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Simple Modules of the Quantum Double of a Nichols Algebra of Unidentified Diagonal Type

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Resumo

Nesta tese está a classificação de todas as 47 possíveis representações irredutíveis do duplo quântico de uma álgebra de Hopf associada a álgebra de Nichols da álgebra de tipo não identificado de menor dimensão (144).

Abstract

In this thesis we classify all 47 possible irreducible representations of the quantum double of a pointed Hopf algebras attached to the Nichols algebra of the unidentified algebra of smallest dimension (144).

Contents

Chapter 1

Introduction

Finite-dimensional Nichols algebras were classified by Heckenberger in [H2] and we can consider three families: standard braidings, (that were introduced in [AA]); braidings of super type, ([AAY]) and a finite list of braidings whose connected components have rank less than eight that are called unidentified. A Nichols algebra of a braided vector space (V, c) is a quotient of its tensor algebra by a suitable ideal I(V) then a important question about Nichols algebras is to obtain a minimal set of relations generating I(V). For the first family this is in [A4] and for the second family the problem is solved for the generic case in [Y], and for the non-generic case, except by some considerations for small orders on the entries of the braiding matrix, in [AAY]. A complete list of relations satisfied by the generators of the Nichols algebras, depending on the matrix entries can be found in [A4]. Angiono in [A3] gave a complete list of relations generating the defining ideal for the Nichols algebra of each braiding of this kind and also the list of positive roots for each case and the dimension for the small ranks.

In this thesis we compute and describe all the irreducible representations (and their dimensions) of a finite-dimensional pointed Hopf algebra, which is the Drinfeld double of a Nichols algebra of unidentified type of smallest dimension (144). There are 47 different cases according with the sets of factors of the Shapovalov determinant who are annhilated. For that purpose, we compute the lattices of submodules of the Verma modules. The parametrization of the simple modules of the mentioned Hopf algebras is deduced from a result of Radford and Schneider ([RS]) which generalizes the method employed in the representation theory of finite-dimensional semisimple Lie algebras and comes from the consideration of the generalized version of the Shapovalov determinant, introduced by Heckenberger and Yamane for these Drinfeld doubles of Nichols algebras ([HY]). This determinant has a factorization, and the Verma module is irreducible if no one of these factors is zero. This factorization also helps to describe the other 46 cases, when either one or two of the factors are 0, generating on most cases explicit relations on the module. We describe explicitly the submodules on each case. Cases 2-10 has one of the factors equal to 0 and the other cases have exactly two factors equal to 0, and we compute the results of the relations obtaining the basis to the module, but this is not always simple and easy. Therefore, we also related the Cases 11-47, with each other in two possible ways, using a morphism between submodules as in Lemma 5.2.5 and this provides relations between the diagrams of the module that we exemplify in Appendix B.

Chapter 2

Preliminaries

2.1 Notation

The base field $\mathbf{k}$ is algebraically closed of characteristic zero; we set $\mathbf{k}^\times = \mathbf{k} - 0$. For each integer $N > 1$, $\mathbb{G}_N$ denotes the group of N-roots of unity in $\mathbf{k}$, and $\mathbb{G}'_N$ is the corresponding subset of primitive roots of order N. If G is a group, then we denote by $\widehat{G}$ the group of multiplicative characters (i. e., one-dimensional representations) of G; and by $Z(G)$ the center of G.

We shall use the notation for q-factorial numbers: for each $q \in \mathbf{k}^\times$, $n \in \mathbb{N}$, $0 \le k \le n$,

$$(n)_q = 1 + q + \ldots + q^{n-1}, \qquad (n)_q! = (1)_q(2)_q \cdots (n)_q.$$

A *braided monoidal category* is a collection $(\mathcal{C}, \otimes, \mathbf{1}, a, r, l, c)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor called tensor product; $\mathbf{1}$ is an object in $\mathcal{C}$; $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $r_X : X \to X \otimes \mathbf{1}$, $l_X : X \to \mathbf{1} \otimes X$, $c_{X,Y} : X \otimes Y \to Y \otimes X$ are natural families of isomorphisms that satisfies some suitable coherence diagrams [M, page 252]. In particular, for $X \in \mathcal{C}$, the map $c = c_{X,X}$ is called a *braiding* and satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c). \tag{2.1}$$

2.2 Hopf algebras

Definition 2.2.1. Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. Then H is a *Hopf algebra* if there exists an element $\mathcal{S} \in Hom_k(H, H)$ which is an inverse to id_H under convolution. $\mathcal{S}$ is called an *antipode* for H.

Note that $\mathcal{S}$ satisfies

$$\sum (\mathcal{S}h_{(1)})h_{(2)} = \varepsilon(h)(1)_H = \sum h_{(1)}(\mathcal{S}h_{(2)})$$

We use the Heyneman-Sweedler notation $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$; the summation sign will be often omitted. The composition inverse of $\mathcal{S}$ is denoted by $\overline{\mathcal{S}}$.

Let H be a Hopf algebra. There are a left and a right action of H^* on H given by

$$f \rightharpoonup h = h_{(1)} f(h_{(2)}), \qquad h \leftharpoonup f = f(h_{(1)}) h_{(2)}, \qquad h \in H, f \in H^*. \tag{2.2}$$

We denote by $G(H)$ the set of grouplike elements of H. The tensor category of finite-dimensional representations of H is denoted $\operatorname{Rep} H$.

A *left integral in* H is an element $\Upsilon \in H$ such that $h\Upsilon = \varepsilon(h)\Upsilon$ for all $h \in H$; a *right integral in* H is an element $\Lambda \in H$ such that $\Lambda h = \varepsilon(h)\Lambda$ for all $h \in H$. The space of left, respectively right, integrals is denoted $I_l(H)$, respectively $I_r(H)$. Assume that H is finite-dimensional. Then $\dim I_l(H) = 1 = \dim I_r(H)$. The *distinguished grouplike* elements of H and H^* are the (unique) $\alpha_H \in G(H^*)$, $g_H \in G(H)$ such that

$$\Upsilon a = \alpha_H(a)\Upsilon, \qquad p\upsilon = p(g_H)\upsilon, \qquad \text{for all } a \in H, p \in H^*, \tag{2.3}$$

where Υ, respectively υ, is an arbitrary non-zero left integral in H, respectively non-zero right integral in H^*.

Example 2.2.2. Let G be a finite group and $\mathbf{k}G$ its group algebra, with basis $(h)_{h\in G}$. Then $(\mathbf{k}G)^* \simeq \mathbf{k}^G$, with basis $(\delta_h)_{h\in G}$, δ_h being the function that is 1 in h, 0 elsewhere. Then

$$⨍_G = \sum_{h\in G} h \in I_l(\mathbf{k}G) = I_r(\mathbf{k}G), \qquad \delta_e \in I_l(\mathbf{k}^G) = I_r(\mathbf{k}^G). \tag{2.4}$$

Hence the distinguished grouplike elements are trivial. Alternatively, if $G = \Gamma$ is abelian, then $\delta_e = |\Gamma|^{-1} \sum_{\chi\in\widehat{\Gamma}} \chi$.

2.3 Yetter-Drinfeld modules

Definition 2.3.1. Let G be a finite group. A *Yetter-Drinfeld module over* $\mathbf{k}G$ is a G-graded vector space $M = \bigoplus_{t\in G} M_t$ provided with a linear action of G such that $t \cdot M_h = M_{tht^{-1}}$ for any $t, h \in G$; morphisms of Yetter-Drinfeld modules are linear maps preserving the action and the grading.

The category ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ of Yetter-Drinfeld modules over G is semisimple. Moreover, let $M \in {}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$, $t \in G$ and $v \in M_t$. If there exists $\chi \in \widehat{G}$ such that $h \cdot v = \chi(h)v$, for all $h \in G$, then we say that $v \in M_t^\chi$; necessarily, $t \in Z(G)$. Furthermore, if $G = \Gamma$ is abelian, then any $M \in {}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ satisfies $M = \bigoplus_{t\in\Gamma,\chi\in\widehat{\Gamma}} M_t^\chi$.

The category ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ is a braided tensor category with the usual tensor product of gradings and actions, and where $c_{X,Y} : X \otimes Y \to Y \otimes X$ is given by

$$c(x \otimes y) = t \cdot y \otimes x, \qquad x \in X_t,\ t \in G,\ y \in Y. \tag{2.5}$$

2.4 Braided Hopf algebras

Definition 2.4.1. Let G be a finite group. A *braided Hopf algebra* in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ is a collection $(R, \cdot, \Delta)$, where

- $R \in {}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$;
- $(R, \cdot)$ is an algebra such that $\cdot$ and the unit are morphisms in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$;
- (R, Δ) is a coalgebra such that Δ and the counit ε are morphisms in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$;
- Δ is an algebra map in the sense $\Delta \circ m = (m \otimes m)(\mathrm{id} \otimes c_{R,R} \otimes \mathrm{id})(\Delta \otimes \Delta)$;
- R has an antipode $\mathcal{S}_R$, i. e. a convolution inverse of the identity of R.

Let A, H be Hopf Algebras and $\pi : A \to H$ and $\iota : H \to A$ Hopf algebra homomorphisms. Assume that $\pi\iota = id_H$, so that π is surjective, and ι is injective. Then

$$R := A^{co\pi} = \{a \in A : (id \otimes \pi)\Delta(a) = a \otimes 1\}$$

is a braided Hopf algebra in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ with the following structure:

- The action $\cdot$ of H on R is the restriction of the adjoint action composed with ι.
- The coaction is $(\pi \otimes id)\Delta$.
- R is a subalgebra of A.
- The comultiplication is $\Delta_R(r) = r_{(1)}\iota\pi\mathcal{S}(r_{(2)}) \otimes r_{(3)}$, for all $r \in R$.

Definition 2.4.2. Let R be braided Hopf algebra in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$. The vector space $R \otimes \mathbf{k}G$ whose multiplication and comultiplication are given by

$$\begin{aligned}(r\#h)(s\#f) &= r(h_{(1)} \cdot s)\#h_{(2)}f,\\ \Delta(r\#h) &= r^{(1)}\#(r^{(2)})_{(-1)}h_{(1)} \otimes (r^{(2)})_{(0)}\#h_{(2)},\end{aligned}$$

is a Hopf algebra, called the *bosonization, or bicrossproduct, of* R *by* $\mathbf{k}G$, and denoted $R\#\mathbf{k}G$,

Let R be a finite-dimensional braided Hopf algebra in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$. A *left integral in* R is a $\Upsilon \in R$ such that $r\Upsilon = \varepsilon(r)\Upsilon$ for all $r \in R$. Right integrals are defined similarly. The space of left integrals, respectively right, is denoted $I_l(R)$, respectively $I_r(R)$. Then $I_l(R) \in {}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ and $I_r(R^*) \in {}^{\mathbf{k}^G}_{\mathbf{k}^G}\mathcal{YD}$ have dimension 1; hence

$$I_l(R) = I_l(R)^{\gamma}_{z}, \qquad \text{for some } z \in Z(G),\ \gamma \in \widehat{G}; \tag{2.6}$$

$$I_r(R^*) = I_r(R^*)^{\ell}_{\mu}, \qquad \text{for some } \ell \in G,\ \mu \in \widehat{G}. \tag{2.7}$$

See [T] for more details.

Let $\Upsilon \in I_l(R)$, $\upsilon \in I_r(R^*)$. The relation with the integrals of the bosonization is given by the following result. Recall $\int_G$, δ_e from (2.4).

Lemma 2.4.3. *[R1, Bu] (i)* $\Upsilon\#(\int_G \leftharpoonup \gamma) = \int_G \Upsilon \in I_l(R\#\mathbf{k}G)$.
(ii) $\upsilon\#\delta_e \in I_r((R\#\mathbf{k}G)^*)$. □

The distinguished grouplike elements of R and R^* are the (unique) $\alpha_R \in G(R^*)$, $g_R \in G(R)$ such that $\Upsilon r = \alpha_R(r)\Upsilon$, $p\upsilon = p(g_R)\upsilon$, for all $r \in R$, $p \in R^*$. We give next the distinguished grouplike elements of the bosonization $R\#\mathbf{k}G$.

Theorem 2.4.4. *[Bu, 4.8, 4.10] Let R be a finite-dimensional braided Hopf algebra in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$, the distinguished grouplike elements of $H = R\#\mathbf{k}G$ are $\alpha_H = \alpha_R\#\gamma^{-1}$ and $g_H = g_R\ell$.* □

Remark 2.4.5. [AG] Let $R = \sum_{i=0}^{N} R_i$ be a finite-dimensional graded braided Hopf algebra in ${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$, with $R_0 = \mathbf{k}$ and $R_N \neq 0$. Then $R_N = I_l(R) = I_r(R)$. Thus R, and similarly R^*, are unimodular. Hence $\alpha_H = \gamma^{-1}$ and $g_H = \ell$, by Theorem 2.4.4.

2.5 Braidings of diagonal type

Let $\theta \in \mathbb{N}$ and $\mathbb{I} = \{1, 2, \dots, \theta\}$.
Let $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}} \in \mathbf{k}^{\mathbb{I}\times\mathbb{I}}$ such that

$$q_{ij} \text{ are roots of 1 for all } i, j \in \mathbb{I}, \tag{2.8}$$
$$q_{ii} \neq 1 \text{ for all } i \in \mathbb{I}. \tag{2.9}$$

Let $\widetilde{q_{ij}} := q_{ij}q_{ji}$. The *generalized Dynkin diagram* of the matrix $\mathbf{q}$ is a graph with θ vertices, the vertex i labeled with q_{ii}, and an arrow between the vertices i and j only if $\widetilde{q_{ij}} \neq 1$, labeled with $\widetilde{q_{ij}}$. For instance, given $\zeta \in \mathbb{G}'_{12}$ and η a square root of ζ, the matrices $\begin{pmatrix} \zeta^4 & 1 \\ \zeta^{11} & -1 \end{pmatrix}$, $\begin{pmatrix} \zeta^4 & \eta^{11} \\ \eta^{11} & -1 \end{pmatrix}$ have the diagram:

$$\overset{\zeta^4}{\circ} \overset{\zeta^{11}}{\text{——}} \overset{-1}{\bullet}\ . \tag{2.10}$$

where we indicate the vertices $1, 2$ by $\circ$, $\bullet$, respectively.
Let V be a vector space with a basis $X = \{x_i : i \in \mathbb{I}\}$. Define $c : V\otimes V \to V\otimes V$ by $c(x_i \otimes x_j) = q_{ij}\, x_j \otimes x_i$, $i, j \in \mathbb{I}$. Then c is a solution of the braid equation (2.1). The pair (V, c) is called a *braided vector space of diagonal type*; such braided vector spaces are related with Yetter-Drinfeld modules over group algebras of finite abelian groups.

2.6 Nichols algebras

Definition 2.6.1. Let (V, c) be a braided vector space of diagonal type attached to a matrix $\mathbf{q}$ as in the previous subsection. A braided graded Hopf algebra $R =$

$\bigotimes_{n\geq 0} R(n) \in {}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$ is called *a Nichols algebra of* V and denoted by $\mathcal{B}(V)$ if $\mathbf{k} \simeq R(0)$ and $V \simeq R(1) \in {}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$, $P(R) = R(1)$ and R is generated as an algebra by $R(1)$. See [AS3] for various precise alternative definitions and its rol in the classification of pointed Hopf algebras.

Let $(\alpha_i)_{i\in\mathbb{I}}$ be the canonical basis of $\mathbb{Z}^{\mathbb{I}}$. Since c is of diagonal type, $T(V)$ admits a unique $\mathbb{Z}^{\mathbb{I}}$-graduation such that $\deg x_i = \alpha_i$; then $I(V)$ is a $\mathbb{Z}^{\mathbb{I}}$-homogeneous ideal and $\mathcal{B}(V)$ is $\mathbb{Z}^{\mathbb{I}}$-graded, see [AS3, Proposition 2.10], [L, Proposition 1.2.3].

Remark 2.6.2. Two braided vector spaces of diagonal type with the same generalized Dynkin diagram are called *twist equivalent*; if this is the case, then the corresponding Nichols algebras are isomorphic as graded vector spaces [AS3, Proposition 3.9].

We now list some notation for elements in $T(V)$ or $\mathcal{B}(V)$.

- $[x, y]_c :=$ product $\circ$ $(\mathrm{id} - c)$ $(x \otimes y)$, for x, y in $T(V)$ or $\mathcal{B}(V)$.
- $\mathrm{ad}_c\, x(y) := [x, y]_c$, in case $x \in V$ and y in $T(V)$ or $\mathcal{B}(V)$.
- $x_{i_1 i_2 \cdots i_k} = (\mathrm{ad}_c\, x_{i_1}) \cdots (\mathrm{ad}_c\, x_{i_{k-1}})\,(x_{i_k})$, $i_1, i_2, \cdots, i_k \in \mathbb{I}$.

Let Δ_+^V be the set of degrees of PBW generators of $\mathcal{B}(V)$, counted with their multiplicities [H1]. We can see that it does not depend on the PBW basis, [H1, AA].

2.7 The Drinfeld double

A *quasitriangular* (QT for short) Hopf algebra is a pair $(A, \mathcal{R})$, where A is a Hopf algebra and $\mathcal{R} = \sum_i a_i \otimes b_i$ is an invertible element in $A \otimes A$ such that for all $h \in A$,

$$\Delta^{\mathrm{cop}}(h) = \mathcal{R}\Delta(h)\mathcal{R}^{-1}, \quad (\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}. \tag{2.11}$$

Here $\mathcal{R}^{12} = \mathcal{R} \otimes 1$, $\mathcal{R}^{23} = 1 \otimes \mathcal{R}$, $\mathcal{R}^{13} = \sum_i a_i \otimes 1 \otimes b_i$. The Drinfeld element of $(A, \mathcal{R})$ is $\mathbf{u} = \sum_i \mathcal{S}(b_i)a_i$; it is invertible with $\mathbf{u}^{-1} = \sum_i b_i\mathcal{S}^2(a_i)$. Then

$$\begin{aligned} w := \mathbf{u}\mathcal{S}(\mathbf{u})^{-1} &= \mathcal{S}(\mathbf{u})^{-1}\mathbf{u} \in G(A), & \mathbf{u}\mathcal{S}(\mathbf{u}) &\in Z(A); \\ \mathcal{S}^2(h) &= \mathbf{u}h\mathbf{u}^{-1}, & \mathcal{S}^4(h) &= whw^{-1}, \qquad h \in A. \end{aligned}$$

Let $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R}$; then $\Delta(\mathbf{u}) = \mathcal{Q}^{-1}(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \otimes \mathbf{u})\mathcal{Q}^{-1}$.

Definition 2.7.1. [RT1] A QT Hopf algebra $(H, \mathcal{R})$ is *ribbon* if there exists $\mathbf{v} \in Z(H)$, called the ribbon element, such that

$$\mathbf{v}^2 = \mathbf{u}\mathcal{S}(\mathbf{u}), \qquad \mathcal{S}(\mathbf{v}) = \mathbf{v}, \qquad \Delta(\mathbf{v}) = \mathcal{Q}^{-1}(\mathbf{v} \otimes \mathbf{v}).$$

Then $\omega = \mathbf{u}\mathbf{v}^{-1} \in G(H)$ and $\mathcal{S}^2(h) = \omega h \omega^{-1}$ for all $h \in H$.

Remark 2.7.2. If $(A, \mathcal{R})$ is a QT Hopf algebra and $\pi : A \to B$ is a surjective morphism of Hopf algebras, then $(B, (\pi \otimes \pi)(\mathcal{R}))$ is a QT Hopf algebra. Clearly, the Drinfeld element of B is $\pi(\mathbf{u})$. Hence, if A has a ribbon element $\mathbf{v}$, then $(B, (\pi \otimes \pi)(\mathcal{R}))$ is ribbon with ribbon element $\pi(\mathbf{v})$.

By a celebrated construction of Drinfeld, every finite-dimensional Hopf algebra H gives rise to a QT Hopf algebra. For this, we first recall the *left and right coadjoint actions* of H on H^* given by

$$h \twoheadrightarrow f = h_{(1)} \rightharpoonup f \leftharpoonup \overline{\mathcal{S}}h_{(2)}, \quad f \twoheadleftarrow h = \overline{\mathcal{S}}h_{(1)} \rightharpoonup f \leftharpoonup h_{(2)}, \quad h \in H, f \in H^*. \quad (2.12)$$

If H is finite-dimensional, consider the left coadjoint action of H on H^*, respectively the right coadjoint action of H^* on H; these actions make $H^{*\,\mathrm{cop}}$ into a left H-module coalgebra, and respectively H into a right $H^{*\,\mathrm{cop}}$-module coalgebra.

Definition 2.7.3. The *Drinfeld double* $D(H) := H^{*\,\mathrm{cop}} \bowtie H$ is the following Hopf algebra: as a coalgebra, this is $H^{*\,\mathrm{cop}} \otimes H$; the algebra structure and antipode are given by

$$(f \bowtie h)(f' \bowtie h') = f(h_{(1)} \twoheadrightarrow f'_{(2)}) \bowtie (h_{(2)} \twoheadleftarrow f'_{(1)})h' \quad (2.13)$$

$$1_{D(A)} = 1_{A^*} \bowtie 1_A = \varepsilon_A \bowtie 1_A \quad (2.14)$$

$$\mathcal{S}(f \bowtie h) = \mathcal{S}(h_{(2)}) \rightharpoonup \mathcal{S}(f_{(1)}) \bowtie f_{(2)} \rightharpoonup \mathcal{S}(h_{(1)}) \quad (2.15)$$

for all $f, f' \in A^*$, $h, h' \in A$.

Let $\{h_i\}$ be a basis of H, $\{f_i\}$ its dual basis of H^* and $\mathcal{R} := \sum_i (\varepsilon \bowtie h_i) \otimes (f_i \bowtie 1)$.

Theorem 2.7.4. *[Dr] If H is a finite-dimensional Hopf algebra, then $(D(H), \mathcal{R})$ is a QT Hopf algebra.* □

We now give an alternative description of the Drinfeld double as a cocycle deformation. Let B be a bialgebra. An invertible bilinear form $\sigma : B \otimes B \to \mathbf{k}$ is a *2-cocycle* if

$$\sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}), \qquad \forall x, y, z \in B. \quad (2.16)$$

Then the *cocycle deformation* of B by σ is the bialgebra B^σ, where $B^\sigma = B$ as coalgebra, with product defined by $x \cdot y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)})$, for $x, y \in B$, and with the same identity as B [D]. If B is a Hopf algebra, then so is B^σ, with antipode

$$\mathcal{S}^\sigma(x) = \sigma(x_{(1)}, \mathcal{S}(x_{(2)}))\, \mathcal{S}(x_{(3)})\, \sigma^{-1}(\mathcal{S}(x_{(4)}), x_{(5)}), \qquad x \in B.$$

Theorem 2.7.5. *[DT, Remark 2.3] If H is a finite-dimensional Hopf algebra, then its Drinfeld double is a cocycle deformation of $H^{*\,\mathrm{cop}} \otimes H$ by σ, where*

$$\sigma(f \otimes h, f' \otimes h') = \varepsilon(f)\, \langle h, f' \rangle\, \varepsilon(h'), \qquad h, h' \in H,\ f, f' \in H^*. \quad \square \quad (2.17)$$

We state a criterium from [KR] to decide whether a Drinfeld double is ribbon.

Theorem 2.7.6. *[KR, Theorem 3] The Drinfeld double $(D(H), \mathcal{R})$ is ribbon iff there exist $k \in G(H)$, $\beta \in G(H^*)$ such that*

$$k^2 = g_H, \quad \beta^2 = \alpha_H, \quad \mathcal{S}^2(h) = k\,(\beta \rightharpoonup h \leftharpoonup \beta^{-1})\, k^{-1} \quad \forall h \in H. \quad \square \quad (2.18)$$

2.8 Spherical Hopf algebras

Definition 2.8.1. A spherical Hopf algebra is a pair (H, ω), where H is a Hopf algebra and $\omega \in G(H)$, called the pivot, such that
(i) $\mathcal{S}^2(x) = \omega x \omega^{-1}$, $x \in H$,
(ii) $tr_V(\vartheta\omega) = tr_V(\vartheta\omega^{-1})$, $\vartheta \in \mathrm{End}_H(V)$, for all $V \in \mathrm{Rep}\, H$.

Let (H, ω) be a spherical Hopf algebra. The quantum dimension of $M \in \mathrm{Rep}\, H$ is

$$\mathrm{qdim}\, M = \mathrm{tr}_M(\omega) = \mathrm{tr}_M(\omega^{-1}). \tag{2.19}$$

Theorem 2.8.2. *[BaW1] If $(H, \mathcal{R})$ is a ribbon Hopf algebra then (H, ω) is spherical with pivot given by $\omega = \mathbf{u}\mathbf{v}^{-1}$ where $\mathbf{u}, \mathbf{v}$ are the ribbon and Drinfeld elements.* $\square$

Chapter 3

Doubles of Nichols algebras

3.1 Hopf algebras attached to reduced data

Let $\mathbf{q} = (q_{ij})_{i,j\in\mathbb{I}}$ be a matrix of elements in $\mathbf{k}^\times$ satisfying (2.8), (2.9) and

$$\dim \mathcal{B}(V) < \infty. \tag{3.1}$$

Definition 3.1.1. [ARS] Let Γ be a finite abelian group. A *reduced YD-datum* (for $\mathbf{q}$ over Γ) is a collection $\mathcal{D}_{red} = ((L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ where $K_i,\ L_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ for $i \in \mathbb{I}$, such that

$$q_{ij} = \chi_j(K_i) = \chi_i(L_j) \qquad \text{for all } i,j \in \mathbb{I}, \tag{3.2}$$

$$K_iL_i \neq 1 \qquad \text{for all } i \in \mathbb{I}. \tag{3.3}$$

We attach Yetter-Drinfeld modules V and W to the reduced datum $\mathcal{D}_{red}$ by

$$V = \oplus_{i\in\mathbb{I}}\mathbf{k}v_i \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}, \qquad \text{with basis } v_i \in V^{\chi_i}_{K_i}, i \in \mathbb{I}, \tag{3.4}$$

$$W = \oplus_{i\in\mathbb{I}}\mathbf{k}w_i \in {}^{\mathbf{k}\Gamma}_{\mathbf{k}\Gamma}\mathcal{YD}, \qquad \text{with basis } w_i \in W^{\chi_i^{-1}}_{L_i}, i \in \mathbb{I}. \tag{3.5}$$

Definition 3.1.2. [ARS] Let $\mathcal{D}_{red} = ((L_i)_{i\in\mathbb{I}}, (K_i)_{i\in\mathbb{I}}, (\chi_i)_{i\in\mathbb{I}})$ be a reduced YD-datum. We define $\mathcal{U}(\mathcal{D}_{red})$ as the quotient of the biproduct $T(V \oplus W)\#\mathbf{k}\Gamma$ modulo the ideal generated by

$$I(V), \tag{3.6}$$

$$I(W), \tag{3.7}$$

$$v_iw_j - \chi_j^{-1}(K_i)w_jv_i - \delta_{ij}(K_iL_i - 1) \text{ for all } 1 \leq i,j \leq \theta. \tag{3.8}$$

it is clear that $\mathcal{U}(\mathcal{D}_{red})$ is a Hopf algebra quotient of $T(V \oplus W)\#\mathbf{k}\Gamma$.

The structure of Hopf algebra is given by

$$\begin{aligned} &\Delta(v_i) = v_i \otimes 1 + K_i \otimes v_i, && \Delta(w_i) = w_i \otimes 1 + L_i \otimes w_i, && \Delta(g) = g \otimes g, \quad g \in \Gamma,\\ &\mathcal{S}(v_i) = -K_i^{-1}v_i, && \mathcal{S}(w_i) = -L_i^{-1}w_i, && \mathcal{S}(g) = g^{-1}, \quad g \in \Gamma. \end{aligned}$$

Example 3.1.3. Let Λ be a finite abelian group and $(g_i)_{i\in\mathbb{I}}$, $(\sigma_i)_{i\in\mathbb{I}}$ where $g_i \in \Lambda$, $\sigma_i \in \widehat{\Lambda}$, $i \in \mathbb{I}$, such that $q_{ij} = \sigma_i(g_j)$ for $i, j \in \mathbb{I}$. Then we have a reduced datum $\mathcal{D}_{red}$ for $\Gamma = \Lambda \times \widehat{\Lambda}$ given by

$$K_i = g_i, \qquad L_i = \sigma_i, \qquad \chi_i = (\sigma_i, g_i), \qquad i \in \mathbb{I}.$$

Theorem 3.1.4. *In the context of Example 3.1.3, $\mathcal{U}(\mathcal{D}_{red})$ is isomorphic to , therefore it is a QT Hopf algebra.*

Proof. We argue as in [ARS, Theorem 3.7]. First, the V in (3.4) also belongs to ${}^{\mathbf{k}\Lambda}_{\mathbf{k}\Lambda}\mathcal{YD}$ by $v_i \in V_{g_i}^{\sigma_i}$, $i \in \mathbb{I}$. Similarly the W in (3.5) belongs to ${}^{\mathbf{k}\widehat{\Lambda}}_{\mathbf{k}\widehat{\Lambda}}\mathcal{YD}$ by $w_i \in W_{\sigma_i}^{g_i^{-1}}$, $i \in \mathbb{I}$.

Let $H = \mathcal{B}(V)\#\mathbf{k}\Lambda$ and $U = \mathcal{B}(W)\#\mathbf{k}\widehat{\Lambda}$. The coproduct and the antipode are determined by

$$\Delta(v_i) = v_i \otimes 1 + g_i \otimes v_i, \quad \Delta(z_i) = z_i \otimes \sigma_i^{-1} + 1 \otimes z_i, \quad \mathcal{S}(v_i) = -g_i^{-1}v_i, \quad \mathcal{S}(z_i) = -z_i\sigma_i.$$

Define $(\,|\,) : H \otimes U \to \mathbf{k}$ the bilinear form

$$(v_i|z_j) = \delta_{ij}, \quad (v_i|\chi) = 0, \quad (g|z_j) = 0, \quad (g|\chi) = \chi(g), \quad g \in \Lambda, \chi \in \widehat{\Lambda}, i, j \in \mathbb{I}.$$

It is a non-degenerate skew-Hopf bilinear form since (for example)

$$\begin{aligned}
(\mathcal{S}v_i|z_j) &= -(g_i^{-1}v_i|z_j)\\
&= -(g_i^{-1}|z_j)(v_i|\sigma_j^{-1}) - (g_i^{-1}|1)(v_i|z_j)\\
&= -(g_i^{-1}|1)(v_i|z_j)\\
&= -\delta_{ij}\\
&= -(v_i|z_j)(1|\sigma_j)\\
&= -(v_i|z_j\sigma_j) = (v_i|\mathcal{S}z_j).
\end{aligned}$$

This imply that $H^{*\,\mathrm{cop}} \simeq U$ as Hopf algebras.

By [ARS, Theorem 3.7] there exist a unique 2-cocycle $\sigma : (H^{*\,\mathrm{cop}} \otimes H) \otimes (H^{*\,\mathrm{cop}} \otimes H) \to \mathbf{k}$ such that $\mathcal{U}(\mathcal{D}_{red}) \simeq (H^{*\,\mathrm{cop}} \otimes H)_\sigma$ given by $\sigma(f \otimes h, f' \otimes h') = \varepsilon(f)(h|f')\varepsilon(h')$ for $f, f' \in H^*$ and $h, h' \in H$. Therefore $\mathcal{U}(\mathcal{D}_{red})$ is a cocycle deformation of H. By Theorem 2.7.5 $\mathcal{U}(\mathcal{D}_{red})$ is the Drinfeld double of H, hence $\mathcal{U}(\mathcal{D}_{red})$ is QT by Theorem 2.7.4 and Remark 2.7.2. □

Let $g \in \Lambda$ and $\alpha \in \widehat{\Lambda}$ be the distinguished grouplike elements of H and H^*.

Corollary 3.1.5. *If there exist $k \in \Lambda$, $\beta \in \widehat{\Lambda}$ such that*

$$k^2 = g, \qquad \beta^2 = \alpha, \qquad \mathcal{S}^2(h) = k\,(\beta \rightharpoonup h \leftharpoonup \beta^{-1})\,k^{-1} \qquad \forall h \in H, \tag{3.9}$$

then $\mathcal{U}(\mathcal{D}_{red})$ is ribbon.

Proof. This follows from Theorem 2.7.6 and Remark 2.7.2. □

We look for conditions on Λ for the existence of $(k,\beta)\in\Lambda\times\widehat{\Lambda}$ satisfing equation (3.9). Continue in the context of example 3.1.3, let $V=\bigoplus_{i\in\mathbb{I}}\mathbf{k}v_i$ and suppose that $\Delta_+^V=\{\beta_1,\dots,\beta_M\}$ is finite, following [A2, Th 3.9] we have that

$$\{v_{\beta_M}^{n_M}\cdots v_{\beta_1}^{n_1}|0\le n_j<N_{\beta_j}\}\tag{3.10}$$

is a basis of $\mathcal{B}(V)$ where $N_\beta=\operatorname{ord}(q_\beta)=\operatorname{h}(v_\beta)$ for $\beta\in\Delta_+^V$. The fact that $\mathcal{B}(V)$ is a finite-dimensional graded braided Hopf algebra implies that $\Upsilon=v_{\beta_M}^{N_{\beta_M}-1}\cdots v_{\beta_1}^{N_{\beta_1}-1}\in I_l(\mathcal{B}(V))=I_r(\mathcal{B}(V))$ and $\upsilon=w_{\beta_M}^{N_{\beta_M}-1}\cdots w_{\beta_1}^{N_{\beta_1}-1}\in I_l(\mathcal{B}(W))=I_r(\mathcal{B}(W))$. Remember that $\mathcal{B}(W)\simeq\mathcal{B}(V)^*$ as braided Hopf algebras.

Let $\lambda_\beta\in\widehat{\Lambda}$ and $\mu_\beta\in\Lambda$ such that $g\cdot v_\beta=\lambda_\beta(g)v_\beta$ and $f\cdot w_\beta=f(\mu_\beta)w_\beta$ for $g\in\Lambda$, $f\in\widehat{\Lambda}$ and $\beta\in\Delta_+^V$.

Remark 3.1.6. Let $\{\alpha_i\}_{i\in\mathbb{I}}$ be the canonical basis of $\mathbb{Z}^{\mathbb{I}}$. If $\beta=p\alpha_i+m\alpha_j\in\Delta_+^V$ for $i,j\in\mathbb{I}$ then $\lambda_\beta=\sigma_i^p\sigma_j^m$ and $\mu_\beta=g_i^pg_j^m$.

Lemma 3.1.7. *γ and ℓ of equations (2.6) and (2.7) are given by*

$$\gamma=\lambda_{\beta_M}^{N_{\beta_M}-1}\cdots\lambda_{\beta_1}^{N_{\beta_1}-1}\qquad\qquad \ell=\mu_{\beta_M}^{N_{\beta_M}-1}\cdots\mu_{\beta_1}^{N_{\beta_1}-1}.\tag{3.11}$$

In particular, the distinguished grouplike elements of H are $\alpha=\lambda_{\beta_M}^{-N_{\beta_M}+1}\cdots\lambda_{\beta_1}^{-N_{\beta_1}+1}$ and $g=\mu_{\beta_M}^{N_{\beta_M}-1}\cdots\mu_{\beta_1}^{N_{\beta_1}-1}$.

Proof. Let $g\in\Lambda$, from the graduation of $\mathcal{B}(V)$ we have that $\Upsilon=v_{\beta_M}^{N_{\beta_M}-1}\cdots v_{\beta_1}^{N_{\beta_1}-1}$ so $g\cdot\Upsilon=(g\cdot v_{\beta_M})^{N_{\beta_M}-1}\cdots(g\cdot v_{\beta_1})^{N_{\beta_1}-1}=\lambda_{\beta_M}(g)^{N_{\beta_M}-1}\cdots\lambda_{\beta_1}(g)^{N_{\beta_1}-1}\Upsilon$, this implies that $\gamma=\lambda_{\beta_M}^{N_{\beta_M}-1}\cdots\lambda_{\beta_1}^{N_{\beta_1}-1}$. In a similar way we obtain that $\ell=\mu_{\beta_M}^{N_{\beta_M}-1}\cdots\mu_{\beta_1}^{N_{\beta_1}-1}$. Last part of the statement follows by Remark 2.4.5. $\square$

Chapter 4

Unidentified Nichols algebras

We now consider a matrix $\mathbf{q} = (q_{ij})_{1\le i,j\le 2} \in \mathbf{k}^{2\times 2}$ such that its associated generalized Dynkin diagram is given by (2.10), that is $\overset{\zeta^4}{\circ} \overset{\zeta^{11}}{\rule[.5ex]{3em}{.4pt}} \overset{-1}{\bullet}$ where $\zeta \in \mathbb{G}'_{12}$; here we indicate the vertices $1, 2$ by $\circ$, $\bullet$, respectively. Let V be the associated braided vector space of diagonal type, with basis E_1, E_2. According to [H2] the corresponding Nichols algebra is finite-dimensional; this is the smallest Nichols algebra of unidentified type (up to Weyl equivalence) in the sense of [A3]. Recall the notation in page 12. By [A3], a consequence of [A1, A2], we know that the Nichols algebra $\mathcal{B}(V)$ has a presentation by generators E_1, E_2 and relations

$$E_1^3 = E_2^2 = [E_{11212}, E_{12}]_c = 0, \tag{4.1}$$

where $E_{11212} = [E_{112}, E_{12}]_c$.

The set of positive roots is $\Delta_+^V = \{\alpha_1, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$, and the corresponding PBW-basis is

$$\big\{E_2^{a_2} E_{12}^{a_{12}} E_{11212}^{a_{11212}} E_{112}^{a_{112}} E_1^{a_1} | \quad 0 \le a_2, a_{11212} \le 1; \quad 0 \le a_{12} \le 3; \quad 0 \le a_{112}, a_1 \le 2\big\}. \tag{4.2}$$

We obtain a new PBW-basis by reordering the PBW-generators:

$$\big\{E_1^{a_1} E_{112}^{a_{112}} E_{11212}^{a_{11212}} E_{12}^{a_{12}} E_2^{a_2} | \quad 0 \le a_2, a_{11212} \le 1; \quad 0 \le a_{12} \le 3; \quad 0 \le a_{112}, a_1 \le 2\big\}. \tag{4.3}$$

Thus $\dim \mathcal{B}(V) = 2^4 3^2 = 144$ and $\Upsilon = E_2 E_{12}^3 E_{11212} E_{112}^2 E_1^2 \in I_l(\mathcal{B}(V))$.

Lemma 4.0.1. *The following relations are valid in* $\mathcal{B}(V)$

$$E_1E_2 = E_{12} + q_{12}E_2E_1, \qquad E_1E_{12} = E_{112} + q_{12}\zeta^4E_{12}E_1,$$
$$E_1E_{112} = q_{12}\zeta^8E_{112}E_1, \qquad E_1E_{11212} = q_{12}^2E_{11212}E_1 + q_{12}\zeta^7(1+\zeta)E_{112}^2,$$
$$E_1E_{12}^2 = E_{11212} + q_{12}\zeta(1+\zeta^3)E_{12}E_{112} + q_{12}^2\zeta^8E_{12}^2E_1, \qquad E_1E_{12}^3 = q_{12}\zeta^{10}E_{12}E_{11212} + q_{12}^2\zeta^5E_{12}^2E_{112} + q_{12}^3E_{12}^3E_1,$$
$$E_1^2E_2 = E_{112} + q_{12}^2\zeta^2E_{12}E_1 + q_{12}^2E_2E_1^2, \qquad E_1^2E_{12} = -q_{12}^2E_{112}E_1 + q_{12}^2\zeta^8E_{12}E_1^2,$$
$$E_{112}E_2 = -q_{12}^2E_2E_{112} + q_{12}\zeta^8E_{12}^2, \qquad E_{112}E_{12} = E_{11212} + q_{12}\zeta E_{12}E_{112},$$
$$E_{112}E_{12}^2 = -q_{12}\zeta^4(1+\zeta^3)E_{12}E_{11212} + q_{12}^2\zeta^2E_{12}^2E_{112}, \qquad E_{112}E_{12}^3 = q_{12}^2\zeta^{11}E_{12}^2E_{11212} + q_{12}^3\zeta^3E_{12}^3E_{112},$$
$$E_{11212}E_{12} = q_{12}\zeta^{10}E_{12}E_{11212}, \qquad E_{112}E_{11212} = q_{12}\zeta^9E_{11212}E_{112},$$
$$E_{11212}E_2 = q_{12}^3E_2E_{11212} + q_{12}^2\zeta^2(1+\zeta)E_{12}^3, \qquad E_{12}E_2 = -q_{12}E_2E_{12}.$$

Proof. It follows from the defining relations of $\mathcal{B}(V)$ that

$$\begin{aligned}
E_{12} &= [E_1, E_2] = E_1E_2 + \sigma_1(g_2)E_2E_1,\\
E_{112} &= [E_1, E_{12}] = E_1E_{12} + \sigma_1(g_1g_2)E_{12}E_1\\
E_{11212} &= [E_{112}, E_{12}] = E_{112}E_{12} + \sigma_1^2(g_1g_2)\sigma_2(g_1g_2)E_{12}E_{112}\\
[E_2, E_{12}] &= [E_{12}, E_{11212}] = [E_{11212}, E_{112}] = [E_{112}, E_1] = 0
\end{aligned}$$

so, applying repeatedly E_i, $i = 1, 2$ to these relations we obtain all the enunciated relations. □

Remark 4.0.2. By [A1, Theorem 4.9], we also have $E_{112}^3 = E_{11212}^2 = E_{12}^4 = 0$.

4.1 Ribbon structure

We now consider the Hopf algebra $\mathcal{U} := \mathcal{U}(\mathcal{D}_{red})$ within the context of Example 3.1.3. Thus Λ is a finite abelian group provided with g_1, $g_2 \in \Lambda$, σ_1, $\sigma_2 \in \widehat{\Lambda}$ such that $\begin{pmatrix}\sigma_1(g_1) & \sigma_2(g_1)\\ \sigma_1(g_2) & \sigma_2(g_2)\end{pmatrix} = \begin{pmatrix}\zeta^4 & q_{12}\\ q_{21} & -1\end{pmatrix}$; recall that $\zeta \in \mathbb{G}'_{12}$. Then $\mathcal{U} \simeq D(H)$ where $H = \mathcal{B}(V)\#\mathbf{k}\Lambda$. We need the explicit relations in $\mathcal{U}$. As in [ARS, H3, HY] we set

$$E_i = v_i, \qquad F_i = w_i\sigma_i^{-1} \qquad \text{in } \mathcal{U} \text{ for } i = 1, 2. \tag{4.4}$$

Let $\mathcal{U}^-$ (respectively $\mathcal{U}^+$) be the subalgebra of $\mathcal{U}$ generated by F_1, F_2 (respectively E_1, E_2). Recall the notation listed in the Subsection 2.6.

Lemma 4.1.1. *The following equalities hold:*

$$F_{12} = q_{21}w_{12}\sigma_1^{-1}\sigma_2^{-1}, \qquad F_{112} = \zeta^4q_{21}^2w_{112}\sigma_1^{-2}\sigma_2^{-1},$$
$$F_{11212} = \zeta^5q_{21}^4w_{11212}\sigma_1^{-3}\sigma_2^{-2}, \qquad [F_{11212}, F_{12}] = 0.$$

Proof. By (4.1), $w_2^2 = 0 = w_1^3 = [w_{11212}, w_{12}]_c$. Also $g_j w_i = q_{ji}^{-1} w_i g_j$, $\sigma_j w_i = q_{ij}^{-1} w_i \sigma_j$, $1 \le i, j \le 2$, hence $F_2^2 = F_1^3 = 0$. For the remaining equality, we first compute

$$\begin{aligned} F_{12} &= F_1F_2 - q_{21}F_2F_1 \\ &= w_1\sigma_1^{-1}w_2\sigma_2^{-1} - q_{21}w_2\sigma_2^{-1}w_1\sigma_1^{-1} \\ &= w_1q_{21}w_2\sigma_1^{-1}\sigma_2^{-1} - q_{21}w_2w_1\sigma_1^{-1}\sigma_2^{-1} \\ &= q_{21}w_{12}\sigma_1^{-1}\sigma_2^{-1}. \end{aligned}$$

In the same way we prove that $F_{112} = F_1F_{12} - \zeta^4 q_{21}F_{12}F_1 = \zeta^4 q_{21}^2 w_{112}\sigma_1^{-2}\sigma_2^{-1}$ and $F_{11212} = F_{112}F_{12} - \zeta q_{21}F_{12}F_{112} = \zeta^5 q_{21}^4 w_{11212}\sigma_1^{-3}\sigma_2^{-2}$. Finally

$$\begin{aligned} [F_{11212}, F_{12}] &= F_{11212}F_{12} - \zeta^4 q_{21}F_{12}F_{11212} \\ &= (\zeta^5 q_{21}^5 w_{11212}w_{12} - \zeta^9 q_{21}^5 w_{12}w_{11212})\sigma_1^{-4}\sigma_2^{-3} = 0. \end{aligned}$$

□

Theorem 4.1.2. *[ARS, Section 3.2] [H3, Proposition 5.6] The Hopf algebra $\mathcal{U}$ is presented by generators $g \in \Lambda$, $\sigma \in \widehat{\Lambda}$, E_1, E_2, F_1, F_2 and relations for $1 \le i, j, k \le 2$*

$$\begin{aligned} &E_1^2 = E_2^2 = [E_{11212}, E_{12}] = 0, && gE_i = \chi_i(g)E_ig, && \sigma E_i = \sigma(g_i)E_i\sigma, \\ &F_1^2 = F_2^2 = [F_{11212}, F_{12}] = 0, && gF_i = \chi_i^{-1}(g)F_ig, && \sigma F_i = \sigma(g_i^{-1})F_i\sigma, \\ &E_kF_i - F_iE_k = \delta_{ki}(g_i - \sigma_i^{-1}), && g\sigma = \sigma g, && \end{aligned}$$

and the relations defining Λ, $\widehat{\Lambda}$. □

Lemma 4.1.3. *The following equalities hold:*

$$\begin{aligned} &F_1E_{12} = E_{12}F_1 + q_{12}(\zeta - 1)E_2\sigma_1^{-1}, && F_2E_{12} = E_{12}F_2 + (\zeta^{11} - 1)E_1g_2, \\ &F_1E_{112} = E_{112}F_1 + q_{12}\zeta^8(1 + \zeta^3)E_{12}\sigma_1^{-1}, && F_2E_{112} = E_{112}F_2 - (3)_{\zeta^7}E_1^2g_2, \\ &F_1E_{11212} = E_{11212}F_1 + q_{12}^2(\zeta^5 - 1)E_{12}^2\sigma_1^{-1}, && F_2E_{11212} = E_{11212}F_2 - E_{112}E_1g_2, \\ &F_1E_{112}^2 = E_{112}^2F_1 - q_{12}(1 + \zeta^3)(E_{11212}\sigma_1^{-1} + && F_2E_{12}^2 = E_{12}^2F_2 + q_{21}(1 + \zeta^5)E_{112}g_2 - \\ &\quad \zeta^4E_{112}E_{12}\sigma_1^{-1}), && \quad (3)_{\zeta^7}E_{12}E_1g_2, \\ &F_1E_{12}^2 = E_{12}^2F_1 + q_{12}^2(3)_{\zeta^5}E_2E_{12}\sigma_1^{-1}, && F_2E_{112}^2 = E_{112}^2F_2 + (3)_{\zeta^7}\zeta^4E_{112}E_1^2g_2, \end{aligned}$$

$$\begin{aligned} &F_1E_{12}^3 = E_{12}^2F_1 + q_{12}^3\zeta^3(\zeta - 1)E_2E_{12}^2\sigma_1^{-1}, && F_2E_{12}^3 = E_{12}^3F_2 + \zeta^8(1 - \zeta)(E_{12}^2E_1g_2 - \\ &F_{11212}E_{11212} = E_{11212}F_{11212} + \sigma_1^{-3}\sigma_2^{-2} - g_1^3g_2^2, && \quad q_{21}\zeta^3E_{12}E_{112}g_2 + q_{21}^2\zeta^3E_{11212}g_2), \\ &F_{12}E_2 = E_2F_{12} + (1 - \zeta^{11})F_1\sigma_2^{-1}, && F_{12}E_1 = E_1F_{12} + q_{21}(1 - \zeta)F_2g_1, \\ &F_{12}E_{12} = E_{12}F_{12} + \sigma_1^{-1}\sigma_2^{-1} - g_1g_2, && F_{12}E_{11212} = E_{11212}F_{12} + \zeta^{11}E_{112}g_1g_2, \\ &F_{12}E_{112} = E_{112}F_{12} + \zeta^3(3)_{\zeta^7}E_1g_1g_2, && F_{112}E_{112} = E_{112}F_{112} + \sigma_1^{-2}\sigma_2^{-1} - g_1^2g_2, \\ &F_{12}E_{112}^2 = E_{112}^2F_{12} + \zeta^{11}(3)_{\zeta^7}E_{112}E_1g_1g_2, && F_{112}E_2 = E_2F_{112} + (\zeta - 1)F_1^2\sigma_2^{-1}. \end{aligned}$$

Proof. Using Theorem 4.1.2 and Lemma 4.0.1, we have that

$$\begin{aligned}
F_1E_{12} &= F_1(E_1E_2 - q_{12}E_2E_1) = F_1E_1E_2 - q_{12}F_1E_2E_1 \\
&= (E_1F_1 - (g_1 - \sigma_1^{-1})E_2 - q_{12}E_2F_1E_1 \\
&= E_1E_2F_1 - q_{12}E_2g_1 + q_{21}^{-1}E_2\sigma_1^{-1} - q_{12}E_2(E_1F_1 - (g_1 - \sigma_1^{-1}) \\
&= E_1E_2F_1 - q_{12}E_2g_1 + q_{21}^{-1}E_2\sigma_1^{-1} - q_{12}E_2E_1F_1 + q_{12}E_2g_1 - q_{12}E_2\sigma_1^{-1} \\
&= E_{12}F_1 + (q_{21}^{-1} - q_{12})E_2\sigma_1^{-1} = E_{12}F_1 + q_{12}(\zeta - 1)E_1\sigma_1^{-1}.
\end{aligned}$$

The other relations come from analogue computation. □

By Theorem 2.7.6, $\mathcal{U}$ would be ribbon if and only if there exist $k \in G(H)$, $\beta \in G(H^*)$ related to the distinguished grouplike elements α_H and g_H of H by (3.9), that is

$$k^2 = g_H, \qquad \beta^2 = \alpha_H, \qquad \mathcal{S}^2(h) = k(\beta \rightharpoonup h \leftharpoonup \beta^{-1})k^{-1}, \qquad \forall h \in H.$$

By Lemma 3.1.7, $\alpha_H = \sigma_1^{-12}\sigma_2^{-8}$ and $g_H = g_1^{12}g_2^8$, only depending on g_1, g_2, σ_1, σ_2.

So, given Γ abelian group, g_1, $g_2 \in \Gamma$ and σ_1, $\sigma_2 \in \widehat{\Gamma}$ such that $\sigma_i(g_j) = q_{ij}$, we choose

$$\beta = \sigma_1^{-6}\sigma_2^{-4}, \quad k = g_1^6g_2^4$$

and we get $\beta^2 = \sigma_1^{-12}\sigma_2^{-8}$, $k^2 = g_1^{12}g_2^8$. Moreover, to verify the third equality it is enough to check it on the generators g_i, E_j: $\mathcal{S}^2(g_i) = g_i = k(\beta \rightharpoonup g_i \leftharpoonup \beta^{-1})k^{-1}$ and $\mathcal{S}^2(E_i) = g_i^{-1}E_ig_i = q_{ii}^{-1}E_i$.

Now

$$\begin{aligned}
k(\beta \rightharpoonup E_i \leftharpoonup \beta^{-1})k^{-1} &= \sigma_i(k)\beta(g_i)E_i = q_{1i}^6q_{2i}^4q_{i1}^{-6}q_{i2}^{-4}E_i, \\
k(\beta \rightharpoonup E_1 \leftharpoonup \beta^{-1})k^{-1} &= \zeta^8E_i = (\zeta^4)^{-1}E_1, \\
k(\beta \rightharpoonup E_2 \leftharpoonup \beta^{-1})k^{-1} &= \zeta^6E_2 = -E_2.
\end{aligned}$$

So, by Theorem 2.7.6 $\mathcal{U}$ *is ribbon.*

Example 4.1.4. We take $\Lambda = \mathbb{Z}_{12} = \langle g_2 \rangle$ and define $g_1 = g_2^8$ and $\sigma_1, \sigma_2 \in \widehat{\Lambda}$ such that

$$\sigma_1(g_2) = \zeta^{11}, \qquad \sigma_2(g_2) = -1; \qquad \text{hence} \qquad \sigma_1(g_1) = \zeta^4, \qquad \sigma_2(g_1) = 1. \tag{4.5}$$

In particular $\mathrm{ord}(g_1) = 3$, $\mathrm{ord}(\sigma_1) = 12$, $\mathrm{ord}(\sigma_2) = 2$ and $\sigma_2 = \sigma_1^6$. It satisfies the conditions of Example 3.1.3. In such a case, $\alpha_H = \epsilon$ and $g_H = g_2^8 = g_1$.

Chapter 5

Representations of $\mathcal{U}$

We construct and classify the irreducible representations of the Drinfeld double of previous section. They are quotient of Verma modules and depend on the values on $g_i\sigma_i$, $i = 1, 2$.

5.1 Verma modules

We keep the setting from the previous Section; recall that $\Gamma = \Lambda \times \widehat{\Lambda}$. The algebra $\mathcal{U}$ has a triangular decomposition $\mathcal{U} \simeq \mathcal{U}^+ \otimes \mathbf{k}\Gamma \otimes \mathcal{U}^-$. Let $\lambda \in \widehat{\Gamma}$ and extend it to an algebra map $\mathbf{k}\Gamma \otimes \mathcal{U}^- \to \mathbf{k}$ by annihilating the elements of $\mathcal{U}^-$; the corresponding module is denoted by $\mathbf{k}_\lambda$. The *Verma module* $M(\lambda)$ associated to λ is the induced module

$$M(\lambda) = \operatorname{Ind}_{\mathbf{k}\Gamma\otimes\mathcal{U}^-}^{\mathcal{U}} \mathbf{k}_\lambda \simeq \mathcal{U}/\big(\mathcal{U}F_1 + \mathcal{U}F_2 + \sum_{g\in\Gamma} \mathcal{U}(g - \lambda(g))\big). \tag{5.1}$$

Let v_λ be the residue class of 1 in $M(\lambda)$; then $1 \mapsto v_\lambda$ extends to an isomorphism of $\mathcal{U}^+$-modules $\mathcal{U}^+ \simeq M(\lambda)$ by using the triangular decomposition. In what follows

$$m(a, b, c, d, e) := E_2^a E_{12}^b E_{11212}^c E_{112}^d E_1^e \cdot v_\lambda, \quad n(a, b, c, d, e) := E_1^e E_{112}^d E_{11212}^c E_{12}^b E_2^a \cdot v_\lambda$$

for $a, b, c, d, e \in \mathbb{Z}$. Then $m(a, b, c, d, e), n(a, b, c, d, e) \neq 0$ if and only if $a, c \in \{0, 1\}$, $b \in \{0, 1, 2, 3\}$, $d, e \in \{0, 1, 2\}$, $v_\lambda = m(0, 0, 0, 0, 0) = n(0, 0, 0, 0, 0)$ and $m(a, b, c, d, e)$, $n(a, b, c, d, e)$ are bases of $M(\lambda)$.

Lemma 5.1.1. *Set* $W_1(\lambda) = span\{m(a, b, c, d, e) \mid e \neq 0\}$, $W_2(\lambda) = span\{m(a, b, c, d, 2)\}$, $W(\lambda) = span\{n(1, b, c, d, e)\}$. *Then*

a) $F_2 \cdot W_i(\lambda) \subseteq W_i(\lambda)$,

$$F_1 \cdot m(a, b, c, d, i) \in \lambda(\sigma_1^{-1})(i)_{\zeta^4}(\zeta^{i-8} - \lambda(g_1\sigma_1))m(a, b, c, d, i - 1) + W_i(\lambda).$$

b) $F_1 \cdot W(\lambda) \subseteq W(\lambda)$,

$$F_2 \cdot n(1, b, c, d, e) \in \lambda(\sigma_2^{-1})(1 - \lambda(g_2\sigma_2))n(0, b, c, d, e) + W(\lambda).$$

In particular,

- $W_1(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda(g_1\sigma_1) = 1$*;*
- $W_2(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda(g_1\sigma_1) = \zeta^8$*;*
- $W(\lambda)$ *is a* $\mathcal{U}$*-submodule if and only if* $\lambda(g_2\sigma_2) = 1$*.*

Proof. It follows by direct computation. □

5.2 Irreducible modules

Now we consider quotients of Verma modules as in [HY, Section 5], [RS, Section 2].

The $\mathbb{Z}^2$-grading on $\mathcal{U}$ induce a $\mathbb{Z}^2$-grading on $M(\lambda)$ such that

$$M(\lambda)_\beta = \mathcal{U}_\beta \cdot v_\lambda, \qquad \beta \in \mathbb{Z}^2.$$

Thus $M(\lambda)_0 = \mathbf{k}v_\lambda$, $\mathcal{U}_\beta \cdot M(\lambda)_\gamma \subset M(\lambda)_{\beta+\gamma}$ for all $\beta, \gamma \in \mathbb{Z}^2$.

Remark 5.2.1. Let $v \in M(\lambda)$ be such that $F_i \cdot v = 0$ for $i = 1, 2$. By the triangular decomposition of $\mathcal{U}$, $\mathcal{U} \cdot v = \mathcal{U}^+ \cdot v$. In particular, if $v \in M(\lambda)_\alpha$, $\alpha \neq 0$, then $\mathcal{U} \cdot v$ is a submodule such that $\mathcal{U} \cdot v \cap \mathbf{k}v_\lambda = 0$.

The family of $\mathcal{U}$-submodules of $M(\lambda)$ contained in $\sum_{\beta\neq 0} M(\lambda)_\beta$ has a unique maximal element $N(\lambda)$. The *highest weight module* of weight λ is the quotient

$$L(\lambda) = M(\lambda)/N(\lambda).$$

The maximality of $N(\lambda)$ guaranties that it is $\mathbb{Z}^2$-homogeneous, so the quotient $L(\lambda)$ inherits the $\mathbb{Z}^2$-grading of $M(\lambda)$. Moreover, as $\mathcal{U}$ is finite-dimensional, a $\mathcal{U}$-module L is irreducible if and only if it is irreducible in the category of $\mathbb{Z}^2$-graded $\mathcal{U}$-modules.

Remark 5.2.2. Let M be a finite dimensional simple $\mathcal{U}$-module. By [RS, Corollary 2.6] M is irreducible if and only if $M \simeq L(\lambda)$ for some $\lambda \in \widehat{\Gamma}$.

A general diagram with the homogeneous component of a finite dimensional simple $\mathcal{U}$-module on each rank $(a, b) = a\alpha_1 + b\alpha_2$ is given in B.1

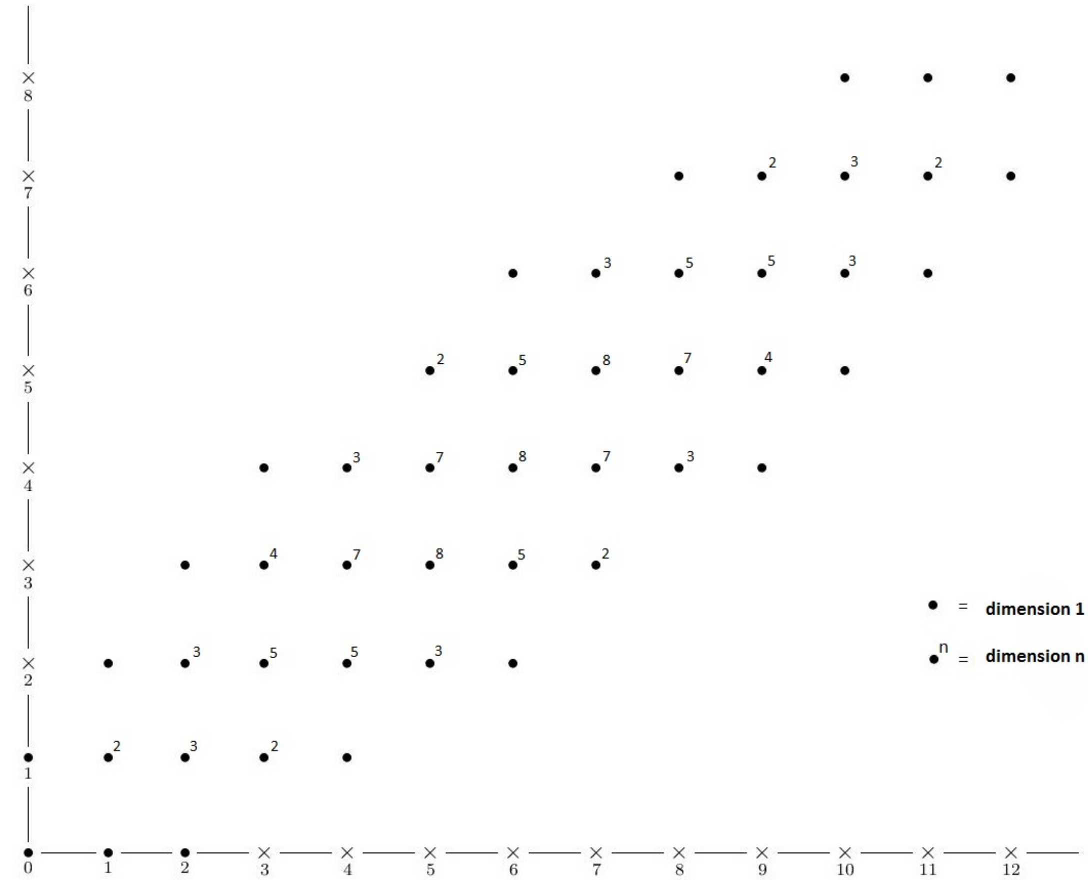

Figure 5.1:

Let A be an associative algebra, V an A-module, and $\phi : A \to A$ an algebra automorphism. Then we consider the A-module V^ϕ, where $V^\phi = V$ as vector space and the A-action is given by $a \triangleright v = \phi(a) \cdot v$, $v \in V$, $a \in A$. Then $W \subset V$ is an A-submodule if and only if $W^\varphi \subset V^\varphi$ is an A-submodule.

Example 5.2.3. There exists an algebra automorphism $\varphi : \mathcal{U} \to \mathcal{U}$, such that $\varphi(K_i) = K_i^{-1}$, $\varphi(L_i) = L_i^{-1}$, $\varphi(E_i) = F_i L_i^{-1}$, $\varphi(F_i) = K_i^{-1} E_i$ [H3, Proposition 4.9 (4)]. Therefore, if $v \in V$ has weight λ, then $v \in V^\varphi$ has weight λ^{-1}.

As application we have the following results.

Lemma 5.2.4. *Let $X(\lambda) := \{x \in L(\lambda) : E_i x = 0$ for all $i\}$. Then $X(\lambda)$ is a one-dimensional subspace and there exists $\mu \in \widehat{\Gamma}$ such that $X(\lambda) \subset L(\lambda)_\mu$, $L(\lambda)^\varphi \simeq L(\mu^{-1})$.*

Proof. $X(\lambda)$ is a non-trivial subspace since there exists $\beta \in \mathbb{N}_0^2$ maximal such that $L(\lambda)_\beta \neq 0$. Let $x \in X(\lambda)_\beta$, $x \neq 0$, so the action of Γ is given by some $\mu \in \widehat{\Gamma}$. Thus there exists a $\mathcal{U}$-module map $\pi : (\mathcal{U}x)^\phi \twoheadrightarrow L(\mu^{-1})$, $x \mapsto v_{\mu^{-1}}$. But $\mathcal{U}x = L(\lambda)$ and π is an isomorphism since $L(\lambda)$ is irreducible. Finally, $\pi(X(\lambda)) \subseteq \{x \in L(\lambda) : F_i x = 0$ for all $i\} = \mathbf{k} v_\lambda$. $\square$

Lemma 5.2.5. *Let $v \in V$, with V a $\mathcal{U}$-module, be such that $E_i v = 0$, $i = 1, 2$, $v \in V^\mu$. If $\overline{m(a,b,c,d,e)} \neq 0$ in $L(\mu^{-1})$ then $F_2^a F_{12}^b F_{11212}^c F_{112}^d F_1^e v \neq 0$ in V.*

Proof. Let $W = \mathcal{U}v \subseteq V$. Then W^φ is a highet weight $\mathcal{U}$-module since

$$F_i \triangleright v = \varphi(E_i L_i^{-1}) \triangleright v = E_i L_i^{-1} v = \mu(L_i^{-1}) E_i v = 0.$$

Thus there exists a unique $\mathcal{U}$-module map $\pi : W^\varphi \to L(\mu^{-1})$ such that $\pi(v) = v_\lambda$. Now $\pi(F_2^a F_{12}^b F_{11212}^c F_{112}^d F_1^e v)$ is, up to a non-zero scalar,

$$E_2^a E_{12}^b E_{11212}^c E_{112}^d E_1^e \pi(v) = \overline{m(a,b,c,d,e)} \neq 0,$$

so $F_2^a F_{12}^b F_{11212}^c F_{112}^d F_1^e v \neq 0$ in V. □

Remark 5.2.6. There is an analogue result for $\overline{n(a,b,c,d,e)}$.

5.3 The case $\dim V = 1$

We consider for a moment the algebra corresponding to a braided vector space of dimension 1. In this case the reduced datum consists of elements $g \in \Lambda$, $\sigma \in \widehat{\Lambda}$. Set $q = \sigma(g)$, and $N \in \mathbb{N}$ its order. The algebra $\mathcal{U} := \mathcal{U}(\mathcal{D}_{red})$ is close to $\mathfrak{u}_q(\mathfrak{sl}_2)$. It has a presentation by generators $g \in \Lambda$, $\sigma \in \widehat{\Lambda}$, E, F with relations

$$\begin{aligned} &E^N = F^N = 0, && gE = \chi(g)Eg, && \sigma E = \sigma(g)E\sigma, \\ &EF - FE = g - \sigma^{-1}, && hF = \chi^{-1}(h)Fh, && \tau F = \tau(g^{-1})F\tau, \end{aligned}$$

and $h\tau = \tau h$ for $h \in \Lambda$, $\tau \in \widehat{\Lambda}$, and the relations defining Λ, $\widehat{\Lambda}$. Thus

$$E^j F - F E^j = (j)_q E^{j-1}(g - q^{1-j}\sigma^{-1}), \qquad j \in \mathbb{N}. \tag{5.2}$$

Lemma 5.3.1. *Let $\lambda \in \widehat{\Gamma}$, $n = \dim L(\lambda)$.*

1. *If $\lambda(g\sigma) = q^{1-j}$ for some $j \in \{1, \dots, n-1\}$, then $N = j$.*
2. *If $\lambda(g\sigma) \notin \{q^j | j = 0, 1, \dots, n-2\}$, then $n = N$.*

Moreover $L(\lambda)$ has a basis $v_0, \dots, v_{n-1}$ such that

$$Ev_i = v_{i+1}, \quad Fv_i = (i)_q(q^{1-i}\lambda(\sigma_1^{-1}) - \lambda(g_1))v_{i-1}, \quad h\tau v_i = \lambda(h\tau)\sigma^i(h)\tau(g^i)v_i. \tag{5.3}$$

Proof. Same argument as for $\mathfrak{u}_q(\mathfrak{sl}_2)$. □

Corollary 5.3.2. *Let M be an $\mathcal{U}$-module, $\lambda \in \widehat{\Gamma}$. If $v \in M_\lambda - 0$ satisfies $Fv = 0$, then there exists n such that $v, Ev, \dots, E^{n-1}v$ are linearly independent, where*

1. *either $n = j$ if $\lambda(g\sigma) = q^{1-j}$ for some $j \in \{1, \dots, N-1\}$,*
2. *or else $n = N - 1$ if $\lambda(g\sigma) \notin \{q^j | j = 0, 1, \dots, n-2\}$.*

Moreover $F^i E^i v = a_i v$ for some $a_i \in \mathbf{k}^\times$.

Proof. Set $v_0 = v$, $v_i = E \cdot v_{i-1}$. Then $v_N = 0$, and the submodule M' generated by v is spanned by $\{v_i\}$. Moreover there exists a surjective $\mathcal{U}$-linear map $M(\lambda) \twoheadrightarrow M'$, $v_\lambda \mapsto v_0$. Therefore it induces a surjective $\mathcal{U}$-linear map $M' \twoheadrightarrow L(\lambda)$. □

5.4 Irreducible modules

We use the following notation: $\lambda_i = \lambda(g_i\sigma_i)$, $i = 1, 2$.

Definition 5.4.1. We set the following subsets of $\widehat{\Gamma}$:

$$\begin{aligned}
\mathfrak{I}_1 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \neq 1, \zeta^8, \lambda_1^2\lambda_2 \neq -1, \zeta^{10}, \lambda_1^3\lambda_2^2 \neq -1, \lambda_1\lambda_2 \neq \zeta, \zeta^4, \zeta^7, \lambda_2 \neq 1\},\\
\mathfrak{I}_2 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 \neq 1, \zeta, \zeta^4, \zeta^7, \zeta^3, \zeta^9, -1, \zeta^{10}\},\\
\mathfrak{I}_3 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 \neq 1, \zeta^5, \zeta^8, \zeta^{11}, \zeta^3, \zeta^9, \zeta^2, -1\},\\
\mathfrak{I}_4 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = -1, \lambda_1 \neq \pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\},\\
\mathfrak{I}_5 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^2\lambda_2 = \zeta^{10}, \lambda_1 \neq \pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\},\\
\mathfrak{I}_6 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1^3\lambda_2^2 = -1, \lambda_1 \neq \pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2\},\\
\mathfrak{I}_7 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta, \lambda_1 \neq 1, \zeta^8, \zeta, \zeta^4, \zeta^9\},\\
\mathfrak{I}_8 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^4, \lambda_1 \neq 1, \zeta^8, \zeta^4, \zeta^2, -1, \zeta^{10}\},\\
\mathfrak{I}_9 &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1\lambda_2 = \zeta^7, \lambda_1 \neq 1, \zeta^8, \zeta^7, \zeta^4, \zeta^{11}\},\\
\mathfrak{I}_{10} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 \notin \mathbb{G}_{12}, \lambda_2 = 1\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{I}_{11} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta\}, & \mathfrak{I}_{12} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^4\},\\
\mathfrak{I}_{13} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^7\}, & \mathfrak{I}_{14} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^3\},\\
\mathfrak{I}_{15} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^9\}, & \mathfrak{I}_{16} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = -1\},\\
\mathfrak{I}_{17} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = \zeta^{10}\}, & \mathfrak{I}_{18} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^5\},\\
\mathfrak{I}_{19} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^8\}, & \mathfrak{I}_{20} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^{11}\},\\
\mathfrak{I}_{21} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^3\}, & \mathfrak{I}_{22} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^9\},\\
\mathfrak{I}_{23} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = \zeta^2\}, & \mathfrak{I}_{24} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = -1\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{I}_{25} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = \zeta^8\}, & \mathfrak{I}_{26} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = \zeta^8\},\\
\mathfrak{I}_{27} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^9\}, & \mathfrak{I}_{28} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = \zeta^4\},\\
\mathfrak{I}_{29} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = -1\}, & \mathfrak{I}_{30} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = \zeta^2\},\\
\mathfrak{I}_{31} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = \zeta^{10}\}, & \mathfrak{I}_{32} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = -1\},\\
\mathfrak{I}_{33} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = -1\}, & \mathfrak{I}_{34} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = \zeta^3\},\\
\mathfrak{I}_{35} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = \zeta^4\},
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{I}_{36} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta, \lambda_2 = 1\}, & \mathfrak{I}_{37} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^2, \lambda_2 = 1\},\\
\mathfrak{I}_{38} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^3, \lambda_2 = 1\}, & \mathfrak{I}_{39} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^4, \lambda_2 = 1\},\\
\mathfrak{I}_{40} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^5, \lambda_2 = 1\}, & \mathfrak{I}_{41} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = -1, \lambda_2 = 1\},\\
\mathfrak{I}_{42} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^7, \lambda_2 = 1\}, & \mathfrak{I}_{43} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^8, \lambda_2 = 1\},\\
\mathfrak{I}_{44} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^9, \lambda_2 = 1\}, & \mathfrak{I}_{45} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{10}, \lambda_2 = 1\},\\
\mathfrak{I}_{46} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = \zeta^{11}, \lambda_2 = 1\}, & \mathfrak{I}_{47} &= \{\lambda \in \widehat{\Gamma} \mid \lambda_1 = 1, \lambda_2 = 1\}.
\end{aligned}
$$

Now we describe the modules $L(\lambda)$ for λ on each subset as above.

Lemma 5.4.2. *If $\lambda \in \mathfrak{I}_1$, then $M(\lambda)$ is simple.*

Proof. This is a consequence of [HY, Proposition 5.16] since if

$$
\begin{aligned}
(\zeta^4\lambda_1^{-1} - \zeta^4)(\zeta^4\lambda_1^{-1} - \zeta^8)(\zeta^2\lambda_1^{-2}\lambda_2^{-1} - \zeta^8)(\zeta^2\lambda_1^{-2}\lambda_2^{-1} - \zeta^4)(\lambda_1^{-3}\lambda_2^{-2} + 1)\\
(\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} - \zeta^9)(\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} + 1)(\zeta^{10}\lambda_1^{-1}\lambda_2^{-1} - \zeta^3)(\lambda_2^{-1} - 1) \neq 0
\end{aligned}
$$

then $M(\lambda)$ is a simple $\mathcal{U}$-module. □

This comes by the generalized version of the Shapovalov determinant, introduced by Heckenberger and Yamane for these Drinfeld doubles of Nichols algebras. This determinant has a factorization, and the Verma module is irreducible if no one of these factors is zero. It also helped on the work of the other cases. Cases 2-10 has one of the factors equal to 0 and the resulting relation are explained in Remarks 5.4.5 -5.4.15, so the simple modules are self-dual and the maximal submodules of the Verma module are cyclic. The other cases have exactly two factors equal to 0, and we compute the results of the relations from the Remarks obtaining what elements are null in the module and what cannot be null. This computation give us basis to the module, but this is not always simple and easy. Therefore, we also related the Cases 11-47, with each other in two possible ways, using a morphism between submodules as in Lemma 5.2.5 and this provides relations between the diagrams of the module that we explain in Appendix B. We observe that the sufficient condition on [HY, Proposition 5.16] for the irreducibility of $M(\lambda)$ is indeed necessary.

Lemma 5.4.3. *If $\lambda \in \mathfrak{I}_2$, then $\dim L(\lambda) = 48$. A basis of $L(\lambda)$ is given by*

$$
\mathrm{B}_2 = \{\overline{m(a,b,c,d,0)}\}.
$$

Proof. Define $L'(\lambda) = M(\lambda)/\mathcal{U}^+m(0,0,0,0,1)$. Notice that $\mathcal{U}^+m(0,0,0,0,1) = W_1(\lambda)$ is a proper submodule of $M(\lambda)$ by Lemma 5.1.1(a), and $\{\overline{m(a,b,c,d,0)}\}$ is a basis for $L'(\lambda)$. We claim that $L'(\lambda)$ is simple, so $L(\lambda) = L'(\lambda)$ and we finish the proof. Let W be a non-zero submodule of $L'(\lambda)$. Then set $w \in W - 0$, which is a linear combination of $\{\overline{m(a,b,c,d,0)}\}$. Fix a minimal element $\overline{m(a,b,c,d,0)}$ with non-zero coefficient. Here we take the lexicographical order: $\overline{m(a,b,c,d,0)} < \overline{m(a',b',c',d',0)}$ iff $a < a'$, or $a = a'$, $b < b'$, etc. Then $E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{3-b}E_2^{1-a}w$ is equal to $\overline{m(1,3,1,2,0)}$, up to a non-zero scalar, so $\overline{m(1,3,1,2,0)} \in W$.

As $\lambda(g_2\sigma_2)\neq 1$, we have that $\overline{m(1,0,0,0,0)}\neq 0$. Now E_{12}, F_{12} and g,σ spam a subalgebra as in Subsection 5.3 and

$$F_{12}\overline{m(1,0,0,0,0)}=0, \qquad g_1g_2\sigma_1\sigma_2\overline{m(1,0,0,0,0)}=\lambda(g_2\sigma_2)\zeta^{11}\overline{m(1,0,0,0,0)},$$
$$E_{12}^3\overline{m(1,0,0,0,0)}=\chi_1^3\chi_2^3(g_2^{-1})\overline{m(1,3,0,0,0)}.$$

By Corollary 5.3.2, $\overline{m(1,3,0,0,0)}\neq 0$ since $\lambda(g_2\sigma_2)\zeta^{11}\neq\pm 1,\zeta^3$ (that is, $\lambda(g_2\sigma_2)\neq\zeta,\zeta^4,\zeta^7$). Similarly, as

$$F_{11212}\overline{m(1,3,0,0,0)}=0, \quad g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(1,3,0,0,0)}=-\lambda(g_2\sigma_2)^2\overline{m(1,3,0,0,0)},$$
$$E_{11212}\overline{m(1,3,0,0,0)}=\chi_1^3\chi_2^2(g_1^{-3}g_2^{-4})\overline{m(1,3,1,0,0)},$$

and $-\lambda(g_2\sigma_2)^2\neq 1$ (since $\lambda(g_2\sigma_2)\neq\zeta^3,\zeta^9$), we have that $\overline{m(1,3,1,0,0)}\neq 0$. Again, as

$$F_{112}\overline{m(1,3,1,0,0)}=0, \quad g_1^2g_2\sigma_1^2\sigma_2\overline{m(1,3,1,0,0)}=-\lambda(g_2\sigma_2)\overline{m(1,3,1,0,0)},$$
$$E_{112}^2\overline{m(1,3,1,0,0)}=\chi_1^4\chi_2^2(g_1^{-6}g_2^{-6})\overline{m(1,3,1,2,0)},$$

and $-\lambda(g_2\sigma_2)\neq 1,\zeta^4$ (since $\lambda(g_2\sigma_2)\neq -1,\zeta^{10}$), we have that $\overline{m(1,3,1,2,0)}\neq 0$.

Moreover Corollary 5.3.2 also implies that $F_2F_{12}^3F_{11212}F_{112}^3\overline{m(1,3,1,2,0)}$ is v_λ, up to a non-zero scalar. Therefore $v_\lambda\in W$, so $W=L'(\lambda)$ and $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.4. *If $\lambda\in\mathfrak{I}_3$, then $\dim L(\lambda)=96$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_3=\{\overline{m(a,b,c,d,e)}|e=0,1\}.$$

Proof. Define $L'(\lambda)=M(\lambda)/\mathcal{U}^+m(0,0,0,0,2)$, so $\{\overline{m(a,b,c,d,e)}|e=0,1\}$ is a basis for $L'(\lambda)$. We claim that $L'(\lambda)$ is simple. Let W be a non-zero submodule of $L'(\lambda)$; we have that $\overline{m(1,3,1,2,1)}\in W$.

We apply now Corollary 5.3.2 repeatedly. First $\overline{m(1,0,0,0,0)}\neq 0$, since $\lambda(g_2\sigma_2)\neq 1$. Now E_{12}, F_{12} and g,σ spam a subalgebra as in Subsection 5.3 and

$$F_{12}\overline{m(1,0,0,0,0)}=0, \qquad g_1g_2\sigma_1\sigma_2\overline{m(1,0,0,0,0)}=\lambda(g_2\sigma_2)\zeta^{7}\overline{m(1,0,0,0,0)},$$
$$E_{12}^3\overline{m(1,0,0,0,0)}=\chi_1^3\chi_2^3(g_2^{-1})\overline{m(1,3,0,0,0)}.$$

Then $\overline{m(1,3,0,0,0)}\neq 0$ since $\lambda(g_2\sigma_2)\zeta^7\neq\pm 1,\zeta^3$ (that is, $\lambda(g_2\sigma_2)\neq\zeta^5,\zeta^8,\zeta^{11}$). As

$$F_{11212}\overline{m(1,3,0,0,0)}=0, \quad g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(1,3,0,0,0)}=-\lambda(g_2\sigma_2)^2\overline{m(1,3,0,0,0)},$$
$$E_{11212}\overline{m(1,3,0,0,0)}=\chi_1^3\chi_2^2(g_1^{-3}g_2^{-4})\overline{m(1,3,1,0,0)},$$

and $-\lambda(g_2\sigma_2)^2\neq 1$ (since $\lambda(g_2\sigma_2)\neq\zeta^3,\zeta^9$), we have that $\overline{m(1,3,1,0,0)}\neq 0$. Again, as

$$F_{112}\overline{m(1,3,1,0,0)}=0, \quad g_1^2g_2\sigma_1^2\sigma_2\overline{m(1,3,1,0,0)}=\zeta^{10}\lambda(g_2\sigma_2)\overline{m(1,3,1,0,0)},$$
$$E_{112}^2\overline{m(1,3,1,0,0)}=\chi_1^4\chi_2^2(g_1^{-6}g_2^{-6})\overline{m(1,3,1,2,0)},$$

and $\zeta^{10}\lambda(g_2\sigma_2) \neq 1, \zeta^4$ (since $\lambda(g_2\sigma_2) \neq \zeta^2, -1$), we have that $\overline{m(1,3,1,2,0)} \neq 0$. Finally $F_1\overline{m(1,3,1,2,0)} = 0$, $g_1\sigma_1\overline{m(1,3,1,2,0)} = \zeta^8\overline{m(1,3,1,2,0)}$, so $\overline{m(1,3,1,2,1)} \neq 0$. Moreover Corollary 5.3.2 also implies that $F_2F_{12}^3F_{11212}F_{112}^2F_1\overline{m(1,3,1,2,1)}$ is v_λ, up to a non-zero scalar. Therefore $v_\lambda \in W$, so $W = L'(\lambda)$ and $L'(\lambda)$ is irreducible. □

Remark 5.4.5. Set $w = m(0,0,0,1,2)$, if $\lambda_1^2\lambda_2 = -1$, then $F_1w = F_2w = 0$.

Proof. According to the Lemma 4.1.3, $F_{112}E_{112} = E_{112}F_{112} + \sigma_1^{-2}\sigma_2^{-1} - g_1^2g_2$, so

$$F_{112}E_{112}E_1^2v_\lambda = (\sigma_1^{-2}\sigma_2^{-1} - g_1^2g_2)E_1^2v_\lambda = \lambda(\sigma_1^{-2}\sigma_2^{-1})q_{21}^2\zeta^4(\lambda_1^2\lambda_2 + 1)E_1^2v_\lambda.$$

As $M(\lambda)_{4\alpha_1} = M(\lambda)_{3\alpha_1} = 0$, we have that $F_2E_{112}E_1^2v_\lambda = F_1E_{112}E_1^2v_\lambda = 0$, so

$$0 = F_{112}E_{112}E_1^2v_\lambda = \zeta^8q_{12}^2F_2F_1^2E_{112}E_1^2v_\lambda.$$

□

Lemma 5.4.6. *If* $\lambda \in \mathfrak{I}_4$, *then* $\dim L(\lambda) = 48$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_4 = \{\overline{m(a,b,c,0,e)}\}.$$

Proof. Let $w = F_1^2E_{112}E_1^2v_\lambda$. Explicitly,

$$w = (\zeta^8 - \lambda_1)(1 - \lambda_1)E_{112}v_\lambda + \frac{q_{21}^2\zeta^2(3)_\zeta}{2}E_2E_1^2v_\lambda - \frac{q_{21}\zeta^8(\zeta^8 - \lambda_1)(1 + \zeta^3)}{2}E_{12}E_1v_\lambda.$$

By Remark 5.4.5 w generates a proper submodule. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$: we claim that it is irreducible. Note that $\mathbf{k}E_1^{2-e}E_{11212}^{1-c}E_{12}^{3-b}E_2^{1-a}\overline{m(a,b,c,0,e)} = \mathbf{k}\overline{m(1,3,1,0,2)}$, $E_i\overline{m(1,3,1,0,2)} = 0$, $i = 1,2$, and then $\{\overline{m(a,b,c,0,e)}\}$ is a basis of $L'(\lambda)$. From the hypothesis on λ_1, λ_2 and

$$\begin{aligned} F_{12}\overline{n(1,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(1,0,0,0,0)} &= \zeta^5\lambda_1^{-1}\overline{n(1,0,0,0,0)}, \\ F_{11212}\overline{n(1,3,0,0,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(1,3,0,0,0)} &= -\lambda_1^{-1}\overline{n(1,3,0,0,0)}, \\ F_1\overline{n(1,3,1,0,0)} &= 0, & g_1\sigma_1\overline{n(1,3,1,0,0)} &= -\lambda_1\overline{n(1,3,1,0,0)}, \end{aligned}$$

we deduce successively that $\overline{n(1,3,0,0,0)} \neq 0$, $\overline{n(1,3,1,0,0)} \neq 0$, $\overline{n(1,3,1,0,2)} \neq 0$, using Corollary 5.3.2.

Notice that $\mathbf{k}\overline{m(1,3,1,0,2)} = \mathbf{k}\overline{n(1,3,1,0,2)}$, so $\overline{m(1,3,1,0,2)} \neq 0$ and there exists $F \in \mathcal{U}$ such that $F\overline{m(1,3,1,0,2)} = v_\lambda$. Then we argue as in previous Lemmas and the claim follows. □

Remark 5.4.7. Set $w = m(0,0,0,2,2)$, if $\lambda_1^2\lambda_2 = \zeta^{10}$, then $F_1w = F_2w = 0$.

Proof.

$$F_{112}E_{112}^2E_1^2v_\lambda = \lambda(\sigma_1^{-2}\sigma_2^{-1})q_{21}^2\zeta^8(\lambda_1^2\lambda_2 - \zeta^{10})E_1^2v_\lambda.$$

As $M(\lambda)_{6\alpha_1+\alpha_2} = M(\lambda)_{5\alpha_1+\alpha_2} = 0$, we have that $F_2E_{112}^2E_1^2v_\lambda = F_2F_1E_{112}^2E_1^2v_\lambda = 0$, so

$$0 = F_{112}E_{112}^2E_1^2v_\lambda = \zeta^8q_{12}^2F_2F_1^2E_{112}^2E_1^2v_\lambda.$$

□

Lemma 5.4.8. *If $\lambda \in \mathfrak{I}_5$, then* $\dim L(\lambda) = 96$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_5 = \{\overline{m(a,b,c,d,e)} | d \neq 2\}.$$

Proof. Let $w = F_1^2 E_{112}^2 E_1^2 v_\lambda$. Explicitly,

$$\begin{aligned} w = m(0,0,0,2,0) &- \frac{q_{12}^3 \zeta^8 (1+\zeta^3)}{(1-\lambda_1)(\zeta^8 - \lambda_1)} n(0,2,0,0,2) - \frac{q_{12}^2 \zeta^5 (1+\zeta^3)}{1-\lambda_1} n(0,1,0,1,1) \\ &+ \frac{q_{12}\zeta^2(1+\zeta^3)}{1-\lambda_1} n(0,0,1,0,1) - \frac{q_{12}^3 \zeta^4 (3)_{\zeta^7}}{(1-\lambda_1)(\zeta^8-\lambda_1)} n(1,0,0,1,2). \end{aligned}$$

By Remark 5.4.7 $\mathcal{U}w$ is a proper submodule. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$: we claim that it is irreducible. Note that $\mathbf{k}E_1^{2-e} E_{112}^{1-d} E_{11212}^{1-c} E_{12}^{3-b} E_2^{1-a} \overline{m(a,b,c,d,e)} = \mathbf{k}\overline{m(1,3,1,1,2)}$, $E_i \overline{n(1,3,1,1,2)} = 0$, $i = 1,2$, and then $\{\overline{m(a,b,c,d,e)} | d \neq 2\}$ is a basis of $L'(\lambda)$. From the hypothesis on λ_1, λ_2 and

$$\begin{aligned} F_{12}\overline{n(1,0,0,0,0)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{n(1,0,0,0,0)} &= \zeta^9 \lambda_1^{-1} \overline{n(1,0,0,0,0)}, \\ F_{11212}\overline{n(1,3,0,0,0)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{n(1,3,0,0,0)} &= \zeta^2 \lambda_1^{-1} \overline{n(1,3,0,0,0)}, \\ F_{112}\overline{n(1,3,1,0,0)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{n(1,3,1,0,0)} &= \zeta^4 \overline{n(1,3,1,0,0)}, \\ F_1 \overline{n(1,3,1,1,0)} &= 0, & g_1 \sigma_1 \overline{n(1,3,1,1,0)} &= \zeta^9 \lambda_1 \overline{n(1,3,1,1,0)}, \end{aligned}$$

we deduce successively that $\overline{n(1,3,0,0,0)} \neq 0$, $\overline{n(1,3,1,0,0)} \neq 0$, $\overline{n(1,3,1,1,0)} \neq 0$, $\overline{n(1,3,1,1,2)} \neq 0$, using Corollary 5.3.2.

Notice that $\mathbf{k}\overline{m(1,3,1,1,2)} = \mathbf{k}\overline{n(1,3,1,1,2)}$, so $\overline{m(1,3,1,1,2)} \neq 0$ and there exists $F \in \mathcal{U}$ such that $F\overline{m(1,3,1,1,2)} = v_\lambda$. Then we argue as in previous Lemmas and the claim follows. □

Remark 5.4.9. Notice that $F_2 w = F_{12} w = 0$ since $\mathcal{U}_{9\alpha_1+3\alpha_2} = \mathcal{U}_{8\alpha_1+3\alpha_2} = 0$. As

$$F_2 F_1^2 F_{112}^2 = q_{12}^4 \zeta^4 F_1 F_{112} F_{11212} + q_{12}^4 \zeta^8 F_1 F_{112}^2 F_{12} + q_{12}^3 \zeta^7 F_1^2 F_{112} F_{12}^2 + q_{12}^4 \zeta^4 F_1^2 F_{112}^2 F_2,$$

we have that

$$\begin{aligned} \lambda(\sigma_1^6 \sigma_2^3) F_2 w &= \lambda(\sigma_1^6\sigma_2^3) q_{12}^4 \zeta^4 F_1 F_{112} F_{11212} E_{11212} E_{112}^2 E_1^2 v_\lambda \\ &= q_{21}^4 (1+\lambda_1^3\lambda_2^2)(\zeta^{10} - \lambda_1^2 \lambda_2)\big(q_{12}\zeta^2(1+\zeta^3) E_{12} E_1^2 v_\lambda + (1-\zeta^4\lambda_1) E_{112} E_1 v_\lambda\big) = 0, \end{aligned}$$

so $F_2 w = 0$. As $F_1^3 = 0$, we have that $F_1 w = 0$.

Thus $w = F_1^2 F_{112}^2 E_{11212} E_{112}^2 E_1^2 v_\lambda$ satisfies that $F_2 w = 0$, $F_1 w = 0$ if $\lambda_1^3 \lambda_2^2 = -1$.

Lemma 5.4.10. *If $\lambda \in \mathfrak{I}_6$, then* $\dim L(\lambda) = 72$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_6 = \{\overline{m(a,b,0,d,e)}\}.$$

Proof. Set w as in Remark 5.4.9, so it generates a proper submodule. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$: we claim that it is irreducible.

Notice that $L'(\lambda)$ is spanned by $\{\overline{m(a,b,0,d,e)}\}$, $E_i\overline{m(1,3,0,2,2)} = 0$, $i = 1,2$, and $\mathbf{k}E_1^{2-e}E_{112}^{2-d}E_{12}^{3-b}E_2^{1-a}\overline{m(a,b,c,d,e)} = \mathbf{k}\overline{m(1,3,0,2,2)}$. From the hypothesis on λ_1, λ_2 and

$$\begin{aligned} F_{12}\overline{m(1,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{m(1,0,0,0,0)} &= \zeta^{11}\lambda_1\lambda_2\overline{m(1,0,0,0,0)}, \\ F_{112}\overline{m(1,3,0,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(1,3,0,0,0)} &= \zeta\lambda_1^2\lambda_2\overline{m(1,3,0,0,0)}, \\ F_1\overline{m(1,3,0,2,0)} &= 0, & g_1\sigma_1\overline{m(1,3,0,2,0)} &= \zeta^2\lambda_1\overline{m(1,3,0,2,0)}, \end{aligned}$$

we deduce successively that $\overline{m(1,3,0,0,0)} \neq 0$, $\overline{m(1,3,0,2,0)} \neq 0$, $\overline{m(1,3,0,2,2)} \neq 0$, using Corollary 5.3.2. Moreover there exists $F \in \mathcal{U}$ such that $F\overline{m(1,3,0,2,2)} = v_\lambda$. Then we argue as in previous Lemmas and the claim follows. □

Remark 5.4.11. Set $w = n(1,1,0,0,0)$, if $\lambda_1\lambda_2 = \zeta$, then $F_1w = F_2w = 0$.

Proof.

$$F_{12}E_2E_{12}v_\lambda = \lambda(\sigma_1^{-1}\sigma_2^{-1})q_{12}(\lambda_1\lambda_2 - \zeta)E_2v_\lambda.$$

As $M(\lambda)_{2\alpha_2} = 0$, we have that $F_1E_2E_{12}v_\lambda = 0$, so

$$F_{12}E_2E_{12}v_\lambda = F_1F_2E_2E_{12}v_\lambda.$$

□

Lemma 5.4.12. *If* $\lambda \in \mathfrak{I}_7$, *then* $\dim L(\lambda) = 36$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_7 = \{\overline{n(a,0,c,d,e)}\}.$$

Proof. Let $w = F_2E_2E_{12}v_\lambda = (\lambda_2^{-1} - 1)E_{12} - (\zeta^{11} - 1)E_1E_2$. By Remark 5.4.11, $\mathcal{U}w \subsetneq M(\lambda)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$: we claim that it is irreducible. First notice that $\{\overline{n(a,0,c,d,e)}\}$ is a basis of $L'(\lambda)$, $E_i\overline{n(1,0,1,2,2)} = 0$, $i = 1,2$, and $\mathbf{k}E_1^{2-e}E_{112}^{2-d}E_{11212}^{1-c}E_2^{1-a}\overline{n(a,0,c,d,e)} = \mathbf{k}\overline{n(1,0,1,2,2)}$. Moreover

$$\begin{aligned} F_{112}\overline{m(0,0,0,0,2)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,2)} &= \zeta^7\lambda_1\overline{m(0,0,0,0,2)}, \\ F_{11212}\overline{m(0,0,0,2,2)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,2,2)} &= \zeta^8\lambda_1\overline{m(0,0,0,2,2)}, \\ F_2\overline{m(0,0,1,2,2)} &= 0, & g_2\sigma_2\overline{m(0,0,1,2,2)} &= \zeta^3\lambda_2\overline{m(0,0,1,2,2)}, \end{aligned}$$

so successively we prove that $\overline{m(0,0,0,2,2)} \neq 0$, $\overline{m(0,0,1,2,2)} \neq 0$, $\overline{m(1,0,1,2,2)} \neq 0$, using Corollary 5.3.2.

Notice that $\mathbf{k}\overline{m(1,0,1,2,2)} = \mathbf{k}\overline{n(1,0,1,2,2)}$, so $\overline{n(1,0,1,2,2)} \neq 0$ and there exists $F \in \mathcal{U}$ such that $F\overline{n(1,0,1,2,2)} = v_\lambda$. Then we argue as in previous Lemmas and the claim follows. □

Remark 5.4.13. Set $w = n(1,2,0,0,0)$, if $\lambda_1\lambda_2 = \zeta^4$, then $F_1w = F_2w = 0$.

Proof.

$$F_{12}E_2E_{12}^2v_\lambda = (2)_{\zeta^9}\lambda(\sigma_1^{-1}\sigma_2^{-1})q_{12}(\lambda_1\lambda_2-\zeta^4)E_2E_{12}v_\lambda.$$

As $M(\lambda)_{\alpha_1+3\alpha_2}=0$, we have that $F_1E_2E_{12}^2v_\lambda=0$, so

$$F_{12}E_2E_{12}^2v_\lambda = F_1F_2E_2E_{12}^2v_\lambda.$$

□

Lemma 5.4.14. *If $\lambda\in\mathfrak{I}_8$, then* $\dim L(\lambda)=72$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_8=\{\overline{n(a,b,c,d,e)}|b\le 1\}.$$

Proof. Let $w=F_2E_2E_{12}^2v_\lambda$. Explicitly,

$$w=(\lambda_2^{-1}-1)E_{12}^2-q_{21}\zeta^2(1-\zeta)E_{112}E_2+q_{21}\zeta^3(3)_{\zeta^7}E_1E_{12}E_2.$$

By Remark 5.4.13, $\mathcal{U}w\subsetneq M(\lambda)$. Let $L'(\lambda)=M(\lambda)/\mathcal{U}w$: we claim that it is irreducible. First notice that $\{\overline{n(a,b,c,d,e)}|b=0,1\}$ is a basis of $L'(\lambda)$.

Then $E_i\overline{n(1,0,1,2,2)}=0$, $i=1,2$, and $\mathbf{k}E_1^{2-e}E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{1-b}E_2^{1-a}\overline{n(a,b,c,d,e)}=\mathbf{k}\overline{n(1,1,1,2,2)}$. From the hypothesis on λ_1,λ_2 and

$$\begin{aligned}
F_{112}\overline{m(0,0,0,0,2)}&=0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,2)}&=\zeta^{10}\lambda_1\overline{m(0,0,0,0,2)},\\
F_{11212}\overline{m(0,0,0,2,2)}&=0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,2,2)}&=\zeta^{2}\lambda_1\overline{m(0,0,0,2,2)},\\
F_{12}\overline{m(0,0,1,2,2)}&=0, & g_1g_2\sigma_1\sigma_2\overline{m(0,0,1,2,2)}&=\zeta^{3}\overline{m(0,0,1,2,2)},\\
F_{2}\overline{m(0,1,1,2,2)}&=0, & g_2\sigma_2\overline{m(0,1,1,2,2)}&=\zeta^{2}\lambda_2\overline{m(0,1,1,2,2)},
\end{aligned}$$

we deduce successively that $\overline{m(0,0,0,2,2)}\neq 0$, $\overline{m(0,0,1,2,2)}\neq 0$, $\overline{m(0,1,1,2,2)}\neq 0$, and finally $\overline{m(1,1,1,2,2)}\neq 0$, using Corollary 5.3.2.

Notice that $\mathbf{k}\overline{m(1,1,1,2,2)}=\mathbf{k}\overline{n(1,1,1,2,2)}$, so $\overline{n(1,1,1,2,2)}\neq 0$ and there exists $F\in\mathcal{U}$ such that $F\overline{n(1,1,1,2,2)}=v_\lambda$. Then we argue as in previous Lemmas and the claim follows. □

Remark 5.4.15. Set $w=n(1,3,0,0,0)$, if $\lambda_1\lambda_2=\zeta^7$, then $F_1w=F_2w=0$.

Proof.

$$F_{12}E_2E_{12}^3v_\lambda = (3)_{\zeta^9}\lambda(\sigma_1^{-1}\sigma_2^{-1})q_{12}(\lambda_1\lambda_2-\zeta^7)E_2E_{12}^2v_\lambda.$$

As $M(\lambda)_{2\alpha_1+4\alpha_2}=0$, we have that $F_1E_2E_{12}^3v_\lambda=0$, so

$$F_{12}E_2E_{12}^3v_\lambda = F_1F_2E_2E_{12}^3v_\lambda.$$

□

Lemma 5.4.16. *If $\lambda\in\mathfrak{I}_9$, then* $\dim L(\lambda)=108$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_9=\{\overline{n(a,b,c,d,e)}|b\neq 3\}.$$

Proof. Let $w = F_2E_2E_{12}^3v_\lambda$. Explicitly,

$$\begin{aligned} w =&(\lambda_2^{-1}-1)E_{12}^3 - q_{21}^2\zeta^2(1-\zeta)E_1E_{12}^2E_2 - q_{21}^2(\zeta^{11}-1)E_{112}E_{12}E_2 \\ &+ q_{21}^2\zeta^2(1-\zeta)E_{11212}E_2. \end{aligned}$$

By Remark 5.4.15, $\mathcal{U}w \subsetneq M(\lambda)$. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w$: we claim that it is irreducible. Note that $\mathbf{k}E_1^{2-e}E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{2-b}E_2^{1-a}\overline{n(a,b,c,d,e)} = \mathbf{k}\overline{n(1,2,1,2,2)}$, $E_i\overline{n(1,2,1,2,2)} = 0$, $i=1,2$, and then $\{\overline{n(a,b,c,d,e)}|b\neq 3\}$ is a basis of $L'(\lambda)$. From the hypothesis on λ_1,λ_2 and

$$\begin{aligned} F_{112}\overline{m(0,0,0,0,2)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,2)} &= \zeta\lambda_1\overline{m(0,0,0,0,2)}, \\ F_{11212}\overline{m(0,0,0,2,2)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,2,2)} &= \zeta^8\lambda_1\overline{m(0,0,0,2,2)}, \\ F_{12}\overline{m(0,0,1,2,2)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{m(0,0,1,2,2)} &= -\overline{m(0,0,1,2,2)}, \\ F_2\overline{m(0,2,1,2,2)} &= 0, & g_2\sigma_2\overline{m(0,2,1,2,2)} &= \zeta\lambda_2\overline{m(0,2,1,2,2)}, \end{aligned}$$

we deduce successively that $\overline{m(0,0,0,2,2)}\neq 0$, $\overline{m(0,0,1,2,2)}\neq 0$, $\overline{m(0,2,1,2,2)}\neq 0$, and finally $\overline{m(1,2,1,2,2)}\neq 0$, using Corollary 5.3.2.

Notice that $\mathbf{k}\overline{m(1,2,1,2,2)} = \mathbf{k}\overline{n(1,2,1,2,2)}$, so $\overline{n(1,2,1,2,2)}\neq 0$ and there exists $F\in\mathcal{U}$ such that $F\overline{n(1,2,1,2,2)} = v_\lambda$. Then we argue as in previous Lemmas and the claim follows. □

Lemma 5.4.17. *If $\lambda\in\mathfrak{I}_{10}$, then* $\dim L(\lambda) = 72$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{10} = \{\overline{n(0,b,c,d,e)}\}.$$

Proof. Note that $W(\lambda) = \mathcal{U}^+n(1,0,0,0,0)$ is a proper submodule of $M(\lambda)$ by Lemma 5.1.1(b). Define $L'(\lambda) = M(\lambda)/W(\lambda)$. Then $\{\overline{n(0,b,c,d,e)}\}$ is a basis for $L'(\lambda)$. We claim that $L'(\lambda)$ is simple, so $L(\lambda) = L'(\lambda)$ and we finish the proof. Let W be a non-zero submodule of $L'(\lambda)$; we may assume it is $\mathbb{Z}^2$-graded. Then set $w\in W-0$, which is a linear combination of $\{\overline{n(0,b,c,d,e)}\}$'s. Fix a minimal element $\overline{n(0,b,c,d,e)}$ with non-zero coefficient. Here we take the lexicographical order: $\overline{n(0,b,c,d,e)} < \overline{n(0,b',c',d',e')}$ iff $e<e'$, or $e=e'$, $d<d'$, etc. Then $E_{12}^{3-b}E_{11212}^{1-c}E_{112}^{2-d}E_1^{2-e}w$ is equal to $\overline{n(0,3,1,2,2)}$, up to a non-zero scalar, so $\overline{n(0,3,1,2,2)}\in W$. From the hypothesis on λ_1,λ_2 and

$$\begin{aligned} F_{112}\overline{n(0,0,0,0,2)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(0,0,0,0,2)} &= -\lambda(g_1\sigma_1)^2\overline{n(0,0,0,0,2)}, \\ F_{11212}\overline{n(0,0,0,2,2)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(0,0,0,2,2)} &= -\lambda(g_1\sigma_1)^3\overline{n(0,0,0,2,2)}, \\ F_{12}\overline{n(0,0,1,2,2)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,1,2,2)} &= \lambda(g_1\sigma_1)\zeta^{11}\overline{n(0,0,1,2,2)}, \end{aligned}$$

we deduce successively that $\overline{m(0,0,0,2,2)}\neq 0$, $\overline{m(0,0,1,2,2)}\neq 0$, $\overline{m(0,3,1,2,2)}\neq 0$ by Corollary 5.3.2, and there exists $F\in\mathcal{U}$ such that $F\overline{n(1,2,1,2,2)} = v_\lambda$. Then we argue as in previous Lemmas and the claim follows. □

In some of next Lemmas we will describe explicitly the action of $\mathcal{U}$ on a fixed basis. Elements $v_{i,j}$ are homogeneous of degree $i\alpha_1 + j\alpha_2$.

Lemma 5.4.18. *If $\lambda \in \mathfrak{I}_{11}$, then* $\dim L(\lambda) = 11$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{11} = \{\overline{m(a,b,0,d,0)} | b \leq 1\} - \{\overline{m(1,1,0,0,0)}\}.$$

The action of E_i, F_i, $i = 1, 2$ is described in Table A.1.

Proof. $\mathcal{U}m(0,0,0,0,1) = W_1(\lambda)$ is a $\mathcal{U}$-submodule by Lemma 5.1.1(a) [1]. We claim that $L'(\lambda) = M(\lambda)/(\mathcal{U}m(1,1,0,0,0) + W_1(\lambda))$ is simple. Note that

$$F_1 m(1,1,0,0,0) = 0, \quad F_2 m(1,1,0,0,0) = (\zeta^{11} - 1)\lambda(g_2) m(1,0,0,0,1) \in W_1(\lambda).$$

From here we have that $\mathcal{U}m(1,1,0,0,0) + W_1(\lambda) \subseteq \sum_{\beta \neq 0} M(\lambda)_\beta$ by Remark 5.2.1, and therefore $\mathcal{U}m(1,1,0,0,0) + W_1(\lambda) \subseteq N(\lambda)$. The canonical projection $M(\lambda) \twoheadrightarrow L(\lambda)$ induces a surjective map $L'(\lambda) \twoheadrightarrow L(\lambda)$ of $\mathcal{U}$-modules, but if $L'(\lambda)$ is simple, then this projection has a trivial kernel. Set the following elements of $L'(\lambda)$:

$$\begin{aligned}
v_{0,0} &= \overline{m(0,0,0,0,0)}, & v_{0,1} &= \overline{m(1,0,0,0,0)}, & v_{1,1} &= \overline{m(0,1,0,0,0)},\\
v_{2,1} &= \overline{m(0,0,0,1,0)}, & v_{2,2} &= \overline{m(1,0,0,1,0)}, & v_{3,2} &= \overline{m(0,1,0,1,0)},\\
v_{4,2} &= \overline{m(0,0,0,2,0)}, & v_{3,3} &= \overline{m(1,1,0,1,0)}, & v_{4,3} &= \overline{m(1,0,0,2,0)},\\
v_{5,3} &= \overline{m(0,1,0,2,0)}, & v_{5,4} &= \overline{m(1,1,0,2,0)}.
\end{aligned}$$

Notice that $v_{i,j} \in L'(\lambda)_{i\alpha_1 + j\alpha_2}$. Those vectors satisfy the relations in Table A.1 by direct computation. The formulae prove that the quotient is spanned by these 11 vectors since they are obtained by applying repeatedly E_1, E_2 over $v_\lambda = v_{0,0}$ and $E_1 v_{5,4} = E_2 v_{5,4} = 0$.

From Table A.1 there exist elements $E_{i,j} \in \mathcal{U}^+_{(5-i)\alpha_1 + (4-j)\alpha_2}$, $F_{5,4} \in \mathcal{U}^-_{-5\alpha_1 - 4\alpha_2}$ such that $E_{i,j} v_{i,j} = v_{5,4}$, $F_{5,4} v_{5,4} = v_\lambda$. Assume now that V is non-zero submodule of $L'(\lambda)$. Then take $v \in V$, $v \neq 0$. As $L'(\lambda)$ is spanned by the vectors $v_{i,j}$ (in fact, this set of vectors is a basis), there exists $E \in \mathcal{U}^+$ such that $Ev = v_{5,4}$. But $\mathcal{U}v_{5,4} = L'(\lambda)$ since $v_\lambda \in \mathcal{U}v_{5,4}$. Then $V = L'(\lambda)$ and $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.19. *If $\lambda \in \mathfrak{I}_{12}$, then* $\dim L(\lambda) = 11$. *A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{12} = \{\overline{m(a,b,0,d,0)} : b, d \leq 1\} \cup \{\overline{m(0,1,1,0,0)}, \overline{m(1,0,1,1,0)}, \overline{m(0,0,1,1,0)}\}.$$

The action of E_i, F_i, $i = 1, 2$ is described in Table A.2.

Proof. Again $\mathcal{U}m(0,0,0,0,1) = W_1(\lambda)$ is a $\mathcal{U}$-submodule by Lemma 5.1.1(a). By Remark 5.4.13 $w = F_2 E_2 E_{12}^2 v_\lambda$ satisfies the equations $F_1 w = F_2 w = 0$, so $\mathcal{U}w + W_1(\lambda)$ is a proper submodule of $M(\lambda)$ by Remark 5.2.1. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U}w + W_1(\lambda)$; it is enough to prove that $L'(\lambda)$ is simple. We label the elements of B_{12} as follows:

$$\begin{aligned}
v_{0,0} &= m(0,0,0,0,0), & v_{0,1} &= m(1,0,0,0,0), & v_{1,1} &= m(0,1,0,0,0),\\
v_{2,1} &= m(0,0,0,1,0), & v_{2,2} &= m(1,0,0,1,0), & v_{1,2} &= m(1,1,0,0,0),\\
v_{3,2} &= m(0,1,0,1,0), & v_{3,3} &= m(1,1,0,1,0), & v_{4,3} &= m(0,1,1,0,0),\\
v_{5,3} &= m(0,0,1,1,0), & v_{5,4} &= m(1,0,1,1,0).
\end{aligned}$$

[1] Here $\lambda_1 \lambda_2 = \zeta$, but the relation in Remark 5.4.11 becomes trivial.

Those vectors satisfy the relations in the Table A.2 by direct computation. We claim that B_{12} is a basis of $L'(\lambda)$. Applying repeatedly E_1, E_2 over w we obtain

$$\overline{m(0,2,0,0,0)} = q_{12}\zeta^4(1-\zeta)\overline{m(1,0,0,1,0)}, \qquad \overline{m(0,1,0,1,0)} = \zeta^{10}q_{21}\overline{m(0,0,1,0,0)},$$
$$\overline{m(0,2,1,0,0)} = \zeta^{11}q_{12}^2(1+\zeta^2)\overline{m(1,0,1,1,0)}, \quad \overline{m(0,0,0,2,0)} = \overline{m(1,1,1,0,0)} = 0.$$

Notice that $\overline{m(a,b,c,2,0)} = \overline{m(a,b+1,1,1,0)} = 0$ for all a,b,c, since $\overline{m(0,1,1,0,0)} = \overline{m(0,0,0,2,0)} = 0$. As also $0 = \overline{m(1,2,1,0,0)} = \overline{m(1,2,0,1,0)} = \overline{m(0,3,1,0,0)}$, we conclude that $L'(\lambda)$ is spanned by B_{12}.

By Corollary 5.3.2 we have that $\overline{m(1,0,1,1,0)}, \overline{m(0,0,1,1,0)}, \overline{m(0,0,0,1,0)} \neq 0$, since

$$\begin{aligned} F_{112}\overline{m(0,0,0,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,0)} &= \zeta^4\overline{m(0,0,0,0,0)},\\ F_{11212}\overline{m(0,0,0,1,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,1,0)} &= \zeta\overline{m(0,0,0,1,0)}\\ F_2\overline{m(0,0,1,1,0)} &= 0, & g_2\sigma_2\overline{m(0,0,1,1,0)} &= \zeta^{11}\overline{m(0,0,1,1,0)}. \end{aligned}$$

Moreover there exists $F \in \mathcal{U}^-$ such that $F\overline{m(1,0,1,1,0)} = v_\lambda$.

Also, $E_i\overline{m(1,0,1,1,0)} = 0$, $i = 1,2$. Indeed, the case $i = 1$ follows from the previous relations, and the case $i = 2$ is direct.

Now suppose that B is not linearly independent. Fix a non-trivial linear combination $\mathtt{S}$ which is zero. Between the elements of minimal N_0-degree with non-trivial coefficient, take the element $\overline{m} = \overline{m(a,b,c,d,0)}$ minimal for the lexicographical order. If $b = 0$, then $E_2^{1-a}E_{11212}^{1-c}E_{112}^{1-d}\overline{m}$ gives $\overline{m(1,0,1,1,0)}$ up to a non-zero scalar. If $b = 1$, then $E_2^{1-a}E_{11212}^{1-c}E_1\overline{m}$ is also $\overline{m(1,0,1,1,0)}$ up to a non-zero scalar. By the minimality of $\overline{m}$, we have that $E\mathtt{S} = \overline{m(1,0,1,1,0)}$ up to a non-zero scalar, where E is either $E_2^{1-a}E_{11212}^{1-c}E_{112}^{1-d}$ or else $E_2^{1-a}E_{11212}^{1-c}E_1$, which is a contradiction. Therefore B is a basis of $L'(\lambda)$.

Let W be a non-zero submodule of $L'(\lambda)$, $w \in W - 0$. By a similar argument there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{m(1,0,1,1,0)}$, so $\overline{m(1,0,1,1,0)} \in W$, but then $v_\lambda \in W$, so $W = L'(\lambda)$ and $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.20. *If $\lambda \in \mathfrak{I}_{13}$, then* $\dim L(\lambda) = 23$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{13} = \{\overline{m(a,b,0,d,0)} | b \leq 2\} \cup \{\overline{m(a,0,1,0,0)}, \overline{m(0,3,0,d,0)}, \overline{m(1,3,0,1,0)} | d \geq 1\}.$$

Proof. $W_1(\lambda)$ is a is a $\mathcal{U}$-submodule by Lemma 5.1.1(a). Also $w = F_2E_2E_{12}^3v_\lambda$ satisfies $F_1w = F_2w = 0$ by Remark 4.22, so w generates a proper submodule of $M(\lambda)$ by Remark 5.2.1. We have that $N(\lambda) = \mathcal{U}w + W_1(\lambda)$ and $L'(\lambda) = M(\lambda)/(\mathcal{U}w + W_1(\lambda))$ is simple and B_{13} is a basis of $L'(\lambda)$.

Consider $W = M(\lambda)/W_1(\lambda)$ and $v = \overline{m(1,3,1,2,0)}$ then $E_iv = 0$, $i = 1,2$, $g_1\sigma_1v = v$ and $g_2\sigma_2v = \zeta^9v$, so, using Lemma 5.2.4, $(\mathcal{U}v)^\varphi$ projects over a simple module $L(\mu)$ with μ corresponding to Case 14; in particular there exists $F \in \mathcal{U}_{-7\alpha_1-5\alpha_2}$ such that $Fv \neq 0$. Also $g_1\sigma_1w = \zeta^9w$, $g_2\sigma_2w = \zeta^4w$, so, using Lemma 5.2.4, $\mathcal{U}w$ projects over a simple module $L(\nu)$ with ν as in case 28. In particular there exists $E \in \mathcal{U}_{7\alpha_1+5\alpha_2}$ such that $Ew \neq 0$, so we may assume that $Ew = v$ since

$W_{10\alpha_1+8\alpha_2} = \mathbf{k}v$. Thus $\mathcal{U}v \subseteq \mathcal{U}w$, and then $Fv \in (\mathcal{U}v)_{3\alpha_1+3\alpha_2} \subseteq (\mathcal{U}w)_{3\alpha_1+3\alpha_2} = \mathbf{k}w$. As $Fv \neq 0$, we conclude that $\mathcal{U}v = \mathcal{U}w$. For any $v' \in W$, $v' \neq 0$, there exists $E' \in \mathcal{U}$ such that $E'v' = v$. Thus we conclude that $\mathcal{U}w$ is simple, and then $\dim L'(\lambda) = 48 - 25 = 23$.

From w and applying repeatedly E_1, E_2 over w we obtain

$$\begin{aligned}
\overline{m(0,3,0,0,0)} &= \frac{q_{12}^2\zeta^5(4)_{\zeta^7}}{3}\overline{m(1,1,0,1,0)} + \frac{q_{12}\zeta(1+\zeta^2)(3)_{\zeta^5}}{3}\overline{m(1,0,1,0,0)},\\
\overline{m(0,1,1,0,0)} &= q_{12}(1+\zeta^3)\overline{m(0,2,0,1,0)} + q_{12}^2\overline{m(1,0,0,2,0)},\\
\overline{m(0,0,1,1,0)} &= q_{12}\zeta^7(1+\zeta^2)\overline{m(0,1,0,2,0)},\\
\overline{m(0,2,1,0,0)} &= q_{12}^3\zeta^{10}(1-\zeta)\overline{m(1,1,0,2,0)},\\
\overline{m(0,3,1,0,0)} &= \zeta^4 q_{12}^4(1-\zeta)\overline{m(1,2,0,2,0)},\\
\overline{m(0,2,1,1,0)} &= q_{12}\zeta^7(1+\zeta^2)\overline{m(0,3,0,2,0)}.
\end{aligned}$$

Now we apply Corollary 5.3.2 to prove that $\overline{m(0,3,0,2,0)}$, $\overline{m(0,0,0,2,0)} \neq 0$, since

$$\begin{aligned}
F_{112}\overline{m(0,0,0,0,0)} &= 0, & g_1^2 g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,0)} &= \zeta^7\overline{m(0,0,0,0,0)},\\
F_{12}\overline{m(0,0,0,2,0)} &= 0, & g_1 g_2\sigma_1\sigma_2\overline{m(0,0,0,2,0)} &= \zeta^9\overline{m(0,0,0,2,0)}.
\end{aligned}$$

Moreover there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,3,0,2,0)} = v_\lambda$. Note that

$$E_2\overline{m(0,3,0,2,0)} = \overline{m(1,3,0,2,0)} = 0$$

since $0 = E_{12}\overline{m(0,3,1,0,0)}$ and $\mathbf{k}\overline{m(1,2,1,1,0)} = \mathbf{k}\overline{m(1,3,0,2,0)}$.

Also $E_1\overline{m(0,3,0,2,0)}$ is a non-zero scalar multiple of $\overline{m(0,1,1,2,0)} = 0$ since $0 = \overline{m(0,0,1,2,0)}$. Using this fact and the previous relations, B_{13} spans $L'(\lambda)$, but as B_{13} has 23 elements, it is a basis.

Let W be a non-zero submodule of $L'(\lambda)$, $w \in W - 0$. Arguing as in previous cases, there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{m(0,3,0,2,0)}$, so $\overline{m(0,3,0,2,0)} \in W$, but then $v_\lambda \in W$, so $W = L'(\lambda)$; and $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.21. *If $\lambda \in \mathfrak{I}_{14}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{14} = \{\overline{m(a,b,0,d,0)}\} \cup \{\overline{m(0,0,1,0,0)}, \overline{m(0,0,1,2,0)}\} - \{\overline{m(1,3,0,2,0)}\}.$$

Proof. $W_1(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a) [2]. Notice that

$$w = (1+\zeta^3)m(1,0,1,0,0) + q_{12}\zeta^3(1+\zeta)m(1,1,0,1,0)$$

satisfies the equations $F_1 w = F_2 w = 0$ by direct computation, so w generates a proper submodule of $M(\lambda)$ by Remark 5.2.1. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U}w + W_1(\lambda)$ is simple.

[2] Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

Applying repeatedly E_1, E_2 over w we obtain

$$\begin{aligned}
\overline{m(0,1,1,0,0)} &= \frac{q_{12}^2(1-\zeta^3)\zeta^2}{2}\overline{m(1,0,0,2,0)} + \frac{q_{12}\zeta^7(3)_\zeta}{2}\overline{m(0,2,0,1,0)},\\
\overline{m(0,0,1,1,0)} &= q_{12}\zeta\overline{m(0,1,0,2,0)},\\
\overline{m(1,2,1,0,0)} &= q_{12}\zeta^7(3)_\zeta\overline{m(0,2,0,2,0)},\\
\overline{m(1,0,1,2,0)} &= \zeta q_{21}\overline{m(0,3,0,2,0)},\\
\overline{m(1,3,1,0,0)} &= \overline{m(0,1,1,2,0)} = \overline{m(0,3,1,1,0)} = 0.
\end{aligned}$$

These relations imply that $L'(\lambda)$ is spanned by B_{14}. Now we apply Corollary 5.3.2 to prove that $\overline{m(1,0,1,2,0)} \neq 0$, since

$$\begin{aligned}
F_{112}\overline{m(0,0,0,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,0)} &= \zeta^3\overline{m(0,0,0,0,0)},\\
F_{11212}\overline{m(0,0,0,2,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,2,0)} &= \zeta^4\overline{m(0,0,0,2,0)},\\
F_2\overline{m(0,0,1,2,0)} &= 0, & g_2\sigma_2\overline{m(0,0,1,2,0)} &= \zeta^8\overline{m(0,0,1,2,0)}.
\end{aligned}$$

Moreover there exists $F \in \mathcal{U}^-$ such that $F\overline{m(1,0,1,2,0)} = v_\lambda$.

Now $E_2\overline{m(1,0,1,2,0)} = 0$, and $E_1\overline{m(1,0,1,2,0)}$ is a non-zero scalar multiple of $\overline{m(0,1,1,2,0)} = 0$.

Suppose that B_{14} is not linearly independent. Fix a non-trivial linear combination $\mathtt{S}$ which is zero. Between the elements of minimal N_0-degree with non-trivial coefficient, take the element $\overline{m} = \overline{m(a,b,c,d,0)}$ minimal for the lexicographical order; we may assume that $\overline{m}$ has coefficient 1. If $a = 1$, then $b \neq 3$ and $E_{12}^{2-b}E_{112}^{2-d}E_1\overline{m}$ gives $\overline{m(0,3,0,2,0)}$ up to a non-zero scalar. If $a = c = 0$, then $E_{12}^{3-b}E_{112}^{2-d}\overline{m}$ gives $\overline{m(0,3,0,2,0)}$ up to a non-zero scalar. If $a = 0$, $c = 1$, $E_2E_{112}^{2-d}\overline{m}$ gives $\overline{m(1,0,1,2,0)}$ up to a non-zero scalar. In any case we obtain $\overline{m(1,0,1,2,0)}$ up to a non-zero scalar, using the relation above. In any case there exists $E \in \mathcal{U}$ such that $0 = E\mathtt{S} = \overline{m(1,0,1,2,0)}$, which is a contradiction. Therefore B_{14} is a basis of $L'(\lambda)$.

Let $W \neq 0$ be a submodule of $L'(\lambda)$. Given $w \in W - 0$, there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{m(1,0,1,2,0)}$, so $\overline{m(1,0,1,2,0)} \in W$. Then $v_\lambda \in W$, so $W = L'(\lambda)$. $\square$

Lemma 5.4.22. *If $\lambda \in \mathfrak{I}_{15}$, then* $\dim L(\lambda) = 37$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{15} = \{\overline{m(a,b,c,d,0)}\} - \{\overline{m(a,b,1,d,0)} | b \geq 2, (a,b,d) \neq (0,2,2)\}.$$

Proof. $W_1(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a) [3]. Let $W = M(\lambda)/\mathcal{U}E_1v_\lambda$ then $w' = \overline{m(1,3,1,2,0)}$ satisfies $E_iw' = 0$, $i = 1,2$, $g_1\sigma_1w' = w'$ $g_2\sigma_2w' = \zeta^{11}w'$, so, using Lemma 5.2.4, $(\mathcal{U}w')^\varphi$ projects over an irreducible module $L(\nu)$ as in case 11. Thus $w = F_2F_{12}F_{112}^2w' \neq 0$ in W, by Lemma 5.2.5. Now $F_2w = 0$ and by direct computation

$$\begin{aligned}
F_1w = F_{12}^2F_{112}^2w' &= \lambda(g_1^2g_2)\zeta^4(1+\zeta)F_{12}^2F_{112}\overline{m(1,3,1,1,0)}\\
&= \lambda(g_1^4g_2^2)\zeta^4(1+\zeta)(\zeta^3-1)F_{12}^2\overline{m(1,3,1,0,0)} = 0.
\end{aligned}$$

[3]Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

Thus w satisfies

$$F_1 w = F_2 w = 0, \qquad g_1\sigma_1 w = 1, \qquad g_2\sigma_2 w = \zeta^4,$$

so $\mathcal{U}w$ projects over an irreducible module $L(\mu)$, for μ as in case 12.

As $w = Fw'$ for some $F \in \mathcal{U}$, $\mathcal{U}w \subset \mathcal{U}w'$. For any $0 \neq v \in W$, there exist $E_v \in \mathcal{U}$ such that $E_v v = w'$, so in particular $E_w w = w'$. In other words, $\mathcal{U}w = \mathcal{U}w'$. If $W' \subset W$ is a non-trivial submodule of $\mathcal{U}w$, then $w' \in \mathcal{U}w$ and $W' = \mathcal{U}w$. Thus $\mathcal{U}w = L(\mu)$.

Let $L'(\lambda) = M(\lambda)/W_1(\lambda) + \mathcal{U}w \simeq W/\mathcal{U}w$. Then $\dim L'(\lambda) = 37$, B_{15} is a basis of $L'(\lambda)$ since it is the image of a basis of a complement of $\mathcal{U}w$ in W. Notice that

$$\begin{aligned} F_{112}\overline{m(0,0,0,,0)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{m(0,0,0,0,0)} &= \zeta^9 \overline{m(0,0,0,0,0)}, \\ F_{11212}\overline{m(0,0,0,2,0)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{m(0,0,0,2,0)} &= \zeta^4 \overline{m(0,0,0,2,0)}, \\ F_{12}\overline{m(0,0,1,2,0)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{m(0,0,1,2,0)} &= \zeta^6 \overline{m(0,0,1,2,0)}, \end{aligned}$$

so by Corollary 5.3.2, $\overline{m(0,2,1,2,0)} \neq 0$ and there exist F such that $F\overline{m(0,2,1,2,0)} = v_\lambda$. Now for any $b \in \mathrm{B}_{15}$ there exist $E_b \in \mathcal{U}$ such that $E_b b = \overline{m(0,2,1,2,0)}$; as in the previous cases, $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.23. *If $\lambda \in \mathfrak{I}_{16}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{16} = \{\overline{m(a,b,c,d,0)}\} - \Big(\{\overline{m(a,3,c,d,0)} | d \geq 1\} \\ \cup \{\overline{m(1,2,1,2,0)}, \overline{m(0,2,1,2,0)}, \overline{m(1,2,0,2,0)}\}\Big).$$

Proof. $W_1(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a)[4]. Set $W = M(\lambda)/\mathcal{U}E_1 v_\lambda$, then $w' = \overline{m(1,3,1,2,0)}$ satisfies $E_i w' = 0$, $i = 1,2$, $g_1\sigma_1 w' = w'$, $g_2\sigma_2 w' = \zeta^8 w'$. By Lemma 5.2.4, $(\mathcal{U}w')^\varphi$ projects over a simple module $L(\mu)$, μ as in case 12. Thus $w = F_2 F_{11212} F_{112} w' \neq 0$ by Lemma 5.2.5. By direct computation

$$F_2 w = 0, \qquad F_1 w = F_{12} F_{11212} F_{112} \overline{m(1,3,1,2,0)} = 0,$$

so $\mathcal{U}w$ projects over an irreducible module $L(\mu)$, μ as in case 11.

Notice that $\mathcal{U}w \subseteq \mathcal{U}w'$. For any $v \in W$, $v \neq 0$, there exists E_v such that $E_v v = w'$; in particular $E_w w = w'$, and then $\mathcal{U}w = \mathcal{U}w'$. If $W' \subseteq W$ is a nontrivial submodule of $\mathcal{U}w$, then $w' \in \mathcal{U}w$ implies that $W' = \mathcal{U}w$. Thus $\mathcal{U}w$ is simple, so $\mathcal{U}w \simeq L(\mu)$. Let $L'(\lambda) = M(\lambda)/(\mathcal{U}E_1 v_\lambda + \mathcal{U}w) \cong W/\mathcal{U}w$. Then $\dim L'(\lambda) = \dim W - \dim \mathcal{U}w = 37$, and B_{16} is a basis of $L'(\lambda)$, since B_{16} spans a linear complement of $\mathcal{U}w$ in W. From case 11, $\mathcal{U}w$ has nontrivial components of dimension 1 in degrees $(5,4)$, $(5,5)$, $(6,5)$, $(7,5)$, $(7,6)$, $(8,6)$, $(9,6)$, $(8,7)$, $(9,7)$, $(10,7)$ and $(10,8)$. Thus $\overline{m(1,1,1,2,0)} \neq 0$ and $E_i\overline{m(1,1,1,2,0)} = 0$, $i = 1,2$. For each $b \in \mathrm{B}_{16}$ there exists E_b such that $E_b b = \overline{m(1,1,1,2,0)}$ by direct computation. Arguing as in the previous cases, we conclude that $L'(\lambda)$ is irreducible. $\square$

[4]Here $\lambda_1^2\lambda_2 = -1$, but the relation in Remark 5.4.5 becomes trivial.

Lemma 5.4.24. *If $\lambda \in \mathfrak{I}_{17}$, then* $\dim L(\lambda) = 47$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{17} = \{\overline{m(a,b,c,d,0)} | (a,b,c,d) \neq (1,3,1,2)\}.$$

Proof. $W_1(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a)[5].

Let $w = E_2 E_{12}^3 E_{11212} E_{112}^2 v_\lambda$; as $F_1 w \in M(\lambda)_{9\alpha_1 + 8\alpha_2}$ we have $F_1 w = 0$. Now we compute

$$\begin{aligned} F_2 w =& (\sigma_2^{-1} - g_2)\overline{m(0,3,1,2,0)} - E_2 E_{12}^3 (E_{112} E_1 g_2) E_{112}^2 v_\lambda + \\ & E_2 E_{12}^3 E_{11212} (3)_{\zeta^7} \zeta^4 E_{112} E_1^2 g_2 v_\lambda + E_2 \big((\zeta^{11} - 1)(3)_{\zeta^9} E_{12}^2 E_1 g_2 \\ & + q_{21} (\zeta^{11} - 1)\zeta^3 E_{12} E_{112} g_2 + q_{21}^2 (\zeta^{11} - 1) E_{11212} g_2 \big) E_{11212} E_{112}^2 v_\lambda \in W_1(\lambda). \end{aligned}$$

Therefore $W_1(\lambda) + \mathcal{U} w = N'(\lambda)$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/N'(\lambda)$ is irreducible. Notice that B_{17} is a basis of $L'(\lambda)$. As

$$\begin{aligned} F_{112}\overline{m(0,0,0,0,0)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{m(0,0,0,0,0)} &= \zeta^{10}\overline{m(0,0,0,0,0)}, \\ F_{11212}\overline{m(0,0,0,2,0)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{m(0,0,0,2,0)} &= \zeta^{10}\overline{m(0,0,0,2,0)}, \\ F_{12}\overline{m(0,0,1,2,0)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{m(0,0,1,2,0)} &= \zeta^{11}\overline{m(0,0,1,2,0)}, \end{aligned}$$

we have that $\overline{m(0,3,1,2,0)} \neq 0$ and there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,3,1,2,0)} = v_\lambda$, by Corollary 5.3.2. Moreover,

$$\begin{aligned} & E_{112}^{2-d} E_{11212}^{1-c} E_{12}^{3-b} \overline{m(0,b,c,d,0)}, & & \\ & E_{112}^{2-d} E_{11212}^{1-c} E_{12}^{2-b} E_1 \overline{m(1,b,c,d,0)}, & & \text{for } b < 3, \\ & E_{112}^{1-d} E_1^2 \overline{m(1,3,1,d,0)}, & & \text{for } d < 2, \end{aligned}$$

gives $\overline{m(0,3,1,2,0)}$ up to non-zero scalar. From here, every $w \in L'(\lambda)$, $w \neq 0$ generates $L'(\lambda)$, so $L'(\lambda)$ is irreducible. □

Lemma 5.4.25. *If $\lambda \in \mathfrak{I}_{18}$, then* $\dim L(\lambda) = 11$. *A basis of $L(\lambda)$ is given by given by*

$$\begin{aligned} \mathrm{B}_{18} = & \{\overline{m(a,b',1,0,1)}, \overline{m(0,b,0,0,e)} | e, b' \leq 1\} \cup \{\overline{m(1,0,0,0,0)}\} \\ & - \{\overline{m(1,1,1,0,1)}, \overline{m(0,3,0,0,1)}\}. \end{aligned}$$

The action of E_i, F_i, $i = 1,2$ is described in Table A.3.

Proof. $W_2(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a). Set $w = F_2 E_2 E_{12}$ then it satisfies the equations $F_1 w = F_2 w = 0$ by Remark 5.4.11, and $\mathcal{U} w + W_2(\lambda)$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U} w + W_2(\lambda)$ is irreducible. We fix the following notation for the elements of B_{18}:

$$\begin{aligned} v_{0,0} &= \overline{m(0,0,0,0,0)}, & v_{1,0} &= \overline{m(0,0,0,0,1)}, & v_{0,1} &= \overline{m(1,0,0,0,0)}, \\ v_{1,1} &= \overline{m(0,1,0,0,0)}, & v_{2,1} &= \overline{m(0,1,0,0,1)}, & v_{2,2} &= \overline{m(0,2,0,0,0)}, \\ v_{3,2} &= \overline{m(0,2,0,0,1)}, & v_{4,2} &= \overline{m(0,0,1,0,1)}, & v_{3,3} &= \overline{m(0,3,0,0,0)}, \\ v_{4,3} &= \overline{m(1,0,1,0,1)}, & v_{5,3} &= \overline{m(0,1,1,0,1)}. & & \end{aligned}$$

[5] Here $\lambda_1^2 \lambda_2 = \zeta^{10}$, but the relation in Remark 5.4.7 becomes trivial.

Notice that $v_{i,j} \in L'(\lambda)_{i\alpha_1+j\alpha_2}$. From $w = E_1w = E_2E_1w = E_{12}w = E_1^2w = 0$ we have

$$\overline{m(1,0,0,0,1)} = -q_{21}(4)_\zeta v_{1,1}, \qquad \overline{m(0,0,0,1,0)} = \frac{q_{12}\zeta^8(1+\zeta^3)(1+\zeta^2)}{3}v_{2,1},$$
$$\overline{m(1,0,0,1,0)} = \frac{q_{12}\zeta^8(1+\zeta^3)(1+\zeta^2)}{3}v_{2,2}, \quad \overline{m(1,1,0,0,1)} = q_{21}^2\zeta^4(4)_\zeta v_{2,2},$$
$$\overline{m(0,0,0,1,1)} = 0.$$

From $\mathcal{U}_{2\alpha_1+\alpha_2}w = 0$ we obtain

$$\overline{m(1,0,0,1,1)} = \overline{m(0,0,1,0,0)} = 0, \quad \overline{m(0,1,0,1,0)} = \frac{q_{12}\zeta^8(1+\zeta^3)(1+\zeta^2)}{3}v_{3,2}.$$

And the following relations also hold:

$$\overline{m(0,3,0,0,1)} = q_{12}(\zeta^{11}-1)\zeta^7 v_{4,3}, \qquad \overline{m(1,2,0,0,1)} = q_{12}(4)_\zeta\zeta^{10}v_{3,3}$$
$$\overline{m(0,2,0,1,0)} = \frac{q_{12}\zeta^{11}(1+\zeta^2)(3)_{\zeta^7}}{3}v_{4,3}.$$

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,1,1,0,1)} = v_\lambda$, since

$$F_{11212}\overline{m(0,0,0,0,1)} = 0, \qquad g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,0,1)} = \zeta^8\overline{m(0,0,0,0,1)},$$
$$F_{12}\overline{m(0,0,1,0,1)} = 0, \qquad g_1g_2\sigma_1\sigma_2\overline{m(0,0,1,0,1)} = \zeta^3\overline{m(0,0,1,0,1)}.$$

Notice that $\overline{m(0,0,0,1,1)} = 0$, so $E_1\overline{m(0,1,1,0,1)} = 0$. As $E_{12}E_{11212}w = 0$, we also have $E_2\overline{m(0,1,1,0,1)} = \overline{m(1,1,1,0,1)} = 0$.

We claim that B_{18} is a basis of $L'(\lambda)$. Using the relations above we prove that $L'(\lambda)$ is spanned by B_{18}. From Table A.3 there exist $E_{i,j} \in \mathcal{U}^+_{(5-i)\alpha_1+(3-j)\alpha_2}$, $F_{5,3} \in \mathcal{U}^-_{-5\alpha_1-3\alpha_2}$ such that $E_{i,j}v_{i,j} = v_{5,3}$, $F_{5,3}v_{5,3} = v_\lambda$. Assume that there exists a non-trivial linear combination $\mathtt{S}$ which is zero. If $v_{i,j}$ is of minimal degree with non-trivial coefficient, then $E_{i,j}\mathtt{S} = v_{5,3}$, a contradiction. Finally $L'(\lambda)$ is irreducible by a similar argument. $\square$

Lemma 5.4.26. *If $\lambda \in \mathfrak{I}_{19}$, then* $\dim L(\lambda) = 35$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{19} = \{\overline{m(0,b,0,d,e)} | e \le 1\} \cup \{\overline{m(1,b,0,0,e)} |\ b,e \le 1\} \cup \{\overline{m(0,b,1,0,0)} |\ b \ge 1\}$$
$$\cup \{\overline{m(1,b,0,0,1)} \mid b \ge 2,3\} \cup \{\overline{m(1,0,0,1,1)}, \overline{m(0,0,1,1,0)}\}.$$

Proof. $W_2(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a). Set $W = M(\lambda)/W_2(\lambda)$. By Remark 5.4.13 $w = F_2E_2E_{12}^2v_\lambda$ satisfies the equations $F_1w = F_2w = 0$. As also $g_1\sigma_1w = \zeta^{10}w$, $g_2\sigma_2w = -w$, so $\mathcal{U}w$ projects over $L(\mu)$ for μ as in case 32. Thus $E_2E_{12}^3E_{112}^2E_1^2w \neq 0$ by Lemma 5.2.5, but this vector is $\overline{m(1,3,1,2,1)}$ up to a non-zero scalar since $W_{11\alpha_1+8\alpha_2} = \mathbf{k}\overline{m(1,3,1,2,1)}$, and then $\mathcal{U}w = \mathcal{U}\overline{m(1,3,1,2,1)}$. Moreover, there exists $F \in \mathcal{U}$ such that $F\overline{m(1,3,1,2,1)} = w$, so $\mathcal{U}w \subseteq \mathcal{U}\overline{m(1,3,1,2,1)}$. Moreover, for any $v \in W$, $v \neq 0$, there exists $E_v \in \mathcal{U}$ such that $E_vv = \overline{m(1,3,1,2,1)}$, so if $V \subset \mathcal{U}w$ is a submodule, $V \neq 0$, then $\overline{m(1,3,1,2,1)} \in V$ and this implies that

$V = \mathcal{U}w$. That is, $\mathcal{U}w$ is irreducible, so $\mathcal{U}w \simeq L(\mu)$. Set $L'(\lambda) = W/\mathcal{U}w = M(\lambda)/\mathcal{U}w + W_2(\lambda)$, so $\dim L'(\lambda) = 96 - 61 = 35$ and B_{19} is a basis of $L'(\lambda)$, since it spans a complement of $\mathcal{U}w$ in W.

Notice that $E_i\overline{m(0,3,0,2,1)} = 0$, $i = 1,2$, and for any $b \in \mathrm{B}_{19}$ there exists $E_b \in \mathcal{U}$ such that $E_b b = \overline{m(0,3,0,2,1)}$ up to a non-zero scalar., so arguing as in the previous cases, $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.27. *If $\lambda \in \mathfrak{I}_{20}$, then $\dim L(\lambda) = 71$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{20} =&\{\overline{m(a,b,c,d,e)}|e \le 1\} - \Big(\{\overline{m(1,b,1,d,e)}|e \le 1, (b,d,e) \neq (2,2,1)\}\\ &\cup \{\overline{m(1,0,0,2,1)}, \overline{m(1,3,0,0,0)}\}\Big).\end{aligned}$$

Proof. $W_2(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a). Set $w = F_2E_{12}^3E_2v_\lambda$, so $F_1w = F_2w = 0$ by Remark 5.4.15 and $N'(\lambda) = W_2(\lambda) + \mathcal{U}w$ is a proper submodule. Set $L'(\lambda) = M(\lambda)/N'(\lambda)$. We claim that $L'(\lambda)$ is irreducible. First we prove that B_{20} generates the module. From $E_1^e w = 0$, $e = 0,1,2$, respectively, we obtain

$$\begin{aligned}\overline{m(1,0,1,0,0)} =&q_{12}^2\zeta^5\overline{m(1,2,0,0,1)} + q_{12}\zeta^{10}\overline{m(1,1,0,1,0)} + q_{21}\zeta^9(4)_\zeta\overline{m(0,3,0,0,0)}\\ \overline{m(1,0,1,0,1)} =&q_{12}\zeta^9\overline{m(1,1,0,1,1)} + q_{21}\zeta^8(1+\zeta)\overline{m(0,3,0,0,1)} + q_{21}\zeta^2\overline{m(1,0,0,2,0)}\\ &+ q_{21}^2\zeta^2(1+\zeta)\overline{m(0,2,0,1,0)} + q_{21}^3\zeta(1+2\zeta^3)\overline{m(0,1,1,0,0)}\\ \overline{m(1,0,0,2,1)} =&q_{21}\zeta^7\overline{m(0,2,0,1,1)} + q_{21}^2(\zeta^2+\zeta^5+2)\overline{m(0,1,1,0,1)}\\ &+ q_{21}^3\zeta(\zeta^9+2\zeta^{10}+2)\overline{m(0,1,0,2,0)} + q_{21}^4\zeta^9(\zeta+\zeta^2+2)\overline{m(0,0,1,1,0)}.\end{aligned}$$

We apply $E_{12}^bE_{112}^dE_1^e$, with $(b,d,e) \neq (2,2,1)$ to w and obtain $\overline{m(1,b,1,d,e)}$ as a linear combination of elements of B_{20}; and applying E_2 to w we see that $\overline{m(1,3,0,0,0)} = 0$.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,3,1,2,1)} = v_\lambda$, since

$$\begin{aligned}F_{112}\overline{m(0,0,0,0,1)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,1)} &= -\overline{m(0,0,0,0,1)},\\ F_{11212}\overline{m(0,0,0,2,1)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,2,1)} &= -\overline{m(0,0,0,2,1)},\\ F_{12}\overline{m(0,0,1,2,1)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{m(0,0,1,2,1)} &= \zeta^{11}\overline{m(0,0,1,2,1)}.\end{aligned}$$

Note that $E_1\overline{m(0,3,1,2,1)} = 0$, and $\overline{m(1,3,1,2,1)} = 0$ by computing $E_{12}^3E_{112}^2E_1w$, so $E_2\overline{m(0,3,1,2,1)} = \overline{m(1,3,1,2,1)} = 0$. Suppose that B_{20} is not linearly independent. Fix $\mathtt{S} = 0$ a non-trivial linear combination, and consider the minimal element $\overline{m(a,b,c,d,e)}$ among those with non trivial coefficient and minimal $\mathbb{N}_0$-degree. If it is $\overline{m(1,2,1,2,1)}$, applying E_1 to $\mathtt{S}$ we obtain $\overline{m(0,3,1,2,1)}$. If $a = 1$, $c = 0$, then $E_{12}^{2-b}E_{112}^{2-d}E_1^{1-e}E_1E_{112}\mathtt{S}$ gives $\overline{m(0,3,1,2,1)}$ up to a non-zero scalar; for the other cases we use $E_{12}^{3-b}E_{11212}^eE_{112}^{2-d}E_1^{1-e}\mathtt{S}$ to obtain the same conclusion. In any case we have $\overline{m(0,3,1,2,1)} = 0$, which is a contradiction. Therefore B_{20} is a basis of $L'(\lambda)$. Let $W \neq 0$ be a submodule of $L'(\lambda)$, $w \in W - 0$. By a similar argument there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{m(0,3,1,2,1)}$, so $\overline{m(0,3,1,2,1)} \in W$. Then $v_\lambda \in W$ and $W = L'(\lambda)$ and $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.28. *If $\lambda \in \mathfrak{I}_{21}$, then* $\dim L(\lambda) = 61$. *A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{21} =&\{\overline{m(a,b,c,d,e)} \mid b,e \leq 1\} \cup \{\overline{m(a,2,c,0,e)}, \mid e \leq 1\}\\ &\cup \{\overline{m(1,3,0,0,e)} \mid e \leq 1\} \cup \{\overline{m(a,3,1,0,1)}, \overline{m(0,2,0,1,0)}\}.\end{aligned}$$

Proof. If μ is in case 19, then $v = \overline{m(0,3,0,2,1)}$ satisfies $E_i v = 0$, $i = 1,2$, $g_1\sigma_1 v = \zeta^7 v$, $g_2\sigma_2 v = 1$ so $L(\mu)^\varphi \simeq L(\lambda)$ by Lemma 5.2.5. In particular $\dim L(\lambda) = 61$.

$W_2(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a) [6]. Set $W = M(\lambda)/W_2(\lambda)$, $w' = \overline{m(1,3,1,2,1)}$. Notice that $E_i w' = 0$, $i = 1,2$, $g_1\sigma_1 w' = \zeta^4 w'$, $g_2\sigma_2 w' = \zeta^4 w'$, so $(\mathcal{U}w')^\varphi$ projects over an irreducible $L(\nu)$ for ν as in Case 19 by Lemma 5.2.4. We claim that $\mathcal{U}w'$ is a proper submodule. Assume on the contrary that $\mathcal{U}w' = W$. For any $v \in W$, $v \neq 0$, there exists $E_v \in \mathcal{U}$ such that $E_v v = w'$, so if $V \subset W$ is a non-zero submodule, then $w' \in V$ and thus $V = W$. Then W is irreducible and $W^\varphi \twoheadrightarrow L(\nu)$, so $W^\varphi \simeq L(\nu)$, but they have different dimension, a contradiction. Let $L'(\lambda) = W/\mathcal{U}w'$. Then

$$\dim L(\lambda) \leq \dim L'(\lambda) = \dim W - \dim \mathcal{U}w' \leq \dim W - \dim L(\nu) = 96 - 35 = 61,$$

so $L'(\lambda) = L(\lambda)$ and $\mathcal{U}w' \simeq L(\nu)$. Moreover $\mathcal{U}w' = \mathcal{U}w$ for $w = F_1F_{11212}F_{12}w'$.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(1,1,1,2,1)} = v_\lambda$, since

$$\begin{aligned}
F_{112}\overline{m(0,0,0,0,1)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,0,0,1)} &= \zeta^{10}\overline{m(0,0,0,0,1)},\\
F_{11212}\overline{m(0,0,0,2,1)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,2,1)} &= \zeta^{2}\overline{m(0,0,0,2,1)},\\
F_{12}\overline{m(0,0,1,2,1)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{m(0,0,1,2,1)} &= \zeta^{3}\overline{m(0,0,1,2,1)},\\
F_{2}\overline{m(0,1,1,2,1)} &= 0, & g_2\sigma_2\overline{m(0,1,1,2,1)} &= -\overline{m(0,1,1,2,1)}.
\end{aligned}$$

Note that $E_2\overline{m(1,1,1,2,1)} = 0$. From $E_{11212}E_{112}E_1^2 w = 0$ we get $\overline{m(0,2,1,2,1)} = 0$, so $E_1\overline{m(1,1,1,2,1)} = \overline{m(0,2,1,2,1)} = 0$. Suppose that B_{21} is not linearly independent. Take a non-trivial linear combination $\mathtt{S}$ which is zero, and take the minimal element $\overline{m(a,b,c,d,e)}$ among those with non trivial coefficient, between the elements of minimal $\mathbb{N}_0$-degree. If $b = 3$, then $d = 0$ and $E_1^{1-e}E_{11212}^{1-c}E_2^{1-a}E_1^2\overline{m(a,3,c,0,e)}$ gives $\overline{m(1,1,1,2,1)}$ up to a non-zero scalar, since

$$E_1^2E_{12}^2 = \zeta^4 q_{12}^4 E_{12}^2E_1^2 + q_{12}^2E_{11212}E_1 - q_{12}E_{112}^2 + q_{12}^3\zeta(1+\zeta^3)E_{12}E_{112}E_1.$$

If $b = 2$, then either $d = 0$ and $E_{112}E_1^{2-e}E_{11212}^{1-c}E_2^{1-a}\overline{m(a,2,c,0,e)}$ gives $\overline{m(1,1,1,2,1)}$ up to a non-zero scalar, or else $E_2E_1E_{11212}E_1\overline{m(0,2,0,1,0)}$ also gives $\overline{m(1,1,1,2,1)}$ up to a non-zero scalar. Otherwise $E_2^{1-a}E_{12}^{1-b}E_{11212}^{1-c}E_{112}^{2-d}E_1^{1-e}\overline{m(a,b,c,d,e)}$ gives $\overline{m(1,1,1,2,1)}$ up to a non-zero scalar. In any case we conclude that $\overline{m(1,1,1,2,1)} = 0$ up to multiply $\mathtt{S}$ by an appropriate element of $\mathcal{U}^+$, a contradiction. Therefore B_{21} is a basis of $L'(\lambda)$. $\square$

Lemma 5.4.29. *If $\lambda \in \mathfrak{I}_{22}$, then* $\dim L(\lambda) = 49$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{22} =\{\overline{m(a,b,c,d,e)} | \ d,e \neq 2\} - \{\overline{m(a,b',1,0,0)}, \overline{m(1,3,1,1,1)}, \overline{m(a,b,1,1,0)} \mid b \neq 0\}.$$

[6] Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

Proof. $W_2(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a) [7]. Set $W = M(\lambda)/W_2(\lambda)$, then $\{\overline{m(a,b,c,d,e)} \mid e \neq 2\}$ is a basis of W. Let $w = F_1^2 E_{112}^2 E_1 v_\lambda$, now $F_1 w = 0$,

$$\begin{aligned} F_2 w =& q_{11}^{-1} q_{12}^{-2} F_{112} E_{112}^2 E_1 v\lambda = \zeta^{10} q_{21}^2 (1+\zeta^8) E_{112} (\zeta^4 \sigma_{112}^{-1} - g_{112}) E_1 v_\lambda \\ &+ q_{21}^2 \zeta^{10} \lambda(\sigma_{112}^{-1})(\zeta^4 q_{11}^{-2} q_{12}^{-1} - \lambda_1^2 \lambda_2 q_{11}^2 q_{21}) E_{112} E_1 v\lambda = 0, \end{aligned}$$

so $\mathcal{U}w$ is a proper submodule. Consider $L'(\lambda) = W/\mathcal{U}w$.

From $\underline{E_2^a E_{12}^b E_{11212}^c E_1^e w}$ with $e \neq 0$, $\underline{E_1^a E_{12}^b w}$, $\underline{E_2^a E_{12}^b w}$, with $b \neq 3$ and $\underline{E_{12}^3 E_{11212} w}$, we write $\overline{m(a,b,c,2,e-1)}$, $\overline{m(a,b,1,0,0)}$, $\overline{m(a,b+1,1,1,0)}$ and $\overline{m(1,3,1,1,1)}$ as a linear combination of elements of B_{22}.

Now there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,3,1,1,1)} = v_\lambda$ by Corollary 5.3.2, since

$$\begin{aligned} F_{112}\overline{m(0,0,0,0,1)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{m(0,0,0,0,1)} &= \zeta^4 \overline{m(0,0,0,0,1)}, \\ F_{11212}\overline{m(0,0,0,1,1)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{m(0,0,0,1,1)} &= \zeta^9 \overline{m(0,0,0,1,1)}, \\ F_{12}\overline{m(0,0,1,1,1)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{m(0,0,1,1,1)} &= \zeta^8 \overline{m(0,0,1,1,1)}. \end{aligned}$$

As $\overline{m(0,2,1,2,1)} = \overline{m(1,3,1,1,1)} = 0$ by applying $E_{12}^3 E_{11212}$ and $E_{12}^2 E_{11212} E_1^2$ to w, respectively, we have that

$$E_1\overline{m(0,3,1,1,1)} = q_{12}^3 \zeta^2 \overline{m(0,2,1,2,1)} = 0, \quad E_2\overline{m(0,3,1,1,1)} = \overline{m(1,3,1,1,1)} = 0.$$

Suppose that B_{22} is not linearly independent. Let $\mathtt{S}$ be a non-trivial linear combination which is zero, and take the minimal element $\overline{m(a,b,c,d,e)}$ among those with non-zero coefficient and minimal $\mathbb{N}_0$-degree. If it is $\overline{m(1,b,c,d,e)}$, $b \leq 2$, then $E_1^{1-e} \underline{E_{112}^{1-d} E_{11212}^{1-c} E_{12}^{2-b}} E_1 \mathtt{S}$ is $\overline{m(0,3,1,1,1)}$. If it is $\overline{m(1,3,0,0,e)}$, then $E_1^{2-e} E_{12} E_{112} E_1 \mathtt{S}$ gives $\overline{m(0,3,1,1,1)}$ up to a non-zero scalar, since

$$E_1\overline{m(1,3,0,0,e)} = q_{12}^2 \zeta^{10} \overline{m(1,1,1,0,e)} + q_{12}^3 \zeta^5 \overline{m(1,2,0,1,e)} + q_{12}^4 \overline{m(1,3,0,0,e+1)}.$$

Otherwise $E_{12}^{3-b} E_{11212}^{1-c} \underline{E_{112}^{1-d} E_1^{1-e} \mathtt{S}}$ gives again $\overline{m(0,3,1,1,1)}$ up to a non-zero scalar In any case we have that $\overline{m(0,3,1,1,1)} = 0$, which is a contradiction. Therefore B_{22} is a basis of $L'(\lambda)$. Moreover $L'(\lambda)$ is irreducible by an argument as in the previous cases. □

Lemma 5.4.30. *If $\lambda \in \mathfrak{I}_{23}$, then* $\dim L(\lambda) = 47$. *A basis of $L(\lambda)$ is given by*

$$\begin{aligned} \mathrm{B}_{23} =& \Big(\{\overline{m(a,b,0,d,e)} | e \leq 1\} \cup \{\overline{m(a,b,1,0,0)} \mid b \leq 1\} \\ &\cup \{\overline{m(0,2,1,0,0)}, \overline{m(1,3,1,0,0)}\} \Big) \\ &- \Big(\{\overline{m(1,b,0,1,e)} | b \leq 2, e \leq 1\} \cup \{\overline{m(0,2,0,2,0)}\} \Big). \end{aligned}$$

Proof. $W_2(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a). Set $w = F_2 E_2 E_{12}^2 v_\lambda$, then $F_1 w = F_2 w = 0$, by Remark 5.4.13, so $N'(\lambda) = W_2(\lambda) + \mathcal{U}w$ is a proper submodule. Let $L'(\lambda) = M(\lambda)/N'(\lambda)$. We claim that $L'(\lambda)$ is irreducible and B_{23} is a basis of

[7] Here $\lambda_1^2 \lambda_2 = -1$, but the relation in Remark 5.4.5 becomes trivial.

$L'(\lambda)$. First we prove that $L'(\lambda)$ is spanned by B_{23}. From $E_1^e w = 0$, $e = 0,1,2$, $E_{12}E_1 w = 0$ and $\mathcal{U}_{4\alpha_1+2\alpha_2} w = 0$ we obtain the relations

$$
\begin{aligned}
\overline{m(1,0,0,1,0)} &= \frac{q_{12}\zeta^2(1+\zeta^3)(\zeta^4-1)}{3}\overline{m(1,1,0,0,1)},\\
\overline{m(1,0,0,1,1)} &= q_{21}^2(4)_\zeta\overline{m(0,1,0,1,0)} + q_{21}\zeta^2(3)_\zeta\overline{m(0,2,0,0,1)},\\
\overline{m(0,0,1,0,1)} &= \frac{q_{12}(\zeta^9-1)}{2}\overline{m(0,1,0,1,1)} + \frac{q_{21}(\zeta^8-1)(1+\zeta^3)\zeta^8}{2}\overline{m(0,0,0,2,0)},\\
\overline{m(1,1,0,1,1)} &= q_{21}^3\zeta^7(4)_\zeta\overline{m(0,1,0,1,0)} + q_{21}^2\zeta^9(3)_\zeta\overline{m(0,2,0,0,1)},\\
\overline{m(0,2,0,2,0)} &= 2q_{12}^2\zeta^{10}\overline{m(0,3,0,1,1)}.
\end{aligned}
$$

We write $\overline{m(a,b,1,d,1)}$, $\overline{m(a,b,1,1,0)}$, $\overline{m(a,b,1,2,0)}$ and $\overline{m(1,b,0,1,e)}$, $b \geq 2$, as a linear combination of elements of B_{23} by applying $E_2^a E_{12}^b E_{112}^d$ to the third relation, $E_2^a E_{12}^b E_{112}$ to the fourth relation, $E_2^a E_{12}^{b-1} E_{11212} E_{112} E_1$ to the first relation and $E_{12}^b E_{112} E_1^e$ to w, respectively.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(1,3,0,2,1)} = v_\lambda$ since

$$
\begin{aligned}
F_{112}\overline{m(0,0,0,0,1)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{m(0,0,0,0,1)} &= \zeta^9 \overline{m(0,0,0,0,1)},\\
F_{12}\overline{m(0,0,0,2,1)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{m(0,0,0,2,1)} &= \zeta^7 \overline{m(0,0,0,2,1)},\\
F_2\overline{m(0,3,0,2,1)} &= 0, & g_2 \sigma_2 \overline{m(0,3,0,2,1)} &= -\overline{m(0,3,0,2,1)}.
\end{aligned}
$$

Note that $E_1\overline{m(1,3,0,2,1)} = q_{12}\overline{m(1,3,0,2,2)} = 0$ and $E_2\overline{m(1,3,0,2,1)} = 0$. Suppose that B_{23} is not linearly independent. Fix $\mathtt{S} = 0$ a non-trivial linear combination, and consider the minimal element $\overline{m(a,b,c,d,e)}$ among those with non trivial coefficient and minimal $\mathbb{N}_0$-degree. If $c = 1$, then $E_2^{1-a}E_{12}^{3-b}E_1^{1-e}\overline{m(a,b,1,d,e)}$ gives $\overline{m(1,3,0,2,1)}$ up to a non-zero scalar by using the third relation. For the other cases $E_2^a E_{12}^{3-b} E_{112}^{2-d} E_1^{1-e}\overline{m(a,b,0,d,e)}$ gives the same conclusion. In any case we have $\overline{m(1,3,0,2,1)} = 0$, which is a contradiction, so B_{23} is a basis of $L'(\lambda)$. Let W be a non-zero submodule of $L'(\lambda)$, $w \in W - 0$. By a similar argument there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{m(1,3,0,2,1)}$, so $\overline{m(1,3,0,2,1)} \in W$, but then $v_\lambda \in W$ and $W = L'(\lambda)$ and $L'(\lambda)$ is irreducible. □

Lemma 5.4.31. *If* $\lambda \in \mathfrak{I}_{24}$, *then* $\dim L(\lambda) = 85$. *A basis of* $L(\lambda)$ *is given by*

$$
\mathrm{B}_{24} = \{\overline{m(a,b,c,d,e)} | e \leq 1\} - \{\overline{m(a,3,c,2,e)}, \overline{m(1,3,c,1,1)}, \overline{m(0,3,1,1,1)}\}.
$$

Proof. If $\mu \in \mathfrak{I}_{35}$, then $w' = \overline{n(1,2,1,2,1)}$ satisfies $E_i w' = 0$, $i = 1,2$, $g_1\sigma_1 w' = \zeta^4 w'$, $g_2\sigma_2 w' = -w'$, so $L(\mu)^\varphi$ is isomorphic to $L(\lambda)$ by Lemma 5.2.4. In particular $\dim L(\lambda) = 85$.

Note that $W_2(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(a) [8]. Set $W = M(\lambda)/W_2(\lambda)$ and $w' = \overline{m(1,3,1,2,1)}$. Then $E_i w' = 0$, $i = 1,2$, $g_1\sigma_1 w' = \zeta^4 w'$, $g_2\sigma_2 w' = \zeta^7 w'$, so by Lemma 5.2.4 $(\mathcal{U}w')^\varphi$ projects over an irreducible module $L(\nu)$, $\nu \in \mathfrak{I}_{18}$.

[8] Here $\lambda_1^2\lambda_2 = \zeta^{10}$, but the relation in Remark 5.4.7 becomes trivial.

$\mathcal{U}w'$ is a proper submodule; otherwise $\mathcal{U}w' = W$ is irreducible, so $L(\nu) \simeq (\mathcal{U}w')^{\varphi} = W$, a contradiction since they have different dimension. Set $L'(\lambda) = M(\lambda)/W_2(\lambda) + \mathcal{U}w$. Then

$$85 = \dim L(\lambda) \leq \dim L(\lambda) \leq \dim W - \dim L(\nu) = 96 - 11 = 85,$$

so $L'(\lambda) = L(\lambda)$ and $L(\nu) \simeq (\mathcal{U}w')^{\varphi}$. In particular $w = F_{12}F_{11212}F_1\overline{m(1,3,1,2,1)}$ satisfies that $F_1 w = F_2 w = 0$.

We claim that B_{24} is a basis of $L'(\lambda)$. From $w = E_1 w = 0$, $E_{12}E_1 w = 0$ and $\mathcal{U}_{4\alpha_1+2\alpha_2} w = 0$ we obtain the relations

$$\begin{aligned}\overline{m(1,3,0,1,1)} =& q_{21}\zeta^4\overline{m(1,2,1,0,1)} + \frac{q_{21}^2(1+\zeta^2)(1+\zeta^3)}{2}\overline{m(1,2,0,2,0)},\\ \overline{m(0,3,0,2,0)} =& \frac{q_{12}^5\zeta(4)_\zeta(\zeta+1)}{3}\overline{m(1,2,0,2,1)} - q_{12}\overline{m(1,0,1,2,0)}\\ &- \frac{q_{12}^4(1+\zeta^2)(3)_\zeta}{3}\overline{m(1,1,1,1,1)}.\end{aligned}$$

Then we can use $E_2^a E_{11212}^c E_1^{e+1} w = 0$, $E_{112}E_1 w$ and $E_{11212}w = 0$ to write $\overline{m(a,3,c,2,e)}$, $\overline{m(0,3,1,1,1)}$ and $\overline{m(1,3,1,1,1)}$ as a linear combination of elements of B_{24}. Thus $L'(\lambda)$ is spanned by B_{24}, and then B_{24} is a basis of $L'(\lambda)$ since it has 85 elements. $\square$

Lemma 5.4.32. *If $\lambda \in \mathfrak{I}_{25}$, then* $\dim L(\lambda) = 37$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{25} = \{\overline{m(a,b,c,0,e)}\} - \Big(\{\overline{m(0,3,0,0,e)} \mid e \leq 1\} \cup \{\overline{m(1,3,c,0,e)}, \overline{m(1,2,1,0,e)}\}\Big).$$

Proof. Let $w_1 = F_1^2 E_{112} E_1^2 v_\lambda$. By Remark 5.4.5, $F_i w_1 = 0$, $i = 1,2$. Set $W = M(\lambda)/\mathcal{U}w_1$, so $\mathrm{B}' = \{\overline{m(a,b,c,0,e)}\}$ is a basis of W. In particular, $W_{a\alpha_1+b\alpha_2} = 0$ if either $a \geq 9$ or else $b \geq 7$, and $W_{8\alpha_1+6\alpha_2} = \mathbf{k}\{\overline{m(1,3,1,0,2)}\}$. By Lemma 5.2.4 $w_2 = E_2 E_{12}^3 v_\lambda$ satisfies $F_i w_2 = 0$, $i,j = 1,2$. As $g_1\sigma_1 w_2 = \zeta^8 w_2$, $g_2\sigma_2 w_2 = \zeta^5 w_2$, $\mathcal{U}w_2$ projects over $L(\nu)$, where ν is as in case 38. By Lemma 5.2.5 $E_{12}E_{11212}E_1 w_2 \neq 0$, and this vector is $\{\overline{m(1,3,1,0,2)}\}$ up to non-zero scalar. Moreover, there exists F such that $F\overline{m(1,3,1,0,2)} = w_2$, and then $\mathcal{U}w_2 = \mathcal{U}\overline{m(1,3,1,0,2)}$. For each $v \in W$, $v \neq 0$, there exists $E_v \in \mathcal{U}$ such that $E_v v = \overline{m(1,3,1,0,2)}$. From here we conclude that $\mathcal{U}w_2$ is simple, so $\mathcal{U}w_2 \cong L(\nu)$. Thus $L'(\lambda) = W/\mathcal{U}w_2$ has dimension $48 - 11 = 37$ and B_{25} is a basis of $L'(\lambda)$ since it spans a linear complement of $\mathcal{U}w_2$ in W.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,3,1,0,2)} = v_\lambda$, since

$$\begin{aligned} F_{11212}\overline{m(0,0,0,0,2)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{m(0,0,0,0,2)} &= \zeta^9 \overline{m(0,0,0,0,2)},\\ F_{12}\overline{m(0,0,1,0,2)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{m(0,0,1,0,2)} &= \zeta^4 \overline{m(0,0,1,0,2)}.\end{aligned}$$

Notice that $E_i\overline{m(0,3,1,0,2)} = 0$, for $i = 1,2$ since $L'(\lambda)_{8\alpha_1+5\alpha_2+\alpha_i} = 0$.

For each $v = \overline{m(a,b,c,0,e)} \in \mathrm{B}_{25}$ there exists $E_v \in \mathcal{U}$ such that $E_v v = \overline{m(0,3,1,0,2)}$. Indeed, if $a = 1$, then $b \leq 2$ and $E_1^{2-e}E_{11212}^{1-c}E_{12}^{2-b}E_1 v$ gives $\overline{m(0,3,1,0,2)}$ up to a non-zero scalar. Otherwise $E_1^{2-e}E_{11212}^{1-c}E_{12}^{3-b} v$ gives the same conclusion. Arguing as above, $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.33. *If* $\lambda \in \mathfrak{I}_{26}$*, then* $\dim L(\lambda) = 25$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{26} = \{\overline{m(0,b,c,0,e)}\} \cup \{\overline{m(1,0,0,0,0)}, \overline{m(1,0,0,0,2)}\} - \{\overline{m(0,3,1,0,0)}\}.$$

Proof. Set $w_1 = F_2E_2E_{12}v_\lambda$, $w_2 = F_1^2E_{112}E_1^2v_\lambda$. By Remarks 5.4.11 and 5.4.5, $F_iw_j = 0$, $i,j = 1,2$, so $\mathcal{U}w_1 + \mathcal{U}w_2$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$ is irreducible. First we prove that $L'(\lambda)$ is spanned by B_{26}. As $w_1 = w_2 = 0$,

$$\overline{m(1,0,0,0,1)} = q_{21}(1+\zeta)^2\zeta^8\overline{m(0,1,0,0,0)},$$
$$\overline{m(0,0,0,1,0)} = q_{12}(3)_{\zeta^7}\overline{m(0,1,0,0,1)}.$$

As $E_{12}^2E_{11212}w_1 = E_2E_{12}E_{11212}E_1w_1 = 0$, we have $\overline{m(0,3,1,0,0)} = \overline{m(1,2,1,0,1)} = 0$. We write $\overline{m(a,b,c,d,e)}$, $d \geq 1$, and $\overline{m(1,b,c,d,e)}$, $e \geq 1$, as a linear combination of elements of B_{26} by applying $E_2^aE_{12}^bE_{11212}^cE_{112}^{d-1}E_1^e$ to the second relation and $E_{12}^bE_{11212}^cE_{112}^dE_1^{e-1}$ to the first relation, respectively. We express $\overline{m(1,b,c,0,0)}$, $b \geq 1$, and $\overline{m(1,b,c,0,0)}$, $c \geq 1$, as a linear combination of the elements of B by applying $E_2E_{12}^{b-1}E_{11212}^cE_1$ and $E_2E_{12}^bE_{112}$ to w_1, respectively. Then $L'(\lambda)$ is spanned by B_{26}. By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,3,1,0,2)} = v_\lambda$, since

$$F_{11212}\overline{m(0,0,0,0,2)} = 0, \qquad g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{m(0,0,0,0,2)} = \zeta\overline{m(0,0,0,0,2)},$$
$$F_{12}\overline{m(0,0,1,0,2)} = 0, \qquad g_1g_2\sigma_1\sigma_2\overline{m(0,0,1,0,2)} = \zeta^{10}\overline{m(0,0,1,0,2)}.$$

As $\mathcal{U}_{7\alpha_1+4\alpha_2}w_2 = 0$, we have that $E_1\overline{m(0,3,1,0,2)} = 0$, and from $E_{12}^3E_{11212}E_1w_1 = 0$, $E_2\overline{m(0,3,1,0,2)} = \overline{m(1,3,1,0,2)} = 0$.

Suppose that B_{26} is not linearly independent. Fix $\mathrm{S} = 0$ a non-trivial linear combination, and consider the minimal element $\overline{m(a,b,c,0,e)}$ among those with non trivial coefficient and minimal $\mathbb{N}_0$-degree. If $a = 0$, then $E_1^{2-e}E_{11212}^{1-c}E_{12}^{3-b}\mathrm{S}$ is equal to $\overline{m(0,3,1,0,2)}$ up to a non-zero scalar. Otherwise $E_1^{2-e}E_{11212}^{1-c}E_{12}^{2-b}E_1\mathrm{S}$ gives the same conclusion. This is a contradiction, since $\overline{m(0,3,1,0,2)} \neq 0$. Thus B_{26} is a basis of $L'(\lambda)$. By a similar argument $L'(\lambda)$ is irreducible. □

Lemma 5.4.34. *If* $\lambda \in \mathfrak{I}_{27}$*, then* $\dim L(\lambda) = 35$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{27} = \{\overline{n(a,0,c,d,e)} \mid (a,c,d,e) \neq (0,1,2,2)\}.$$

Proof. Set $w_1 = F_2E_{12}E_2v_\lambda$, $w_2 = \overline{n(0,0,1,2,2)}$ [9]. By Remark 5.4.11, $F_iw = 0$, $i = 1,2$. Then $\mathcal{U}w_1$ is a proper submodule and $\{\overline{n(a,0,c,d,e)}\}$ is a basis of $W = M(\lambda)/\mathcal{U}w_1$, since $w_1 = \overline{n(0,1,0,0,0)} - \frac{\zeta^3(3)_{\zeta^7}}{2}\overline{n(1,0,0,0,1)}$. Now $F_2w_2 = 0$ since $\mathcal{U}_{9\alpha_1+3\alpha_2} = 0$, and

$$\begin{aligned} F_1w_2 =& E_1^2(-q_{12}\zeta^4(1+\zeta^3)E_{112}E_{12}\sigma_1^{-1})E_{11212}v_\lambda + E_1^2E_{112}^2q_{12}^2(\zeta^5-1)E_{12}^2\sigma_1^{-1}v_\lambda \\ &\in \mathcal{U}_{7\alpha_1+3\alpha_2}\overline{n(1,0,0,0,1)} \subseteq \mathcal{U}_{8\alpha_1+3\alpha_2}\overline{n(1,0,0,0,0)}, \end{aligned}$$

[9] Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

so $F_1 w_2 = 0$, since $\mathcal{U}_{8\alpha_1+3\alpha_2} = 0$. Also, $E_1 w_2 = 0$ and

$$\begin{aligned}
E_2 w_2 &= (q_{21}^2\zeta^2 E_1^2 E_2 - q_{21}^2 E_1 E_{12}) E_{112}^2 E_{11212} v_\lambda \\
&= q_{21}^2\zeta^2 E_1^2 (q_{21}^3\zeta^7 E_{112} E_{12}^2 + q_{21}^4\zeta^4 E_{112}^2 E_2) E_{11212} v_\lambda - q_{21}^2 E_1 (q_{21}^2 E_{112}^2 E_{12}) E_{11212} v_\lambda \\
&= q_{21}^7\zeta^3 \overline{n(0,2,1,1,2)} - q_{21}^5\zeta^3 \overline{n(1,0,1,2,2)} + q_{21}^7\zeta^3(\zeta+1)\overline{n(0,3,0,2,2)} \\
&= \frac{\zeta^3(3)_{\zeta^7}}{2}\Big(q_{21}^7\zeta^3 E_1^2 E_{112} E_{11212} E_{12} E_1 E_2 + q_{21}^3\zeta^9 E_1 E_{112}^2 E_{11212} E_1 E_2 \\
&\quad + q_{21}^7\zeta^3(1+\zeta) E_1^2 E_{112}^2 E_{12}^2 E_1 E_2\Big) v_\lambda + q_{21}^9\zeta^9 \overline{n(1,0,1,2,2)} \\
&= q_{21}^9\Big(\frac{\zeta^3(3)_{\zeta^7}}{2}(\zeta^{10}+\zeta^{11}+1)+\zeta^9\Big)\overline{n(1,0,1,2,2)} = 0.
\end{aligned}$$

Set $L'(\lambda) = W/\mathcal{U}w_2 = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$, so B_{27} is a basis of $L'(\lambda)$ and $\dim L'(\lambda) = 27$. By Corollary 5.3.2 there exists $F \in \mathcal{U}$ such that $F\overline{n(1,0,1,2,2)} = v_\lambda$ since

$$\begin{aligned}
F_{11212}\overline{n(1,0,0,0,0)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{n(1,0,0,0,0)} &= -\overline{n(1,0,0,0,0)}, \\
F_{112}\overline{n(1,0,1,0,0)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{n(1,0,1,0,0)} &= \zeta^8 \overline{n(1,0,1,0,0)}, \\
F_1\overline{n(1,0,1,2,0)} &= 0, & g_1\sigma_1 \overline{n(1,0,1,2,0)} &= \zeta^7 \overline{n(1,0,1,2,0)}.
\end{aligned}$$

For each $b = \overline{n(a,0,c,d,e)} \in \mathrm{B}_{27}$, $E_2^{1-a} E_{11212}^{1-c} E_{112}^{2-d} E_1^{2-e} b$ gives $\overline{n(1,0,1,2,2)}$ up to a non-zero scalar. Arguing as in the previous cases, $L'(\lambda)$ is irreducible. □

Lemma 5.4.35. *If $\lambda \in \mathfrak{I}_{28}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{28} = \{\overline{n(a,0,c,d,e)}\} - \Big(\{\overline{n(0,0,1,1,e)}, \overline{n(0,0,c,2,e)}\} \cup \{\overline{n(1,0,1,2,e)} \mid e \neq 0\}\Big).$$

Proof. If $\mu \in \mathfrak{I}_{14}$, then $w' = \overline{m(1,0,1,2,0)}$ satisfies $E_i w' = 0$, $i = 1,2$, $g_1\sigma_1 w' = \zeta^3 w'$, $g_2\sigma_2 w' = \zeta^8 w'$, so $L(\mu)^\varphi$ is isomorphic to $L(\lambda)$ by Lemma 5.2.4. In particular $\dim L(\lambda) = 25$.

Set $w_1 = F_2 E_2 E_{12} v_\lambda$, $w_2 = F_1^2 E_1^2 E_{112}^2 v_\lambda$. By Remark 5.4.11, $F_i w_1 = 0$, $i = 1,2$, so $\mathcal{U}w_1$ is a proper submodule. By a direct computation, $\mathrm{B} = \{\overline{n(a,0,c,d,e)}\}$ is a basis of $W = M(\lambda)/\mathcal{U}w_1$. By Remark 5.4.7, $F_i w_2 = 0$, $i = 1,2$ and as $g_1\sigma_1 w_2 = \zeta^3 w_2$, $g_2\sigma_2 w_2 = w_2$, $\mathcal{U}w_2$ projects over $L(\nu)$, ν as in Case 38. If $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2 \simeq W/\mathcal{U}w_2$, then

$$25 = \dim L(\lambda) \leq \dim L'(\lambda) = 36 - \dim \mathcal{U}w_2 \leq 36 - \dim L(\nu) = 25,$$

so $L'(\lambda) = L(\lambda)$, $\mathcal{U}w_2 = L(\nu)$. Now B_{28} is a basis of $L'(\lambda)$ since B_{28} spans a linear complement of $\mathcal{U}w_2$ in W. Here we use the basis B_{38} of $L(\nu)$ in Lemma 5.4.45 to compute a basis of $\mathcal{U}w_2$. □

Lemma 5.4.36. *If $\lambda \in \mathfrak{I}_{29}$, then $\dim L(\lambda) = 47$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{29} = \{\overline{m(a,b,c,0,e)} \mid (a,b,c,e) \neq (1,3,1,0)\}.$$

Proof. Set $w_1 = F_1^2 E_{112} E_1^2 v_\lambda$, $w_2 = m(1,3,1,0,0)$ [10]. By Remark 5.4.5 $F_1 w = F_2 w = 0$, so $\mathcal{U} w_1$ is a proper submodule. Notice that $\{\overline{m(a,b,c,0,e)}\}$ is basis of $W = M(\lambda)/\mathcal{U} w_1$. Now $F_1 w_2 = 0$ since $M(\lambda)_{5\alpha_1+6\alpha_2} = 0$ and

$$\begin{aligned} F_2 w_2 &= E_2\Big((1-\zeta)\zeta^8 E_{12}^2 E_1 g_2 + q_{21}(\zeta^{11}-1)\zeta^3 E_{12} E_{112} g_2\Big) E_{11212} v_\lambda - E_2 E_{12}^3 E_{112} E_1 g_2 v_\lambda \\ &= \lambda(g_2) E_2 E_{12}((\zeta-1) q_{21} \overline{m(0,1,1,0,1)}) + q_{21}^2 \overline{m(0,1,0,2,0)} \\ &\quad + q_{21}^3 \zeta^{10}(1-\zeta)\overline{m(0,0,1,1,0)} - \overline{m(0,2,0,1,1)} = 0, \end{aligned}$$

by direct computation. Also $E_2 w_2 = 0$ since $M(\lambda)_{6\alpha_1+7\alpha_2} = 0$ and

$$\begin{aligned} E_1 w_2 &= q_{12} E_2 E_1 E_{12}^3 E_{11212} v_\lambda \\ &= q_{12} E_2 (q_{12}^2 \zeta^5 E_{12}^2 E_{112} + q_{12}^3 E_{12}^3 E_1) E_{11212} v_\lambda \\ &= q_{12}^4 E_2 E_{12}^2 \big(\zeta^2 \overline{m(0,0,1,1,0)} + q_{12}^2 \overline{m(0,1,1,0,1)} - q_{12}\zeta(1+\zeta)\overline{m(0,1,0,2,0)}\big) = 0, \end{aligned}$$

again by direct computation. Set $L'(\lambda) = M(\lambda)/\mathcal{U} w_1 + \mathcal{U} w_2$, so B_{29} is basis of $L'(\lambda)$.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(1,3,1,0,2)} = v_\lambda$ since

$$\begin{aligned} F_{11212}\overline{m(0,0,0,0,2)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{m(0,0,0,0,2)} &= \zeta^2 \overline{m(0,0,0,0,2)}, \\ F_{12}\overline{m(0,0,1,0,2)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{m(0,0,1,0,2)} &= \zeta^9 \overline{m(0,0,1,0,2)}, \\ F_2\overline{m(0,3,1,0,2)} &= 0, & g_2 \sigma_2 \overline{m(0,3,1,0,2)} &= \zeta^{10} \overline{m(0,3,1,0,2)}. \end{aligned}$$

Set $w_0 = \overline{m(1,3,1,0,2)}$. Note that $E_2 w_0 = 0$, and $E_1 w_0 = 0$ since $L'(\lambda)_{9\alpha_1+6\alpha_2} = 0$. Thus there exist a map $\pi : (\mathcal{U} w_0)^\phi \twoheadrightarrow L(\nu)$ for ν as in case 23, see Lemma 5.2.4. Then $\dim L'(\lambda)_\alpha \geq \dim(\mathcal{U} w_0)^\phi_\alpha \geq \dim L(\nu)_{8\alpha_1+6\alpha_2-\alpha}$ but we have an equality for

$$\alpha \in P_{29} := \{a\alpha_1+6\alpha_2 | a = 7,8\} \cup \{b\alpha_1+5\alpha_2 | b = 5,6,7,8\} \cup \{c\alpha_1+4\alpha_2 | c = 3,4,5,6,7\},$$

since $\dim L'(\lambda)_\alpha = \begin{cases} 1, & \alpha = 8\alpha_1+6\alpha_2, 7\alpha_1+6\alpha_2, 8\alpha_1+5\alpha_2, 5\alpha_1+5\alpha_2, \\ 2, & \alpha = 7\alpha_1+5\alpha_2, 6\alpha_1+5\alpha_2, 6\alpha_1+4\alpha_2, 4\alpha_1+4\alpha_2, \\ 3, & \alpha = 7\alpha_1+5\alpha_2. \end{cases}$ Thus for each $0 \neq v \in L'(\lambda)_\alpha$, $\alpha \in P_{29}$, there exist $E \in \mathcal{U}$ such that $Ev = w_0$. For each $b \in B$ of degree $\alpha \notin P_{29}$ there exist $E_b' \in \mathcal{U}$ such that $E_b' b \neq 0$ and has a degree $a\alpha_1 + 4\alpha_2$, so finally there exist $E_b \in \mathcal{U}$ such that $E_b b = w_0$. Arguing as in previous cases, $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.37. *If $\lambda \in \mathfrak{I}_{30}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{30} = \{\overline{m(a,b,c,0,e)}\} - \{\overline{m(1,b,c,0,e)} \mid b \geq 2, (b,c,e) \neq (3,1,2)\}.$$

Proof. By Lemma 5.4.23, if ν is as in Case 16, then $\widetilde{w} = \overline{m(1,1,1,2,0)}$ in $L(\nu)$ satisfies $E_i = 0$, $i = 1,2$, $g_1\sigma_1\widetilde{w} = \zeta 10\widetilde{w}$, $g_2\sigma_2\widetilde{w} = \zeta^{10}\widetilde{w}$. Thus $L(\mu)^\varphi$ is an irreducible module as in Case 30 by Lemma 5.2.4. In particular, $\dim L(\lambda) = \dim L(\nu) = 37$.

Set $w_1 = F_1^2 E_{112} E_1^2 v_\lambda$, $w_2 = E_2 E_{12}^2 v_\lambda$ [11]. By Remark 5.4.5, $F_i w_1 = 0$, $i = 1,2$, so $\mathcal{U} w_1$ is a proper submodule. Let $W = M(\lambda)/\mathcal{U} w_1$, then $\{\overline{m(a,b,c,0,e)}\}$

[10] Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

[11] Here $\lambda_1\lambda_2 = \zeta^4$, but the relation in Remark 5.4.13 becomes trivial.

is a basis of W. By direct computation, $F_i w_2 = 0$, $i = 1,2$, $g_1\sigma_1 w_2 = \zeta^3 w_2$, $g_2\sigma_2 w_2 = w_2$, so $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, μ as in Case 38. Let $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2 \simeq W/\mathcal{U}w_2$, so

$$37 = \dim L(\lambda) \leq \dim L'(\lambda) = \dim W - \dim \mathcal{U}w_2 \leq \dim W - \dim L(\mu) = 48 - 11 = 37,$$

so $L'(\lambda) = L(\lambda)$ and $\mathcal{U}w_2 \simeq L(\mu)$. Then B_{30} is a basis of $L(\lambda)$ since the subspace spanned by B_{30} is a complement of $\mathcal{U}w_2$ in W. Here we use the basis B_{38} of $L(\mu)$ given in Lemma 5.4.45 to compute $\mathcal{U}w_2$. $\square$

Lemma 5.4.38. *If $\lambda \in \mathfrak{I}_{31}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{31} = &\{\overline{n(a,b,c,d,e)} \mid b \leq 1\} - \big(\{\overline{n(0,0,0,2,e)} \mid e \leq 1\} \\ &\cup \{\overline{n(0,0,1,1,e)}, \overline{n(0,0,1,2,e)}, \overline{n(0,1,1,2,e)}\}\big).\end{aligned}$$

Proof. Set $w_1 = F_2 E_2 E_{12}^2 v_\lambda$ [12]. By Remark 5.4.13, $F_i w_1 = 0$, $i = 1,2$, so $\mathcal{U}w_1$ is a proper submodule. Set $W = M(\lambda)/\mathcal{U}w_1$, thus $\{\overline{n(a,b,c,d,e)} | b \leq 1\}$ is a basis of W. Let

$$w_2 = \overline{n(0,0,0,2,1)} + \frac{q_{21}}{3}\zeta(1+\zeta^3)(1+\zeta^2)\big(\overline{n(0,0,1,0,2)} + \zeta^4\overline{n(0,1,0,1,2)}\big).$$

By direct computation $F_i w_2 = 0$, $i = 1,2$, $g_1\sigma_1 w_2 = \zeta^8 w$, $g_2\sigma_2 w_2 = \zeta^5 w_2$, so $\mathcal{U}w$ projects over $L(\nu)$ for ν as in case 18. In particular, $E_{12}E_{11212}E_1 w_2 \neq 0$ by Lemma 5.2.5, so it is $\overline{n(0,1,1,2,2)}$ up to non-zero scalar since $W_{10\alpha_1+5\alpha_2} = \mathbf{k}\overline{n(0,1,1,2,2)}$. By the same result there exists $F \in \mathcal{U}$ such that $F\overline{n(0,1,1,2,2)} = w_2$, so $\mathcal{U}w_2 \subseteq \mathcal{U}\overline{n(0,1,1,2,2)}$. Given $v \in W$, $v \neq 0$, there exists $E_v \in \mathcal{U}$ such that $E_v v = \overline{n(0,1,1,2,2)}$; from here $\mathcal{U}w_2 = \mathcal{U}\overline{n(0,1,1,2,2)}$ and $\mathcal{U}w_2$ is irreducible, by the same argument of the previous cases. Thus $\mathcal{U}w_2 \simeq L(\nu)$. Then $L'(\lambda) = W/\mathcal{U}w_2 = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w_2$ has dimension $72 - 11 = 61$ and B_{31} is a basis of $L'(\lambda)$ since it spans a complement of $\mathcal{U}w_2$ in W. Here we use the basis B_{18} of $L(\nu)$ from Lemma 5.4.25 to compute $\mathcal{U}w_2$.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{n(1,1,1,2,2)} = v_\lambda$, since

$$\begin{aligned}
F_{12}\overline{n(1,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(1,0,0,0,0)} &= \zeta^3\overline{n(1,0,0,0,0)},\\
F_{11212}\overline{n(1,1,0,0,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(1,1,0,0,0)} &= -\overline{n(1,1,0,0,0)},\\
F_{112}\overline{n(1,1,1,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(1,1,1,0,0)} &= \zeta^2\overline{n(1,1,1,0,0)},\\
F_1\overline{n(1,1,1,2,0)} &= 0, & g_1\sigma_1\overline{n(1,1,1,2,0)} &= \zeta^4\overline{n(1,1,1,2,2)}.
\end{aligned}$$

Notice that $E_i\overline{n(1,1,1,2,2)} = 0$, $i = 1,2$, and if $v = \overline{n(a,b,c,d,e)} \in \mathrm{B}_{31}$, then $E_2^{1-a}E_{12}^{1-b}E_{11212}^{1-c}E_{112}^{2-d}E_1^{2-e}v$ gives $\overline{n(1,1,1,2,2)}$ up to a non-zero scalar. Let $W \neq 0$ be a submodule of $L'(\lambda)$, $w \in W - 0$. Then there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{n(1,1,1,2,2)}$, so $\overline{n(1,1,1,2,2)} \in W$, but then $v_\lambda \in W$ and $W = L'(\lambda)$. Therefore $L'(\lambda)$ is irreducible. $\square$

[12]Here $\lambda_1^2\lambda_2 = -1$, but the relation in Remark 5.4.7 becomes trivial.

Lemma 5.4.39. *If* $\lambda \in \mathfrak{I}_{32}$*, then* $\dim L(\lambda) = 61$*. A basis of* $L(\lambda)$ *is given by*

$$\begin{aligned}\mathrm{B}_{32} =&\{\overline{n(a,b,c,d,e)}|b \leq 1\} - \Big(\{\overline{n(a,b,1,d,2)} \mid b \geq 1, d \neq 0\} \\ &\cup \{\overline{n(a,0,1,0,2)n(1,0,0,2,2)}\}\Big).\end{aligned}$$

Proof. If $\mu \in \mathfrak{I}_{21}$, then $w' = \overline{m(1,1,1,2,1)}$ satisfies $E_i w' = 0$, $i = 1,2$, $g_1\sigma_1 w' = \zeta^2 w'$, $g_2\sigma_2 w' = -w'$, so $L(\mu)^{\varphi}$ is isomorphic to $L(\lambda)$ by Lemma 5.2.4. In particular $\dim L(\lambda) = 61$.

Set $w_1 = F_2E_2E_{12}^2 v_\lambda$ [13]. Then $F_i w_1 = 0$, $i = 1,2$ by Remark 5.4.13, and $\mathrm{B} = \{\overline{n(a,b,c,d,e)}|b \leq 1\}$ is a basis of $W = M(\lambda)/\mathcal{U}w_1$. Moreover $w = \overline{n(1,1,1,2,2)} \in V_{10\alpha_1+6\alpha_2}$ satisfies that $E_1 w = E_2 w = 0$, $g_1\sigma_1 w = w$, $g_2\sigma_2 w = \zeta^8 w$. By Lemma 5.2.4 $(\mathcal{U}w)^{\varphi}$ projects over $L(\nu)$, $\nu \in \mathfrak{I}_{12}$. Arguing as in some previous cases we conclude that $\mathcal{U}w$ is a proper submodule. Then set $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}w$. Notice that

$$61 = \dim L(\lambda) \leq \dim L'(\lambda) = \dim W - \dim \mathcal{U}w \leq \dim W - \dim L(\nu) = 61,$$

so $L(\lambda) = L'(\lambda)$ and $\mathcal{U}w \simeq L(\nu)^{\varphi}$. In particular $w_2 := F_2F_{11212}F_{112}w \neq 0$, $F_i w_2 = 0$ and $\mathcal{U}w_2 = \mathcal{U}w$. Moreover B_{32} is a basis of $L(\lambda)$ since it spans a linear complement of $\mathcal{U}w$ in W. Here we use the basis B_{12} of $L(\nu)$ in Lemma 5.4.19 to compute $\mathcal{U}w$. □

Lemma 5.4.40. *If* $\lambda \in \mathfrak{I}_{33}$*, then* $\dim L(\lambda) = 71$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{33} = \{\overline{m(a,b,c,d,e)}, \overline{m(1,3,0,0,0)} \mid b \neq 3, d \neq 2\} - \{\overline{m(0,0,1,0,0)}, \overline{m(1,2,0,1,2)}\}.$$

Proof. Let $w_1 = F_1^2E_{112}^2E_1^2 v_\lambda$. By Remark 5.4.7, $F_1 w_1 = F_2 w_1 = 0$. As also $g_1\sigma_1 w_1 = \zeta^9 w_1$, $g_2\sigma_2 w_1 = \zeta^2 w_1$, $\mathcal{U}w_1$ projects over $L(\mu)$, for μ as in Case 23. We claim that $\mathrm{B}' = \{\overline{m(a,b,c,d,e)} \mid d \neq 2\} \cup \{\overline{m(0,0,0,2,2)}\}$ is a basis of $W' = M(\lambda)/\mathcal{U}w_1$. Indeed $\overline{m(a,b,c,2,e)}$ appears with non-zero coefficient in $E_1^eE_{11212}E_{12}^bE_2^a w_1$ if $(a,b,c,e) \neq (0,0,0,2)$, but $E_1^2 w_1 = 0$ by direct computation, so B' is linearly independent in W'. It is a basis since $\dim W' = 144 - \dim \mathcal{U}w_1 \geq 144 - \dim L(\mu) = 97$.

Now $F_2\overline{m(0,0,0,2,2)} = E_1\overline{m(0,0,0,2,2)} = 0$, since $\mathcal{U}_{6\alpha_1+3\alpha_2} = \mathcal{U}_{7\alpha_1+4\alpha_2} = 0$ and $F_1\overline{m(0,0,0,2,2)}$, $E_2\overline{m(0,0,0,2,2)} \in \mathcal{U}w_1$ by direct computation, so $\mathcal{U}\overline{m(0,0,0,2,2)} = \mathbf{k}\overline{m(0,0,0,2,2)}$ in W'. Let $W = W'/\mathbf{k}\overline{m(0,0,0,2,2)}$; $\mathrm{B} = \{\overline{m(a,b,c,d,e)} \mid d \neq 2\}$ is a basis of W.

Set $w_2 = F_1^2F_{112}^2E_{11212}E_{112}^2E_1^2 v_\lambda$. By Remark 5.4.9, $F_i w_2 = 0$, $i = 1,2$, and as $g_1\sigma_1 w_2 = w_2$, $g_2\sigma_2 w_2 = \zeta^3 w_2$, $\mathcal{U}w_2$ projects over $L(\nu)$, for ν as in Case 14. Set $L'(\lambda) = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}\overline{m(0,0,0,2,2)} + \mathcal{U}w_2 \simeq W/\mathcal{U}w_2$. From $E_2^aE_1^eE_{112}^dE_{11212}^cE_2w_2$ we write $\overline{m(a,3,c,d,e)}$ as a linear combination of elements of B_{33}; from $w_2 = 0$ we write $\overline{m(0,0,1,0,0)}$ as a linear combination of elements of B_{33}, and the same happens for $\overline{m(1,2,0,1,2)}$, since $\mathcal{U}_{3\alpha_1+2\alpha_2}w_2 = \mathcal{U}_{2\alpha_1+2\alpha_2}w_1 = 0$. Thus B_{33} spans $L'(\lambda)$. As $\mathcal{U}w_2$ projects over $L(\mu)$, we have that $\dim(\mathcal{U}w_2)_\alpha \geq 1$ if $\alpha \in \{9\alpha_1+6\alpha_2, 10\alpha_1+6\alpha_2, 8\alpha_1+7\alpha_2, 9\alpha_1+7\alpha_2, 10\alpha_1+7\alpha_2\}$, so $L'(\lambda)_\alpha = 0$ for each $\alpha \neq 9\alpha_1+6\alpha_2$ in this

[13] Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

set, and $\dim L'(\lambda)_{9\alpha_1+6\alpha_2} \leq 1$. By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(1,2,1,1,2)} = v_\lambda$ since

$$\begin{aligned}
F_{112}\overline{m(0,0,0,0,2)} &= 0, & g_1^2 g_2 \sigma_1^2 \sigma_2 \overline{m(0,0,0,0,2)} &= \zeta^4 \overline{m(0,0,0,0,2)},\\
F_{12112}\overline{m(0,0,0,1,2)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{m(0,0,0,1,2)} &= \zeta^7 \overline{m(0,0,0,1,2)},\\
F_{12}\overline{m(0,0,1,1,2)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{m(0,0,1,1,2)} &= -\overline{m(0,0,1,1,2)},\\
F_{2}\overline{m(0,2,1,1,2)} &= 0, & g_2 \sigma_2 \overline{m(0,2,1,1,2)} &= \zeta^9 \overline{m(0,2,1,1,2)},
\end{aligned}$$

so $\overline{m(1,2,1,1,2)} \neq 0$ and then $\dim L'(\lambda)_{9\alpha_1+6\alpha_2} = 1$. By an argument as in the previous cases we prove that B_{33} is linearly independent, and $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.41. *If $\lambda \in \mathfrak{I}_{34}$, then $\dim L(\lambda) = 71$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{34} = \{\overline{n(a,b,c,d,e)}, \overline{n(0,0,0,2,e)} \mid b \neq 3, d \neq 2\} - \{\overline{n(0,0,1,0,e)}, \overline{n(0,1,1,1,0)}\}.$$

Proof. Let $\mu \in \mathfrak{I}_{33}$. By Lemma 5.4.40 $E_i\overline{m(1,2,1,1,2)} = 0$, $i = 1,2$, $g_1\sigma_1\overline{m(1,2,1,1,2)} = \zeta^{10}\overline{m(1,2,1,1,2)}$, $g_2\sigma_2\overline{m(1,2,1,1,2)} = -\overline{m(1,2,1,1,2)}$ in $L(\mu)$, so $L(\mu)^\varphi$ is isomorphic to $L(\lambda)$ by Lemma 5.2.4. In particular, $\dim L(\lambda) = 71$.

Set $w_1 = F_2E_{12}^3E_2v_\lambda$. By Remark 5.4.15, $F_iw_1 = 0$, $i = 1,2$ and as $g_1\sigma_1w_1 = \zeta w_1$, $g_2\sigma_2w_1 = w_1$, $\mathcal{U}w_1$ projects over $L(\nu)$, for ν as in Case 36. Notice that

- $\overline{n(0,3,c,d,e)}$ has non-zero coefficient in $E_1^eE_{112}^dE_{11212}^cw_1$;
- $\overline{n(1,3,c,d,e)}$ has non-zero coefficient in $E_2E_1^eE_{112}^dE_{11212}^cw_1$, if $c+d+e \neq 0$;
- at least one vector $\overline{n(a,3,c,d,e)}$ has non-zero coefficient in $E_2^aE_1^eE_{112}^dE_{11212}^cE_{12}^bw_1$ for any $(a,b,c,d,e) \neq (1,0,0,0,0)$, but $E_2w_1 = 0$;
- $\{E_2^aE_1^eE_{112}^dE_{11212}^cE_{12}^b\}$ is a basis of $\mathcal{B}(V)$ (Equation 2.14, in [HY]).

Thus $\mathrm{B}' = \{\overline{n(a,b,c,d,e)} \mid b \neq 3\} \cup \{\overline{n(1,3,0,0,0)}\}$ is a basis of $W' = M(\lambda)/\mathcal{U}w_1$. By direct computation, $F_i\overline{n(1,3,0,0,0)} = 0 = E_2\overline{n(1,3,0,0,0)}$, and $E_1\overline{n(1,3,0,0,0)} \in \mathcal{U}w_1$, so $\mathcal{U}w_1 + \mathbf{k}n(1,3,0,0,0)$ is a proper $\mathcal{U}$-submodule, and $\mathrm{B} = \{\overline{n(a,b,c,d,e)} \mid b \neq 3\}$ is a basis of $W = M(\lambda)/\mathcal{U}w_1 + \mathcal{U}n(1,3,0,0,0) \simeq W'/\mathbf{k}\overline{n(1,3,0,0,0)}$.

Set $w_2 = F_1^2F_{112}^2E_{11212}E_{112}^2E_1^2v_\lambda$. By Remark 5.4.9, $F_iw_2 = 0$, $i = 1,2$ and as $g_1\sigma_1w_2 = \zeta^2w_2$, $g_2\sigma_2w_2 = w_2$, $\mathcal{U}w_2$ projects over $L(\nu')$, ν' as in Case 37. Let $L'(\lambda) = W/\mathcal{U}w_2$, then

$$71 = \dim L(\lambda) \leq \dim L'(\lambda) = 108 - \dim \mathcal{U}w_2 \leq 108 - \dim L(\nu') = 71,$$

so $\mathcal{U}w_2 \simeq L(\nu')$ and $L'(\lambda)$ is irreducible. Now B_{34} is a basis of $L'(\lambda) = L(\lambda)$ since it spans a linear complement of $\mathcal{U}w_2$ in W. Here we use the basis B_{37} of $L(\nu')$ in Lemma 5.4.44 to compute $\mathcal{U}w_2$. $\square$

Lemma 5.4.42. *If $\lambda \in \mathfrak{I}_{35}$, then $\dim L(\lambda) = 85$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}
\mathrm{B}_{35} = &\{\overline{n(a,b,c,d,e)} \mid b \neq 3\}\\
&- \Big(\{\overline{n(0,b,c,2,e)} \mid b \neq 3\} \cup \{\overline{n(1,2,1,2,2)}, \overline{n(1,0,0,2,2)}, \overline{n(1,0,1,2,e)}\}\Big).
\end{aligned}$$

Proof. Set $w_1 = F_2E_2E_{12}^3 v_\lambda$, $w_2 = F_1^2E_{112}^2E_1^2 v_\lambda$. Then $F_i w_j = 0$, $i, j = 1, 2$, by Remarks 5.4.15 and 5.4.7. Let $W = M(\lambda)/\mathcal{U}w_1$. Then $\mathrm{B} = \{\overline{n(a,b,c,d,e)} \mid b \neq 3\}$ is a basis of W. As $g_1\sigma_1 w_2 = \zeta^9 w_1$, $g_2\sigma_2 w_2 = w_2$, $\mathcal{U}w_2$ projects over a simple module $L(\mu)$, $\mu \in \mathfrak{I}_{44}$ and then $E_{112}^2E_{12}^3 w_2 \neq 0$. Thus $E_{112}^2E_{12}^3 w_2$ is $\overline{n(1,2,1,2,2)}$ up to non-zero scalar and there exists $F \in \mathcal{U}$ such that $F\overline{n(1,2,1,2,2)} = w_2$. For each $v \in W$, $v \neq 0$, there exists $E_v \in \mathcal{U}$ such that $E_v v = \overline{n(1,2,1,2,2)}$. As in the previous cases we prove that $\mathcal{U}w_2 = \mathcal{U}\overline{n(1,2,1,2,2)}$, and this submodule is irreducible.

Set $L'(\lambda) = M(\lambda)/N'(\lambda)$. We claim that $L'(\lambda)$ is irreducible. Notice that B_{35} is a basis of $L'(\lambda)$. Here we use the basis B_{44} of $L(\mu)$ in Lemma 5.4.51 to compute a basis of $\mathcal{U}w_2$.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{n(1,2,1,2,1)} = v_\lambda$, since

$$\begin{aligned}
F_{12}\overline{n(1,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(1,0,0,0,0)} &= -\overline{n(1,0,0,0,0)},\\
F_{11212}\overline{n(1,2,0,0,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(1,2,0,0,0)} &= \zeta^4\overline{n(1,2,0,0,0)},\\
F_{112}\overline{n(1,2,1,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(1,2,1,0,0)} &= \zeta^3\overline{n(1,2,1,0,0)},\\
F_1\overline{n(1,2,1,2,0)} &= 0, & g_1\sigma_1\overline{n(1,2,1,2,0)} &= \zeta^8\overline{n(1,2,1,2,0)}.
\end{aligned}$$

Suppose that B_{35} is not linearly independent. Fix $\mathtt{S} = 0$ a non-trivial linear combination, and consider the minimal element $\overline{n(a,b,c,d,e)}$ among those with non trivial coefficient and minimal $\mathbb{N}_0$-degree. Then if $e = 2$, then $E_2^{1-a}E_{12}^{1-b}E_{11212}^{1-c}E_{112}^{2-d}E_2$ gives $\overline{n(1,2,1,2,1)}$ up to a non-zero scalar.

Otherwise, $E_2^{1-a}E_{12}^{2-b}E_{11212}^{1-c}E_{112}^{2-d}E_1^{1-e}\overline{n(a,b,c,d,e)}$, gives the same conclusion. But this is a contradiction, so B_{35} is a basis of $L'(\lambda)$. Arguing as in the other cases, $L'(\lambda)$ is irreducible. □

Lemma 5.4.43. *If $\lambda \in \mathfrak{I}_{36}$, then* $\dim L(\lambda) = 35$. *A basis of $L(\lambda)$ is given by*

$$\begin{aligned}
\mathrm{B}_{36} =& \{\overline{n(0,b,0,d,e)}, \overline{n(0,0,1,2,e)}, \overline{n(0,0,1,0,e)}\}\\
&- \{\overline{n(0,1,0,1,e)}, \overline{n(0,2,0,2,e)}, \overline{n(0,1,0,0,2)}\}.
\end{aligned}$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b) [14]. Let $W = M(\lambda)/W(\lambda)$, $w = \overline{n(0,1,0,0,2)}$. Notice that $F_i w = 0$, $i = 1, 2$, $g_1\sigma_1 w = w$, $g_2\sigma_2 = \zeta^9 w$, so $\mathcal{U}w$ projects over an irreducible module $L(\mu)$, $\mu \in \mathfrak{I}_{15}$. Thus $E_{12}^2E_{11212}E_{112}^2 w \neq 0$, so it is $\overline{n(0,3,1,2,2)}$ up to non-zero scalar, and moreover there exist $F \in \mathcal{U}$ such that $F\overline{n(0,3,1,2,2)} = w$. It implies that $\mathcal{U}w = \mathcal{U}\overline{n(0,3,1,2,2)}$. For any $v \neq 0$ there exist E_v such that $E_v v = \overline{n(0,3,1,2,2)}$, so $\mathcal{U}w$ is irreducible, and then $\mathcal{U}w \simeq L(\mu)$.

Let $L'(\lambda) = M(\lambda)/\mathcal{U}E_2v_\lambda + \mathcal{U}w \simeq W/\mathcal{U}w$, so $\dim(L'(\lambda)) = 35$ and B_{36} is a basis of $L'(\lambda)$, since it spans a complement of $\mathcal{U}w$ in W. Here we use the basis B_{15} of $L(\mu)$ in Lemma 5.4.22 to compute $\mathcal{U}w$.

By Corollary 5.3.2 there exist $F \in \mathcal{U}$ such that $F\overline{n(0,3,0,2,2)} = v_\lambda$ since

$$\begin{aligned}
F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,0,0,0)} &= \zeta\overline{n(0,0,0,0,0)},\\
F_{112}\overline{n(0,3,0,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(0,3,0,0,0)} &= \zeta^{11}\overline{n(0,3,0,0,0)},\\
F_1\overline{n(0,3,0,2,0)} &= 0, & g_1\sigma_1\overline{n(0,3,0,2,0)} &= \zeta^4\overline{n(0,3,0,2,0)}.
\end{aligned}$$

[14] Here $\lambda_1\lambda_2 = \zeta$, but the relation in Remark 5.4.11 becomes trivial.

As $E_i\overline{n(0,3,0,2,2)} = 0$ for $i = 1,2$, we prove that for each $v \neq 0$, there exists E_v such that $E_v v = \overline{n(0,3,0,2,2)}$. Arguing as in the previous cases, $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.44. *If $\lambda \in \mathfrak{I}_{37}$, then* $\dim L(\lambda) = 37$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{37} = \{\overline{n(0,b,0,d,e)}, \overline{n(0,0,1,0,0)}, \overline{n(0,3,1,0,e)}\} - \{\overline{n(0,3,0,2,e)}\}.$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b) [15]. Set

$$w = n(0,1,0,1,1) - \zeta n(0,2,0,0,2) - \zeta^{10}(1-\zeta)^2 n(0,0,1,0,1).$$

Then $F_i w = 0$ in $M(\lambda)/W(\lambda)$, so $\mathcal{U}w + W(\lambda)$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/W(\lambda) + \mathcal{U}w$ is simple. First we prove that $L'(\lambda)$ is spanned by B_{37}. This follows from the following relations, obtained from $E_2 w = E_{12} w = E_1 E_{12}^3 w = 0$:

$$\begin{aligned}
&\overline{n(0,1,1,0,0)} = (1+\zeta)\zeta^5\overline{n(0,3,0,0,1)} + (1+\zeta)\overline{n(0,2,0,1,0)},\\
&\overline{n(0,0,1,1,0)} = (1+\zeta)^2\left(q_{21}\zeta^2\overline{n(0,2,0,1,1)} - \overline{n(0,1,0,2,0)} - q_{21}(1+\zeta)\overline{n(0,3,0,0,2)}\right),\\
&\overline{n(0,3,0,2,0)} = \zeta^2(1-\zeta)\overline{n(0,3,1,0,1)}.
\end{aligned}$$

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{m(0,3,1,0,2)} = v_\lambda$, since

$$\begin{aligned}
F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{n(0,0,0,0,0)} &= \zeta^2\overline{n(0,0,0,0,0)},\\
F_{11212}\overline{n(0,3,0,0,0)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{n(0,3,0,0,0)} &= \zeta^3\overline{n(0,3,0,0,0)},\\
F_1\overline{n(0,3,1,0,0)} &= 0, & g_1\sigma_1\overline{n(0,3,1,0,0)} &= \zeta^9\overline{n(0,3,1,0,0)}.
\end{aligned}$$

Using the previous relations and Lemma 4.0.1,

$$E_2\overline{n(0,3,1,0,2)} = q_{21}^2\zeta^4 E_{112}E_{11212}E_{12}^3 v_\lambda = q_{21}^4\zeta^{10}E_{112}E_{12}^2\overline{n(0,1,1,0,0)} = 0,$$

and also $E_1\overline{n(0,3,1,0,2)} = 0$. Suppose that B_{37} is not linearly independent. Fix $\mathtt{S} = 0$ a non-trivial linear combination, let $\overline{n(0,b,c,d,e)}$ be the element with not zero coefficient minimal for the lexicographical order between the elements of minimal $\mathbb{N}_0$-degree. By multiplying $\mathtt{S}$ either by $E_1^{1-e}E_{112}^{2-d}E_{12}^{3-b}$ if $c = 0$, or else by $E_1^{2-e}E_{12}^{3-b}$ if $c = 1$, we obtain $\overline{n(0,3,1,0,2)}$ up to a non-zero scalar, a contradiction. Therefore B_{37} is a basis of $L'(\lambda)$.

Let W be a non-zero submodule of $L'(\lambda)$, $w \in W - 0$. By a similar argument there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{n(0,3,1,0,2)}$, so $\overline{n(0,3,1,0,2)} \in W$, but then $v_\lambda \in W$, so $W = L'(\lambda)$ and $L'(\lambda)$ is irreducible. $\square$

Lemma 5.4.45. *If $\lambda \in \mathfrak{I}_{38}$, then* $\dim L(\lambda) = 11$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{38} = \{\overline{n(0,b,c,0,d)} | b \le 1\} - \{\overline{n(0,1,1,0,2)}\}.$$

The action of E_i, F_i, $i = 1,2$ is described in Table A.4.

[15] Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b). Set $w = F_1^2 E_{112} E_1^2 v_\lambda$, so $F_1 w = F_2 w = 0$ by Remark 5.4.5. Then $\mathcal{U}w + W(\lambda)$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/(\mathcal{U}w + W(\lambda))$ is simple. We label the elements of B_{38} as follows:

$$\begin{aligned} &v_{0,0} = \overline{n(0,0,0,0,0)}, && v_{1,0} = \overline{n(0,0,0,0,1)}, && v_{2,0} = \overline{n(0,0,0,0,2)}, && v_{1,1} = \overline{n(0,1,0,0,0)},\\ &v_{2,1} = \overline{n(0,1,0,0,1)}, && v_{3,2} = \overline{n(0,0,1,0,0)}, && v_{3,1} = \overline{n(0,1,0,0,2)}, && v_{4,2} = \overline{n(0,0,1,0,1)},\\ &v_{4,3} = \overline{n(0,1,1,0,0)}, && v_{5,2} = \overline{n(0,0,1,0,2)}, && v_{5,3} = \overline{n(0,1,1,0,1)}. \end{aligned}$$

Notice that $v_{i,j} \in L'(\lambda)_{i\alpha_1 + j\alpha_2}$. The following relations hold in $L'(\lambda)$:

$$\begin{aligned} &\overline{n(0,0,0,1,0)} = (1-\zeta^3)v_{2,1}, && \overline{n(0,1,0,1,0)} = \zeta^3 v_{3,2}\\ &\overline{n(0,0,0,2,0)} = \zeta^8 q_{21}(1-\zeta^3)v_{4,2}, && \overline{n(0,2,0,0,0)} = \overline{n(0,1,1,0,2)} = 0. \end{aligned}$$

Then we prove that the $v_{i,j}$ satisfy equations in Table A.4 and $L'(\lambda)$ is spanned by B_{38}. By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $Fv_{5,3} = v_\lambda$, since

$$\begin{aligned} F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1 g_2 \sigma_1 \sigma_2 \overline{n(0,0,0,0,0)} &= \zeta^3 \overline{n(0,0,0,0,0)},\\ F_{11212}\overline{n(0,1,0,0,0)} &= 0, & g_1^3 g_2^2 \sigma_1^3 \sigma_2^2 \overline{n(0,1,0,0,0)} &= \zeta^4 \overline{n(0,1,0,0,0)},\\ F_1\overline{n(0,1,1,0,0)} &= 0, & g_1 \sigma_1 \overline{n(0,1,1,0,0)} &= \zeta^8 \overline{n(0,1,1,0,0)}. \end{aligned}$$

Notice that $E_i v_{5,3} = 0$, $i = 1,2$, and for $(i,j) \neq (5,3)$ there exists $E_{i,j} \in \mathcal{U}_{(5-i)\alpha_1 + (3-j)\alpha_2}$ such that $E_{i,j} v_{i,j} = v_{5,3}$. Now suppose that B_{38} is not linearly independent. Fix a non-trivial linear combination $\mathtt{S} = 0$. If (i,j) is a minimal element with non-trivial coefficient, then we may assume that this coefficient is 1, so $E_{i,j}\mathtt{S} = v_{5,3}$, which is a contradiction. Then B_{38} is a basis of $L'(\lambda)$.

Let $W \neq 0$ be a submodule of $L'(\lambda)$, $w \in W - 0$. By a similar argument there exists $E \in \mathcal{U}^+$ such that $Ew = v_{5,3}$. Then $v_{5,3} \in W$, so $W = L'(\lambda)$ and $L'(\lambda)$ is irreducible. □

Lemma 5.4.46. *If $\lambda \in \mathfrak{I}_{39}$, then $\dim L(\lambda) = 61$. A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{39} = \{\overline{n(0,b,c,d,e)}\} - \Big(\{\overline{n(0,3,c,2,e)}, \overline{n(0,2,1,2,e)}\} \cup \{\overline{n(0,2,0,2,e)} \mid e \geq 1\}\Big).$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b) [16]. Set $W = M(\lambda)/W(\lambda)$ and $w' = \overline{n(0,3,1,2,2)}$. Notice that $E_i w' = 0$, $i = 1,2$, $g_1\sigma_1 w' = \zeta^9 w'$, $g_2 \sigma_2 w' = w'$, so $(\mathcal{U}w')^\varphi$ projects over an irreducible $L(\nu)$, $\nu \in \mathfrak{I}_{38}$. We claim that $\mathcal{U}w'$ is a proper submodule. Otherwise $\mathcal{U}w' = W$ is irreducible since for any $v \in W$, $v \neq 0$, there exists $E_v \in \mathcal{U}$ such that $E_v v = w'$. If $V \subset W$ is non-zero submodule, then $w' \in V$ and $\mathcal{U}v = W$. But then $W^\varphi \simeq L(\nu)$, a contradiction since they have different dimension.

[16] Here $\lambda_1\lambda_2 = \zeta^4$, but the relation in Remark 5.4.13 becomes trivial.

If $\mu \in \mathfrak{I}_{31}$, then $v = \overline{n(1,1,1,2,2)}$ satisfies $E_i v = 0$, $i = 1,2$, $g_1\sigma_1 v = \zeta^9 v$, $g_2\sigma_2 v = 1$ so $L(\mu)^\varphi \simeq L(\lambda)$ by Lemma 5.2.5, and then $\dim L(\lambda) = 61$. Let $L'(\lambda) = W/\mathcal{U}w'$. Then

$$61 = \dim L(\lambda) \leq \dim L'(\lambda) \leq \dim W - \dim L(\nu) = 61,$$

so $L'(\lambda) = L(\lambda)$ and $\mathcal{U}w' \simeq L(\nu)^\varphi$. Thus $\mathcal{U}w' = \mathcal{U}w$ for $w = F_1F_{11212}F_{12}w'$ and B_{39} is a basis of $L'(\lambda)$ since it spans a complement of $\mathcal{U}w$ in W. Here we use the basis B_{38} of $L(\nu)$ in Lemma 5.4.45 to compute $(\mathcal{U}w')^\varphi$. □

Lemma 5.4.47. *If $\lambda \in \mathfrak{I}_{40}$, then $\dim L(\lambda) = 35$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{40} =& \{\overline{n(0,b,c,0,e)}\} \cup \{\overline{n(0,b,c,1,e)} \mid b \leq 1\} \cup \{\overline{n(0,3,0,2,e)} \mid e \leq 1\} \\ &- \{\overline{n(0,3,1,0,e)}\}.\end{aligned}$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b). Set $W = M(\lambda)/W(\lambda)$. By Remark 5.4.7, $w = F_1^2E_{112}^2E_1^2v_\lambda$ satisfies $F_iw = 0$, $i = 1,2$. As $g_1\sigma_1 w = \zeta^{11}w$, $g_2\sigma_2 w = \zeta^8 w$, $\mathcal{U}w$ projects over $L(\mu)$, $\mu \in \mathfrak{I}_{25}$. Thus $E_{12}^3E_{11212}E_1^2 w \neq 0$, but this vector is $\overline{n(0,3,1,2,2)}$ up to a non-zero scalar since $W_{12\alpha_1+7\alpha_2} = \mathbf{k}\overline{n(0,3,1,2,2)}$. Moreover, there exists $F \in \mathcal{U}$ such that $F\overline{n(0,3,1,2,2)} = w$, so $\mathcal{U}w \subseteq \mathcal{U}\overline{n(0,3,1,2,2)}$. As also $E_2E_{12}^3E_{11212}E_1^2w = \zeta^8q_{12}^4\overline{n(0,3,1,2,2)}$, we have that $\mathcal{U}w = \mathcal{U}\overline{n(0,3,1,2,2)}$. For any $v \in W$, $v \neq 0$, there exists $E_v \in \mathcal{U}$ such that $E_v v = \overline{n(0,3,1,2,2)}$, so if $V \subset \mathcal{U}w$ is a submodule, $V \neq 0$, then $\overline{n(0,3,1,2,2)} \in V$ and then $V = \mathcal{U}w$. Thus $\mathcal{U}w \simeq L(\mu)$.

Set $L'(\lambda) = W/\mathcal{U}w = M(\lambda)/\mathcal{U}w + W(\lambda)$, so $\dim L'(\lambda) = 72 - 37 = 35$. Notice B_{40} is a basis of $L'(\lambda)$, since it spans a complement of $\mathcal{U}w$ in W. Here we use the basis B_{25} of $L(\nu)$ in Lemma 5.4.32 to compute $\mathcal{U}w$.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{n(0,3,0,2,1)} = v_\lambda$, since

$$\begin{aligned} F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,0,0,0)} &= \zeta^5\overline{n(0,0,0,0,0)}, \\ F_{112}\overline{n(0,3,0,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(0,3,0,0,0)} &= \zeta\overline{n(0,3,0,0,0)}, \\ F_1\overline{n(0,3,0,2,0)} &= 0, & g_1\sigma_1\overline{n(0,3,0,2,0)} &= \zeta^8\overline{n(0,3,0,2,0)}. \end{aligned}$$

Note that $E_i\overline{n(0,3,0,2,1)} = 0$, $i = 1,2$. For any $b \in \mathrm{B}_{40}$ there exists $E_b \in \mathcal{U}$ such that $E_b b = \overline{n(0,3,0,2,1)}$. Indeed we choose $E_b = E_1^{1-e}$ if $b = \overline{n(0,3,0,2,e)}$; $E_b = E_{12}^{3-b}E_{11212}^{1-c}E_1^{2-e}$ if $b = \overline{n(0,b,c,0,e)}$ and use that $\overline{n(0,3,1,0,2)}$ is $\overline{n(0,3,0,2,1)}$ up to a non-zero scalar; $E_b = E_{12}^{1-b}E_{11212}^{1-c}E_2E_1^{2-e}$ if $b = \overline{n(0,b,c,1,e)}$ and use that $\overline{n(0,1,1,2,0)}$ is $\overline{n(0,3,0,2,1)}$ up to a non-zero scalar. Arguing as in previous cases, $L'(\lambda)$ is irreducible. □

Lemma 5.4.48. *If $\lambda \in \mathfrak{I}_{41}$, then $\dim L(\lambda) = 37$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned}\mathrm{B}_{41} =& \{\overline{n(0,b,c,d,0)} | b \leq 2\} \cup \{\overline{n(0,b,c,d,e)} |\ b \leq 1, e \neq 0\} \\ &- \{\overline{n(0,1,c,d,2)}, \overline{n(0,0,1,2,2)} | d \neq 0\}.\end{aligned}$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b).

Set $w = F_1^2F_{112}^2E_{11212}E_{112}^2E_1^2v_\lambda$, so it satisfies the equations $F_1w = F_2w = 0$ by Remark 5.4.9. Then $\mathcal{U}w + W(\lambda)$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U}w + W(\lambda)$ is irreducible. From $w = E_2w = E_1^2w = E_{12}E_{11212}E_1^2w = 0$:

$$\overline{n(0,2,0,0,1)} = \overline{n(0,0,1,0,0)} + \zeta^3\overline{n(0,1,0,1,0)}, \qquad \overline{n(0,3,0,0,0)} = 0,$$
$$\overline{n(0,1,0,1,0)} = \zeta^3\overline{n(0,0,1,0,2)}, \qquad \overline{n(0,0,1,2,2)} = 0.$$

From the first relation we write $\overline{n(0,b,c,d,e)}$, $b \geq 2$, $e \neq 0$, as a linear combination of elements of B_{41}. From the second and the fourth relations we know that $\overline{n(0,3,c,d,e)} = \overline{n(0,0,1,2,2)} = 0$. From the third relation we write $\overline{n(0,1,c,d,2)}$, $d \neq 0$, as a linear combination of elements of B_{41}. Then $L'(\lambda)$ is spanned by B_{41}.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{n(0,2,1,2,0)} = v_\lambda$, since

$$\begin{aligned}
F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,0,0,0)} &= -\overline{n(0,0,0,0,0)},\\
F_{11212}\overline{n(0,2,0,0,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(0,2,0,0,0)} &= \zeta^8\overline{n(0,2,0,0,0)},\\
F_{112}\overline{n(0,2,1,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(0,2,1,0,0)} &= \zeta^7\overline{n(0,2,1,0,0)}.
\end{aligned}$$

Notice that

$$\begin{aligned}
E_1\overline{n(0,2,1,2,0)} &= \overline{n(0,2,1,2,1)} = q_{12}^4\zeta^4E_{12}\overline{n(0,1,1,2,1)} = 0,\\
E_2\overline{n(0,2,1,2,0)} &= q_{21}^9\zeta^9\overline{n(1,2,1,2,0)} = 0.
\end{aligned}$$

so $E_i\overline{n(0,2,1,2,0)} = 0$, $i = 1,2$. Now suppose that B_{41} is not linearly independent. Fix a non-trivial linear combination $\mathtt{S}$ which is zero. Between the elements of minimal $\mathbb{N}_0$-degree with non-trivial coefficient, take the element $\overline{n(0,b,c,d,e)}$ minimal for the lexicographical order. If $e = 2$ and $c = 1$, then $d \leq 1$ and $E_{112}^{1-d}E_{12}^{2-b}E_2\mathtt{S}$ gives $\overline{n(0,2,1,2,0)}$ up to a non-zero scalar. If $e = 2$ and $c = 0$, then $E_{112}^{2-d}E_{12}^{2-b}E_2\mathtt{S}$ gives $\overline{n(0,2,1,2,0)}$ up to a non-zero scalar. If $e = 1$, then $E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{1-b}E_2\mathtt{S}$ gives $\overline{n(0,2,1,2,0)}$ up to a non-zero scalar. If $e = 0$, then $E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{2-b}\mathtt{S}$ gives $\overline{n(0,2,1,2,0)}$ up to a non-zero scalar. In any case we obtain a contradiction. Therefore B_{41} is a basis of $L'(\lambda)$.

Let $W \neq 0$ be a submodule of $L'(\lambda)$, $w \in W - 0$. By a similar argument there exists $E \in \mathcal{U}^+$ such that $Ew = \overline{n(0,2,1,2,0)}$. Then $\overline{n(0,2,1,2,0)} \in W$, so $v_\lambda \in W$ and $W = L'(\lambda)$. Therefore $L'(\lambda)$ is irreducible. □

Lemma 5.4.49. *If $\lambda \in \mathfrak{I}_{42}$, then* $\dim L(\lambda) = 71$. *A basis of $L(\lambda)$ is given by*

$$\mathrm{B}_{42} = \{\overline{n(0,b,c,d,e)} | (b,c,d,e) \neq (3,1,2,2)\}.$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b)[17]. Let $w = n(0,3,1,2,2)$, then $E_1w = 0$, $E_2w \in W(\lambda)$, $F_2w = 0$ and

$$\begin{aligned}
F_1w &= (1+\zeta^4)E_1(\zeta^8\sigma_1^{-1} - g_1)\overline{n(0,3,1,2,0)}\\
&= \zeta^2\lambda(\sigma_1)q_{21}^{-7}(\zeta^8q_{11}^{-10} - q_{11}^{10}q_{12}^7q_{21}^7\lambda_1)\overline{n(0,3,1,2,1)} = 0,
\end{aligned}$$

[17]Here $\lambda_1\lambda_2 = \zeta^7$, but the relation in Remark 5.4.15 becomes trivial.

so $\mathcal{U}w = \mathbf{k}w$. Thus $W(\lambda) + \mathcal{U}w$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/W(\lambda) + \mathcal{U}w$ is irreducible. Note that B_{42} is a basis of $L'(\lambda)$. By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{n(0,3,1,2,1)} = v_\lambda$, since

$$\begin{aligned} F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,0,0,0)} &= \zeta^7\overline{n(0,0,0,0,0)}, \\ F_{11212}\overline{n(0,3,0,0,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(0,3,0,0,0)} &= -\overline{n(0,3,0,0,0)}, \\ F_{112}\overline{n(0,3,1,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(0,3,1,0,0)} &= \zeta^{10}\overline{n(0,3,1,0,0)}, \\ F_1\overline{n(0,3,1,2,0)} &= 0, & g_1\sigma_1\overline{n(0,3,1,2,0)} &= \zeta^8\overline{n(0,3,1,2,0)}, \end{aligned}$$

Also, for any $\overline{n(0,b,c,d,e)} \in \mathrm{B}_{42}$, if $e \neq 2$, we have that $E_1^{1-e}E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{3-b}\overline{n(0,b,c,d,e)}$ gives $\overline{n(0,3,1,2,1)}$ up to non-zero scalar. And if $e = 2$ we have that:

$$\begin{aligned} E_{112}^{2-d}E_{11212}^{1-c}E_{12}^{2-b}E_2\overline{n(0,b,c,d,2)} ifb \neq 3 \\ E_{112}^{1-d}E_{11212}^{1-c}E_{12}^{3-b}E_12\overline{n(0,b,c,d,2)} ifd \neq 2 \\ E_1E_{112}^{2-d}E_{12}^{3-b}E_112\overline{n(0,b,c,d,2)} ifc \neq 1 \end{aligned}$$

gives $\overline{n(0,3,1,2,1)}$ up to non-zero scalar. From here, every $w \in L'(\lambda)$, $w \neq 0$ generates $L'(\lambda)$, so $L'(\lambda)$ is irreducible. □

Lemma 5.4.50. *If $\lambda \in \mathfrak{I}_{43}$, then $\dim L(\lambda) = 25$. A basis of $L(\lambda)$ is given by*

$$\begin{aligned} \mathrm{B}_{43} = \{(\overline{n(0,b,c,d,e)} | e \neq 2)\} - \Big(&\{\overline{n(0,2,1,2,0)}\} \\ &\cup \{\overline{n(0,b,c,d,1)} | b \geq 1\} \cup \{\overline{n(0,3,c,d,0)} | d \geq 1\}\Big). \end{aligned}$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b). Note that $w = \overline{n(0,0,0,0,2)}$ satisfies $F_iw = 0$, $i = 1,2$, $g_1\sigma_1w = w$, $g_2\sigma_2w = \zeta^{10}w$, so $\mathcal{U}w$ projects over a module $L(\nu)$, where ν corresponds to the case 17. Then $w' := E_{12}^3E_{11212}E_{112}^2w \neq 0$ by Lemma 5.2.5, so w' is $\overline{n(0,3,1,2,2)}$ up to a non-zero scalar and there exists $F \in \mathcal{U}$ such that $Fw' = w$. For any $0 \neq v \in W$ there exists $E_v \in \mathcal{U}$ such that $E_vv = w'$. Therefore, if $0 \neq V \subset \mathcal{U}w$ is a submodule then $w' \in \mathcal{U}w$. This implies $V = \mathcal{U}w$, and $\mathcal{U}w \simeq L(\nu)$.

Let $L'(\lambda) = M(\lambda)/W(\lambda) + W_2(\lambda)$. Thus B_{43} is a basis of $L'(\lambda)$ since it spans a linear complement of $\mathcal{U}w$ in W; here we use the basis B_{17} from Lemma 5.4.24 to describe a basis of $\mathcal{U}w$. By Corollary 5.3.2 there exists $F \in \mathcal{U}$ such that $F\overline{n(0,3,1,0,2)} = v_\lambda$ since

$$\begin{aligned} F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,0,0,0)} &= \zeta^3\overline{n(0,0,0,0,0)}, \\ F_{11212}\overline{n(0,3,0,0,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(0,3,0,0,0)} &= \zeta^9\overline{n(0,3,0,0,0)}, \\ F_1\overline{n(0,3,1,0,0)} &= 0, & g_1\sigma_1\overline{n(0,3,1,0,0)} &= \zeta^3\overline{n(0,3,1,0,0)}. \end{aligned}$$

Notice that $\overline{n(0,3,1,0,2)}$ is $\overline{n(0,1,1,2,0)}$ up to a non-zero scalar, and for each $b \in B$ there exists $E_b \in \mathcal{U}$ such that $E_bb = \overline{n(0,1,1,2,0)}$. Arguing as in the previous cases, $L'(\lambda)$ is irreducible. □

Lemma 5.4.51. *If* $\lambda \in \mathfrak{I}_{44}$*, then* $\dim L(\lambda) = 23$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{44} = \{\overline{n(0,b,0,d,e)}, \overline{n(0,0,0,0,2)} | e \le 1\} - \{\overline{n(0,3,0,1,1)}, \overline{n(0,3,0,2,1)}\}.$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b) [18]. Set $w = \zeta^4 n(0,0,0,1,1) + n(0,1,0,0,2)$. Then $F_2 w = 0$ and $F_1 w \in W(\lambda)$ by direct computation, so $\mathcal{U}w + W(\lambda)$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/W(\lambda) + \mathcal{U}w$ is simple and B_{44} is a basis of $L'(\lambda)$. Applying repeatedly E_1, E_2 over w we obtain

$$\begin{aligned}
\overline{n(0,0,1,0,0)} &= (1+\zeta^{11})\overline{n(0,1,0,1,0)} + \zeta^5\overline{n(0,2,0,0,1)}),\\
\overline{n(0,0,0,1,2)} &= \overline{n(0,2,1,0,2)} = \overline{n(0,3,0,1,2)} = 0,\\
\overline{n(0,2,0,0,2)} &= -\overline{n(0,1,0,1,1)},\\
\overline{n(0,1,1,0,0)} &= (1-\zeta)\zeta^4\overline{n(0,3,0,0,1)} - \zeta^4\overline{n(0,2,0,1,0)}\\
\overline{n(0,0,1,1,0)} &= q_{21}\zeta^{10}\overline{n(0,2,0,1,1)} + (1+\zeta^{11})\overline{n(0,1,0,2,0)},\\
\overline{n(0,3,0,0,2)} &= q_{12}(\zeta-1)\overline{n(0,1,0,2,0)} + q_{12}\zeta^{11}(3)_{\zeta^7}\overline{n(0,2,0,1,1)},\\
\overline{n(0,2,1,0,0)} &= \zeta^9\overline{n(0,3,0,1,0)},\\
\overline{n(0,1,1,1,1)} &= \zeta^{10}\overline{n(0,2,0,2,1)}.
\end{aligned}$$

Using these relations we prove that $L'(\lambda)$ is spanned by B_{44}. By Corollary 5.3.2 and

$$\begin{aligned}
F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,0,0,0)} &= \zeta^9\overline{n(0,0,0,0,0)},\\
F_{112}\overline{n(0,3,0,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(0,3,0,0,0)} &= \zeta^9\overline{n(0,3,0,0,0)},
\end{aligned}$$

there exists $F \in \mathcal{U}^-$ such that $F\overline{n(0,3,0,2,0)} = v_\lambda$. From $\mathcal{U}_{5\alpha_1+4\alpha_2}w = 0$ we have that $E_1\overline{n(0,3,0,2,0)} = \overline{n(0,3,0,2,1)} = 0$, and by Lemma 4.0.1 $E_2\overline{n(0,3,0,2,0)} = 0$. Now suppose that B_{44} is not linearly independent. Take a non-trivial linear combination $\mathtt{S}$ which is zero, and take the minimal element $\overline{n(0,b,0,d,e)}$ among those with non trivial coefficient, between the elements of minimal $\mathbb{N}_0$-degree. If this element is $\overline{n(0,0,0,0,2)}$, then compute $E_{112}^{1-d}E_{12}^{3-b}E_2\mathtt{S}$. If it is $\overline{n(0,b,0,d,1)}$, then compute $E_{112}^{1-d}E_{12}^{3-b}E_2E_1\mathtt{S}$. Finally if it is $\overline{n(0,b,0,d,0)}$, then compute $E_{112}^{2-d}E_{12}^{3-b}\mathtt{S}$. In any case we obtain $\overline{n(0,3,0,2,0)}$ up to a non-zero scalar, so we have a contradiction. Therefore B_{44} is a basis of $L'(\lambda)$.

Let $W \neq 0$ be a submodule of $L'(\lambda)$, $w \in W - 0$. By a similar argument, $Ew = \overline{n(0,3,0,2,0)}$ for some $E \in \mathcal{U}^+$, so $\overline{n(0,3,0,2,0)} \in W$, but then $v_\lambda \in W$, so $W = L'(\lambda)$; then $L(\lambda)$ is irreducible. □

Lemma 5.4.52. *If* $\lambda \in \mathfrak{I}_{45}$*, then* $\dim L(\lambda) = 49$*. A basis of* $L(\lambda)$ *is given by*

$$\mathrm{B}_{45} = \{\overline{n(0,b,c,d,e)}\} - \{\overline{n(0,b,c,2,e)}, \overline{n(0,0,1,2,e)}, \overline{n(0,0,1,0,2)}, \overline{n(0,3,1,1,2)} \mid b \neq 0\}.$$

Proof. Let $\nu \in \mathfrak{I}_{22}$. By Lemma 5.4.29, $\widetilde{w} = \overline{m(0,3,1,1,1)} \in L(\nu)$ satisfies $E_i\widetilde{w} = 0$, $i = 1,2$, $g_1\sigma_1\widetilde{w} = \zeta^2\widetilde{w}$, $g_2\sigma_2\widetilde{w} = \widetilde{w}$; by Lemma 5.2.4, $L(\nu)^\varphi = L(\lambda)$. In particular, $\dim L(\lambda) = 49$.

[18] Here $\lambda_1^2\lambda_2 = -1$, but the relation in Remark 5.4.5 becomes trivial.

$W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b) [19]. Set $W = \overline{M(\lambda)/W(\lambda)}$, so $\dim W = 72$ since $\{\overline{n(0,b,c,d,e)}\}$ is a basis of W. Set $w = \overline{n(0,1,0,1,2)} - \zeta^{11}(3)_{\zeta^7}\overline{n(0,0,1,0,2)}$. As $F_i w = 0$, $i = 1,2$, $g_1\sigma_1 w = w$, $g_2\sigma_2 w = \zeta^7 w$, $\mathcal{U}w$ projects over $L(\mu)$, $\mu \in \mathfrak{I}_{13}$. Set $L'(\lambda) = W/\mathcal{U}$. Note that $49 = \dim L(\lambda) \leq \dim L'(\lambda) = \dim W - \dim \mathcal{U}w \leq \dim W - \dim L(\mu) = 49$, so $L(\lambda) = L(\lambda')$ is irreducible, and $\mathcal{U}w \simeq L(\mu)$. Now B_{45} is a basis of $L'(\lambda)$ since it spans a linear complement of $\mathcal{U}w$ in W. Here we use the basis B_{13} of $L(\mu)$ in Lemma 5.4.20 to compute $\mathcal{U}w$. $\square$

Lemma 5.4.53. *If* $\lambda \in \mathfrak{I}_{46}$, *then* $\dim L(\lambda) = 47$. *A basis of* $L(\lambda)$ *is*

$$\begin{aligned}\mathrm{B}_{46} = \{&\overline{n(0,b,c,d,e)}, \overline{n(0,1,0,2,0)}, \overline{n(0,3,1,2,0)} | d \leq 1\} \\ &- \{\overline{n(0,1,1,0,2)}, \overline{n(0,3,0,0,1)}, \overline{n(0,1,1,0,1)}\}.\end{aligned}$$

Proof. $W(\lambda)$ is a submodule of $M(\lambda)$ by Lemma 5.1.1(b). By Remark 5.4.7, $w = F_1^2 E_{112}^2 E_1^2 v_\lambda$ satisfies that $F_1 w = F_2 w = 0$. Then $\mathcal{U}w + W(\lambda)$ is a proper submodule. We claim that $L'(\lambda) = M(\lambda)/\mathcal{U}w + W(\lambda)$ is irreducible.

Applying repeteadly E_1, E_2 over w we obtain:

$$\begin{aligned}
\overline{n(0,0,0,2,0)} &= q_{21}\zeta^5\overline{n(0,0,1,0,1)} + q_{21}\zeta^9\overline{n(0,1,0,1,1)} + q_{21}(1+\zeta^5)\overline{n(0,2,0,0,2)},\\
\overline{n(0,3,0,0,1)} &= \zeta^{10}(\zeta^2+1)\overline{n(0,1,1,0,0)} + \zeta(\zeta^2+1)\overline{n(0,2,0,1,0)},\\
\overline{n(0,1,1,0,1)} &= \zeta^9\overline{n(0,2,0,1,1)} + \zeta^7\overline{n(0,3,0,0,2)},\\
\overline{n(0,2,0,2,0)} &= q_{21}\frac{(\zeta^2+1)\zeta^2}{3}\overline{n(0,3,0,1,1)} - \zeta^9\overline{n(0,1,1,1,0)},\\
\overline{n(0,1,0,2,2)} &= \zeta^2\overline{n(0,0,1,1,2)},\\
\overline{n(0,0,1,2,0)} &= q_{21}^3(\zeta-1)\overline{n(0,2,1,0,2)} - q_{21}^3\frac{(\zeta^2+1)}{3}\overline{n(0,3,0,1,2)},\\
\overline{n(0,3,0,2,0)} &= \zeta^{11}(4)_{\zeta^7}\overline{n(0,2,1,1,0)} - q_{21}\zeta^2(1-\zeta)\overline{n(0,3,1,0,1)},\\
\overline{n(0,1,1,2,0)} &= q_{21}^2(\zeta^{11}-1)\overline{n(0,2,1,1,1)} - \frac{q_{21}^3(4)_{\zeta^7}}{3}\overline{n(0,3,0,2,1)},\\
\overline{n(0,2,1,2,0)} &= \frac{q_{21}^2(\zeta^8-1)}{3}\overline{n(0,3,1,1,1)}.
\end{aligned}$$

Thus $L'(\lambda)$ is spanned by B_{46}, and then $\dim L'(\lambda) \leq 47$.

By Corollary 5.3.2 there exists $F \in \mathcal{U}^-$ such that $F\overline{n(0,3,1,2,0)} = v_\lambda$, since

$$\begin{aligned}
F_{12}\overline{n(0,0,0,0,0)} &= 0, & g_1g_2\sigma_1\sigma_2\overline{n(0,0,0,0,0)} &= \zeta^{11}\overline{n(0,0,0,0,0)},\\
F_{11212}\overline{n(0,3,0,0,0)} &= 0, & g_1^3g_2^2\sigma_1^3\sigma_2^2\overline{n(0,3,0,0,0)} &= -\overline{n(0,3,0,0,0)},\\
F_{112}\overline{n(0,3,1,0,0)} &= 0, & g_1^2g_2\sigma_1^2\sigma_2\overline{n(0,3,1,0,0)} &= -\overline{n(0,3,1,0,0)}.
\end{aligned}$$

Notice that $w' = \overline{n(0,3,1,2,0)}$ satisfies $E_1 w' = E_2 w' = 0$, $g_1\sigma_1 w' = w'$, $g_2\sigma_2 w' = \zeta^2 w'$, so $(\mathcal{U}w')^\phi$ projects over a simple module $L(\nu)$, $\nu \in \mathfrak{I}_{17}$. Then $\dim L'(\mu) \geq \dim \mathcal{U}w' \geq \dim L(\nu) = 47$. Thus $\dim L(\lambda) = 47$ and B_{46} is a basis of $L'(\lambda)$. Let

[19]Here $\lambda_1^3\lambda_2^2 = -1$, but the relation in Remark 5.4.9 becomes trivial.

$W \neq 0$ be a submodule of $L'(\lambda)$, $v \in W - 0$. Arguing as in previous cases, there exists $E \in \mathcal{U}^+$ such that $Ev = \overline{m(0,3,1,2,0)}$, so $\overline{m(0,3,1,2,0)} \in W$. Then $v_\lambda \in W$ and $W = L'(\lambda)$, so $L'(\lambda)$ is irreducible. □

Lemma 5.4.54. *If $\lambda \in \mathfrak{I}_{47}$, then $\dim L(\lambda) = 1$ and $E_i v_\lambda = 0$, $F_i v_\lambda = 0$, $g\sigma v_\lambda = \lambda(g\sigma)v_\lambda$.*

Proof. Let $N'(\lambda) = W(\lambda) + W_1(\lambda)$. By Corollary 5.1.1 $N'(\lambda)$ is a proper $\mathcal{U}$-submodule. By direct computation, $N'(\lambda) = \sum_{\beta \neq 0} M(\lambda)_\beta$. Therefore $L'(\lambda) = M(\lambda)/N'(\lambda)$ is one-dimensional and irreducible. □

We recall that $\lambda_i = \lambda(g_i \sigma_i)$, $i = 1, 2$.

This way we enunciate the main result of this work:

Theorem 5.4.55.

1. *The map $\Lambda \mapsto L(\Lambda)$ gives a bijective correspondence between $\widehat{\Gamma}$ and the irreducible representations of $\mathcal{U}$.*

2. *The structure of $L(\lambda)$ depends on the values of λ_i, $i = 1, 2$. The dimension and the maximal degree of $L(\lambda)$ appear in Table 5.1 and a basis description is given in Lemmas 5.4.2-5.4.54.*

Table 5.1: Dimensions and highest degrees of irreducible modules

Case	Conditions on λ_i	$\dim L(\lambda)$	max. degree	$L(\lambda)^{\varphi}$
1	$\lambda_1 \neq 1, \zeta^8$, $\lambda_1^2\lambda_2 \neq -1, \zeta^{10}$, $\lambda_1^3\lambda_2^2 \neq -1$, $\lambda_1\lambda_2 \neq \zeta, \zeta^4, \zeta^7$, $\lambda_2 \neq 1$	144	$(12, 8)$	Case 1
2	$\lambda_1 = 1, \lambda_2 \neq 1, \zeta, \zeta^4, \zeta^7, \zeta^3, \zeta^9, -1, \zeta^{10}$	48	$(10, 8)$	Case 2
3	$\lambda_1 = \zeta^8, \lambda_2 \neq 1, \zeta^5, \zeta^8, \zeta^{11}, \zeta^3, \zeta^9, \zeta^2, -1$	96	$(11, 8)$	Case 3
4	$\lambda_1^2\lambda_2 = -1, \lambda_1 \neq \pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2$	48	$(8, 6)$	Case 4
5	$\lambda_1^2\lambda_2 = \zeta^{10}, \lambda_1 \neq \pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2$	96	$(10, 7)$	Case 5
6	$\lambda_1^3\lambda_2^2 = -1, \lambda_1 \neq \pm 1, \zeta^8, \zeta^{10}, \zeta^4, \zeta^2$	72	$(9, 6)$	Case 6
7	$\lambda_1\lambda_2 = \zeta, \lambda_1 \neq 1, \zeta^8, \zeta, \zeta^4, \zeta^9$	36	$(9, 5)$	Case 7
8	$\lambda_1\lambda_2 = \zeta^4, \lambda_1 \neq 1, \zeta^8, \zeta^4, \zeta^2, -1, \zeta^{10}$	72	$(10, 6)$	Case 8
9	$\lambda_1\lambda_2 = \zeta^7, \lambda_1 \neq 1, \zeta^8, \zeta^7, \zeta^4, \zeta^{11}$	108	$(11, 7)$	Case 9
10	$\lambda_1 \notin \mathbb{G}_{12}, \lambda_2 = 1$	72	$(12, 7)$	Case 10
11	$\lambda_1 = 1,\ \lambda_2 = \zeta$	11	$(5, 4)$	Case 12
12	$\lambda_1 = 1,\ \lambda_2 = \zeta^4$	11	$(5, 4)$	Case 11
13	$\lambda_1 = 1,\ \lambda_2 = \zeta^7$	23	$(7, 5)$	Case 44
14	$\lambda_1 = 1,\ \lambda_2 = \zeta^3$	25	$(7, 5)$	Case 28
15	$\lambda_1 = 1,\ \lambda_2 = \zeta^9$	37	$(9, 6)$	Case 41
16	$\lambda_1 = 1,\ \lambda_2 = -1$	37	$(8, 6)$	Case 30
17	$\lambda_1 = 1,\ \lambda_2 = \zeta^{10}$	47	$(10, 7)$	Case 46
18	$\lambda_1 = \zeta^8,\ \lambda_2 = \zeta^5$	11	$(5, 3)$	Case 38
19	$\lambda_1 = \zeta^8,\ \lambda_2 = \zeta^8$	35	$(8, 5)$	Case 40
20	$\lambda_1 = \zeta^8,\ \lambda_2 = \zeta^{11}$	71	$(11, 7)$	Case 42
21	$\lambda_1 = \zeta^8,\ \lambda_2 = \zeta^3$	61	$(9, 6)$	Case 32
22	$\lambda_1 = \zeta^8,\ \lambda_2 = \zeta^9$	49	$(9, 6)$	Case 45
23	$\lambda_1 = \zeta^8,\ \lambda_2 = \zeta^2$	47	$(8, 6)$	Case 29
24	$\lambda_1 = \zeta^8,\ \lambda_2 = -1$	85	$(10, 7)$	Case 35
25	$\lambda_1 = \zeta^{11},\ \lambda_2 = \zeta^8$	37	$(8, 5)$	Case 37
26	$\lambda_1 = \zeta^5,\ \lambda_2 = \zeta^8$	25	$(8, 5)$	Case 43
27	$\lambda_1 = \zeta^4,\ \lambda_2 = \zeta^9$	35	$(9, 5)$	Case 36
28	$\lambda_1 = \zeta^9,\ \lambda_2 = \zeta^4$	25	$(7, 5)$	Case 14
29	$\lambda_1 = -1,\ \lambda_2 = -1$	47	$(8, 6)$	Case 23
30	$\lambda_1 = \zeta^2,\ \lambda_2 = \zeta^2$	37	$(8, 6)$	Case 16
31	$\lambda_1 = -1,\ \lambda_2 = \zeta^{10}$	61	$(10, 6)$	Case 39
32	$\lambda_1 = \zeta^{10},\ \lambda_2 = -1$	61	$(9, 6)$	Case 21
33	$\lambda_1 = \zeta^2,\ \lambda_2 = -1$	71	$(9, 6)$	Case 34
34	$\lambda_1 = \zeta^4,\ \lambda_2 = \zeta^3$	71	$(9, 6)$	Case 33
35	$\lambda_1 = \zeta^3,\ \lambda_2 = \zeta^4$	85	$(10, 7)$	Case 24
36	$\lambda_1 = \zeta,\ \lambda_2 = 1$	35	$(9, 5)$	Case 27
37	$\lambda_1 = \zeta^2,\ \lambda_2 = 1$	37	$(8, 5)$	Case 25
38	$\lambda_1 = \zeta^3,\ \lambda_2 = 1$	11	$(5, 3)$	Case 18
39	$\lambda_1 = \zeta^4,\ \lambda_2 = 1$	61	$(10, 6)$	Case 31

Case	Conditions on λ_i	$\dim L(\lambda)$	max. degree	$L(\lambda)^{\varphi}$
40	$\lambda_1 = \zeta^5$, $\lambda_2 = 1$	35	$(8, 5)$	Case 19
41	$\lambda_1 = -1$, $\lambda_2 = 1$	37	$(9, 6)$	Case 15
42	$\lambda_1 = \zeta^7$, $\lambda_2 = 1$	71	$(11, 7)$	Case 20
43	$\lambda_1 = \zeta^8$, $\lambda_2 = 1$	25	$(8, 5)$	Case 26
44	$\lambda_1 = \zeta^9$, $\lambda_2 = 1$	23	$(7, 5)$	Case 13
45	$\lambda_1 = \zeta^{10}$, $\lambda_2 = 1$	49	$(9, 6)$	Case 22
46	$\lambda_1 = \zeta^{11}$, $\lambda_2 = 1$	47	$(10, 7)$	Case 17
47	$\lambda_1 = 1$, $\lambda_2 = 1$	1	$(0, 0)$	Case 47

Proof. The algebra $\mathcal{U}$ satisfies the conditions on [RS, Section 2], so [RS, Corollary 2.6] applies and (1) follows since all the modules $L(\lambda)$, $\lambda \in \widehat{\Gamma}$, are finite-dimensional. For (2) we use Lemmas 5.4.2-5.4.54. □

Example 5.4.56. Applying Theorem 5.4.55 to the Example 4.1.4 we get that:

- There are 67 simple modules of dimension 144.
- There are 7 simple modules of dimension 108.
- There are 10 simple modules of dimension 96.
- There are 2 simple modules of dimension 85.
- There are 6 simple modules of dimension 72.
- There are 4 simple modules of dimension 71.
- There are 4 simple modules of dimension 61.
- There are 2 simple modules of dimension 49.
- There are 10 simple modules of dimension 48.
- There are 4 simple modules of dimension 47.
- There are 6 simple modules of dimension 37.
- There are 7 simple modules of dimension 36.
- There are 4 simple modules of dimension 35.
- There are 4 simple modules of dimension 25.
- There are 2 simple modules of dimension 23.
- There are 4 simple modules of dimension 11.
- There is one simple module of dimension 1.

Note that $\mathfrak{I}_6$ and $\mathfrak{I}_{10}$ are empty.

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Appendix A

Explicit formulas for some irreducible $\mathcal{U}$-modules

Table A.1: Irreducible modules for $\lambda \in \mathfrak{I}_{11}$

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	0	$v_{0,1}$	0	0
$v_{0,1}$	$v_{1,1}$	0	0	$(\zeta^{11}-1)v_{0,0}$
$v_{1,1}$	$v_{2,1}$	0	$q_{12}(\zeta-1)v_{0,1}$	0
$v_{2,1}$	0	$v_{2,2}$	$q_{12}\zeta^8(1+\zeta^3)v_{1,1}$	0
$v_{2,2}$	$v_{3,2}$	0	0	$q_{21}^2(1-\zeta)v_{2,1}$
$v_{3,2}$	$v_{4,2}$	$v_{3,3}$	$q_{12}^2(\zeta^2-1)v_{2,2}$	0
$v_{4,2}$	0	$v_{4,3}$	$2q_{12}^2(\zeta^2-1)v_{3,2}$	0
$v_{3,3}$	$q_{12}\frac{\zeta^8(\zeta^3-1)}{2}v_{4,3}$	0	0	$q_{21}^3(\zeta^2-1)v_{3,2}$
$v_{4,3}$	$v_{5,3}$	0	$2q_{12}^2(\zeta^2-1)v_{3,3}$	$q_{21}^4(\zeta^3-1)v_{4,2}$
$v_{5,3}$	0	$v_{5,4}$	$q_{12}^3\zeta^8(1-\zeta^{11})v_{4,3}$	0
$v_{5,4}$	0	0	0	$q_{21}^5(\zeta^{11}+1)v_{5,3}$

Table A.2: Irreducible modules for $\lambda \in \mathfrak{I}_{12}$

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	0	$v_{0,1}$	0	0
$v_{0,1}$	$v_{1,1}$	0	0	$(\zeta^{10}+1)v_{0,0}$
$v_{1,1}$	$v_{2,1}$	$v_{1,2}$	$q_{12}(\zeta-1)v_{0,1}$	0
$v_{2,1}$	0	$v_{2,2}$	$q_{12}\zeta^8(1+\zeta^3)v_{1,1}$	0
$v_{1,2}$	$\zeta^{11}(1+\zeta^3)q_{12}v_{2,2}$	0	0	$q_{21}(1+\zeta^3)\zeta^4 v_{1,1}$
$v_{2,2}$	$v_{3,2}$	0	$q_{12}(\zeta^3+1)\zeta^8 v_{1,2}$	$-q_{2,1}^2 v_{2,1}$
$v_{3,2}$	0	$v_{3,3}$	$q_{12}^2\zeta^{10}v_{2,2}$	0
$v_{3,3}$	0	0	0	$q_{21}^3\zeta^3(1-\zeta)v_{3,2}$
$v_{4,3}$	$\zeta^9 q_{12}v_{5,3}$	0	$q_{12}^4\zeta(3)_{\zeta^{11}}v_{3,3}$	0
$v_{5,3}$	0	$v_{5,4}$	$-q_{12}^2(1+\zeta^3)v_{4,3}$	0
$v_{5,4}$	0	0	0	$q_{21}^5(1-\zeta)\zeta^4 v_{5,3}$

Table A.3: Irreducible modules for $\lambda \in \mathfrak{I}_{18}$

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(\sigma_1)F_1 \cdot w$	$\lambda(g_2)^{-1}F_2 \cdot w$
$v_{0,0}$	$v_{1,0}$	$v_{0,1}$	0	0
$v_{1,0}$	0	$q_{21}\zeta^9(4)_\zeta v_{1,1}$	$(1+\zeta^2)v_{0,0}$	0
$v_{0,1}$	$\zeta^8(4)_\zeta v_{1,1}$	0	0	$(\zeta^7-1)v_{0,0}$
$v_{1,1}$	$\frac{q_{12}\zeta^4(4)_{\zeta^7}}{3}v_{2,1}$	0	$q_{12}(\zeta-1)v_{0,1}$	$(\zeta^{11}-1)v_{1,0}$
$v_{2,1}$	0	$q_{21}^2\zeta^{10}(4)_\zeta v_{2,2}$	$(1-\zeta^4)v_{1,1}$	0
$v_{2,2}$	$(1-\zeta^4)v_{3,2}$	0	0	$\frac{-(1+\zeta^2)(3)_{\zeta^7}}{3}v_{2,1}$
$v_{3,2}$	$v_{4,2}$	$q_{12}\zeta^{10}(4)_\zeta v_{3,3}$	$\zeta^{10}(4)_\zeta v_{2,2}$	0
$v_{4,2}$	0	$v_{4,3}$	$q_{12}^2\zeta(\zeta+1)v_{3,2}$	0
$v_{3,3}$	$\frac{q_{12}^4\zeta^7(4)_\zeta}{3}v_{4,3}$	0	0	$\frac{\zeta^8-1}{3}v_{3,2}$
$v_{4,3}$	$v_{5,3}$	0	$q_{12}^3(\zeta^{11}+1)(4)_\zeta^2 v_{3,3}$	$q_{21}^4(\zeta^{11}-1)v_{4,2}$
$v_{5,3}$	0	0	$q_{12}^3\zeta^4 v_{4,3}$	0

Table A.4: Irreducible modules for $\lambda \in \mathfrak{I}_{38}$

w	$E_1 \cdot w$	$E_2 \cdot w$	$\lambda(g_1^{-1})F_1 \cdot w$	$\lambda(g_2^{-1})F_2 \cdot w$
$v_{0,0}$	$v_{1,0}$	0	0	0
$v_{1,0}$	$v_{2,0}$	$\zeta^7 q_{21} v_{1,1}$	$(1-\zeta^3)v_{0,0}$	0
$v_{2,0}$	0	$\zeta^8 q_{21}^2(1+\zeta^3)v_{2,1}$	$\zeta^7(1+\zeta)v_{1,0}$	0
$v_{1,1}$	$v_{2,1}$	0	0	$(\zeta^{11}-1)v_{1,0}$
$v_{2,1}$	$v_{3,1}$	0	$q_{12}\zeta^8 v_{1,1}$	$(\zeta^{11}-1)v_{2,0}$
$v_{3,1}$	0	$q_{21}^2\zeta v_{3,2}$	$q_{12}\zeta^2 v_{2,1}$	0
$v_{3,2}$	$v_{4,3}$	0	0	$q_{21}\zeta^{11}(1-\zeta^3)v_{3,1}$
$v_{4,2}$	$v_{5,2}$	$q_{21}^2\zeta^{10}v_{4,3}$	$q_{12}^2(\zeta^{11}-1)v_{3,2}$	0
$v_{5,2}$	0	$q_{21}^3(3)_\zeta v_{5,3}$	$q_{12}^2\zeta^8(1+\zeta)v_{4,2}$	0
$v_{4,3}$	$v_{5,3}$	0	0	$q_{21}^2\zeta^{10}(3)_{\zeta^{11}}v_{4,2}$
$v_{5,3}$	0	0	$q_{12}^3\zeta^8(1+\zeta^2)v_{4,3}$	$q_{21}^2\zeta^{10}(3)_{\zeta^{11}}v_{5,2}$

Appendix B

Diagrams for some irreducible $\mathcal{U}$-modules

B.1

We give some diagrams that refer to the rank lattice of homogeneous elements for some irreducible $\mathcal{U}$-modules to explicit the relations between dual modules and understand the action of φ on the homogeneous elements on each rank $(a, b) = a\alpha_1 + b\alpha_2$. The basic diagram of possible rank distribution is given in B.1

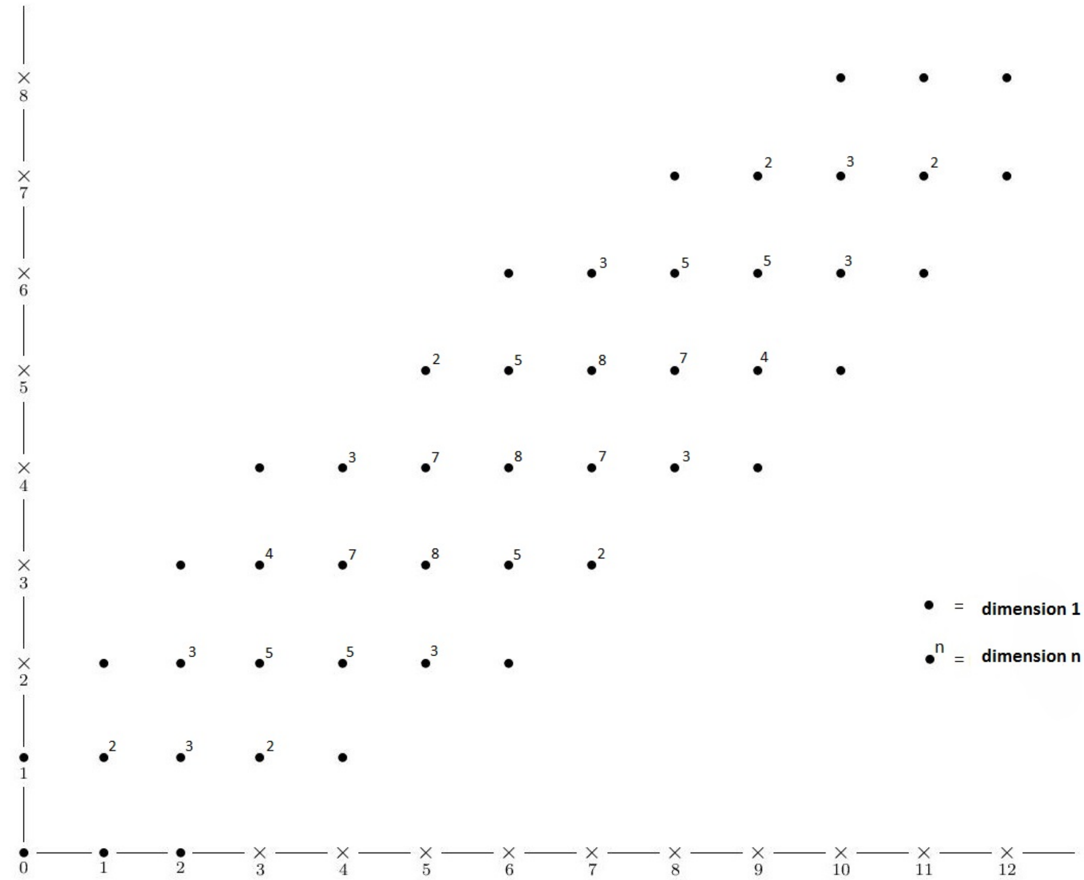
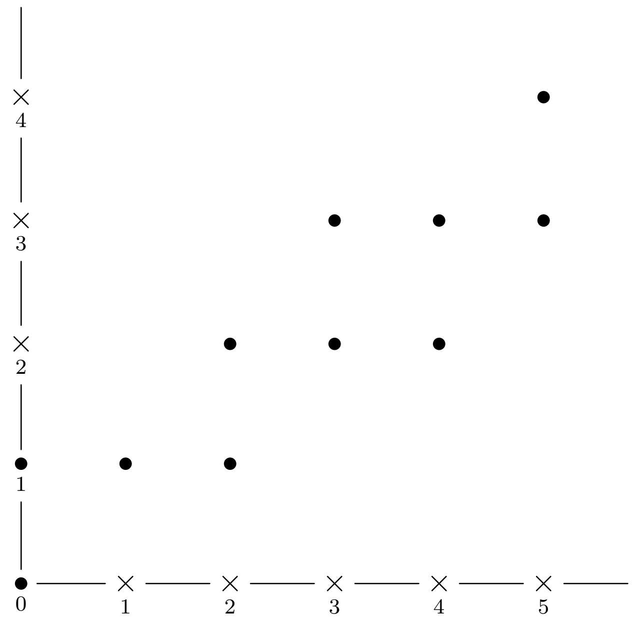


Figure B.1:

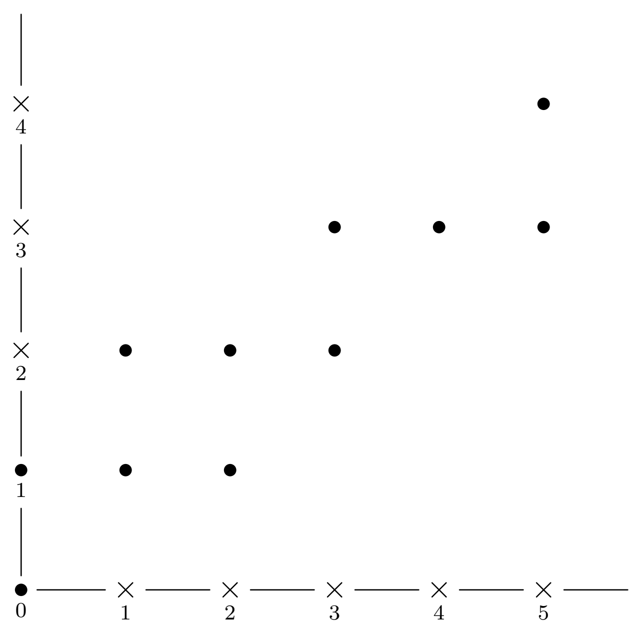
By Lemma 5.2.5, the modules U and U^{φ} have the same maximal element and their rank lattice are a 180° rotation of each other, as we can see in next example.

Example B.1.1. If $\lambda \in \mathfrak{I}_{11}$, then $\dim L(\lambda) = 11$ and $L(\lambda)^{\varphi}$ is as in Case 12. The rank diagram of Cases 11 and 12 are given below.

Case 11, $\lambda_1 = 1$, $\lambda_2 = \zeta$



Case 12, $\lambda_1 = 1$, $\lambda_2 = \zeta^4$



There is another relation we can identify on lattices, given by next example.

Example B.1.2. If $\lambda \in \mathfrak{I}_{15}$, we have that $M(\lambda)/\mathcal{U}E_1v_\lambda = W$, because $\lambda_1 = 1$. Besides that, we can see in 5.4.22, using Lemma 5.2.4, that $(\mathcal{U}w')^\varphi$ projects over an irreducible module $L(\nu)$ as in case 11, with $w' = \overline{m(1,3,1,2,0)}$. Therefore we can have the rank lattice of this Case in the following way: B.2 shows the basic rank lattice for $W = M(\lambda)/\mathcal{U}E_1 v_\lambda$ and using the diagram from Case 11 it annihilates elements from the top, using a 180° rotation of Case 11's rank lattice as we can see in B.3.

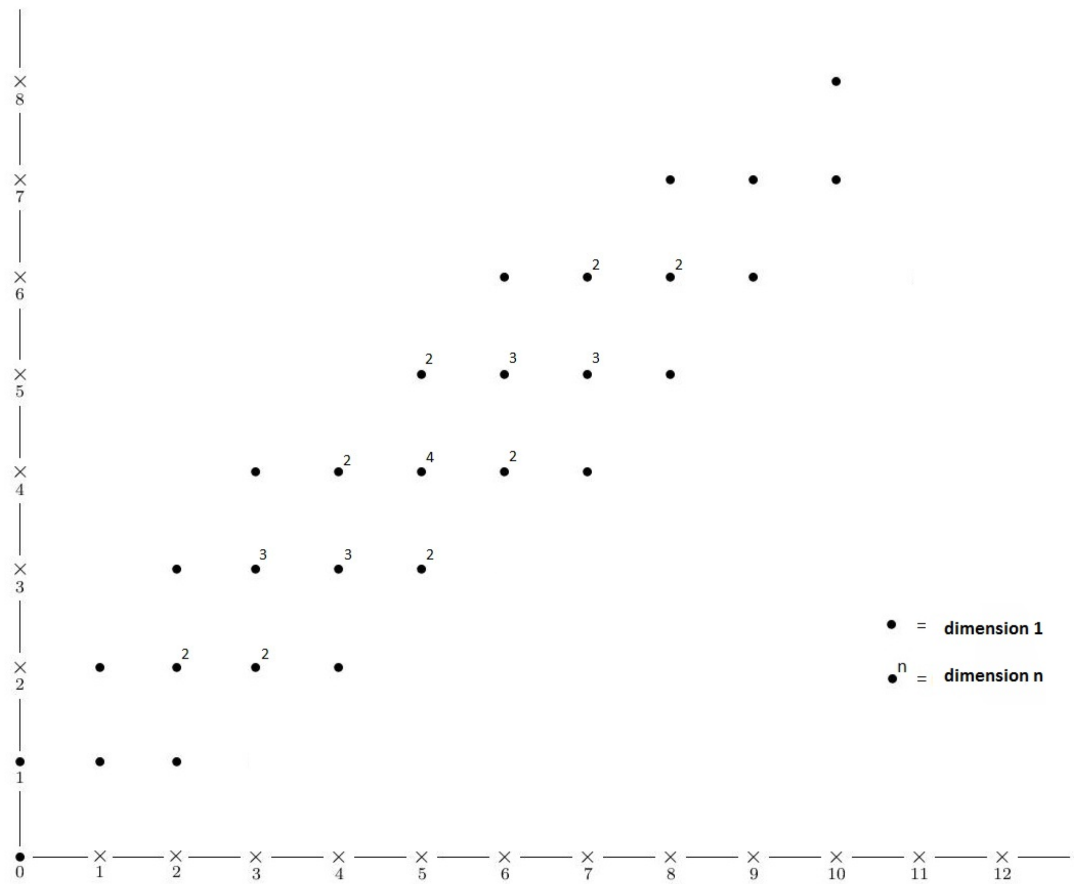


Figure B.2:

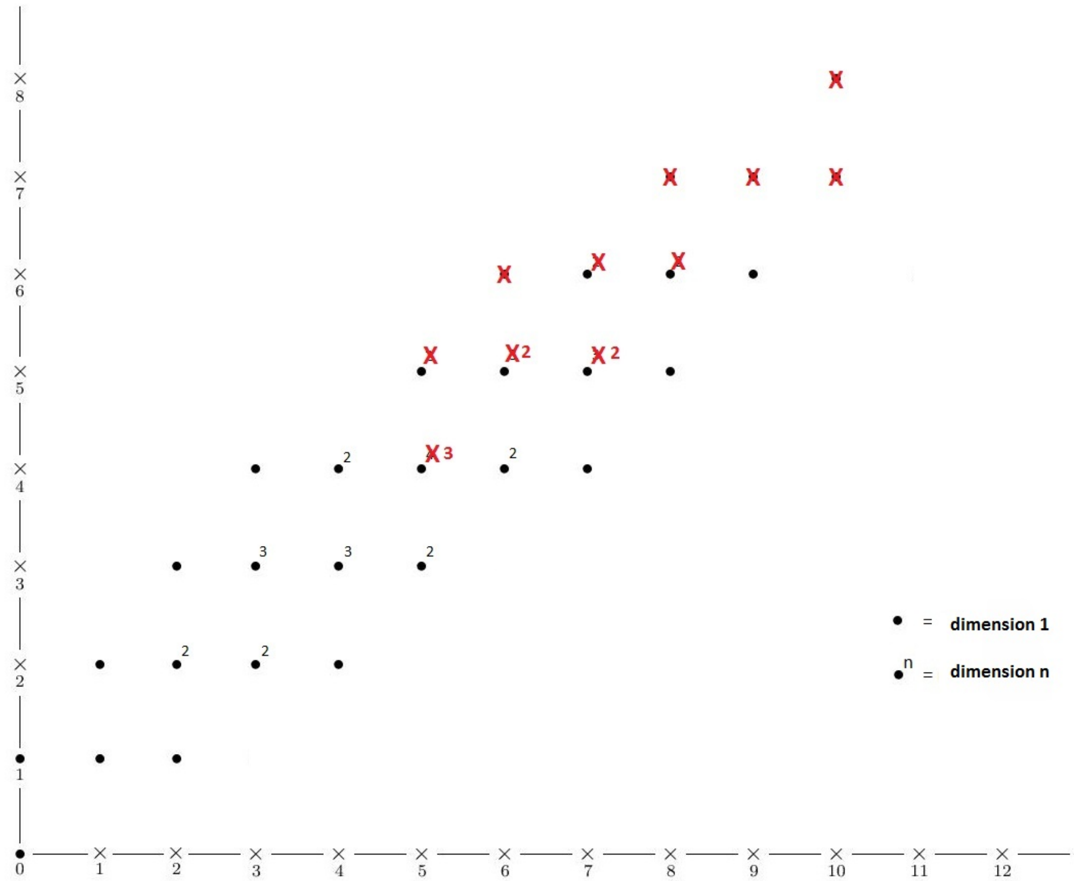


Figure B.3:

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Appendix C

Glossary

Symbol	**Description**
$\mathbf{k}$	Algebraically closed field of characteristic zero
$\mathbb{G}_N$	Group of N-roots of unity in $\mathbf{k}$
$\mathbb{G}'_N$	Subset of $\mathbb{G}_N$ of primitive roots of order N
$\widehat{G}$	Group of multiplicative characters of a group G
$Z(G)$	Center of G
$\mathcal{S}$	Antipode for a Hopf Algebra
$G(H)$	Set of grouplike elements of H
$\operatorname{Rep} H$	Tensor category of finite-dimensional representations of H
${}^{\mathbf{k}G}_{\mathbf{k}G}\mathcal{YD}$	Category of Yetter-Drinfeld modules over G
$\mathcal{B}(V)$	Nichols algebra of V
$D(H)$	Drinfeld double of H
$\mathcal{U} = \mathcal{U}(\mathcal{D}_{red})$	Drinfeld double of $\mathcal{B}(V)\#\mathbf{k}\Lambda$
$M(\lambda)$	Verma module associated to λ