Technical Report

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An Interval Fixed Point Theorem*

Paulo Werlang de Oliveira
Dalcidio Moraes Claudio
Instituto de Informática da UFRGS
P.O.BOX: 15.064 – ZIP: 91.501-970 – Porto Alegre/RS – Brazil
Phone: (051) 316.61.68 – Fax: (051) 319.15.76
pwerlang@inf.ufrgs.br — dalcidio@music.pucrs.br

Abstract

In this paper we will present an interval version to the Fixed Point Theorem. Such theorem offers a practical method (the 'sucessive approximations method') which serves to the interval fixed point equation root compute. We will also present a criterion that allows to define easily the interval semi-plain regions which can hold such roots. Finally, we will do a practical application, showing in what manner to compute the polynomial interval function fixed points.

Key Words

Interval Arithmetic, Fixed Point Theorem.

1. Introduction

In this paper we will present an interval version to the Fixed Point Theorem. Such theorem offers a practical method (the 'sucessive approximations method') to compute the interval fixed point equation roots.

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We will start presenting some interval arithmetic basic concepts which will be necessary to our study. After this we will present and demonstrate the Fixed Point Theorem interval version. We will also show a criterion that allows to define easily the regions of the interval semi-plain which can hold such roots. At last, we will make a practical application of such theorem, showing in what manner to compute the polynomial interval functions fixed points such as $f(X) = A.X^2 + B.X + C$, where $| A | < 1$.

2. Basic Concepts

Follows we will present some interval arithmetic basic concepts.

Definition 1 (Interval)

Let be $\mathbb{R}$ the real number set and $a, b \in \mathbb{R}$ two points such that $a \leqslant b$. Then, the set $\{x \in \mathbb{R}/a \leqslant x \leqslant b\}$ is a "interval of real numbers" (or just "interval") and it will be denoted by $X = [a;\ b]$.

Definition 2 ($\mathbb{IR}$ Set)

We define and denote by $\mathbb{IR}$ the set of all intervals, that is $\mathbb{IR} = \{[a;\ b]/a, b \in \mathbb{R}, a \leqslant b\}$.

Definition 3 (Arithmetic Operations on $\mathbb{IR}$)

Let $A, B \in \mathbb{IR}$ two intervals. The operations "sum", subtraction", "product" and "division" in $\mathbb{IR}$ are defined by $A * B = \{a * b/a \in A, b \in B\}$, in which $* \in \{+, -, ., /\}$ is any one of the four arithmetic operations. We need assume that $0 \notin B$, in the division.

Definition 4 (Distance between two intervals)

Let be $A = [a;\ b]$ and $B = [c;\ d]$ two intervals.

We define the **distance** *from A to B by $\delta = max\{| a - c |, | b - d |\}$.*

Notation: $dist(A, B) = dist([a;\ b], [c;\ d]) = max\{| a - c |, | b - d |\} \geqslant 0$.

Corollary 1

$A = B \Leftrightarrow dist(A, B) = 0$.

Definition 5 (Modulus of an interval)

Let $A = [a;\ b] \in \mathbb{IR}$ *an interval.*

We define the **modulus** *of the interval* A *by* $\mu = dist(A, 0)$, *which corresponds to the distance from* A *to zero.*

Notation: $\mid A \mid = \mid [a;\ b] \mid = dist(A, 0) = max\{\mid a \mid, \mid b \mid\} \geqslant 0.$

Corollary 2 (Modulus Properties)

1. $\mid X \mid = 0 \Leftrightarrow X = 0;$
2. $\mid X + Y \mid \leqslant \mid X \mid + \mid Y \mid;$
3. $\mid X.Y \mid \leqslant \mid X \mid . \mid Y \mid.$

Theorem 1

Let $A, B, C, D \in \mathbb{IR}$ *intervals. Then:*

1. $dist(A + B, A + C) = dist(B, C);$
2. $dist(A.B, A.C) \leqslant \mid A \mid .dist(B, C);$
3. $dist(A + B, C + D) \leqslant dist(A, C) + dist(B, D);$
4. $dist(A.B, C.D) \leqslant \mid B \mid .dist(A, C) + \mid C \mid .dist(B, D);$
5. $dist(A, B) \leqslant \mid A \mid . \mid B \mid .dist(\frac{1}{A}, \frac{1}{B})$, *if* $0 \notin A$, $0 \notin B$;
6. $\mid X^n \mid = \mid X \mid^n .$

Definition 6 (Interval Sequence)

An **interval sequence** *is a function*

$$\mathcal{X} : \mathbb{N} \rightarrow \mathbb{IR}$$
$$n \mapsto X(n) = X_n$$

that associates to each natural number n *an interval* $X(n)$ *in* $\mathbb{IR}$.

The interval $X(n)$ *will be represented by* X_n *and the sequence* $\mathcal{X} = (X_1, X_2, X_3, \ldots, X_n, \ldots)$ *will be denoted by* $(X_n)_{n \in \mathbb{N}}$ *or simply by* $(X_n)_n$.

Definition 7 (Limit of a Sequence)

Let $\mathcal{X} = (X_n)_n$ *an interval sequence.*

We say that the interval $\mathcal{L}$ *is the* **limit** *of the sequence* $\mathcal{X} = (X_n)_n$ *if the terms* X_n *tends to* $\mathcal{L}$, *that is* $\mathcal{L}$ *is the limit of sequence* $\mathcal{X} = (X_n)_n$ *if*

$$(\forall \varepsilon > 0)(\exists\, n_0 \in \mathbb{N})/\{dist(X_n, \mathcal{L}) < \varepsilon\}$$

always that $n \geqslant n_0$. *Simbolically we have:*

$\lim_{n \to \infty} X_n = \mathcal{L} \Leftrightarrow (\forall \varepsilon > 0)(\exists\, n_0 \in \mathbb{N})/n > n_0 \Rightarrow dist(X_n, \mathcal{L}) < \varepsilon$

<u>Notation</u>: $\mathcal{L} = \lim_{n\to\infty} X_n$ or $\mathcal{L} = \lim X_n$

Definition 8 (Convergent Sequence)
Let $\mathcal{X} = (X_n)_n$ *an interval sequence.* $\mathcal{X} = (X_n)_n$ *is a* **convergent sequence** *if exist* $\mathcal{L} \in \mathbb{IR}$ *such that* $\lim_{n\to\infty} X_n = \mathcal{L}$.

In this case, we say that X_n *converges to* $\mathcal{L}$ *and we denote by* $X_n \to \mathcal{L}$ *to* $n \to \infty$.

Definition 9 (Interval DIGSEs)
Let $X = [\underline{x}; \overline{x}]$ *the exact solution for an interval problem and let* $Y = [\underline{y}; \overline{y}]$ *an approximation for such solution* X. *We say that* Y *has* α **exact significative digits** *(***DIGSEs***) in respect to* X *if* $\underline{y}$ *as well as* $\overline{y}$ *have* α *exact significative digits respecting to* $\underline{x}$ *and* $\overline{x}$*, respectively. The value of* α *is given by:*

$$\alpha = DIGSE(Y, X) = min\{-\log_{10}(2.\frac{\mid \underline{y} - \underline{x} \mid}{\mid \underline{x} \mid}), -\log_{10}(2.\frac{\mid \overline{y} - \overline{x} \mid}{\mid \overline{x} \mid})\}$$

If $Y = X$ *then , by definition,* $\alpha = +\infty$*, that is, all the digits of* Y *are corret.*

Example 1 *Let* $X = [e, \pi]$ *and* $Y = [2.7182818, 3.1415999]$.
We have $DIGSE(Y, X) = DIGSE([2.7182818, 3.1415999], [e, \pi]) =$
$min\{-\log_{10}(2.\frac{|2.7182818 - e|}{|e|}), -\log_{10}(2.\frac{|3.1415999 - \pi|}{|\pi|})\} =$
$min\{-\log_{10}(2.0527 * 10^{-8}), -\log_{10}(4.6132 * 10^{-6})\} = min\{7.687, 5.336\} =$
5.33.

Thus, Y *has 5 corret digits in respect to* X.

Definition 10 (Complete Metric Space)
We say that (S, ρ) *is a* **Complete Metric Space** *if for any Cauchy's sequence* $\mathcal{X} = (X_n)_n$ *in* S*, the value* $\mathcal{L}$ *of the limit is also an element that belongs to the set* S.

Theorem 2 (IR is a Complete Metric Space)
The $\mathbb{IR}$ *set supplied function* $\rho(X, Y) = dist(X, Y)$ *is a Complete Metric Space.*

Proof: Omitted. The detailed proof can be found in [EDG 90] or in [ALE 83].

Definition 11 (Closed Ball)

The **closed ball** *centered in* $A \in \mathbb{IR}$ *with radius* $r \geqslant 0$ *is the set of the points* $X \in \mathbb{IR}$ *in which the distance to point* A *is smaller or equal to* r. *Such set will be denoted by* $\mathcal{F}_r(A)$.

Thus, $\mathcal{F}_r(A) = \{X \in \mathbb{IR} / dist(X, A) \leqslant r\}$.

Definition 12 (Open Ball)

The **open ball** *centered in* $A \in \mathbb{IR}$ *with radius* $r > 0$ *is the set of the points* $X \in \mathbb{IR}$ *in which the distance to point* A *is smaller to* r. *Such set will be denoted by* $\mathcal{B}_r(A)$.

Thus, $\mathcal{B}_r(A) = \{X \in \mathbb{IR} / dist(X, A) < r\}$.

Definition 13 (Accumulation Point)

Let $\mathbb{X} \subseteq \mathbb{IR}$ *a subset of* $\mathbb{IR}$.

We say that $A \in \mathbb{IR}$ *is an* **accumulation point** *of* $\mathbb{X}$ *if any open ball centered in* A *holds some point of* $\mathbb{X}$ *different from* A, *that is,*

$$(\forall \varepsilon > 0)(\exists X \in \mathbb{X}) / X \in (\mathcal{B}_\varepsilon(A) - \{A\})$$

In other words, A *is an accumulation point of* $\mathbb{X}$ *if* $(\forall \varepsilon > 0)(\exists X \in \mathbb{X})$ *such that* $0 < dist(X, A) < \varepsilon$.

Definition 14 (Interval Function)

Let $f : \begin{array}{ccc} \mathbb{X} & \mapsto & \mathbb{Y} \\ X & \mapsto & f(X) \end{array}$ *a function* .

If $\mathbb{X} = Dom(f) \subseteq \mathbb{IR}$ *and* $\mathbb{Y} = Cod(f) \subseteq \mathbb{IR}$ *then we say that* f *is an* **interval function of an interval variable.**

Definition 15 (Limit of a Function)

Let $f : \mathbb{X} \rightarrow \mathbb{IR}$ *a interval function defined in a subset* $\mathbb{X} \subseteq \mathbb{IR}$ *and let* $A \in \mathbb{IR}$ *an accumulation point of* $\mathbb{X}$.

The interval L *is the* **limit of function** $f(X)$ *when* X *tends to* A *if for each real number* $\varepsilon > 0$ *arbitrary, exists a real number* $\delta > 0$ *such that* $dist(f(X), L) < \varepsilon$ *always that* $X \in \mathbb{X}$ *and* $0 < dist(X, A) < \delta$.

Notation

$$\lim_{X\to A} f(X) = L$$

It must be observed that expression $\lim_{X\to A} f(X) = L$ is an abbreviation to the following affirmation:

$$(\forall\, \varepsilon > 0)\ \ (\exists\, \delta > 0)/X \in \mathbb{X},\ \ 0 < dist(X, A) < \delta \Rightarrow dist(f(X), L) < \varepsilon$$

or that, it always possible to become $f(X)$ arbitrarily near L, since that we take $X \in \mathbb{X}$ suficiently near A, but $X \neq A$.

Intuitively, a function $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ is continuous in $A \in \mathbb{X}$ when it is possible to become $f(X)$ arbitrarily near of $f(A)$ since that we take X suficiently near of A.

To be more precise, we have the following definition:

Definition 16 (Continuous Interval Function)

Let $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ a interval function.

Given $A \in \mathbb{X}$ a point in the domain of the function, we say that f is a **continuous interval function** *in A if*

$$(\forall\, \varepsilon > 0)(\exists\, \delta > 0)/X \in \mathbb{X} \quad and \quad dist(X, A) < \delta \Rightarrow dist(f(X), f(A)) < \varepsilon$$

That is, f is continuous in A if for each $\varepsilon > 0$ given arbitrarily we can find $\delta > 0$ such that given $X \in \mathbb{X}$ we have that the distance from $f(X)$ to $f(A)$ is smaller than ε always that the distance from X to A is smaller than δ.

We say simply that $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ is **continuous** when f is continuous in all the points $X \in \mathbb{X}$.

As we saw in the previous definition, a function $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ is continuous in $\mathbb{X}$ if $(\forall\, A \in \mathbb{X})$ and $(\forall\, \varepsilon > 0)(\exists\, \delta > 0)/X \in \mathbb{X}$ and $dist(X, A) < \delta \Rightarrow dist(f(X), f(A)) < \varepsilon$. In this case, it must be observed that the value of δ depends on ε as well as the point A chosen in $\mathbb{X}$.

In general, it is not possible to obtain from $\varepsilon > 0$ given, an only $\delta > 0$ that serves for all the points A from $\mathbb{X}$. If it to happen, then f will be said a continuous uniform function.

Definition 17 (Continuous Uniform Function)

Let $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ a interval function. Then f is a **continuous uniform function** *if $(\forall\, \varepsilon > 0)(\exists\, \delta > 0)$ such that $\forall\, X, Y \in \mathbb{X}$ with $dist(X, Y) < \delta$ we have that $dist(f(X), f(Y)) < \varepsilon$.*

Theorem 3

Let $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ a continuous uniform function. If $\mathcal{X} = (X_n)_n$ is a Cauchy's sequence in $\mathbb{X}$ then $Y_n = f(X_n)$ is a Cauchy's sequence.

Corollary 3

Let $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ a continuous uniform function. Then for all $A \in \mathbb{X}'$ exists the limit $\lim_{X \to A} f(X)$.

Theorem 4

A function $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ is a continuous uniform function iff $(X_n)_n$ and $(Y_n)_n$ in $\mathbb{X}$ with $\lim_{n\to\infty} dist(X_n, Y_n) = 0$ we have that $\lim_{n\to\infty} dist(f(X_n), f(Y_n)) = 0$.

Definition 18 (Lipschitz's Function)

Let $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ a interval function. Then f is a **Lipschitz's function** *if exist $c > 0$ such that $dist(f(X), f(Y)) \leqslant c.dist(X, Y)$, for all $X, Y \in \mathbb{X}$.*

Remark: All Lipschitz's function is continuous, because given $A \in \mathbb{X}$ and for all $\varepsilon > 0$ we take $\delta = \frac{\varepsilon}{c} > 0$ and thus $dist(X, A) < \delta \Rightarrow dist(f(X), f(A)) \leqslant c.dist(X, A) = c.\frac{\varepsilon}{c} = \varepsilon$.

Definition 19 (Contraction)

Let $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ a Lipschitz's function.

Then f is a **λ-contraction** *(or simply* **contraction***) if exist $\lambda \in \mathbb{R}$ such that $0 \leqslant \lambda < 1$ so that $dist(f(X), f(Y)) \leqslant \lambda.dist(X, Y)$, for any $X, Y \in \mathbb{X}$.*

Example 2

$$\begin{array}{rccl} f: & \mathbb{IR} & \to & \mathbb{IR} \\ & X & \mapsto & f(X) = [\frac{1}{10};\ \frac{1}{2}].X + [-3;\ 5] \end{array}$$

Remark: All contraction is a continuous function, because it is Lipschitz.

Definition 20 (Fixed Point)

Let $f : \mathbb{X} \subseteq \mathbb{IR} \to \mathbb{IR}$ a interval function.

Then $X_* \in \mathbb{X}$ is a **fixed point** *of the function f if $X_* = f(X_*)$.*

There are various theorems about fixed points, however the theorem as follows is one of the usefuler theorems because beyond its demonstration be enough simple, it serves to prove the existence of a fixed point, to guarantee its unicity and still to furnish an iterative method to compute the fixed point in question.

Then, we will see the theorem and its demonstration.

Theorem 5 (Successive Approximations Method)
Let $\mathbb{F} \subseteq \mathbb{IR}$ *a closed set and* $f : \mathbb{F} \to \mathbb{F}$ *a contraction. Given any point* $X_0 \in \mathbb{F}$*, the interval sequence* $\mathcal{X} = (X_n)_n$ *defined by* $X_1 = f(X_0), X_2 = f(X_1), \ldots, X_{n+1} = f(X_n), \ldots$ *converges for a point* $X_* \in \mathbb{F}$*, that is the unique fixed point of* f *in* $\mathbb{F}$.

Proof:

As f is a contraction, it follows that exist $\lambda \in \mathbb{R}$ such that $0 \leqslant \lambda < 1$, with $dist(f(X), f(Y)) \leqslant \lambda . dist(X, Y)$.

Thus, $dist(X_{k+1}, X_k) \leqslant \lambda^k . dist(X_1, X_0)$, because we have:
$dist(X_2, X_1) = dist(f(X_1), f(X_0)) \leqslant \lambda . dist(X_1, X_0)$.

Supposing that $dist(X_{k+1}, X_k) \leqslant \lambda^k . dist(X_1, X_0)$ and, by induction, we just need to show that $dist(X_{k+2}, X_{k+1}) \leqslant \lambda^{k+1} . dist(X_1, X_0)$.

But $dist(X_{k+2}, X_{k+1}) = dist(f(X_{k+1}), f(X_k)) \leqslant \lambda . dist(X_{k+1}, X_k) \leqslant \lambda . \lambda^k . dist(X_1, X_0) = \lambda^{k+1} . dist(X_1, X_0)$.

Soon $n \in \mathbb{N}$ we have that $dist(X_{n+1}, X_n) \leqslant \lambda^n . dist(X_1, X_0)$.

Now we need to show that the interval sequence $X_{k+1} = f(X_k)$ is a Cauchy's sequence, that is, that $dist(X_m, X_n) \to 0$ to $n, m \to \infty$, or equivalently that $\lim_{k \to \infty} dist(X_{k+p}, X_k) = 0$, for all $p \in \mathbb{N}$.

But $dist(X_{k+p}, X_k) \leqslant dist(X_{k+p}, X_{k+1}) + dist(X_{k+1}, X_k) \leqslant dist(X_{k+p}, X_{k+2}) + dist(X_{k+2}, X_{k+1}) + dist(X_{k+1}, X_k) \leqslant \ldots$

$$\vdots$$

$\ldots \leqslant dist(X_{k+p}, X_{k+p-1}) + dist(X_{k+p-1}, X_{k+p-2}) + \ldots + dist(X_{k+1}, X_k) \leqslant$
$\leqslant \lambda^{k+p-1} .dist(X_1, X_0) + \ldots + \lambda^{k+1} .dist(X_1, X_0) + \lambda^k .dist(X_1, X_0) =$
$= dist(X_1, X_0)[\lambda^{k+p-1} + \ldots + \lambda^{k+1} + \lambda^k] = dist(X_1, X_0). \sum_{i=0}^{p-1} \lambda^{k+i} =$
$= dist(X_1, X_0). \sum_{i=0}^{p-1} \lambda^k . \lambda^i = dist(X_1, X_0). \lambda^k . \sum_{i=0}^{p-1} \lambda^i \leqslant$
$dist(X_1, X_0). \lambda^k . \sum_{i=0}^{\infty} \lambda^i = dist(X_1, X_0). \lambda^k . \frac{1}{1-\lambda} = \frac{\lambda^k}{1-\lambda}.dist(X_1, X_0)$, because $0 \leqslant \lambda < 1$.

Summarizing, $0 \leqslant dist(X_{k+p}, X_k) \leqslant \frac{\lambda^k}{1-\lambda}.dist(X_1, X_0)$.

Thus, $0 \leqslant dist(X_{k+p}, X_k) \leqslant \frac{\lambda^k}{1-\lambda}.dist(X_1, X_0) \to 0$, ao $k \to \infty$.

That is, $\lim_{k\to\infty} dist(X_{k+p}, X_k) = 0$, for all $p \in \mathbb{N}$, whence it follows that $\mathcal{X} = (X_n)_n$ is a Cauchy's sequence.

It follows that exist $X_* \in \mathbb{R}$ such that $\lim_{k\to\infty} X_k = X_*$ and as $\mathbb{F}$ is a closed set, it follows that $X_* \in \mathbb{F}$.

As f is a continuous function, we have:

$X_* = \lim_{k\to\infty} X_k = \lim_{k\to\infty} X_{k+1} = \lim_{k\to\infty} f(X_k) = f(X_*)$.

That is, X_* is a fixed point of f because $X_* = f(X_*)$.

Now we just need to show the unicity of X_*.

Supposing that exists another point $A \in \mathbb{F}$ such that $A = f(A)$.

Thence we have $0 \leqslant dist(X_*, A) = dist(f(X_*), A) = dist(f(X_*), f(A)) \leqslant \lambda .dist(X_*, A)$, or be, $dist(X_*, A) \leqslant \lambda .dist(X_*, A)$, or still that $(1-\lambda).dist(X_*, A) \leqslant 0$.

But $(1-\lambda) > 0$ because $0 \leqslant \lambda < 1$.

Thus $0 \leqslant (1-\lambda).dist(X_*, A) \leqslant 0 \Leftrightarrow dist(X_*, A) = 0$.

That is, $A = X_*$. Soon the fixed point X_* of f is unique.

Remark: It is immediate of the previous theorem that if $\mathbf{X} \subseteq \mathbb{R}$ is a compact set and the function $f : \mathbf{X} \to \mathbf{X}$ performs the condition $dist(f(X), f(Y)) \leqslant dist(X, Y)$ for all pair of the point $X \neq Y$ in $\mathbf{X}$ then the function f has an unique fixed point in $\mathbf{X}$. But, if $\mathbf{X} \subseteq \mathbb{R}$ is an any set, then in order to we can guarantee that a contraction $f : \mathbf{X} \to \mathbb{R}$ has an unique fixed point $X_* \in \mathbf{X}$ it is necessary to find a subset $\mathbb{F} \subseteq \mathbf{X}$ such that $f(\mathbb{F}) \subseteq \mathbb{F}$, in which $\mathbb{F}$ is a closed set in $\mathbb{R}$. The following theorem is frequently utilized to this end.

Theorem 6

Let $f : \mathbf{X} \subseteq \mathbb{R} \to \mathbb{R}$ a λ-contraction. If $\mathbf{X}$ holds the closed ball $\mathcal{F}_r(A)$ such that $dist(f(A), A) \leqslant (1-\lambda).r$ then f admits a fixed point X_* in $\mathcal{F}_r(A)$.

Proof:

It is enough to prove that $f(\mathcal{F}_r(A)) \subseteq \mathcal{F}_r(A)$.

Given any $X \in \mathcal{F}_r(A)$ we show that $f(X) \in \mathcal{F}_r(A)$.

But $X \in \mathcal{F}_r(A) \Leftrightarrow dist(X, A) \leqslant r$.

Thus $dist(f(X), A) \leqslant dist(f(X), f(A)) + dist(f(A), A) \leqslant \lambda . dist(X, A) + (1-\lambda).r \leqslant \lambda . r + (1-\lambda).r = r$.

Whence $dist(f(X), A) \leqslant r$ or be $f(X) \in \mathcal{F}_r(A)$.

3. Aplications

In this section we will examine some practical applications to the Fixed Point Interval Method.

Example 3 *Compute the fixed point of the function*

$$\begin{array}{rcl} f: \mathbb{IR} & \rightarrow & \mathbb{IR} \\ X & \mapsto & f(X) = [\frac{1}{10};\ \frac{1}{2}]X + [-3;\ 5] \end{array}$$

Solution:

Firstly, we need to show that f is a contraction. After, we must find a region that holds the fixed point X_* and finally, we must compute such fixed point through the sequence limit $X_{n+1} = f(X_n)$, in which X_0 is an interval token inside of the convergence region.

1. f is a contraction because $dist(f(X), f(Y)) = dist([\frac{1}{10};\ \frac{1}{2}]X + [-3;\ 5], [\frac{1}{10};\ \frac{1}{2}]Y + [-3;\ 5]) = dist([\frac{1}{10};\ \frac{1}{2}]X, [\frac{1}{10};\ \frac{1}{2}]Y) \leqslant \mid [\frac{1}{10};\ \frac{1}{2}] \mid . dist(X, Y) = \frac{1}{2}. dist(X, Y)$, in which $\lambda = \frac{1}{2}$.

2. We need to define the value of R of such way that $f(\mathcal{F}_R(0)) \subseteq \mathcal{F}_R(0)$, that is, if $X \in \mathcal{F}_R(0)$ then $f(X) \in \mathcal{F}_R(0)$.

Thence $f(X) \in \mathcal{F}_R(0) \Leftrightarrow \mid f(X) \mid \leqslant R$.

But $\mid f(X) \mid = \mid [\frac{1}{10};\ \frac{1}{2}]X + [-3;\ 5] \mid \leqslant \mid [\frac{1}{10};\ \frac{1}{2}] \mid . \mid X \mid + \mid [-3;\ 5] \mid \leqslant \frac{1}{2}. \mid X \mid + 5 \leqslant \frac{1}{2}.R + 5$ because $\mid X \mid \leqslant R$.

Thence $\frac{1}{2}.R + 5 \leqslant R \Leftrightarrow R \geqslant 10$.

Thus, $f(\mathcal{F}_R(0)) \subseteq \mathcal{F}_R(0) \Leftrightarrow R \geqslant 10$.

In this manner, it is enough we take any X_0 such that $\mid X_0 \mid \geqslant 10$.

At last, for we obtain the value of X_* it is enough iterates the sequence $X_{n+1} = f(X_n) = [\frac{1}{10};\ \frac{1}{2}]X_n + [-3;\ 5]$, taking as X_0 any $X \in \mathbb{IR}$ such that $|\ X\ | \geqslant 10$.

For example, taking $X_0 = [15;\ 15]$, we have as follow the table with the computed values to the sequence $X_{n+1} = f(X_n) = [\frac{1}{10};\ \frac{1}{2}]X_n + [-3;\ 5]$:

Table 1: Sequence Values $X_{n+1} = f(X_n) = [\frac{1}{10};\ \frac{1}{2}].X_n + [-3;\ 5]$

n	X_n	$DIGSE(X_n, X_{n+1})$
0	[15 15]	-1.34
1	[-1.5 12.5]	-0.08
2	[-3.75 11.25]	0.34
3	[-4.875 10.625]	0.68
4	[-5.4375 10.3125]	1.01
5	[-5.71875 10.15625]	1.32
6	[-5.859375 10.078125]	1.62
7	[-5.9296875 10.0390625]	1.93
8	[-5.96484375 10.01953125]	2.23
9	[-5.982421875 10.009765625]	2.53
10	[-5.9912109375 10.0048828125]	2.83
20	[-5.99999141693116 10.0000047683716]	5.84
30	[-5.9999999916181 10.0000000046567]	8.85

Thus, the limit is $[-6;\ 10]$ and $[-6;\ 10] = [\frac{1}{10};\ \frac{1}{2}].[-6;\ 10] + [-3;\ 5]$.

Example 4 *Compute the fixed point of the function*

$$f:\ \mathbb{IR} \rightarrow \mathbb{IR}$$
$$X \mapsto f(X) = [\tfrac{1}{10};\ \tfrac{1}{4}]X^2 + [\tfrac{1}{20};\ \tfrac{1}{4}]X + [\tfrac{7}{45};\ \tfrac{5}{16}]$$

Solution:

Firstly we have to show that f is a a contraction, that is, that $dist(f(X),f(Y)) \leqslant \lambda.dist(X,Y)$.

But $dist(f(X),f(Y)) =$

$$dist([\tfrac{1}{10};\ \tfrac{1}{4}]X^2 + [\tfrac{1}{20};\ \tfrac{1}{4}]X + [\tfrac{7}{45};\ \tfrac{5}{16}], [\tfrac{1}{10};\ \tfrac{1}{4}]Y^2 + [\tfrac{1}{20};\ \tfrac{1}{4}]Y + [\tfrac{7}{45};\ \tfrac{5}{16}]) =$$
$$dist([\tfrac{1}{10};\ \tfrac{1}{4}]X^2 + [\tfrac{1}{20};\ \tfrac{1}{4}]X, [\tfrac{1}{10};\ \tfrac{1}{4}]Y^2 + [\tfrac{1}{20};\ \tfrac{1}{4}]Y) \leqslant$$
$$dist([\tfrac{1}{10};\ \tfrac{1}{4}]X^2, [\tfrac{1}{10};\ \tfrac{1}{4}]Y^2) + dist([\tfrac{1}{20};\ \tfrac{1}{4}]X, [\tfrac{1}{20};\ \tfrac{1}{4}]Y) \leqslant$$
$$\mid [\frac{1}{10};\ \frac{1}{4}] \mid .dist(X^2, Y^2) + \mid [\frac{1}{20};\ \frac{1}{4}] \mid .dist(X, Y) =$$
$$\frac{1}{4}.dist(X^2, Y^2) + \frac{1}{4}.dist(X, Y) = \frac{1}{4}(dist(X^2, Y^2) + dist(X, Y)) \leqslant$$
$$\frac{1}{4}(dist(X^2, X.Y) + dist(X.Y, Y^2) + dist(X, Y)) \leqslant$$
$$\frac{1}{4}(\mid X \mid .dist(X, Y) + \mid Y \mid .dist(X, Y) + dist(X, Y)) =$$
$$\frac{\mid X \mid + \mid Y \mid +1}{4}.dist(X, Y) \leqslant dist(X, Y) \Leftrightarrow$$
$$\frac{\mid X \mid + \mid Y \mid +1}{4} < 1 \Leftrightarrow \mid X \mid + \mid Y \mid \leqslant 3.$$

Thus, the first condition that needs to be satisfied in order to $f(X)$ be a contraction is that $\mid X \mid + \mid Y \mid \leqslant 3$, for any X, Y taken in the region of convergence, which still needs to be defined.

Convergence region determination:

Let $\mathbb{X} = \mathcal{F}_R(0)$. By the theorem 6, we need to show that $f(\mathbb{X}) \subseteq \mathbb{X}$. But $f(\mathbb{X}) \subseteq \mathbb{X} \Leftrightarrow \mid f(X) \mid \leqslant R, \quad \forall X, \in \mathbb{X}$.

We have $\mid f(X) \mid = \mid [\tfrac{1}{10};\ \tfrac{1}{4}]X^2 + [\tfrac{1}{20};\ \tfrac{1}{4}]X + [\tfrac{7}{45};\ \tfrac{5}{16}] \mid \leqslant$ $\mid [\tfrac{1}{10};\ \tfrac{1}{4}] \mid . \mid X \mid^2 + \mid [\tfrac{1}{20};\ \tfrac{1}{4}] \mid . \mid X \mid + \mid [\tfrac{7}{45};\ \tfrac{5}{16}] \mid = \tfrac{1}{4}. \mid X \mid^2 + \tfrac{1}{4}. \mid X \mid + \tfrac{5}{16} \leqslant$ $\tfrac{1}{4}.R^2 + \tfrac{1}{4}.R + \tfrac{5}{16}$ because $\mid X \mid \leqslant R$ and, by hypothesis, $X \in \mathbb{X}$.

Thus, $\mid f(X) \mid \leqslant R \Leftrightarrow \tfrac{1}{4}.R^2 + \tfrac{1}{4}.R + \tfrac{5}{16} \leqslant R \Leftrightarrow \tfrac{1}{4}.R^2 - \tfrac{3}{4}.R + \tfrac{5}{16} \leqslant 0 \Leftrightarrow$ $4R^2 - 12R + 5 \leqslant 0 \Leftrightarrow R \in [\tfrac{1}{2};\ \tfrac{5}{2}]$.

Thence, we have that $X_* \in \{X \in \mathbb{IR} / \tfrac{1}{2} \leqslant \mid X \mid \leqslant \tfrac{5}{2}\}$.

Finally, to obtain the value of X_* , it is enough to take $X_0 \in \mathbb{IR}$ such that $\tfrac{1}{2} \leqslant \mid X_0 \mid \leqslant \tfrac{5}{2}$ and then we will have X_* as the limit of the sequence $X_{n+1} = f(X_n) = [\tfrac{1}{10};\ \tfrac{1}{4}]X_n^2 + [\tfrac{1}{20};\ \tfrac{1}{4}]X_n + [\tfrac{7}{45};\ \tfrac{5}{16}]$, according as we examine in the tables as follows:

Table 2: Sequence Values $X_{n+1} = [\frac{1}{10};\ \frac{1}{4}]X_n^2 + [\frac{1}{20};\ \frac{1}{4}]X_n + [\frac{7}{45};\ \frac{5}{16}]$

n	X_n	$DIGSE(X_n, X_{n+1})$
0	[0.5 1]	-0.46
1	[0.205555555555556 0.8125]	0.38
2	[0.170058641975309 0.6806640625]	0.56
3	[0.16695048182537 0.598491907119751]	0.77
4	[0.166690325984997 0.551671117501897]	1.02
5	[0.166668638332504 0.526503034846922]	1.29
6	[0.166666830972542 0.513427120137486]	1.58
7	[0.166666680358826 0.50675863195754]	1.87
8	[0.16666666780768 0.503390735755254]	2.17
9	[0.166666666761752 0.501698242149868]	2.47
10	[0.166666666674591 0.500849842081534]	2.77
20	[0.166666666666667 0.500000830628922]	5.78
30	[0.166666666666667 0.500000000811162]	8.79
35	[0.166666666666667 0.500000000025349]	10.29

Table 3: Sequence Values $X_{n+1} = [\frac{1}{10};\ \frac{1}{4}]X_n^2 + [\frac{1}{20};\ \frac{1}{4}]X_n + [\frac{7}{45};\ \frac{5}{16}]$

n	X_n	$DIGSE(X_n, X_{n+1})$
0	[0.5 2.5]	-0.46
1	[0.205555555555556 2.5]	0.38
2	[0.170058641975309 2.5]	1.43
3	[0.16695048182537 2.5]	2.51
4	[0.166690325984997 2.5]	3.58
5	[0.166668638332504 2.5]	4.66
6	[0.166666830972542 2.5]	5.74
7	[0.166666680358826 2.5]	6.82
8	[0.16666666780768 2.5]	7.90
9	[0.166666666761752 2.5]	8.98
10	[0.166666666674591 2.5]	10.06
20	[0.166666666666667 2.5]	15.00

4. Conclusions

In this paper we prove an interval version to the Fixed Point Theorem. Such theorem provides a new method (the 'Successive Approximations Method') which we use to compute the roots of interval fixed point equations. This method, which is legitimately interval, can be applied for any Lipschitz's function of constant $\lambda < 1$, but here we just did the interval polynomial root compute. In order to we can apply such method, we need to show that exist a region in the interval semi-plain which holds all the probable roots. For this, we proved the theorem 6, that beyond defining such regions, it also serves to decide about the existence or not of the solutions (if we do not get to find a region which satisfyes the theorem 6 then, certainly, the equation does not have fixed points). Thence, taking any point X_0 in the region of convergence, we will obtain the fixed point value that iterate the function, that is, computing $X_{n+1} = f(X_n)$, whence $X_* = \lim_{n\to\infty} X_n$.

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