Artigo Técnico

Revista de
Informática
Teórica e Aplicada

A two-sided method for nonlinear equations with cubic convergence

Dalcidio M. Claudio and Paulo R. Zingano *
PGCC/II - Universidade Federal do Rio Grande do Sul

Sumário

É apresentado um método tipo intervalar para cálculo de raizes reais de equações não lineares, baseado nos métodos de Newton-Raphson e Regula Falsi, gerando duas seqüências de aproximações que convergem para a raiz da equação sem o uso da aritmética intervalar. Para raizes simples a ordem de convergência é cúbica e com condições mais restritivas a convergência é monotônica.

Abstract

A two-sided method for finding a zero of a real-valued function on a given interval is presented and its convergence features are analysed. This method combines in a simple way the well known schemes of Newton-Raphson and Regula Falsi to produce two sequences of approximations to the root of the equation. For simple roots this convergence turns out to be of third order, and under more restrictive conditions it is also monotonic. Though oriented toward interval methods, no use of interval arithmetic is made.

Key Words: Cubic Convergence, Bracketed Zero, multiple roots, nonlinear equations, analytic complexity.

1. Introduction

In a previous paper [2], D.M.Claudio suggested an iteration method for solving nonlinear equations which showed to be very efficient. Oriented toward interval methods, this algorithm (henceforth called the *HIM method*, following [2]) produces for simple roots two cubically convergent sequences of approximations to the root of the equation. Under appropriate convexity requirements, this convergence is of monotonic type and yields both lower and upper bounds to

*This work was sponsored by CNPq–Conselho Nacional de Desenvolvimento Científico e Tecnológico–under grant no. 300196/80.

the desired root. The construction is based on a combination of the standard Regula Falsi and Newton-Raphson methods. Combination of lower order iteration schemes is a general procedure which has been investigated in connection with the problem of computational complexity of nonlinear solving processes [1,5]. The convergence rate of the resulting algorithm is intrinsically bounded by the corresponding behavior of the basic algorithms employed in its construction [7,10]. On these grounds, the *HIM* method is almost optimal. The scheme itself is particularly much like some methods recently proposed [6,11]; however, its most similar counterparts seem to be the classical constructions of J.B.J. Fourier [4] and G.P. Dandelin [3].

2. The HIM Method

Let f denote a real-valued function defined on a closed interval $[a, b]$ with a second derivative continuous there. Moreover, f is supposed to satisfy the following set of ***Fourier conditions***:

(1) $f(a)f(b) < 0$

(2) $f'(x) \neq 0$ for all x in $[a, b]$

(3) $f''(x) \neq 0$ for all x in $[a, b]$

(4) if x_0 denotes the end point of $[a, b]$ such that $f(x_0)f''(x_0) > 0$ and y_0 is the other extreme, there holds $|f'(y_0)| \geq |f(y_0)|/|y_0 - x_0|$.

The problem is to find the (unique) root ζ of f on $[a, b] = [x_0, y_0]$, where $[a_1, a_2, \cdots, a_n]$ stands for the smallest closed interval which contains all of the given points $a_1, a_2, \cdots, a_n$. We observe that condition (4) is always satisfied if y_0 is taken sufficiently close to ζ. Conditions (3),(4) together strainghter the graph of f up enough to guarantee convergence (of monotonic type) of the *HIM* scheme.

Given x_0, y_0 as stated above, we take

$$y_1 = y_0 - f(y_0)[y_0 - x_0]/[f(y_0) - f(x_0)] \qquad (RegulaFalsi)$$

and then

$$x_1 = y_1 - f(y_1)/f'(y_1) \qquad (Newton - Raphson)$$

from the Fourier conditions, it can be easily checked that

$$[x_1, y_1] \subseteq [x_0, y_0]$$

and so x_1, y_1 are better approximations to ζ than x_0, y_0; moreover, the Fourier conditions also hold on $[x_1, y_1]$. Thus, we can iterate again to obtain improved estimations x_2, y_2, repeating the process until the error bound $|x_n - y_n|$ is sufficiently small. In general, we obtain x_{n+1}, y_{n+1} from x_n, y_n by

$$y_{n+1} = y_n - f(y_n)[y_n - x_n]/[f(y_n) - f(x_n)] \qquad (RegulaFalsi)$$

and

$$x_{n+1} = y_{n+1} - f(y_{n+1})/f'(y_{n+1}) \qquad (Newton - Raphson)$$

which requires two evaluations of f and one of f' per step. It is straightforward to conclude

Theorem 1 *the sequences $(x_n), (y_n)$ constructed above are strictly monotone and converge to the root ζ. More precisely, we have either $x_n \nearrow \zeta, y_n \searrow \zeta$ or $x_n \searrow \zeta, y_n \nearrow \zeta$, according to $x_0 = a$ or $x_0 = b$, respectively.*

We turn now to the question of how fast this convergence is by making some asymptotic estimates, in much the same way as A.M.Ostrowski [8].

3. Rate of convergence

From the analysis of Newton's method [8] we know that

$$x_n - \zeta \sim \kappa[y_n - \zeta]^2$$

whereas from linear interpolation theory [8]

$$y_{n+1} - \zeta \sim \kappa[y_n - \zeta][x_n - \zeta]$$

where

$$\kappa \equiv f''(\zeta)/[2f'(\zeta)]$$

and $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$.

Now

$$[y_{n+1} - \zeta]/[y_n - \zeta] =$$

$$= [y_n - x_n][f(y_n) - f(x_n)]^{-1} \int_0^1 [f'(y_n + t(x_n - y_n)) - f'(y_n + t(\zeta - y_n))]dt$$

$$= [y_n - x_n][x_n - \zeta][f(y_n) - f(x_n)]^{-1} \int_0^1 \int_0^1 tf''(y_n + t(x_n - y_n) + ut(\zeta - x_n))dudt$$

implies

$$y_{n+1} - \zeta \sim \kappa[y_n - \zeta]^3 f'(\zeta)^{-1} \int_0^1 \int_0^1 tf''(\zeta)dudt,$$

that is,

$$y_{n+1} - \zeta \sim \kappa^2[y_n - \zeta]^3.$$

So, $y_n \rightarrow \zeta$ with cubic convergence. We now show that the same is true for the approximations x_n and the with of the intervals $[x_n, y_n]$; in fact, observing that

$$y_n - x_n = [1-\mu][y_n - \zeta],$$

where

$$\mu_n \equiv [x_n - \zeta][y_n - \zeta]^{-1},$$

there follows

$$\begin{aligned} 1 &\sim [1-\mu_{n+1}][1-\mu_n]^{-3} = \\ &[y_{n+1} - x_{n+1}][y_n - \zeta]^3 / \left[[y_{n+1} - \zeta][y_n - x_n]^3\right] \end{aligned}$$

that is,

$$[y_{n+1} - x_{n+1}]/[y_n - x_n]^3 \sim [y_{n+1} - \zeta]/[y_n - \zeta]^3 \sim \kappa^2.$$

On the other hand,

$$\begin{aligned} \mu_{n+1}/\mu_n^3 &\sim \kappa^{-1}[x_{n+1} - \zeta][y_n - \zeta]/\left[[y_{n+1} - \zeta][x_n - \zeta]^2\right] \sim \\ &\sim [y_{n+1} - \zeta][y_n - \zeta]/[x_n - \zeta]^2 = \\ &= [y_{n+1} - \zeta][y_n - \zeta]^{-1}[x_n - \zeta]^{-1}[y_n - \zeta]^2[x_n - \zeta]^{-1} \sim 1, \end{aligned}$$

that is,

$$\mu_{n+1} \sim \mu_n^3.$$

Thus

$$[x_{n+1} - \zeta][x_n - \zeta]^{-3} = [y_{n+1} - \zeta][y_n - \zeta]^{-3}\mu_{n+1}\mu_n^{-3} \sim$$

$$\sim [y_{n+1} - \zeta][y_n - \zeta]^{-3} \sim \kappa^2.$$

Summing up these results, we have proved

Theorem 2 *Under the notation and conditions above, there holds*

$$\begin{aligned} y_{n+1} - \zeta &\sim \kappa^2[y_n - \zeta]^3 \\ x_{n+1} - \zeta &\sim \kappa^2[x_n - \zeta]^3 \\ y_{n+1} - x_{n+1} &\sim \kappa^2[y_n - x_n]^3 \end{aligned}$$

where

$$\kappa \equiv f''(\zeta)/[2f'(\zeta)].$$

Clearly, theorem 2 is also true if conditions (3),(4) are dropped, provided only that x_0,y_0 be taken sufficiently close to the root ζ.

4. Behavior in case of multiple roots

We consider now the more delicate case in which $f'(\zeta) = 0$. One obvious approach is to work out with the function $g \equiv f/f'$ instead, since $g(x) = 0$ admits ζ as a simple root. It is certainly the best way to follow if we are not concerned with avoiding making appeal to the second derivative of f.

Anyway, a question which naturally arises is: What happens to the *HIM* method when ζ is an isolated root of f with a finite multiplicity other than one, say p? In order to attain supralinear convergence, we first substitute Newton-Raphson's iteration procedure by Newton-Schröder's formula [9]

$$x_{n+1} = y_{n+1} - pf(y_{n+1})/f'(y_{n+1})$$

where y_{n+1} is given by an extended Regula Falsi method such as

$$y_{n+1} = y_n - qf(y_n)[y_n - x_n]/[f(y_n) - f(x_n)]$$

for some convenient parameter q which will be determined shortly. Convergence of this scheme is assured by taking x_0, y_0 sufficiently close to ζ, so far as x_n, y_n can be calculated.

If $f^{(p+1)} \equiv d^{p+1}f/dx^{p+1}$ is continuous, it is easy to see that

$$x_n - \zeta \sim \gamma[y_n - \zeta]^2$$

where

$$\gamma \equiv f^{(p+1)}(\zeta) / \left[p(p+1)f^{(p)}(\zeta)\right].$$

It then follows

$$y_n - x_n \sim y_n - \zeta,$$

$$f(y_n) - f(x_n) \sim f(y_n) \sim f^{(p)}(\zeta)[y_n - \zeta]^p/p!,$$

from which it can be easily seen that the best choice for q above is to take $q = 1$. In this case,

$$\begin{aligned}
&[y_{n+1} - \zeta]/[y_n - \zeta] = \\
&= f(y_n)[x_n - \zeta][f(y_n) - f(x_n)]^{-1}[y_n - \zeta]^{-1} - f(x_n)[f(y_n) - f(x_n)]^{-1} \\
&\sim [x_n - \zeta]/[y_n - \zeta] \\
&\sim \gamma[y_n - \zeta],
\end{aligned}$$

that is,

$$y_{n+1} - \zeta \sim \gamma[y_n - \zeta]^2.$$

Finally, from

$$y_{n+1} - \zeta \sim x_n - \zeta$$

there follows

$$[x_{n+1} - \zeta]/[x_n - \zeta]^2 \sim [x_{n+1} - \zeta]/[y_{n+1} - \zeta]^2 \sim \gamma$$

and

$$[y_{n+1} - x_{n+1}]/[y_n - x_n]^2 \sim [y_{n+1} - \zeta]/[y_n - \zeta]^2 \sim \gamma.$$

So, in case of multiple roots the *HIM* method converges only quadratically if no appeal to higher order derivatives is made. This decrease in the order of convergence is due to the Regula Falsi part of the method, which fails to converge supralinearly if the root is not simple. Another practical difficulty in this case is that computations are more subject to rounding error and careful procedures must be followed to guarantee the accuracy of the numerical results. However, we are not going into these details in this paper.

References

[1] BRENT, R.P. A class of optimal-order zerofinding methods using derivative evaluations. In:TRAUB, J.F.(Ed.). Analityc Computational Complexity. Academic Press, New york, 1976, 59-74.

[2] CLAUDIO, D.M. An algorithm for solving nonlinear equations based on the Regula Falsi and Newton methods. *ZAMM*, 64 (1984), 407-408.

[3] DANDELIN, G.P. Recherches sur la résolution des équations numériques. *Mém. de l'Acad. Bruxelles*, 3(1826), 7-71, 153-159.

[4] DARBOUX, G. (Ed.). Ouevres de Fourier. Gauthier-Villars, Paris, 1890, Vol.II.

[5] JARRAT, P. Some efficient fourth-order multipoint methods for solving equations. *BIT*, 9 (1969), 119-124.

[6] KING, R.F. Tangent methods for nonlinear equations. *Num. Math.*, 18(1972), 298-304.

[7] MEERSMAN, R. Optimal use of information in certain iterative processes. In:TRAUB, J.F.(Ed.). Analityc Computational Complexity. Academic Press, New york, 1976, 109-126.

[8] OSTROWSKI, A.M. Solution of Equations and Systems of Equations. Academic Press, New york, 1960.

[9] SCHRÖDER, E. Über unendlich viele Algorithmen zur Auflösung der Gleichungen. *Math. Ann.*, 2(1870), 317-365.

[10] TRAUB, J.F. Iterative Methods for the Solution of Equations. Prentice-Hall, Englewood Cliffs, 1964.

[11] WERNER, W. Über ein Verfahren der Ordnung $1+\sqrt{2}$ zur Nullstellenbestimmung. *Num. Math.*, 32(1979), 333-342.

Curriculum vitae (Dalcidio M. Claudio)

Received the B.S. degree in Mathematics from the Universidade Federal do Rio Grande do Sul, Porto Alegre, Brasil in 1974 and the Ph.D degree in 1979 from the Universität Karlsruhe, Karlsruhe, Germany. Currently, he is Professor Adjunto in the Instituto de Informática, Universidade Federal do Rio Grande do Sul, Brasil. His research interests include computer arithmetics, nonlinear equations, computer on educations, and desgin and analysis of efficient algorithms. Dr. Claudio is a member of ACM-SIGNUM and Gesellschaft für Angewandte Mathematik und Mechanik - GAMM.

Curriculum vitae (Paulo R. Zingano)

Received the B.S. degree in Mathematics from the Universidade Federal do Rio Grande do Sul, Porto Alegre, Brasil in 1982 and the Msc degree from the Universidade de Brasília, Brasil, in 1986. Currently, he is Professor Assistente in the Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Brasil. His research interests include differential equations, and desgin and analysis of efficient algorithms.