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Semilinear wave equation with time-dependent exponential speed of propagation, mass and dissipation

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Abstract. In order to prove global existence (in time) of small data energy solutions to the semilinear Cauchy problem for the wave equation with time-dependent exponential speed of propagation mass and dissipation we will derive sharp linear estimates for the solution and its derivatives.

Keywords: Semilinear evolution equations, linear estimates, critical exponent, global existence (in time), small data energy solutions.

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1 Introduction

Let us consider the semilinear Cauchy problem for the wave models with exponential time-dependent speed of propagation, mass and dissipation

$$\phi_{tt} - e^{2t}\Delta\phi + n\phi_t + m\phi = |\phi|^p, \;\; \phi(0,x) = f(x), \;\; \phi_t(0,x) = g(x), \quad (1.1)$$

in $[0,\infty) \times \mathbb{R}^d$, where m and n are a positive coefficient for the mass and dissipation terms, respectively. The main goal of this note is to prove global existence (in time) of small data energy solutions for all $p > 1$. To achieve this goal, we shall derive sharp linear estimates for the solutions of the linear parameterized problem associated with the Cauchy problem (1.1).

The asymptotic behavior of the solution of (1.1) depends on the interaction between the dissipative and mass terms. So, in order to derive our results we shall propose the following classification for the semilinear Cauchy problem (1.1): Applying the transformation (see [4])

$$\phi(t,x) = e^{rt}u(t,x) \tag{1.2}$$

to the Cauchy problem (1.1) we get

$$u_{tt} - e^{2t}\Delta u + (2r+n)u_t + \tilde{s}u = e^{(p-1)rt}|u|^p, \tag{1.3}$$

in $[0,\infty) \times \mathbb{R}^d$, where $\tilde{s} = r^2 + nr + m$ and initial datas $u(0,x) = f(x)$, $u_t(0,x) = g(x) - rf(x)$. Some results for the constant speed of propagation case of (1.3) can be found in [6]. Therefore we may have three possibilities for the model (1.3):

1.1 Model with predominant dissipation: case $4m \in (0, n^2)$

If we choose

$$2r := \mu - n, \quad \mu := \sqrt{n^2 - 4m}, \tag{1.4}$$

in (1.3), then $\tilde{s} = 0$ and we obtain the wave model with exponential speed of propagation and **predominant dissipation**:

$$u_{tt} - e^{2t}\Delta u + \mu u_t = e^{\frac{\mu-n}{2}(p-1)t}|u|^p, \tag{1.5}$$

in $[0,\infty) \times \mathbb{R}^d$.

In order to feel some influence of the dissipative term, we shall assume $4m < n^2$. In the paper [2] was introduced a classification of damping terms for the Cauchy problem (1.5).

1.2 Model with predominant mass: case $4m \in (n^2, \infty)$

If we choose

$$2r = -n \tag{1.6}$$

in (1.3), then we obtain the wave model with exponential speed of propagation and **predominat mass**

$$u_{tt} - e^{2t}\Delta u + \left(m - \frac{n^2}{4}\right) u = e^{-\frac{n}{2}(p-1)t}|u|^p, \tag{1.7}$$

in $[0, \infty) \times \mathbb{R}^d$.

In order to feel some influence of the mass term, we shall assume $4m > n^2$. In the paper [3] was introduced a classification of mass terms for the Cauchy problem (1.7).

1.3 Model with a balance between mass and dissipation: case $4m = n^2$

If we choose

$$2r = -n \quad \text{and} \quad 4m = n^2 \tag{1.8}$$

in (1.3), then we obtain the **balance between mass and dissipation**

$$u_{tt} - e^{2t}\Delta u = e^{-\frac{n}{2}(p-1)t}|u|^p, \tag{1.9}$$

in $[0, \infty) \times \mathbb{R}^d$.

According with the change of variable (1.2) and the possible choices of r, (1.4),(1.6), (1.8), to prove global existence (in time) for the semilinear Cauchy problem (1.1) it is equivalent to prove global existence (in time) for the semilinear Cauchy problems (1.5), (1.7) and (1.9).

2 Main Results

In this section we list the main results of this paper that are related to the global (in time) existence of small data energy solutions for the Cauchy problem (1.1).

Theorem 2.1. *Let us consider the Cauchy problem* (1.1) *with initial datas* $f \in H^1(\mathbb{R}^d)$ *and* $g \in L^2(\mathbb{R}^d)$. *Suppose that* $4m \in (0, n^2 - 1)$, $p > 1$ *and* $p \leq \frac{d}{d-2}$ *for* $d \geq 3$. *Then, there exists a constant* $\varepsilon_0 > 0$ *such that, for every small data satisfying*

$$\|f\|_{H^1} + \|g\|_{L^2} \leq \varepsilon, \ \textit{for } \varepsilon \leq \varepsilon_0,$$

there exists a uniquely determined global (in time) energy solution

$$\phi \in \mathcal{C}([0,\infty), H^1(\mathbb{R}^d)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^d)).$$

Moreover, the solution ϕ *satisfies the energy estimate*

$$\|\phi(t,\cdot)\|_{L^2} + e^t\|\nabla\phi(t,\cdot)\|_{L^2} + \|\phi_t(t,\cdot)\|_{L^2} \lesssim e^{\frac{\mu-n}{2}t}(\|f\|_{H^1} + \|g\|_{L^2}).$$

Remark 2.2. The condition $4m < n^2 - 1$, implies that $\mu > 1$.

Remark 2.3. If $4m \in (n^2 - 1, n^2)$, then $\mu < 1$. This condition implies that we are in the case where the predominant dissipation is non-effective (see [2]). In this situation the size of the dissipation coefficient μ has a strong influence on the linear estimates, see Remark 3.5. It is expected a existence of a $p_c \geq 1$, such that there exists global (in time) small data energy solutions for $p > p_c$. Although, the techniques used in this paper, in general, does not give us sharp p_c for the non-effective dissipation case.

Theorem 2.4. *Let us consider the Cauchy problem* (1.1) *with initial datas* $f \in H^1(\mathbb{R}^d)$ *and* $f \in L^2(\mathbb{R}^d)$. *Suppose that* $n > 1$, $4m \in (n^2, \infty)$, $p > 1$ *and* $p \leq \frac{d}{d-2}$ *for* $d \geq 3$. *Then, there exists a constant* $\varepsilon_0 > 0$ *such that, for every small data satisfying*

$$\|f\|_{H^1} + \|g\|_{L^2} \leq \varepsilon, \ \textit{for } \varepsilon \leq \varepsilon_0,$$

there exist a uniquely determined global (in time) energy solution

$$\phi \in \mathcal{C}([0,\infty), H^1(\mathbb{R}^d)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^d)).$$

Moreover, the solution ϕ *satisfies the following decay estimate:*

$$\|\phi(t,\cdot)\|_{L^2} + e^t\|\nabla\phi(t,\cdot)\|_{L^2} + \|\phi_t(t,\cdot)\|_{L^2} \lesssim e^{-\frac{n-1}{2}t}(\|f\|_{H^1} + \|g\|_{L^2}).$$

Theorem 2.5. *Let us consider the Cauchy problem* (1.1) *with initial datas* $f \in H^1(\mathbb{R}^d)$ *and* $f \in L^2(\mathbb{R}^d)$*. Suppose that* $4m = n^2$*,* $p > 1$ *and* $p \leq \frac{d}{d-2}$ *for* $d \geq 3$*. Then, there exists a constant* $\varepsilon_0 > 0$ *such that, for every small data satisfying*

$$\|f\|_{H^1} + \|g\|_{L^2} \leq \varepsilon, \ \textit{for } \varepsilon \leq \varepsilon_0,$$

there exist a uniquely determined global (in time) energy solution

$$\phi \in \mathcal{C}([0,\infty), H^1(\mathbb{R}^d)) \cap \mathcal{C}^1([0,\infty), L^2(\mathbb{R}^d)).$$

Moreover, the solution ϕ *satisfies the following decay estimate:*

$$(1+t)^{-1}\|\phi(t,\cdot)\|_{L^2} + e^{\frac{t}{2}}\|\nabla\phi(t,\cdot)\|_{L^2} + e^{-t}\|\phi_t(t,\cdot)\|_{L^2} \lesssim e^{-\frac{n}{2}t}(\|f\|_{H^1} + \|g\|_{L^2}).$$

3 Linear estimatives

Consider the Cauchy problem with time-dependent exponential speed of propagation, mass and dissipation (1.3). According to Duhamel's principle, a solution of (1.3) satisfies the non-linear integral equation

$$\begin{aligned} u(t,x) = {} & K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x) \\ & + \int_0^t K_1(t,s,x) *_{(x)} e^{(p-1)rs}|u(s,x)|^p ds, \end{aligned} \tag{3.1}$$

where $K_j(t,0,x) *_{(x)} u_j(x)$, $j = 0,1$, are the solutions to the corresponding linear Cauchy problem

$$u_{tt} - e^{2t}\Delta u + (2r+n)u_t + \tilde{s}u = 0, \ u(0,x) = \delta_{0j}u_0(x), \ u_t(0,x) = \delta_{1j}u_1(x), \tag{3.2}$$

where $(t,x) \in [0,\infty) \times \mathbb{R}^d$ and $\delta_{kj} = 1$, if $k = j$ and zero otherwise. The term $K_1(t,s,x) *_{(x)} f(s,x)$ is the solution of the correspondent linear parameter dependent Cauchy problem

$$u_{tt} - e^{2t}\Delta u + (2r+n)u_t + \tilde{s}u = 0, \ u(s,x) = 0, \ u_t(s,x) = f(s,x), \tag{3.3}$$

where $(t,x) \in [s,\infty) \times \mathbb{R}^d$. Therefore, by Duhamel's principle, to understand the solution for the semi-linear Cauchy problem (1.3) we shall take account the solutions to a family of parameter-dependent Cauchy problem (3.3).

3.1 Model with dominant dissipation: case $4m \in (0, n^2)$

Let us consider the linear parameter-dependent Cauchy problem for the damped wave equation with exponential speed of propagation

$$u_{tt} - e^{2t}\Delta u + \mu u_t = 0,\ \ u(s,x) = \varphi(s,x),\ \ u_t(s,x) = \psi(s,x) \tag{3.4}$$

where $(t,x) \in [s,\infty) \times \mathbb{R}^d$ and $\mu > 0$ is defined in (1.4).

Applying the partial Fourier transform with respect to the spatial variable to (3.4) we get

$$\widehat{u}_{tt} + \mu\widehat{u}_t + e^{2t}|\xi|^2\widehat{u} = 0,\ \ \widehat{u}(s,x) = \widehat{\varphi}(s,\xi),\ \ \widehat{u}_t(s,\xi) = \widehat{\psi}(s,\xi). \tag{3.5}$$

Performing the change of variable $v(\tau) = \widehat{u}(t,\xi),\ \ \tau = e^t|\xi|$, we arrive at the following Cauchy problem:

$$v_{\tau\tau} + \frac{1+\mu}{\tau}v_\tau + v = 0,\ \ v(|\xi|e^s) = \widehat{\varphi}(s,\xi),\ \ v_\tau(|\xi|e^s) = \frac{\widehat{\psi}(s,\xi)}{|\xi|e^s}.$$

Setting $w(z) = e^{\frac{z}{2}}v(\tau),\ \ z = 2i\tau = 2i|\xi|e^t$, we get the confluent hypergeometric equation

$$zw_{zz} + (1+\mu-z)w_z - \frac{1+\mu}{2}w = 0, \tag{3.6}$$

with the initial condition:

$$w(z_0) = e^{i|\xi|e^s}\widehat{\varphi}(s,\xi),\ \ w_z(z_0) = \frac{e^{i|\xi|e^s}}{2}\left(\widehat{\varphi}(s,\xi) + \frac{1}{i|\xi|e^s}\widehat{\psi}(s,\xi)\right),$$

where $z_0 = z_0(s,\xi) = 2i|\xi|e^s$.

If $\mu \notin \mathbb{Z}$, then due [1] the general solution to (3.6) is given by

$$w(z) = c_1(s,\xi)w_1(z) + c_2(s,\xi)w_2(z),$$

where $w_1(z)$ and $w_2(z)$ are two independent solutions given by

$$w_1(z) = \Phi\left(\frac{1+\mu}{2}, 1+\mu; z\right),\ \ w_2(z) = z^{-\mu}\Phi\left(\frac{1-\mu}{2}, 1-\mu; z\right).$$

Here Φ is the Kummer's function given by

$$\Phi(\alpha,\beta;z)=\sum_{n=0}^{\infty}\frac{\alpha^{(n)}z^n}{\beta^{(n)}n!},\quad \gamma^{(n)}=\prod_{k=0}^{n-1}(\gamma+k),\quad \gamma^{(0)}=1. \tag{3.7}$$

Forthermore, the representation formula to $c_j(s,\xi)$ are given by

$$c_j(s,\xi)=(-1)^{3-j}\frac{w(z_0)(d_z w_{3-j})(z_0)-(d_z w)(z_0)w_{3-j}(z_0)}{W(w_1,w_2)(z_0)},\quad \text{for } j=1,2, \tag{3.8}$$

where $W(w_1,w_2)$ is the Wronskian of the two linear independent solutions and it satisfies

$$W(w_1,w_2)(z)=w_1\frac{d}{dz}w_2-w_2\frac{d}{dz}w_1=-\mu z^{-1-\mu}e^z.$$

Finally we can conclude the following representation for the solution $\widehat{u}$ to (3.5):

$$\widehat{u}(t,\xi)=e^{-i|\xi|e^t}\left[c_1(s,\xi)\Phi_1+c_2(s,\xi)(2i|\xi|e^t)^{-\mu}\Phi_2\right], \tag{3.9}$$

here $c_j(s,\xi)$, for $j=1,2$ are given by (3.8) with $z_0=2i|\xi|e^s$ and $\Phi_1=\Phi\left(\frac{1+\mu}{2},1+\mu,2i|\xi|e^t\right)$ and $\Phi_2=\Phi\left(\frac{1-\mu}{2},1-\mu,2i|\xi|e^t\right)$.

Therefore, to give the asymptotic behaviour of the solution $\widehat{u}$ we shall use the following properties of the Kummer's functions, that can be found in [1]:

Lemma 3.1. *Let α and β fixed parameters in $\mathbb{C}$ with $\beta\notin\mathbb{Z}$. Then the function $\Phi=\Phi(\alpha,\beta;z)$ satisfies the following properties:*

1. *Φ is an entire function with respect to z;*
2. $\frac{d}{dz}\Phi(\alpha,\beta;z)=\frac{\alpha}{\beta}\Phi(\alpha+1,\beta+1;z)$;
3. *For large $|z|$ the asymptotic behavior of the Kummer's function is given by*

$$|\Phi(\alpha,\beta;z)|\le C_{\alpha,\beta}|z|^{\max\{Re(\alpha-\beta),-Re(\alpha)\}}.$$

Remark 3.2. Note that in our case we have $\beta = 2\alpha$. In that way, for large $|z|$ the property (3) of the Lemma 3.1 can be written as

$$|\Phi(\alpha, \beta; z)| \leq C_{\alpha,\beta}|z|^{-\mathrm{Re}(\alpha)}.$$

According to the previous Lemma it is usefull to split the extended phase space into zones to analyze the behavior of the Kummer's function for small and large frequencies. Let us consider the following zones:

$$Z_1(t) = \{\xi : |\xi|e^t \leq N\}, \;\; Z_2(s,t) = \{\xi : Ne^{-t} \leq |\xi| \leq Ne^{-s}\} \\ \text{and} \;\; Z_3(s) = \{\xi : |\xi|e^s \geq N\}. \tag{3.10}$$

Using the properties of the confluent hypergeometric functions it is possible derive the following results:

Proposition 3.3. *Suppose that* $\varphi(0,\cdot) \in \dot{H}^\gamma(\mathbb{R}^d)$*, with* $\gamma \geq 0$ *and* $\psi(0,\cdot) \equiv 0$ *in the Cauchy problem* (3.4)*. Then the following estimative hold for* $t \geq 0$ *and for* $\mu \in (0,n)$

$$\|K_0(t,0,x) *_{(x)} \varphi(0,x)\|_{\dot{H}^\gamma} \lesssim \|\varphi(0,x)\|_{\dot{H}^\gamma}. \tag{3.11}$$

For the kinect energy we have for $\mu \geq 1$

$$\|\partial_t K_0(t,0,x) *_{(x)} \varphi(0,x)\|_{\dot{H}^{\gamma-1}} \lesssim \|\varphi(0,x)\|_{\dot{H}^\gamma}, \;\; \textit{for } \gamma \geq 1. \tag{3.12}$$

The above estimatives can also be founded in [2].

Proposition 3.4. *Suppose that* $\psi(s,\cdot) \in \dot{H}^\gamma(\mathbb{R}^d)$*, with* $\gamma \geq 0$ *for* $s \geq 0$ *and* $\varphi(s,\cdot) \equiv 0$*. Then the following estimatives holds for* $\mu \in [1,n)$ *and* $t \geq s$:

$$\|(K_1(t,s,x) *_{(x)} \psi(s,x), \partial_t K_1(t,s,x) *_{(x)} \psi(s,x))\|_{\dot{H}^\gamma} \lesssim \|\psi(s,x)\|_{\dot{H}^\gamma},$$

and

$$\|\nabla K_1(t,s,x) *_{(x)} \psi(s,x)\|_{\dot{H}^\gamma} \lesssim e^{-t}\|\psi(s,x)\|_{\dot{H}^\gamma}.$$

If $s = 0$ we recover the results from [2].

Remark 3.5. If $\mu < 1$, then the decay of the solutions and its derivatives depends strongly on μ. For instance, in the hypothesis of Proposition 3.4, the following estimatives holds for $\mu < 1$

$$\|(K_1(t,s,x) *_{(x)} \psi(s,x), \partial_t K_1(t,s,x) *_{(x)} \psi(s,x))\|_{\dot{H}^\gamma} \lesssim e^{(t-s)\frac{1-\mu}{2}} \|\psi(s,x)\|_{\dot{H}^\gamma}.$$

Proof. **Consideration in $Z_1(s)$:**
Using the properties (1) and (2) of the Lemma 3.1 we get

$$\begin{aligned}|c_1(s,\xi)| &\lesssim (|\xi|e^s)^{\mu+1}\left(|\widehat{\varphi}(s,\xi)|(|\xi|e^s)^{-\mu-1} + \left(|\widehat{\varphi}(s,\xi)| + \frac{|\widehat{\psi}(s,\xi)|}{|\xi|e^s}\right)(|\xi|e^s)^{-\mu}\right)\\ &\lesssim |\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|\end{aligned}$$

and

$$\begin{aligned}|c_2(s,\xi)| &\lesssim (|\xi|e^s)^{\mu+1}\left(|\widehat{\varphi}(s,\xi)| + \frac{|\widehat{\psi}(s,\xi)|}{|\xi|e^s}\right)\\ &\lesssim (|\xi|e^s)^{\mu+1}|\widehat{\varphi}(s,\xi)| + (|\xi|e^s)^{\mu}|\widehat{\psi}(s,\xi)|.\end{aligned}$$

Then, for all $t \geq s$ we obtain

$$\begin{aligned}|\widehat{u}(t,\xi)| &\lesssim \left((|\xi|e^s)^{\mu+1}|\widehat{\varphi}(s,\xi)| + (|\xi|e^s)^{\mu}|\widehat{\psi}(s,\xi)|\right)(|\xi|e^t)^{-\mu}\\ &\lesssim |\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|,\end{aligned}$$

and, more general,

$$|\xi|^\gamma|\widehat{u}(t,\xi)| \lesssim |\xi|^\gamma|\widehat{\varphi}(s,\xi)| + |\xi|^{\gamma_1}e^{-\gamma_2 t}|\widehat{\psi}(s,\xi)|,$$

for $\gamma, \gamma_1, \gamma_2 \geq 0$ and $\gamma_1 + \gamma_2 = \gamma$.

Similarly we have for $\gamma \geq 1$ the estimatives:

$$|\xi|^{\gamma-1}|\widehat{u}_t(t,\xi)| \lesssim |\xi|^{\gamma-1}\left(|\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|\right).$$

Consideration in $Z_3(s)$:
In this zone we have $|\xi|e^s \geq N$ and $|\xi|e^t \geq N$. Using the properties (2) and (3) of the Lemma 3.1, we arrive at

$$\begin{aligned}|c_j(s,\xi)| &\lesssim (|\xi|e^s)^{\mu+1}\left(|\widehat{\varphi}(s,\xi)|(|\xi|e^s)^{-\frac{\mu+1}{2}} + |\widehat{\psi}(s,\xi)|(|\xi|e^s)^{-\frac{\mu+3}{2}}\right)\\ &\lesssim |\widehat{\varphi}(s,\xi)|(|\xi|e^s)^{\frac{\mu+1}{2}} + |\widehat{\psi}(s,\xi)|(|\xi|e^s)^{\frac{\mu-1}{2}},\end{aligned} \tag{3.13}$$

for $j=1,2$. Therefore, by using the above estimatives for $c_{j's}$, the definition of the zone and the property (3) of Lemma 3.1 we can conclude that:

$$\begin{aligned}
|\xi|^{\gamma}|\widehat{u}(t,\xi)| &\lesssim |\xi|^{\gamma}(|\widehat{\varphi}(s,\xi)|(|\xi|e^{s})^{\frac{\mu+1}{2}}+|\widehat{\psi}(s,\xi)|(|\xi|e^{s})^{\frac{\mu-1}{2}})(|\xi|e^{t})^{-\frac{\mu+1}{2}}\\
&\lesssim |\xi|^{\gamma}\left(|\widehat{\varphi}(s,\xi)|e^{s\frac{\mu+1}{2}}+\frac{|\widehat{\psi}(s,\xi)|}{|\xi|}e^{s\frac{\mu-1}{2}}\right)e^{-t\frac{\mu+1}{2}}\\
&\lesssim |\xi|^{\gamma}(e^{(s-t)\frac{\mu+1}{2}}|\widehat{\varphi}(s,\xi)|+e^{(s-t)\frac{\mu-1}{2}}|\widehat{\psi}(s,\xi)|),
\end{aligned}$$

for $\gamma\geq 0$. If $\psi\in\dot{H}^{\gamma-1}$, but does not belong to $\psi\in\dot{H}^{\gamma}$, we derive

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)|\lesssim e^{(s-t)\frac{\mu+1}{2}}|\xi|^{\gamma}|\widehat{\varphi}(s,\xi)|+e^{s\frac{\mu-1}{2}}e^{-t\frac{\mu+1}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(s,\xi)|.$$

Similarly, we conclude for $\gamma\geq 1$ the estimate

$$|\xi|^{\gamma-1}|\widehat{u}_{t}(t,\xi)|\lesssim e^{t}e^{(s-t)\frac{(\mu+1)}{2}}|\xi|^{\gamma}|\widehat{\varphi}(s,\xi)|+e^{(s-t)\frac{(\mu-1)}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(s,\xi)|.$$

Consideration in $Z_2(s)$:

Using the definition of the zone $Z_2(s)$ and (3.13), it follows that

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)|\lesssim e^{(s-t)\frac{\mu+1}{2}}|\xi|^{\gamma}|\widehat{\varphi}(s,\xi)|+e^{s\frac{\mu-1}{2}}e^{-t\frac{\mu+1}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(s,\xi)|.$$

Similarly, we conclude for $\gamma\geq 1$ the estimate

$$|\xi|^{\gamma-1}|\widehat{u}_{t}(t,\xi)|\lesssim e^{t}e^{(s-t)\frac{(\mu+1)}{2}}|\xi|^{\gamma}|\widehat{\varphi}(s,\xi)|+e^{(s-t)\frac{(\mu-1)}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(s,\xi)|.$$

This completes the prove of the Propositions 3.3 and 3.4. □

3.2 Model with dominant mass: case $4m\in(n^2,\infty)$

Let us consider the linear parameter-dependent Cauchy problem for the Klein-Gordon type equation with exponential speed of propagation

$$u_{tt}-e^{2t}\Delta u+\tilde{m}u=0,\ \ u(s,x)=\varphi(s,x),\ u_t(s,x)=\psi(s,x), \tag{3.14}$$

where $(t,x)\in[s,\infty)\times\mathbb{R}^d$ and $\tilde{m}=\left(m-\frac{n^2}{4}\right)>0$.

Applying the partial Fourier transform we have

$$\widehat{u}_{tt} + \left(\tilde{m} + e^{2t}|\xi|^2\right)\widehat{u} = 0, \;\; \widehat{u}(s,x) = \widehat{\varphi}(s,\xi), \;\; \widehat{u}_t(s,\xi) = \widehat{\psi}(s,\xi).$$

Introducing the change of variable $v(\tau) := \widehat{u}(t,\xi)$ with $\tau = |\xi|e^t$ and $\tau_0 = |\xi|e^s$ we have

$$v_{\tau\tau} + \frac{1}{\tau}v_\tau + \left(1 + \frac{\tilde{m}}{\tau^2}\right)v = 0.$$

Defining $v(\tau) = \tau^\rho \widetilde{v}(\tau)$, then after choosing $\rho = i\sqrt{\tilde{m}}$ we arrive at

$$\tau\widetilde{v}_{\tau\tau} + (2\rho+1)\widetilde{v}_\tau + \tau\widetilde{v} = 0. \tag{3.15}$$

The equation (3.15) can be reduced to a confluent hypergeomtric equation performing the change of variable $z := 2i\tau$ and $w(z) := e^{i\tau}\widetilde{v}(\tau)$,

$$zw_{zz} + (1+2\rho-z)w_z - \frac{1+2\rho}{2}w = 0 \tag{3.16}$$

with the initial condition at $z_0 := 2i|\xi|e^s$

$$w(z_0) = e^{i|\xi|e^s}\frac{\widehat{\varphi}(s,\xi)}{(|\xi|e^s)^\rho}, \;\; w'(z_0) = \frac{e^{i|\xi|e^s}}{2}\left(\frac{\widehat{\varphi}(s,\xi)}{(|\xi|e^s)^\rho} + i\frac{\rho\widehat{\varphi}(s,\xi) + \widehat{\psi}(s,\xi)}{(|\xi|e^s)^{\rho+1}}\right).$$

The solution to (3.16) is given by the representation (see [1])

$$w(z) = c_1(s,\xi)w_1(z) + c_2(s,\xi)w_2(z),$$

where w_1 and w_2 are two linear independent solutions given by

$$w_1(z) = \Phi\left(\frac{1+2\rho}{2}, 1+2\rho; z\right) \text{ and } w_2(z) = z^{-2\rho}\Phi\left(\frac{1-2\rho}{2}, 1-2\rho; z\right),$$

where Φ is the confluent hypergeometric function and the constants $c_{j's}$ are given by

$$c_j(s,\xi) = (-1)^{3-j}\frac{w(z_0)(d_z w_{3-j})(z_0) - (d_z w)(z_0)w_{3-j}(z_0)}{W(w_1,w_2)(z_0)}, \tag{3.17}$$

for $j = 1, 2$, where $W(w_1, w_2)$ is the Wronskian of the two linear independent solutions and it satisfies

$$W(w_1, w_2)(z) = w_1 \frac{d}{dz} w_2 - w_2 \frac{d}{dz} w_1 = -2\rho z^{-2\rho-1} e^z.$$

From the representation above for $W(w_1, w_2)$ and since z is a pure imaginary number, it follows that $|W(w_1, w_2)| \sim |z|^{-1}$.

Therefore the WKB representation for $\widehat{u}$ is given by

$$\widehat{u}(t, \xi) = e^{\rho \ln(|\xi| e^t) - i|\xi| e^t} \left(c_1(s, \xi) \Phi_3 + c_2(s, \xi)(2i|\xi| e^t)^{-2\rho} \Phi_4 \right), \tag{3.18}$$

with $\Phi_3 = \Phi\left(\frac{1+2\rho}{2}, 1 + 2\rho; z\right)$ and $\Phi_4 = \Phi\left(\frac{1-2\rho}{2}, 1 - 2\rho; z\right)$.

Analogously the previous section we shall divide the extended phase space into zones like in (3.10). Using the properties of the confluent hypergeometric functions it is possible derive the following results:

Proposition 3.6. *Suppose that $\varphi(0, \cdot) \in \dot{H}^\gamma(\mathbb{R}^d)$, with $\gamma \geq 0$ and $\psi(0, \cdot) \equiv 0$. Then the following estimatives holds for $t \geq 0$ and for $4m \in (n^2, \infty)$*

$$\|K_0(t, 0, x) *_{(x)} \varphi(0, x)\|_{\dot{H}^\gamma} \lesssim \|\varphi(0, x)\|_{\dot{H}^\gamma} \ \textit{and} \tag{3.19}$$

$$\|\partial_t K_0(t, 0, x) *_{(x)} \varphi(0, x)\|_{\dot{H}^{\gamma-1}} \lesssim e^{\frac{t}{2}} \|\varphi(0, x)\|_{\dot{H}^\gamma}, \ \textit{for } \gamma \geq 1. \tag{3.20}$$

The above estimatives can also be founded in [3].

Proposition 3.7. *Suppose that $\psi(s, \cdot) \in \dot{H}^\gamma(\mathbb{R}^d)$, with $\gamma \geq 0$ for $s \geq 0$ and $\varphi(s, \cdot) \equiv 0$. Then the following estimatives holds for $t \geq s$ and for $4m \in (n^2, \infty)$*

$$\|(K_1(t, s, x) *_{(x)} \psi(s, x), \partial_t K_1(t, s, x) *_{(x)} \psi(s, x))\|_{\dot{H}^\gamma} \lesssim e^{\frac{t-s}{2}} \|\psi(s, x)\|_{\dot{H}^\gamma}$$

and

$$\|\nabla K_1(t, s, x) *_{(x)} \psi(s, x)\|_{\dot{H}^\gamma} \lesssim e^{-\frac{s+t}{2}} \|\psi(s, x)\|_{\dot{H}^\gamma}.$$

If $s = 0$ we recover the results from [3].

Proof. **Considerations in Z_1:** Using the properties of the confluent hypergeometric functions described in Lemma 3.1, we have

$$|c_1(s,\xi)| \lesssim |\xi|e^s \left(\frac{|\widehat{\varphi}(s,\xi)|}{|\xi|e^s} + |\widehat{\varphi}(s,\xi)| + \frac{|\widehat{\psi}(s,\xi)|}{|\xi|e^s} \right) \lesssim |\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|$$

and

$$|c_2(s,\xi)| \lesssim |\xi|e^s \left(|\widehat{\varphi}(s,\xi)| + \frac{|\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|}{|\xi|e^s} \right) \lesssim |\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|.$$

If $\gamma \geq 0$ we have

$$|\xi|^\gamma |\widehat{u}(t,\xi)| \lesssim |\xi|^\gamma |(|\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|)$$

and if $\psi \in \dot{H}^{\gamma-1}$ but $\psi \notin \dot{H}^\gamma$, we can obtain the estimate

$$|\xi|^\gamma |\widehat{u}(t,\xi)| \lesssim |\xi|^\gamma |\widehat{\varphi}(s,\xi)| + e^{-t}|\xi|^{\gamma-1}|\widehat{\psi}(s,\xi)|.$$

Analogously we derive for $\gamma \geq 1$ the estimate:

$$|\xi|^{\gamma-1} |\widehat{u}_t(t,\xi)| \lesssim |\xi|^{\gamma-1}(|\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|).$$

Considerations in Z_3: In this zone we have that $|\xi|e^t \geq |\xi|e^s \geq N$. Then,

$$\begin{aligned} |c_j(s,\xi)| &\lesssim |\xi|e^s \left(|\widehat{\varphi}(s,\xi)| + \frac{|\widehat{\varphi}(s,\xi)| + |\widehat{\psi}(s,\xi)|}{|\xi|e^s} \right) (|\xi|e^s)^{-\frac{1}{2}} \\ &\lesssim (|\xi|e^s)^{\frac{1}{2}}|\widehat{\varphi}(s,\xi)| + (|\xi|e^s)^{-\frac{1}{2}}|\widehat{\psi}(s,\xi)|, \end{aligned}$$

for $j = 1, 2$. Therefore, using the above estimatives, the definiton of the zone and Lemma 3.1 we may conclude

$$\begin{aligned} |\xi|^\gamma |\widehat{u}(t,\xi)| &\lesssim |\xi|^\gamma ((|\xi|e^s)^{\frac{1}{2}}|\widehat{\varphi}(s,\xi)| + (|\xi|e^s)^{-\frac{1}{2}}|\widehat{\psi}(s,\xi)|)(|\xi|e^t)^{-\frac{1}{2}} \\ &\lesssim |\xi|^\gamma (e^{\frac{s-t}{2}}|\widehat{\varphi}(s,\xi)| + e^{\frac{s-t}{2}}|\widehat{\psi}(s,\xi)|). \end{aligned}$$

Similarly,

$$\begin{aligned} |\xi|^{\gamma-1} |\widehat{u}_t(t,\xi)| &\lesssim |\xi|^{\gamma-1}((|\xi|e^s)^{\frac{1}{2}}|\widehat{\varphi}(s,\xi)| + (|\xi|e^s)^{-\frac{1}{2}}|\widehat{\psi}(s,\xi)|)(|\xi|e^t)^{\frac{1}{2}} \\ &= e^{\frac{s+t}{2}}|\xi|^\gamma |\widehat{\varphi}(s,\xi)| + e^{\frac{t-s}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(s,\xi)|. \end{aligned}$$

Considerations in Z_2: In Z_2 we have

$$|\xi|^\gamma |\widehat{u}(t,\xi)| \lesssim |\xi|^\gamma (e^{\frac{s-t}{2}} |\widehat{\varphi}(s,\xi)| + e^{\frac{t-s}{2}} |\widehat{\psi}(s,\xi)|)$$

and

$$|\xi|^{\gamma-1} |\widehat{u}_t(t,\xi)| \lesssim e^{\frac{t}{2}} |\xi|^\gamma |\widehat{\varphi}(s,\xi)| + e^{\frac{t-s}{2}} |\xi|^{\gamma-1} |\widehat{\psi}(s,\xi)|.$$

This completes the prove of the Propositions 3.6 and 3.7. □

3.3 Model with balanced dissipation and mass: case $4m = n^2$

Let us consider the linear parameter-dependent Cauchy problem for the wave equation with exponential speed of propagation

$$u_{tt} - e^{2t}\Delta u = 0, \quad u(s,x) = \varphi(s,x),\, u_t(s,x) = \psi(s,x), \tag{3.21}$$

where $(t,x) \in [s,\infty) \times \mathbb{R}^d$. Applying the partial Fourier transform we have

$$\widehat{u}_{tt} + e^{2t}|\xi|^2 u = 0, \;\; \widehat{u}(s,x) = \widehat{\varphi}(s,\xi), \;\; \widehat{u}_t(s,\xi) = \widehat{\psi}(s,\xi). \tag{3.22}$$

Introducing the change of variable $v(\tau) := \widehat{u}(t,\xi)$ with $\tau = |\xi|e^t$ and $\tau_0 = |\xi|e^s$ we have

$$v_{\tau\tau} + \frac{1}{\tau} v_\tau + v = 0,$$

with initial datas $v(|\xi|e^s) = \widehat{\varphi}(s,\xi)$ and $v_\tau(|\xi|e^s) = \frac{\widehat{\psi}(s,\xi)}{|\xi|e^s}$. The above equation is well known as Bessel equantion of order zero.

The general solution of the Bessel equation of order zero is given for $\tau > 0$ by (see [1])

$$v(\tau) = c_1(\xi) J_0(\tau) + c_2(\xi) Y_0(\tau),$$

where J_ν and Y_ν are Bessel functions of order ν, of first and second kind, respectively. The Wronskian satisfies

$$W(J_0(\tau), Y_0(\tau)) = \frac{2}{\pi\tau}.$$

We obtain the following representation for the solution $\widehat{u}$ to (3.22):

$$\begin{aligned}\widehat{u}(t,\xi) = \frac{\pi}{2}|\xi|e^s \left(Y_0'(|\xi|e^s)J_0(|\xi|e^t) - J_0'(|\xi|e^s)Y_0(|\xi|e^t)\right)\widehat{\varphi}(s,\xi) \\ +\frac{\pi}{2}\left(J_0(|\xi|e^s)Y_0(|\xi|e^t) - Y_0(|\xi|e^s)J_0(|\xi|e^t)\right)\widehat{\psi}(s,\xi),\end{aligned} \tag{3.23}$$

see [5] for $s = 0$.

In order to describe the behaviour of $\widehat{u}$ we will use the properties of the functions J_0 and Y_0:

Lemma 3.8. *Consider the Bessel function of order zero J_0 and Y_0. Then the follwing properties holds true:*

1. *J_0 is an entire function analytic function, whereas Y_0 has a logarithmic singularity at $\tau = 0$;*
2. *$J_0'(\tau) = -J_1(\tau)$;*
3. *For small τ we have the asymptotic behaviour:*
$$|J_0(\tau)| \lesssim 1,\ \ |J_0'(\tau)| \lesssim \tau,\ \ |Y_0(\tau)| \lesssim |\log(\tau)|,\ \ |Y_0'(\tau)| \lesssim \tau^{-1};$$
4. *For large τ we have the asymptotic behaviour:*
$$|J_0^{(k)}(\tau)| \lesssim \tau^{-\frac{1}{2}},\ \ |Y_0^{(k)}(\tau)| \lesssim \tau^{-\frac{1}{2}},\ \ k = 0, 1.$$

According to the previous Lemma it is usefull to split the extended phase space into zones to analyze the behavior of the Bessel's function for small and large frequencies. Let us consider zones like in (3.10).

Proposition 3.9. *Suppose that $\varphi(0,\cdot) \in \dot{H}^\gamma(\mathbb{R}^d)$, with $\gamma \geq 0$ and $\psi(0,\cdot) \equiv 0$. Then the following estimatives holds for $t \geq 0$ and $4m = n^2$:*

$$\|K_0(t,0,x) *_{(x)} \varphi(0,x)\|_{\dot{H}^\gamma} \lesssim (1+t)\|\varphi(0,x)\|_{\dot{H}^\gamma}, \tag{3.24}$$

$$\|\partial_t K_0(t,0,x) *_{(x)} \varphi(0,x)\|_{\dot{H}^{\gamma-1}} \lesssim e^t\|\varphi(0,x)\|_{\dot{H}^\gamma},\ \ \textit{for}\ \gamma \geq 1. \tag{3.25}$$

Proposition 3.10. *Suppose that $\psi(s,\cdot)\in \dot{H}^{\gamma}(\mathbb{R}^d)$, for $s\geq 0$ and $\varphi(s,\cdot)\equiv 0$. Then the following estimatives holds for $t\geq s$ and $4m=n^2$: If $\gamma\geq 0$,*

$$\|K_1(t,s,x)*_{(x)}\psi(s,x)\|_{\dot{H}^{\gamma}}\lesssim (1+t)\|\psi(s,x)\|_{\dot{H}^{\gamma}},$$

$$\|\nabla K_1(t,s,x)*_{(x)}\psi(s,x)\|_{\dot{H}^{\gamma}}\lesssim (1+s)e^{-\frac{s+t}{2}}\|\psi(s,x)\|_{\dot{H}^{\gamma}},$$

and for $\gamma\geq 1$

$$\|\partial_t K_1(t,s,x)*_{(x)}\psi(s,x)\|_{\dot{H}^{\gamma}}\lesssim (1+s)e^{\frac{t-s}{2}}\|\psi(s,x)\|_{\dot{H}^{\gamma}}.$$

Proof. **Considerations in Z_1:** Using the representation of the solution (3.23) and the property (3) of the Lemma 3.8 we have

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)|\lesssim |\xi|^{\gamma}(1+t)\left(|\widehat{\varphi}(t,\xi)|+|\widehat{\psi}(t,\xi)|\right).$$

If $\psi\in\dot{H}^{\gamma-1}$ but $\psi\notin\dot{H}^{\gamma}$, we obtain:

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)|\lesssim (1+t)\left(|\xi|^{\gamma}|\widehat{\varphi}(t,\xi)|+e^{-t}|\xi|^{\gamma-1}|\widehat{\psi}(t,\xi)|\right)$$

and

$$|\xi|^{\gamma-1}|\widehat{u}_t(t,\xi)|\lesssim e^{t}|\xi|^{\gamma}|\widehat{\varphi}(t,\xi)|+(1+s)|\xi|^{\gamma-1}|\widehat{\psi}(t,\xi)|.$$

Considerations in Z_3: In Z_3 we have $|\xi|e^t\geq|\xi|e^s\geq N$ and by using (4) of the Lemma 3.8 we have

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)|\lesssim e^{\frac{s-t}{2}}|\xi|^{\gamma}\left(|\widehat{\varphi}(t,\xi)|+|\widehat{\psi}(t,\xi)|\right).$$

If $\psi\in\dot{H}^{\gamma-1}$ but $\psi\notin\dot{H}^{\gamma}$, we obtain:

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)|\lesssim e^{\frac{s-t}{2}}|\xi|^{\gamma}|\widehat{\varphi}(t,\xi)|+e^{-\frac{s+t}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(t,\xi)|$$

and

$$|\xi|^{\gamma-1}|\widehat{u}_t(t,\xi)|\lesssim e^{\frac{s+t}{2}}|\xi|^{\gamma}|\widehat{\varphi}(t,\xi)|+e^{\frac{t-s}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(t,\xi)|.$$

Considerations in Z_2: In Z_3 we have $|\xi|e^t\geq N$ and $|\xi|e^s\leq N$. By using (3) and (4) of the Lemma 3.8 we have

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)|\lesssim |\xi|^{\gamma}\left(|\widehat{\varphi}(t,\xi)|+(1+s)|\widehat{\psi}(t,\xi)|\right).$$

If $\psi \in \dot{H}^{\gamma-1}$ but $\psi \notin \dot{H}^{\gamma}$, we obtain:

$$|\xi|^{\gamma}|\widehat{u}(t,\xi)| \lesssim |\xi|^{\gamma}|\widehat{\varphi}(t,\xi)| + (1+s)e^{-\frac{s+t}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(t,\xi)|,$$

and

$$|\xi|^{\gamma-1}|\widehat{u}_t(t,\xi)| \lesssim e^{t}|\xi|^{\gamma}|\widehat{\varphi}(t,\xi)| + (1+s)e^{\frac{t-s}{2}}|\xi|^{\gamma-1}|\widehat{\psi}(t,\xi)|.$$

This completes the prove of the Propositions 3.9 and 3.10. □

4 Global existence of small data solutions

In this section we will give the proof of the main results of this paper Theorems 2.1, 2.4 and 2.5. For this reason, we will use Duhamel's principle and the linear estimates derived in the last section.

4.1 Proof of Theorem 2.1:

First we are interested to obtain global (in time) energy solutions to (1.1) for $4m \in (0, n^2-1)$. For this it is enough to prove global existence of small data energy solutions to (1.5).

In order to use the Banach's fixed point theorem we shall define a suitable operator in a function space which satisfies some conditions. According with the observations made in the Section 3, let us consider r as in (1.4) and define the space

$$X(t) := \{u \in \mathcal{C}([0,t], H^1(\mathbb{R}^d)) \cap \mathcal{C}^1([0,t], L^2(\mathbb{R}^d))\},$$

with the norm

$$\|u\|_{X(t)} := \sup_{\tau\in[0,t]}\{\|u(\tau,\cdot)\|_{L^2} + e^{\tau}\|\nabla u(\tau,\cdot)\|_{L^2} + \|u_\tau(\tau,\cdot)\|_{L^2}\}.$$

For $u \in X(t)$ we define the following linear operator

$$\begin{aligned} Pu(t,x) := {}& K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x) \\ &+ \int_0^t e^{(p-1)rs}K_1(t,s,x) *_{(x)} |u(s,x)|^p ds. \end{aligned}$$

We will prove that there exists a fixed point for the operator defined above, for this we will show that

$$\|Pu\|_X \le C\|u\|_X^p\,, \tag{4.1}$$
$$\|Pu - Pv\|_X \le C\|u-v\|_X\big(\|u\|_X^{p-1} + \|v\|_X^{p-1}\big)\,. \tag{4.2}$$

By the Propositions 3.3 and 3.4 for $s=0$ we arrive at

$$\|K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x)\|_{X(t)} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2}.$$

Applyng Minkowski's integral inequality and Proposition 3.4 we have

$$\left\|\int_0^t e^{(p-1)rs} K_1(t,s,x) *_{(x)} |u(s,x)|^p ds\right\|_{L^2} \lesssim \int_0^t e^{(p-1)rs}\||u(s,x)|^p\|_{L^2} ds.$$

By Gagliardo-Niremberg inequality and the definition of the norm of $X(t)$ we derive the estimative

$$\||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{L^2}^{p(1-\theta)}\|\nabla u(s,\cdot)\|_{L^2}^{p\theta} \lesssim \|u\|_{X(s)}^p,$$

with

$$\theta = d\left(\frac{1}{2} - \frac{1}{2p}\right),\ \ 2p \le \begin{cases} \infty, & \text{if } d \le 2 \\ \frac{2d}{d-2}, & \text{if } d \ge 3 \end{cases}\,. \tag{4.3}$$

Therefore,

$$\left\|\int_0^t e^{(p-1)rs} K_1(t,s,x) *_{(x)} |u(s,x)|^p ds\right\|_{L^2} \lesssim \|u\|_{X(t)}^p \int_0^t e^{(p-1)rs} ds \lesssim \|u\|_{X(t)}^p,$$

once $r<0$ due to $\mu \in (1,n)$ and $p>1$. Analousgly, it is possible to prove the desired estimatives for the kinect and elastic energies. In that way, it follows (4.1).

To derive the Lipschitz condition, notice that:

$$\begin{aligned} Pu - Pv &= \int_0^t e^{(p-1)rs} K_1(t,s,x) *_{(x)} (|u(s,x)|^p - |v(s,x)|^p) ds \\ &= p\int_0^t e^{(p-1)rs} K_1 *_{(x)} \ \int_0^t |v+\tau(u-v)|^{p-2}(v+\tau(u-v)) d\tau (u-v) ds. \end{aligned}$$

Applying Hölder and Gagliardo-Niremberg inequalities, it is possible to obtain (4.2) analousgly to the previous case.

Then we conclude by Banach's fixed point theorem the global existence (in time) for the Cauchy problem (1.1) with sufficiently small initial datas. The decay rate comes from the change of variable $\phi(t,x) = e^{\frac{-n+\mu}{2}t}u(t,x)$.

4.2 Proof of Theorem 2.4:

We are interested to obtain global (in time) energy solutions to (1.1) for $4m \in (n^2, \infty)$. For this it is enough to prove global existence of small data energy solutions to (1.7).

Let us define the space

$$X(t) := \{u \in \mathcal{C}([0,t], H^1(\mathbb{R}^d)) \cap \mathcal{C}^1([0,t], L^2(\mathbb{R}^d))\},$$

with the norm

$$\|u\|_{X(t)} := \sup_{\tau\in[0,t]} \{e^{-\frac{\tau}{2}}\|u_\tau(\tau,\cdot)\|_{L^2} + e^{-\frac{\tau}{2}}\|u(\tau,\cdot)\|_{L^2} + e^{\frac{\tau}{2}}\|\nabla u(\tau,\cdot)\|_{L^2}\}.$$

For $u \in X(t)$ we define the linear operator

$$\begin{aligned} Pu(t,x) &:= K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x) \\ &\quad + \int_0^t e^{-\frac{n}{2}(p-1)s} K_1(t,s,x) *_{(x)} |u(s,x)|^p ds. \end{aligned}$$

By the Propositions 3.6 and 3.7 for $s = 0$ we arrive at

$$\|K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x)\|_{X(t)} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2}.$$

Applyng Minkowski's integral inequality Proposition 3.7 we have

$$\begin{aligned} &\left\| \int_0^t e^{-\frac{n}{2}(p-1)s} K_1(t,s,x) *_{(x)} |u(s,x)|^p ds \right\|_{L^2} \\ &\qquad \lesssim e^{\frac{t}{2}} \int_0^t e^{-\frac{n}{2}(p-1)s-\frac{s}{2}} \||u(s,x)|^p\|_{L^2} ds. \end{aligned}$$

By Gagliardo-Niremberg inequality and the definition of the norm of $X(t)$ we derive the estimative

$$\||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{L^2}^{p(1-\theta)}\|\nabla u(s,\cdot)\|_{L^2}^{p\theta} \lesssim e^{\frac{s}{2}p}\|u\|_{X(t)}^p,$$

with θ like in (4.3). Therefore,

$$\begin{aligned}\left\|\int_0^t e^{-\frac{n}{2}(p-1)s}K_1(t,s,x)*_{(x)}|u(s,x)|^p ds\right\|_{L^2} &\lesssim e^{\frac{t}{2}}\|u\|_{X(t)}^p\int_0^t e^{\frac{1}{2}(1-n)(p-1)s}ds\\ &\lesssim e^{\frac{t}{2}}\|u\|_{X(t)}^p,\end{aligned}$$

once $n,p>1$. Analousgly, it is possible to prove the desired estimatives for the kinect and elastic energies. In that way, it follows (4.1).

The Lipschitz condition follow as the proof of the Theorem 2.1. In that way, we conclude by Banach's fixed point theorem the global existence (in time) for the Cauchy problem (1.1) with sufficiently small datas. The decay rate comes from the change of variable $\phi(t,x) = e^{-\frac{n}{2}t}u(t,x)$.

4.3 Proof of Theorem 2.5:

We are interested to obtain global (in time) energy solutions to (1.1) for $4m = n^2$. For this it is enough to prove global existence of small data energy solutions to (1.9).

Let us define the space

$$X(t) := \{u \in \mathcal{C}([0,t], L^2(\mathbb{R}^d)) \cap \mathcal{C}^1([0,t], H^1(\mathbb{R}^d))\},$$

with the norm

$$\|u\|_{X(t)} := \sup_{\tau\in[0,t]}\{(1+\tau)^{-1}\|u(\tau,\cdot)\|_{L^2} + e^{-\tau}\|u_\tau(\tau,\cdot)\|_{L^2} + e^{\frac{\tau}{2}}\|\nabla u(\tau,\cdot)\|_{L^2}\}.$$

For $u \in X(t)$ we define the linear operator

$$\begin{aligned}Pu(t,x) := K_0(t,0,x)*_{(x)}u_0(x) + K_1(t,0,x)*_{(x)}u_1(x)\\ + \int_0^t e^{-\frac{n}{2}(p-1)s}K_1(t,s,x)*_{(x)}|u(s,x)|^p ds.\end{aligned}$$

By the Propositions 3.9 and 3.10 for $s = 0$ we arrive at

$$\|K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x)\|_{X(t)} \lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2}.$$

Applyng Minkowski's integral inequality and Proposition 3.10 we have

$$\begin{aligned} &(1+t)^{-1} \left\| \int_0^t e^{-\frac{n}{2}(p-1)s} K_1(t,s,x) *_{(x)} |u(s,x)|^p ds \right\|_{L^2} \\ &\lesssim (1+t)^{-1} \int_0^t e^{-\frac{n}{2}(p-1)s} (1+s) \||u(s,x)|^p\|_{L^2} ds \\ &\lesssim \int_0^t e^{-\frac{n}{2}(p-1)s} \||u(s,x)|^p\|_{L^2} ds. \end{aligned}$$

By Gagliardo-Niremberg inequality and the definition of the norm of $X(t)$ we derive the estimative

$$\||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|^p_{L^{2p}} \lesssim \|u(s,\cdot)\|^{p(1-\theta)}_{L^2} \|\nabla u(s,\cdot)\|^{p\theta}_{L^2} \lesssim (1+s)^p \|u\|^p_{X(t)},$$

with θ like in (4.3). Therefore,

$$\begin{aligned} &(1+t)^{-1} \left\| \int_0^t e^{-\frac{n}{2}(p-1)s} K_1(t,s,x) *_{(x)} |u(s,x)|^p ds \right\|_{L^2} \\ &\lesssim \|u\|^p_{X(t)} \int_0^t e^{-\frac{n}{2}(p-1)s} (1+s)^p ds \lesssim \|u\|^p_{X(t)}, \end{aligned}$$

once $p > 1$. Analousgly it is possible to prove the desired estimatives for the kinect and elastic energies. In that way, it follows (4.1).

The Lipschitz condition follow as the proof of the Theorem 2.1. In that way, we conclude by Banach's fixed point theorem the global existence (in time) for the Cauchy problem (1.1) with sufficiently small datas. The decay rate comes from the change of variable $\phi(t,x) = e^{-\frac{n}{2}t} u(t,x)$.

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