2\theta)^{1/2}. \tag{29}$$

Figure 2 plots both Eqs. (28) and (29) vs θ, for $\omega_{pe}/|\Omega_e| = 5$ and 2. As Fig. 2 shows, the agreement is excellent.

Making use of the solution given by Eq. (29) as the basis, we add the thermal correction

$$0 = \omega^2 - \Omega_e^2 - \omega_{pe}^2\left(1 - \frac{\Omega_e^2}{\omega^2}\frac{k_\parallel^2}{k^2}\right) - k^2 v_{Te}^2\left(1 - \frac{\Omega_e^2}{\omega^2}\frac{k_\parallel^2}{k^2}\right). \tag{30}$$

We write down the approximate solution by

$$\omega = \omega_{\mathbf{k}}^U = (\omega_{pe}^2 + \Omega_e^2 \sin^2\theta + k^2 v_{Te}^2)^{1/2}. \tag{31}$$

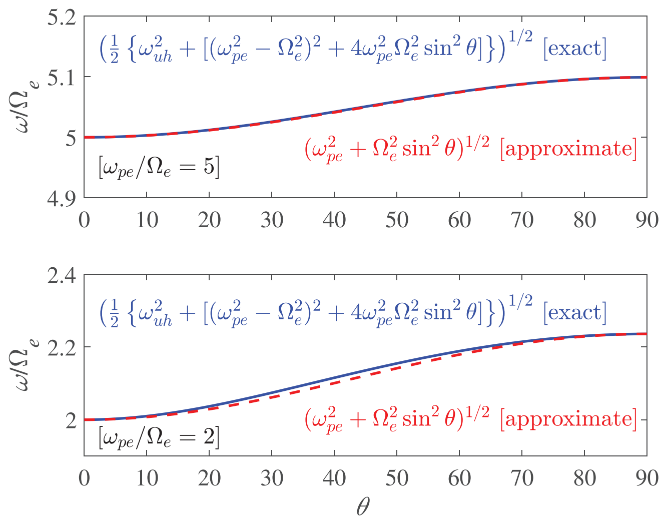

FIG. 2. Comparison between the exact cold-plasma solution given by Eq. (28) vs approximate solution given by Eq. (29) for $\omega_{pe}/|\Omega_e| = 5$ and 2.

We have superposed this solution to the numerical solution in the top-left panel of Fig. 1, and as already noted, the two overlap almost completely. In short, the dispersion relation for the Langmuir/upper-hybrid wave, which we simply call "upper-hybrid" or U mode, is given by Eq. (31). The approximate solution given by Eq. (31) amounts to replacing the dielectric constant for the U mode as follows:

$$\mathrm{Re}\,\varepsilon(\mathbf{k},\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2\sin^2\theta - k^2 v_{Te}^2}, \tag{32}$$

where $\omega = \sigma\omega_{\mathbf{k}}^U$. This also means that the derivative is given by

$$\varepsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^U) = \frac{\partial\,\mathrm{Re}\,\varepsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^U)}{\partial(\sigma\omega_{\mathbf{k}}^U)} = \frac{2\sigma\omega_{\mathbf{k}}^U}{\omega_{pe}^2}. \tag{33}$$

Next, we consider the low-frequency ion-acoustic mode. For this mode, we ignore the thermal and magnetic effects. We also assume

$$k^2 v_{Te}^2 \gg \omega^2 \tag{34}$$

for the electrons. This leads to the approximate dispersion relation

$$\mathrm{Re}\,\varepsilon(\mathbf{k},\omega) \approx \frac{\omega_{pe}^2}{k^2 v_{Te}^2} - \frac{\omega_{pi}^2}{\omega^2} = 0. \tag{35}$$

From this, we obtain

$$\begin{aligned} &\omega = \omega_{\mathbf{k}}^S = kc_S, \quad c_S = \sqrt{\frac{T_e}{m_i}}, \\ &\mathrm{Re}\,\varepsilon(\mathbf{k},\omega) = \frac{\omega_{pe}^2}{k^2 v_{Te}^2} - \frac{\omega_{pi}^2}{\omega^2}, \quad \omega = \sigma\omega_{\mathbf{k}}^S, \\ &\varepsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^S) = \frac{2}{\sigma\omega_{\mathbf{k}}^S}\frac{\omega_{pe}^2}{k^2 v_{Te}^2}. \end{aligned} \tag{36}$$

We have also superposed this solution to the bottom-left panel of Fig. 1 and found that the comparison with the numerical solution was excellent.

B. Wave kinetic equations for upper-hybrid/ion-sound turbulence

We now write down the wave kinetic equation for $\alpha = U$ and $\alpha = S$, respectively. In doing so, we note that only the three-wave interaction of the type $U + S \leftrightarrow U$ is allowed from the viewpoint of wave energetics and that we only retain the induced scattering of the type $U + i \leftrightarrow U$ in the wave kinetic equation for the U mode, while we ignore the induced scattering for the S mode. Such considerations are a direct analogy with the case of unmagnetized Langmuir/ion-acoustic turbulence situation. We thus write down the specific wave kinetic equations for the U and S modes as follows:

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma U}}{\partial t} &= -\sigma\omega_{\mathbf{k}}^{U}\mu_{\mathbf{k}}^{U}\,\mathrm{Im}\,\varepsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{U})\,I_{\mathbf{k}}^{\sigma U} - 2\pi\sigma\omega_{\mathbf{k}}^{U}\mu_{\mathbf{k}}^{U}\\ &\times\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,|\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{U}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S})|^2\\ &\times\Big(\sigma'\omega_{\mathbf{k}'}^{U}\mu_{\mathbf{k}'}^{U}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''S}I_{\mathbf{k}}^{\sigma U} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S}\mu_{\mathbf{k}-\mathbf{k}'}^{S}I_{\mathbf{k}'}^{\sigma' U}I_{\mathbf{k}}^{\sigma U}\\ &- \sigma\omega_{\mathbf{k}}^{U}\mu_{\mathbf{k}}^{U}I_{\mathbf{k}'}^{\sigma' U}I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}\Big)\delta(\sigma\omega_{\mathbf{k}}^{U} - \sigma'\omega_{\mathbf{k}'}^{U} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S})\\ &- 4\sigma\omega_{\mathbf{k}}^{U}\mu_{\mathbf{k}}^{U}\,\mathrm{Im}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,I_{\mathbf{k}'}^{\sigma' U}I_{\mathbf{k}}^{\sigma U}\\ &\times\mathscr{P}\,\frac{\{\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{U}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{U}-\sigma'\omega_{\mathbf{k}'}^{U})\}^2}{\varepsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{U}-\sigma'\omega_{\mathbf{k}'}^{U})},\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} &= -\sigma\omega_{\mathbf{k}}^{S}\mu_{\mathbf{k}}^{S}\,\mathrm{Im}\,\varepsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{S})\,I_{\mathbf{k}}^{\sigma S}\\ &- \pi\sigma\omega_{\mathbf{k}}^{S}\mu_{\mathbf{k}}^{S}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,|\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{U}|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{U})|^2\\ &\times\Big(\sigma'\omega_{\mathbf{k}'}^{U}\mu_{\mathbf{k}'}^{U}I_{\mathbf{k}-\mathbf{k}'}^{\sigma''U}I_{\mathbf{k}}^{\sigma S} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{U}\mu_{\mathbf{k}-\mathbf{k}'}^{U}I_{\mathbf{k}'}^{\sigma' U}I_{\mathbf{k}}^{\sigma S}\\ &- \sigma\omega_{\mathbf{k}}^{S}\mu_{\mathbf{k}}^{S}I_{\mathbf{k}'}^{\sigma' U}I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' U}\Big)\delta(\sigma\omega_{\mathbf{k}}^{S} - \sigma'\omega_{\mathbf{k}'}^{U} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{U}),\end{aligned} \tag{38}$$

where we have defined

$$\mu_{\mathbf{k}}^{U} = \frac{\omega_{pe}^2}{(\omega_{\mathbf{k}}^{U})^2}, \qquad \mu_{\mathbf{k}}^{S} = \frac{k^2 v_{Te}^2}{\omega_{pe}^2}. \tag{39}$$

A quantity of relevance is the nonlinear susceptibility, which is determined entirely by the electron response. This is because the lighter and more mobile electrons readily respond to the perturbation, while the heavier ions remain much less mobile. We make note of the fact that we are generally concerned with the weakly magnetized situations exemplified by the condition $\omega_{pe}^2 \gg \Omega_e^2$. We also note that the U mode is a fast mode, while the S mode is a slow mode in the following sense:

$$\begin{aligned}\omega_{\mathbf{k}}^{U} &\gg kv_{Te}, \quad \frac{|\Omega_e|}{\omega_{\mathbf{k}}^{U}} \ll 1,\\ \omega_{\mathbf{k}}^{S} &\ll kv_{Te}, \quad \frac{|\Omega_e|}{\omega_{\mathbf{k}}^{S}} \gg 1.\end{aligned} \tag{40}$$

Because of this, let us approximate the susceptibility by first making use of the relative magnitudes of η, η', and η''; but in doing so, we retain terms that survive in the unmagnetized limit. Thus, we approximate the following depending on various limits:

$$\begin{aligned}&\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\ &= -\frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2}{kk'|\mathbf{k}-\mathbf{k}'|R_{\mathbf{k},\omega}R_{\mathbf{k}',\omega'}R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}\\ &\times\Big(\omega'(\omega-\omega')\Big\{(\mathbf{k}\cdot\mathbf{k}')\Big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}') - \eta''^2k'_\parallel(k_\parallel-k'_\parallel)\Big]\\ &+\Big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}') - \eta''^2k_\parallel(k_\parallel-k'_\parallel)\Big]\Big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')\Big]\Big\}\\ &+\omega(\omega-\omega')k'^2\Big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}') - \eta''^2k_\parallel(k_\parallel-k'_\parallel)\Big]\\ &+\omega\omega'\Big[(\mathbf{k}-\mathbf{k}')^2 - \eta''^2(k_\parallel-k'_\parallel)^2\Big](\mathbf{k}\cdot\mathbf{k}')\Big)\end{aligned} \tag{41}$$

for $\eta \ll 1$, $\eta' \ll 1$, $\eta'' \gg 1$,

$$\begin{aligned}&\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\ &= -\frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2}{kk'|\mathbf{k}-\mathbf{k}'|R_{\mathbf{k},\omega}R_{\mathbf{k}',\omega'}R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}\\ &\times\Big(\omega'(\omega-\omega')\Big\{(\mathbf{k}\cdot\mathbf{k}' - \eta'^2k_\parallel k'_\parallel)\big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')\big]\\ &+\big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\big]\Big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}') - \eta'^2k'_\parallel(k_\parallel-k'_\parallel)\Big]\Big\}\\ &+\omega(\omega-\omega')(k'^2-\eta'^2k'^2_\parallel)\big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\big]\\ &+\omega\omega'(\mathbf{k}-\mathbf{k}')^2(\mathbf{k}\cdot\mathbf{k}' - \eta'^2k_\parallel k'_\parallel)\Big)\end{aligned} \tag{42}$$

for $\eta \ll 1$, $\eta' \gg 1$, $\eta'' \ll 1$, and

$$\begin{aligned}&\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\ &= -\frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2}{kk'|\mathbf{k}-\mathbf{k}'|R_{\mathbf{k},\omega}R_{\mathbf{k}',\omega'}R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}\\ &\times\Big(\omega'(\omega-\omega')\Big\{(\mathbf{k}\cdot\mathbf{k}' - \eta^2k_\parallel k'_\parallel)\big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')\big]\\ &+\Big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}') - \eta^2k_\parallel(k_\parallel-k'_\parallel)\Big]\big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')\big]\Big\}\\ &-\omega\eta^2\Big\{(\omega-\omega')k'^2\big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\big]+\omega'(\mathbf{k}-\mathbf{k}')^2(\mathbf{k}\cdot\mathbf{k}')\Big\}\Big)\end{aligned} \tag{43}$$

for $\eta \gg 1$, $\eta' \ll 1$, $\eta'' \ll 1$.

Next, we retain the dominant terms in relations to the magnitudes of ω, ω', and $\omega-\omega'$, i.e.,

$$\begin{aligned}&\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\ &= -\frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2(\mathbf{k}\cdot\mathbf{k}')\omega\omega'\Big[(\mathbf{k}-\mathbf{k}')^2 - \eta''^2(k_\parallel-k'_\parallel)^2\Big]}{kk'|\mathbf{k}-\mathbf{k}'|R_{\mathbf{k},\omega}R_{\mathbf{k}',\omega'}R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}\end{aligned} \tag{44}$$

for $\omega \gg kv_{Te}$, $\omega' \gg k'v_{Te}$, $\omega-\omega' \ll |\mathbf{k}-\mathbf{k}'|v_{Te}$

$$\begin{aligned}&\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\ &= -\frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2\big[\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')\big]\omega(\omega-\omega')(k'^2-\eta'^2k'^2_\parallel)}{kk'|\mathbf{k}-\mathbf{k}'|R_{\mathbf{k},\omega}R_{\mathbf{k}',\omega'}R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}\end{aligned} \tag{45}$$

for $\omega \gg kv_{Te}$, $\omega' \ll k'v_{Te}$, $\omega-\omega' \gg |\mathbf{k}-\mathbf{k}'|v_{Te}$, and

$$\begin{aligned}&\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega')\\ &= -\frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2\big[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')\big]\omega'(\omega-\omega')(k^2-\eta^2k^2_\parallel)}{kk'|\mathbf{k}-\mathbf{k}'|R_{\mathbf{k},\omega}R_{\mathbf{k}',\omega'}R_{\mathbf{k}-\mathbf{k}',\omega-\omega'}}\end{aligned} \tag{46}$$

for $\omega \ll kv_{Te}$, $\omega' \gg k'v_{Te}$, $\omega-\omega' \gg |\mathbf{k}-\mathbf{k}'|v_{Te}$.

Next, we approximate the resonant denominators accordingly as follows:

$$\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega') = \frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2}{\omega\omega'}\frac{\mathbf{k}\cdot\mathbf{k}'}{kk'|\mathbf{k}-\mathbf{k}'|} \times \frac{(\mathbf{k}-\mathbf{k}')^2 - \eta''^2(k_\parallel - k'_\parallel)^2}{\Omega_e^2 + \left[(\mathbf{k}-\mathbf{k}')^2 - \eta''^2(k_\parallel - k'_\parallel)^2\right]v_{Te}^2} \tag{47}$$

for $\omega \gg kv_{Te}$, $\eta \ll 1$, $\omega' \gg k'v_{Te}$, $\eta' \ll 1$, $\omega - \omega' \ll |\mathbf{k}-\mathbf{k}'|v_{Te}$, $\eta'' \gg 1$

$$\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega') = \frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2}{\omega(\omega-\omega')}\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{k}')}{kk'|\mathbf{k}-\mathbf{k}'|} \times \frac{(k'^2-\eta'^2k_\parallel'^2)}{\Omega_e^2 + (k'^2-\eta'^2k_\parallel'^2)v_{Te}^2} \tag{48}$$

for $\omega \gg kv_{Te}$, $\eta \ll 1$, $\omega' \ll k'v_{Te}$, $\eta' \gg 1$, $\omega - \omega' \gg |\mathbf{k}-\mathbf{k}'|v_{Te}$, $\eta'' \ll 1$, and

$$\chi_e(\mathbf{k}',\omega'|\mathbf{k}-\mathbf{k}',\omega-\omega') = \frac{i}{2}\frac{e}{m_e}\frac{\omega_{pe}^2}{\omega'(\omega-\omega')}\frac{\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')}{kk'|\mathbf{k}-\mathbf{k}'|} \times \frac{k^2-\eta^2k_\parallel^2}{\Omega_e^2 + (k^2-\eta^2k_\parallel^2)v_{Te}^2} \tag{49}$$

for $\omega \ll kv_{Te}$, $\eta \gg 1$, $\omega' \gg k'v_{Te}$, $\eta' \ll 1$, $\omega - \omega' \gg |\mathbf{k}-\mathbf{k}'|v_{Te}$, $\eta'' \ll 1$.

Making use of all this, we now approximate the following nonlinear susceptibilities of interest:

$$|\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^U|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)|^2 = \frac{1}{4}\frac{e^2}{m_e^2}\frac{\mu_\mathbf{k}^U\mu_{\mathbf{k}'}^U(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \times \left|\frac{(\mathbf{k}-\mathbf{k}')^2-(\eta_{\mathbf{k}-\mathbf{k}'}^S)^2(k_\parallel-k'_\parallel)^2}{\Omega_e^2+\left[(\mathbf{k}-\mathbf{k}')^2-(\eta_{\mathbf{k}-\mathbf{k}'}^S)^2(k_\parallel-k'_\parallel)^2\right]v_{Te}^2}\right|^2, \quad \eta_{\mathbf{k}-\mathbf{k}'}^S = \frac{|\Omega_e|}{\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S}, \tag{50}$$

$$\{\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^U|\mathbf{k}-\mathbf{k}',\sigma\omega_\mathbf{k}^U-\sigma\prime\omega_{\mathbf{k}'}^U)\}^2 = -\frac{1}{4}\frac{e^2}{m_e^2}\frac{\mu_\mathbf{k}^U\mu_{\mathbf{k}'}^U(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \times \left|\frac{(\mathbf{k}-\mathbf{k}')^2-(\eta_{\mathbf{k},\mathbf{k}'}^U)^2(k_\parallel-k'_\parallel)^2}{\Omega_e^2+\left[(\mathbf{k}-\mathbf{k}')^2-(\eta_{\mathbf{k},\mathbf{k}'}^U)^2(k_\parallel-k'_\parallel)^2\right]v_{Te}^2}\right|^2, \quad \eta_{\mathbf{k},\mathbf{k}'}^U = \frac{|\Omega_e|}{\sigma\omega_\mathbf{k}^U-\sigma'\omega_{\mathbf{k}'}^U}, \tag{51}$$

$$|\chi(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^U|\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^U)|^2 = \frac{1}{4}\frac{e^2}{m_e^2}\frac{\mu_{\mathbf{k}'}^U\mu_{\mathbf{k}-\mathbf{k}'}^U\left[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')\right]^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \times \left|\frac{k^2-(\eta_\mathbf{k}^S)^2k_\parallel^2}{\Omega_e^2+\left[k^2-(\eta_\mathbf{k}^S)^2k_\parallel^2\right]v_{Te}^2}\right|^2, \quad \eta_\mathbf{k}^S = \frac{|\Omega_e|}{\sigma\omega_\mathbf{k}^S}. \tag{52}$$

This leads to the following provisional weak turbulence wave kinetic equations, where the imaginary parts related to the linear dielectric constant are yet to be determined:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{I_\mathbf{k}^{\sigma U}}{\mu_\mathbf{k}^U} &= -\frac{\omega_{pe}^2}{\sigma\omega_\mathbf{k}^U}\,\mathrm{Im}\,\varepsilon(\mathbf{k},\sigma\omega_\mathbf{k}^U)\,\frac{I_\mathbf{k}^{\sigma U}}{\mu_\mathbf{k}^U} \\ &\quad + 2\sigma\omega_\mathbf{k}^U\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^U\left[\sigma\omega_\mathbf{k}^U\frac{I_{\mathbf{k}'}^{\sigma'U}}{\mu_{\mathbf{k}'}^U}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''S}}{\mu_{\mathbf{k}-\mathbf{k}'}^S} - \left(\sigma'\omega_{\mathbf{k}'}^U\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''S}}{\mu_{\mathbf{k}-\mathbf{k}'}^S} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S\frac{I_{\mathbf{k}'}^{\sigma'U}}{\mu_{\mathbf{k}'}^U}\right)\frac{I_\mathbf{k}^{\sigma U}}{\mu_\mathbf{k}^U}\right] \\ &\quad \times \delta(\sigma\omega_\mathbf{k}^U - \sigma'\omega_{\mathbf{k}'}^U - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S) \\ &\quad + \sigma\omega_\mathbf{k}^U\,\mathrm{Im}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,\frac{I_{\mathbf{k}'}^{\sigma'U}}{\mu_{\mathbf{k}'}^U}\frac{I_\mathbf{k}^{\sigma U}}{\mu_\mathbf{k}^U} \times \mathscr{P}\frac{M_{\mathbf{k},\mathbf{k}'}}{\varepsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_\mathbf{k}^U-\sigma'\omega_{\mathbf{k}'}^U)},\end{aligned} \tag{53}$$

$$\begin{aligned}\frac{\partial}{\partial t}\frac{I_\mathbf{k}^{\sigma S}}{\mu_\mathbf{k}^S} &= -\sigma\omega_\mathbf{k}^S\,\mu_\mathbf{k}^S\,\mathrm{Im}\,\varepsilon(\mathbf{k},\sigma\omega_\mathbf{k}^S)\,\frac{I_\mathbf{k}^{\sigma S}}{\mu_\mathbf{k}^S} \\ &\quad + \sigma\omega_\mathbf{k}^S\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^S\left[\sigma\omega_\mathbf{k}^S\frac{I_{\mathbf{k}'}^{\sigma'U}}{\mu_{\mathbf{k}'}^U}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''U}}{\mu_{\mathbf{k}-\mathbf{k}'}^U} - \left(\sigma'\omega_{\mathbf{k}'}^U\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma''U}}{\mu_{\mathbf{k}-\mathbf{k}'}^U} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^U\frac{I_{\mathbf{k}'}^{\sigma'U}}{\mu_{\mathbf{k}'}^U}\right)\frac{I_\mathbf{k}^{\sigma S}}{\mu_\mathbf{k}^S}\right] \\ &\quad \times \delta(\sigma\omega_\mathbf{k}^S - \sigma'\omega_{\mathbf{k}'}^U - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^U),\end{aligned} \tag{54}$$

where

$$\begin{aligned} V_{\mathbf{k},\mathbf{k}'}^U &= \frac{\pi}{4}\frac{e^2}{m_e^2}\frac{(\mu_\mathbf{k}^U)^2(\mu_{\mathbf{k}'}^U)^2\mu_{\mathbf{k}-\mathbf{k}'}^S(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \times \left|\frac{(\mathbf{k}-\mathbf{k}')^2-\eta_{\mathbf{k}-\mathbf{k}'}^S(k_\parallel-k'_\parallel)^2}{\Omega_e^2+[(\mathbf{k}-\mathbf{k}')^2-\eta_{\mathbf{k}-\mathbf{k}'}^S(k_\parallel-k'_\parallel)^2]v_{Te}^2}\right|^2, \\ V_{\mathbf{k},\mathbf{k}'}^S &= \frac{\pi}{4}\frac{e^2}{m_e^2}\frac{(\mu_\mathbf{k}^U)^2(\mu_{\mathbf{k}-\mathbf{k}'}^U)^2\mu_\mathbf{k}^S[\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \times \left|\frac{k^2-\eta_\mathbf{k}^Sk_\parallel^2}{\Omega_e^2+[k^2-\eta_\mathbf{k}^Sk_\parallel^2]v_{Te}^2}\right|^2, \\ M_{\mathbf{k},\mathbf{k}'} &= \frac{e^2}{m_e^2}\frac{(\mu_\mathbf{k}^U)^2(\mu_{\mathbf{k}'}^U)^2(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2|\mathbf{k}-\mathbf{k}'|^2} \times \left|\frac{(\mathbf{k}-\mathbf{k}')^2-\eta_{\mathbf{k},\mathbf{k}'}^U(k_\parallel-k'_\parallel)^2}{\Omega_e^2+\{(\mathbf{k}-\mathbf{k}')^2-\eta_{\mathbf{k},\mathbf{k}'}^U(k_\parallel-k'_\parallel)^2\}v_{Te}^2}\right|^2. \end{aligned} \tag{55}$$

Note that the nonlinear terms in Eqs. (53) and (54) associated with the three-wave resonance delta function condition represent the decay interactions. In the U mode wave kinetic equation, the last term on the right-hand side of Eq. (53) denotes the induced scattering terms. The linear terms associated with the imaginary parts of the dielectric constant in Eqs. (53) and (54) correspond to the quasilinear growth/damping (or induced emission) terms.

Let us consider the inverse dielectric constant with shifted argument

$$\frac{1}{\varepsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^U - \sigma'\omega_{\mathbf{k}'}^U)}. \tag{56}$$

Since the shifted frequency $\sigma\omega_{\mathbf{k}}^U - \sigma'\omega_{\mathbf{k}'}^U$ is small, the low-frequency expression of the real part is applicable, i.e.,

$$\begin{aligned}\varepsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^U - \sigma'\omega_{\mathbf{k}'}^U) \\ \approx \frac{\omega_{pe}^2}{|\mathbf{k}-\mathbf{k}'|^2 v_{Te}^2} + i\,\mathrm{Im}\,\varepsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^U - \sigma'\omega_{\mathbf{k}'}^U).\end{aligned} \tag{57}$$

Thus, we have

$$\begin{aligned}\mathrm{Im}\,\mathscr{P}\,\frac{1}{\varepsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^U - \sigma'\omega_{\mathbf{k}'}^U)} \\ = -(\mu_{\mathbf{k}-\mathbf{k}'}^S)^2\,\mathrm{Im}\,\varepsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^U - \sigma'\omega_{\mathbf{k}'}^U).\end{aligned} \tag{58}$$

Consequently, the induced scattering term in the U mode wave equation can be alternatively written as

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{\text{ind.scatt.}}\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^U} &= -\sigma\omega_{\mathbf{k}}^U\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,M'_{\mathbf{k},\mathbf{k}'} \\ &\quad\times \mathrm{Im}\,\varepsilon(\mathbf{k}-\mathbf{k}', \sigma\omega_{\mathbf{k}}^U - \sigma'\omega_{\mathbf{k}'}^U)\frac{I_{\mathbf{k}'}^{\sigma' U}}{\mu_{\mathbf{k}'}^U}\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^U}, \\ M'_{\mathbf{k},\mathbf{k}'} &= \frac{e^2}{m_e^2}\frac{|\mathbf{k}-\mathbf{k}'|^2}{\omega_{pe}^4}\frac{(\mu_{\mathbf{k}}^U)^2(\mu_{\mathbf{k}'}^U)^2(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2} \\ &\quad\times\left|\frac{(\mathbf{k}-\mathbf{k}')^2 v_{Te}^2 - \eta_{\mathbf{k},\mathbf{k}'}^U(k_\parallel - k'_\parallel)^2 v_{Te}^2}{\Omega_e^2 + \{(\mathbf{k}-\mathbf{k}')^2 - \eta_{\mathbf{k},\mathbf{k}'}^U(k_\parallel-k'_\parallel)^2\}v_{Te}^2}\right|^2.\end{aligned} \tag{59}$$

The provisional wave kinetic equation given by Eqs. (53) and (54) with the modified induced scattering term given by Eq. (59) are almost complete, except that the imaginary part of the dielectric constant is undetermined. Under the strict warm two-fluid formalism, the dielectric constant is purely real. However, the present warm two-fluid formalism does not provide a complete description of the electrostatic weak turbulence in magnetized plasmas. We thus turn to kinetic theory in order to supplement the missing information as far as the warm two-fluid approach goes. Also, the wave kinetic equation must be solved in conjunction with the particle kinetic equation. For that, we resort to the quasilinear theory.

C. Adding kinetic effects

We start from the Vlasov–Poisson system of equations given by

$$\begin{aligned}\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_a}{m_a}\left(\mathbf{E}(\mathbf{r},t) + \frac{\mathbf{v}}{c}\times\mathbf{B}_0\right)\cdot\frac{\partial}{\partial\mathbf{v}}\right]f_a(\mathbf{r},\mathbf{v},t) = 0, \\ \nabla\cdot\mathbf{E}(\mathbf{r},t) = 4\pi\sum_a e_a n_a\int d\mathbf{v}\,f_a(\mathbf{r},\mathbf{v},t).\end{aligned} \tag{60}$$

Separating the physical quantities into average and fluctuating parts, $f_a(\mathbf{r},\mathbf{v},t) = F_a(\mathbf{v},t) + \delta f_a(\mathbf{r},\mathbf{v},t)$ and $\mathbf{E}(\mathbf{r},t) = \delta\mathbf{E}(\mathbf{r},t)$, and considering only linear equation for the perturbation, we obtain

$$\begin{aligned}\frac{\partial F_a}{\partial t} &= \frac{e_a}{m_a}\frac{\partial}{\partial v_i}\int d\mathbf{k}\int d\omega\,\frac{k_i}{k}\langle E_{-\mathbf{k},-\omega}f_{\mathbf{k}\omega}^a\rangle, \\ E_{\mathbf{k}\omega} &= -\frac{4\pi i}{k}\sum_a e_a n_0\int d\mathbf{v}\,f_{\mathbf{k}\omega}^a, \\ \frac{\partial f_{\mathbf{k}\omega}^a}{\partial\varphi} + \frac{i(\omega-\mathbf{k}\cdot\mathbf{v})}{\Omega_a}f_{\mathbf{k}\omega}^a &= \frac{e_a}{m_a\Omega_a}E_{\mathbf{k}\omega}\frac{k_i}{k}\frac{\partial F_a}{\partial v_i},\end{aligned} \tag{61}$$

where we have assumed the gyrotropy for F_a, and we have expressed the results in spectral representation.

Solving for the perturbed distribution function—third equation in Eq. (61)—following the standard textbook method, we have

$$\begin{aligned}f_{\mathbf{k},\omega}^a &= \frac{-ie_a}{m_a k}\sum_{n=-\infty}^{\infty}\frac{J_n(b)e^{ib\sin\varphi - in\varphi}}{\omega - k_\parallel v_\parallel - n\Omega_a + i0} \\ &\quad\times\left(\frac{n\Omega_a}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel}\right)F_a E_{\mathbf{k},\omega}.\end{aligned} \tag{62}$$

Substituting Eq. (62) to the Poisson equation—second equation in Eq. (61), we have the kinetic version of the wave dispersion relation, together with the kinetic definition for the dielectric response function

$$\begin{aligned}0 &= \varepsilon(\mathbf{k},\omega)E_{\mathbf{k},\omega}, \\ \varepsilon(\mathbf{k},\omega) &= 1 + \sum_a\frac{\omega_{pa}^2}{k^2}\int d\mathbf{v}\sum_{n=-\infty}^{\infty}\frac{J_n^2(b)}{\omega - k_\parallel v_\parallel - n\Omega_a + i0} \\ &\quad\times\left(\frac{n\Omega_a}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel}\right)F_a.\end{aligned} \tag{63}$$

From this, we obtain the desired expression for the imaginary part of dielectric constant given by

$$\begin{aligned}\mathrm{Im}\,\varepsilon(\mathbf{k},\omega) &= -\sum_a\frac{\pi\omega_{pa}^2}{k^2}\int d\mathbf{v}\sum_{n=-\infty}^{\infty}J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right) \\ &\quad\times\delta(\omega - k_\parallel v_\parallel - n\Omega_a) \\ &\quad\times\left(\frac{n\Omega_a}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel}\right)F_a.\end{aligned} \tag{64}$$

Inserting Eq. (62) to the particle kinetic equation in Eq. (61), we also obtain the desired quasilinear velocity–space diffusion equation as follows:

$$\begin{aligned}\frac{\partial F_a}{\partial t} &= \frac{\pi e_a^2}{m_a^2}\int d\mathbf{k}\int d\omega\sum_n\left(\frac{n\Omega_a}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel}\right) \\ &\quad\times J_n^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\delta(\omega - k_\parallel v_\parallel - n\Omega_a) \\ &\quad\times\frac{\langle\delta E^2\rangle_{\mathbf{k},\omega}}{k^2}\left(\frac{n\Omega_a}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel}\right)F_a.\end{aligned} \tag{65}$$

D. Summary of equations

We are now in a situation to write down the complete set of equations that can describe electrostatic turbulence in magnetized plasmas. The onset of turbulence may be initiated by some free energy source associated with the particles. For instance, an electron beam traveling along the ambient magnetic field may excite primarily Langmuir instability, but during the course of nonlinear mode coupling, the beam-generated Langmuir waves may undergo backscattering and decay that involves upper-hybrid waves and low-frequency ion-sound waves. The set of equations to be summarized here can be solved either by analytical means or by fully numerical means to describe such processes. To present the final result, we take the expression for the imaginary part of dielectric constant computed from kinetic theory, namely, Eq. (64). We also make use of the particle kinetic equation (65) to provide a self-consistent dynamical description of the particle distribution function. We next discuss the incorporation of these kinetic effects.

First, we note that the upper-hybrid mode is a high-frequency mode. As such, we may ignore ions in the linear growth/damping (or induced emission) term given by

$$\gamma_{\mathbf{k}}^{\sigma U} = \sigma\omega_{\mathbf{k}}^{U}\,\mu_{\mathbf{k}}^{U}\,\frac{\pi\omega_{pe}^2}{2k^2}\int d\mathbf{v}\sum_{n=-\infty}^{\infty}J_n^2\left(\frac{k_\perp v_\perp}{|\Omega_e|}\right) \times \delta(\sigma\omega_{\mathbf{k}}^{U} - k_\parallel v_\parallel - n|\Omega_e|) \times \left(\frac{n|\Omega_e|}{v_\perp}\frac{\partial}{\partial v_\perp} + k_\parallel\frac{\partial}{\partial v_\parallel}\right)F_e. \tag{66}$$

The Langmuir/upper-hybrid mode is characterized by $\omega_{\mathbf{k}}^{U} \sim \omega_{pe}, \omega_{uh}$. Consequently, the resonance condition leads to $v_\parallel \sim (\sigma\omega_{\mathbf{k}}^{U} - n|\Omega_e|)/k_\parallel$. If the instability is driven by the electron beam, then it is seen that the harmonic mode number corresponding to

$$n \sim \frac{\sigma\omega_{\mathbf{k}}^{U} - k_\parallel v_b}{|\Omega_e|} \sim \left[\sigma\left(\frac{\omega_{pe}^2}{\Omega_e^2} + \frac{k_\perp^2}{k^2} + \frac{k^2 v_{Te}^2}{\Omega_e^2}\right)^{1/2} - \frac{k_\parallel v_b}{\Omega_e^2}\right], \tag{67}$$

where v_b is the average electron beam speed, is expected to make the most important contribution. If $\omega_{pe}/|\Omega_e|$ is, say, 3 or so, then n could also be close to 3. If $\omega_{pe}/|\Omega_e| \sim 5$, then $n \sim 5$, etc. On the other hand, the Bessel function factor $J_n^2(k_\perp v_\perp/|\Omega_e|)$ decreases for increasing n. This means that there is a trade-off between the resonance condition and the Bessel function multiplicative factor. In general, many harmonic terms need to be included in the summation.

For the S mode, on the other hand, the frequency is low so that one may keep only the lowest harmonic term in the electron Bessel function series. In fact, only the $n=0$ term (the Landau resonance) will be sufficient. The S mode, however, is affected by the ions as well, but since many higher-harmonics of ion terms need to be kept, we approximate the problem by treating the ions as unmagnetized. Thus, the S mode damping rate can be approximated by

$$\gamma_{\mathbf{k}}^{\sigma S} = \sigma\omega_{\mathbf{k}}^{S}\,\mu_{\mathbf{k}}^{S}\,\frac{\pi\omega_{pe}^2}{2k^2}\int d\mathbf{v}\,J_0^2\left(\frac{k_\perp v_\perp}{|\Omega_e|}\right) \times \delta(\sigma\omega_{\mathbf{k}}^{S} - k_\parallel v_\parallel)\,k_\parallel\,\frac{\partial F_e}{\partial v_\parallel} + \sigma\omega_{\mathbf{k}}^{S}\mu_{\mathbf{k}}^{S}\,\frac{\pi\omega_{pe}^2}{2k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^{S} - \mathbf{k}\cdot\mathbf{v})\,\mathbf{k}\cdot\frac{\partial F_i}{\partial\mathbf{v}}. \tag{68}$$

For the induced scattering (nonlinear Landau damping) term in the U mode wave equation, we retain only the ion (proton) contribution. As such, we also replace the expression by its unmagnetized counterpart given by

$$\left.\frac{\partial}{\partial t}\right|_{\text{ind.scatt.}}\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^{U}} \to \sigma\omega_{\mathbf{k}}^{U}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,U_{\mathbf{k},\mathbf{k}'}\int d\mathbf{v}\,\frac{I_{\mathbf{k}'}^{\sigma' U}}{\mu_{\mathbf{k}'}^{U}}\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^{U}} \times \delta\left[\sigma\omega_{\mathbf{k}}^{U} - \sigma'\omega_{\mathbf{k}'}^{U} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}\right](\mathbf{k}-\mathbf{k}')\cdot\frac{\partial F_i}{\partial\mathbf{v}}, \tag{69}$$

where

$$U_{\mathbf{k},\mathbf{k}'} = \frac{m_a}{m_e}\frac{\pi}{\omega_{pe}^2}\frac{e^2}{m_e^2}(\mu_{\mathbf{k}}^{U})^2(\mu_{\mathbf{k}'}^{U})^2\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2} \times \left|\frac{(\mathbf{k}-\mathbf{k}')^2 v_{Te}^2 - \eta_{\mathbf{k},\mathbf{k}'}^{U}(k_\parallel - k_\parallel')^2 v_{Te}^2}{\Omega_e^2 + \{(\mathbf{k}-\mathbf{k}')^2 - \eta_{\mathbf{k},\mathbf{k}'}^{U}(k_\parallel - k_\parallel')^2\}v_{Te}^2}\right|^2. \tag{70}$$

Finally, we treat ions as a stationary background so that we only solve for the electron velocity–space diffusion equation. For the electrons, only the U mode waves contribute to the velocity–space diffusion, since the low-frequency S mode is generally unimportant for the electrons especially in the range of electron beam.

The final set of equations are thus summarized as follows: The wave kinetic equation for the U mode is given by

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^{U}} = 2\gamma_{\mathbf{k}}^{\sigma U}\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^{U}} + 2\sigma\omega_{\mathbf{k}}^{U}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^{U}\left[\sigma\omega_{\mathbf{k}}^{U}\frac{I_{\mathbf{k}'}^{\sigma' U}}{\mu_{\mathbf{k}'}^{U}}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}^{S}} - \left(\sigma'\omega_{\mathbf{k}'}^{U}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}{\mu_{\mathbf{k}-\mathbf{k}'}^{S}} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S}\frac{I_{\mathbf{k}'}^{\sigma' U}}{\mu_{\mathbf{k}'}^{U}}\right)\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^{U}}\right] \times \delta(\sigma\omega_{\mathbf{k}}^{U} - \sigma'\omega_{\mathbf{k}'}^{U} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{S}) + \sigma\omega_{\mathbf{k}}^{U}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,U_{\mathbf{k},\mathbf{k}'}\int d\mathbf{v}\,\frac{I_{\mathbf{k}'}^{\sigma' U}}{\mu_{\mathbf{k}'}^{U}}\frac{I_{\mathbf{k}}^{\sigma U}}{\mu_{\mathbf{k}}^{U}} \times \delta\left[\sigma\omega_{\mathbf{k}}^{U} - \sigma'\omega_{\mathbf{k}'}^{U} - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}\right](\mathbf{k}-\mathbf{k}')\cdot\frac{\partial F_i}{\partial\mathbf{v}}, \tag{71}$$

where $\gamma_{\mathbf{k}}^{\sigma U}$ is given by Eq. (66); $V_{\mathbf{k},\mathbf{k}'}^{U}$ is defined in Eq. (55); the dispersion relations $\omega_{\mathbf{k}}^{U}$ and $\omega_{\mathbf{k}}^{S}$ are defined in Eqs. (31) and (36), respectively; the quantities $\mu_{\mathbf{k}}^{U}$ and $\mu_{\mathbf{k}}^{S}$ are given in Eq. (39); and the coefficient $U_{\mathbf{k},\mathbf{k}'}$ is defined by Eq. (70).

The S mode wave kinetic equation is a slight modification of Eq. (54) where

$$\frac{\partial}{\partial t}\frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}^{S}} = 2\gamma_{\mathbf{k}}^{\sigma S}\frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}^{S}} + \sigma\omega_{\mathbf{k}}^{S}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^{S}\left[\sigma\omega_{\mathbf{k}}^{S}\frac{I_{\mathbf{k}'}^{\sigma' U}}{\mu_{\mathbf{k}'}^{U}}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' U}}{\mu_{\mathbf{k}-\mathbf{k}'}^{U}} - \left(\sigma'\omega_{\mathbf{k}'}^{U}\frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' U}}{\mu_{\mathbf{k}-\mathbf{k}'}^{U}} + \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{U}\frac{I_{\mathbf{k}'}^{\sigma' U}}{\mu_{\mathbf{k}'}^{U}}\right)\frac{I_{\mathbf{k}}^{\sigma S}}{\mu_{\mathbf{k}}^{S}}\right] \times \delta(\sigma\omega_{\mathbf{k}}^{S} - \sigma'\omega_{\mathbf{k}'}^{U} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{U}), \tag{72}$$

where $\gamma_{\mathbf{k}}^{\sigma S}$ is defined in Eq. (68), and $V_{\mathbf{k},\mathbf{k}'}^{S}$ is defined in the same manner as in Eq. (55). The ions are treated as quasistationary, but the electron distribution F_e evolves according to the dictates of the quasilinear velocity–space diffusion equation given by

$$\frac{\partial F_e}{\partial t} = \frac{\pi e^2}{m_e^2} \sum_{\sigma=\pm 1} \int \frac{d\mathbf{k}}{k^2} \sum_n \left(\frac{n|\Omega_e|}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right) \times J_n^2 \left(\frac{k_\perp v_\perp}{|\Omega_e|} \right) \delta(\sigma\omega_{\mathbf{k}}^U - k_\parallel v_\parallel - n|\Omega_e|) I_{\mathbf{k}}^{\sigma U} \times \left(\frac{n|\Omega_e|}{v_\perp} \frac{\partial}{\partial v_\perp} + k_\parallel \frac{\partial}{\partial v_\parallel} \right) F_e. \quad (73)$$

This completes the formulation of weak turbulence in magnetized plasmas under the assumption of electrostatic interaction. If we eliminate the correction that arises from the presence of ambient magnetic field, then the set of equations that we have derived thus far reduce to that of unmagnetized plasmas, which had been derived with kinetic theory and solved for one- and two-dimensional (or three-dimensions with azimuthal symmetry) situations.[33,84–95]

V. SUMMARY AND DISCUSSIONS

To summarize the present paper, we have made use of a hybrid technique that involves a warm two-fluid theory to compute the linear dispersion relation, nonlinear susceptibility, and the basic form of nonlinear wave equation under the weak turbulence ordering. We have then formulated the general weak turbulence analysis to derive the wave kinetic equation that describes linear wave–particle interaction, or induced emission, nonlinear wave–wave interaction, or decay/coalescence, and nonlinear wave–particle interaction, or induced scattering process. Among these, the decay term is adequately described by the warm two-fluid approach, but the processes that involve particles cannot be discussed with macroscopic theory. We have thus employed the linear and quasilinear kinetic theory to provide the mathematical expressions for induced emission and induced scattering terms. We have also derived the quasilinear diffusion equation for the particles, thereby completing the formalism.

As noted in Sec. I, the present state of matter regarding the weak turbulence theory in magnetized plasmas is not at a mature state. Instead, weak turbulence theory for magnetized plasmas is discussed under various simplifying assumptions. Despite some early efforts,[49–51] completely general kinetic theory of weak turbulence in magnetized plasmas is not practical. The purpose of the present paper has been to derive a reduced theory of weak turbulence in relatively weakly magnetized plasmas under the assumption of electrostatic interaction. Unlike the early works,[49–51] the present paper has taken a more pragmatic approach in that, we started from the warm two-fluid theory. Recently, one of us (P.H.Y.) demonstrated that the warm two-fluid theory is capable of partially reproducing the weak turbulence wave equation for unmagnetized plasmas, which is normally derived from full kinetic theory.[81] The present paper adopted such an approach, which is combined with quasilinear kinetic theory, and succeeded in formulating the weak turbulence theory for magnetized plasmas under the assumption of electrostatic interaction.

In the future, the set of equations derived in this paper will be analyzed/solved for a practical problem. Further, the present formalism will be extended to a fully electromagnetic formalism. Such tasks are beyond the scope of the present work, however.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The theoretical plots are in normalized units, and the equations are clearly explained in the text. Therefore, no actual data are generated by the theory.

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