Matemática
Contemporânea
Vol. 56, 20–30
http://doi.org/10.21711/231766362023/rmc563

Some properties of solutions of advection-diffusion equations in $\mathbb{R}$

Patrícia Lisandra Guidolin [1], **Lineia Schütz** [1] and **Juliana Sartori Ziebell** [1]

[1]Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9.500, Porto Alegre, CEP:91509-900, Brazil

Abstract. In this paper is provided a proof, using a technique based on energy methods, of the continuity of bounded solutions for the advection-diffusion equations $u_t + (b(x,t)u^{k+1})_x = \mu(t)u_{xx}$ $\forall x \in \mathbb{R}, t > 0$, with initial data $u(\cdot,0) = u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In respect of the arbitrary advective speed term, it is only assumed that $b(x,t)$ is limited. Also, some known results about existence of solutions of this problem are revised and a discussion of some open problems is presented .

Keywords: Advection-diffusion equations, continuity on L^p norm, global existence

2020 Mathematics Subject Classification: 35A01, 35K15.

1 Introduction

Consider the following initial value problem

$$u_t + (b(x,t)u^{k+1})_x = \mu(t)u_{xx} \quad \forall x \in \mathbb{R},\ t > 0,$$
$$u(\cdot,0) = u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{1.1}$$

e-mail: julianaziebell@ufrgs.br

for some positive function $\mu(t) \in C^0([0,\infty))$ and $k \geq 0$ constant. Here b is an arbitrary continuously differentiable advection fields satisfying

$$b(\cdot,t) \in L^\infty_{loc}([0,\infty), L^\infty(\mathbb{R})) \;\; \forall x \in \mathbb{R},\; t \geq 0, \tag{1.2}$$

that is, $|b(x,t)| < B(t)$ $\forall x \in \mathbb{R}$ and $t \geq 0$ for some $B \in C^0[0,\infty)$.

The main objective of this work is to emphasize some known results about existence of solutions of (1.1), prove the continuity of bounded solutions to this initial value problem in L^q norm, and discuss some open problems for this equation. Here, by a bounded solution of (1.1) - (1.2) in $[0,T_*)$, for some $T_* < \infty$, we mean $u(\cdot,t) \in L^\infty_{loc}([0,T_*), L^\infty(\mathbb{R}))$ that solves the equation (1.1) in the classical sense and satisfies $u(\cdot,t) \to u(\cdot,0)$ in $L^1_{loc}(\mathbb{R})$, as $t \to 0$.

Equations as (1.1) that combines effects of the advective and diffusive terms have been widely studied due to their applications in several areas [2, 3, 5, 8, 9, 10, 12, 14, 15].

We start observing that, if $k = 1$ and the function b is constant in the problem (1.1), the Hopf-Cole transformation transforms the nonlinear Burger's equation in the linear heat equation associated, thus obtaining an explicit solution for it [7]. The Burger's equation is a simplified version of the Navier-Stokes equation and it has been used to model gas dynamics and traffic flow as one of the simplest nonlinear model equation for analyzing combined effect of nonlinear advection and diffusion. In this work, we allow the advective term to explicitly depends on x. This detail makes the analysis more interesting and the asymptotic behaviour of solutions and their properties difficult to describe [2, 8].

Before presenting the result of this work, some comments of what is known about the existence of solutions will be discussed. For the local (in time) existence of solutions to the problem (1.1) - (1.2), see e.g. [11], [13], Ch. 7. When b does not explicitly depend on x or, more generally, when b depends on x with $\partial b(x,t)/\partial x \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$, and $k = 0$ in (1.1), it has already been proven that for each $1 \leq p_0 \leq p \leq \infty$, $\|u(\cdot,t)\|_{L^p(\mathbb{R})}$ is monotonically decreasing on t, and $\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C(p_0)$

$\|u(\cdot,t)\|_{L^{p_0}(\mathbb{R})}t^{-1/2p_0}$. In this case, the solutions will not only be defined for all time, but they will also decay as t goes to infinity, [1, 6, 3, 12]. Besides that, the global existence of solutions to the problem (1.1) with $k = 0$ and $b = b(x,t)$ limited (not assuming $\partial b(x,t)/\partial x \geq 0$), was proved in [2]. Moreover, in a recent work [15], the global existence for solutions of (1.1) - (1.2) with $0 \leq k < 1/n$, where n is the dimension of the spatial variable was demonstrated. In [8], for $0 \leq k < 2$, and only assuming that $b(x,t)$ and $\partial b(x,t)/\partial x$ are limited, was proved that any given solution to problem (1.1) is globally defined. Moreover, it should be noted that the results presented in [8], up to section 3, are also valid for the problem defined in $\mathbb{R}^n$, with $0 \leq k < 2/n$ in (1.1).

In this paper, following the lines of [2], [3], and [15], supposing that $u(\cdot,t)$ is a bounded solution to the problem (1.1) and just assuming that the advection fields b is limited (1.2), we are interested in qualitative properties, such as derivative regularity and the continuity of $u(\cdot,t)$ on the L^p norm, for all $k \geq 0$, and $p \geq 1$. These results are important to obtain further properties of solutions to the equations considered here or spacial cases of such equations. It is also relevant to mention that the results present here are also valid for the problem defined in $\mathbb{R}^n$ and the proof is similar.

This article is organized as follows. In Section 2, we introduce some definitions, notations and results that will be used in this paper. Moreover, we show that $u(\cdot,t)$ satisfies $\|u(\cdot,t)\|_{L^p(\mathbb{R})} \leq K(p,t)\|u(\cdot,0)\|_{L^p(\mathbb{R})}$, for $p \geq 1$, where $K(p,t)$ is a constant that only depends of p and t and $\int_{t_0}^t \mu(\tau)\int_{\mathbb{R}} |u_x(\cdot,t)|^2 dx < \infty$. In Section 3, we present the proof that $u(\cdot,t) \in C^0([0,T^*), L^p(\mathbb{R}))$, for all $k \geq 0$, $p \geq 1$. In Section 4, we list some open problems related to the problem (1.1).

2 Preliminary Tools

2.1 Cut-off and regularizing functions

To prove the results present in this paper it will also be necessary to define some cut-off functions and regularizing functions. Then, given $R > 0$, $\epsilon > 0$, let $\zeta_{R,\epsilon} \in C^2(\mathbb{R})$ be the cut-off function

$$\zeta_{R,\epsilon}(x) = e^{-\epsilon\sqrt{1+x^2}} - e^{-\epsilon\sqrt{1+R^2}} \qquad \text{if } |x| < R \tag{2.1}$$

and $\zeta_{R,\epsilon}(x) = 0$ if $|x| \geq R$. Also let $R, S > 0$ be given, define $\zeta_{R,S} \in C^2(\mathbb{R})$ be a cut-off function satisfying: $0 \leq \zeta_{R,S} \leq 1$ everywhere, and

$$\zeta_{R,S} = \begin{cases} 0 & \text{if} \quad |x| < \dfrac{R}{2} \\ 1 & \text{if} \quad R < |x| < R + S \\ 0 & \text{if} \quad |x| > R + 2S \end{cases} \tag{2.2}$$

with $|\zeta'_{R,S}(x)| \leq C/R$ if $|x| < R$ and $|\zeta'_{R,S}(x)| \leq C/S$ if $R + S < |x| < R + 2S$ for some constant C independent of R, S. Finally, giving $R > 0$, let

$$\zeta_R(x) = \zeta(x/R), \tag{2.3}$$

where $\zeta \in C^2(\mathbb{R})$ is such that $\zeta(x) = 1$ if $|x| \leq 1/2$, $\zeta(x) = 0$ if $|x| > 1$, $0 \leq \zeta(x) \leq 1, \forall x \in \mathbb{R}$. It will be important to note that

$$|\zeta'_R(x)| < \frac{C}{R}, C \in \mathbb{R} \ ; \ |\zeta'_R(x)| \to 0 \ \text{ and } \ \zeta_R(x) \to 1 \text{ as } R \to \infty, \ \forall x \in \mathbb{R}.$$

In order to define one regularizing function, we consider a function $W \in C^1(\mathbb{R})$ such that $W'(v) \geq 0 \ \forall \ v \in \mathbb{R}$, $W(0) = 0$ and $W(v) = sgn(v)$, for $|v| \geq 1$. For each $\delta > 0$, we define the regularizing function $W_\delta(v) = W\left(\frac{v}{\delta}\right)$. Let $L_\delta \in C^2(\mathbb{R})$ be the regularized absolute value function

$$L_\delta(u) = \int_0^u W\left(\frac{v}{\delta}\right) dv. \tag{2.4}$$

Observe that

$$0 \leq L_\delta(u) \leq |u|, \ L'_\delta(u) \leq C\frac{|u|}{\delta}, \ |L'_\delta(u)| \leq 1, \text{ and } 0 \leq L''_\delta(u) \leq \frac{C}{\delta},$$

where $C \in \mathbb{R}$ and, as $\delta \to 0$,

$$L'_\delta(u) \to sgn(u), \ |u| L''_\delta(u) \to 0 \text{ and } L_\delta(u) \to |u|, \text{ uniformly in } u.$$

2.2 Fundamental properties

In this subsection, some preliminary required results to ensure our main result will be presented.

Proposition 2.1. *Let $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*), L^{\infty}(\mathbb{R}))$ be any given solution to problem* (1.1) *under hypothesis* (1.2) *and let $p \geq 1$. Then,*

$$\|u(\cdot,t)\|_{L^P(\mathbb{R})} \leq exp\left\{\frac{(p-1)}{2} S_{\infty}^{2k}(t) \int_0^t \frac{B(\tau)^2}{\mu(\tau)} d\tau\right\} \|u(\cdot,0)\|_{L^p(\mathbb{R})},$$

for $S_{\infty}(t) = sup\{\|u(\cdot,\tau)\|_{L^{\infty}(\mathbb{R})}, 0 \leq \tau \leq t\}$.

Proof. Let $0 < t_0 < t < T_*$ and consider $\zeta_{R,\epsilon}$ and $L_{\delta}(u)$ defined in (2.1) and (2.4). Consider $p \geq 1$. Then, multiplying equation (1.1) by $\Phi'_{\delta}(u)\zeta_{R,\epsilon}(x)$, for $\Phi_{\delta}(u) = L^p_{\delta}(u)$ and integrating the result in $[t_0,t] \times B_R$, where $B_R = \{x \in \mathbb{R} \,|\, |x| < R\}$, by Fubini's theorem, integration by parts and Young's inequality, letting $t_0 \to 0$, $\delta \to 0$, and $R \to \infty$ we

$$\|u(\cdot,t)\|^p_{L^P(\mathbb{R})} \leq \|u(\cdot,0)\|^p_{L^P(\mathbb{R})} + \frac{p(p-1)}{2} S_{\infty}^{2k}(t) \int_0^t \frac{B(\tau)^2}{\mu(\tau)} \int_{\mathbb{R}} |u(\cdot,0)|^p dx d\tau,$$

where $B(t)$ is given in (1.2).

By Gronwall's Lemma, we get the result. □

Remark 2.2. Note that, if $p = 1$, $\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u(\cdot,0)\|_{L^1(\mathbb{R})}$.

Proposition 2.3. *Let $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*), L^{\infty}(\mathbb{R}))$ be any given solution to problem* (1.1) *under hypothesis* (1.2)*. Then*

$$\int_0^t \mu(\tau) \int_{\mathbb{R}} |u_x(\cdot,\tau)|^2 dx d\tau < \infty$$

for all $0 < t < T_*$.

Proof. Let $0 < t_0 < t < T_*$, and consider the functions $\zeta_R(x)$ defined in (2.3). Then, multiplying equation (1.1) by $2u\zeta_R$, integrating the result in $[t_0,t] \times B_R$, where $B_R = \{x \in \mathbb{R} | |x| < R\}$, and applying integration by parts, we obtain

$$
\begin{aligned}
&\int_{B_R} \zeta_R(x) u(x,t)^2 dx + 2\int_{t_0}^{t} \mu(\tau) \int_{B_R} \zeta_R(x) u_x^2 dx d\tau \\
&+ 2\int_{t_0}^{t} \mu(\tau) \int_{B_R} \zeta_R'(x) u_x u dx d\tau \\
&\leq 2\int_{t_0}^{t} \int_{B_R} \zeta_R'(x) u^{(k+2)} b dx d\tau + 2\int_{t_0}^{t} \int_{B_R} \zeta_R(x) |u_x| |u^{k+1}| |b(x,\tau)| dx d\tau \\
&+ \int_{B_R} \zeta_R(x) |u(x,t_0)|^2 dx.
\end{aligned}
$$

From Young's inequality,

$$
\begin{aligned}
&\int_{B_R} \zeta_R(x) u(x,t)^2 dx + \int_{t_0}^{t} \mu(\tau) \int_{B_R} \zeta_R(x) |u_x(x,\tau)|^2 dx d\tau \\
&+ 2\int_{t_0}^{t} \mu(\tau) \int_{B_R} \zeta_R'(x) u_x u dx d\tau \\
&\leq 2\int_{t_0}^{t} \int_{B_R} \zeta_R'(x) u^{(k+2)} b dx d\tau + \int_{t_0}^{t} \mu(\tau)^{-1} \int_{B_R} \zeta_R(x) |u|^{2(k+1)} |b|^2 dx d\tau \\
&+ \int_{B_R} \zeta_R(x) |u(x,t_0)|^2 dx.
\end{aligned}
$$

Then, by the definitions (2.3), and taking $R \to \infty$, and $t_0 \to 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}} |u(x,t)|^2 dx + \int_{0}^{t} \mu(\tau) \int_{\mathbb{R}} |u_x|^2 dx d\tau \leq \int_{\mathbb{R}} |u(x,0)|^2 dx \\
&+ S_\infty^{2k}(t) \int_{t_0}^{t} \frac{B(\tau)^2}{\mu(\tau)} \int_{\mathbb{R}} |u(x,0)|^2 dx.
\end{aligned}
$$

Thereby,

$$
\int_0^t \mu(\tau) \int_{\mathbb{R}} |u_x(\cdot,\tau)|^2 dx d\tau < \infty, \qquad \forall t \in [0, T_*)
$$

as we claim. □

Remark 2.4. We also note that is valid

$$
\int_0^t \mu(\tau) \int_{\mathbb{R}} |u(\cdot,\tau)|^{p-2} |u_x(\cdot,\tau)|^2 dx d\tau < \infty,
$$

for $p > 2$. The proof is in [15].

3 Main theorem

Theorem 3.1. *Let $u(\cdot,t) \in L^{\infty}_{loc}([0,T_*), L^{\infty}(\mathbb{R}))$ be any solution of (1.1) that satisfies (1.2). Then $u(\cdot,t) \in C^0([0,T_*), L^p(\mathbb{R}))$, for $p \geq 1$*

Proof. The proof of Theorem 3.1 will first be done for $p = 1$. Note that is sufficient to prove that, given $\epsilon > 0$ arbitrary, we can find $R = R(\epsilon, T) \gg 1$ in order to have $\|u(\cdot,t)\|_{L^1(|x|>R)} < \epsilon$ for all $0 < t \leq T < T_*$. So, let $0 < T < T_*$ be given and take $\zeta_{R,S} \in C^2(\mathbb{R})$ as in (2.2). Given $0 < t_0 < t \leq T$, $\delta > 0$, let $L_\delta \in C^2(\mathbb{R})$ be the regularized absolute value function introduced in equation (2.4).

Multiplying the pde in (1.1) by $L'_\delta(u(x,t))\zeta_{R,S}$ and integrating the result in $(t_0,t) \times B_{R,S}$, we have

$$\int_{t_0}^{t}\int_{B_{R,S}} u_\tau L'_\delta(u)\zeta_{R,S}(x)dxd\tau + \int_{t_0}^{t}\int_{B_{R,S}} (b(x,\tau)u^{k+1})_x(L'_\delta(u)\zeta_{R,S})dxd\tau$$
$$= \int_{t_0}^{t}\mu(\tau)\int_{B_{R,S}} u_{xx}L'_\delta(u)\zeta_{R,S}dxd\tau,$$

where $B_{R,S} = \{x \in \mathbb{R} : \ R/2 < |x| < R + 2S\}$. By Fubini's Theorem and applying integration by parts,

$$\begin{aligned}
&\int_{B_{R,S}} L_\delta(u(x,t))\zeta_{R,S}(x)dx + \int_{t_0}^{t}\mu(\tau)\int_{B_{R,S}} L''_\delta(u)u_x^2\zeta_{R,S}dxd\tau \\
&= \int_{B_{R,S}} L_\delta(u(x,t_0))\zeta_{R,S}(x)dx + \int_{t_0}^{t}\int_{B_{R,S}} b(x,\tau)u^{k+1}L''_\delta(u)u_x\zeta_{R,S}dxd\tau \\
&+ \int_{t_0}^{t}\int_{B_{R,S}} b(x,\tau)u^{k+1}L'_\delta(u)\zeta'_{R,S}dxd\tau - \int_{t_0}^{t}\mu(\tau)\int_{B_{R,S}} L'_\delta(u)u_x\zeta'_{R,S}dxd\tau.
\end{aligned} \tag{3.1}$$

Applying Young's Inequality in the second term of the right side of equation (3.1) we obtain

$$\int_{t_0}^{t}\int_{B_{R,S}} b(x,\tau)u^{k+1}L''_\delta(u)u_x\zeta_{R,S}dxd\tau \leq \frac{1}{2}\int_{t_0}^{t}\mu(\tau)\int_{B_{R,S}}|L''_\delta(u)|u_x^2\zeta_{R,S}dxd\tau$$

$$+\frac{1}{2}\int_{t_0}^{t}\mu(\tau)^{-1}B(\tau)\int_{B_{R,S}}|L''_\delta(u)||u|^{2(k+1)}\zeta_{R,S}dxd\tau.$$

Noting that the second term on the left side of equation (3.1) is positive, we obtain

$$\begin{aligned}\int_{B_{R,S}}L_\delta(u(x,t))\zeta_{R,S}(x)dx \le &\int_{B_{R,S}}L_\delta(u(x,t_0))\zeta_{R,S}(x)dx\\ &+\frac{1}{2}\int_{t_0}^{t}\mu(\tau)^{-1}B(\tau)\int_{B_{R,S}}|L''_\delta(u)||u|^{2(k+1)}\zeta_{R,S}dxd\tau\\ &+\int_{t_0}^{t}|B(\tau)|^2\int_{B_{R,S}}|u|^{k+1}|L'_\delta(u)||\zeta'_{R,S}|dxd\tau\\ &+\underbrace{\int_{t_0}^{t}\mu(\tau)\int_{R/2<|x|<R}|L'_\delta(u)||u_x||\zeta'_{R,S}|dxd\tau}_{J_1}\\ &+\underbrace{\int_{t_0}^{t}\mu(\tau)\int_{R+S<|x|<R+2S}|L'_\delta(u)||u_x||\zeta'_{R,S}|dxd\tau}_{J_2}.\end{aligned}$$

Let $\epsilon > 0$ be given. Then, by Young's inequality, (J1) and (J2) can be written as

$$\begin{aligned}|J_1| \le &\frac{\epsilon}{2}\frac{C^2}{R^2}\int_{t_0}^{t}\mu(\tau)Area(\{x\in\mathbb{R}:\frac{R}{2}\le|x|\le R\})d\tau\\ &+\frac{\epsilon^{-1}}{2}\int_{t_0}^{t}\mu(\tau)\int_{\frac{R}{2}<|x|<R}|u_x|^2dxd\tau\\ &\le\frac{\epsilon}{2}C^2\frac{3\pi}{4}\int_{t_0}^{t}\mu(\tau)d\tau+\frac{\epsilon^{-1}}{2}\int_{t_0}^{t}\mu(\tau)\int_{\frac{R}{2}<|x|<R}|u_x|^2dxd\tau\end{aligned}$$

and

$$\begin{aligned}|J_2| &\le \frac{\epsilon}{2}\frac{C^2}{S^2}\int_{t_0}^{t}\mu(\tau)Area(B_{R,S})d\tau+\frac{\epsilon^{-1}}{2}\int_{t_0}^{t}\mu(\tau)\int_{R+S<|x|<R+2S}|u_x|^2dxd\tau\\ &\le\frac{\epsilon}{2}\frac{C^2}{S^2}\pi\left(\frac{3R^2}{4}+2RS+3S^2\right)\int_{t_0}^{t}\mu(\tau)d\tau\\ &+\frac{\epsilon^{-1}}{2}\int_{t_0}^{t}\mu(\tau)\int_{B_{R,S}}|u_x|^2dxd\tau.\end{aligned}$$

Then, letting $t_0 \to 0$, $\delta \to 0$ and $S \to \infty$ (in this order) we have

$$\int_{|x|>R} |u(x,t)|dx \le \int_{|x|>R/2} |u_0|dx + \frac{\epsilon^{-1}}{2}\int_0^t \mu(\tau)\int_{\frac{R}{2}\le|x|\le R} |u_x|^2 dxd\tau + \frac{\epsilon}{2}\mathcal{C}\int_{t_0}^t \mu(\tau)d\tau,$$

where $\mathcal{C}$ is a constant.

Therefore, as $u_0 \in L^\infty(\mathbb{R})$, by (1.2) and Propositions 2.1 and 2.3, there exists $R > 0$ sufficiently large (depending on ϵ, T) so that

$$\int_{|x|>R} |u(x,t)|dx \le \epsilon\left(2 + \frac{\mathcal{C}}{2}\int_0^t \mu(\tau)d\tau\right)$$

for all $0 < \tau \le t$. As ϵ can be taking arbitrary small, $u \in C^0([0,T_*), L^1(\mathbb{R}))$.

Now, multiplying the pde in (1.1) by $\Phi'_\delta(u)\zeta_{R,S}$, where $\Phi(u) = L_\delta(u)^p$, and following the same steps as above, by Proposition 2.1 and Remark 2.4, we obtain that $u \in C^0([0,T_*), L^p(\mathbb{R}))$, for $p > 1$, and the proof is now complete. □

4 Open questions

We close our discussion of problem (1.1) with some open questions:

1. Is it possible to guarantee global existence for solutions of the problem (1.1) when $k \ge 2$?

2. For which $k \ge \dfrac{1}{n}$ it is possible to guarantee global existence for solutions of the problem (1.1) in $\mathbb{R}^n$ just assuming that the $|b(x,t)| < B(t)$.

We believe that the previous questions may play an important role in the continuation of the research in this area.

References

[1] C. J. Amick, J. L. Bona, and M. E. Schonbek, Decay of solutions of some nonlinear wave equations, *J. Differential Equations* **81** (1989), No. 1, 1–49.

[2] J. A. Barrionuevo, L. S. Oliveira and P. R. Zingano, General asymptotic supnorm estimates for solutions of one-dimensional advection-diffusion equations in heterogeneous media, *Intern. J. Partial Diff. Equations* **2014** (2014), Article ID 450417, 1–8.

[3] P. Braz e Silva, L. Schütz and P. R. Zingano, On some energy inequalities and supnorm estimates for advection-diffusion equations in $\mathbb{R}^n$, *Nonlinear Anal.* **93** (2013), 90–96.

[4] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.

[5] N. M. L. Diehl, L. Fabris, and J. S. Ziebell, *Decay estimates for solutions of porous medium equations with advection*, *Acta. App. Math.* **165** (2019), 149–162.

[6] Escobedo, M. and Zuazua, E., Large time behavior for convection-diffusion equations in $\mathbb{R}^n$, *J. Funct. Anal.* **100** (1991), No. 1, 119–161.

[7] L. C. Evans, *Partial Differencial Equations*, American Mathematical Society, Graduate Studies in Mathematics, Vol. 19, 2010.

[8] P. L. Guidolin, L. Schütz, J. S. Ziebell , J. P. Zingano, Global existence results for solutions of general conservative advection-diffusion equations in $\mathbb{R}$, *J. Math. Anal. Appl.* **515** (2022), 126361.

[9] R. Guterres, C. J. Niche, C. Perusato, and P. Zingano, Upper and lower $\dot{H}^m$ estimates for solutions to parabolic equations, *J. Differential Equations* **356** (2023), 407–431.

[10] J. Q. Chagas, P. L. Guidolin, and P. Zingano, Global solvability results for parabolic equations with p-Laplacian type diffusion, *J. Math. Anal. Appl.* **458** (2018), 860-874.

[11] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[12] M. E. Schonbek, Uniform decay rates for parabolic conservation laws, *Nonlinear Anal.* **10** (1986), No. 9, 943–956.

[13] D. Serre, *Systems of Conservation Laws*, Cambridge University Press, Vol. 1, Cambridge, 1999.

[14] M. Winkler. *A Gagliardo-Nirenberg-type inequality and its applications to decay estimates for solutions of a degenerate parabolic equation*, *Adv. Math.* **357** (2019), 106823.

[15] P.R. Zingano, *Dois Problemas em Equações Diferenciais Parciais* ArXiv e-prints, 1801.04361v1, 2018.