UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
INSTITUTO DE MATEMATICA E ESTATÍSTICA
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

THE DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION ON UNBOUNDED HELICOIDAL DOMAINS OF $\mathbb{R}^m$

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Porto Alegre/RS

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Tese apresentada ao Programa de Pós-Gradua-ção em Matemática, área de Geometria Diferencial, da Universidade Federal do Rio Grande do Sul (UFRGS, RS), como requisito parcial para obtenção do título de Doutor em Matemática.

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Abstract

We consider a helicoidal group G in $\mathbb{R}^{n+1}$ and unbounded G-invariant $C^{2,\alpha}$-domains $\Omega \subset \mathbb{R}^{n+1}$ whose helicoidal projections are exterior domains in $\mathbb{R}^n$, $n \geq 2$. We show that for all $s \in \mathbb{R}$, there exists a G-invariant solution $u_s \in C^{2,\alpha}\left(\overline{\Omega}\right)$ of the Dirichlet problem for the minimal surface equation with zero boundary data which satisfies $\sup_{\partial\Omega} |\operatorname{grad} u_s| = |s|$. Additionally, we provide further information on the behavior of these solutions at infinity.

Keywords: Dirichlet problem; invariant domain; unbounded domains; Invariant solutions; Helicoidal group

1 Introduction

The Dirichlet problem for the minimal surface equation (mse) in $\mathbb{R}^m$, $m \geq 2$, namely,

$$\begin{cases} \mathcal{M}(u) := \operatorname{div}\left(\frac{\operatorname{grad} u}{\sqrt{1+|\operatorname{grad} u|^2}}\right) = 0 \text{ in } \Omega, \ u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right) \\ u|_{\partial\Omega} = \varphi \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^m$ is a C^2-domain and $\varphi \in C^0(\partial\Omega)$ is given a priori, has been intensively explored in the last decades. One of the most general answers to the Dirichlet problem (1) for bounded domain was given by H. Jenkins and J. Serrin in [10]. They showed that (1) is solvable for arbitrary $\varphi \in C^0(\partial\Omega)$ if only if Ω is mean convex. Moreover, they noted that if $\varphi \in C^2(\partial\Omega)$, a bound on the oscillation of φ in terms of the second order norm of φ should be enough to ensure the solvability of (1) on arbitrary bounded domains (Theorem 2 of [10]).

The study of the Dirichlet problem for the mse on unbounded domains began with J. C. C. Nitsche in the so called *exterior domains* that is, when $\mathbb{R}^m - \Omega$ is relatively compact (Section 4 of [16]). Since then several authors continue the investigation of the Dirichlet problem for the mse in exterior domains ([17], [14], [15], [3],[13] [18], [20] [1]).

The Dirichlet problem (1) for more general unbounded domains reduces, to authors knowledge, to few works: when $m = 2$, Rosenberg and Sa Earp ([7]) proved that (1) has a solution if $\Omega \subset \mathbb{R}^2$ is convex subset distinct of a half plane for any continuous boundary data φ. In the half plane case, in the case that φ is bounded, Collin and Krust ([3]) proved that if Ω is a convex set distinct of a half plane, then the solution is unique and, if Ω is a half plane then there is a unique solution with linear growth. For an arbitrary dimension m, Z. Jin, in [11], proved that (1) has a solution if Ω is a mean convex domain contained in some special like parabola-shape region or in the complement of a cone in $\mathbb{R}^m$. More recently N. Edelen and Z. Wang proved that if $\Omega \subsetneq \mathbb{R}^n$ is an open convex domain (e.g. a half-space) and $\varphi \in C^0(\partial\Omega)$ is a linear function, then any solution of (1) must also be linear.

In our work we obtain an extension of the exterior Dirichlet problem for the minimal surface equation in $\mathbb{R}^m$, $m \geq 3$, in the following sense: we say that a domain is $k-$bounded, $0 \leq k \leq m$, if it is bounded in k directions of the space $\mathbb{R}^m$ (as a direction we mean an equivalence class of parallel lines of $\mathbb{R}^m$). As so, a domain of $\mathbb{R}^m$ is relatively compact if and only if it is $m-$bounded. In our main results we study the Dirichlet problem for the mse on certain domains Ω of $\mathbb{R}^m$, $m \geq 3$, which complement $\mathbb{R}^m - \Omega$ is $(m-1)-$bounded, and for zero boundary data. We recall that Theorem 3.5 of [3] proves the existence of solutions of the Dirichlet problem for the mse on a strip, a special $1-$bounded domain of $\mathbb{R}^2$, for arbitrary continuous bounded boundary data.

To state precisely our theorems we need to recall a result of the third author and F. Tomi (Theorem 2 of [20]) which asserts that if Ω is G-invariant $C^{2,\alpha}$-domain for $m \geq 3$, where G is a subgroup of $ISO(\mathbb{R}^m)$ that acts freely and properly on $\mathbb{R}^m$, such that $P(\Omega)$ is a bounded and mean convex domain, then (1) has an unique G-invariant solution for any $G-$invariant boundary data $\varphi \in C^{2,\alpha}(\partial\Omega)$, where $P : \mathbb{R}^m \longrightarrow \mathbb{R}^m/G$ is the projection through the orbits of G and $\mathbb{R}^m/G$ is endowed with a metric such that P becomes a

Riemannian submersion.

Related to the above result, we would like to mention that the use of a Lie group of symmetries to study minimal surfaces was first considered by Wu-ye Hsiang and Blaine Lawson in [8]. Although proving distinct facts, we can say that Proposition 3 of [20] has the same spirit of Theorem 2 of [8]. Also related to these results, there is the idea of using Lie groups of symmetries to the study of minimal graphs (constant mean curvature graphs and more general PDE's too), as Killing graphs in warped products. This technique was first considered by Marcos Dajczer and the third author of this work in [4] and, since then, many works have been done extending and generalizing the results of [4], as [5], [6] and [9].

Let $\lambda \in \mathbb{R}$, $a \in \mathbb{R} - \{0\}$ and $i, j, k \in \{1, ..., n+1\}$ be given with any two i, j and k distinct. Consider the helicoidal like group $G \equiv G_{\lambda,a}^{i,j,k}$ in $\mathbb{R}^{n+1}$, $n \geq 2$, determined by the one parameter subgroup of isometries $G = \{\varphi_t\}_{t\in\mathbb{R}}$, where $\varphi_t : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is given by

$$\varphi_t(p) = \alpha(p)\, e_i + \beta(p)\, e_j + \gamma(p)\, e_k + \sum_{l \neq i,j,k} x_l e_l, \tag{2}$$

where $p = (x_1, ..., x_{n+1})$, $\{e_i\}_{i=1}^{n+1}$ is usual orthonormal basis of $\mathbb{R}^{n+1}$,

$$\alpha(p) = x_i \cos\lambda t + x_j \sin\lambda t,\ \beta(p) = x_j \cos\lambda t - x_i \sin\lambda t$$

and $\gamma(p) = x_k + at$.

Let $\pi : \mathbb{R}^{n+1} \longrightarrow \{x_k = 0\} \equiv \mathbb{R}^n$ be the *helicoidal* projection determined by G, that is,

$$\pi(p) = \widehat{\alpha}(p)\, e_i + \widehat{\beta}(p)\, e_j + \sum_{l \neq i,j,k} x_l e_l, \tag{3}$$

where

$$\widehat{\alpha}(p) = x_i \cos\frac{\lambda x_k}{a} - x_j \sin\frac{\lambda x_k}{a},\ \widehat{\beta}(p) = x_j \cos\frac{\lambda x_k}{a} + x_i \sin\frac{\lambda x_k}{a}.$$

Set

$$M := (\mathbb{R}^n, \langle , \rangle_G), \tag{4}$$

where $\langle\ ,\ \rangle_G$ is the metric such that π becomes a Riemannian submersion (clearly G acts freely and properly in $\mathbb{R}^{n+1}$ and $\{x_k = 0\} \equiv \mathbb{R}^n$ is a slice relatively to the orbits $Gp = \{\varphi_t(p),\, t \in \mathbb{R}\}$). One may see that the map $\psi : \mathbb{R}^{n+1}/G \to \mathbb{R}^n$ given by $\psi = \pi \circ P^{-1}$ is well defined and is an isometry with the metrics mentioned above. From the isometric submersion theory, it follows that M is complete.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a G-invariant domain of class $C^{2,\alpha}$ and set $\Lambda = \pi(\Omega)$. Let $d_E(p) = d_E(p, \partial\Omega)$, $p \in \Omega$, be the (Euclidean) distance in $\mathbb{R}^{n+1}$ to $\partial\Omega$ and let $d(q) = d_M(q, \partial\Lambda)$, $q \in \Lambda$, be the distance in M to $\partial\Lambda$. Given $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ and $\varphi \in C^0(\partial\Omega)$, G-invariant functions, that is, $u = v \circ \pi$ for some $v \in C^2(\Lambda) \cap C^0\left(\overline{\Lambda}\right)$ and $\varphi = \psi \circ \pi$ for some $\psi \in C^0(\partial\Lambda)$, it follows from Proposition 3 of [20] that u is solution of (1) (relatively to $m = n+1$) if, and only if,

$$\begin{cases} \mathfrak{M}(v) := \operatorname{div}_M\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) - \frac{1}{\sqrt{1+|\nabla v|^2}} \langle \nabla v, J\rangle_M = 0 \text{ on } \Lambda \\ v|_{\partial\Lambda} = \psi \end{cases} \tag{5}$$

where ∇ and div_M are the gradient and divergence in M, respectively, and

$$J\left(\pi\left(p\right)\right)=d\pi_p\left(\overrightarrow{H}_{Gp}\left(p\right)\right),\ p\in\mathbb{R}^{n+1}, \tag{6}$$

where $\overrightarrow{H}_{Gp}$ is the mean curvature vector of the 1-dimensional submanifold Gp of $\mathbb{R}^{n+1}$. Moreover, $\left|\overline{\nabla}u\right|=\left|\nabla v\right|\circ\pi$, where $\overline{\nabla}$ denotes the gradient in $\mathbb{R}^{n+1}$.

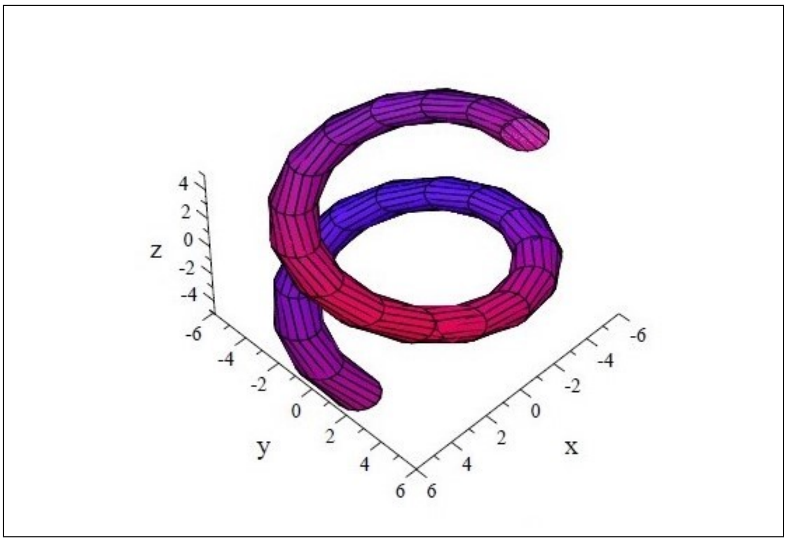

Figure 1: A G-invariant domain in $\mathbb{R}^3$

([1]) When Λ is a bounded and mean convex domain is proved in [20], as mentioned above, that there is an unique G-invariant solution $u\in C^{2,\alpha}\left(\overline{\Omega}\right)$ of (1) for all G-invariant $\varphi\in C^{2,\alpha}\left(\partial\Omega\right)$. In this note we work with the case where $\Lambda=\pi\left(\Omega\right)$ is an exterior domain in $\mathbb{R}^n$ and the boundary data is zero.

[1]The white region in Figure 1 is a G-invariant domain $\Omega\subset\mathbb{R}^3$ (with $\lambda=a=1$), whose boundary is the surface $\Psi:[0,\pi]\times\mathbb{R}\longrightarrow\mathbb{R}^3$ given by

$$\Psi\left(t,z\right)=\left(\cos t\cos z+\left(5+\sin t\right)\sin z,\left(5+\sin t\right)\cos z-\cos t\sin z,z\right).$$

Note that $\mathbb{R}^3-\overline{\Omega}$ is an example of a 2-bounded domain in $\mathbb{R}^3$.

We observe that Λ is an exterior domain in M if and only if Ω is $n-$bounded.

Regarding the condition of zero boundary data, we recall an old but quite suggestive result to our work of Osserman that proves that in $\mathbb{R}^2$ with the Euclidean metric, there is a boundary data on a disk D for which there is no solution to the exterior Dirichlet problem in $\mathbb{R}^2 - D$. This strongly suggests, since $K_M > 0$, that the zero boundary data condition can not also be dropped out in our case. But we don't have a counter example.

In order to state our main results, we remember that, relatively to an exterior domain $\mathbb{R}^n - \overline{\mathfrak{B}}_\rho\left(p_0\right)$ in $\mathbb{R}^n$, $n \geq 2$, where $\mathfrak{B}_\rho\left(p_0\right)$ is an open ball of $\mathbb{R}^n$ centered at $p_0 \in \mathbb{R}^n$ and of radius $\rho > 0$, the function

$$v_\rho\left(p\right) := \rho \int_1^{\frac{\tau}{\rho}} \frac{dt}{\sqrt{t^{2(n-1)} - 1}},\ \tau = \left|p - p_0\right|,\ p \in \mathbb{R}^n - \mathfrak{B}_\rho\left(p_0\right), \tag{7}$$

is the solution relatively to the Dirichlet problem (1) which satisfies

$$\lim_{p \to \partial\mathfrak{B}_\rho(p_0)} \left|\overline{\nabla} v_\rho\left(p\right)\right| = \infty,\ \lim_{|p| \to \infty} \left|\overline{\nabla} v_\rho\left(p\right)\right| = 0$$

(a half-catenoid). If $n \geq 3$, v_ρ is bounded and its height at infinity, which we denote by $h\left(n, \rho\right)$, is

$$h\left(n, \rho\right) = \rho h\left(n, 1\right) = \rho \int_1^{\infty} \frac{dt}{\sqrt{t^{2(n-1)} - 1}}. \tag{8}$$

In all of the results from now on, $G \equiv G_{\lambda,a}^{i,j,k}$ is as defined in (2), with λ, a, i, j, k fixed.

We prove:

Theorem 1 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a G-invariant and $C^{2,\alpha}$-domain such that $\pi(\Omega)$ is an exterior domain of $\mathbb{R}^n = \{x_k = 0\}$, where G and π are as defined in (2) and (3), respectively. Let $\varrho > 0$ be the radius of smallest geodesic ball of $\mathbb{R}^n$ which contain $\partial\pi(\Omega)$, centered at origin of $\mathbb{R}^n$ if $\lambda \neq 0$. Then, given $s \in \mathbb{R}$, there is a G-invariant solution $u_s \in C^{2,\alpha}\left(\overline{\Omega}\right)$ of the Dirichlet problem (1) with $u_s|_{\partial\Omega} = 0$, such that:*

i) $\sup_{\overline{\Omega}} \left|\overline{\nabla} u_s\right| = \sup_{\partial\Omega} \left|\overline{\nabla} u_s\right| = |s|$*;*

ii) u_s is unbounded if $n = 2$ and $s \neq 0$ and, if $n \geq 3$, either

$$\sup |u_s| \leq h(n, \varrho)$$

or there is a complete, non-compact, properly embedded n-dimensional submanifold $N \subset \Omega$, such that

$$\lim_{d_E(p)\to+\infty} u_s|_N (p) = h(n, \varrho),$$

where $h(n, \varrho)$ is given by (8);

iii) u_s satisfies

$$\lim_{d_E(p)\to\infty} \left|\overline{\nabla} u_s(p)\right| = 0$$

if $\lambda = 0$ or $s = 0$ or, if $\lambda \neq 0$, $3 \leq n \leq 6$ and $u_s$is bounded.

Additional informations relatively to set of G-invariant solutions of the Dirichlet problem (1) also are obtained under the assumption that $M - \pi(\Omega)$ satisfies the interior sphere condition of radius $r > 0$, that is, for each $q \in \partial\pi(\Omega)$, there is a geodesic sphere S_q of M of radius r contained in $M - \pi(\Omega)$ such that S_q is tangent to $\partial\pi(\Omega)$ at q and r is maximal with this property.

Given $a \in \mathbb{R} - \{0\}$, $\lambda \in \mathbb{R}$, $n \geq 3$ and $r > 0$ set

$$C = C(r, n, \lambda, a) := \frac{2|a|(n-1) + |\lambda| r}{2|a| r}. \tag{9}$$

Let $\varsigma > C$ be the solution of the equation

$$\cosh\left(\frac{\mu}{\sqrt{\mu^2 - C^2}}\right) = \frac{\mu}{C},\ \mu > C, \tag{10}$$

and set

$$\mathcal{L} = \mathcal{L}(r, n, \lambda, a) := \begin{cases} \frac{1}{\sqrt{\varsigma^2 - C^2}} \text{ if } \lambda \neq 0 \\ h(n, r) \text{ if } \lambda = 0 \end{cases}, \tag{11}$$

where $h(n, r)$ is given by (8).

Theorem 2 *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 3$, be a G-invariant and $C^{2,\alpha}$-domain such that $\pi(\Omega)$ is an exterior domain of $\mathbb{R}^n = \{x_k = 0\}$, where G and π are as defined in (2) and (3), respectively. Assume that $M - \pi(\Omega)$ satisfies the interior sphere condition of radius $r > 0$, where M is the n-dimensional Riemannian manifold given by (4). Let $\mathcal{L} = \mathcal{L}(r, n, \lambda, a)$ be as defined in (11), where λ and a are given in the definition of G. Then, given $c \in [0, \mathcal{L}]$, there is a G-invariant solution $u_c \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of (1) with $u_c|_{\partial\Omega} = 0$, such that*

$$\lim_{d_E(p) \to \infty} u_c(p) = c.$$

In particular, if $c \in [0, \mathcal{L})$ then $u_c \in C^{2,\alpha}(\overline{\Omega})$.

We note that our approach is not applicable for general boundary data.

Moreover, we were not able to prove that the solutions u_s obtained in Theorem 1 have a limit at infinity and this could be the subject of a future research.

2 Preliminaries

We first observe that, relatively to the PDE given in (5), the maximum and comparison principles work (see Section 3 of [20]).

In this section, we give further informations on M and we provide the basic results to construct barriers relative to the Dirichlet problem (5) when the boundary data is zero. We shall use the meaning of the indexes i and j as defined in (2).

Lemma 3 *Let G be as defined in (2). Given $p = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}$, the orbit Gp has constant curvature*

$$H_{Gp} = \frac{\lambda^2 \sqrt{x_i^2 + x_j^2}}{\lambda^2 (x_i^2 + x_j^2) + a^2}. \tag{12}$$

In particular,

$$\sup_{p \in \mathbb{R}^{n+1}} H_{Gp} = \frac{|\lambda|}{2\,|a|}, \tag{13}$$

and the supremum is attended, if $\lambda \neq 0$, at those orbits through the points p in $\mathbb{R}^{n+1}$ such that $|(x_i, x_j)| = |a| \,/\, |\lambda|$.

Proof. An arch length parametrization of Gp is given by

$$\gamma_p(s) = F\,(p)\, e_i + G\,(p)\, e_j + H\,(p)\, e_k + \sum_{l \neq i,j,k} x_l e_l.$$

where

$$F\,(p) = x_i \cos(A\,(p)\, s) + x_j \sin(A\,(p)\, s),\ G\,(p) = x_j \cos(A\,(p)\, s) - x_i \sin(A\,(p)\, s)$$

and

$$H\left(p\right)=x_{k}+\frac{aA\left(p\right)s}{\lambda},$$

with

$$A(p)=\frac{\lambda}{\sqrt{\lambda^{2}(x_{i}^{2}+x_{j}^{2})+a^{2}}}. \tag{14}$$

The mean curvature vector of Gp is then

$$\overrightarrow{H}_{Gp}(\gamma_{p}(s))=\gamma_{p}''(s)=-A^{2}\left(p\right)\left[F\left(p\right)e_{i}+G\left(p\right)e_{j}\right], \tag{15}$$

and, consequently, the mean curvature of Gp in $\mathbb{R}^{n+1}$ is given by (12). Since the mean curvature of the orbit Gp only depends on the Euclidean distance of $x_{i}e_{i}+x_{j}e_{j}$ to the origin, setting $\sigma\left(p\right)=\left|(x_{i},x_{j})\right|$, we have that $\xi:[0,\infty)\longrightarrow\mathbb{R}$ given by

$$\xi\left(\sigma\right)=\frac{\lambda^{2}\sigma}{\lambda^{2}\sigma^{2}+a^{2}}, \tag{16}$$

has a maximal absolute point at $\sigma_{0}=\left|a\right|/\left|\lambda\right|$ if $\lambda\neq0$ and the result follows.

■

Since $\pi:\mathbb{R}^{n+1}\longrightarrow M$ is a Riemannian submersion, given two orthogonal vector fields $X,Y\in\chi\left(M\right)$ and their respective horizontal lift $\overline{X},\overline{Y}$, we know that

$$K_{M}\left(X,Y\right)=K_{\mathbb{R}^{n+1}}\left(\overline{X},\overline{Y}\right)+\frac{3}{4}\left|\left[\overline{X},\overline{Y}\right]^{v}\right|_{\mathbb{R}^{n+1}}^{2}$$

where K and $\left[\overline{X},\overline{Y}\right]^{v}$ means, respectively, the sectional curvature and the vertical component of $\left[\overline{X},\overline{Y}\right]$, that is, the component which is tangent to the orbits Gp, $p\in\mathbb{R}^{n+1}$. As $K_{\mathbb{R}^{n+1}}\left(\overline{X},\overline{Y}\right)=0$, it follows that $K_{M}\left(X,Y\right)\geq0$

and, therefore, $Ric_M \geq 0$ (straightforward, but quite extensive calculations, give us that, in fact, $K_M > 0$ with $K_M \rightarrow 0$ at infinity).

Lemma 4 *Let Λ be an exterior domain in M. Denote by ν the horizontal lift of ∇d, where $d = d_M(., \partial\Lambda)$. Then*

$$\langle \nabla d, J \rangle_M \circ \pi = \left\langle \nu, \overrightarrow{H}_G \right\rangle \tag{17}$$

where $\langle , \rangle$ is the Euclidean metric and J is given by (6). In particular

$$-H_{Gp}(p) \leq \langle \nabla d, J \rangle_M (\pi(p)),$$

for all $p \in \pi^{-1}(\Lambda)$, where H_{Gp} is given by (12).

Proof. As $\overrightarrow{H}_G$ and ν are the horizontal lift of J and ∇d respectively, we have $J(\pi(p)) = d\pi_p\left(\overrightarrow{H}_{Gp}(p)\right)$ and $\nabla d(\pi(p)) = d\pi_p(\nu(p))$. Since π is a Riemannian submersion,

$$d\pi_p|_{[T_p\mathbb{R}^{n+1}]^h} : \left[T_p\mathbb{R}^{n+1}\right]^h \longrightarrow T_{\pi(p)}M$$

is an isometry, where $\left[T_p\mathbb{R}^{n+1}\right]^h$ means the horizontal vector space relatively to Gp at p and, from this, we have (17). In particular, $\left\langle \nu, \overrightarrow{H}_{Gp} \right\rangle$ is constant along Gp. Note that $|\nu| = 1$ since $|\nabla d| = 1$. Thus

$$-H_{Gp}(p) = -\left|\overrightarrow{H}_{Gp}(p)\right| \leq \left\langle \nu, \overrightarrow{H}_{Gp} \right\rangle (p)$$

and the result follows. ∎

Proposition 5 *Let G, π and $M = (\mathbb{R}^n, \langle, \rangle_G)$ as defined in (2), (3) and (4), respectively, and assume $n \geq 3$ and $\lambda \neq 0$. Let o be an arbitrary but fixed point of M and let $\Lambda = M - \overline{B_r(o)}$, where $B_r(o)$ is the open geodesic ball of M of radius r centered at o. Let $b \in \mathbb{R}$ satisfying $b > C$, where C is given by (9). Consider $\psi : [0, \infty) \longrightarrow \mathbb{R}$ given by*

$$\psi(t) = \frac{1}{b} \cosh^{-1}(1 + bt) \tag{18}$$

and $\Lambda_0 := \{q \in \Lambda; d(q) \leq t_0\}$, where

$$t_0 = \frac{b - C}{bC}. \tag{19}$$

Then $w = \psi \circ d : \Lambda_0 \to \mathbb{R}$ is such that $w \in C^2(\Lambda_0) \cap C^0\left(\overline{\Lambda}_0\right)$, $w|_{\partial\Lambda} = 0$, $w > 0$ on Λ_0,

$$\lim_{d(q)\to 0} |\nabla w(q)| = +\infty$$

and $\mathfrak{M}(w) \leq 0$ on Λ_0, where $\mathfrak{M}$ is the operator defined in (5).

Proof. Let $\varphi \in C^2((0, \infty)) \cap C^0([0, \infty))$ to be determined *a posteriori* and consider the function $w : \Lambda \subset M \longrightarrow \mathbb{R}$ given by $w(q) = (\varphi \circ d)(q)$. Straightforward calculations give us that, in Λ,

$$div_M \left(\frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} \right) = g(d) \Delta d + g'(d)$$

and

$$\frac{1}{\sqrt{1 + |\nabla w|^2}} \langle \nabla w, J \rangle_M = g(d) \langle \nabla d, J \rangle_M,$$

where

$$g\left(d\right) := \frac{\varphi'\left(d\right)}{\sqrt{1+\left[\varphi'\left(d\right)\right]^{2}}}, \tag{20}$$

Δ is the Laplacian in M and "$'$" means $\frac{\partial}{\partial d}$. Thus, $\mathfrak{M}\left(w\right) \leq 0$ in Λ if and only if

$$g\left(d\right)\Delta d + g'\left(d\right) - g\left(d\right)\left\langle \nabla d, J\right\rangle_{M} \leq 0 \text{ in } \Lambda. \tag{21}$$

From the Laplacian's Comparison Theorem, since $Ric_{M} \geq 0$ and $\dim M = n$, we have

$$\Delta d\left(q\right) \leq \frac{n-1}{d\left(q\right)+r} \leq \frac{n-1}{r},\ q \in \Lambda. \tag{22}$$

Now, we assume that our φ satisfies $\varphi\left(0\right) = 0$ and $\varphi'\left(d\right) > 0$ for $d > 0$ (consequently, $\varphi\left(d\right) > 0$ for $d > 0$). From (20), it follows that $g\left(d\right) > 0$ for $d > 0$ and, from (22), we conclude that if

$$\frac{\left(n-1\right)g\left(d\right)}{r} + g'\left(d\right) - g\left(d\right)\left\langle \nabla d, J\right\rangle_{M} \leq 0 \text{ in } \Lambda \tag{23}$$

then we have (21). From Lemma 4 and (13), we see that

$$-\frac{\left|\lambda\right|}{2\left|a\right|} \leq -H_{Gp}\left(p\right) \leq \left\langle \nabla d, J\right\rangle_{M} \circ \pi\left(p\right)$$

and, then, if

$$\frac{g'\left(d\right)}{g\left(d\right)} \leq -C \text{ in } \Lambda, \tag{24}$$

where C is given by (9), then we have (23). From (20), we see that (24) is equivalent to

$$\frac{\varphi''\left(d\right)}{\varphi'\left(d\right)} \leq -C\left(1+\left[\psi'\left(d\right)\right]^{2}\right) \text{ in } \Lambda. \tag{25}$$

We will assume from now on that our φ satisfies

$$\lim_{d\to 0}\varphi'(d) = +\infty.$$

A function φ which satisfies all the requirements demanded until now is given by

$$\varphi(t) = \alpha \cosh^{-1}(1+bt)$$

$t \geq 0$, with α, b positive constants to be determinate, where, here, $t = d(q)$, $q \in \Lambda$. Assuming such one φ, we see that (25) is equivalent to

$$\frac{-b(1+bt)}{\left((1+bt)^2 - 1 + \alpha^2 b^2\right)} \leq -C. \tag{26}$$

We assume α, b such that $\alpha b = 1$. Thus, we have (26) if

$$t \leq \frac{b-C}{bC},\ C < b.$$

Thus, assuming $b > C$, setting

$$t_0 := \frac{b-C}{bC},$$

considering the neighborhood of $\partial\Lambda$ in Λ given by

$$\Lambda_0 := \{q \in \Lambda; d(q) \leq t_0\},$$

and the function

$$\psi(t) = \frac{1}{b}\cosh^{-1}(1+bt),\ t \geq 0,$$

we have that

$$w(q) = \psi \circ d(q), \; q \in \Lambda_0, \tag{27}$$

satisfies

$$\mathfrak{M}(w) \leq 0$$

in Λ_0 (note that $\exp_{\partial\Lambda} : \partial\Lambda \times [0, t_0] \longrightarrow \Lambda_0$, $\exp_{\partial\Lambda}(p,t) = \exp_p t\eta(p)$, where η is the unit vector field normal to $\partial\Lambda$ that points to Λ, is a diffeomorphism, since Λ is the exterior of a geodesic ball in M). The others conclusion follow directly from the definition of ψ. ■

Corollary 6 *Assume the same hypotheses of Proposition 5. Let $\varsigma > C$ be the solution of the equation (10), where $C = C(r, n, \lambda, a)$ is given by (9). The function w given in Proposition 5 with the biggest height at $\partial\Lambda_0 - \partial\Lambda$ is obtained taking $b = \varsigma$. In particular, for such w,*

$$\sup_{\overline{\Lambda}_0} w = \sup_{\partial\Lambda_0 - \partial\Lambda} w = \mathcal{L},$$

where $\mathcal{L}$ is given by (11) and, setting $W : \overline{\Lambda} \subset M \longrightarrow \mathbb{R}$ given by

$$W(q) = \begin{cases} w(q), \; \text{if } q \in \Lambda_0 \\ \mathcal{L} \; \text{if } q \in \Lambda - \Lambda_0 \end{cases}, \tag{28}$$

we have $W \in C^0\left(\overline{\Lambda}\right)$ and radial with relation to the point o, $\mathfrak{M}(W) = 0$ on $\Lambda - \partial\Lambda_0$, $W|_{\partial\Lambda} = 0$, with

$$\lim_{d(q) \to 0} |\nabla W(q)| = +\infty.$$

Proof. Given $\mu > C$, take b as in Proposition 5 given by $b = \mu$. The correspondent t_0 is

$$t_0 = \frac{\mu - C}{\mu C}$$

and, at t_0, from (27), we have $\psi(t_0) = \mu^{-1}\cosh^{-1}(\mu C^{-1})$. The function

$$f(\mu) = \frac{1}{\mu}\cosh^{-1}\left(\frac{\mu}{C}\right),\ \mu > C,$$

clearly satisfies $f(\mu) \to 0$ when $\mu \to C$ and $f(\mu) \to 0$ when $\mu \to +\infty$ and the absolute maximal point of f in (C, ∞) is reach at the point ς which is solution of (10) and $f(\varsigma) = (\varsigma^2 - C^2)^{-1/2}$. The other conclusions follow from the definition of W and from the fact that Λ is the exterior of a geodesic ball of M center at o. $\blacksquare$

Remark 7 *We observe that if $\lambda = 0$, then G is the group of the translations in e_k- direction. In this case, we have $M \equiv \mathbb{R}^n$ and the domain Λ, as in the hypothesis of Proposition 5, is the exterior of the geodesic ball of $\mathbb{R}^n$ of radius r. Then v_r given by (7) is a solution of (5) if the boundary data is zero. In particular, if $n \geq 3$, its height at infinity is $h(n, r)$, where $h(n, r)$ is given by (8).*

Proposition 8 *Let G, π and $M = (\mathbb{R}^n, \langle , \rangle_G)$, $n \geq 2$, as defined in (2), (3) and (4), respectively. Let $\Lambda_\rho := M - \overline{\mathfrak{B}_\rho(0)}$, where $\mathfrak{B}_\rho(0)$ is the open geodesic ball of $\mathbb{R}^n$ of radius ρ centered at origin of $\mathbb{R}^n$. Then $v_\rho \in C^2(\Lambda_\rho) \cap C^0(\Lambda_\rho)$ given by (7) is a non-negative solution of the Dirichlet (5) relatively to Λ_ρ*

with $v_\rho|_{\partial\Lambda_\rho} = 0$*, which is unbounded if* $n = 2$ *and satisfies* $|\nabla v_\rho| \circ \pi = |\overline{\nabla} v_\rho|$*,*

$$\lim_{d(q)\to 0} |\nabla v_\rho(q)| = \infty,\ \lim_{d(q)\to\infty} |\nabla v_\rho(q)| = 0 \tag{29}$$

and

$$\lim_{d(q)\to\infty} v_\rho(q) = h(n,\rho) \ \ \textit{if } n \geq 3 \tag{30}$$

where $d = d_M(.,\partial\Lambda_\rho)$*,* $h(n,\rho)$ *is given by (8).*

Proof. Let $v_\rho : \mathbb{R}^n - \mathfrak{B}_\rho(0) \to \mathbb{R}$ be the function given by (7), $\mathbb{R}^n \equiv \{x_k = 0\}$. As the graph of v_ρ is a minimal graph in $\mathbb{R}^{n+1}$ it follows that the graph of

$$u_\rho(x_1, ..., x_{n+1}) := v_\rho(x),\ x = \sum_{i\neq k} x_i e_i \in \mathbb{R}^n - \mathfrak{B}_\rho(0),$$

is a minimal graph in $\mathbb{R}^{n+2}$. In particular $|\overline{\nabla} u_\rho(x_1, ..., x_{n+1})| = |\overline{\nabla} v_\rho(x)|$. Note that, setting $\Omega_\rho \subset \mathbb{R}^{n+1}$ the domain of u_ρ, we have Ω_ρ a G-invariant domain such that its helicoidal projection $\pi(\Omega)$ on $\mathbb{R}^n$ coincides with the image of the its orthogonal projection on $\mathbb{R}^n$ in this case. From (3) we see that $|x| = |\pi(x_1, ..., x_{n+1})|$. As v_ρ is radial, it follows that

$$u_\rho(x_1, ..., x_{n+1}) = (v_\rho \circ \pi)(x_1, ..., x_{n+1})$$

and, then, u_ρ is a G-invariant function with $u_\rho|_{\partial\Omega_\rho} = 0$. It follows from Proposition 3 of [20] that $v_\rho \in C^2(\Lambda_\rho) \cap C^0(\Lambda_\rho)$ is a solution of the Dirichlet problem (5) relatively to Λ_ρ with $v_\rho|_{\partial\Lambda_\rho} = 0$ and, moreover,

$$|\nabla v_\rho| \circ \pi = |\overline{\nabla} u_\rho|.$$

The other conclusions follow immediately from the definition of u_ρ and from the properties satisfied by v_ρ, taking into account that $d\,(q) \to 0$ $(d\,(q) \to \infty)$ if only if $d_E\,(q, \partial\Lambda_\rho) \to 0$ $(d_E\,(q, \partial\Lambda_\rho) \to \infty)$. $\blacksquare$

3 Proof of the main results

The proof of Theorem 1 and Corollary 2 follow directly from the results of this section, by using Proposition 3 of [20].

Proposition 9 *Let G, π and $M = (\mathbb{R}^n, \langle , \rangle_G)$ as defined in (2), (3) and (4), respectively. Let U be an exterior $C^{2,\alpha}$-domain in M, $U = \pi(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is a G-invariant $C^{2,\alpha}$-domain and let $d = d_M(., \partial U)$. Let $\varrho > 0$ be the radius of the smallest geodesic ball of $\mathbb{R}^n$ which contain $M - U$, centered at origin of $\mathbb{R}^n$ if $\lambda \neq 0$. Given $s \geq 0$, there is a non-negative solution $\vartheta_s \in C^{2,\alpha}\left(\overline{U}\right)$ of the Dirichlet problem (5) with $\vartheta_s|_{\partial U} = 0$, such that*

$$\sup_{\overline{U}} |\nabla \vartheta_s| = \sup_{\partial U} |\nabla \vartheta_s| = s,$$

ϑ_s is unbounded if $n = 2$ and $s \neq 0$ and, if $n \geq 3$, either

$$\sup_{\overline{U}} \vartheta_s \leq h(n, \varrho) \tag{31}$$

or there is a complete, non-compact, properly embedded $(n-1)$-dimensional submanifold Σ of M, $\Sigma \subset U$, such that

$$\lim_{d(q) \to +\infty} \vartheta_s|_\Sigma (q) = h(n, \varrho), \tag{32}$$

where $h(n, \varrho)$ is given by (8). Moreover,

$$\lim_{d(q) \to \infty} |\nabla \vartheta_s(q)| = 0$$

if $\lambda = 0$ or $s = 0$ or, if $\lambda \neq 0$, $3 \leq n \leq 6$ and ϑ_s is bounded.

Proof. As mentioned in Remark 7, if $\lambda = 0$ then $M \equiv \mathbb{R}^n$ and the result is already contemplate in Theorem 1 of [1] for $n \geq 3$ and in Theorem 1 of [19] if $n = 2$. The case $s = 0$ is trivial. Assume then $s > 0$ and $\lambda \neq 0$.

Let $\rho > 0$ be such that $\partial U \subset \mathfrak{B}_\rho(0)$, where $\mathfrak{B}_\rho := \mathfrak{B}_\rho(0)$ is the open geodesic ball of $\mathbb{R}^n$, centered at origin of $\mathbb{R}^n$ and of radius ρ. Let $\Lambda_\rho = M - \overline{\mathfrak{B}_\rho(0)}$. From Proposition 8 , there is $v_\rho \in C^\infty(\Lambda_\rho)$ solution of (5) relatively to Λ_ρ, with $v_\rho|_{\partial\Lambda_\rho} = 0$, satisfying what is stated in (29). Since

$$\lim_{d_M(q,\partial\Lambda_\rho)\to\infty} |\nabla v_\rho(q)| = 0, \tag{33}$$

we can choose $k > \rho$ such that

$$|\nabla v_\rho|_{\partial\mathfrak{B}_k(0)} \leq \frac{s}{2}, \tag{34}$$

where $\mathfrak{B}_k(0)$ is the open geodesic ball of $\mathbb{R}^n$ centered at origin and of radius k. Let $U_k = \mathfrak{B}_k(0) \cap U$ and define

$$T_k := \left\{ \begin{array}{c} t \geq 0\, ; \exists\ w_t \in C^{2,\alpha}\left(\overline{U}_k\right); \mathfrak{M}(w_t) = 0 \\ \sup_{\overline{U}_k} |\nabla w_t| \leq s,\ w_t|_{\partial U} = 0, w_t|_{\partial\mathfrak{B}_k(0)} = t \end{array} \right\}. \tag{35}$$

Note that the constant function $w_0 \equiv 0$ on $\overline{U}_k$ satisfies all the condition in (35), then $T_k \neq \emptyset$. Moreover, $\sup T_k < \infty$ since

$$\sup_{\overline{U}_k} |\nabla w_t| \leq s$$

for all $t \in T_k$. Now, since the maximum principle and comparison principle are applicable relatively to the operator $\mathfrak{M}$, we can use the same approach used in the proof of Theorem 1 of [1] to show that $t_k := \sup T_k \in T_k$,

$$\sup_{\overline{U}_k} \|\nabla w_{t_k}\| = s, \ \ \sup_{\partial \mathfrak{B}_k} |\nabla w_{t_k}| \leq s/2$$

and, since $\partial U_k = \partial U \cup \partial \mathfrak{B}_k (0)$, to conclude that

$$\sup_{\overline{U}_k} |\nabla w_{t_k}| = \sup_{\partial U} |\nabla w_{t_k}| = s. \tag{36}$$

The proof of these facts follow essentially the same steps of the aforementioned theorem (see p. 3067 and 3068 of [1]), and then we will not do it here. Now, taking $k \to \infty$ and from diagonal method, we obtain a subsequence of (w_{t_k}) which converges uniformly in the C^2 norm in compact subsets of $\overline{U}$ to a function $\vartheta_s \in C^{2,\alpha}\left(\overline{U}\right)$ satisfying $\mathfrak{M}(\vartheta_s) = 0$ in U, $\vartheta_s|_{\partial U} = 0$, which is non-negative and such that $\sup_{\overline{U}} |\nabla \vartheta_s| = \sup_{\partial U} |\nabla \vartheta_s| = s$. In particular, from regularity elliptic PDE theory ([12]), we have $\vartheta_s \in C^\infty (U)$.

We will show now that if $n = 2$ then ϑ_s is unbounded and, if $n \geq 3$, we have either (31) or (32).

Let $\varrho > 0$ be the radius of the smallest open geodesic ball of $\mathbb{R}^n$ which contain $M - U$, centered at origin of $\mathbb{R}^n$ and denote such ball by $\mathfrak{B}_\varrho (0)$. We have $\partial U \subset \overline{\mathfrak{B}_\varrho (0)}$ and we can conclude that $\partial U \cap \partial \mathfrak{B}_\varrho \neq \emptyset$. Let $\Lambda_\varrho = M - \overline{\mathfrak{B}_\varrho (0)}$. From Proposition 8 , there is $v_\varrho \in C^\infty (\Lambda_\varrho)$, $v_\rho|_{\partial \Lambda_\varrho} = 0$, solution of (5), satisfying what is stated in (29) and (30) relatively to Λ_ϱ if $n \geq 3$. Otherwise, if $n = 2$, v_ϱ is unbounded and satisfies the equalities in (29).

Let $q_0 \in \partial U \cap \partial \mathfrak{B}_\varrho$. Since

$$\lim_{d_M(q,\partial\Lambda_\varrho)\to 0} |\nabla v_\varrho (q)| = +\infty, \sup_{\overline{U}} |\nabla \vartheta_s| = \sup_{\partial U} |\nabla \vartheta_s| = s < +\infty$$

and $v_\varrho (q_0) = \vartheta_s (q_0) = 0$, it follows that there is an open set V_{q_0} in $U \cap \Lambda_\varrho$, with $q_0 \in \partial V_{q_0}$, such that $\vartheta_s < v_\varrho$ in V_{q_0}. We claim that V_{q_0} is unbounded. Suppose that V_{q_0} is bounded. Since $\vartheta_s|_{\partial V_{q_0}} = v_\varrho|_{\partial V_{q_0}}$, it follows that $\vartheta_s|_{\overline{V}_{q_0}}$ and $v_\varrho|_{\overline{V}_{q_0}}$ are distinct solutions to the Dirichlet problem

$$\begin{cases} \mathfrak{M}(f) = 0 \text{ in } V_{q_0},\ f \in C^2(V_{q_0}) \cap C^0(\overline{V}_{q_0}) \\ f|_{\partial V_{q_0}} = \vartheta_s|_{\partial V_{q_0}} \end{cases}, \tag{37}$$

a contradiction, since a solution of (37) is unique if the domain is bounded. It follows that V_{q_0} is unbounded. Note that we can have two possibility for ∂V_{q_0}: either ∂V_{q_0} is bounded (in this case $V_{q_0} = \Lambda_\varrho$) or ∂V_{q_0} is unbounded (in this case, setting $\Sigma = \partial V_{q_0}$, we have $\Sigma \subset \overline{U} \cap \overline{\Lambda}_\varrho$ a complete $(n-1)$-dimensional manifold of M).

Assume first that $n \geq 3$. In this case v_ϱ is bounded. If ∂V_{q_0} is bounded, as in this case $V_{q_0} = \Lambda_\varrho$, this means that $\vartheta_s < v_\varrho$ on Λ_ϱ and so we have (31). If ∂V_{q_0} is unbounded, as $\vartheta_s|_{\partial V_{q_0}} = v_\varrho|_{\partial V_{q_0}}$, we conclude that

$$\lim_{d(q)\to+\infty} \vartheta_s|_{\partial V_{q_0}} (q) = h(n, \varrho).$$

Assume now that $n = 2$. Note first that on $\Lambda_\varrho \subset \mathbb{R}^2$ it is well know that, for all $s > 0$, there is a half catenoid $v_{\varrho,s} \in C^2\left(\overline{\Lambda}_\varrho\right)$ which is unbounded (logarithmic growth), which satisfies $v_{\varrho,s}|_{\partial\Lambda_\varrho} = 0$ and $\left|\overline{\nabla} v_{\varrho,s}\right| = s$ on $\partial\Lambda_\varrho$

$(= \partial\mathfrak{B}_\varrho)$. The same arguments used in Proposition 8 give us that $v_{\varrho,s}$ is solution for the Dirichlet problem (5) for zero boundary data relatively to Λ_ϱ and satisfies $|\nabla v_{\varrho,s}| = s$ on $\partial\Lambda_\varrho$, since $\mathfrak{B}_\varrho$ is centered at origin of $\mathbb{R}^2$. As $\sup_{\partial U} |\nabla\vartheta_s| = s$ and $\partial U \cap \partial\mathfrak{B}_\varrho \neq \emptyset$, there is $0 < s' < s$ such that $v_{\varrho,s'} \in C^2\left(\overline{\Lambda}_\varrho\right)$ as described above satisfies $v_{\varrho,s'} < \vartheta_s$ in some open set $V \subset U \cap \Lambda_\varrho$. The same arguments used before to prove that V_{q_0} is unbounded give us that V is unbounded and we see that if ∂V is bounded then $V = \Lambda_\varrho$. Thus, if ∂V is bounded, we have $v_{\varrho,s'} < \vartheta_s$ on Λ_ϱ and then, ϑ_s is unbounded. If ∂V is unbounded, as $v_{\varrho,s'}|_{\partial V} = \vartheta_s|_{\partial V}$, we have

$$\lim_{d(q)\to+\infty} \vartheta_s|_{\partial V}(q) = +\infty$$

since

$$\lim_{d(q)\to+\infty} v_{\varrho,s'}|_{\partial V}(q) = +\infty.$$

and, therefore, ϑ_s is unbounded.

Now we will prove the last affirmation of the proposition.

As $\vartheta_s \in C^{2,\alpha}\left(\overline{U}\right)$ is a solution of the Dirichlet problem (5) with $\vartheta_s|_{\partial U} = 0$, it follows from Proposition 3 of [20] that $u = \vartheta_s \circ \pi \in C^{2,\alpha}\left(\overline{\Omega}\right)$ satisfies $\mathcal{M}(u) = 0$ in $\overline{\Omega}$ with $u|_{\partial\Omega} = 0$ and $\left|\overline{\nabla} u(p)\right| = |\nabla(\vartheta_s \circ \pi)(p)|$, $p \in \Omega$, where $\mathcal{M}$ is the operator defined in (1). Note that we have necessarily Ω unbounded with $d_E(p) \to \infty$ if only if $d(\pi(p)) \to \infty$, where d_E is the Euclidean distance in $\mathbb{R}^{n+1}$ to $\partial\Omega$. Suppose that

$$\lim_{d_E(p)\to\infty} \left|\overline{\nabla} u(p)\right| \neq 0.$$

Then there is $\varepsilon > 0$ and a sequence (p_n) in Ω, with $d_E(p_n) \to \infty$ when $n \to \infty$ such that $\left|\overline{\nabla} u(p_n)\right| \geq \varepsilon$ for all n large enough, $n \geq n_0$. For each $n \in \mathbb{N}$, define

$$\Omega_n = \left\{p \in \mathbb{R}^{n+1}; p + p_n \in \Omega\right\}$$

and consider the sequence of functions $u_n : \Omega_n \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ given by $u_n(p) = u(p + p_n)$. Note that $0 \in \Omega_n$ for all n since $(p_n) \subset \Omega$. Also

$$\mathbb{R}^{n+1} = \bigcup_{n \in \mathbb{N}} \Omega_n.$$

Indeed, given $w \in \mathbb{R}^{n+1}$, if the sequence $(w+p_n)_n$ were contained in $\mathbb{R}^{n+1} - \Omega$, since $\pi(\mathbb{R}^{n+1} - \Omega)$ is compact, we would have $d_E(w + p_n) \leq R$ for all n, for some $R > 0$, a contradiction, since $d_E(p_n) \to \infty$. It follows that, as $u_n(0) = u(p_n)$, for all $n \geq n_0$ we have $\left|\overline{\nabla} u_n(0)\right| \geq \varepsilon$. Note that (u_n) is uniformly bounded since, by hypothesis, $n \geq 3$ and ϑ_s is bounded. Then (u_n) has a subsequence (u_{n_k}) which converges uniformly on compact subsets of $\mathbb{R}^{n+1}$ to a bounded function $\widetilde{u}$ defined on the whole $\mathbb{R}^{n+1}$ which satisfies $\mathcal{M}(\widetilde{u}) = 0$. Assume that dimension of M satisfies $3 \leq n \leq 6$. Then $\Omega \subset \mathbb{R}^m$, $4 \leq m \leq 7$. From Bersntein Theorem extended to $\mathbb{R}^m$, $2 \leq m \leq 7$, by Simons ([21]) - which is false for $m \geq 8$ ([2]) - it follows that $\widetilde{u}$ has to be constant. Therefore, we cannot have $\left|\overline{\nabla} u_{n_k}(0)\right| \geq \varepsilon$ for all $n_k \geq n_0$, a contradiction. Hence

$$\lim_{d_E(p) \to \infty} \left|\overline{\nabla} u(p)\right| = 0.$$

From Proposition 3 of [20] we have $\left|\overline{\nabla}u\right| = |\nabla\vartheta_s| \circ \pi$ and, therefore

$$\lim_{d(q)\to\infty} |\nabla\vartheta_s(q)| = 0.$$

■

Remark 10 *If $\lambda = 0$ and $n = 2$, it was proved in [19] that, setting*

$$\vartheta_\infty(p) := \lim_{s\to\infty} \vartheta_s(p), p \in U, \tag{38}$$

$\vartheta_\infty \in C^2(U) \cap C^0\left(\overline{U}\right)$ and is an unbounded solution of the Dirichlet problem (5) with $\vartheta_\infty|_{\partial U} = 0$, which satisfies

$$\lim_{d_E(p)\to\infty} \left|\overline{\nabla}u(p)\right| = 0.$$

If $\lambda = 0$ and $n \geq 3$, it was proved in [1] that ϑ_∞, as defined in (38), is in $C^2(U)$, is a bounded solution of the Dirichlet problem (5) and its graph is contained in a $C^{1,1}$-manifold $\Upsilon \subset \overline{U} \times \mathbb{R}$ such that $\partial\Upsilon = \partial\Omega$.

Proposition 11 *Let G, π and $M = (\mathbb{R}^n, \langle,\rangle_G)$, $n \geq 3$, as defined in (2), (3) and (4), respectively. Let U be an exterior $C^{2,\alpha}$-domain in M, $U = \pi(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is a G-invariant $C^{2,\alpha}$-domain and let $d = d_M(., \partial U)$. Assume that $M - U$ satisfies the interior sphere condition of radius $r > 0$. Let $\mathcal{L} = \mathcal{L}(r, n, \lambda a)$ be as given in (11), where λ and a are given in the definition of G. Then, given $c \in [0, \mathcal{L}]$, there is $w_c \in C^2(U) \cap C^0\left(\overline{U}\right)$ solution of the*

Dirichlet problem (5) relatively to U, with $w_c|_{\partial U} = 0$, which satisfies

$$\lim_{d(q)\to\infty} w_c(q) = c.$$

In particular, if $c \in [0, \mathcal{L})$, then $w_c \in C^{2,\alpha}\left(\overline{U}\right)$.

Proof. If $\lambda = 0$ the result is already contemplate in Theorem 1 of [1]. Assume $\lambda \neq 0$. Given $c \in [0, \mathcal{L})$, define

$$\mathcal{F} = \left\{ \begin{array}{c} f \in C^0\left(\overline{U}\right);\ f \text{ is subsolution relative to } \mathfrak{M} \\ f|_{\partial U} = 0 \text{ and } \lim\limits_{d(q)\to\infty} \sup f(q) \leq c \end{array} \right\}. \tag{39}$$

Note that $\mathcal{F} \neq \emptyset$ since $f_0 \equiv 0 \in \mathcal{F}$. From comparison principle we have $f \leq c$ for all $f \in \mathcal{F}$. From Perron method applied relatively to operator $\mathfrak{M}$ ([12] - Section 2.8, [20] - Section 3), we conclude that

$$w_c(q) := \sup\{f(q);\ f \in \mathcal{F}\},\ q \in \overline{U},$$

is in $C^\infty(U)$ and satisfies $\mathfrak{M}(w_c) = 0$ in U. We will show now that

$$\lim_{d(q)\to\infty} w_c(q) = c. \tag{40}$$

Consider $\alpha > 0$ such that $\mathfrak{B}_\alpha(0)$, the geodesic ball of $\mathbb{R}^n$ center at origin of $\mathbb{R}^n$, of radius $\alpha > 0$, contain $M - U$ and be such that $v_\alpha(\infty) > c$, where v_α is as defined in (7) (note that $v_\alpha(\infty) = \alpha h(n, 1)$ and $h(n, 1) > 0$). Define now $f \in C^0\left(\overline{U}\right)$ by

$$
f\left(q\right) = \begin{cases} 0 \text{ if } q \in \overline{U} \cap \mathfrak{B}_{\alpha}\left(0\right) \\ \max\{0, v_{\alpha}\left(q\right) - \left(v_{\alpha}\left(\infty\right) - c\right)\}, \text{ if } q \in M - \mathfrak{B}_{\alpha}\left(0\right). \end{cases}
$$

From Proposition 8 we have f a non-negative (generalized) subsolution to the Dirichlet problem (5) (for zero boundary data) relatively to U, which satisfies

$$
\lim_{d(q)\to\infty} f\left(q\right) = c. \tag{41}
$$

It follows that $f \in \mathcal{F}$ and that $f \leq w_c \leq c$. Then we have (40).

Given $q_0 \in \partial U$, by hypothesis there is a geodesic open ball of M, say B_r, of radius $r > 0$, contained in $M - U$ and such that ∂B_r is tangent to ∂U $(= \partial\left(M - U\right))$ at q_0. From Corollary 6, there is $W \in C^0\left(M - B_r\right)$ a (generalized) supersolution relatively to the operator $\mathfrak{M}$ on $M - B_r$ such that $c \leq W\left(\infty\right) = \mathcal{L}$, with $W|_{\partial B_r} = 0$, which is C^1 in a neighborhood of ∂B_r in $M - B_r$ and such that

$$
\lim_{d_M(q,\partial B_r)\to 0} \left|\nabla W\left(q\right)\right| = +\infty.
$$

From the comparison principle, since $\overline{U} \subset M - B_r$, it follows that on $\overline{U}$, we have $0 \leq w_c \leq W$. As q_0 is arbitrary, we conclude that $w_c \in C^0\left(\overline{U}\right)$ with $w_c|_{\partial U} = 0$.

Assume that $0 \leq c < \mathcal{L}$. Let $\delta = \left(c + \mathcal{L}\right)/2$. Let $L\left(\sigma\right) := \mathcal{L}\left(\sigma, n, \lambda, a\right)$, $\sigma \in (0, r]$, where $\mathcal{L}$ is given by (11). Since $L \in C^0(0, r]$, either there is $\sigma_0 \in (0, r)$ such that $L\left(\sigma_0\right) = \delta$, or $\delta < L(\sigma)$ for all $\sigma \in (0, r)$. Take $r' = \sigma_0$

in the first case and r' any point in $(0,r)$ in the second case. Let $B_{r'}$ be the open geodesic ball of M, $B_{r'} \subset B_r$, with the same center of B_r. Consider the correspondent $t_0 > 0$ and the function $w \in C^2(\Lambda_0) \cap C^0\left(\overline{\Lambda}_0\right)$, $w|_{\partial B_{r'}} = 0$, where $\Lambda_0 = \{q \in M - B_{r'}; d_M(q, \partial B_{r'}) \leq t_0\}$ is such that $\mathfrak{M}(w) \leq 0$, as given in Corollary 6. If the second case occurs, the height of this w is greater than δ at its correspondent distance t_0 to $\partial B_{r'}$ and, as w is radial with respect to the center of $B_{r'}$, there is $t_0' < t_0$ such that, at the distance t_0' of $\partial B_{r'}$, the height of w is δ. In any case, there is $0 < t_0' \leq t_0$ such that, setting

$$W_{r'}(q) = \begin{cases} w(q) \text{ if } q \in \Lambda_0' \\ \delta \text{ if } q \in M - \Lambda_0' \end{cases}$$

where

$$\Lambda_0' := \{q \in \Lambda; d(q) \leq t_0'\}.$$

satisfies the same properties of the function W as given in Corollary 6. Note that it is possible to translate the graph of $W_{r'}$ in the $\partial/\partial t$-direction (e_k-direction) in a way that its height at infinity is in $[c, \delta)$ and such that Γ, the intersection of the hypersurface resulting of this displacement with $\{t \geq 0\}$ is such that

$$\partial\Gamma = \Gamma \cap B_r = \partial B_{r''},$$

with $B_{r''} \subset B_r$ a geodesic open ball of M with the same center of B_r and radius r'', being $r' < r'' < r$. Now, move Γ keeping $\partial\Gamma$ on M and the center of $\partial\Gamma$ on the geodesic of M linking the center of B_r to $q_0 \in \partial U$, until $\partial\Gamma$ touches ∂U at q_0 and call $\widetilde{\Gamma}$ this final hypersurface. Observe that such displacement is an isometry in $M \times \mathbb{R}$. Denote by $\widetilde{B}_{r''}$ the geodesic ball contained in B_r of radius r'' such that $\partial \widetilde{B}_{r''} = \partial \widetilde{\Gamma}$. We have then that $W_{r''} : M - \widetilde{B}_{r''} \longrightarrow \mathbb{R}$ is a (generalized) supersolution relatively to $\mathfrak{M}$. Moreover, since our translation in $\partial/\partial t$ direction is small enough, $W_{r''}$ satisfies $\mathfrak{M}(W_{r''}) = 0$ in $\overline{\Lambda}_0''$, where Λ_0'' is a neighborhood in U such that $q_0 \in \partial \Lambda_0''$. In particular, $W_{r''} \in C^\infty\left(\overline{\Lambda}_0''\right)$. From comparison principle, since $0 \leq w_c|_{\partial U} \leq W_{r''}|_{\partial U}$ and $c \leq W_{r''}(\infty)$, we conclude that $0 \leq w_c \leq W_{r''}$ on $\overline{U}$. As $w_c(q_0) = W_{r''}(q_0)$ and $W_{r''} \in C^\infty\left(\overline{\Lambda}_0''\right)$ it follows that $w_c \in C^1\left(\overline{U}\right)$ and, from elliptic PDE regularity theory ([12]), it follows that $w_c \in C^{2,\alpha}\left(\overline{U}\right) \cap C^\infty(U)$. ■

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