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Large Deviations for the SSEP with slow boundary: the non-critical case

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Abstract. We prove a large deviations principle for the empirical measure of the one dimensional symmetric simple exclusion process in contact with reservoirs. The dynamics of the reservoirs is slowed down with respect to the dynamics of the bulk of the system, that is, the rate at which the system exchanges particles with the boundary reservoirs is of order $n^{-\theta}$, where n is number of sites in the system, θ is a non negative parameter, and the system is taken in the diffusive time scaling tn^2. Two regimes are studied here, the subcritical $\theta \in (0,1)$ whose hydrodynamic equation is the heat equation with Dirichlet boundary conditions and the supercritical $\theta \in (1,+\infty)$ whose hydrodynamic equation is the heat equation with Neumann boundary conditions. In the subcritical case $\theta \in (0,1)$, the rate function that we obtain matches with the rate function corresponding to the case $\theta = 0$ which was derived on previous works, see Bertini et al. (2009); Farfan et al. (2011). In the supercritical case $\theta \in (1,+\infty)$, the rate function is equal to infinity outside the set of trajectories that preserve the total mass, meaning that, despite the discrete system exchanges particles with the reservoirs, this phenomenon has super-exponentially small probability in the diffusive scaling limit.

Contents

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1. Introduction

Due to its special features and simplicity, the exclusion process became a prototype interacting particle system in Probability and Statistical Mechanics: on one hand it presents an interaction among particles (the hard-core interaction) describing many physical phenomena of interest. On the other hand, it is a mathematically treatable model, allowing rigorous proofs of those phenomena Kipnis and Landim (1999).

In plain words, the exclusion process is described by independent random walks on some graph under the constraint that at most one particle is allowed to occupy each vertex of the graph. Variations of the exclusion dynamics then lead to many different physical situations. One of the most common and relevant is to put the exclusion process in contact with reservoirs, and this has been widely studied in the literature, see for instance the seminal paper Eyink et al. (1990). In particular, the symmetric exclusion in contact with reservoirs is the subject of study in this paper.

Recently, in Baldasso et al. (2017) it was derived the hydrodynamic limit of the one-dimensional symmetric exclusion process on the box with n sites and in contact with slow reservoirs. That is, the dynamics is given by a superposition of a Kawasaki dynamics and a Glauber dynamics at the end points of the box. More precisely, the symmetric simple exclusion dynamics acts on the bulk, that is the set of points $\{1, \ldots, n-1\}$ and, at the sites 1 and $n-1$, particles can be injected/removed to/from the bulk at a rate which is slowed down with respect to the bulk dynamics. More precisely, particles enter (respectively, leave) the system through the left boundary at rate α/n^θ (respectively, $(1-\alpha)/n^\theta$) and particles enter (respectively, leave) the system through the right boundary at rate β/n^θ (respectively, $(1-\beta)/n^\theta$). Here, $0 < \alpha, \beta < 1$ and $\theta \geq 0$ are fixed parameters.

Given that the hydrodynamic limit has been established, which is, in some sense, a law of large numbers for the density of particles, since its limit is deterministic, it is quite natural to ask about its large deviations. That is, the asymptotic probability to observe rare events (which, *grosso modo*, goes exponentially fast to zero for events that do not contain the expected limit from the law of large numbers). This is precisely what we do here: in this paper we analyze the large deviations of the model studied in Baldasso et al. (2017), in both the subcritical case $\theta \in (0,1)$ and the supercritical case $\theta \in (1,+\infty)$. The critical case, $\theta = 1$, was recently analyzed in Franco et al. (2022).

We describe next the main features of this work, starting with some words about the super-exponential replacement lemmas which are of fundamental importance in the derivation of our results. For $\theta \in (0,1)$ since we are in the regime of Dirichlet boundary conditions, we need to replace the value of the empirical measure at the left (resp. right) boundary by the value α (resp. β). This can be achieved by taking as reference measure a product measure associated to a continuous profile $g_{\alpha,\beta}$ which is locally constant equal to α at the left boundary and locally constant equal to β at the right boundary. For $\theta \in (1,+\infty)$, we need to assure that profiles not preserving the total mass of the system have a super-exponentially small probability. These facts help solving the elliptic equation (associated to the weakly asymmetric system), and this provides the correct perturbation in order to observe a given profile.

For $\theta \in (0,1)$, the large deviations rate function that we obtained coincides with the large deviations rate function of many previous works, as Bertini et al. (2003), or Farfan et al. (2011) in dimension one and parameter $a = 0$, or in Bodineau and Lagouge (2012) if we do not consider the reaction dynamics as they do. We stress that despite having the same large deviations rate function, the case $\theta \in (0,1)$ is not a particular case of those aforementioned works, since many of the estimates that we need are harder to obtain. Nevertheless, their exchange rates at the boundary corresponds to taking $\theta = 0$ in our rates. At the end, we prove that slowing down the exchange rate of the boundary by $n^{-\theta}$, with $\theta \in (0,1)$ the large deviations behave as in the case $\theta = 0$.

For $\theta \in (1,+\infty)$, contrarily to the case $\theta \in (0,1)$, the large deviations rate function depends on the value of the density profile at the boundary. The rate functions for the cases $\theta \in (0,1)$ and $\theta \in (1,+\infty)$ are then written in the following succinct form, as the supremum over the set of possible perturbations H of the price function $J_H^\theta(\rho)$ to be precisely defined in Subsection 2.5, restricting the set of reachable profiles ρ to distinct sets in each case. In other words, the rate functions for $\theta \in (0,1)$ and $\theta \in (1,+\infty)$ are quite similar, in their form, but they have, as natural, different attainable profiles. The constraint $\rho_t(0) = \alpha$ and $\rho_t(1) = \beta$ defines the set of reachable profiles for $\theta \in (0,1)$, which corresponds to the Dirichlet case. On the other hand, for $\theta \in (1,+\infty)$, reachable trajectories must have constant mass in time, which corresponds to the Neumann case. Both sets of reachable profiles are natural if we take into consideration that the corresponding hydrodynamic equations have Dirichlet and Neumann boundary conditions, respectively.

In neither the cases $\theta \in (0,1)$ and $\theta \in (1,+\infty)$ the current through the boundary plays any role. This can be explained as follows. For $\theta \in (0,1)$, in the same spirit of Bertini et al. (2003); Bodineau and Lagouge (2012); Farfan et al. (2011), a super-exponential replacement lemma at the boundary holds, meaning that the exchange of particles is fast enough to not allow any large deviations in the diffusive time scaling. On the other hand, for $\theta \in (1,+\infty)$, the exchange of particles is so slow that any large deviations of the current through the boundary have no strength to interfere in the large deviations of the density. That is, the current through the boundary disappears super-exponentially fast in the diffusive time scaling. This phenomenon does not appear in the case $\theta = 1$. In that case, the current has an impact in the large deviations functional, as we can see in equations (2.10)-(2.12) in Franco et al. (2022) and because of this, different techniques have to be derived in order to prove the large deviations principle.

The paper is structured as follows: In Section 2 we give definitions and we state our main results. Section 3 contains the necessary super-exponential replacement lemmas which are crucial along the arguments. In Section 4 we study the hydrodynamic limit of the associated weakly asymmetric process. In Sections 5 and 6 it is presented the large deviations upper bound and lower bound, respectively.

2. Statement of results

2.1. *The model.* Given $n \geq 1$, denote $\Sigma_n = \{1, \dots, n-1\}$ and consider the state space $\Omega_n := \{0,1\}^{\Sigma_n}$. Configurations on this state space Ω_n will be denoted by η so that, for $x \in \Sigma_n$, $\eta(x) = 0$ means that the site x is vacant while $\eta(x) = 1$ means that the site x is occupied. We define the infinitesimal generator $\mathcal{L}_n = \mathcal{L}_{n,0} + n^{-\theta}\mathcal{L}_{n,b}$ as follows. For any function $f : \Omega_n \to \mathbb{R}$,

$$(\mathcal{L}_{n,0}f)(\eta) \;=\; \sum_{x=1}^{n-2} \Big(f(\eta^{x,x+1}) - f(\eta) \Big)\,, \tag{2.1}$$

$$(\mathcal{L}_{n,b}f)(\eta) \;=\; \sum_{x\in\{1,n-1\}} \Big[r_x(1-\eta(x)) + (1-r_x)\eta(x) \Big] \Big(f(\sigma^x\eta) - f(\eta) \Big)\,, \tag{2.2}$$

with $r_1 = \alpha$ and $r_{n-1} = \beta$. Above, for $x \in \{1, \dots, n-2\}$, the configuration $\eta^{x,x+1}$ is obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$, i.e.,

$$(\eta^{x,x+1})(y) \;=\; \begin{cases} \eta(x+1)\,, & \text{if } \; y = x\,, \\ \eta(x)\,, & \text{if } \; y = x+1\,, \\ \eta(y)\,, & \text{otherwise,} \end{cases} \tag{2.3}$$

and for $x \in \{1, n-1\}$ the configuration $\sigma^x\eta$ is obtained from η by flipping the occupation variable $\eta(x)$, i.e,

$$(\eta^{x})(y) \;=\; \begin{cases} 1-\eta(y)\,, & \text{if } \; y = x\,, \\ \eta(y)\,, & \text{otherwise.} \end{cases} \tag{2.4}$$

The dynamics of this model can be described in words in the following way. In the bulk, particles move according to continuous time symmetric random walks under the exclusion rule: whenever a particle tries to jump to an occupied site, such jump is suppressed. Additionally, at the left boundary, particles can be created (resp. removed) at rate α/n^θ (resp. at rate $(1-\alpha)/n^\theta$) and at the right boundary, particles can be created (resp. removed) at rate β/n^θ (resp. at rate $(1-\beta)/n^\theta$), see Figure 2.1 for an illustration. When $\alpha = \beta = \rho$, for which there is no external current induced by

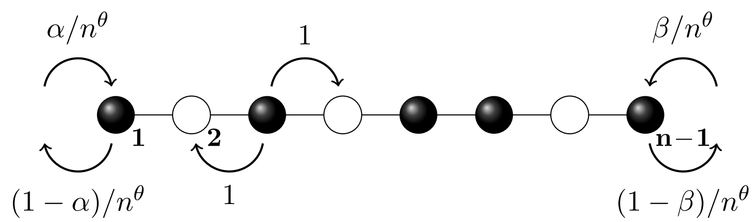

FIGURE 2.1. Illustration of jump rates. The leftmost and rightmost rates are the entrance/exiting rates.

the reservoirs, the Bernoulli product measures given by $\nu_\rho\{\eta : \eta(x) = 1\} = \rho$ are invariant. However, when $\alpha \neq \beta$, this is no longer true. Nevertheless, for $\alpha \neq \beta$, there is a unique stationary measure of the system, that we denote by μ_{ss}, which is not a product measure. For further properties on this measure we refer the reader to Derrida (2007), for instance. In Baldasso et al. (2017, Theorem 2.2), it is shown that this measure is associated to a profile $\bar\rho(\cdot)$ which is stationary with respect to the corresponding hydrodynamic equation.

Fix, once and for all, a time horizon $T > 0$. We denote by $\{\eta_t : t \in [0,T]\}$ the Markov process with generator $n^2\mathcal{L}_n$, omitting the dependence on n to shorten notation. This family of Markov processes $\{\eta_t : t \in [0,T]\}$ indexed on $n \in \mathbb{N}$ is what we will call the *Exclusion Process with Slow Boundary* (EPSB).

2.2. *Empirical measure.* The so-called *empirical measure*, which represents the spatial density of particles in the system, is defined by

$$\pi^n(du) \;=\; \pi^n(\eta, du) \;:=\; \frac{1}{n}\sum_{x=1}^{n-1} \eta(x)\,\delta_{\frac{x}{n}}(du)\,, \tag{2.5}$$

where $\delta_{\frac{x}{n}}$ is the Dirac-measure at $x/n \in [0,1]$ and $\eta \in \Omega_n$. Note that the empirical measure is a random positive measure on $[0,1]$ with total mass bounded by one. Let

$$\mathcal{M} \;=\; \{\, \mu \text{ is a positive measure on } [0,1] : \mu([0,1]) \leq 1\}\,, \tag{2.6}$$

hence $\pi^n \in \mathcal{M}$. The integral of a function $f : [0,1] \to \mathbb{R}$ with respect to the empirical measure is denoted by $\langle \pi^n, f\rangle = \int_0^1 f(u)\,\pi^n(du) = \frac{1}{n}\sum_{x=1}^{n-1}\eta(x)\,f(\frac{x}{n})$, for which we will write $\langle \pi^n, f\rangle$. The time evolution of the density of particles can be represented by the time evolution of the empirical measure as

$$\pi_t^n(du) \;=\; \pi^n(\eta_t, du) \;:=\; \frac{1}{n}\sum_{x=1}^{n-1} \eta_t(x)\,\delta_{\frac{x}{n}}(du)\,,$$

where $\{\eta_t \,:\, t \in [0,T]\}$ is the EPSB. This is the object we are concerned with in this work.

2.3. *Notations.* In what follows we present notations to be used everywhere in this paper and we also recall some classical spaces from Analysis.

• We will write $\langle \cdot, \cdot\rangle$ to denote both an integral of a function f with respect to a measure μ, that is, $\langle \mu, f\rangle = \int_0^1 f(u)\,d\mu(u)$, and to denote the inner product on $L^2(0,1)$ given by $\langle f, g\rangle = \int_0^1 f(u)\,g(u)\,du$ and the corresponding norm is denoted by $\|\cdot\|_{L^2}$. The double bracket $\langle\!\langle \cdot, \cdot\rangle\!\rangle$ denotes the inner product in $L^2([0,T]\times(0,1))$ and the corresponding norm is denoted by $\|\cdot\|_{L^2(0,T;(0,1))}$.

• Recall Ω_n from the beginning of the Section 2. Let $\mathcal{D}_{\Omega_n} = \mathcal{D}([0,T], \Omega_n)$ be the space of trajectories that are right continuous, with left limits and taking values in Ω_n. Denote by $\mathbb{P}_{\mu_n}$ the probability measure on $\mathcal{D}_{\Omega_n}$ induced by $\{\eta_t \,:\, t \in [0,T]\}$ and by the initial measure μ_n, and let $\mathbb{E}_{\mu_n}$ be the expectation with respect to $\mathbb{P}_{\mu_n}$.

Denote by $\mathcal{D}_{\mathcal{M}} = \mathcal{D}([0,T], \mathcal{M})$ the space of trajectories that are right continuous, with left limits and taking values in $\mathcal{M}$, which was defined in (2.6). Denote by $\mathbb{Q}_{\mu_n}$ the probability on $\mathcal{D}_{\mathcal{M}}$ induced by $\{\pi_t^n \,:\, t \in [0,T]\}$ and by the initial measure μ_n on Ω_n.

Denote by $\mathcal{D}_{\mathcal{M}_0}$ the subset of $\mathcal{D}_{\mathcal{M}}$ consisting of trajectories taking values on measures which have a density ρ with respect to the Lebesgue measure such that $0 \leq \rho \leq 1$.

• We will denote by $C^{i,j} := C^{i,j}([0,T]\times[0,1])$ the set of functions with i derivatives in time, j derivatives in space with all partial derivatives in $C^0([0,T]\times[0,1])$. By $C^j := C^j([0,1])$ we denote the set of functions which are C^j in space. When a *subindex* 0 appears, it will restrict the considered sets to functions which vanish at the boundary of $[0,1]$. When a subindex c appears, it will restrict the considered set to functions of support compact in $(0,1)$. For example, by $C_c^{1,2}$ we mean the subset of $C^{1,2}$ of functions with compact support in $[0,T]\times(0,1)$ and by $C_0^{1,2}$ we mean the subset of $C^{1,2}$ composed by functions H such that $H(t,0) = H(t,1) = 0$ for all $t \geq 0$.

Define then

$$\mathbf{C}_\theta \;=\; \begin{cases} C_0^{1,2}, \text{ if } \theta \in (0,1), \\ C^{1,2}, \text{ if } \theta \in (1,+\infty). \end{cases} \tag{2.7}$$

• Given a function $g : [0,T]\times[0,1]$, we sometimes use $g_t(u)$ to denote $g(t,u)$. It should not be confounded with the notation $\partial_t g(t,u)$ for the time derivative.

• The notation $g(n) = O(f(n))$ means $g(n)$ is bounded from above by $Cf(n)$, where the constant $C > 0$ does not depend on n. A presence of *subindexes* in the $O(\cdot)$ means that the constant may depend on those subindexes. Equivalently, $f \lesssim g$ will stand for $f = O(g)$. The notation $g(n) = o(f(n))$ will stand for $\lim_{n\to\infty} g(n)/f(n) = 0$.

- The indicator function of a set A will be written as $\mathbf{1}_A(u)$, which is one if $u \in A$ and zero otherwise.
- The discrete derivatives and the discrete Laplacian are defined by

$$\nabla_n^+ H_n(\tfrac{x}{n}) \;=\; n\Big[H(\tfrac{x+1}{n}) - H(\tfrac{x}{n})\Big]\,, \qquad \nabla_n^- H_n(\tfrac{x}{n}) \;=\; n\Big[H(\tfrac{x}{n}) - H(\tfrac{x-1}{n})\Big]\,, \tag{2.8}$$

$$\Delta_n H_n(\tfrac{x}{n}) \;=\; n^2\Big[H(\tfrac{x+1}{n}) + H(\tfrac{x-1}{n}) - 2H(\tfrac{x}{n})\Big]\,. \tag{2.9}$$

Definition 2.1 (Sobolev Space)**.** Let $\mathcal{H}^1$ be the set of all locally summable functions $\zeta : (0,1) \to \mathbb{R}$ such that there exists a function $\partial_u \zeta \in L^2$ satisfying $\langle \partial_u G, \zeta\rangle = -\langle G, \partial_u \zeta\rangle$, for all $G \in C_c^\infty$. For $\zeta \in \mathcal{H}^1$, we define the norm

$$\|\zeta\|_{\mathcal{H}^1} \;:=\; \Big(\|\zeta\|_{L^2}^2 + \|\partial_u \zeta\|_{L^2}^2\Big)^{1/2}\,.$$

Let $L^2(0,T;\mathcal{H}^1)$ be the space of all measurable functions $\xi : [0,T] \to \mathcal{H}^1$ such that

$$\|\xi\|_{L^2(0,T;\mathcal{H}^1)}^2 \;:=\; \int_0^T \|\xi_t\|_{\mathcal{H}^1}^2\, dt \;<\; \infty\,.$$

Remark 2.2. An equivalent definition for the Sobolev space $L^2(0,T;\mathcal{H}^1)$ is the set of bounded functions $\xi : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that there exists a function $\partial\xi \in L^2([0,T]\times\mathbb{T})$ satisfying

$$\langle\!\langle \partial_u H, \xi\rangle\!\rangle \;=\; -\langle\!\langle H, \partial\xi\rangle\!\rangle\,,$$

for all functions $H \in C_c^{0,1}$.

2.4. *Hydrodynamic limit.* Fix a measurable profile $\gamma : [0,1] \to [0,1]$. For each $n \in \mathbb{N}$, let μ_n be a probability measure on Ω_n. We say that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is *associated* to the profile $\gamma : [0,1] \to [0,1]$ if, for any $\delta > 0$ and any $f \in C^0$, the following limit holds:

$$\lim_{n\to\infty} \mu_n\Big[\eta : \Big|\langle \pi_0^n, f\rangle - \langle \gamma, f\rangle\Big| > \delta\Big] \;=\; 0\,. \tag{2.10}$$

From Baldasso et al. (2017) we have the following result:

Theorem 2.3 (Hydrodynamic limit for the EPSB, c.f. Baldasso et al., 2017)**.** *Suppose that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is associated to a measurable profile $\gamma : [0,1] \to [0,1]$ in the sense of* (2.10). *Then, for each $t \in [0,T]$, for any $\delta > 0$ and any continuous function $f : [0,1] \to \mathbb{R}$,*

$$\lim_{n\to+\infty} \mathbb{P}_{\mu_n}\Big[\eta_\cdot : \Big|\langle \pi_t^n,\, f\rangle - \langle \rho_t,\, f\rangle\Big| > \delta\Big] \;=\; 0\,,$$

where $\rho(t,\cdot)$ is:

- *If $0 < \theta < 1$, the unique weak solution of the heat equation with Dirichlet boundary conditions*

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u)\,, & \text{for } t > 0\,,\ u \in (0,1)\,,\\ \rho(t,0) = \alpha\,,\ \rho(t,1) = \beta & \text{for } t > 0\,,\\ \rho(0,u) = \gamma(u)\,, & \text{for } u \in [0,1]\,. \end{cases} \tag{2.11}$$

- *If $\theta > 1$, the unique weak solution of the heat equation with Neumann boundary conditions*

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u)\,, & \text{for } t > 0\,,\ u \in (0,1)\,,\\ \partial_u \rho(t,0) = \partial_u \rho(t,1) = 0\,, & \text{for } t > 0\,,\\ \rho(0,u) = \gamma(u)\,, & u \in [0,1]\,. \end{cases} \tag{2.12}$$

In Baldasso et al. (2017) the authors prove that the sequence of probability measures $\{\mathbb{Q}_{\mu_n}\}_{n\in\mathbb{N}}$ converges weakly to $\mathbb{Q}$ as $n\to+\infty$, where $\mathbb{Q}$ is the probability measure on $\mathcal{D}_{\mathcal{M}}$ which gives mass 1 to the path $\pi(t,du)=\rho_t(u)du$, $\rho_t(\cdot)$ being the unique weak solution of (2.11). Observe that Theorem 2.3 is a corollary of this result.

2.5. *Large Deviations Principle.* We start by recalling the notion of *energy* similarly to Bertini et al. (2003); Farfan et al. (2011); Franco and Neumann (2017) and many other related papers.

Definition 2.4. For $H\in C_c^{0,1}$, define $\mathcal{E}_H:\mathcal{D}_{\mathcal{M}}\to\mathbb{R}\cup\{+\infty\}$ by

$$\mathcal{E}_H(\pi)\;=\;\begin{cases}\langle\langle\partial_u H,\rho\rangle\rangle-2\langle\langle H,H\rangle\rangle\,, & \text{if }\;\pi\in\mathcal{D}_{\mathcal{M}_0}\text{ and }\pi_t(du)=\rho_t(u)\,du\,,\\ \infty\,, & \text{otherwise}\,.\end{cases}$$

The energy functional $\mathcal{E}:\mathcal{D}_{\mathcal{M}}\to\mathbb{R}_+\cup\{\infty\}$ is then defined as

$$\mathcal{E}(\pi)\;=\;\sup_{H\in C_c^{0,1}}\mathcal{E}_H(\pi)\,.$$

By the Riesz Representation Theorem, it is well-known that $\mathcal{E}(\pi)<\infty$ implies $\pi_t=\rho_t(u)du$ with ρ belonging to the Sobolev space $L^2(0,T;\mathcal{H}^1)$, see Franco and Neumann (2017, Proposition 3.10) for instance.

Given a profile $\rho\in L^2(0,T;\mathcal{H}^1)$, which is bounded away from 0 and 1, and a measurable profile $\gamma:[0,1]\to[0,1]$, we define the linear functional $\ell_H^\theta(\rho|\gamma)$ acting on $H\in C^{1,2}$ as

$$\ell_H^\theta(\rho|\gamma)\;=\;\langle\rho_T,H_T\rangle-\langle\gamma,H_0\rangle-\int_0^T\langle\rho_s,(\partial_s+\Delta)H_s\rangle\,ds+\int_0^T\Big(\beta\partial_u H_s(1)-\alpha\partial_u H_s(0)\Big)\,ds,$$

if $\theta\in(0,1)$, and

$$\begin{aligned}\ell_H^\theta(\rho|\gamma)\;&=\;\langle\rho_T,H_T\rangle-\langle\gamma,H_0\rangle-\int_0^T\langle\rho_s,(\partial_s+\Delta)H_s\rangle\,ds+\int_0^T\Big(\rho_s(1)\partial_u H_s(1)-\rho_s(0)\partial_u H_s(0)\Big)ds\\ &=\langle\rho_T,H_T\rangle-\langle\gamma,H_0\rangle-\int_0^T\langle\rho_s,\partial_s H_s\rangle\,ds+\int_0^T\langle\partial_u\rho_s,\partial_u H_s\rangle\,ds,\end{aligned}$$

if $\theta\in(1,+\infty)$. Let $\Phi_H(\rho)$ be the non-negative convex functional acting on $H\in C^{1,2}$ as

$$H\;\mapsto\;\Phi_H(\rho)\;=\;\int_0^T\langle\chi(\rho_s),(\partial_u H_s)^2\rangle\,ds \tag{2.13}$$

where $\chi(u)=u(1-u)$ is the so-called *static compressibility* of the system. Given $H\in C^{1,2}$, we define the functional $J_H^\theta:\mathcal{D}_{\mathcal{M}}\to\mathbb{R}\cup\{+\infty\}$ by

$$J_H^\theta(\pi|\gamma)\;=\;\begin{cases}\ell_H^\theta(\rho|\gamma)-\Phi_H(\rho)\,, & \text{if }\pi\in\mathcal{F}_\theta\text{ and }\mathcal{E}(\pi)<\infty\text{ with }\pi_t=\rho_t(u)du\,,\\ +\infty\,, & \text{otherwise,}\end{cases} \tag{2.14}$$

where

$$\mathcal{F}^\theta\;:=\;\begin{cases}\mathcal{D}_{\mathcal{M}}, & \text{if }\theta\in(0,1),\\ \big\{\pi\in\mathcal{D}_{\mathcal{M}}:\langle\pi_t,1\rangle=\langle\pi_0,1\rangle\,,\,\forall t\in[0,T]\big\}, & \text{if }\theta\in(1,+\infty).\end{cases} \tag{2.15}$$

We point out that, for $\theta\in(1,+\infty)$, $\mathcal{F}^\theta$ is the set of trajectories whose total mass is constant in time. Note that the boundary integrals in $\ell_H^\theta(\rho|\gamma)$ are well-defined due to the assumption $\rho\in L^2(0,T;\mathcal{H}^1)$ and the notion of *trace* of a Sobolev space, see for instance Evans (1998).

We study in this paper the large deviations of the empirical measure starting the system from a deterministic configuration η^n, such that the sequence $\{\eta^n\}_{n\in\mathbb{N}}$ of deltas of Dirac is associated to the profile γ, where $\gamma:[0,1]\to[0,1]$ is a measurable profile bounded away from 0 and 1. The probability and expectation of the process starting from a delta of Dirac measure at η^n will be denoted by $\mathbb{P}_{\delta_{\eta^n}}$ and $\mathbb{E}_{\delta_{\eta^n}}$, respectively. We define next the large deviations rate function.

Definition 2.5. Recall from (2.7) the definition of $\mathbf{C}_\theta$. Let $\mathbf{I}_T^\theta(\cdot\,|\gamma):\mathcal{D}_\mathcal{M}\to\mathbb{R}_+\cup\{+\infty\}$ be defined by

$$\mathbf{I}_T^\theta(\pi|\gamma) \;=\; \sup_{H\in\mathbf{C}_\theta} \; J_H^\theta(\pi|\gamma)\,. \tag{2.16}$$

The rate functional (2.16) is lower semi-continuous with compact level sets in both cases $\theta\in(0,1)$ and $\theta\in(1,+\infty)$. The proof of this fact can be readily adapted from Landim and Tsunoda (2018, Theorem 4.7) taking into account that the set of trajectories with constant mass is a closed set in $\mathcal{D}_\mathcal{M}$.

We are now in position to state the main result of this paper. Let $\mathbb{Q}_{\delta_{\eta^n}}$ be the probability measure induced by the empirical measure when we start the system from η^n, where $\{\eta^n\}_{n\in\mathbb{N}}$ is a sequence of deterministic configurations associated to the measurable profile $\gamma:[0,1]\to[0,1]$, which is bounded away from 0 and 1.

Theorem 2.6. *The sequence of probability measures $\{\mathbb{Q}_{\delta_{\eta^n}}\}_{n\geq 1}$ satisfies the following large deviations principle:*

a) **(Upper bound)** *For any closed subset $\mathcal{C}$ of $\mathcal{D}_\mathcal{M}$,*

$$\varlimsup_{n\to\infty}\frac{1}{n}\log\mathbb{Q}_{\delta_{\eta^n}}\big[\,\mathcal{C}\,\big] \;\leq\; -\inf_{\pi\in\mathcal{C}}\mathbf{I}_T^\theta(\pi|\gamma)\,.$$

b) **(Lower bound)** *For any open subset $\mathcal{O}$ of $\mathcal{D}_\mathcal{M}$,*

$$\varliminf_{n\to\infty}\frac{1}{n}\log\mathbb{Q}_{\delta_{\eta^n}}\big[\,\mathcal{O}\,\big] \;\geq\; -\inf_{\pi\in\mathcal{O}}\mathbf{I}_T^\theta(\pi|\gamma)\,.$$

3. Super-exponential Replacement Lemmas

We start this section by stating some important estimates on entropy bounds and Dirichlet forms. For technical reasons, it will be important to fix a particular profile and we choose one which is locally constant equal to α near zero, locally constant equal to β near one, and linearly interpolated elsewhere. We denote once and for all this profile by $g_{\alpha,\beta}:[0,1]\to[0,1]$ and we illustrate it in Figure 3.2.

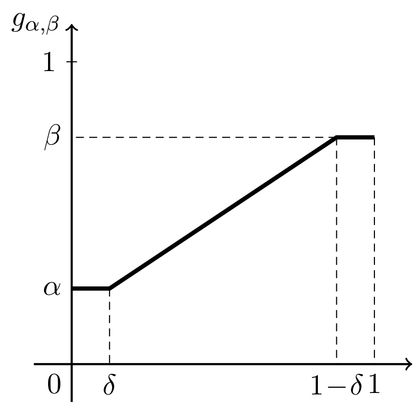

FIGURE 3.2. Profile $g_{\alpha,\beta}$. Note that it depends on δ, which is fixed and whose specific value does not play any role.

Let $\nu^n_{g_{\alpha,\beta}(\cdot)}$ be the *slow varying Bernoulli product measure* on Ω_n with parameters given by the profile $g_{\alpha,\beta}$, that is,

$$\nu^n_{g_{\alpha,\beta}(\cdot)}\{\eta\in\Omega_n\,:\,\eta(x)=1\text{ for all }x\in D\}\;=\;\prod_{x\in D}g_{\alpha,\beta}\big(\tfrac{x}{n}\big)\,,\quad\forall\,D\subset\Sigma_n\,. \tag{3.1}$$

3.1. *Entropy bounds and estimates on Dirichlet forms.* For a density function $f:\Omega_n\to[0,\infty)$ with respect to $\nu^n_{g_{\alpha,\beta}(\cdot)}$ we define

$$D_n(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})\;:=\;D_{n,0}(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+D_{n,b}(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})\,,$$

where

$$D_{n,0}(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})\;:=\;\sum_{x\in\Sigma_n}\Big\langle 1,\,\Big(\sqrt{f(\eta^{x,x+1})}-\sqrt{f(\eta)}\Big)^2\Big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\,, \tag{3.2}$$

$$D_{n,b}(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})\;:=\;\frac{1}{n^\theta}\sum_{x\in\{1,n-1\}}\Big\langle r_x(1-\eta(x))+(1-r_x)\eta(x),\Big(\sqrt{f(\eta^x)}-\sqrt{f(\eta)}\Big)^2\Big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} \tag{3.3}$$

where r_x was defined in (2.2). Our first goal is to express a relationship between the Dirichlet form defined by $\langle\mathcal{L}_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}$ and $D_n(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})$. We claim that

$$\begin{aligned}\langle\mathcal{L}_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\;&\lesssim\;-D_n(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+\sum_{x=1}^{n-1}\Big(g_{\alpha,\beta}(\tfrac{x+1}{n})-g_{\alpha,\beta}(\tfrac{x}{n})\Big)^2\\&\lesssim\;-D_n(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+\frac{1}{n}\,.\end{aligned} \tag{3.4}$$

The second inequality above is readily deduced from the definition of $g_{\alpha,\beta}$. To prove the first inequality, we recall the following lemma from Bernardin et al. (2019); Gonçalves (2019).

Lemma 3.1. *Let $T:\Omega_n\to\Omega_n$ be a map and let $c:\eta\to c(\eta)$ be a positive local function. Let f be a density with respect to a probability measure μ on Ω_n. Then*

$$\begin{aligned}\Big\langle c(\eta)[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}]\,,\ \sqrt{f(\eta)}\Big\rangle_\mu\;&\lesssim\;-\int c(\eta)\left(\Big[\sqrt{f(T(\eta))}\Big]-\Big[\sqrt{f(\eta)}\Big]\right)^2d\mu\\+\int\frac{1}{c(\eta)}\left[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\right]^2&\left(\Big[\sqrt{f(T(\eta))}\Big]+\Big[\sqrt{f(\eta)}\Big]\right)^2d\mu\,.\end{aligned} \tag{3.5}$$

As a consequence of the previous lemma, taking $\mu=\nu^n_{g_{\alpha,\beta}(\cdot)}$ we have that

$$\begin{aligned}\Big\langle\mathcal{L}_{n,0}\sqrt{f},\sqrt{f}\Big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\;&\lesssim\;-D_{n,0}(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+\sum_{x=1}^{n-1}\Big(g_{\alpha,\beta}(\tfrac{x}{n})-g_{\alpha,\beta}(\tfrac{x+1}{n})\Big)^2\\\Big\langle\mathcal{L}_{n,b}\sqrt{f},\sqrt{f}\Big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\;&\lesssim\;-D_{n,b}(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+\frac{1}{n^\theta}\Big\{\Big(g_{\alpha,\beta}(\tfrac{1}{n})-\alpha\Big)^2+\Big(g_{\alpha,\beta}(\tfrac{n-1}{n})-\beta\Big)^2\Big\}\end{aligned}$$

for any density f with respect to $\nu^n_{g_{\alpha,\beta}(\cdot)}$. We leave of details of deriving the above inequalities to the reader. We stress that the profile $g_{\alpha,\beta}(\cdot)$ is assumed to satisfy the conditions described below (3.1), hence the error coming from the bulk dynamics is of order $O(\frac{1}{n})$. In the case $\theta\in(1,+\infty)$ we do not need to impose any extra condition on the profile $g_{\alpha,\beta}(\cdot)$ in order to have the bound given by (3.4) since the factor $\frac{1}{n^\theta}$ is enough to control this term. On the other hand, in the case $\theta\in(0,1)$ we choose the profile $g_{\alpha,\beta}(\cdot)$ as being equal to α (resp. β) at 0 (resp. 1) and locally constant in a neighborhood of the boundary so that we can control the error in the previous display.

3.2. *Replacement lemmas and energy estimates.* In this section we prove the replacement lemmas required to write down the Radon-Nikodym derivative as a function of the empirical measure, as well as some energy estimates. Before proceeding, we introduce the notion of the empirical average on a box around x. By abuse of notation, let εn denotes $\lfloor \varepsilon n \rfloor$, the integer part of εn.

Definition 3.2. For any $x \in \Sigma_n$ and $\varepsilon > 0$ that satisfy $x + \varepsilon n \in \Sigma_n$ we denote by $\eta^{\varepsilon n}(x)$ the centred average on a box of size εn situated to the right or to the left of the site $x \in \Sigma_n$, that is,

$$\eta^{\varepsilon n}(x) = \begin{cases} \dfrac{1}{\varepsilon n} \displaystyle\sum_{z=x+1}^{x+\varepsilon n} \eta(z)\,, & \text{if } x \in \{1, \dots, n-1-\varepsilon n\}\,, \\ \dfrac{1}{\varepsilon n} \displaystyle\sum_{z=x-\varepsilon n}^{x-1} \eta(z)\,, & \text{if } x \in \{n-1-\varepsilon n, \dots, n-1\}\,. \end{cases} \tag{3.6}$$

Lemma 3.3. *Let $\psi = \psi_{x,\varepsilon,n} : \Omega_n \to \mathbb{R}$ be a uniformly bounded function on n and ε which is invariant for the map $\eta \mapsto \eta^{y,y+1}$ for any $y \in \{x+1, \dots, x+\varepsilon n\}$, that is, $\psi(\eta) = \psi(\eta^{y,y+1})$ for any $y \in \{x+1, \dots, x+\varepsilon n\}$. Then, for any density f with respect to $\nu^n_{g_{\alpha,\beta}(\cdot)}$, for any $n \geq 1$, for any $\varepsilon > 0$ and for any positive constant A, it holds that*

$$\Big| \big\langle \psi(\eta)\big[\eta(x) - \eta^{\varepsilon n}(x)\big], f \big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} \Big| \lesssim \tfrac{1}{A} D_n(\sqrt{f}, \nu^n_{g_{\alpha,\beta}(\cdot)}) + A\varepsilon n + \varepsilon\,.$$

Proof: We present the proof only for the case $x \in \{1, ..., n-1-\varepsilon n\}$ since the remaining case is analogous. Note that

$$\eta(x) - \eta^{\varepsilon n}(x) \;=\; \frac{1}{\varepsilon n} \sum_{y=x+1}^{x+\varepsilon n} \sum_{z=x}^{y-1} \eta(z) - \eta(z+1)\,.$$

and

$$\begin{aligned} \big\langle \psi(\eta)\big(\eta(z+1) - \eta(z)\big), f(\eta) \big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} &= \frac{1}{2} \big\langle \psi(\eta)\big(\eta(z+1) - \eta(z)\big), f(\eta) - f(\eta^{z,z+1}) \big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} \\ &+ \frac{1}{2} \big\langle \psi(\eta)\big(\eta(z+1) - \eta(z)\big), f(\eta) + f(\eta^{z,z+1}) \big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\,. \end{aligned}$$

By using the fact that for any $a, b \geq 0$, $(a - b) = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$, from Young's inequality, for any positive constant A, it holds that

$$\begin{aligned} &\Big| \big\langle \psi(\eta)\big(\eta(x) - \eta^{\varepsilon n}(x)\big), f \big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} \Big| \\ &\lesssim \frac{A}{\varepsilon n} \sum_{y=x+1}^{x+\varepsilon n} \sum_{z=x}^{y-1} \Big\langle \big(\eta(z+1) - \eta(z)\big)^2, \Big(\sqrt{f(\eta)} + \sqrt{f(\eta^{z,z+1})}\Big)^2 \Big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} \\ &+ \frac{1}{A\varepsilon n} \sum_{y=x+1}^{x+\varepsilon n} \sum_{z=x}^{y-1} \Big\langle 1, \Big(\sqrt{f(\eta)} - \sqrt{f(\eta^{z,z+1})}\Big)^2 \Big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} \\ &+ \frac{1}{\varepsilon n} \sum_{y=x+1}^{x+\varepsilon n} \left| \sum_{z=1}^{y-1} \big\langle \psi(\eta)\big(\eta(z+1) - \eta(z)\big), f(\eta) + f(\eta^{z,z+1}) \big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}} \right|. \end{aligned} \tag{3.7}$$

Note that the second term on the right-hand side of last display is bounded from above by $\frac{1}{A} D_n(\sqrt{f}, \nu^n_{g_{\alpha,\beta}(\cdot)})$. Since there is at most one particle per site and since f is a density, the first term at the right-hand side of last display is bounded from above by $A\varepsilon n$. Finally, to estimate the third term on the right-hand side of last display, we note that, since $g_{\alpha,\beta}(\cdot)$ is Lipschitz and there is at most a particle per site, it is not complicated to show that

$$\sum_{z=1}^{y-1}\Big|\big\langle\psi(\eta)\big(\eta(z+1)-\eta(z)\big),f(\eta)+f(\eta^{z,z+1})\big\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\Big| \;\lesssim\; \sum_{z=1}^{y-1}\Big|g_{\alpha,\beta}\big(\tfrac{z+1}{n}\big)-g_{\alpha,\beta}\big(\tfrac{z}{n}\big)\Big| \;\lesssim\; \frac{y}{n}\,,$$

from where the proof ends. □

In what follows φ^n is a sequence of functions in $C^{0,0}$ with uniformly bounded supremum norm. Define, for all $\theta\geq 0$,

$$V^{\theta,\varphi^n}_{\varepsilon,0}(\eta_s,s) \;=\; \frac{1}{n}\sum_{x=1}^{n-2}\varphi^n_s\big(\tfrac{x}{n}\big)\bigg\{\frac{\big(\eta_s(x)-\eta_s(x+1)\big)^2}{2}-\chi\big(\eta^{\varepsilon n}_s(x)\big)\bigg\}\,, \tag{3.8}$$

where χ is the static compressibility of the system defined below (2.13). Although this expression does not depend on θ, we keep θ in the notation to make short some statements in the sequel. For $x\in\{1,n-1\}$, let

$$V^{\theta,\varphi^n}_{\varepsilon,x}(\eta_s,s) \;=\; \begin{cases}\varphi^n_s\big(\tfrac{x}{n}\big)\,\big[\eta_s(x)-r_x\big]\,, & \text{if } \theta\in(0,1)\,,\\ \varphi^n_s\big(\tfrac{x}{n}\big)\,\big[\eta_s(x)-\eta^{\varepsilon n}_s(x)\big]\,, & \text{if } \theta\in(1,+\infty)\,,\end{cases} \tag{3.9}$$

where $r_1=\alpha$, $r_{n-1}=\beta$ and $\eta^{\varepsilon n}(x)$ was defined in (3.6).

Proposition 3.4. *For any $t\in[0,T]$, any $\theta\geq 0$ and any $x=0,1,n-1$, we have that*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\bigg[\Big|\int_0^t V^{\theta,\varphi^n}_{\varepsilon,x}(\eta_s,s)\,ds\Big|>\delta\bigg] \;=\; -\infty\,,$$

for all $\delta>0$.

Proof: Note that, for $a_n\to+\infty$ and $b_n,c_n>0$,

$$\varlimsup_{n\to+\infty}\frac{1}{a_n}\log(b_n+c_n) \;=\; \max\Big\{\varlimsup_{n\to+\infty}\frac{1}{a_n}\log b_n,\varlimsup_{n\to+\infty}\frac{1}{a_n}\log c_n\Big\}\,. \tag{3.10}$$

Using this fact, in order to prove (3.14) it is enough to show that estimate without the absolute value. By the exponential Chebychev's inequality, this probability (without the absolute value) is bounded from above by

$$\exp\{-C\delta n\}\,\mathbb{E}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\bigg[\exp\Big\{Cn\int_0^t V^{\theta,\varphi^n}_{\varepsilon,x}(\eta_s,s)\,ds\Big\}\bigg]\,,$$

for any $C>0$. From Feynman-Kac's formula, last expectation is bounded from above by

$$\exp\bigg\{\int_0^t\sup_f\Big\{\langle CnV^{\theta,\varphi^n}_{\varepsilon,x}(\eta,s),f\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}+n^2\langle\mathcal{L}_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\Big\}\,ds\bigg\}\,,$$

where the supremum is carried over all the densities f with respect to $\nu^n_{g_{\alpha,\beta}(\cdot)}$. Up to here we have

$$\begin{aligned}&\frac{1}{n}\log\mathbb{P}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\bigg[\Big|\int_0^t V^{\theta,\varphi^n}_{\varepsilon,x}(\eta_s,s)\,ds\Big|>\delta\bigg]\\ &\qquad\leq\; -C\delta+\int_0^t\sup_f\Big\{\langle CV^{\theta,\varphi^n}_{\varepsilon,x}(\eta,s),f\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}+n\langle\mathcal{L}_n\sqrt{f},\sqrt{f}\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\Big\}\,ds\,.\end{aligned} \tag{3.11}$$

Due to (3.4), the last expression is bounded from above by a constant times

$$-\,C\delta+\int_0^t\sup_f\Big\{\langle CV^{\theta,\varphi^n}_{\varepsilon,x}(\eta,s),f\rangle_{\nu^n_{g_{\alpha,\beta}(\cdot)}}-nD_n(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+1\Big\}\,ds\,. \tag{3.12}$$

The next step is to obtain a relationship between the two first parcels inside the supremum above, which has been provided by Lemma 3.3. Last display can be bounded from above by

$$-C\delta+t\sup_{f}\Big\{\frac{C}{A}D_n(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+CA\varepsilon n+C\varepsilon-nD_n(\sqrt{f},\nu^n_{g_{\alpha,\beta}(\cdot)})+1\Big\}. \tag{3.13}$$

Choosing $A=\frac{C}{n}$ on the previous expression, we get $-C\delta+t(\varepsilon C^2+\varepsilon C+1)$, so taking $\varepsilon\to 0$, we get $-C\delta+t$. And then taking $C\to+\infty$ we conclude the proof, because $t\in[0,T]$ and $\delta>0$ are fixed. $\square$

In possession of the previous results, it is a standard procedure to derive the (super-exponential) energy estimate as written below. One can follow the arguments of Franco and Neumann (2017), for instance.

Proposition 3.5. *For a function $H\in C^{0,1}_c$ and $\ell\in\mathbb{R}$ fixed, the following inequality holds:*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\Big[\mathcal{E}_H(\pi^n*\iota_\varepsilon)\geq\ell\Big]\;\leq\;-\ell\,.$$

Corollary 3.6. *For $k\in\mathbb{N}$, for functions $\{H_j\}_{1\leq j\leq k}$ in C^1_c, and $\ell\in\mathbb{R}$ fixed, we have*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}\Big[\max_{1\leq j\leq k}\mathcal{E}_{H_j}\big(\pi^n*\iota_\varepsilon\big)\;\geq\;\ell\Big]\;\leq\;-\ell\,.$$

We can now move towards super-exponential replacement lemmas for the system starting from the configuration η^n associated to the profile γ. Since

$$\frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}}\;=\;\frac{\mathbf{d}\delta_{\eta^n}}{\mathbf{d}\nu^n_{g_{\alpha,\beta}(\cdot)}}\;=\;\mathbf{1}_{\eta^n}(\eta)\prod_{x=1}^{n-1}\Big(g_{\alpha,\beta}\big(\tfrac{x}{n}\big)\Big)^{\eta(x)}\Big(1-g_{\alpha,\beta}\big(\tfrac{x}{n}\big)\Big)^{1-\eta(x)},$$

we deduce that there exists a constant $c_{\alpha,\beta}>0$ such that

$$\left|\frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}}\right|\;\leq\;e^{c_{\alpha,\beta}n}\,.$$

From the inequality above, we have $\mathbb{P}_{\delta_{\eta^n}}[\cdot]\leq\exp\{c_{\alpha,\beta}n\}\mathbb{P}_{\nu^n_{g_{\alpha,\beta}(\cdot)}}[\cdot]$. Then, from Proposition 3.4, Proposition 3.5 and Corollary 3.6 we obtain the analogous results when the system starts from η^n, that is:

Proposition 3.7. *For any $t\in[0,T]$, any $\theta\geq 0$ and any $x=0,1,n-1$, we have that*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\Big|\int_0^t V^{\theta,\varphi^n}_{\varepsilon,x}(\eta_s,s)\,ds\Big|>\delta\Big]\;=\;-\infty\,, \tag{3.14}$$

for all $\delta>0$.

Proposition 3.8. *For a function $H\in C^{0,1}_c$ and $\ell\in\mathbb{R}$ fixed, the following inequality holds:*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\mathcal{E}_H(\pi^n*\iota_\varepsilon)\geq\ell\Big]\;\leq\;-\ell+c_{\alpha,\beta}\,.$$

Corollary 3.9. *For $k\in\mathbb{N}$, for functions $\{H_j\}_{1\leq j\leq k}$ in C^1_c, and $\ell\in\mathbb{R}$ fixed, we have*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\max_{1\leq j\leq k}\mathcal{E}_{H_j}\big(\pi^n*\iota_\varepsilon\big)\;\geq\;\ell\Big]\;\leq\;-\ell+c_{\alpha,\beta}\,.$$

4. Perturbed Process

In order to derive a large deviations principle, it is natural to start with a class of perturbations of the original process, which leads the system to converge in the hydrodynamic limit to any given profile, or at least to any profile in a dense set.

A priori, it is not clear what is the natural set of perturbations of the system that one has to consider. For this reason, it makes sense to study at first a quite general set of perturbations. We will decide *a posteriori* which one is the correct set of perturbations based on the following criterion: the Radon-Nikodym derivative should be (close to) a function of the empirical measure and the elliptic equation associated to the perturbed process must have a solution. This will be made clear along the text. Of course, we could have started from the correct set of perturbations, but we chose not doing so for the sake of clarity.

Fix two functions H and G. The general perturbed process we consider is the weakly asymmetric exclusion process with slow boundary (WAEPSB), which we define through the generator $\mathcal{L}_{n,t}^{H,G} = \mathcal{L}_{n,0}^{H,t} + n^{-\theta}\mathcal{L}_{n,b}^{G,t}$ acting on functions $f : \Omega_n \to \mathbb{R}$ as:

$$(\mathcal{L}_{n,0}^{H,t} f)(\eta) = \sum_{x=1}^{n-2} e^{(\eta(x)-\eta(x+1))\left(H_t(\frac{x+1}{n})-H_t(\frac{x}{n})\right)}\Big(f(\eta^{x,x+1})-f(\eta)\Big), \tag{4.1}$$

$$(\mathcal{L}_{n,b}^{G,t} f)(\eta) = \sum_{x\in\{1,n-1\}} \Big[e^{G_t(\frac{x}{n})}r_x(1-\eta(x)) + e^{-G_t(\frac{x}{n})}(1-r_x)\,\eta(x)\Big]\Big(f(\eta^{x})-f(\eta)\Big), \tag{4.2}$$

where $\eta^{x,x+1}$ was defined in (2.3), $r_1 = \alpha$, $r_{n-1} = \beta$ and η^x was defined in (2.4). The role of the functions H and G is to introduce weak asymmetries at the bulk and at the boundary, respectively. We assume here that $H \in C^{1,2}$ and that G is C^1 in time.

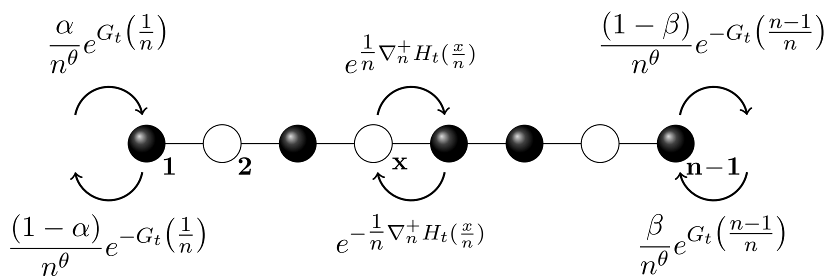

FIGURE 4.3. Illustration of jump rates for the perturbed process.

The general formula for the Radon-Nikodym derivative between two time inhomogeneous Markov processes $\mathbb{P}$ and $\overline{\mathbb{P}}$ can be found in equation (A.6) of Bertini et al. (2002), and it is given by

$$\frac{\mathbf{d}\mathbb{P}}{\mathbf{d}\overline{\mathbb{P}}}\Bigg|_{\mathcal{F}_t} = \exp\left\{-\Big(\int_0^t \big[\lambda_s(X_s)-\overline{\lambda}_s(X_s)\big]ds - \sum_{s\leq t}\log\frac{\lambda_s(X_{s^-})p_s(X_{s^-},X_s))}{\overline{\lambda}_s(X_{s^-})\overline{p}_s(X_{s^-},X_s)}\Big)\right\}, \tag{4.3}$$

where λ_s and $\overline{\lambda}_s$ are the waiting times and $p_s(\cdot,\cdot)$ and $\overline{p}_s(\cdot,\cdot)$ are the transition probabilities of $\mathbb{P}$ and $\overline{\mathbb{P}}$, respectively. Above $\mathcal{F}_t$ stands for the natural filtration. In what follows we compute the Radon-Nikodym derivative $\frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}^{H,G}_{\delta_{\eta^n}}}\Big|_{\mathcal{F}_t}$, where:

- The measure $\mathbb{P}_{\delta_{\eta^n}}$ is induced by the Markov process with infinitesimal generator $\mathcal{L}_n = \mathcal{L}_{n,0} + n^{-\theta}\mathcal{L}_{n,b}$, see (2.1) and (2.2), starting from the configuration η^n.
- The measure $\mathbb{P}^{H,G}_{\eta^n}$ is induced by the Markov process with infinitesimal generator $\mathcal{L}_{n,t}^{H,G} = \mathcal{L}_{n,0}^{H,t} + n^{-\theta}\mathcal{L}_{n,b}^{G,t}$, see (4.1) and (4.2), starting from the configuration η^n.

Having the expression (4.3) for the Radon-Nikodym derivative between two processes, we first deal with the sum

$$-\sum_{s\leq t}\log\frac{\lambda_s(X_{s^-})p_s(X_{s^-},X_s)}{\overline{\lambda}_s(X_{s^-})\overline{p}_s(X_{s^-},X_s)}\,. \tag{4.4}$$

Evaluating the parameters $\lambda_s, \bar{\lambda}_s, p_s, \bar{p}_s$ for our model, (4.4) becomes equal to

$$\begin{aligned}
&\sum_{s\leq t}\Big[G_s(\tfrac{1}{n})\Big\{\mathbf{1}_{\{\eta_{s^-}(1)=0,\eta_s(1)=1,\eta_{s^-}(2)=\eta_s(2)\}}-\mathbf{1}_{\{\eta_{s^-}(1)=1,\eta_s(1)=0,\eta_{s^-}(2)=\eta_s(2)\}}\Big\}\\
&+G_s(\tfrac{n-1}{n})\Big\{\mathbf{1}_{\{\eta_{s^-}(n-1)=0,\eta_s(n-1)=1,\eta_{s^-}(n-2)=\eta_s(n-2)\}}-\mathbf{1}_{\{\eta_{s^-}(n-1)=1,\eta_s(n-1)=0,\eta_{s^-}(n-2)=\eta_s(n-2)\}}\Big\}\\
&+\sum_{x=1}^{n-2}\tfrac{1}{n}\nabla_n^+H_s(\tfrac{x}{n})\Big\{\mathbf{1}_{\{\eta_{s^-}(x)=1,\eta_s(x)=0,\eta_{s^-}(x+1)=0,\eta_s(x+1)=1\}}\\
&-\mathbf{1}_{\{\eta_{s^-}(x)=0,\eta_s(x)=1,\eta_{s^-}(x+1)=1,\eta_s(x+1)=0\}}\Big\}\Big]\,,
\end{aligned}\tag{4.5}$$

where ∇_n^+ is the discrete derivative defined in (2.8). To shorten the expression above, we define now some currents.

For $x\in\{1,\dots,n-2\}$, denote by $J^n_{x,x+1}(t)$, the current through the edge $\{x,x+1\}$, that is, the total number of particles that have jumped from x to $x+1$ minus the total number of particles that have jumped from $x+1$ to x up to time t. The quantity $J^n_{0,1}(t)$ denotes the current at site 1, that is, the total number of particles created at the site 1 minus the total number of particles destroyed at the site 1 up to time t, while $J^n_{n-1,n}(t)$ denotes the current at site $n-1$, that is, the total number of particles destroyed at the site $n-1$ minus the total number of particles created at the site $n-1$ up to time t. These notions allow to rewrite the expression (4.5) simply as

$$\int_0^t\Big\{G_s(\tfrac{1}{n})\partial_sJ^n_{0,1}(s)-G_s(\tfrac{n-1}{n})\partial_sJ^n_{n-1,n}(s)+\sum_{x=1}^{n-2}\tfrac{1}{n}\nabla_n^+H_s(\tfrac{x}{n})\partial_sJ^n_{x,x+1}(s)\Big\}\,ds\,. \tag{4.6}$$

From an integration by parts in time, a summation by parts in space and the conservation law $\eta_t(x)-\eta_0(x)=J^n_{x-1,x}(t)-J^n_{x,x+1}(t)$, we infer that (4.6) is the same as

$$\begin{aligned}
&n\Big\{\langle\pi^n_t,H_t\rangle-\langle\pi^n_0,H_0\rangle-\int_0^t\langle\pi^n_s,\partial_sH_s\rangle\,ds\\
&+\big(G_t(\tfrac{1}{n})-H_t(\tfrac{1}{n})\big)\tfrac{1}{n}J^n_{0,1}(t)-\int_0^t(\partial_sG_s(\tfrac{1}{n})-\partial_sH_s(\tfrac{1}{n}))\tfrac{1}{n}J^n_{0,1}(s)\,ds\\
&+\big(H_t(\tfrac{n-1}{n})-G_t(\tfrac{n-1}{n})\big)\tfrac{1}{n}J^n_{n-1,n}(t)-\int_0^t(\partial_sH_s(\tfrac{n-1}{n})-\partial_sG_s(\tfrac{n-1}{n}))\tfrac{1}{n}J^n_{n-1,n}(s)\,ds\Big\}\,.
\end{aligned}\tag{4.7}$$

On the other hand, the integral term on the Radon-Nikodym derivative (4.3) is given by

$$\begin{aligned}
\int_0^t\big[\lambda_s(X_s)-\overline{\lambda}_s(X_s)\big]ds \;=\; &n\Big\{-\int_0^t\langle\pi^n_s,\Delta_nH_s\rangle\,ds-\int_0^t\langle\chi^n_s,(\nabla_n^+H_s)^2\rangle ds\\
&+\int_0^t\Big[\eta_s(n-1)\nabla_n^-H_s(\tfrac{n-1}{n})-\eta_s(1)\nabla_n^+H_s(\tfrac{1}{n})\Big]\,ds+O_H(\tfrac{1}{n})\\
&+\sum_{x\in\{1,n-1\}}n^{1-\theta}\int_0^t\Big[r_x(1-e^{G_s(\frac{x}{n})})(1-\eta_s(x))+(1-r_x)(1-e^{-G_s(\frac{x}{n})})\eta_s(x)\Big]\,ds\Big\}\,,
\end{aligned}$$

where the discrete derivatives $\nabla_n^+ H$ and $\nabla_n^- H$ and the discrete Laplacian $\Delta_n H$ have been defined in (2.8) and (2.9) and

$$\chi_s^n(du) \;=\; \frac{1}{2n}\sum_{x=1}^{n-2}\Big(\eta_s(x)-\eta_s(x+1)\Big)^2\delta_{\frac{x}{n}}(du)\,. \tag{4.8}$$

Putting all together, the Radon-Nikodym derivative is given by

$$\begin{aligned}
\frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}^{H,G}_{\delta_{\eta^n}}}\bigg|_{\mathcal{F}_t} \;=\; &\exp\Bigg\{-n\,\Bigg[\langle \pi_t^n,H_t\rangle-\langle\pi_0^n,H_0\rangle-\int_0^t\langle\pi_s^n,(\partial_s+\Delta_n)H_s\rangle\,ds\\
&-\int_0^t\langle\chi_s^n,(\nabla_n^+H_s)^2\rangle ds+\int_0^t\Big[\eta_s(n-1)\nabla_n^-H_s(\tfrac{n-1}{n})-\eta_s(1)\nabla_n^+H_s(\tfrac{1}{n})\Big]\,ds+O_H(\tfrac{1}{n})\\
&-\sum_{x\in\{1,n-1\}}n^{1-\theta}\int_0^t\Big[r_x(e^{G_s(\frac{x}{n})}-1)(1-\eta_s(x))+(1-r_x)(e^{-G_s(\frac{x}{n})}-1)\eta_s(x)\Big]\,ds\\
&+\big(G_t(\tfrac{1}{n})-H_t(\tfrac{1}{n})\big)\tfrac{1}{n}J_{0,1}^n(t)-\int_0^t(\partial_sG_s(\tfrac{1}{n})-\partial_sH_s(\tfrac{1}{n}))\tfrac{1}{n}J_{0,1}^n(s)\,ds\\
&+\big(H_t(\tfrac{n-1}{n})-G_t(\tfrac{n-1}{n}))\big)\tfrac{1}{n}J_{n-1,n}^n(t)-\int_0^t(\partial_sH_s(\tfrac{n-1}{n})-\partial_sG_s(\tfrac{n-1}{n}))\tfrac{1}{n}J_{n-1,n}^n(s)\,ds\Bigg]\Bigg\}.
\end{aligned} \tag{4.9}$$

At this point we impose that $G=H$, that is, we pick G as $G(\frac{1}{n})=H(\frac{1}{n})$ and $G(\frac{n-1}{n})=H(\frac{n-1}{n})$. The reason for such a choice is explained below.

For $\theta\in(0,1)$, as shown in Proposition 3.7 (see also (3.9) for $\theta\in(0,1)$), the time integral of the occupation variables $\eta_s(1)$ and $\eta_s(n-1)$ can be replaced by α and β, respectively. This situation lies in the same scenario of Farfan et al. (2011) for $\theta=0$ and no perturbation over the current is required.

For $\theta\in(1,+\infty)$, we need a spoiler: Lemma 5.2 will assure that the normalized currents $\frac{1}{n}J^n_{n-1,n}$ and $\frac{1}{n}J^n_{0,1}$ are super-exponentially small. Hence, no perturbation at the boundary would contribute in the limit, and the choice $G=H$ takes place for sake of simplicity.

Finally, we justify why we did not start *a priori* with the choice $G=H$. First, for pedagogical reasons: the most natural form of the Radon-Nikodym derivative is given by (4.9), including the current at the boundary. Second, but not less important, to be sure that only one perturbation is enough.

Now, as usual, we replace the discrete Laplacian by the continuous Laplacian, the discrete derivative by the continuous derivative and the values of H at $1/n$ and $(n-1)/n$ by the values of H at 0 and 1, respectively. These changes can be done by paying a price of order $o(1)$, because $H\in C^{1,2}$ and we are working on a compact space. Because of the choice $G=H$, the Radon-Nikodym derivative (4.9) can be rewritten as

$$\begin{aligned}
\frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}^{H}_{\delta_{\eta^n}}}\bigg|_{\mathcal{F}_t} = &\exp\Bigg\{-n\,\Bigg[\langle \pi_t^n,H_t\rangle-\langle\pi_0^n,H_0\rangle-\int_0^t\langle\pi_s^n,(\partial_s+\Delta)H_s\rangle\,ds\\
&-\int_0^t\langle\chi_s^n,(\partial_uH_s)^2\rangle ds+\int_0^t\Big[\eta_s(n-1)\partial_uH_s(1)-\eta_s(1)\partial_uH_s(0)\Big]\,ds+o_H(\tfrac{1}{n})\\
&-\sum_{x\in\{1,n-1\}}n^{1-\theta}\int_0^t\Big[r_x(e^{H_s(u_x)}-1)(1-\eta_s(x))+(1-r_x)(e^{-H_s(u_x)}-1)\eta_s(x)\Big]\,ds\Bigg]\Bigg\},
\end{aligned} \tag{4.10}$$

where $u_1=0$ and $u_{n-1}=1$.

Note that for $\theta \in (0,1)$, the last sum in (4.10) above may explode, motivating us to additionally assume $H_s(0) = H_s(1) = 0$ for all $s \in [0,T]$, also in agreement with Farfan et al. (2011). In Section 5 this Radon-Nikodym derivative will be further studied.

4.1. *Hydrodynamic limit for the perturbed process.* Recall the definition of the empirical measure from (2.5). Let μ_n be a measure in Ω_n associated to a measurable profile $\gamma(\cdot)$. Denote by $\mathbb{P}^H_{\mu_n}$ the measure on $\mathcal{D}([0,T],\mathcal{M})$ induced by the Markov process with infinitesimal generator $n^2\mathcal{L}_n^{H,t}$ and the initial measure μ_n and denote by $\mathbb{Q}^H_{\mu_n}$ the probability on $\mathcal{D}([0,T],\mathcal{M})$ induced by $\{\pi^n_t;\ t\in[0,T]\}$ and the initial measure μ_n. Recall the definition (2.7) for $\mathbf{C}_\theta$ and keep in mind that we additionally assume $H_s(0) = H_s(1) = 0$ for all $s \in [0,T]$, when $\theta \in (0,1)$.

Theorem 4.1. *Suppose that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is associated with a measurable profile $\gamma(\cdot)$ in the sense of (2.10). Then, for each $t\in[0,T]$, for any $\delta>0$ and any function $f\in C^0$,*

$$\lim_{n\to+\infty} \mathbb{P}^H_{\mu_n}\Big[\,\eta_\cdot : \Big|\langle \pi^n_t, f\rangle - \langle \rho^H_t, f\rangle\Big| > \delta\,\Big] \;=\; 0\,,$$

where $\rho^H\in L^2(0,T;\mathcal{H}^1)$ and

- *If $\theta\in(0,1)$, then ρ^H is the unique solution of the integral equation*

$$\begin{aligned}\mathcal{F}_{\mathrm{Dir}}(t,f,\rho^H) \;:=&\; \langle \rho^H_t, f_t\rangle - \langle \gamma, f_0\rangle - \int_0^t \langle \rho^H_s, (\partial_s+\Delta) f_s\rangle\, ds\\ &+ \int_0^t \Big[\beta\,\partial_u f_s(1) - \alpha\,\partial_u f_s(0)\Big]\, ds - 2\int_0^t \langle \chi(\rho^H_s)\,\partial_u H_s, \partial_u f_s\rangle\, ds \;=\; 0\,,\end{aligned} \tag{4.11}$$

for all $t\geq 0$ and for all $f\in\mathbf{C}_\theta$.

- *If $\theta\in(1,\infty)$, then ρ^H is the unique solution of the integral equation*

$$\begin{aligned}\mathcal{F}_{\mathrm{Neu}}(t,f,\rho^H) \;:=&\; \langle \rho^H_t, f_t\rangle - \langle \gamma, f_0\rangle - \int_0^t \langle \rho^H_s, (\partial_s+\Delta) f_s\rangle\, ds\\ &+ \int_0^t \Big[\rho^H_s(1)\partial_u f_s(1) - \rho^H_s(0)\partial_u f_s(0)\Big]\, ds - 2\int_0^t \langle \chi(\rho^H_s)\,\partial_u H_s, \partial_u f_s\rangle\, ds \;=\; 0\,,\end{aligned} \tag{4.12}$$

for all $t\geq 0$ and $f\in\mathbf{C}_\theta$.

The classical counterpart of (4.11) is the partial differential equation

$$\begin{cases} \partial_t\rho \;=\; \Delta\rho - 2\,\partial_u\big(\chi(\rho)\partial_u H\big)\\ \rho_t(0) \;=\; \alpha\,, \quad \forall\, t\in(0,T]\\ \rho_t(1) \;=\; \beta\,, \quad \forall\, t\in(0,T]\\ \rho(0,\cdot) \;=\; \gamma(\cdot)\end{cases} \tag{4.13}$$

while the classical counterpart of (4.12) is

$$\begin{cases} \partial_t\rho \;=\; \Delta\rho - 2\,\partial_u\big(\chi(\rho)\partial_u H\big)\\ \partial_u\rho_t(0) \;=\; 2\,\chi\big(\rho_t(0)\big)\,\partial_u H_t(0)\,, \quad \forall\, t\in(0,T]\\ \partial_u\rho_t(1) \;=\; 2\,\chi\big(\rho_t(1)\big)\,\partial_u H_t(1)\,, \quad \forall\, t\in(0,T]\\ \rho(0,\cdot) \;=\; \gamma(\cdot)\end{cases} \tag{4.14}$$

that is, ρ^H in each case is a weak solution of the respective PDE above.

Remark 4.2. As the reader can observe, the PDE (4.13) has Dirichlet boundary conditions, while the PDE (4.14) has Robin boundary conditions. At a first glance, the fact that the PDE (4.14) has Robin boundary conditions may look as a contradiction, since the corresponding PDE (2.12) in the symmetric case has boundary conditions. This apparent contradiction is due to the fact that such

PDE is not the heat equation, but *the heat equation with a non linear drift*. By taking $f \equiv 1$ in (4.12) we can see that the total mass of the solution ρ of (4.14) is time-invariant, which characterizes it as very close to the symmetric case with Neumann boundary conditions.

The outline of the proof of Theorem 4.1 goes as follows. As usual, the proof is split into tightness of the sequence $\{\mathbb{Q}^H_{\mu_n}\}_{n\geq 1}$ and the characterization of limit points of this sequence. Let us denote such a limit point by $\mathbb{Q}^H$. By Prohorov's Theorem, the two last results imply the convergence of $\{\mathbb{Q}^H_{\mu_n}\}_{n\geq 1}$ to $\mathbb{Q}^H$ as $n\to\infty$.

In Subsection 4.2 we deal with the tightness, while in Subsection 4.4 we characterize the limit point $\mathbb{Q}^H$ being supported on trajectories of measures with a density $\rho^H_t(\cdot)$ which is a weak solution of the corresponding hydrodynamic equation. By the uniqueness of weak solutions of the hydrodynamic equations proved in Subsection 4.5, we conclude that $\{\mathbb{Q}_{\mu_n}\}_{n\geq 1}$ has a unique limit point $\mathbb{Q}$, which yields the convergence of the whole sequence to $\mathbb{Q}^H$.

4.2. *Tightness.* In this section we show that the sequence of probability measures $\{\mathbb{Q}^H_{\mu_n}\}_{n\geq 1}$ is tight in the Skorohod space $\mathcal{D}_{\mathcal{M}}$. By Kipnis and Landim (1999, Proposition 1.7, Chapter 4) it is enough to show that for every test function f in a dense subset of C^0 with respect to the uniform topology, the sequence of measures that corresponds to the real processes $\langle \pi^n_t, f\rangle$ is tight. The prove this last claim, we will use the Aldous' Criterion, see Aldous (1978).

Lemma 4.3 (Aldous' Criterion)**.** *Let (S,d) be a Polish metric space. A sequence $\{P_n\}_{n\geq 1}$ of probability measures defined on a Skorohod space $\mathcal{D}_S$ is tight if the two conditions below hold:*

(a) For every $t\in[0,T]$ and every $\varepsilon>0$, there exists a compact set $K^t_\varepsilon\subset\mathcal{M}$ such that

$$\sup_{n\geq 1} P_n\Big(\zeta_t\notin K^t_\varepsilon\Big)\;\leq\;\varepsilon\,.$$

(b) For every $\varepsilon>0$,

$$\lim_{\gamma\downarrow 0}\,\overline{\lim_{n\to\infty}}\,\sup_{\substack{\tau\in\mathcal{T}_T\\ \theta\leq\gamma}} P_n\Big(d(\zeta_{(\tau+\theta)\wedge T},\zeta_\tau)>\varepsilon\Big)\;=\;0\,,$$

where $\mathcal{T}_T$ denotes the set of stopping times with respect to the canonical filtration, bounded by T, and ζ_t denotes the value of $\zeta\in\mathcal{D}_S$ at time t.

The condition (a) above in our setting can be translated into

$$\lim_{A\to+\infty}\,\overline{\lim_{n\to+\infty}}\,\mathbb{P}^H_{\mu_n}\Big(|\langle\pi^n_t,f\rangle|>A\Big)\;=\;0$$

which follows from Chebychev's inequality and the fact there is at most one particle per site. Now we show condition (b), which in this context, asks that for all $\varepsilon>0$ and any function f in a dense subset of C^0, with respect to the uniform topology,

$$\lim_{\delta\downarrow 0}\,\overline{\lim_{n\to\infty}}\,\sup_{\tau\in\mathcal{T}_T,r\leq\delta}\mathbb{P}^H_{\mu_n}\Big(\eta.:\big|\langle\pi^n_{\tau+r},f\rangle-\langle\pi^n_\tau,f\rangle\big|>\varepsilon\Big)\;=\;0\,. \tag{4.15}$$

The verification of condition (b) in our setting requires two different dense sets with respect to C^0. Namely, the space $C^2_c(0,1)$ for $\theta<1$ and the space C^2 for $\theta\in(1,+\infty)$. These approximations are in L^1 for $\theta<1$ and in the uniform topology for $\theta\in(1,+\infty)$.

Given $f:\Omega_n\to\mathbb{R}$, we know by Dynkin's formula (see Lemma A1.5.1 of Kipnis and Landim, 1999) that

$$M^{n,H}_t(f)\;=\;\langle\pi^n_t,f\rangle-\langle\pi^n_0,f\rangle-\int_0^t n^2\mathcal{L}^{H,s}_n\langle\pi^n_s,f\rangle\,ds \tag{4.16}$$

is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}=\{\sigma(\eta_s): s\leq t\}_{t\geq 0}$. By a simple computation, for $\eta\in\Omega_n$, for $x\in\Sigma_n$ and for $s\in[0,t]$, we have that $\mathcal{L}_n^{H,s}\eta_s(x)=j^{H_s}_{x-1,x}(\eta_s)-j^{H_s}_{x,x+1}(\eta_s)$, where the instantaneous current $j^{H_s}_{x,x+1}(\eta_s)$ is given, for $x\in\{1,\dots,n-2\}$, by

$$j^{H_s}_{x,x+1}(\eta_s) \;=\; e^{(\eta_s(x)-\eta_s(x+1))\frac{1}{n}\nabla_n^+ H_s(\frac{x}{n})}\big(\eta_s(x)-\eta_s(x+1)\big) \tag{4.17}$$

and by

$$j^{H_s}_{0,1}(\eta_s) \;=\; \frac{1}{n^\theta}\Big(e^{H_s(\frac{1}{n})}\alpha(1-\eta_s(1))-e^{-H_s(\frac{1}{n})}(1-\alpha)\eta_s(1)\Big)\,, \tag{4.18}$$

$$j^{H_s}_{n-1,n}(\eta_s) \;=\; \frac{1}{n^\theta}\Big(-e^{H_s(\frac{n-1}{n})}\beta(1-\eta_s(n-1))+e^{-H_s(\frac{n-1}{n})}(1-\beta)\eta_s(n-1)\Big)\,. \tag{4.19}$$

at the boundary. Moreover, the martingale $M_t^{n,H}(f)$ can be rewritten as

$$\langle\pi_t^n,f\rangle-\langle\pi_0^n,f\rangle-\int_0^t\sum_{x=1}^{n-2}\nabla_n^+f(\tfrac{x}{n})j^{H_s}_{x,x+1}(\eta_s)ds-\int_0^t\Big[nf(\tfrac{1}{n})j^{H_s}_{0,1}(\eta_s)-nf(\tfrac{n-1}{n})j^{H_s}_{n-1,n}(\eta_s)\Big]ds\,. \tag{4.20}$$

We start with the case $\theta\in(1,+\infty)$ and prove (4.15) directly for functions $f\in C^2$. By the triangular inequality and an union bound, the probability in (4.15) is equal or less than

$$\mathbb{P}^H_{\mu_n}\Big(\eta_\cdot:\Big|M_\tau^{n,H}(f)-M_{\tau+r}^{n,H}(f)\Big|>\frac{\varepsilon}{2}\Big)+\mathbb{P}^H_{\mu_n}\Big(\eta_\cdot:\Big|\int_\tau^{\tau+r}n^2\mathcal{L}_n^{H,s}\langle\pi_s^n,f\rangle\,ds\Big|>\frac{\varepsilon}{2}\Big)\,.$$

Applying Chebychev's inequality in the term on the left-hand side of last display and Markov's inequality in the term on the right-hand side of last display, the proof ends as long as we show that

$$\lim_{\delta\downarrow 0}\varlimsup_{n\to\infty}\sup_{\tau\in\mathcal{T}_T,r\leq\delta}\mathbb{E}^H_{\mu_n}\Big[\Big|\int_\tau^{\tau+r}n^2\mathcal{L}_n^{H,s}\langle\pi_s^n,f\rangle ds\Big|\Big] \;=\; 0 \tag{4.21}$$

and

$$\lim_{\delta\downarrow 0}\varlimsup_{n\to\infty}\sup_{\tau\in\mathcal{T}_T,r\leq\delta}\mathbb{E}^H_{\mu_n}\Big[\Big(M_\tau^{n,H}(f)-M_{\tau+r}^{n,H}(f)\Big)^2\Big] \;=\; 0 \tag{4.22}$$

where $\mathbb{E}^H_{\mu_n}$ denotes the expectation with respect to $\mathbb{P}^H_{\mu_n}$. Now we prove (4.21) and for that purpose recall (4.16), which is equal to (4.20) as mentioned above. A computation, based on the Taylor expansion of the exponential function and the fact that $H\in C^{1,2}$, permits to rewrite

$$\sum_{x=1}^{n-2}\nabla_n^+f(\tfrac{x}{n})j^{H_s}_{x,x+1}(\eta_s) \tag{4.23}$$

as

$$\nabla_n^+f(0)\eta_s(1)-\nabla_n^-f(1)\eta_s(n-1)+\frac{1}{n}\sum_{x=1}^{n-1}\Delta_nf(\tfrac{x}{n})\eta_s(x)$$

plus terms of order $O_f(1)$. Since $f\in C^2$ and due to the fact that the number of particles per site is at most one, the last expression is also of order $O_f(1)$.

Now we analyze the boundary terms in (4.20). Since $f\in C^2$, these terms are of order $O(n^{1-\theta})$. Since $\theta\in(1,+\infty)$ we conclude that $n^2\mathcal{L}_n^H(\langle\pi_s^n,f\rangle)$ is bounded by a constant. Note that for $\theta\in(0,1)$, since we consider $f\in C_c^2$, all the boundary terms that appear in the expression for $n^2\mathcal{L}_n^{H,s}\langle\pi_s^n,f\rangle$ vanish and the previous bound also shows (4.21) for the case $\theta\in(0,1)$, provided the test functions are in C_c^2.

Now we prove (4.22). The quadratic variation of the martingale $M_t^{n,H}$ is given by

$$\langle M^{n,H}(f)\rangle_t \;=\; \int_0^t\Big[n^2\mathcal{L}_n^{H,s}\langle\pi_s^n,f_s\rangle^2-2\langle\pi_s^n,f_s\rangle n^2\mathcal{L}_n^{H,s}\langle\pi_s^n,f_s\rangle\Big]\,ds\,.$$

Some computations show that the contribution from the bulk dynamics in the previous expression writes as

$$\int_0^t \frac{1}{n^2}\sum_{x=1}^{n-1}\Big(\nabla_n^+ f(\tfrac{x}{n})\Big)^2\Big(e^{\frac{1}{n}\nabla_n^+ H_s(\frac{x}{n})}\eta_s(x)(1-\eta_s(x+1))+e^{-\frac{1}{n}\nabla_n^+ H_s(\frac{x}{n})}\eta_s(x+1)(1-\eta_s(x))\Big)ds \tag{4.24}$$

while the contribution from the boundary dynamics writes as

$$\int_0^t \frac{1}{n^\theta}\sum_{x\in\{1,n-1\}} f^2(\tfrac{x}{n})\Big[e^{H_s(\frac{x}{n})}r_x(1-\eta(x))+e^{-H_s(\frac{x}{n})}(1-r_x)\,\eta(x)\Big]ds\,. \tag{4.25}$$

Since $H\in C^{1,2}$, $f\in C^2$ and due to the fact that there is at most one particle per site, we conclude that the quadratic variation of the martingale $M_t^{n,H}(f)$ is of order $O(\frac{1}{n}+\frac{1}{n^\theta})$, which vanishes as $n\to+\infty$. Since C^2 is a dense subset of C, with respect to the uniform topology, the proof of tightness in the case $\theta\in(1,+\infty)$ ends. Now let us go back to the case $\theta\in(0,1)$. Recall that we have already seen above that for test functions in C_c^2 the limit in (4.21) is true. It remains to show (4.22). But as in the case $\theta\in(1,+\infty)$ we can conclude that the quadratic variation of the corresponding martingale is of order $O(\frac{1}{n})$ and again it vanishes as $n\to+\infty$. This ends the proof of tightness.

4.3. *Replacement lemmas and energy estimates.* In this section we state the replacement lemmas that we need in order to recognize the density profile as a weak solution of the corresponding hydrodynamic equation. At the end of this section we prove that the profile belongs to the Sobolev space given in Definition 2.1. We start with a replacement lemma which suits all cases of θ. Recall (3.8) and (3.9). In what follows $\varphi\in C^{0,0}$.

Lemma 4.4. *For any $t\in[0,T]$, for any θ and for $x=0,1,n-1$ we have that*

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\mathbb{E}_{\mu_n}\Big[\Big|\int_0^t V_{\varepsilon,x}^{\theta,\varphi^n}(\eta_s,s)\,ds\Big|\Big] \;=\; 0\,.$$

From the super-exponential replacement lemma stated in Proposition 3.7 together with the fact that the Radon-Nikodym derivative is bounded and an entropy estimate (needed in order to change measures), we obtain all the replacement lemmas stated above. For this reason we omit their proofs and leave the gaps to the reader. Finally, we note that the density $\rho_t^H(u)$ belongs to $L^2(0,T;\mathcal{H}^1)$, see Definition 2.1. For that purpose, let us define the linear functional ℓ_{ρ^H} on $C_c^{0,1}$ by

$$\ell_{\rho^H}(f) \;=\; \langle\langle \partial_u f,\rho^H\rangle\rangle \;=\; \int_0^T\int_0^1 \partial_u f_s(u)\pi_s^H(du)ds\,.$$

Lemma 4.5. *The following inequality holds:*

$$\mathbb{E}_{\mathbb{Q}^H}\Big[\sup_{f\in C_c^{0,1}}\Big\{\ell_{\rho^H}(f)-2\|f\|^2_{L^2(0,T;(0,1))}\Big\}\Big] \;\lesssim\; 1\,.$$

From the last result it follows that ℓ_{ρ^H} is $\mathbb{Q}^H$ almost surely continuous, so that this linear functional can be extended to $L^2([0,T]\times(0,1))$. Then, by the Riesz's Representation Theorem, we can find $\zeta\in L^2([0,T]\times(0,1))$ such that $\ell_{\rho^H}(f)=-\langle\langle f,\zeta\rangle\rangle$ for all $f\in C_c^{0,1}$, which implies $\rho^H\in L^2(0,T;\mathcal{H}^1)$.

4.4. *Characterization of limit points.* Since we allow at most one particle per site, any limit point of the sequence $\{\mathbb{Q}^H_{\mu_n}\}_{n\geq 1}$ is concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure. That is, any limit point $\mathbb{Q}^H$ of the sequence sequence $\{\mathbb{Q}^H_n\}_{n\geq 1}$ is concentrated on trajectories of measures $\pi_t(du)$ such that $\pi_t(du) = \rho_t(u)du$, see Kipnis and Landim (1999, Chapter 4).

Since the initial measure is associated to the profile $\gamma(\cdot)$ we also know that all limit points $\mathbb{Q}^H$ of the sequence $\{\mathbb{Q}^H_{\mu_n}\}_{n\geq 1}$ are concentrated on the initial measure $\pi_0(du) = \gamma(u)du$. Now we prove that all limit points are concentrated on trajectories of measures of the form $\rho_t(u)du$, where $\rho_t(\cdot)$ is a weak solution of the corresponding hydrodynamic equation. For that purpose, let $\mathbb{Q}^H$ be a limit point of the sequence $\{\mathbb{Q}^H_{\mu_n}\}_{n\geq 1}$ and assume, without loss of generality that $\{\mathbb{Q}^H_{\mu_n}\}_{n\geq 1}$ converges weakly to $\mathbb{Q}^H$ as $n \to +\infty$.

Proposition 4.6. *If $\mathbb{Q}^H$ is a limit point of $\{\mathbb{Q}^H_{\mu_n}\}_{n\in\mathbb{N}}$, then*

$$\mathbb{Q}^H\Big(\pi \in \mathcal{D}_{\mathcal{M}} \ : \ \pi_t(du) = \rho_t(u)du \ \textit{and} \ \mathcal{F}_\theta(t,f,\rho) = 0, \forall t \in [0,T]\,, \ \forall f \in \mathbf{C}_\theta\Big) \ = \ 1\,,$$

where $\mathbf{C}_\theta$ has been defined in (2.7), and

$$\mathcal{F}_\theta(t,f,\rho) \ := \ \begin{cases} \mathcal{F}_{\mathrm{Dir}}(t,f,\rho), \ \textit{if } \theta \in (0,1), \\ \mathcal{F}_{\mathrm{Neu}}(t,f,\rho), \ \textit{if } \theta \in (1,+\infty), \end{cases}$$

with $\mathcal{F}_{\mathrm{Dir}}$ and $\mathcal{F}_{\mathrm{Neu}}$ defined in (4.11) and (4.12).

Proof: Let us start with the case $\theta \in (1,+\infty)$. It is enough to check that, for any $\delta > 0$ and any $f \in \mathbf{C}_\theta = C^{1,2}$,

$$\mathbb{Q}^H\Big(\pi \in \mathcal{D}_{\mathcal{M}} \ : \ \sup_{0\leq t\leq T} |\mathcal{F}_{\mathrm{Neu}}(t,f,\rho)| > \delta\Big) \ = \ 0\,. \tag{4.26}$$

For $u \in [0,1]$ and $\varepsilon > 0$, let $\iota_\varepsilon(u) : [0,1] \to \mathbb{R}$ be an approximation of the identity defined as

$$\iota_\varepsilon(u)(v) \ := \ \begin{cases} \varepsilon^{-1}\, \mathbf{1}_{(u,u+\varepsilon)}(v), & \text{if } u \in [0,1-\varepsilon), \\ \varepsilon^{-1}\, \mathbf{1}_{(u-\varepsilon,u)}(v), & \text{if } u \in (1-\varepsilon,1]. \end{cases} \tag{4.27}$$

Note that $\eta^{\varepsilon n}_s(x) = \pi^n_s * \iota_\varepsilon(\frac{x}{n})$ and

$$\pi_s * \iota_\varepsilon(u) \ := \ \begin{cases} \frac{1}{\varepsilon}\int_u^{u+\varepsilon} \rho^H_s(v)dv, & \text{if } u \in [0,1-\varepsilon), \\ \frac{1}{\varepsilon}\int_{u-\varepsilon}^{u} \rho^H_s(v)dv, & \text{if } u \in (1-\varepsilon,1], \end{cases} \tag{4.28}$$

since $\mathbb{Q}^H$ is concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure, that is, $\pi_t(du) = \rho_t(u)du$. By adding and subtracting $\pi_s*\iota_\varepsilon(0)$ and $\pi_s*\iota_\varepsilon(1)$ to $\rho_s(0)$ and to $\rho_s(1)$, respectively, by adding and subtracting $\chi(\pi_s*\iota_\varepsilon(u))$ to $\chi(\rho_s(u))$, and applying the triangular inequality, we can now bound the probability in (4.26) by the sum of the following probabilities:

$$\begin{aligned}\mathbb{Q}^H\bigg(\pi_t(du) = \rho_t(u)du \ : \ \sup_{0\leq t\leq T}\Big|\langle \rho_t, f_t\rangle - \langle\gamma, f_0\rangle - \int_0^t \langle \rho_s, (\partial_s+\Delta)f_s\rangle\, ds \\ - \int_0^t 2\langle \chi(\pi_s*\iota_\varepsilon)\,\partial_u H_s, \partial_u f_s\rangle\, ds + \int_0^t \Big[\pi_s*\iota_\varepsilon(1)\partial_u f_s(1) - \pi_s*\iota_\varepsilon(0)\partial_u f_s(0)\Big]\, ds\Big| > \frac{\delta}{3}\bigg),\end{aligned} \tag{4.29}$$

$$\mathbb{Q}^H\bigg(\pi_t(du) = \rho_t(u)du;: \ \Big|\int_0^t 2\langle(\chi(\rho_s) - \chi(\pi_s*\iota_\varepsilon))\,\partial_u H_s, \partial_u f_s\rangle\, ds\Big| > \frac{\delta}{3}\bigg), \tag{4.30}$$

$$\mathbb{Q}^H\bigg(\pi_t(du) = \rho_t(u)du : \sup_{0\leq t\leq T}\int_0^t \Big[(\rho_s(1) - \pi_s*\iota_\varepsilon(1))\partial_u f_s(1) - (\rho_s(0) - \pi_s*\iota_\varepsilon(0))\partial_u f_s(0)\Big]ds\Big| > \frac{\delta}{3}\bigg). \tag{4.31}$$

Now to control (4.30), observe that, by the triangular inequality and the fact that $\rho_s(\cdot) \leq 1$ for all $s \in [0,T]$, we have that

$$\Big|\chi(\rho_s(u)) - \chi(\pi_s * \iota_\varepsilon)(u)\Big| \;\leq\; C|\rho_s(u) - \pi_s * \iota_\varepsilon(u)| \tag{4.32}$$

and from Lebesgue's differentiation theorem last expression vanishes as $\varepsilon \to 0$, for a.e. $u \in [0,1]$. In a similar way, in order to control (4.31), we just need to use the fact that $\rho \in L^2(0,T;\mathcal{H}^1)$, to show that, for $j \in \{0,1\}$

$$\lim_{\varepsilon\to 0}\Big|\rho_s(j) - (\pi_s * \iota_\varepsilon)(j)\Big| \;=\; 0\,. \tag{4.33}$$

Since $\mathbb{Q}^H$ is the weak limit of $\{\mathbb{Q}^H_{\mu_n}\}_{n\in\mathbb{N}}$, we would like to apply Portmanteau's Theorem to deal with (4.29). However, the function ι_ε is not continuous, so this is, in principle, not possible. However, as in Franco et al. (2013, Proposition A.3), by approximating ι_ε by a continuous function, in such a way that the error vanishes as $\varepsilon \to 0$, we can bound (4.29) from above by

$$\begin{aligned}\varlimsup_{n\to+\infty}\mathbb{Q}^H_{\mu_n}\Bigg(\pi_t(du) = \rho_t(u)du \;:\; \sup_{0\leq t\leq T}\Big|\langle \rho_t, f_t\rangle - \langle \gamma, f_0\rangle - \int_0^t \langle \rho_s, (\partial_s+\Delta)f_s\rangle\, ds\\ - \int_0^t 2\langle \chi(\pi_s * \iota_\varepsilon)\,\partial_u H_s, \partial_u f_s\rangle\, ds\, ds + \int_0^t \Big[\pi_s * \iota_\varepsilon(1)\partial_u f_s(1) - \pi_s * \iota_\varepsilon(0)\partial_u f_s(0)\Big]\, ds\Big| > \frac{\delta}{3}\Bigg),\end{aligned} \tag{4.34}$$

plus a term that vanishes as $\varepsilon \to 0$. Now we make use of the martingale (4.16). Recal that $\mathbb{Q}^H_{\mu_n}$ is induced by $\mathbb{P}^H_{\mu_n}$ and the empirical measure π, that is, $\mathbb{Q}^H_{\mu_n} = \mathbb{P}^H_{\mu_n}\circ\pi^{-1}$. By adding and subtracting $\int_0^t n^2\mathcal{L}^{H,s}_n\langle \pi^n_s, f_s\rangle ds$ to the term inside last probability, we can bound (4.34) from above by the sum of

$$\varlimsup_{n\to\infty}\mathbb{P}^H_{\mu_n}\left(\sup_{0\leq t\leq T}\Big|\mathcal{M}^{n,H}_t(f)\Big| > \frac{\delta}{6}\right), \tag{4.35}$$

and

$$\begin{aligned}\varlimsup_{n\to\infty}\mathbb{P}^H_{\mu_n}\Bigg(\sup_{0\leq t\leq T}\Big|\int_0^t n^2\mathcal{L}^{H,s}_n\langle \pi^n_s, f_s\rangle\, ds - \int_0^t \langle \rho_s, \Delta f_s\rangle\, ds\\ - \int_0^t 2\langle \chi(\pi_s * \iota_\varepsilon)\,\partial_u H_s, \partial_u f_s\rangle\, ds + \int_0^t \Big[\eta^{\varepsilon n}_s(n-1)\partial_u f_s(1) - \eta^{\varepsilon n}_s(1)\partial_u f_s(0)\Big]\, ds\Big| > \frac{\delta}{6}\Bigg).\end{aligned} \tag{4.36}$$

By using Doob's inequality together with (4.24) and (4.25), it is easy to show that (4.35) vanishes as $n \to \infty$. Now, (4.36) can be rewritten as

$$\begin{aligned}\varlimsup_{n\to\infty}\mathbb{P}^H_{\mu_n}\Bigg(\sup_{0\leq t\leq T}\Big|\int_0^t n^2\mathcal{L}^{H,s}_n\langle \pi^n_s, f_s\rangle\, ds - \int_0^t \langle \pi^n_s, \Delta f_s\rangle\, ds\\ - \int_0^t 2\langle \chi(\pi^n_s * \iota_\varepsilon)\,\partial_u H_s, \partial_u f_s\rangle\, ds + \int_0^t \Big[\eta^{\varepsilon n}_s(n-1)\partial_u f_s(1) - \eta^{\varepsilon n}_s(1)\partial_u f_s(0)\Big]\, ds\Big| > \frac{\delta}{6}\Bigg).\end{aligned} \tag{4.37}$$

From the computations right below (4.16), we have that

$$n^2\mathcal{L}^{H,s}_n\langle \pi^n_s, f_s\rangle \;=\; -nf_s(\tfrac{1}{n})j^{H_s}_{0,1} + nf_s(\tfrac{n-1}{n})j^{H_s}_{n-1,n} + \sum_{x=1}^{n-2}\nabla^+_n f_s(\tfrac{x}{n})j^{H_s}_{x,x+1}(\eta_s). \tag{4.38}$$

Recall (4.17). By doing a Taylor expansion on the exponential in $j^{H_s}_{x,x+1}$, the term on the right-hand side of last expression is equal to

$$\sum_{x=1}^{n-2}\nabla_n^+ f_s(\tfrac{x}{n})(\eta_s(x)-\eta_s(x+1))+\frac{1}{n}\sum_{x=1}^{n-2}\nabla_n^+ f_s(\tfrac{x}{n})(\eta_s(x)-\eta_s(x+1))^2\nabla_n^+ H(\tfrac{x}{n})$$

plus a term of order $O_H(\frac{1}{n})$. A summation by parts shows that the term on the right-hand side of last expression can be written as

$$\nabla_n^+ f_s(0)\eta_s(1)-\nabla^+ f_s(\tfrac{n-1}{n})\eta_s(n-1)+\frac{1}{n}\sum_{x=1}^{n-1}\Delta_n f_s(\tfrac{x}{n})\eta_s(x)\,.$$

Then, we can bound from above the probability in (4.36) by the sum of the following terms

$$\mathbb{P}^H_{\mu_n}\left(\sup_{0\le t\le T}\Big|\int_0^t\Big(\frac{1}{n}\sum_{x=1}^{n-1}\Delta_n f_s(\tfrac{x}{n})\eta_s(x)-\langle\pi_s^n,\Delta f_s\rangle\Big)\,ds\Big|>\frac{\delta}{24}\right),\tag{4.39}$$

$$\mathbb{P}^H_{\mu_n}\Big(\sup_{0\le t\le T}\Big|\int_0^t\Big(\frac{1}{n}\sum_{x=1}^{n-2}\nabla_n^+ f_s(\tfrac{x}{n})(\eta_s(x)-\eta_s(x+1))^2\nabla_n^+ H_s(\tfrac{x}{n})-2\langle\chi(\pi_s^n*\iota_\varepsilon)\,\partial_u H_s,\partial_u f_s\rangle\Big)ds\Big|>\frac{\delta}{24}\Big),\tag{4.40}$$

$$\mathbb{P}^H_{\mu_n}\left(\sup_{0\le t\le T}\Big|\int_0^t\Big(\nabla_n^+ f_s(0)\eta_s(1)-\eta_s^{\varepsilon n}(1)\partial_u f_s(0)\Big)\,ds\Big|>\frac{\delta}{24}\right),\tag{4.41}$$

plus terms which are very similar to the previous one but related to the action of the right boundary dynamics, plus other terms that vanish as $n\to+\infty$ due to the fact that $f\in C^{1,2}$. Now, the proof ends by doing the following arguments. From a Taylor expansion on f_s we easily treat the probability in (4.39). From a Taylor expansion on both f_s and H_s, together with Markov's inequality and Lemma 4.4 for the case $x=0$, for $\varphi^n=\partial_u f_s\partial_u H_s$ and $u_x=x/n$ we are able to treat the probability in (4.40). Finally, to treat the probability in (4.41), we just need to apply Taylor expansion to f_s, together with Markov's inequality and Lemma 4.4 for the case $\theta\in(1,+\infty)$, $x=1$, for $\varphi^n=\partial_u f_s$ and $u_x=0$. We leave the details to the reader.

Now we do the sketch of the characterization of limit points in the case $\theta\in(0,1)$. In this case $f\in C_0^{1,2}$ and $\mathcal{F}_{\rm Dir}$ was defined in (4.11). Since $\mathcal{F}_{\rm Dir}$ and $\mathcal{F}_{\rm Neu}$ have a very similar expression, the only difference in the proof now is that the boundary term in (4.31) is replaced by

$$\mathbb{Q}^H\Big(\sup_{0\le t\le T}\int_0^t\Big[(\beta-\pi_s*\iota_\varepsilon(1))\partial_u f_s(1)-(\alpha-\pi_s*\iota_\varepsilon(0))\partial_u f_s(0)\Big]\,ds\Big|>\frac{\delta}{3}\Big).\tag{4.42}$$

All the other terms can be treated exactly as we did in the case $\theta\in(1,+\infty)$. Now, in order to control the last probability we just need to apply Markov's inequality and Lemma 4.4 for the case $\theta\in(0,1)$, $x=1$, for $\varphi^n=\partial_u f_s$ and $u_x=0$. We leave the details to the reader. □

4.5. *Uniqueness of weak solutions.* In this subsection we prove uniqueness of weak solutions of equations (4.11) and (4.12). These proofs are based on the fact that the eigenfunctions of the Laplacian with Neumann (and with Dirichlet) boundary conditions form an orthonormal basis. Recall that if $\{\Psi_k\}_k$ is an orthonormal basis of $L^2(0,1)$, then for all $f\in L^2$,

$$\int f^2\,du\;=\;\sum_{k\ge 0}\langle f,\Psi_k\rangle^2\,.\tag{4.43}$$

4.5.1. *The Neumann case:* $\theta \in (1,+\infty)$. Let ρ^1 and ρ^2 be weak solutions of (4.12) such that $\rho^1_0 = \gamma = \rho^2_0$. Denote $\overline{\rho} = \rho^1 - \rho^2$ and consider the set $\{\psi_k\}_{k\geq 0}$ of eigenfunctions of Laplacian with Neumann boundary conditions, i.e., $\psi_k(u) = \sqrt{2}\cos(k\pi u)$ for $k \geq 1$ and $\psi_0(u) = 1$, which is, in fact, an orthonormal basis of $L^2(0,1)$. Now, define

$$\mathcal{R}(t) \;=\; \sum_{k\geq 0} \frac{1}{2c_k} \left\langle \overline{\rho}_t, \psi_k \right\rangle^2,$$

where $c_k = (k\pi)^2 + 1$. Our goal here is to show that

$$\mathcal{R}'(t) \;\lesssim\; \mathcal{R}(t)\,. \tag{4.44}$$

In fact from last inequality together with Gronwall's inequality we conclude that $\mathcal{R}'(t) \leq 0$, which in turn implies that $\rho^1 = \rho^2$ a.e. Since $\overline{\rho}$ is bounded and $\langle \overline{\rho}_t, \psi_k\rangle$ is differentiable in time (by the definition of weak solution), we may compute the derivative of $\mathcal{R}$, which is given by

$$\mathcal{R}'(t) \;=\; \sum_{k\geq 0} \frac{1}{c_k} \left\langle \overline{\rho}_t, \psi_k \right\rangle \frac{d}{dt}\left\langle \overline{\rho}_t, \psi_k \right\rangle. \tag{4.45}$$

Using the integral equation (4.12), the expression $\frac{d}{dt}\langle \overline{\rho}_t, \psi_k\rangle$ in the last display above is equal to

$$\left\langle \overline{\rho}_t, \Delta\psi_k \right\rangle \;+\; 2\left\langle \overline{\chi}\, \partial_u H_t, \partial_u \psi_k \right\rangle,$$

where $\overline{\chi} = \chi(\rho^1_t) - \chi(\rho^2_t)$. Note that $\langle \overline{\rho}_t, \Delta\psi_k\rangle = -(k\pi)^2 \langle \overline{\rho}_t, \psi_k\rangle$. Plugging this into (4.45), we get

$$\mathcal{R}'(t) \;=\; -\sum_{k\geq 0} \frac{(k\pi)^2}{c_k} \left\langle \overline{\rho}_t, \psi_k \right\rangle^2 + \sum_{k\geq 0} \frac{2}{c_k} \left\langle \overline{\rho}_t, \psi_k \right\rangle \left\langle \overline{\chi} \partial_u H_t, \partial_u \psi_k \right\rangle. \tag{4.46}$$

Now, Young's inequality allows to bound the previous expression by

$$\frac{1}{A} \sum_{k\geq 0} \frac{1}{c_k} \left\langle \overline{\rho}_t, \psi_k \right\rangle^2 + A \sum_{k\geq 0} \frac{1}{c_k} \left\langle \overline{\chi} \partial_u H_t, \partial_u \psi_k \right\rangle^2, \tag{4.47}$$

where the specific value $A > 0$ will be chosen later. Now, observe that $\partial_u \psi_k(u) = -k\pi\, \varphi_k(u)$ with $\varphi_k(u) = \sqrt{2}\sin(k\pi u)$ for $k \geq 1$ and $\varphi_0(u) = 1$. Therefore we can bound the second sum in the display by

$$\sum_{k\geq 0} \frac{(k\pi)^2}{c_k} \left\langle \overline{\chi}\, \partial_u H_t, \varphi_k \right\rangle^2 \;\leq\; \sum_{k\geq 0} \left\langle \overline{\chi}\, \partial_u H_t, \varphi_k \right\rangle^2,$$

because $c_k = (k\pi)^2 + 1$. Since $\{\varphi_k\}_{k\geq 0}$ is an orthonormal basis of $L^2(0,1)$, it is possible to use (4.43) to write the last sum as $\int_0^1 \left(\overline{\chi} \partial_u H_t\right)^2 du$. Using the definition of $\overline{\chi}$ and the fact that χ is a Lipschitz function, we have $\int_0^1 \left(\overline{\chi}\, \partial_u H_t\right)^2 du \;\leq\; C_H \int_0^1 \left(\overline{\rho}_t\right)^2 du$. Then using again (4.43) to rewrite $\int \left(\overline{\rho}_t\right)^2 du$ as $\sum_{k\geq 0} \langle \overline{\rho}_t, \psi_k\rangle^2$, we get that

$$\mathcal{R}'(t) \;\leq\; \sum_{k\geq 0} \Big(-\frac{(k\pi)^2}{c_k} + \frac{1}{A c_k} + C_H A \Big) \left\langle \overline{\rho}_t, \psi_k \right\rangle^2.$$

Now choosing $A = \frac{1}{C_H}$ we finally get (4.44).

4.5.2. *The Dirichlet case:* $\theta \in (0,1)$. This proof in this case is similar to the one above, considering the set $\{\psi_k\}_{k\geq 0}$ of eigenfunctions of the Laplacian with Dirichlet boundary conditions, where $\psi_k(u) = \sqrt{2}\sin(k\pi u)$. Details are omitted here.

5. Large deviations upper bound

In this section we establish the large deviations uper bound, first for compact sets, then to closed sets. To do so, the following notion is relevant. We say a family of sets $\{\Gamma_\lambda\}_\lambda$ is *super-exponentially small* whenever

$$\varlimsup_{\lambda} \frac{1}{\lambda} \log P\big[\Gamma_\lambda\big] \;=\; -\infty$$

where the limsup in λ (or more parameters) depends on the context.

Let us describe the line of ideas for the proof of the upper bound. From the perturbed model presented in Section 4, we have that, for any measurable set $\mathcal{C}$ of trajectories,

$$\mathbb{P}_{\delta_{\eta^n}}\big[\{\pi^n \in \mathcal{C}\} \cap \mathcal{G}\big] \;=\; \mathbb{E}^H_{\delta_{\eta^n}}\Bigg[\mathbf{1}_{\{\pi^n\in\mathcal{C}\}\cap\mathcal{G}} \cdot \frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}^H_{\delta_{\eta^n}}}\bigg|_{\mathcal{F}_T}\Bigg]\,,$$

where the Radon-Nikodym derivative above has been computed in (4.10) and the *good set* $\mathcal{G}$, to be defined in (5.10), is a set such that its complement is super-exponentially small. In Subsection 5.2 we consider this Radon-Nikodym derivative restricted to the *good set* $\mathcal{G}$, obtaining the expression of the large deviations rate functional. Finally, in Subsection 5.3 we prove the upper bound for compact sets, and in Subsection 5.4 we extend it to closed sets by a standard argument based on exponential tightness.

5.1. *Super-exponentially small sets.* Define the set

$$B^{H,\theta}_{\varepsilon,\delta} \;:=\; \Big\{\eta_\cdot \in \mathcal{D}_{\Omega_n} \;:\; \Big|\int_0^T V^{H,\theta}_{\varepsilon,x}(\eta_s,s)\,ds\Big| \leq \delta,\; x=0,1,n-1\Big\}\,, \tag{5.1}$$

where $V^{H,\theta}_{\varepsilon,x}(\eta_s,s)$, as defined in (3.8) and (3.9), is taken under the particular choice $\varphi^n_s(u_x)=\partial_u H_s(\frac{x}{n})$. By Proposition 3.7, we know that

$$\varlimsup_{\varepsilon\downarrow 0}\,\varlimsup_{n\to\infty}\,\tfrac{1}{n}\log \mathbb{P}_{\delta_{\eta^n}}\Big[\big(B^{H,\theta}_{\varepsilon,\delta}\big)^{\complement}\Big] \;=\; -\infty \tag{5.2}$$

for all $\delta,\theta>0$ and $H\in\mathbf{C}_\theta$. Before introducing the next super-exponential small set, which is somewhat technical, let us discuss its rather simple motivation. Keep in mind that our objective is to asymptotically deal with the Radon-Nikodym derivative, which will lead us to the large deviations rate functional.

Recall that $\eta^{\varepsilon n}_s(x)=\pi^n_s*\iota_\varepsilon(\frac{x}{n})$, where the approximation of the identity $\iota_\varepsilon(u)(v)$ has been defined in (4.27). Although important, the extra regularity given by this convolution is not enough to handle limits at the boundaries, since, in general, $\pi*\iota_\varepsilon$ is not a continuous function. To overcome this, we shall (super-exponentially) replace $\pi^N*\iota_\varepsilon$ by $(\pi^N*\iota^{\mathrm{s}}_\tau)*\iota_\varepsilon$, where ι^{s}_τ is a *smooth* approximation of the identity that is defined as follows.

Fix $f:[0,1]\to\mathbb{R}_+$ a continuous function with support contained in $[\frac14,\frac34]$, $0\leq f\leq 4$, $f(0)>0$, $\int f d\lambda=1$ and symmetric around $1/2$, that is, satisfying $f(u)=f(1-u)$ for all $u\in[0,1]$. Define the continuous approximation of the identity ι^{s}_τ by $\iota^{\mathrm{s}}_\tau(u)=\frac{1}{\tau}f(\frac{u}{\tau})$. As in Lemmas 5.1, 5.2 and 5.3 of Franco and Neumann (2017), changing $\pi^n*\iota_\varepsilon$ by $(\pi^n*\iota_\varepsilon)*\iota^{\mathrm{s}}_\tau$ inside the expression of the Radon-Nikodym derivative has a cost of order $O_H(\varepsilon)+O_H(\frac{\tau}{\varepsilon})$.

Recall the notions of $\mathcal{E}$ and $\mathcal{E}_H$ in Definition 2.4. Since the rate functional is equal to infinity on trajectories $\pi\in\mathcal{D}_{\mathcal{M}}$ such that $\mathcal{E}(\pi)=\infty$, another important remark about the double convolution is that $\mathcal{E}((\pi*\iota^{\mathrm{s}}_\tau)*\iota_\varepsilon)<\infty$ for all $\pi\in\mathcal{D}_{\mathcal{M}}$.

The next set what we introduce is the set that handles with trajectories with finite energy, that is, the set $\{\pi\in\mathcal{D}_{\mathcal{M}}\;;\;\mathcal{E}(\pi)<\infty\}$. Since this set is not closed with respect to the Skorohod topology of $\mathcal{D}_{\mathcal{M}}$, this is an obstacle to apply the *Minimax Lemma* (see Kipnis and Landim, 1999, page 364, Lemma 3.3), which is an important device in the proof of the large deviations' upper bound.

To overcome this difficulty, we introduce the following sets. Let $A_{k,l}$ and $A_{k,l}^{\zeta,\tau}$ be the subsets of trajectories given by

$$
\begin{aligned}
A_{k,l} &= \{\pi \in \mathcal{D}_{\mathcal{M}} : \max_{1\le j\le k} \mathcal{E}_{H_j}(\pi) \le l\}\,,\\
A_{k,l}^{\zeta,\tau} &= \{\pi \in \mathcal{D}_{\mathcal{M}} : (\pi * \iota_\tau^{\mathrm{s}}) * \iota_\zeta \in A_{k,l}\}\,,
\end{aligned}
\tag{5.3}
$$

where $\{H_j\}_{j\in\mathbb{N}}$ is a dense set of continuous functions in the supremum norm. It is worth to emphasize that ι_ζ is the approximation of the identity defined in (4.27), where the letter ε has been replaced by ζ for aesthetic reasons. For fixed ζ, τ, k, l, the set $A_{k,l}^{\zeta,\tau}$ is closed because the function $\pi \mapsto \mathcal{E}_H((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\zeta)$ is continuous in the Skorohod topology. Indeed, it follows from the definition of $\mathcal{E}_H$, in (2.4) using the facts that H is fixed and $(\pi * \iota_\tau^{\mathrm{s}}) * \iota_\zeta$ has a density with respect to the Lebesgue measure. We claim that, for fixed k and l,

$$
\varlimsup_{\zeta\downarrow 0}\varlimsup_{\tau\downarrow 0}\varlimsup_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{\delta_{\eta^n}}\Big[\pi^n \in (A_{k,l}^{\zeta,\tau})^{\complement}\Big] \;\le\; -l\,. \tag{5.4}
$$

This is a consequence of Corollary 3.9 and the fact that $(\pi^n * \iota_\tau^{\mathrm{s}}) * \iota_\zeta - \pi^n * \iota_\zeta = O(\frac{\tau}{\zeta})$, see Proposition 5.9 in Franco and Neumann (2017) for details.

Another technical problem that arises in this setting is the fact that the empirical measure does not have a density with respect to the Lebesgue measure. An extra family of sets is then defined to circumvent this issue. Fix a sequence $\{F_i\}_{i\ge1}$ of smooth non-negative functions dense, with respect to the uniform topology, in the subset of non-negative continuous functions. For $m \ge 1$ and $j \ge 1$, define the set

$$
E_m^j \;=\; \Big\{\pi \in \mathcal{D}_{\mathcal{M}}\;;\; 0 \le \langle \pi_t, F_i\rangle \le \int_0^1 F_i(u)\,du + \tfrac{1}{j}\|F_i'\|_\infty,\, 0\le t\le T,\; i=1,\dots,m\Big\}\,. \tag{5.5}
$$

It is a simple task to check that $\mathcal{D}_{\mathcal{M}_0} = \cap_{j\ge1}\cap_{m\ge1} E_m^j$. Given $m \ge 1$ and $j \ge 1$, the following limsup holds:

$$
\varlimsup_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{\delta_{\eta^n}}\big[\pi^n \in (E_m^j)^{\complement}\big] \;=\; -\infty\,. \tag{5.6}
$$

This result is very similar to the one in Farfan et al. (2011, Subsection 6.3) and Franco and Neumann (2017), thus its proofs is omitted. For the case $\theta \in (1,+\infty)$, we also need to assure that trajectories that do not conserve mass are negligible. We thus introduce one more set. For $\lambda > 0$, let

$$
\mathcal{F}_\lambda^\theta \;=\; \begin{cases} \big\{\pi \in \mathcal{D}_{\mathcal{M}} : |\langle \pi_t, 1\rangle - \langle \pi_0, 1\rangle| \le \lambda,\; 0\le t\le T\big\}\,, & \text{if } \theta\in(1,+\infty)\,,\\ \mathcal{D}_{\mathcal{M}}\,, & \text{if } \theta \in (0,1)\,. \end{cases} \tag{5.7}
$$

This is a closed set and below we prove that it is super-exponentially small.

Lemma 5.1. *For all $\theta \in (1,+\infty)$ and all $\lambda > 0$, it holds*

$$
\varlimsup_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{\delta_{\eta^n}}\Big[\pi^n \in (\mathcal{F}_\lambda^\theta)^{\complement}\Big] \;=\; -\infty\,. \tag{5.8}
$$

Proof: Let us appeal to the Harris graphical construction of the process. Let $N_t^{+,1}$ and $N_t^{-,1}$ be the Poisson processes associated to the site $x = 1$, whose parameters are $\alpha n^{2-\theta}$ and $(1-\alpha)n^{2-\theta}$, respectively. At an arrival of the Poisson process $N_t^{+,1}$, if there is no particle at the site 1, a new particle is dropped there. And at an arrival of the Poisson process $N_t^{-,1}$, if there is a particle at the site 1, it leaves the system. Analogously, let $N_t^{+,n-1}$ and $N_t^{-,n-1}$ be the Poisson processes associated to the right site $x = n-1$, whose parameters are $\beta n^{2-\theta}$ and $(1-\beta)n^{2-\theta}$, respectively, with the same action of creation and destruction of particles at the site $x = n-1$. Since each particle contributes with a mass $1/n$ to the empirical measure, we get that

$$\Big[\pi^n \in (\mathcal{F}_\lambda^\theta)^\complement\Big] \;\subset\; \bigcup_{i\in\{+,-\}}\;\bigcup_{j\in\{1,n-1\}} \Big[\exists\, t\in[0,T] : N_t^{i,j} \geq \lambda n\Big]$$
$$\subset\; \bigcup_{i\in\{+,-\}}\;\bigcup_{j\in\{1,n-1\}} \Big[N_T^{i,j} \geq \lambda n\Big]\,.$$

By (3.10), in order to prove (5.8), it is enough to prove that

$$\varlimsup_{n\to\infty} \tfrac{1}{n}\log \mathbb{P}_{\delta_{\eta^n}}\Big[N_T^{i,j}\geq \lambda n\Big] \;=\; -\infty$$

for $i\in\{+,-\}$ and $j\in\{1,n-1\}$.

Let us prove that $\varlimsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\Big[X\geq\lambda n\Big] \;=\; -\infty$ for any $\lambda>0$, where $X\sim \text{Poisson}(cN^{2-\theta})$ with $\theta>1$ and $c>0$, which will finish the proof. By applying the exponential Tchebyshev inequality, we have that, for any $t>0$,

$$\tfrac{1}{n}\log\mathbb{P}[X\geq\lambda n] \;\leq\; \tfrac{1}{N}\log\left[\frac{\mathbb{E}[e^{tX}]}{e^{t\lambda N}}\right] \;=\; N^{1-\theta}(e^t-1)-t\lambda\,.$$

Taking the lim sup in N and recalling that $t>0$ is arbitrary, the proof ends. $\square$

Lemma 5.2. *For all $\theta\in(1,+\infty)$ and all $\lambda>0$, it holds*

$$\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\tfrac{1}{n}J_{0,1}^n(t)>\lambda\Big] \;=\; \varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\tfrac{1}{n}J_{n-1,n}^n(t)>\lambda\Big] \;=\; -\infty\,. \tag{5.9}$$

Proof: The current $J_{0,1}^n(t)$ of particles through the left boundary is stochastically dominated by a Poisson random variable $N_T^{i,j}$ of parameter $cn^{2-\theta}$ for some $c>0$. Then

$$\frac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\tfrac{1}{n}J_{0,1}^n(t)\geq\lambda\Big] \;\leq\; \frac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\tfrac{1}{n}N_T^{i,j}\geq\lambda\Big]$$

leading to (5.9) because of the argument in the proof of the previous lemma. The reasoning for $J_{n-1,n}^n(t)$ is the same. $\square$

To conclude this subsection, define the set

$$\mathcal{G}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}^{\theta,n,k,l,m,j} \;:=\; \{\pi^n\in A_{k,l}^{\zeta,\tau}\cap E_m^j\cap\mathcal{F}_\lambda^\theta\}\cap B_{\delta,\varepsilon}^H \;\subset\; \mathcal{D}_{\Omega_n}\,, \tag{5.10}$$

where the sets $A_{k,l}^{\zeta,\tau}$, E_m^j, $\mathcal{F}_\lambda^\theta$ and $B_{\delta,\varepsilon}^H$ were defined in (5.3), (5.5), (5.7) and (5.1), respectively. Since

$$\begin{aligned}
&\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\left[\left(\mathcal{G}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}^{\theta,n,k,l,m,j}\right)^\complement\right]\\
&\leq \max\bigg\{\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\pi^n\in(A_{k,l}^{\zeta,\tau})^\complement\Big]\,,\;\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\pi^n\in(E_m^j)^\complement\Big]\,,\\
&\qquad\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\pi^n\in(\mathcal{F}_\lambda^\theta)^\complement\Big]\,,\;\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[(B_{\delta,\varepsilon}^H)^\complement\Big]\bigg\}
\end{aligned}$$

and due to (5.2), (5.4), (5.6) and (5.8), we deduce that

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{\zeta\downarrow 0}\varlimsup_{\tau\downarrow 0}\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\left[\left(\mathcal{G}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}^{\theta,n,k,l,m,j}\right)^\complement\right] \leq -l\,. \tag{5.11}$$

5.2. *Radon-Nikodym derivative (continuation).* In order to write the Radon-Nikodym derivative in a proper way we start by introducing some notations. Having in mind that $\mathcal{E}((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\varepsilon) < \infty$ for all $\pi \in \mathcal{D}_{\mathcal{M}}$, we define the functional

$$J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \;=\; \begin{cases} \ell_H^\theta\big((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\varepsilon|\gamma\big) - \Phi_H\big((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\varepsilon\big), & \text{if } \;\pi \in A_{k,l}^{\zeta,\tau} \cap E_m^j \cap \mathcal{F}_\lambda^\theta\,, \\ +\infty, & \text{otherwise}\,. \end{cases} \tag{5.12}$$

The next result establishes the connection between $J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma)$ and the functional $J_H^\theta(\pi|\gamma)$ defined in (2.14).

Proposition 5.3. *For all $\pi \in \mathcal{D}_{\mathcal{M}}$,*

$$\varlimsup_{\varepsilon\downarrow 0}\,\varlimsup_{l\to\infty}\,\varlimsup_{k\to\infty}\,\varlimsup_{\zeta\downarrow 0}\,\varlimsup_{\tau\downarrow 0}\,\varlimsup_{\lambda\downarrow 0}\,\varlimsup_{j\to\infty}\,\varlimsup_{m\to\infty}\, J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \;\geq\; J_H^\theta(\pi|\gamma)\,.$$

Proof: The proof of this proposition is very similar to the proof of Proposition 5.12 of Franco and Neumann (2017), except by the presence of an extra limsup as $\lambda \downarrow 0$. For $\pi \in \mathcal{D}_{\mathcal{M}}$, if $\pi \notin \mathcal{D}_{\mathcal{M}_0}$ then there exist m and j such that $\pi \notin E_m^j$. Therefore,

$$\varlimsup_{j\to\infty}\,\varlimsup_{m\to\infty}\, J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \;=\; \begin{cases} \ell_H^\theta\big((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\varepsilon|\gamma\big) - \Phi_H\big((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\varepsilon\big), & \text{if } \;\pi \in A_{k,l}^{\zeta,\tau} \cap \mathcal{D}_{\mathcal{M}_0} \cap \mathcal{F}_\lambda^\theta, \\ +\infty, & \text{otherwise.} \end{cases}$$

Recall in (5.7) the definition of $\mathcal{F}_\lambda^\theta$ and recall in (2.15) the definition of $\mathcal{F}^\theta$. Taking the limsup as $\lambda \downarrow 0$ we obtain that

$$\varlimsup_{\lambda\downarrow 0}\,\varlimsup_{j\to\infty}\,\varlimsup_{m\to\infty}\, J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \geq \begin{cases} \ell_H^\theta\big((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\varepsilon|\gamma\big) - \Phi_H\big((\pi * \iota_\tau^{\mathrm{s}}) * \iota_\varepsilon\big), & \text{if } \;\pi \in A_{k,l}^{\zeta,\tau} \cap \mathcal{D}_{\mathcal{M}_0} \cap \mathcal{F}^\theta, \\ +\infty, & \text{otherwise.} \end{cases}$$

Recall (5.3). Taking the limsup as $\tau \downarrow 0$ and then as $\zeta \downarrow 0$,

$$\varlimsup_{\zeta\downarrow 0}\,\varlimsup_{\tau\downarrow 0}\,\varlimsup_{\lambda\downarrow 0}\,\varlimsup_{j\to\infty}\,\varlimsup_{m\to\infty}\, J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \;\geq\; \begin{cases} \ell_H^\theta\big(\pi * \iota_\varepsilon|\gamma\big) - \Phi_H\big(\pi * \iota_\varepsilon\big), & \text{if } \pi \in A_{k,l+2} \cap \mathcal{D}_{\mathcal{M}_0} \cap \mathcal{F}^\theta\,, \\ +\infty, & \text{otherwise}\,, \end{cases}$$

see Franco and Neumann (2017) for details on this step. Since $\{\pi : \mathcal{E}(\pi) \leq l+2\} \subset \mathcal{D}_{\mathcal{M}_0}$, taking now the limsup as $k \to \infty$ we obtain that

$$\varlimsup_{k\to\infty}\,\varlimsup_{\zeta\downarrow 0}\,\varlimsup_{\tau\downarrow 0}\,\varlimsup_{\lambda\downarrow 0}\,\varlimsup_{j\to\infty}\,\varlimsup_{m\to\infty}\, J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \geq \begin{cases} \ell_H^\theta\big(\pi * \iota_\varepsilon|\gamma\big) - \Phi_H\big(\pi * \iota_\varepsilon\big), & \text{if } \pi \in \mathcal{F}^\theta \text{ and } \mathcal{E}(\pi) \leq l+2, \\ +\infty\,, & \text{otherwise.} \end{cases}$$

Taking now the limsup as $l \to \infty$, we get

$$\varlimsup_{l\to\infty}\,\varlimsup_{k\to\infty}\,\varlimsup_{\zeta\downarrow 0}\,\varlimsup_{\tau\downarrow 0}\,\varlimsup_{\lambda\downarrow 0}\,\varlimsup_{j\to\infty}\,\varlimsup_{m\to\infty}\, J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \geq \begin{cases} \ell_H^\theta\big(\pi * \iota_\varepsilon|\gamma\big) - \Phi_H\big(\pi * \iota_\varepsilon\big), & \text{if } \pi \in \mathcal{F}^\theta \text{ and } \mathcal{E}(\pi) < \infty, \\ +\infty\,, & \text{otherwise.} \end{cases}$$

For π such that $\mathcal{E}(\pi) < \infty$ it holds that $\pi_t(du) = \rho_t(u)du$, where ρ has well-defined limits at the boundary. Thus, taking the limsup as $\varepsilon \downarrow 0$, we obtain

$$\varlimsup_{\varepsilon\downarrow 0}\,\varlimsup_{l\to\infty}\,\varlimsup_{k\to\infty}\,\varlimsup_{\zeta\downarrow 0}\,\varlimsup_{\tau\downarrow 0}\,\varlimsup_{\lambda\downarrow 0}\,\varlimsup_{j\to\infty}\,\varlimsup_{m\to\infty}\, J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi|\gamma) \;\geq\; J_H^\theta(\pi|\gamma)\,,$$

concluding the proof. □

One ingredient in the proof of large deviations is to restrict the Radon-Nikodym derivative given in (4.10) to the set $\mathcal{G}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}^{\theta,n,k,l,m,j}$ defined in (5.10), which encodes all the sets introduced in Subsection 5.1 and then to show that this "restricted Radon-Nikodym derivative" is close to an exponential of minus n times a functional of the empirical measure, that is, we must assure that

$$\frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}^H}\Bigg|_{\mathcal{F}_T} \cdot\; \mathbf{1}_{\mathcal{G}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}^{\theta,n,k,l,m,j}} \;=\; \mathbf{1}_{\mathcal{G}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}^{\theta,n,k,l,m,j}} \cdot \exp\Big\{ -n\Big[J_{H,\zeta,\tau,\lambda,\varepsilon}^{\theta,k,l,m,j}(\pi^n|\gamma) + \mathrm{err}_{H,\gamma}^\theta(n,\tau,\varepsilon,\delta)\Big]\Big\}, \tag{5.13}$$

with

$$\varlimsup_{\delta\downarrow 0}\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{\tau\downarrow 0}\varlimsup_{n\to\infty}\left|\mathrm{err}^{\theta}_{H,\gamma}(n,\tau,\varepsilon,\delta)\right| \;=\; 0, \tag{5.14}$$

for all $\theta > 0$, $H \in \mathbf{C}_\theta$, where the dependence on T has been omitted. Here we do not present the derivation of (5.13) because it is very similar to what is done in Franco and Neumann (2017, Subsection 5.1). We only advertise that the order of the limits above can not be exchanged. For example, one term of $\mathrm{err}^{\theta}_{H,\gamma}(n,\tau,\varepsilon,\delta)$ is of order $\frac{\tau}{\varepsilon}$. The expression (5.13) is the appropriate form for the Radon-Nikodym derivative to be used in the next subsection. Although the relationship between $J^{\theta,k,l,m,j}_{H,\zeta,\tau,\lambda,\varepsilon}(\pi|\gamma)$ and $J^{\theta}_{H}(\pi|\gamma)$ was presented in Proposition 5.3, we will use $J^{\theta,k,l,m,j}_{H,\zeta,\tau,\lambda,\varepsilon}(\pi|\gamma)$ instead of $J^{\theta}_{H}(\pi|\gamma)$ to allow the application of the Minimax Lemma.

5.3. *Upper bound for compact sets.* To reach the upper bound for compact sets we have to recall the Minimax Lemma, see Kipnis and Landim (1999, page 373, Lemma 3.3). We start with the upper bound for open sets. Let $\mathcal{O} \subseteq \mathcal{D}_{\mathcal{M}}$ be an open set and fix a function $H \in \mathbf{C}_\theta$. By a similar computation presented at the beginning of Section 5, we have, for all $\theta > 0$, $H \in \mathbf{C}_\theta$, $\lambda > 0$, $\delta > 0$, $k,l,m,j \in \mathbb{N}$, $\zeta,\tau,\varepsilon > 0$, that

$$\begin{aligned}
&\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{Q}_{\delta_{\eta^n}}[\mathcal{O}] = \varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}[\pi^n\in\mathcal{O}]\\
&\le \max\left\{\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\{\pi^n\in\mathcal{O}\}\cap\mathcal{G}^{\theta,n,k,l,m,j}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}\Big]\,,\;\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\Big(\mathcal{G}^{\theta,n,k,l,m,j}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}\Big)^{\complement}\Big]\right\},
\end{aligned}\tag{5.15}$$

where

$$\varlimsup_{l\to\infty}\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{\tau\downarrow 0}\varlimsup_{\zeta\downarrow 0}\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\Big(\mathcal{G}^{\theta,n,k,l,m,j}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}\Big)^{\complement}\Big] \;=\; -\infty\,, \tag{5.16}$$

due to (5.11). Now, we use the expression (5.13) of the Radon-Nikodym derivative to estimate the first probability in (5.15), that is:

$$\begin{aligned}
\mathbb{P}_{\delta_{\eta^n}}\Big[\{\pi^n\in\mathcal{O}\}\cap\mathcal{G}^{\theta,n,k,l,m,j}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}\Big] \;&=\; \mathbb{E}^{H}_{\delta_{\eta^n}}\left[\frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}^{H}_{\delta_{\eta^n}}}\Bigg|_{\mathcal{F}_T}\cdot\;\mathbf{1}_{\mathcal{G}^{\theta,n,k,l,m,j}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}}\cdot\mathbf{1}_{\{\pi^n\in\mathcal{O}\}}\right]\\
&=\; \mathbb{E}^{H}_{\delta_{\eta^n}}\left[\mathbf{1}_{\mathcal{G}^{\theta,n,k,l,m,j}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}}\cdot\exp\Big\{-n\big[J^{\theta,k,l,m,j}_{H,\zeta,\tau,\lambda,\varepsilon}(\pi^n|\gamma)+\mathrm{err}^{\theta}_{H,\gamma}(n,\tau,\varepsilon,\delta)\big]\Big\}\cdot\mathbf{1}_{\{\pi^n\in\mathcal{O}\}}\right].
\end{aligned}$$

Therefore,

$$\tfrac{1}{n}\log\mathbb{P}_{\delta_{\eta^n}}\Big[\{\pi^n\in\mathcal{O}\}\cap\mathcal{G}^{\theta,n,k,l,m,j}_{H,\zeta,\tau,\lambda,\delta,\varepsilon}\Big] \;\le\; \sup_{\pi\in\mathcal{O}}\Big\{-J^{\theta,k,l,m,j}_{H,\zeta,\tau,\lambda,\varepsilon}(\pi|\gamma)-\mathrm{err}^{\theta}_{H,\gamma}(n,\tau,\varepsilon,\delta)\Big\}\,.$$

Optimizing over all the parameters $\tau,\varepsilon,\zeta,\delta,\lambda,k,l,m,j,H$, it yields

$$\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{Q}_{\delta_{\eta^n}}[\,\mathcal{O}\,] \le \inf_{\substack{\tau,\varepsilon,\zeta,\delta,\lambda,\\ k,l,m,j,H}}\;\sup_{\pi\in\mathcal{O}}\max\Big\{-J^{\theta,k,l,m,j}_{H,\zeta,\tau,\lambda,\varepsilon}(\pi|\gamma)-\mathrm{err}^{\theta}_{H,\gamma}(n,\tau,\varepsilon,\delta)\,,\;\mathcal{R}^{\theta,k,l,m,j}_{H,\lambda,\delta}(\zeta,\tau,\varepsilon)\Big\}\,. \tag{5.17}$$

To interchange the supremum and the infimum above, we start by observing that for fixed parameters $\tau,\varepsilon,\zeta,\delta,\lambda,k,l,m,j,H$, the functional

$$\pi\mapsto\max\Big\{-J^{\theta,k,l,m,j}_{H,\zeta,\tau,\lambda,\varepsilon}(\pi|\gamma)-\mathrm{err}^{\theta}_{H,\gamma}(n,\tau,\varepsilon,\delta)\,,\;\mathcal{R}^{\theta,k,l,m,j}_{H,\lambda,\delta}(\zeta,\tau,\varepsilon)\Big\}$$

is upper semi-continuous in $\mathcal{D}_{\mathcal{M}}$. The proof of this result is similar to the proof of Proposition 5.11 in Franco and Neumann (2017). Thus, we can apply the Minimax Lemma, see Kipnis and Landim

(1999, page 373, Lemma 3.3), hence interchanging the supremum with the infimum in (5.17), and passing the bound to compacts sets. Then, for all $\mathcal{K} \subset \mathcal{D}_{\mathcal{M}}$ compact,

$$\overline{\lim_{n\to\infty}} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}[\mathcal{K}] \leq \sup_{\pi\in\mathcal{K}} \inf_{\substack{\tau,\varepsilon,\zeta,\delta,\lambda,\\ k,l,m,j,H}} \max\Big\{ \big[-J^{\theta,k,l,m,j}_{H,\zeta,\tau,\lambda,\varepsilon}(\pi|\gamma) - \mathrm{err}^{\theta}_{H,\gamma}(n,\tau,\varepsilon,\delta)\big]\,,\ \mathcal{R}^{\theta,k,l,m,j}_{H,\lambda,\delta}(\zeta,\tau,\varepsilon)\Big\}\,.$$

Putting together Proposition 5.3, (5.16) and (5.14), we deduce:

Proposition 5.4 (Upper bound for compact sets). *For every $\mathcal{K}$ compact subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\overline{\lim_{n\to\infty}} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}[\mathcal{K}] \ \leq\ -\inf_{\pi\in\mathcal{K}} \mathbf{I}^{\theta}_T(\pi|\gamma)\,.$$

5.4. *Upper bound for closed sets.* In Subsection 5.3, we already have the large deviations upper bound for closed sets. The extension to closed sets is a standard routine based on *exponential tightness*. The exponential tightness is defined as the existence of compact sets $K_\ell \subset \mathcal{D}_{\mathcal{M}}$ such that

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}\big[K_\ell^{\complement}\big] \ \leq\ -\ell\,, \qquad \forall\, \ell\in\mathbb{N}\,. \tag{5.18}$$

Let $\mathcal{C} \subset \mathcal{D}_{\mathcal{M}}$ be a closed set. Assuming exponential tightness, we have that

$$\begin{aligned}\limsup_{N\to\infty} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}\big[\mathcal{C}\big] \ &\leq\ \max\Big\{ \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}\big[\mathcal{C}\cap K_\ell\big]\ ,\ \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}\big[K_\ell^{\complement}\big]\Big\}\\ &\leq\ \max\Big\{ \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}\big[\mathcal{C}\cap K_\ell\big]\ ,\ -\ell\Big\}\,.\end{aligned}$$

Hence, since the set $\mathcal{C}\cap K_\ell$ is compact and ℓ is arbitrary, the upper bound for closed sets will follow from the upper bound for compact sets.

The proof of the exponential tightness (5.18) is somewhat technical and follows the same steps of Franco and Neumann (2017, Section 5.3)[1]. For this reason, we discuss only what needs to be checked for our model. With respect to Franco and Neumann (2017, Section 5.3), the only and somewhat crucial point to be adapted is to find a positive mean one martingale with respect to the natural filtration,

$$M^{a,H}_t \ :=\ \exp\Big\{ an\Big[\langle \pi^n_t, H\rangle - \langle \pi^n_0, H\rangle - \int_0^t U^a_n(H,s,\eta_s)\,ds\Big]\Big\}\,,$$

where $|U^a_n(H,s,\eta_s)|$ is uniformly bounded in $n\in\mathbb{N}$. This claim is a consequence of the general fact that the Radon-Nikodym derivative between two Markov processes is a positive mean one martingale with respect to the natural filtration, together with formula (4.10) choosing aH in lieu of H. In resume, we have therefore achieved:

Proposition 5.5 (Upper bound for closed sets). *For every $\mathcal{C}$ closed subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}\big[\mathcal{C}\big] \ \leq\ -\inf_{\pi\in\mathcal{C}} \mathbf{I}^{\theta}_T(\pi|\gamma)\,.$$

6. Large deviations lower bound

The proof of the lower bound in the case $\theta\in(0,1)$ is quite similar to Bertini et al. (2009) or Farfan et al. (2011) (in dimension $d=1$), which correspond to $\theta=0$ in our setting. We henceforth study in detail the case $\theta\in(1,+\infty)$ following the more recent approach of Landim and Tsunoda (2018). Due to the presence of large deviations from the initial measure we are not allowed to apply Jona-Lasinio et al. (1993, Theorem 2.4) and an $\mathbf{I}^{\theta}_T$-density argument is required here as in the framework of Landim and Tsunoda (2018).

[1] In its hand, Franco and Neumann (2017, Section 5.3) is essentially a detailed version of Kipnis and Landim (1999, pp. 271–273).

6.1. *Lower bound for smooth profiles.* The next two propositions are immediate consequences of the definition of J_G^θ and show that solutions of the perturbed partial differential equations (4.13) or (4.14) depending on whether $\theta \in (0,1)$ or $\theta \in (1,+\infty)$ lead to a simpler representation of the rate function.

Proposition 6.1. *Consider $\theta \in (0,1)$ or $\theta \in (1,+\infty)$ and recall the definition of $\mathbf{C}_\theta$. Given $H \in \mathbf{C}_\theta$, let ρ^H be the unique weak solution of (4.13) if $\theta \in (0,1)$ or the unique weak solution of (4.14) if $\theta \in (1,+\infty)$. Then*

$$\begin{aligned}\sup_{G\in\mathbf{C}_\theta} J_G^\theta(\rho^H|\gamma) &= \sup_{G\in\mathbf{C}_\theta}\Big\{\ell_G^\theta(\rho^H|\gamma) - \Phi_G(\rho^H)\Big\}\\ &= \sup_{G\in\mathbf{C}_\theta}\Big\{2\int_0^t \langle \chi(\rho_s^H)\,\partial_u H_s, \partial_u G_s\rangle\,ds - \int_0^T \langle \chi(\rho_s^H), (\partial_u H_s)^2\rangle\,ds\Big\}\\ &= \int_0^T \langle \chi(\rho_s^H), (\partial_u H_s)^2\rangle\,ds\,.\end{aligned}$$

Proposition 6.1 motivates the next definition.

Definition 6.2. Denote by Π the subspace of $\mathcal{D}_{\mathcal{M}_0}$ consisting of all paths $\pi_t(du) = \rho_t(u)\,du$ for which there exists some $H \in \mathbf{C}_\theta$ such that $\rho = \rho^H$ is the unique weak solution of (4.13) if $\theta \in (0,1)$ or the unique weak solution of (4.14) if $\theta \in (1,+\infty)$.

The next two propositions provide conditions to assure that a profile ρ is a solution of the corresponding hydrodynamic equation (according to each regime of θ) for some H. That is, conditions to assure that $\rho \in \Pi$. Proposition 6.3 is well known in the literature and it is included here for sake of completeness.

Proposition 6.3. *Let $\theta \in (0,1)$. Let $\rho \in C^{1,2}$ such that $0 < \varepsilon \le \rho \le 1-\varepsilon$ for some $\varepsilon > 0$. Then, there exists an unique (strong) solution H of the elliptic equation*

$$\begin{cases} \partial_u^2 H_t(u) + \dfrac{\partial_u\big(\chi(\rho_t(u))\big)}{\chi(\rho_t(u))}\,\partial_u H_t(u) = \dfrac{\Delta\rho_t(u) - \partial_t\rho_t(u)}{2\,\chi(\rho_t(u))}\,,\ \forall u \in (0,1) & (6.1)\\ H_t(0) = 0 & (6.2)\\ H_t(1) = 0 & (6.3)\end{cases}$$

Proof: Fix $t \in [0,T]$. Since (6.1) is a linear ODE of second order on H, we solve it, getting

$$\begin{aligned}H_t(u) &= H_t(0) + \big(2\chi(\rho_t(0))\partial_u H_t(0) - \partial_u\rho_t(0)\big)\int_0^u \frac{1}{2\chi(\rho_t(v))}\,dv\\ &\quad + \int_0^u \frac{\partial_u\rho_t(v) - \partial_t\int_0^v \rho_t(w)\,dw}{2\chi(\rho_t(v))}\,dv\,.\end{aligned}\tag{6.4}$$

Taking $u=1$ and then applying the boundary conditions (6.2) and (6.3) in the equality (6.4) above, we get

$$\partial_u H_t(0) = \frac{1}{2\chi(\rho_t(0))}\Big\{\partial_u\rho_t(0) - \frac{\mathbb{I}_t}{I_t}\Big\}\,,\tag{6.5}$$

where

$$I_t := \int_0^1 \frac{1}{2\chi(\rho_t(v))}\,dv \quad\text{and}\quad \mathbb{I}_t := \int_0^1 \frac{\partial_u\rho_t(v) - \partial_t\int_0^v \rho_t(w)\,dw}{2\chi(\rho_t(v))}\,dv\,.\tag{6.6}$$

In other words, (6.5) is the right guess for $\partial_u H_t(0)$ in order to achieve the solution of the elliptic PDE in the statement of the proposition. Coming back to (6.4), we then apply (6.2) and (6.5), which leads us to

$$\begin{aligned} H_t(u) \;&=\; \int_0^u \frac{\frac{-\mathbb{I}_t}{I_t} + \partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w)\,dw}{2\chi(\rho_t(v))}\,dv \\ &=\; -\frac{\mathbb{I}_t}{I_t}\int_0^u \frac{1}{2\chi(\rho_t(v))}\,dv + \int_0^u \frac{\partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w)\,dw}{2\chi(\rho_t(v))}\,dv\,, \end{aligned}$$

and it is straightforward to check that this is the required solution of the elliptic PDE. □

Proposition 6.4. *Let $\theta \in (1,+\infty)$. Consider $\rho \in C^{1,2}$ such that $0 < \varepsilon \le \rho \le 1-\varepsilon$ for some $\varepsilon > 0$ and $\partial_t \int_0^1 \rho_t(z)\,dz = 0$. Then, up to an additive constant, there exists an unique (strong) solution H of the elliptic equation*

$$\begin{cases} \partial_u^2 H_t(u) \;+\; \frac{\partial_u\left(\chi(\rho_t(u))\right)}{\chi(\rho_t(u))}\,\partial_u H_t(u) \;=\; \frac{\Delta\rho_t(u) - \partial_t \rho_t(u)}{2\,\chi(\rho_t(u))}\,,\ \forall u \in (0,1) & (6.7)\\ \partial_u H_t(0) = \frac{1}{2\,\chi(\rho_t(0))}\partial_u \rho_t(0) & (6.8)\\ \partial_u H_t(1) = \frac{1}{2\,\chi(\rho_t(1))}\partial_u \rho_t(1) & (6.9) \end{cases}$$

Proof: Fix $t \in [0,T]$. Solving the linear ODE of second order (6.7), we get

$$H_t(u) := H_t(0) + \int_0^u \frac{2\chi(\rho_t(0))\partial_u H_t(0) - \partial_u \rho_t(0) + \partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w)\,dw}{2\chi(\rho_t(v))}\,dv\,.$$

The boundary condition (6.8) then leads us to

$$H_t(u) \;=\; H_t(0) + \int_0^u \frac{\partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w)\,dw}{2\chi(\rho_t(v))}\,dv\,.$$

Keeping in mind that $\partial_t \int_0^1 \rho_t(z)\,dz = 0$ it is straightforward to check that the expression on the right-hand side of the above expression satisfies (6.9) regardless of the chosen value for $H_t(0)$. □

Proposition 6.5. *Let $\mathcal{O}$ be an open set of $\mathcal{D}_{\mathcal{M}}$. Then*

$$\varliminf_{n\to\infty} \frac{1}{n} \log \mathbb{Q}_{\delta_{\eta^n}}\left[\,\mathcal{O}\,\right] \;\ge\; -\inf_{\pi \in \mathcal{O}\cap\Pi} \mathbf{I}_T^{\theta}(\pi|\gamma)\,.$$

The proof of the inequality above relies on the hydrodynamic limit for the perturbed process and Proposition 6.1. It follows the same lines of Kipnis and Landim (1999, Chapter 10) or Franco and Neumann (2017). Let

$$\boldsymbol{H}\big(\mathbb{P}^H_{\delta_{\eta^n}}|\mathbb{P}_{\delta_{\eta^n}}\big) \;:=\; \mathbb{E}^H_{\delta_{\eta^n}}\Big[\log \frac{\mathbf{d}\mathbb{P}^H_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}}\Big] \;=\; -\mathbb{E}^H_{\delta_{\eta^n}}\Big[\log \frac{\mathbf{d}\mathbb{P}_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}^H_{\delta_{\eta^n}}}\Big] \tag{6.10}$$

be the so-called *relative entropy* of $\mathbb{P}^H_{\delta_{\eta^n}}$ with respect to $\mathbb{P}_{\delta_{\eta^n}}$.

Lemma 6.6. *Let $H \in \mathbf{C}_\theta$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\boldsymbol{H}\big(\mathbb{P}^H_{\delta_{\eta^n}}|\mathbb{P}_{\delta_{\eta^n}}\big) \;=\; \mathbf{I}_T^{\theta}(\rho^H|\gamma)\,,$$

where ρ^H is the unique weak solution of (4.13) if $\theta \in (0,1)$, or the unique weak solution of (4.14) if $\theta \in (1,+\infty)$.

Proof: Recall the definition of $B^{H,\theta}_{\varepsilon,\delta}$ in (5.1), which is super-exponentially small, see (5.2). On the $B^{H,\theta}_{\varepsilon,\delta}$, the Radon-Nikodym derivative $\frac{\mathbf{d}\mathbb{P}^H_{\delta_{\eta^n}}}{\mathbf{d}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}}$ is equal to

$$\exp\Big\{n\Big[J^\theta_H\Big((\pi^n * \iota^{\mathrm{s}}_\tau) * \iota_\varepsilon|\gamma\Big) + O_{H,T,\varepsilon,\gamma}(\tfrac{1}{n}) + O(\delta) + O_H(\varepsilon) + O_H(\tfrac{\gamma}{\varepsilon})\Big]\Big\}\,. \tag{6.11}$$

The proof of the above assertion is technical and follows the same steps of Franco and Neumann (2017). In view of (6.10) for the relative entropy,

$$\frac{1}{n}\boldsymbol{H}\big(\mathbb{P}^H_{\delta_\eta n}|\mathbb{P}_{\delta_\eta n}\big) \;=\; \frac{1}{n}\mathbb{E}^H_{\delta_\eta n}\Big[\log\frac{\mathbf{d}\mathbb{P}^H_{\delta_\eta n}}{\mathbf{d}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}}\mathbf{1}_{B^{H,\theta}_{\varepsilon,\delta}}\Big] + \frac{1}{n}\mathbb{E}^H_{\delta_\eta n}\Big[\log\frac{\mathbf{d}\mathbb{P}^H_{\delta_\eta n}}{\mathbf{d}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}}\mathbf{1}_{(B^{H,\theta}_{\varepsilon,\delta})^\complement}\Big], \tag{6.12}$$

where the $B^{H,\theta}_{\varepsilon,\delta}$ has been defined in (5.10). By (5.11), the complement of this set is super-exponentially small with respect to $\mathbb{P}_{\nu^n_{\gamma(\cdot)}}$. We claim now that the complement is super-exponentially small also with respect to $\mathbb{P}^H_{\delta_\eta n}$. Indeed, by (4.10) there exists a constant $C(H,T)>0$ such that

$$\mathbb{P}^H_{\delta_\eta n}\Big[(B^{H,\theta}_{\varepsilon,\delta})^\complement\Big] \;=\; \mathbb{E}_{\nu^n_{\gamma(\cdot)}}\Big[\frac{\mathbf{d}\mathbb{P}^H_{\delta_\eta n}}{\mathbf{d}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}}\mathbf{1}_{(B^{H,\theta}_{\varepsilon,\delta})^\complement}\Big] \;\leq\; e^{C(H,T)n}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}\Big[(B^{H,\theta}_{\varepsilon,\delta})^\complement\Big]$$

and by (5.2) we get

$$\varlimsup_{\varepsilon\downarrow 0}\varlimsup_{n\to\infty}\tfrac{1}{n}\log\mathbb{P}^H_{\delta_\eta n}\Big[\big(B^{H,\theta}_{\varepsilon,\delta}\big)^\complement\Big] \;=\; -\infty$$

concluding the proof of the claim. By the previous limit and since $\frac{1}{n}\log\frac{\mathbf{d}\mathbb{P}^H_{\delta_\eta n}}{\mathbf{d}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}}$ is bounded, the right-hand side of (6.12) can be written as

$$\frac{1}{n}\mathbb{E}^H_{\delta_\eta n}\Big[\log\frac{\mathbf{d}\mathbb{P}^H_{\delta_\eta n}}{\mathbf{d}\mathbb{P}_{\nu^n_{\gamma(\cdot)}}}\mathbf{1}_{B^{H,\theta}_{\varepsilon,\delta}}\Big] + o_n(1)\,. \tag{6.13}$$

By Theorem 4.1, under $\mathbb{P}^H_{\delta_\eta n}$ the probability concentrates on ρ^H. Since the functional $J^\theta_H((\pi^n*\iota^{\mathrm{s}}_\tau)*\iota_\varepsilon|\gamma)$ is continuous in the Skohorod topology, recalling (6.11) the proof ends. □

6.2. *The $\mathbf{I}^\theta_T$-density.* In the previous subsection we have achieved the lower bound for smooth profiles. Our task now consists on extending it to any profile. We start with the definition of $\mathbf{I}^\theta_T$-density.

Definition 6.7. Let A be a subset of $\mathcal{D}_\mathcal{M}$. The set A is said to be $\mathbf{I}^\theta_T(\cdot|\gamma)$-dense if for any $\pi\in\mathcal{D}_\mathcal{M}$ such that $\mathbf{I}^\theta_T(\pi|\gamma)<\infty$ there exists a sequence $\{\pi_n : n\geq 1\}$ in A such that

$$\pi_n\to\pi \text{ in } \mathcal{D}_\mathcal{M} \qquad\text{and}\qquad \mathbf{I}^\theta_T(\pi_n|\gamma)\to\mathbf{I}^\theta_T(\pi|\gamma)\,.$$

Recall Definition 6.2. The main result to be proved now is:

Theorem 6.8. *The set Π is $\mathbf{I}^\theta_T$-dense.*

The statement above does not involve probability: it is a purely analytical result. Thus, since the $\mathbf{I}^\theta_T$ functional for $\theta\in(0,1)$ coincides with the rate functional of Bertini et al. (2009) under the assumption that the external field there considered is null, we thus may apply Bertini et al. (2009, Theorem 5.1) in this case.

From this point on we will deal only with the case $\theta\in(1,+\infty)$, where the proof of Theorem 6.8 is split into intermediate lemmas. We start with a key technical result, in whose proof is developed by mixing ideas from Farfan et al. (2011) and Landim and Tsunoda (2018).

Proposition 6.9. *Let $\theta\in(1,+\infty)$. There exists a constant $\tilde{C}_0>0$ such that, for any $\rho\in\mathcal{D}_\mathcal{M}$, it holds*

$$\int_0^T\int_0^1\frac{\big(\partial_u\rho_t(u)\big)^2}{\chi(\rho_t(u))}dudt \;\leq\; \tilde{C}_0\big(\mathbf{I}^\theta_T(\rho|\gamma)+1\big)\,. \tag{6.14}$$

Proof: In what follows, assume $\pi \in \mathcal{D}_{\mathcal{M}}$ to be such that $\mathbf{I}_T^{\theta}(\pi|\gamma) < \infty$, otherwise (6.14) is trivial. Since $\mathbf{I}_T^{\theta}(\pi|\gamma) < \infty$, then $\pi(t, du) = \rho(t, u)du$ with $\rho \in L^2(0, T; \mathcal{H}^1)$ and from an integration by parts we have that

$$\mathbf{I}_T^{\theta}(\pi|\gamma) \;=\; \sup_{H\in\mathbf{C}_\theta} J_H^{\theta}(\rho|\gamma) \;=\; \sup_{H\in\mathbf{C}_\theta} \Big\{L_H(\rho|\gamma) + B_H(\rho)\Big\}\,,$$

where

$$L_H(\rho|\gamma) \;=\; \langle \rho_T, H_T\rangle - \langle \gamma, H_0\rangle - \int_0^T \langle \rho_s, \partial_s H_s\rangle ds \qquad \text{and}$$

$$B_H(\rho) \;=\; \int_0^T \langle \partial_u \rho_s, \partial_u H_s\rangle ds - \int_0^T \big\langle \chi\big(\rho_s\big), \big(\partial_u H_s\big)^2\big\rangle\, ds\,.$$

For $a \in (0, 1)$, let $h_a : [0, 1] \to \mathbb{R}$ be the function defined by

$$h_a(x) \;=\; (x + a)\log(x + a) + (1 - x + a)\log(1 - x + a)$$

whose first and second derivatives are, respectively,

$$h_a'(x) \;=\; \log\left(\frac{x + a}{1 - x + a}\right) \qquad \text{and} \qquad h_a''(x) \;=\; \frac{1 + 2a}{(x + a)(1 - x + a)}\,.$$

It is elementary to check that $-\log 2 \le h_a(x) \le \log 4$ for all $x \in (0, 1)$. Let

$$H_\rho \;:=\; h_a'(\rho)\,.$$

Since the space integrals above are with respect to the Lebesgue measure, we can see the integrated functions as functions defined on the continuous torus $\mathbb{T} = [0, 1)$ rather than on the interval $[0, 1]$. Moreover, we extend (on the time parameter) the functions above from $[0, T]$ to some open interval (c, d) containing $[0, T]$ by imposing that the extension is constant on $(c, 0]$ and $[T, d)$, that is, given $f\colon [c, d] \times \mathbb{T} \to \mathbb{R}$, its extension $\overline{f}\colon [c, d] \times \mathbb{T} \to \mathbb{R}$ will be defined by

$$\overline{f}(t, u) \;=\; \begin{cases} f(t, u)\,, & \text{if } (t, u) \in [0, T] \times \mathbb{T}\,,\\ f(0, u)\,, & \text{if } (t, u) \in (c, 0) \times \mathbb{T}\,,\\ f(T, u)\,, & \text{if } (t, u) \in (T, d) \times \mathbb{T}\,. \end{cases}$$

Abusing of notation, let ι_δ and ι_ε be smooth approximations of the identity on $\mathbb{T}$ and (c, d), respectively. Let $H_{\rho^{\varepsilon,\delta}} := h_a'(\rho^{\varepsilon,\delta})$, where $\rho^{\varepsilon,\delta}$ is a convolution in space and in time (on the parameters ε and δ, respectively) of the function ρ, that is,

$$\rho^{\varepsilon,\delta}(u, t) \;:=\; \big(\rho * \iota_\varepsilon * \iota_\delta\big)(u, t) \;=\; \int_{(c,d)}\int_{\mathbb{T}} \rho(s, v)\iota_\varepsilon(u - v)\iota_\delta(t - s)dvds\,.$$

Note now that

$$\begin{aligned}\sup_{H\in\mathbf{C}_\theta} \Big\{L_H(\rho) + B_H(\rho)\Big\} \;&\ge\; L_{H_{\rho^{\varepsilon,\delta}}}(\rho|\gamma) + B_{H_{\rho^{\varepsilon,\delta}}}(\rho)\\ &=\; L_{H_{\rho^{\varepsilon,\delta}}}(\rho^{\varepsilon,\delta}|\gamma) + \Big\{L_{H_{\rho^{\varepsilon,\delta}}}(\rho|\gamma) - L_{H_{\rho^{\varepsilon,\delta}}}(\rho^{\varepsilon,\delta}|\gamma)\Big\} + B_{H_{\rho^{\varepsilon,\delta}}}(\rho)\,.\end{aligned}$$

At this point we must handle each one of the parcels above. By the chain rule and Fubini's Theorem,

$$\begin{aligned}L_{H_{\rho^{\varepsilon,\delta}}}(\rho^{\varepsilon,\delta}|\gamma) \;&=\; \int_0^T \langle \partial_s \rho^{\varepsilon,\delta}, H_{\rho^{\varepsilon,\delta}}\rangle ds \;=\; \int_0^T\int_{\mathbb{T}} \partial_s \rho_s^{\varepsilon,\delta}(u) h_a'\big(\rho_s^{\varepsilon,\delta}(u)\big)duds\\ &=\; \int_{\mathbb{T}}\int_0^T \partial_s\Big(h_a\big(\rho_s^{\varepsilon,\delta}(u)\big)\Big)dsdu \;=\; \int_{\mathbb{T}} \Big\{h_a\big(\rho_T^{\varepsilon,\delta}(u)\big) - h_a\big(\gamma^{\varepsilon,\delta}(u)\big)\Big\}du\end{aligned}$$

and from $-\log 2 \le h_a(\cdot) \le \log 4$ we infer that

$$L_{H_{\rho^{\varepsilon,\delta}}}(\rho^{\varepsilon,\delta}|\gamma) \;\ge\; -(\log 2 + \log 4) \;=\; -3\log 2\,. \tag{6.15}$$

By the same arguments of Landim and Tsunoda (2018, Lemma 4.4), for any fixed $\varepsilon > 0$,

$$\lim_{\delta\searrow 0}\Big\{L_{H_{\rho^{\varepsilon,\delta}}}(\rho|\gamma) - L_{H_{\rho^{\varepsilon,\delta}}}(\rho^{\varepsilon,\delta}|\gamma)\Big\} \;=\; 0\,. \tag{6.16}$$

Finally, $B_{H_{\rho^{\varepsilon,\delta}}}(\rho)$ converges, as ε and δ decrease to zero, to

$$\begin{aligned} B_{H_\rho}(\rho) \;&=\; \int_0^T \langle \partial_u\rho, \partial_u h_a'(\rho)\rangle ds - \int_0^T \langle \chi(\rho), \big(\partial_u h_a'(\rho)\big)^2\rangle ds \\ &\geq\; \int_0^T \Big\langle \partial_u\rho, \frac{(1+2a)\partial_u\rho}{(\rho+a)(1-\rho+a)}\Big\rangle ds - \int_0^T \langle \frac{1}{4}, \frac{(1+2a)^2(\partial_u\rho)^2}{(\rho+a)^2(1-\rho+a)^2}\rangle ds\,. \end{aligned}$$

Taking the lim inf as $a \searrow 0$, applying Fatou's Lemma and recalling (6.15) and (6.16), we are lead to

$$\mathbf{I}_T^\theta(\pi|\gamma) \;\geq\; -3\log 2 + \frac{3}{4}\int_0^T\int_0^1 \frac{\big(\partial_u\rho_t(u)\big)^2}{\chi(\rho_t(u))} du dt$$

finishing the proof. $\square$

Lemma 6.10. *The density ρ of a trajectory $\pi \in \mathcal{D}_{\mathcal{M}_0}$ is the weak solution of hydrodynamic equation (2.12) with initial condition γ if, and only if, $\mathbf{I}_T^\theta(\pi|\gamma) = 0$. Moreover, in such case we have that*

$$\int_0^T\int_0^1 \frac{\big(\partial_u\rho_t(u)\big)^2}{\chi(\rho_t(u))} du dt \;<\; \infty\,. \tag{6.17}$$

Proof: Suppose that the density ρ of a trajectory $\pi \in \mathcal{D}_{\mathcal{M}_0}$ is the weak solution of hydrodynamic equation (2.12) with initial condition γ. Then, for $H \in C^{1,2}$,

$$J_H(\rho|\gamma) \;=\; -\int_0^T \langle \chi(\rho_s), (\partial_u H_s)^2\rangle\, ds \;\leq\; 0\,.$$

Moreover, since ρ is the weak solution of (2.12), it is easy to check that the total mass of $\pi_t(du) = \rho_t(u)du$ is conserved in time, that is, $\pi \in \mathcal{F}^\theta$, see (2.15). This implies that $\mathbf{I}_T^\theta(\pi|\gamma) = 0$.

Suppose now that $\mathbf{I}_T^\theta(\pi|\gamma) = 0$. Therefore $J_{\varepsilon H}(\rho) \leq 0$ for any $H \in C^{1,2}$, which in its turn implies that the derivative of $J_{\varepsilon H}(\rho) \leq 0$ with respect to ε is zero at $\varepsilon = 0$. This permits to conclude that the density ρ is the weak solution of hydrodynamic equation (2.12) with initial condition γ.

Finally, if $\mathbf{I}_T^\theta(\pi|\gamma) < \infty$, then (6.17) holds by Proposition 6.9. $\square$

Let Π_1 be the set of all paths $\pi_t(du) = \rho_t(u)du$ in $\mathcal{D}_{\mathcal{M}_0}$ whose density ρ is a weak solution of the Cauchy problem (2.2) on some time interval $[0,\delta]$, with $\delta > 0$.

Lemma 6.11. *The set Π_1 is $\mathbf{I}_T^\theta$-dense.*

Proof: The proof here follows the same steps of Landim and Tsunoda (2018, Lemma 5.3). Fix $\pi_t = \rho(t,u)du \in \mathcal{D}_{\mathcal{M}_0}$ such that $\mathbf{I}_T^\theta(\pi|\gamma) < \infty$. Let λ be the solution of the hydrodynamic equation (2.12). For $\delta > 0$, let $\pi_t^\delta(du) = \rho_t^\delta(u)du$ where ρ^δ evolves as λ on the time interval $[0,\delta]$, then evolves as λ reversed in time on $[\delta, 2\delta]$ and then evolves as ρ in the remaining time interval, that is,

$$\rho^\delta(t,u) \;=\; \begin{cases} \lambda(t,u) & \text{if } t \in [0,\delta]\,, \\ \lambda(2\delta - t, u) & \text{if } t \in [\delta, 2\delta]\,, \\ \rho(t-2\delta, u) & \text{if } t \in [2\delta, T]\,. \end{cases} \tag{6.18}$$

Since π^δ converges to π in $\mathcal{D}_{\mathcal{M}}$ as $\delta \downarrow 0$ and $\pi^\delta \in \Pi_1$, it only remains to show that $\mathbf{I}_T^\theta(\pi^\delta|\gamma)$ converges to $\mathbf{I}_T^\theta(\pi|\gamma)$ as $\delta \downarrow 0$. By the lower semi-continuity of the rate function, we have $\mathbf{I}_T^\theta(\pi|\gamma) \leq \liminf_{\delta\to 0}\mathbf{I}_T^\theta(\pi^\delta|\gamma)$ hence it is missing to assure that

$$\mathbf{I}_T^\theta(\pi|\gamma) \;\geq\; \limsup_{\delta\to 0}\mathbf{I}_T^\theta(\pi^\delta|\gamma)\,. \tag{6.19}$$

To do so, note that

$$\mathcal{E}_H(\pi^\delta) \;\leq\; 2\mathcal{E}_H(\lambda) + \mathcal{E}_H(\pi) \;<\; \infty\,,$$

where the last inequality above is due to the assumption $\mathbf{I}_T^\theta(\pi|\gamma) < \infty$ and Lemma 6.10. Using this and the fact the profile ρ^δ conserves the total mass we can infer that $\mathbf{I}_T^\theta(\pi^\delta|\gamma) < \infty$ for any δ.

By linearity of integrals, we will analyze separately the contributions on $\mathbf{I}_T^\theta(\pi^\delta|\gamma)$ from the three time intervals of (6.18). The contribution of $[0,\delta]$ is zero by Lemma 6.10.

Since the Neumann boundary conditions are invariant by a time inversion, the profile ρ^δ is a weak solution on the time interval $[\delta, 2\delta]$ of

$$\begin{cases} \partial_t \rho(t,u) = -\partial_u^2 \rho(t,u) \\ \partial_u \rho(t,0) = \partial_u \rho(t,1) = 0 \end{cases}$$

which allows to conclude that the second contribution is given by

$$\sup_{H\in C^{1,2}} \left\{ \int_0^\delta \Big(2\langle \partial_u \lambda_t, \partial_u H\rangle - \langle \chi(\lambda_t), (\partial_u H)^2\rangle \Big) dt \right\}. \tag{6.20}$$

Multiplying and diving the leftmost term inside parenthesis of last expression by $\sqrt{\chi(\lambda_t)}$ and applying Young's inequality $ab \leq a^2/2 + b^2/2$, we can bound the previous expression from above by

$$\int_0^\delta \int_0^1 \frac{\big(\partial_u \lambda_t(u)\big)^2}{\chi(\lambda_t(u))} du dt$$

which goes to zero as $\delta \searrow 0$ by Lemma 6.10 and Dominated Convergence Theorem.

Finally, the third contribution is bounded above by $\mathbf{I}_T^\theta(\pi|\gamma)$ since π^δ on this interval is a time translation of π. Putting all these things together, we are lead to (6.19) and hence we finish the proof.

□

Next, we present the sets Π_2, Π_3 and Π_4. Let Π_2 be the set of all paths $\pi_t(du) = \rho_t(u)du$ in Π_1 with the property that for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\varepsilon \leq \rho_t(u) \leq 1-\varepsilon$ for all $(t,u) \in [\delta, T] \times [0,1]$. Let Π_3 be the set of all paths $\pi_t(du) = \rho_t(u)du$ in Π_2 whose density $\rho(t,u)\,du$ belongs to the space $C^\infty[0,1]$ for any $t \in [0,T]$. Let Π_4 be the set of all paths $\pi_t(du) = \rho_t(u)du$ in Π_3 whose density $\rho_t(u)\,du$ belongs to the space $C^{\infty,\infty}([0,T]\times[0,1])$.

Lemma 6.12. *The sets Π_2, Π_3 and Π_4 are $\mathbf{I}_T^\theta$-dense.*

The proof of the Lemma 6.12 can be promptly adapted from Landim and Tsunoda (2018, Lemmas 5.4, 5.5 and 5.6), and for this reason it is omitted. We thus conclude the proof of the $\mathbf{I}_T^\theta$-density, that is, the proof of Theorem 6.8.

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