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MEAN-CVAR PORTFOLIO: A MIXTURE-COPULA APPROACH

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Trabalho de conclusão submetido ao Curso de Graduação em Ciências Econômicas da Faculdade de Ciências Econômicas da UFRGS, como requisito parcial para obtenção do título Bacharel em Economia.

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"An asset manager should concentrate her efforts on developing a theory rather than on backtesting potential trading rules."

— MARCOS M. LÓPEZ DE PRADO

ABSTRACT

The current study aims to conduct a comparative analysis of portfolio optimization techniques in the context of financial applications. The proposed approach involves the use of mixture-copulas as an alternative to mitigate the inherent risks of investments associated with investments in the stock market, particularly during times of economic crisis. To conduct this research, data from 19 country index ETFs were sourced from *Historical Market Data - Stooq*, spanning the period from 2013 to 2023. The study employed Mean-CVaR portfolio optimization, and the structural dependence between assets was modeled using a mixture of copulas (specifically Clayton-t-Gumbel), with marginal adjusted by an AR(1)-GARCH(1,1) model. The results of simulations based on this strategy were compared with two other benchmark portfolios, including those using Gaussian copulas and equally weighted portfolios, across three distinct time horizons: one, two, and five years. Portfolios generated through simulated returns using the mixture-copulas technique demonstrated superior risk-return performance when contrasted with the benchmark portfolios. Simultaneously, a reduction in financial losses was observed, with equivalent or superior returns in the comparison, particularly over longer time periods where the estimates were more accurate.

JEL classification: G11, G17, G32.

Keywords: Portfolio Optimization. Downside Risk Measurement. Mixture-Copulas. Time-Series Analysis. Uncertainty Modeling. Econometrics.

RESUMO

A presente pesquisa tem como objetivo analisar comparativamente técnicas de otimização de portfólio no contexto de aplicações financeiras. A abordagem de mistura de cópulas é proposta como uma alternativa para mitigar os riscos inerentes aos investimentos realizados na Bolsa de Valores, principalmente em momentos de crise. Para desenvolver a pesquisa, foram utilizados dados de preços de 19 ETFs de índices de países, provenientes do *Historical Market Data - Stooq*, que abrangem o período de 2013 a 2023. Foi empregada uma otimização de portfólio Média-CVaR, e a dependência estrutural entre os ativos foi modelada usando uma mistura de cópulas (Clayton-t-Gumbel), cujas marginais foram ajustadas por um modelo AR(1)-GARCH(1,1). Os resultados das simulações feitas a partir dessa estratégia foram comparados com outros dois portfólios de referência de técnicas mais simples, usando cópulas Gaussianas e igualmente ponderado, em três janelas de tempo: um, dois e cinco anos. As carteiras geradas a partir dos retornos simulados com a técnica de mistura de cópulas apresentaram melhores desempenhos em termos de risco-retorno quando comparada aos portfólios de referência. Ao mesmo tempo, notou-se uma redução das perdas financeiras, inclusive retornos iguais ou superiores na comparação, especialmente nas janelas de tempo maiores, nas quais as estimativas foram mais precisas.

Classificação JEL: G11, G17, G32.

Palavras-chave: Otimização de Carteiras. Medição de Risco Negativo. Cópulas Mistas. Análise de Séries Temporais. Modelagem de Incertezas. Econometria.

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LIST OF ABBREVIATIONS AND ACRONYMS

VaR: Value-at-Risk

CVaR: Conditional Value-at-Risk

LP: Linear Program

MPT: Modern Portfolio Theory

ETF: Exchange Traded Fund

MCP: Mixture of Copulas Mean-CVaR Portfolio

GCP: Gaussian Copulas Mean-CVaR Portfolio

EWP: Equal Weighted Portfolio

CONTENTS

1 INTRODUCTION

The financial market is an environment characterized by uncertainty, where various factors like economic shifts, political events, and unforeseeable circumstances can significantly influence investment performance. The well-known saying "don't put all your eggs in one basket" cautions against concentrating all resources in a single investment. With this in mind, investors often diversify their capital across different asset classes to reduce the risk associated with a single type of asset.

Markowitz (1952), in his seminal work, was the first to demonstrate the benefits of diversification. He emphasized that investors are primarily concerned with two factors: risk and return. Markowitz's Modern Portfolio Theory laid the foundation for understanding the relationship between risk and return in investment portfolios, highlighting the importance of diversification as a means to optimize returns while managing risk [1].

However, understanding financial markets poses a significant challenge due to their highly complex dynamics. In economic studies, a high degree of uncertainty is always taken into account, as agents' behavior is influenced by the information available to them. Fama (1970) addressed this issue by introducing the Efficient Market Hypothesis, stating that under certain conditions, prices incorporate all available information. This hypothesis suggests a market equilibrium determined by expected returns as a function of their associated risk. It operates under the assumption of a "fair game" where the probabilities of winning or losing are equal, leading to two distinct models. The first model assumes that price movement is a martingale, implying an expected return of zero. The second suggests that price series follow a random walk, characterized by independent returns at each point in time and a known distribution.

While considering the Efficient Market Hypothesis as reasonably valid, it is practically impossible, or at least extremely challenging, to predict the prices of financial assets. Nonetheless, this does not imply that a portfolio manager cannot achieve a risk-return relationship that outperforms the overall market composition. A recent study by Prado (2020) emphasizes that "as investors, we have no (legitimate) control over prices, and the key decision we can and must make is to size bets properly." Therefore, it is crucial to focus on studies that provide solutions for the optimal allocation of an investment portfolio.

[1]Harry Markowitz initially focused his attention on passive portfolio management. He believed that investors could identify an optimal portfolio that effectively balanced risk and return, determine the appropriate asset allocation, and then adopt a stance of patience and observation.

Alovisi (2019) presented an empirical study with similar objectives. The primary goal was to construct an investment portfolio capable of protecting against extreme negative market conditions while aiming for positive returns. To achieve this, a portfolio optimization technique based on the mean-CVaR (Conditional Value at Risk) method was employed. The structural dependence among financial assets was modeled using a Clayton-t-Gumbel mixture-copula approach, and the marginal distributions of the assets were adjusted through an AR(1)-GARCH(1, 1) model.

At present, various portfolio optimization techniques are available, some employing risk measures such as standard deviation, initially proposed by Markowitz. However, this metric is not ideal as it encompasses both negative variations (which investors aim to avoid) and positive variations in returns (which they aim to achieve). Therefore, it is essential to focus on measures known as "Downside risk," as defined by Sortino and Meer (1991). CVaR is a preferred choice due to its nature as a coherent risk measure that adheres to four fundamental axioms, as demonstrated by Pflug (2000): (i) monotonicity, (ii) invariance translation, (iii) positive homogeneity, and (iv) subadditivity, as defined by Artzner et al. (1999). CVaR is also convex, as demonstrated by Rockafellar and Uryasev (2000). This allows us to find a unique solution in portfolio optimization problems.

Additionally, some authors argue that the distribution of returns is considered normal, with a mean μ and variance σ^2, $r \sim N(\mu, \sigma^2)$; nevertheless, returns exhibit different stylized facts: (i) the distribution is not normal, (ii) they approximate a random walk, and (iii) there is a positive dependence between absolute (or quadratic) returns, as observed by Taylor (2011).

These atypical behaviors can be partially attributed to the existence of volatility clusters. This means that during periods of crisis or instability, different financial assets tend to move together, resulting in distinct patterns of market behavior. The presence of volatility clustering serves as a prominent indicator of non-normality within the daily return distribution. Therefore, by implementing a mixture probability distribution, we can enhance the model's capacity to effectively capture the stylized facts, a crucial aspect in mitigating extraordinary losses.

Given that we are working with a set of financial instruments, establishing their interrelationships is essential. One straightforward approach involves using linear correlation to model asset dependencies. However, as Pfaff (2012) asserts, this is accurate only if the assets are jointly elliptically distributed, which does not apply to our specific scenario. To overcome this limitation and account for different distribution patterns, copula functions prove useful. They enable a more precise adjustment of relationships between assets, especially when the distribution is non-elliptical, a common occurrence in financial markets. A mixture of copula functions allows modeling a wider range of possible multivariate dependence structures of the assets.

Kakouris and Rustem (2014) proposed the utilization of a mixture of Archimedean copulas. Copulas capture the dependency between the marginal distributions of the random variables rather than the dependency between the variables themselves. A notable advantage of employing copulas is the ability to separate the selection of multivariate dependence from the selection of the univariate distributions. A mixture of copula functions allows modeling a wider range of possible multivariate dependence structures of the assets.

In this work, we apply the Worst Case Mixture-Copula for Mean-CVaR optimization to 19 ETFs different countries indexes from January 2013 to June 2023, contributing to a comprehensive analysis and meaningful insights into the performance of our portfolio optimization strategy. This paper is structured as follows: in Section 2, we provide a comprehensive literature review in portfolio optimization and risk management. Section 3 outlines the methodology, including mathematical techniques employed and the empirical strategy. Section 4 describes the data and results. Finally, in Section 5, we conclude our study, summarizing the findings and discussing their implications for future works.

2 LITERATURE REVIEW

2.1 EVOLUTION OF PORTFOLIO OPTIMIZATION MODELS

While studying information transmission through communication channels, Kelly (1956) provides a practical example addressing the portfolio optimization problem. The example involves a gambler who uses received information to profit from their bets. The results demonstrate that, in each round, the player maximizes the logarithm of the expected capital because in a sequence of repeated bets, where the law of large numbers applies, the gambler's fortune depends only on the predetermined percentage of capital to be wagered.

Markowitz (1952) proposed another approach to analyzing agent preferences. He argues that investors should not only consider the expected return of each asset but also the associated risk and their correlation with one another. Therefore, they should diversify their capital to find the weights of the optimal portfolio, which maximizes return subject to minimum variance [1].

In his analysis, Latane (1959) proposed the use of additional criteria for financial asset allocation strategies. He acknowledges that while Markowitz has developed a method for selecting efficient portfolios, there is still a need to provide objective ways to adjust the risk aversion indicator. His process is divided into three parts, with a primary objective, a secondary objective, and a strategy definition criterion. Based on Bernoulli, the conclusion of the analysis suggests that a portfolio manager maximizes wealth in the long term by: (i) maximizing mathematical expectation, (ii) maximizing expected utility, and (iii) choosing the investment portfolio with the highest probability of success.

The divergences among the presented models have generated different stances within the academic community and the financial industry. According to Samuelson (1963), the use of a logarithmic utility function by economic agents would violate the assumption of rational behavior because, for a given level of risk aversion, the choices made would be inconsistent, resulting in what is known as the "Samuelson Fallacy" concerning the law of large numbers. Although, Thorp (1975) counters this argument by highlighting the distinction between different types of utility theories [2] and asserting that the use

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[1]This optimal portfolio is found on the efficient frontier.

[2]Thorp compares three types of utility theory: Descriptive - when observed data is mathematically adjusted, Predictive - which explains observed data, and Prescriptive - used to guide behavior to achieve a specific goal.

of logarithmic utility is prescriptive, meaning it guides decision-making for groups and institutions aiming to maximize returns. Furthermore, it emphasizes that by maximizing logarithmic utility, portfolios with a higher probability of losses are eliminated, partially reducing the risk.

Recently, Carr and Cherubini (2022) published a study aiming to resolve the debate surrounding the use of logarithmic utility in wealth maximization. They propose a new definition of market equilibrium that does not refer to a specific individual but only to laws governing the excess return of an investment portfolio. By employing power laws to maximize wealth, the authors argue that composition rules cannot be arbitrary, and the event's dynamics do not depend on the investor's behavior, but rather on the nature of the market itself. The economic model developed indicates that price speculation includes a process governing time dynamics and that if trading time is stochastic, the average returns must account for this randomness. Two dynamic models are introduced, with the first one using the Variance Gamma (VG) distribution, yielding the same wealth function proposed by Samuelson but without the inclusion of a utility function. In the second case, a model of Inverse Gaussian (NIG) distribution is used, resulting in a quadratic wealth function similar to Markowitz's approach. In both cases, the impact of optimal portfolio growth is isomorphic to an increase in risk aversion, and the parameter playing the role of the risk aversion index is the variance of the stochastic clock [3].

2.2 FROM VAR TO CVAR IN PORTFOLIO OPTIMIZATION

Over the years, the process of quantifying risk has become an indispensable component of investment decision-making, particularly when confronted with the volatility of stock market crashes, as highlighted by Kakouris and Rustem (2014). The continuously evolving financial landscape has compelled investors to reorient their attention towards effectively gauging risks and potential losses, even in relatively stable market conditions, as articulated by Pfaff (2012).

Prominent financial institutions have pioneered various risk measurement methodologies, one of which is Value at Risk (VaR), introduced by J.P. Morgan in the mid-1980s, as documented in Longerstaey and Spencer (1996), Morgan et al. (1996). While VaR satisfies three out of the four axioms necessary to qualify as a coherent risk measure, it falls

[3] The idea behind this process is to link return volatility to the flow of market information, which is not always uniform over time and often not directly observable. See Geman (2009).

short in subadditivity, as pointed out by Zhu and Fukushima (2009). Consequently, in scenarios involving portfolio optimization with multiple assets under consideration, there is no guarantee that the portfolio's VaR will be lower than the sum of individual VaRs.

This potential outcome may lead to a misleading conclusion that diversification lacks merit. Therefore, as proposed by Rockafellar and Uryasev (2002), it is advisable to conduct optimizations subject to the minimum Conditional Value at Risk (CVaR) [4], also known as Expected Shortfall (ES), elaborated by Szego (2005). Furthermore, it has been established that CVaR is a convex function for both continuous and discrete distributions. This property enables the reduction of optimization problems to linear programming, facilitating the efficient optimization of portfolios with large dimensions.

2.3 ROBUST MODELING TECHNIQUES AND DEPENDENCE STRUCTURES

In the pursuit of constructing investment portfolios that demonstrate resilience in the face of dynamic market conditions, a major shift in portfolio optimization methodology has emerged. Traditional approaches, reliant on single sets of assumptions or historical data, have shown vulnerability to extreme market events, which can severely impact portfolio performance. In response to these challenges, Fabozzi et al. (2007) suggests incorporating a variety of scenarios and potential risks into the portfolio construction process, rather than relying on a single set of assumptions. This approach helps to reduce the impact of crashes or bubbles on the performance of the investment portfolio.

Significant advancements have been made in using econometric models to estimate the volatility of financial assets. An important approach is the ARCH model introduced by Engle (1982) and applied to finance by Bollerslev, Chou and Kroner (1992), which assumes that the variation in returns of a financial asset changes over time and depends on its own past returns. The utility of the ARCH model in finance is widely recognized and has been applied in various areas, including asset pricing, portfolio optimization, and risk management.

In another work, Bollerslev, Engle and Wooldridge (1988) propose a version of the CAPM model that takes into account time-varying covariances between the returns of different assets. Unlike the traditional CAPM, which assumes that covariances between asset returns are constant over time, the model proposed by Bollerslev, Engle, and Wooldridge

[4] CVaR serves as a tail risk metric, representing the anticipated loss when returns dip below the VaR threshold at a specified confidence level.

incorporates the idea that these covariances change over time. This is particularly important because the assumption of constant covariances can lead to imprecise estimates of expected returns and risk. To explain volatility variation, a multivariate GARCH model is used, allowing for the estimation of time-varying covariance matrices between the returns of different assets. Applying this model to a dataset of stock returns revealed that time-varying covariances have significant effects on expected returns and risks.

Moreover, the use of copulas as a way to model the dependence structure between credit risks in a portfolio was proposed by Li (2000). However, this model was criticized for assuming that the dependence structure between credit risks was stable over time, which proved to be a major flaw during the financial crisis when correlations between different assets suddenly became highly correlated. Engle (2002) demonstrated that analyzing dynamic correlations is more suitable, as the return of financial assets is positively correlated with market volatility.

Other authors, such as Cherubini, Luciano and Vecchiato (2004) and McNeil, Frey and Embrechts (2015), have also made significant contributions to the use of copulas in finance. Christoffersen (1998) proposed the use of copulas to estimate risk measures by modeling the dependence between different financial assets. By modeling the dependence structure with copulas, we can estimate the joint distribution of different financial assets and calculate measures such as VaR and ES that take into account the joint behavior of assets. Christoffersen showed that using copulas can lead to more accurate risk estimates compared to traditional

Moreover, a model that aims to address the distribution issue was proposed by Zou and Zhu and Fukushima (2009). The authors applied a robust worst-case technique in this domain. However, they also acknowledged the possibility of assuming a multivariate distribution among the assets. Hu (2006) and Kakouris and Rustem (2014) explain that relying on a Gaussian distribution implies that the probability of losses is equal to the probability of gains. Nevertheless, in the context of financial markets, assets tend to exhibit stronger co-movements during crises. As a result, the use of a combination of Archimedean copulas was suggested to enhance data fitting. A notable advantage of employing copulas is the ability to separate the selection of multivariate dependence from the selection of the univariate distributions, emphasized by Nelsen (2000).

2.4 INNOVATIONS IN ASSET ALLOCATION

New methods for asset selection have continued to be developed based on theoretical frameworks. Luca, Rivieccio and Zuccolotto (2010) devised a heuristic for asset selection that combines machine learning and copulas. Three steps were taken in the article: (i) application of univariate AR-GARCH models, (ii) asset selection using data mining techniques, the random forest algorithm [5], to filter assets with lower lower-tail dependence, and (iii) copulas to estimate lower-tail dependence coefficients.

A set of techniques was proposed to deal with uncertainty in input data by Prado (2016). One of the main challenges in portfolio optimization is that ideal portfolio weights obtained using historical data may not perform well when applied to out-of-sample future data. This is known as overfitting. To address this issue, Lopez de Prado introduced a new method called the Hierarchical Risk Parity (HRP) algorithm, based on machine learning techniques. The HRP algorithm offers several advantages over conventional portfolio optimization methods. In particular, it is less susceptible to fluctuations in input data and can adapt to new market conditions with greater flexibility. Moreover, it tends to generate more broadly diversified portfolios, which can reduce the risk of significant losses.

Finally, in the book "Machine Learning for Asset Managers," Prado (2020) explores various techniques that make use of supervised and unsupervised machine learning algorithms to address issues such as numerical instability, noise elimination resulting from substitute effects and multicollinearity, as well as improving prediction accuracy and identifying the importance of each variable. The book also provides practical guidance for financial professionals to achieve more reliable results in their work, drawing on economic theories.

2.5 EVALUATING INVESTMENT PORTFOLIO PERFORMANCE

Up to this point, we have discussed decision-making methods for investments. Nonetheless, it is equally important to assess whether the choices made were the best possible ones. In this regard, Sharpe (1963) proposes the most famous indicator for evaluating investment portfolios. For example, the Markowitz model indicates that for two portfolios with the same variance, the one with the higher expected return should be se-

[5] This algorithm constructs multiple decision trees using subsets of variables and random observations from the dataset and aggregates the predictions of each decision tree to obtain a final prediction.

lected. Similarly, for two portfolios with equivalent expected returns, the one with lower variance should be chosen. The best combination of assets is the one found on the efficient frontier, tangential to the objective function. This metric, called the Sharpe ratio, not only serves as an evaluation measure but also suggests the percentage that investors should allocate to equities and risk-free assets.

There are other performance metrics available in the literature. For instance, Jensen's alpha measures the excess return of a portfolio over the expected return. Alpha assesses the manager's ability to generate gains independently of market movements. A positive alpha indicates that the manager has added value, while a negative alpha suggests that the results fell short of expectations, as explained by Jensen (1969).

The Treynor ratio adjusts the excess return relative to the risk (measured by beta) assumed by the manager. This measure represents the portfolio's risk that cannot be eliminated through diversification, i.e., systematic risk. A higher Treynor ratio indicates higher returns generated in relation to the amount of assumed systematic risk, as explained by (TREYNOR, 1961).

Both measures align with the famous Capital Asset Pricing Model (CAPM) proposed by Sharpe (1964). This model assumes that investors are rational and risk-averse, demanding compensation for bearing risk. Consequently, the expected return can be explained by the risk-free rate, the risk premium [6], and an idiosyncratic component.

More robust measures have been proposed by Bailey and Prado (2014) to address the issue of selection bias when backtesting potential trading strategies. The authors highlight the dangers of encountering false positives, which can lead investors and researchers to make Type I errors, by not identifying false positives. The proposed indicator, the Deflated Sharpe Ratio, is designed to address several issues commonly encountered when assessing the statistical significance of the Sharpe Ratio. It takes into account factors such as multiple trials, non-normal returns, and shorter sample lengths, which can potentially distort the accuracy of the Sharpe Ratio. Please be aware that the primary goal of this measure is to prevent overfitting.

[6] This relationship is given by: $\beta \times (R_m - R_f)$.

3 METHODOLOGY

CVaR is a widely used risk measure in portfolio optimization, offering a more comprehensive view of risk by considering not only the probability of losses but also their magnitude. By incorporating copulas that capture asset dependencies, we can enhance portfolio modeling, enabling the construction of portfolios that are more resilient to complex and volatile market scenarios. The combination of CVaR and copulas opens up new opportunities for risk management and asset allocation in investment contexts, providing a more sophisticated and robust approach to portfolio optimization.

In this chapter, we will explore the application of CVaR in portfolio optimization, with a focus on integrating copulas to more accurately represent interactions between financial assets. We will delve into the theory behind CVaR and its superiority compared to traditional metrics like Value at Risk (VaR). Additionally, we will examine copulas as an effective tool for modeling asset dependence and discuss how they can be incorporated into the portfolio optimization process. Therefore, we will present a strategy following the steps outlined in Alovisi (2019). Last, we show a bunch of performance measures to evaluate results.

3.1 THEORETICAL FOUNDATION

As previously mentioned, VaR is a common risk measure originally developed by J.P. Morgan to manage financial risks in their investment portfolios. It helps calculate the maximum expected loss over a defined period with a certain level of confidence. However, VaR has its limitations. It doesn't provide information about losses beyond the estimated value at the chosen confidence level, and it is not suitable for optimizing portfolios because, in certain instances, the combined risk of individual assets may be lower than the overall portfolio risk due to the lack of subadditivity, as pointed out by Zhu and Fukushima (2009).

In mathematical terms, VaR is defined as follows:

$$
\begin{aligned}
Pr(x \leq VaR(X)) = 1 - \beta \\
VaR_\beta(X) &= \inf\{x \mid Pr(X > x) \leq 1 - \beta\} \\
&= \inf\{x \mid F_X(x) \geq \beta\}
\end{aligned}
\tag{3.1}
$$

This represents the probability that a random variable X falls below or equals its VaR at a confidence level of $1-\beta$. $VaR_\beta(X)$ is essentially the infimum of x such that the probability of X exceeding x is less than $1-\beta$, a condition which can be equivalently expressed as $F_X(x) \geq \beta$.

On the other hand, CVaR is a measure that derives from VaR but offers several advantages. While VaR provides a single point estimate of the maximum potential loss, CVaR goes a step further by considering the entire tail of the loss distribution beyond the VaR threshold. This means that CVaR takes into account not only the probability of loss exceeding a specific threshold but also the expected magnitude of those losses.

Moreover, CVaR is coherent, meaning it satisfies certain mathematical properties that VaR does not, making it a more suitable choice for risk optimization and management. It respects subadditivity, ensuring that the risk of a portfolio is always lower than or equal to the sum of individual asset risks, addressing one of VaR's limitations in portfolio optimization.

Conditional Value at Risk at a confidence level $1-\beta$, denoted as $CVaR_\beta(X)$, represents a critical risk measure used to gauge the expected loss under extreme scenarios. It is calculated as the expected value of the random variable X conditional on X being less than or equal to its Value at Risk (VaR) at the same confidence level. This can be expressed mathematically as follows:

$$
\begin{aligned}
CVaR_\beta(X) &= \mathbb{E}[X|X \leq VaR_\beta(X)] \\
&= \frac{1}{1-\beta}\int_0^{1-\beta} VaR_\alpha(X)\, d\alpha
\end{aligned}
\tag{3.2}
$$

In essence, $CVaR_\beta(X)$ provides valuable insights into the potential loss magnitude beyond the $VaR_\beta(X)$ threshold, taking into account extreme scenarios with a confidence level of $1-\beta$.

In the context of portfolio optimization, according Pflug (2000), where w represents a vector of portfolio weights, X is a set of feasible portfolios subject to linear constraints, and r is a vector denoting market variables affecting asset losses, a loss function $f(w,r)$ must be introduced. This loss function depends on the decision vector w, a member of any arbitrary subset $X \in \mathbb{R}^m$, and the random vector $r \in \mathbb{R}^m$. It effectively combines the weight vector w and the return vector r to capture the joint impact of portfolio weights and market outcomes. This formalism is integral to various portfolio optimization problems, including the one outlined in Silva and Ziegelmann (2017).

$$CVaR_\beta = \frac{1}{1-\beta} \int_{f(w,r) \geq VaR_\beta(w)} f(w,r)p(r)\,dr \tag{3.3}$$

Let's see the four axiom defined by Artzner et al. (1999) for a risk measure to be coherent. A brief explanation of the axioms is given by Pfaff (2012). Let ρ denote a risk measure and $\rho(L)$ the risk value of a portfolio, where the loss L is a random variable.

a) **Monotonicity:** For two given losses, L_1 and L_2, this principle states that $L_1 \leq L_2 \implies \rho(L_1) \leq \rho(L_2)$.
b) **Translation invariance:** It postulates that the risk measure is specified in the same terms as the losses and is formalized as $\rho(L_1) = \rho(L) + l$, $l \in \mathbb{R}$, where $l \in \mathbb{R}$.
c) **Positive homogeneity:** If $\rho(\lambda L) = \lambda\rho(L), \lambda > 0$, the axiom of positive homogeneity is met. If the size of a portfolio position directly affected how risky it was, this premise would be broken.
d) **Subadditivity:** This axiom asserts that $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$. In other words, due to the advantages of diversification, the portfolio risk must be lower than or equal to the total of the individual risk measurements of the assets included in the portfolio.

It's worth emphasizing that for a risk measure to exhibit convexity, it must respect the axioms of positive homogeneity and subadditivity.

It is obvious that CVaR optimization employs VaR in its definition given equation 3.3. As previously established, VaR is neither convex nor linear. In their major contribution, Rockafellar and Uryasev (2000) defines a more straightforward auxiliary function that may be used to calculate CVaR without first computing VaR:

$$F_\beta(w,\alpha) = \alpha + \frac{1}{1-\beta} \int_{f(w,r) \geq \alpha} (f(w,r) - \alpha)\,p(r)\,dr, \tag{3.4}$$

where $F(w,\alpha)$ is the non-decreasing and right-continuous cumulative distribution function for the loss function $f(w,r)$ with respect to α. It is also shown that $F_\beta(w,\alpha)$ is convex with respect to α that $\min F = \min CVaR$. This implies that minimizing F leads to minimizing CVaR.

Now, suppose we do not have an analytical representation to describe the probability of returns; to address this, we can employ an approximation. Let's consider K different scenarios, which can be historical or simulated returns, represented as r_1, r_2, up to r_K. Equation 3.4 can be approximated using the equation below:

$$F_\beta(w,\alpha) = \alpha + \frac{1}{(1-\beta)K}\sum_{k=1}^{K}(f(w,r_k)-\alpha)^+. \tag{3.5}$$

Equation 3.5 assists in estimating CVaR. Essentially, it involves calculating a weighted average of the discrepancies between $f(w, r_k)$ and a reference value α for each of the scenarios, where $k = 1, 2, ..., K$ represents the complete set of available scenarios.

To handle the distribution of asset returns, we can resort to the use of copulas. To address various distribution patterns, copula functions are valuable. They enable precise adjustment of relationships between assets, particularly when the distribution is non-elliptical, which is common in financial markets.

Copulas are multivariate distribution functions whose marginals are uniformly distributed in [0, 1]. Sklar's Theorem teaches us that we can represent the joint distribution of multiple random variables using a copula function that depends solely on the individual distributions of each variable, referred to as marginals.

Theorem 3.1.1 (Sklar's Theorem) *Let F be an n-dimensional distribution function with margins $F_1, ...F_n$. Then there exists an n-copula C such that for all $x \in R^n$,*

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)). \tag{3.6}$$

Furthermore, if $F_1, ...F_n$ are continuous, then C is unique.

In other words, Sklar's Theorem demonstrates how to characterize the relationships among multiple random variables in terms of copulas, which describe the dependencies between the random variables and their individual distributions. This simplifies our analysis significantly.

Let F be an n-dimensional distribution function with margins $F_1, ...F_n$ and let C be and n-copula. Then, for any $u = (u_1, ..., u_n) \in U[0, 1]^n$,

Corollary 3.1.1.1

$$C(u_1, ...u_n) = F(F_1^{-1}(u_1), ..., F_n^{-1}(u_n)), \tag{3.7}$$

where $F_i^{-1}, i = 1, ..., n$ are the quasi-inverses of the marginals.

Kakouris and Rustem (2014) demonstrated that a relationship between the probability density functions and the copulas can be derived using Theorem 2.3.1 and Corollary 2.3.1.1. An n-copula's copula density is defined as follows:

Definition 3.1.1 *Let f be the multivariate probability density function the probability distribution F and $f_1, ..., f_n$ the univariate probability density functions of the margins $F_1, ..., F_n$. The copulas density function of an n-copula C is the function c: $U[0,1]^n \mapsto [0, \infty)$ such that*

$$\frac{\partial F(x_1, ..., x_n)}{\partial x_1, ..., \partial x_n} = c(u_1, ..., u_n) \prod_{i=1}^{n} f_i(x_i) \tag{3.8}$$

The concept enables us to distinguish between the dependence structure symbolized by C and the modeling of the marginals $F_i(x_i)$. The join probability function is then compared to what it would have been under independence to create the copula probability density function. It is feasible to think of the copula as the adjustment we must do in order to change the independent probability density function into the multivariate density function, according to the Silva and Ziegelmann (2017) interpretation. In other words, copulas break down the joint *p.d.f.* from its margins.

Similar to Hofert et al. (2018b), we adopt a two-step approach for estimating multivariate distributions. Firstly, we determine the marginal distribution for each variable x_i, and then we establish the relationship between the filtered data from step one. This methodology doesn't assume joint behavior among the marginals, allowing us to generate joint distributions independently. Copula function modeling thus offers flexibility for joint distribution modeling, as elaborated in Fan and Patton (2014) and Patton (2008) for econometrics and finance. For simulating joint probability density functions (p.d.fs), we consider two distinct copula families: Archimedean Copulas and Elliptical Copulas as described in Nelsen (2000) and Pfaff (2012).

Elliptical copulas, like Gaussian or t-student copulas, implicitly capture dependence via distribution parameters. While elliptical copulas are easy to simulate, their symmetry may not align with the skewed empirical distributions typically seen in financial data. The Gaussian copula, commonly used in modeling, has zero tail dependency, limiting its use in risk modeling. On the other hand, the Student's t-copula, another elliptical copula, exhibits non-zero tail dependency.

Archimedian copulas are not always symmetric, unlike elliptical copulas, because tail dependency is modeled by the specific copula-generating function. The Clayton Copula and the Gumbel Copula are two practical and well-known Archimedean copulas, each with unique properties. You may find more extensive literature on Archimedian copulas in Cherubini, Luciano and Vecchiato (2004); Hofert et al. (2018b); Nelsen (2006).

We shall now elucidate the attributes of three copulas employed for modeling the interdependence among asset returns:

a) Clayton Copula: To model the dependence between random variables, especially in the lower tails of their distributions. It is particularly useful for capturing dependence in low-probability scenarios, such as extreme events in the financial market.

b) t-Copula: Very useful when aiming to capture dependence in both the lower and upper tails of the distributions of random variables. It is suitable for handling extreme or rare events where the tails of the distribution are heavier than the Normal distribution.

c) Gumbel Copula: A copula function that describes the dependence between random variables, with a focus on the upper tails of their distributions. In contrast to Clayton, it investigates events in the upper tail, significantly above the average.

This approach allows us to explore a wide range of dependence structures among assets, with the goal of more accurately capturing how these individual assets relate to each other. The choice of copulas and their weights is grounded in previous studies, such as those by Pfaff (2012) and Hu (2006).

Kakouris and Rustem (2014) established the framework for utilizing copula functions in conjunction with Conditional Value at Risk (CVaR). Let $w \in W$ represent a decision vector, $u \in U[0,1]^n$ a random vector following a continuous distribution with a copula density function denoted as $c(.)$, and a set of marginal distributions $F(r) = (F_1(r_1), ..., F_n(x_n))$ where $u = F(r)$. The corresponding equation, incorporating copula into equation 3.5, is given by:

$$G_\beta^d(w,\alpha) = \alpha + \frac{1}{(1-\beta)K^i}\sum_{k=1}^{K}(f(w,u_k^i)-\alpha)^+,\ i = 1,2...l. \tag{3.9}$$

Equation 3.9 can then be evaluated using Monte Carlo simulations. This is accomplished by sampling realizations of the copulas $C_i(.)$ using the previously determined marginals.

Finally, following Wuertz et al. (2010), portfolio optimization involves minimizing Conditional Value at Risk (CVaR) under constraints. This involves finding optimal asset weights (w) to achieve a target return (R) while considering a desired CVaR significance level (β). Rockafellar and Uryasev (2000) introduced an approach that simultaneously determines portfolio weights, CVaR, and VaR within a feasible set. When an analytical representation for density is unavailable, they proposed an approximation method. If both the feasible set (X) and the loss function ($f(w, r_j)$) are convex, CVaR optimization can be transformed into a Linear Programming problem, allowing efficient solutions. From equation 3.9 we have:

$$
\begin{aligned}
\min_{w\in\mathbb{R}^n, z\in\mathbb{R}^K, \alpha\in\mathbb{R}} \quad & \alpha + \frac{1}{(1-\beta)K}\sum_{i=1}^{K} z_i, \\
\text{s.a} \quad & z_i \geq f(w, u_k) - \alpha, i = 1, ..., K \\
& z_i \geq 0, i = 1, ..., d, \\
& w \in W, \\
& w^T \hat{\mu} \geq R, \\
& w^T 1 = 1,
\end{aligned}
\tag{3.10}
$$

3.2 OPTIMIZATION AND STRATEGY

We will present a strategy following the steps outlined in Alovisi (2019). Modifications were made in our study to explore the effects of sample size for a period of $T = \{252, 504, 1260\}$ days, representing one year, two years, and five years, respectively. In order to ensure more accurate results and to mitigate bias, transaction costs of 0.0003 or 0.03% were included in our analysis, aligning with the methodology of Frazzini, Israel and Moskowitz (2018).

An optimization with varying sample sizes was conducted, employing the method described by Xi (2014) to estimate our model and capture market dynamics. The portfolio's performance one day ahead was projected through simulations using the estimated model, implementing a rolling window approach. This allowed for insights into future asset performance. To adapt to changing market conditions and align with our objectives, the portfolio was rebalanced daily. This resulted in a total of $L - T$ optimizations, depending on the length of the period.

Here's a breakdown of the optimization process:

a) **Optimization 1:** Data spanning from day 1 to day T was utilized to execute the strategy and determine portfolio weights for day $T + 1$;
b) **Optimization 2:** Data spanning from day 2 to day $T + 1$ was utilized to execute the strategy and determine portfolio weights for day $T + 2$;
c) **Optimization 3:** ...
d) **Optimization** $L - T$**:** Data spanning from day $L - T$ to day $L - 1$ was utilized to execute the strategy and determine portfolio weights for day L;

We were able to efficiently modify our portfolio to varied market conditions and acquire a complete grasp of the asset's performance across diverse time frames by continuously implementing this methodology for different in-sample sizes. The rolling window technique kept our model current and captured the most recent market conditions, allowing us to make informed decisions and improve portfolio performance.

To implement the Worst Case Mixture-Copula Mean-CVaR portfolio optimization, we primarily follow the procedures described in Christoffersen (2011), Pfaff (2012), Hofert et al. (2018a), Hofert et al. (2018b) and Xi (2014). The optimization steps previously demonstrated are reiterated below. Additionally, for comparative purposes, we also consider the estimation of two other portfolios: a Gaussian Copula Portfolio and an Equal Weight Portfolio. These steps are taken into account when conducting benchmarking.

a) First, an AR(1)-GARCH(1,1) model with skewed t-distributed innovations was fitted to each univariate time series.

b) A standardized residuals vector was created for each asset using the estimated parametric model.

$$\frac{\hat{\epsilon}_{t,j}}{\hat{\sigma}_{t,j}}, \quad t = 1, \ldots, (L-T) \quad \text{and} \quad j = 1, \ldots, 19. \tag{3.11}$$

c) The Skewed-t distribution of the GARCH error process was used to parametrically calculate pseudo-uniform variables from the standardized residuals. The empirical distribution functions of the standardized residual vectors could also be used to accomplish this semiparametrically, as shown in Pfaff (2012) and Hofert et al. (2018a).

d) The multivariate Clayton-t-Gumbel Mixture Copula model was calculated using data transformed to $[0, 1]$ margins from the linear combination of copulas,

$$C^{CtG}(\Theta, u) = \pi_1 C^C(\theta_1, u) + \pi_2 C^t(\theta_2, \theta_3, u) + \pi_3 C^G(\theta_4, u), \tag{3.12}$$

where Θ is the vector of pseudo-uniform observations for each asset, π_i is a copula weight parameter such that $\pi_i \in [0, 1]$ and $\sum \pi_i = 1$, and u is the vector of Clayton, t, and Gumbel copula parameters.
The copula parameters and weights were estimated by minimizing the negative log-likelihood of the weighted densities from the Clayton, t, and Gumbel copulas. Copula densities were calculated with Hofert et al. (2018b). Based on the work of Ye

(1987), a general non-linear augmented Lagrange multiplier technique solver was used. This was done in R using *"Rsolnp"* package.

e) $K = 1000$ scenarios of random variates were generated for the pseudo-uniformly distributed variables using the dependence structure provided by the calculated Copula Mixture.

f) For these Monte Carlo drawings, $z_{j,t}$, Skewed-t quantiles for $j = 1, \ldots, 19$ and $t = 1, \ldots, (L - T)$ were computed.

g) K scenarios of simulated daily log-returns for the out-of-sample following day, we are forecasting, were determined for each j asset,

$$r_{j,t} = X_j + \epsilon_{j,t}, \tag{3.13}$$

where X_j is provided by the AR(1) model,

$$X_{j,t} = \epsilon_{j,t} + \phi_{j,i}\, X_{j,t-i} \tag{3.14}$$

and $\epsilon_{j,t}$ is the error term following a GARCH(1,1) process given as

$$\begin{aligned} \epsilon_{j,t} &= \sigma_{j,t}\, z_{j,t} \\ \sigma^2_{j,t} &= \alpha_{j,0} + \alpha_{j,1}\, \epsilon^2_{j,t-1} + \beta_{j,1}\, \sigma^2_{j,t-1} \end{aligned} \tag{3.15}$$

h) Finnaly, portfolio weights were optimized by minimizing CVaR with a confidence level of 2.5%, as proposed by Ramos et al. (2023). The simulated data was used as input, following the works of Wuertz et al. (2010), in which the approach of Rockafellar and Uryasev (2002) had been used to optimize CVaR with a linear program. The optimization ran for target daily returns equal to the transaction costs of 0.0003 or 0.03%, to assess optimization performance.

For each period, same steps are taken to optimize the Gaussian Copula Portfolio. However, because estimating a mixture of copula functions is unnecessary, step (4) of the optimization only fits a Gaussian Multivariate Copula to given pseudo-uniform data using the Hofert et al. (2018b) method.

For each period in $(L - T)$, the respective estimated vector of asset weights and the data-set's log returns are used to construct out-of-sample portfolio returns for mixture of copulas portfolios and Gaussian copulas portfolios, as shown below.

$$R_t^{port} = \sum_{j=1}^{19} w_{j,t}\, r_{j,t}, \quad t = 1, ..., (L - T - 1) \tag{3.16}$$

For EWP, the weight of each asset is simply $1/N$.

Several performance measures were produced using calculated out-of-sample portfolio returns for the Mixture Copula Portfolio, Gaussian Copula Portfolio, and Equal Weighted Portfolio, as shown in Peterson and Carl (2019).

3.3 PERFORMANCE MEASURES

The performance of the investment portfolio is evaluated using various performance measures, as computed in Peterson and Carl (2019) and based on the methodology presented in Bacon (2008). The following performance measures are utilized to assess the portfolio's risk and return characteristics:

1. **Annualized Return**: The annualized return measures the average percentage gain or loss of the investment per year.
2. **Annualized Standard Deviation**: This metric quantifies the volatility or risk of the investment by measuring the dispersion of returns around the mean over a one-year period.
3. **Sharpe Ratio**: The Sharpe ratio evaluates the risk-adjusted return by comparing the portfolio's excess return over the risk-free rate to its standard deviation.
4. **Sortino Ratio**: The Sortino ratio is a modified version of the Sharpe ratio that considers only downside risk, which is calculated using the semi-deviation.
5. **Omega Ratio**: The Omega ratio assesses the probability-weighted ratio of gains to losses, providing a measure of upside potential relative to downside risk.
6. $VaR_{0.975}$ **(Value at Risk)**: The $VaR_{0.975}$ measures the maximum expected loss with a 97.5% confidence level over a specified time horizon.
7. $CVaR_{0.975}$ **(Conditional Value at Risk)**: Also known as Expected Shortfall, $CVaR_{0.975}$ quantifies the average loss beyond the $VaR_{0.975}$ level.
8. **Semi-Deviation**: The semi-deviation calculates the volatility of returns below the mean, focusing on downside risk.

9. **Worst-Drawdown**: The worst-drawdown measures the maximum percentage decline in the portfolio's value from a previous peak to the lowest subsequent point.

These performance measures provide a comprehensive assessment of the investment portfolio's risk and return characteristics, allowing for a better understanding of its performance in different market conditions.

The following performance measures are computed in Peterson and Carl (2019) and based on Bacon (2008): annualized return, annualized standard deviation, annualized Sharpe ratio, Sortino ratio, Omega ratio, $VaR_{0.975}$, $CVaR_{0.975}$, Semi-Deviation and Worst-Drawdown.

The annualized return is a measure used to determine the average rate of return per year over a specific investment period. It is commonly calculated as follows:

$$\text{An. Ret.} = \left(\prod_{i=1}^{252} (1 + R_i)^{\frac{1}{252}} \right) - 1 \tag{3.17}$$

Furthermore, the annualized Standard Deviation is the rescaled daily Standard Deviation and can be calculated as

$$\text{An. Std. Dev.} = \sigma \times \sqrt{252}. \tag{3.18}$$

The Sharpe ratio is a widely used metric in finance to assess the risk-adjusted performance of an investment or portfolio. It quantifies the excess return generated per unit of risk taken, Sharpe (1966). The formula for the Sharpe ratio is given by:

$$\text{SR} = \frac{\mathbb{E}[R] - R_f}{\sigma}. \tag{3.19}$$

The Sortino ratio is a performance metric extensively used in finance to estimate an investment's or portfolio's risk-adjusted return. It is based on the Sharpe ratio concept, but it focuses entirely on downside risk, which is typically evaluated by the standard deviation of negative returns. The Sortino ratio is determined by dividing the investment or portfolio's excess return over a defined target return by the downside risk. It tries to provide a more relevant measure of risk-adjusted performance, especially in instances when investors are concerned about downside volatility, see Sortino and Price (1994).

$$\text{Sortino} = \frac{\mathbb{E}[R] - R_f}{\sigma^-} \tag{3.20}$$

where σ_- is the standard deviation of negative asset returns.

The Omega ratio, on the other hand, is a performance indicator meant to assess the asymmetry of returns and the chance of achieving a specific goal return. It evaluates the full return distribution and assesses the likelihood of returns exceeding a specific target or threshold level, see Keating and Shadwick (2002). The Omega ratio is the probability-weighted average of positive returns divided by the probability-weighted average of negative returns. Beyond typical risk metrics, it provides useful insights into the possible upside and downside of an investment or portfolio.

$$\Omega(\theta) = \frac{\int_{\theta}^{\infty}[1 - F(R)]\, dr}{\int_{-\infty}^{\theta} F(R)\, dr} \tag{3.21}$$

where F is the return's cumulative probability distribution function and θ is the desired return threshold determining what is regarded a gain against a loss. A higher ratio shows that the asset generates more returns relative to losses for certain threshold θ, and hence is favoured by investors.

Var and CVaR have already been defined in previous chapters.

The semi-deviation is calculated as follows:

$$\text{Semi-Deviation} = \sqrt{\frac{\sum_{R_i<\bar{R}}(R_i - \bar{R})^2}{N}},$$

where r_i represents individual returns, $\bar{R}$ is the mean return, and N is the number of observations. The semi-deviation provides valuable insights into the asset's performance during periods of negative returns. Check Bodie, Kane and Marcus (2014)

The worst-drawdown is a measure that quantifies the maximum percentage decline in a portfolio's value from a previous peak to the lowest subsequent point. It helps investors understand the largest loss they might have experienced during a specific investment period. The worst-drawdown can be calculated using the following formula:

$$\text{Worst-Drawdown} = \text{Maximum Drawdown} = \max_{i,j}\left(\frac{V_i - V_j}{V_i}\right) \times 100\%,$$

where V_i is the portfolio's value at time i, V_j is the lowest subsequent value after the peak at time j, and $\max_{i,j}$ represents the maximum value over all peak-to-trough periods. Check Bodie, Kane and Marcus (2014) for more information.

4 DATA AND CASE STUDY

The empirical research on the Worst Case Mixture-Copula Mean-CVaR portfolio, as previously described, utilizes sample data from 19 ETFs available on *Free Historical Market Data - Stooq*. These ETFs were selected based on the inclusion of G20 countries, with the exception of Russia due to the absence of available data. EWA, EXS1, and EXSA may encounter issues related to non-overlapping trading hours due to time zone differences.

The daily log-returns of the selected ETFs were calculated based on closing prices. The study period, spanning from January 2, 2013, to June 30, 2023, was chosen to provide a substantial dataset for robust analysis and to encompass various market conditions. The resulting dataset consists of $L = 2692$ observations, allowing for a comprehensive assessment of portfolio performance and risk measures. Table 4.1 provides a comprehensive list of the selected ETFs, including their respective countries, full names, and exchanges.

ETF	Country	Full Name	Exchange
ARGT	Argentina	Global X MSCI Argentina ETF	NYSE Arca
EWA	Australia	iShares MSCI Australia ETF	ASX
EWZ	Brazil	iShares MSCI Brazil ETF	NYSE Arca
EWC	Canada	iShares MSCI Canada ETF	NYSE Arca
FXI	China	iShares China Large-Cap ETF	NYSE Arca
EWQ	France	iShares MSCI France ETF	NYSE Arca
EXS1	Germany	iShares DAX UCITS ETF	Xetra
INDA	India	iShares MSCI India ETF	NYSE Arca
EIDO	Indonesia	iShares MSCI Indonesia ETF	NYSE Arca
EWI	Italy	iShares MSCI Italy ETF	NYSE Arca
EWJ	Japan	iShares MSCI Japan ETF	NYSE Arca
EWW	Mexico	iShares MSCI Mexico ETF	NYSE Arca
KSA	Saudi Arabia	iShares MSCI Saudi Arabia ETF	NYSE Arca
EZA	South Africa	iShares MSCI South Africa ETF	NYSE Arca
EWY	South Korea	iShares MSCI Korea ETF	NYSE Arca
TUR	Turkey	iShares MSCI Turkey ETF	NYSE Arca
EWU	United Kingdom	iShares MSCI United Kingdom ETF	NYSE Arca
SPY	United States	SPDR S&P 500 ETF Trust	NYSE Arca
EXSA	European Union	iShares STOXX Europe 600 UCITS ETF	Euronext Amsterdam

Table 4.1: ETFs, Countries, Full Names, and Exchanges

4.1 DESCRIPTIVE STATISTICS

From the collected adjusted prices for each ETF, we calculated the logarithmic returns, as depicted in Figure 4.1. Additionally, a summary of statistics for the entire sample period is presented in Table 4.2. These statistics provide valuable insights into the historical distribution of returns for the selected data.

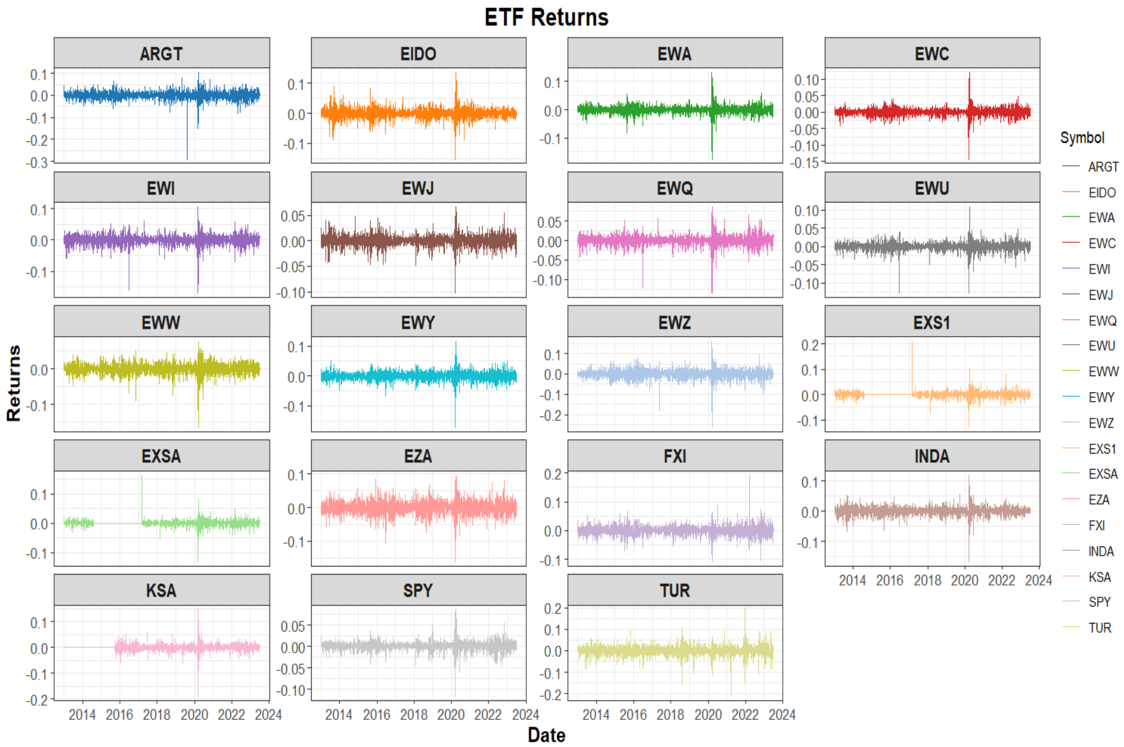

Figure 4.1: Log Returns

Source: Author's own work

Two statistics that deserve special attention in the table are skewness and kurtosis. Notably, negative skewness indicates a return distribution with a longer left tail, suggesting the presence of extremely negative returns. Conversely, a very high kurtosis indicates the existence of heavier tails than the normal distribution, implying that returns have a greater likelihood of reaching extreme values. These characteristics align with the stylized facts of financial asset returns presented by Taylor (2011).

The insights above are crucial for comprehending the risks and potential rewards of investing in financial assets and prompt the question of how these risks are distributed over time. This question finds its answer in the presence of volatility clusters—periods characterized by heightened asset price volatility. During such episodes, diverse assets tend to move in tandem, giving rise to distinctive market behavior patterns. The non-normal distribution of returns, exemplified by skewness and kurtosis, underscores the ne-

ETF	Min.	1st Q.	Median	Mean	3rd Q.	Max.	Skewness	Kurtosis
ARGT	-0.29	-0.0086	0.0006	0.0004	0.0101	0.10	-1.9764	27.6424
EIDO	-0.15	-0.0079	0.0000	-0.0000	0.0082	0.14	-0.5315	9.4203
EWA	-0.18	-0.0064	0.0003	0.0001	0.0071	0.13	-1.1350	22.8411
EWC	-0.14	-0.0052	0.0006	0.0002	0.0058	0.12	-1.1270	24.3630
EWI	-0.17	-0.0069	0.0007	0.0002	0.0083	0.11	-1.4798	15.4850
EWJ	-0.10	-0.0050	0.0004	0.0002	0.0060	0.07	-0.5206	6.3646
EWQ	-0.14	-0.0055	0.0006	0.0003	0.0068	0.09	-1.2070	15.0895
EWU	-0.13	-0.0051	0.0005	0.0001	0.0060	0.11	-1.3594	18.4232
EWW	-0.17	-0.0077	0.0000	0.0000	0.0086	0.08	-1.0387	9.7245
EWY	-0.17	-0.0078	0.0000	0.0001	0.0086	0.12	-0.7468	11.4362
EWZ	-0.26	-0.0116	0.0001	-0.0001	0.0123	0.16	-1.0798	13.1711
EXS1	-0.13	-0.0026	0.0000	0.0003	0.0039	0.21	1.8920	63.3013
EXSA	-0.13	-0.0019	0.0000	0.0003	0.0034	0.16	0.5871	56.3795
EZA	-0.16	-0.0109	0.0000	-0.0001	0.0112	0.10	-0.5740	5.4369
FXI	-0.11	-0.0088	0.0000	-0.0001	0.0089	0.19	0.4166	9.6157
INDA	-0.17	-0.0067	0.0003	0.0002	0.0080	0.12	-1.0409	16.2523
KSA	-0.19	-0.0027	0.0000	0.0002	0.0037	0.15	-0.8038	41.9807
SPY	-0.12	-0.0036	0.0005	0.0005	0.0055	0.09	-0.8170	14.4863
TUR	-0.21	-0.0108	0.0000	-0.0002	0.0120	0.19	-0.5261	8.8241

Table 4.2: Summary Statistics of ETF Returns
Source: Author's own work

cessity of employing risk models and investment strategies that account for these features. For instance, a risk model assuming a normal return distribution may underestimate the risks associated with assets susceptible to volatility clusters, potentially leading to suboptimal investment decisions.

4.2 EMPIRICAL RESULTS

The empirical analysis compared the performance of Mixture Copula for Mean-CVaR portfolios (MCP) against two benchmarks: an Equal Weight Portfolio (EWP) and a Gaussian Copulas Mean-CVaR Portfolios (GCP), considering sample windows of 252, 504, and 1260 days. The results are presented in Table 4.3, indicating that the Mixture Copula portfolios consistently outperformed the benchmarks in terms of return and risk. The metrics include annualized return, annualized standard deviation, Sharpe ratio, Sortino ratio, Omega Sharpe ratio, VaR at the 97.5% level, CVaR at the 97.5% level, semi-deviation, and worst drawdown.

The analysis of the results reveals that portfolios constructed from simulated returns using copula mixture exhibited superior performance when assessed by risk-return

Metrics	MCP 1Y	MCP 2Y	MCP 5Y	GCP 1Y	GCP 2Y	GCP 5Y	EWP
Annualized Return	0.0494	0.0790	**0.1206**	0.0490	0.0692	0.0648	0.0350
Annualized Std. Dev.	**0.1498**	0.1578	0.1730	0.1516	0.1630	0.1710	0.1798
Sharpe Ratio	0.3300	0.5005	**0.6974**	0.3232	0.4245	0.3791	0.1947
Sortino Ratio	0.0337	0.0484	**0.0653**	0.0329	0.0422	0.0383	0.0239
Omega Sharpe Ratio	0.0774	0.1112	**0.1506**	0.0755	0.0993	0.0896	0.0539
VaR (97.5%)	**-0.0195**	-0.0198	-0.0214	-0.0202	-0.0209	-0.0207	-0.0223
CVaR (97.5%)	**-0.0294**	-0.0307	-0.0338	-0.0308	-0.0331	-0.0349	-0.0348
Semi-Deviation	**0.0071**	0.0074	0.0080	0.0073	0.0077	0.0082	0.0085
Worst Drawdown	0.3268	0.3215	**0.3057**	0.3253	0.3372	0.3146	0.4380

Table 4.3: Performance Metrics
Source: Author's own work

metrics such as the Sharpe ratio, Sortino ratio, and Omega Sharpe ratio. These portfolios displayed lower levels of risk and experienced fewer financial losses during the evaluation period, while still managing to achieve comparable or even superior returns compared to other investment strategies.

Another crucial aspect to highlight is the substantial impact of transaction costs on portfolio performance. The costs associated with executing the strategies were considered in the backtest, and the results indicate that effective management of these costs played a crucial role in the overall outcome. This underscores the importance of a strategic approach to handling transaction costs when developing and implementing copula mixture-based investment strategies. In summary, the results suggest that the copula mixture approach can be a valuable tool for investors seeking to optimize the trade-off between risk and return in their portfolios, especially when accompanied by careful consideration of associated costs.

The following charts illustrate the performance of the portfolios under analysis, with each chart addressing specific aspects. The top section of the charts displays the cumulative returns over time, providing an overview of the portfolio's growth or decline. In the middle section, daily returns are shown, allowing for a more detailed analysis of the portfolios' daily fluctuations. Finally, in the bottom part, we find the drawdown chart, which highlights periods when the portfolio experienced declines compared to its previous values.

For the purpose of comparison, the charts are divided into three distinct figures, each corresponding to a different time window: the first figure covers a 1-year window, the second considers a 2-year period, and the third represents a 5-year time horizon. This segmentation enables a comparative analysis of return trajectories and drawdown patterns over these different time intervals, offering a comprehensive view of portfolio performance under various market conditions.

Each chart features three distinct lines: the black line represents the copula mixture portfolio that achieved the best results, the red line corresponds to the Gaussian copulas portfolio, and the green line represents the equally weighted portfolio. This differentiation allows for an immediate visual analysis of performance differences among these strategies over time.

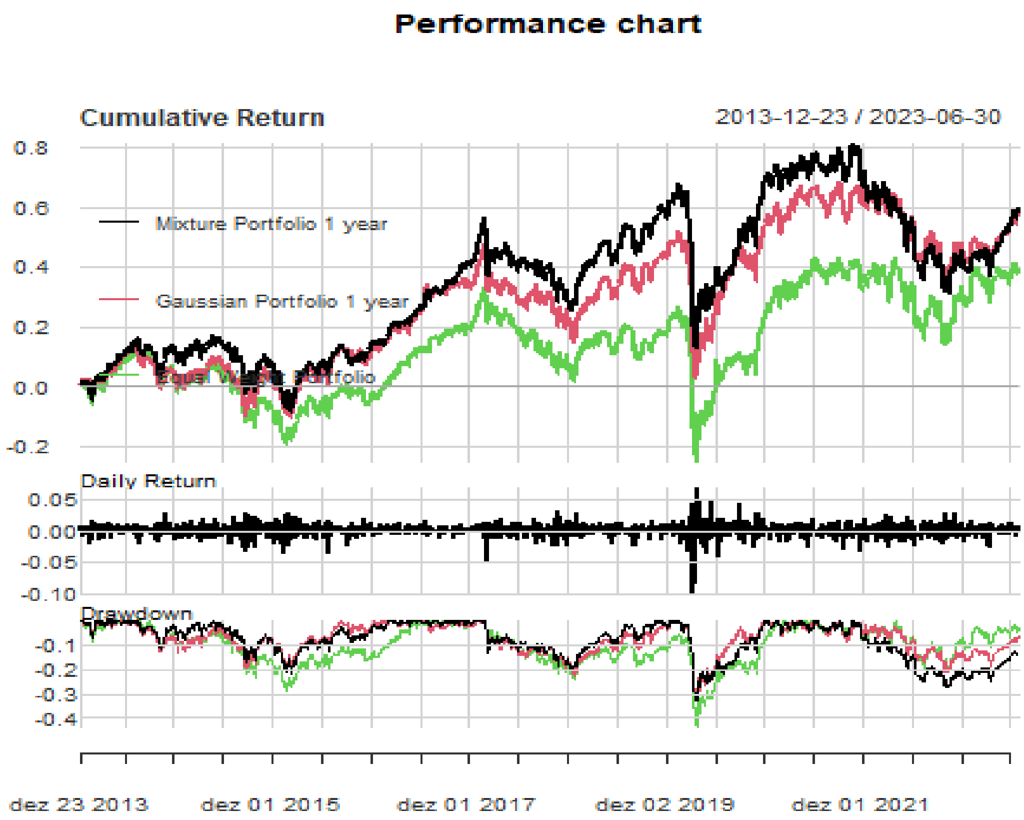

Figure 4.2: Portfolio construction with 1 year sample

Source: Author's own work

Observing Figure 4.2, it becomes evident that the copula mixture portfolio achieved the lowest levels of risk, as indicated by smaller drawdowns in the chart, especially during periods of heightened market volatility. While the red line representing the Gaussian copula portfolio exhibited a final result quite close, it displayed greater volatility and larger drawdowns compared to the copula mixture portfolio. In contrast, the green line representing the equally weighted portfolio showcased an intermediate performance.

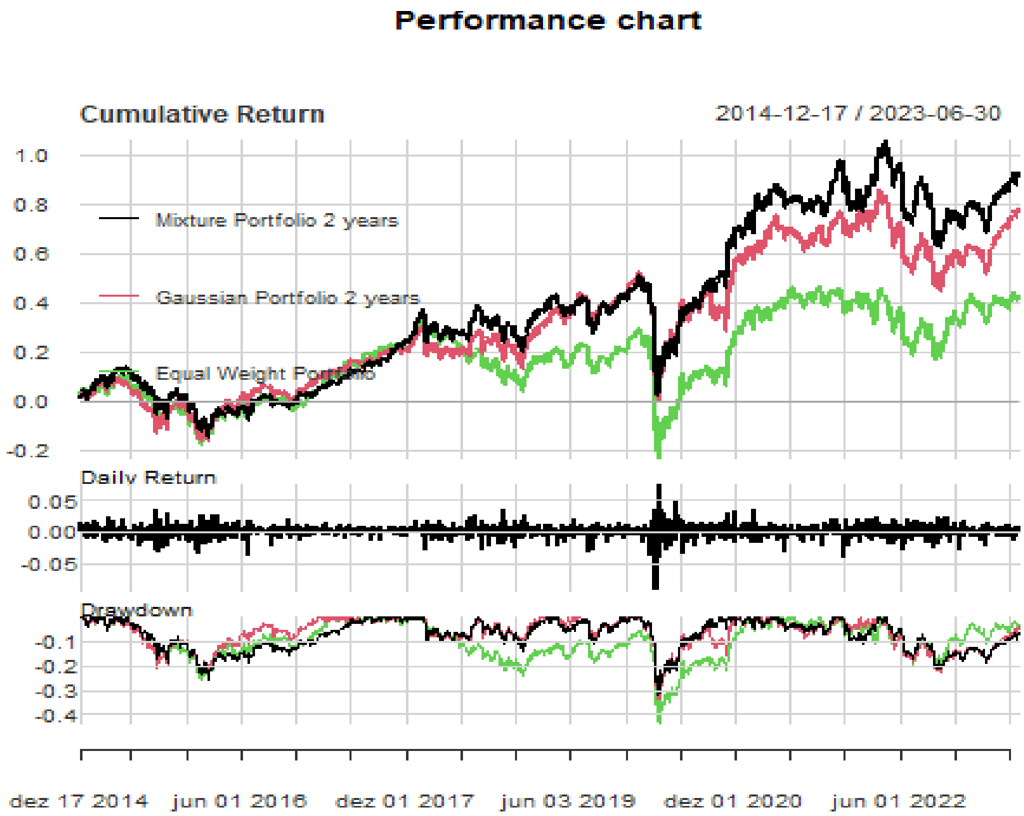

Figure 4.3: Portfolio construction with 2 year sample

Source: Author's own work

In Figure 4.3, we observe a more substantial divergence in outcomes, underscoring the increased effectiveness of copula mixture estimates in capturing market trends and patterns as more information is incorporated. The black line, denoting the copula mixture portfolio, continues to demonstrate the lowest risk levels, characterized by shallower drawdowns, particularly during episodes of elevated market volatility. Consequently, the disparities in portfolio returns become more pronounced, as the copula mixture portfolio appears to outperform both the Gaussian copula and equally weighted portfolios by a more substantial margin in the 2-year sample window.

The disparities become even more pronounced when the portfolio construction window expands to 5 years. As depicted in Figure 4.4, during periods of heightened market volatility, copula mixtures excel in capturing extreme values that contribute to loss mitigation, particularly evident during the COVID-19 crisis. The chart demonstrates that the copula mixture portfolios managed to navigate the turbulent market conditions more effectively, leading to shallower drawdowns and demonstrating resilience during extreme events.

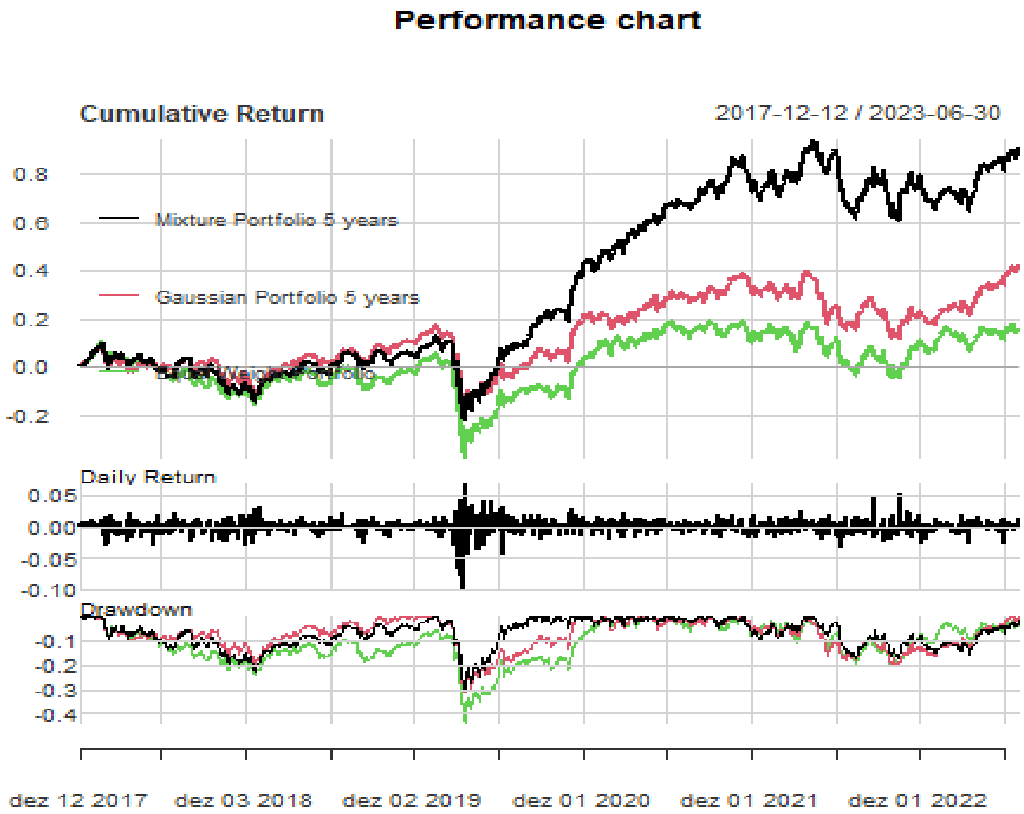

Figure 4.4: Portfolio construction with 5 year sample

Source: Author's own work

Understanding the ramifications of the pandemic-induced drawdown on downside risk and overall risk-adjusted performance is crucial, with a particular focus on the Sortino Ratio and Omega Sharpe Ratio. These metrics shed light on how effectively the portfolios managed losses during the pandemic. The MCP and GCP portfolios endured significant drawdowns, with the 5-year MCP witnessing a worst drawdown of 30.57%, while the 5-year GCP experienced a worst drawdown of 31.46%. This amplified downside volatility had a noticeable impact on the Sortino Ratio and Omega Sharpe Ratio, resulting in diminished values relative to the pre-pandemic period.

The notable superiority of the 5-year portfolio in contrast to the 1-year and 2-year portfolios can be ascribed to the influence of the estimation window's size (sample) on portfolio performance. An extended investment horizon enables the portfolio to capitalize on a more extended period of market observations, thereby mitigating the impact of short-term fluctuations and data noise. This protracted time frame provides a more comprehensive perspective on market trends and economic cycles, enabling investors to discern more substantial underlying patterns and make well-informed decisions.

In conclusion, the empirical analysis strongly suggests that the worst-case Mixture of Copulas Mean-CVaR portfolio is a superior choice for investors seeking improved risk-adjusted performance over different time horizons, especially for longer investment periods. Its ability to capture tail dependence appears to be a significant factor in achieving these favorable outcomes.

5 CONCLUDING REMARKS

This study accomplishes Mean-CVaR portfolio optimization by integrating a copula-based dependence framework and dynamic adjustments for mean and volatility in asset returns through an AR-GARCH model. Our assessment of the designated portfolio's performance draws on data spanning 19 country-specific ETF indexes from 2013 to 2023, comparing it against three benchmarks: a Gaussian Copula Mean-CVaR portfolio and an equally weighted ($1/N$) portfolio.

Our findings unveil that optimizing the Mean-CVaR portfolio while modeling asset dependencies through a blend of Clayton, t, and Gumbel copulas yields a portfolio exhibiting superior downside-risk and drawdown metrics when contrasted with the Gaussian and $1/N$ portfolios, especially concerning the target returns for Mean-CVaR optimization. Moreover, the findings suggest that using longer time windows for estimations played a crucial role in result accuracy. Longer time windows provided a significantly larger amount of data, contributing to a clearer identification of patterns and trends in financial markets.

The first priority for future work is to refine and enhance the data preprocessing stage. This involves organizing the dataset and ensuring that it includes a more extensive historical period and a broader range of financial assets. Improving data quality and expanding the scope of data sources will not only lead to more trustworthy results but also help mitigate survivorship bias, a critical consideration in portfolio optimization.

Secondly, incorporating constraints and conducting comparative analyses with existing studies is a vital avenue for future research. One particular constraint that I intended to include but was time-consuming to implement in this study is the cardinality constraint. This constraint restricts the number of assets in the portfolio with the aim of mitigating over-diversification. For instance, consider a universe of 100 assets; applying a cardinality constraint can help identify an optimal subset of assets for portfolio inclusion. A study that has explored the impact of this constraint on portfolio optimization is the work by Ramos et al. (2023). However, their analysis primarily relied on historical simulations and did not focus extensively on precise return estimations.

Third, the feasibility of incorporating into the same framework an asset selection method that combines copulas and machine learning, as proposed by Luca, Rivieccio and Zuccolotto (2010), could be examined. This approach offers the opportunity to enhance portfolio diversification by leveraging advanced computational approaches and statistical models. By integrating copulas and machine learning, the framework can benefit from more sophisticated asset selection strategies, ensuring that the portfolio includes assets with ideal risk-return profiles. This integration may result in improved risk management and portfolio performance, making it a promising avenue for future research.

The execution of robustness tests and sub-period analysis is crucial. This aims to ensure the reliability of the results and provide insights into the performance observed by comparing performance before and after the COVID-19 pandemic. A widely recognized test that could be used is the Diebold and Mariano test, which is valuable for assessing whether the observed changes in results are statistically significant and represent a real improvement compared to previous performance. This detailed analysis of data over time would contribute to a more robust understanding of the method and its effects in the context of different market scenarios. Data Snooping tests such as Hansen (2005) should be used to assess predictive ability.

Other suggestions include investigating the impact of daily rebalancing on performance, as it was influenced by transaction costs. Additionally, it would be interesting to compare different mixture-copulas, like pairs, triples or quartets. Also, to evaluate return objectives, assessing how the method performs under various performance targets. Furthermore, incorporating dynamic copulas or vine copula models, as proposed in works such as Ausin and Lopes (2010), Xi (2014), could enhance the modeling of dependence and potentially improve the accuracy of portfolio estimates. These are promising areas for future research that can further enhance the method and its application in different financial contexts.

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APPENDIX - GITHUB REPOSITORY

For the sake of transparency and reproducibility, you can find the code used for the calculations presented in this report on the GitHub repository:

<https://github.com/JoaoJungblut/Mean-CVaR_Portfolio_with_Mixture-Copulas>

The repository contains all the necessary scripts and files required to perform the risk optimization calculations discussed in this document. Additionally, it includes any extra code used for data preprocessing, analysis, and visualization. By accessing the GitHub repository, readers can review, validate, and independently execute the code to verify the results or adapt it to their specific requirements.

Please be aware that the GitHub repository may receive updates or improvements over time. If you encounter any issues or have questions regarding the code, please feel free to create an issue on the repository or contact the repository owner.