Federal University of Rio Grande do Sul

Institute of Mathematics and Statistics

Graduate Program in Mathematics

# Regularity of solutions of general obstacle problem arising in American option pricing and Lagrangian structure of relativistic Vlasov systems

Master's thesis

Henrique Borrin de Souza

Porto Alegre, February of 2021.

Thesis submitted by Henrique Borrin de Souza[1] as a partial requisite for obtaining Master's degree in mathematics by Graduate Program in Mathematics of Institute of Mathematics and Statistics of Federal University of Rio Grande do Sul.

Advisor:
Diego Marcon Farias (IME-UFRGS)

Thesis committee:
Leonardo Prange Bonorino (IME-UFRGS)
Gabriela Del Valle Planas (Imecc-Unicamp)
Marcelo Martins Santos (Imecc-Unicamp)
Eduardo V. O. Teixeira (Dept. of Math.-UCF)

[1]Scholarship student of Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - CAPES

# Contents

# Acknowledgements

Firstly, I thank my parents Celita and Jorge for the support during the stressful moments in master's program. I also thank my advisor Diego Marcon for the opportunity of researching alongside him in this four years. Finally, i thank my friends Amadeus and Isabel for keeping (or even increasing) our reunions during the confinement.

I thank in particular my girlfriend Bruna Carlos for accepting living with me during the pandemic and by her (enormous) patience and willingness to adapt in such a short time, especially after the loss of Duca. The highlight of 2020 was the decision of being together from now on (and with lots of cats).

# Introduction

By the different nature between obtained results, we split the introduction in a motivation for modelling American options and a (rough) statement of regularity results and a motivation for Vlasov systems and a (rough) statement of its Lagrangian structure.

## American options

We begin by recalling the Black-Scholes model (see [12], [25]), which states that a (vector) stock price $S_t$ at time $t$ evolves as

$$\mathrm{d}S_t^i = (r - d_i)S_t^i \,\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij} S_t^i \,\mathrm{d}W_t^j,$$

where $W_t$ is the (vector) Brownian motion, $r$ is the short rate, $d$ is the continuously compounded dividend rate of the stock, and $\sigma$ is the volatility matrix of the stock. In this model, $r$ is a non-negative constant, and $\sigma$ is a non-negative constant matrix. Moreover, it is usual to denote $\tau = 0$ as the "now" and $\tau = T$ as the expiring time. It is well known (see [12], [25]) that the price of the option $V_t(S)$ solves the celebrated Black-Scholes equation:

$$\begin{cases} -\partial_\tau V_t - \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij} S_i S_j \partial_{S_i S_j} V_t + \sum_{i=1}^{n} (r - d_i) S_i \partial_{S_i} V_t - r V_t = 0; \\ V_T = \psi, \end{cases}$$

where $\psi$ is a known non-negative function, and it is the payoff. By the change of variables $t = T - \tau$ and $x_i = \log S_i$, we simplify the problem:

$$\begin{cases} \partial_t u - \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij} \partial_{x_i x_j} u + \sum_{i=1}^{n} (r - d_i - \sigma_{ii}/2) \partial_{x_i} u - r u = 0; \\ u(0, \cdot) = \psi, \end{cases}$$

where $u(t,x) = V_\tau(S)$ is the rational price. Notice that the Black-Scholes model does not allow sudden changes in the stock price $S_t$. For this purpose, Merton [28] added a "jump" term in the evolution of $S_t$, that is,

$$\mathrm{d}S_t^i = (r - d_i)S_t^i\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij} S_t^i\,\mathrm{d}W_t^j + Y^i S_t^i\,\mathrm{d}N_t^i,$$

where $N_t^i$ is a Poisson process "counting the jumps" of $S_t^i$ with size $Y^i$. The corresponding PDE is then modified (see [12]) with a non-local term:

$$\begin{cases} \partial_t u - \frac{1}{2}\sigma\colon D^2 u - b\cdot\nabla u - ru - \mathcal{K}u = 0; \\ u(0,\cdot) = \psi, \end{cases} \tag{1}$$

where $b = (d_1 + \sigma_{11}/2 - r, \dots, d_n + \sigma_{nn}/2 - r)$,

$$\mathcal{K}v(t,x) \coloneqq \int_{\mathbb{R}^n} \left[ v(t,x+y) - v(t,x) - \sum_{i=1}^{n} (e^{y_i} - 1)\partial_{x_i} v(t,x) \right] \mu(\mathrm{d}y)$$

and $\mu$ is the associated jump measure. System (1) models European options, since one can only exercise at the expiration date (recall that in $(t,x)$ coordinates, the expiration date is $t = 0$). In contrast, American options do allow exercises prior than the expiration. We may split the domain into $\{u > \psi\}$ and $\{u = \psi\}$, which are known as continuation and exercise regions, respectively. These names suggest that the first time we enter the exercise region, it is optimal to exercise the option, otherwise (that is, when we are in continuation region) we should continue the evolution of $u$ as in (1). This information is encapsulated in the following PDE:

$$\begin{cases} \min\{\partial_t u - \frac{1}{2}\sigma\colon D^2 u - b\cdot\nabla u - ru - \mathcal{K}u, u - \psi\} = 0; \\ u(0,\cdot) = \psi. \end{cases} \tag{2}$$

If there is no jump term, i.e., $\mu \equiv 0$, the regularity of (2) is well-understood (see [25]). We assume that $\sigma \equiv 0$, so that all the regularity comes from the jump term. As in [7], we assume that $\mathrm{d}\mu(y)$ behaves as $|y|^{-n-2s}\mathrm{d}y$ as leading order, so that $\mathcal{K}$ can be written as

$$\mathcal{K}v(t,x) \approx -(-\Delta)^s v(t,x) + \mathcal{I}v(t,x),$$

where $(-\Delta)^s$ is the fractional Laplacian and it is defined by

$$-(-\Delta)^s f(x) := \int_{\mathbb{R}^n} \frac{f(y)-f(x)}{|y-x|^{n+2s}}\,\mathrm{d}y,$$

and $\mathcal{I}$ is a non-local operator of lower order with respect to $(-\Delta)^s$. Notice that under this assumption, the regularity of the solution of (2) depends on $s$:

- if $s < 1/2$, the gradient term is of greater order with respect to $(-\Delta)^s$, and we do not expect any regularity result for $u$;
- if $s = 1/2$, the gradient and the fractional Laplacian have the same order, and the problem becomes very delicate;
- if $s > 1/2$, we expect that diffusion provided by the fractional Laplacian dominates and $u$ is as regular as the solution of the fractional heat equation obstacle problem, i.e., the regularity is the same as the solution of (2) with $b \equiv 0$, $r \equiv 0$, and $\mathcal{I} \equiv 0$.

The main goal of the first part of the thesis is to investigate the expected regularity in the case $s > 1/2$.

Thus, we study in chapter 1 the obstacle problem

$$\begin{cases} \min\{\partial_t u + (-\Delta)^s u - \mathcal{R}u, u - \psi\} = 0; \\ u(0,\cdot) = \psi, \end{cases} \tag{3}$$

where $\mathcal{R} := (r + b\cdot\nabla + \mathcal{I})$ is a lower order operator with respect to $(-\Delta)^s$. We were unable to find the existence of solutions for the obstacle problem (1.1) in the literature and we present a proof in Chapter 1. Nevertheless, in the elliptic case, Petrosyan and Pop [29] prove existence and regularity results when $\mathcal{I} \equiv 0$, $b \in C^s(\mathbb{R}^n;\mathbb{R}^n)$, $r \in C^s(\mathbb{R}^n)$ is negative bounded away from zero, and $\psi \in C^{3s}$.

In order to study (3), we assume some regularity of $\mathcal{I}$, namely, $\mathcal{I}$ is a convex non-local non-linear uniform elliptic operator (see (iv) for a rigorous definition and some examples) and that the payoff (from now on, it will be called obstacle) is globally $W^{2,\infty} \cap C^2$. The main result of chapter 1 is the following (see Theorem 1.32):

**Theorem** (Rough version)**.** *In this setting, there exists a unique solution* $u$ *of* (3)*. Moreover, it is globally Lipschitz in space-time and*

$$\begin{cases}\partial_t u \in C^{\frac{1-s}{2s}-0^+,1-s}_{t,x}((0,T]\times\mathbb{R}^n);\\ (-\Delta)^s u \in C^{\frac{1-s}{2s},1-s}_{t,x}((0,T]\times\mathbb{R}^n).\end{cases}$$

Our regularity in time would be optimal if we did not have $0^+$ in the Hölder exponent of $\partial_t u$. It is natural to expect that

$$\partial_t u \in C^{\frac{1-s}{2s},1-s}_{t,x}((0,T]\times\mathbb{R}^n);$$

however, this is unknown even for the fractional heat operator. In fact, the same type of regularity of solutions has been addressed by Caffarelli and Figalli [7] when $b\equiv 0$, $\mathcal{I}\equiv 0$, and $r\equiv 0$. In the setting of [7], the regularity of the free boundary for the obstacle problem has been investigated by Barrios, Figalli, and Ros-Oton [5], where they show the free boundary is of class $C^{1,\alpha}$ in space-time. Their techniques, however, heavily depend on the scale invariance of the operator, do not readily extend to the general problem (1.1), and can be the subject of a future work.

# Relativistic Vlasov systems

We begin by stating the relativistic (diffusion-free) Boltzmann equation:

$$\partial_t f_t(x,v) + \hat{v}\cdot\nabla_x f_t(x,v) + A_t(x,v)\cdot\nabla_v f_t(x,v) = \partial_t f\Big|_{\text{coll}}(x,v),$$

where $f_t(x,v)$ is the distribution of particles in the phase space $(x,v)$ at time $t$ with acceleration $A_t(x,v)$, $\hat{v} \coloneqq (1+|v|^2)^{-1/2}v$ is the relativistic velocity, and the right hand side term models the collision of the particles (here, we have chosen the speed of light $c=1$). The right hand side should be understood as an operator which is applied into the distribution of particles $f_t$. If we assume that the system is collisionless, that is,

$$\partial_t f_t(x,v) + \hat{v}\cdot\nabla_x f_t(x,v) + A_t(x,v)\cdot\nabla_v f_t(x,v) = 0, \tag{4}$$

we have the Vlasov equation. Notice that the acceleration may be self-consistent, i.e., the particles exert forces between themselves, so that $A_t$

depends on $f_t$, turning the equation into a non-linear system. Classical examples are the Vlasov-Poisson system, where $A_t$ is the electric field given by Coulomb's Law, and Vlasov-Maxwell system, where $A_t$ is determined by Lorentz force law with electromagnetic field given by Maxwell's equations. In Appendix 2, we shall assume that the acceleration is given by

$$A_t(x,v) = g_t(x) + \frac{q}{m}(E_t(x) + \hat{v} \times B_t(x)),$$

where $g_t$, $E_t$, and $B_t$ are the Newtonian gravitational, electric, and magnetic fields, respectively, and $q$ and $m$ are the particle charge and mass. Newtonian gravity implies that $g_t = Gm\nabla(-\Delta)^{-1}\rho_t$, where $G$ is the gravitational constant and $\rho_t$ the density of particles. We now assume that the electromagnetic field satisfies one of quasi static limits of Maxwell's equations (see, for instance, [27] and references therein):

$$\nabla \cdot E_t = \frac{q}{\epsilon_0}\rho_t, \quad \nabla \cdot B_t = 0, \quad \nabla \times E_t = 0, \quad \nabla \times B_t = \frac{q}{\epsilon_0}J_t + \partial_t E, \tag{5}$$

or

$$\nabla \cdot E_t = \frac{q}{\epsilon_0}\rho_t, \quad \nabla \cdot B_t = 0, \quad \nabla \times E_t = -\partial_t B_t, \quad \nabla \times B_t = \frac{q}{\epsilon_0}J_t, \tag{6}$$

where $J_t$ is the relativistic particle current density and it is treated as a given (vectorial) function. Equations (5) and (6) are known as the quasi-electrostatic (QES) and quasi-magnetostatic (QMS) limit, respectively. The solution of (5) is

$$E_t = -\frac{q}{\epsilon_0}\nabla(-\Delta)^{-1}\rho_t, \quad \text{and} \quad B_t = \frac{q}{\epsilon_0}\nabla \times (-\Delta)^{-1}J_t,$$

while the solution of (6) is

$$E_t = -\frac{q}{\epsilon_0}\nabla(-\Delta)^{-1}\rho_t - \frac{q}{\epsilon_0}\partial_t(-\Delta)^{-1}J_t, \quad \text{and} \quad B_t = \frac{q}{\epsilon_0}\nabla \times (-\Delta)^{-1}J_t.$$

Notice that the leading term in QES limit is the electric field, and in QMS is the magnetic field. Hence, if we are in QES case, we can write $A_t$ only in terms of $\rho_t$ and $J_t$:

$$\begin{aligned} A_t(x,v) = &\left(\frac{q^2}{4\pi\,\epsilon_0 m} - Gm\right)\int_{\mathbb{R}^3} \rho_t(y)\frac{x-y}{|x-y|^3}\,\mathrm{d}y \\ &+ \frac{q^2}{4\pi\,\epsilon_0 m}\hat{v} \times \int_{\mathbb{R}^3} J_t(y) \times \frac{x-y}{|x-y|^3}\,\mathrm{d}y, \end{aligned}$$

where $\epsilon_0$ is the electric permittivity. Now, define the critical charge $q_c$ as

$$q_c \coloneqq \pm\sqrt{4\pi\,\epsilon_0 G}\,m.$$

If $q > q_c$, we have that the electric field is stronger, and up to a redefinition of $\rho_t$ and $J_t$, we may write the acceleration as

$$A_t(x,v) = \int_{\mathbb{R}^3} \rho_t(y)K(x-y)\,\mathrm{d}y + \hat{v} \times \int_{\mathbb{R}^3} J_t(y) \times K(x-y)\,\mathrm{d}y,$$

where $K(x) = (4\pi)^{-1}x/|x|^3$. Analogously, if $q < q_c$, we have

$$A_t(x,v) = -\int_{\mathbb{R}^3} \rho_t(y)K(x-y)\,\mathrm{d}y + \hat{v} \times \int_{\mathbb{R}^3} J_t(y) \times K(x-y)\,\mathrm{d}y.$$

In both cases, if we drop the magnetic field (since it is a lower order term), we have the relativistic Vlasov-Poisson system. Moreover, notice that in the critical case $q = q_c$, we only have the magnetic force acting in Vlasov equation, which is exactly the same as if we only considered the leading term in the QMS limit, that is, the relativistic Vlasov-Biot-Savart system.

Thus, we write

$$\begin{cases} \partial_t f_t + \hat{v}\cdot\nabla_x f_t + (E_t + \hat{v}\times B_t)\cdot\nabla_v f_t = 0; \\ \rho_t(x) = \int_{\mathbb{R}^3} f_t(x,v)\,\mathrm{d}v, \quad J_t(x) = \int_{\mathbb{R}^3} \hat{v} f_t(x,v)\,\mathrm{d}v; \\ E_t(x) = \sigma_E \int_{\mathbb{R}^3} \rho_t(y)K(x-y)\,\mathrm{d}y; \\ B_t(x) = \sigma_B \int_{\mathbb{R}^3} J_t(y)\times K(x-y)\,\mathrm{d}y, \end{cases} \tag{7}$$

where $\sigma_E \in \{0, \pm 1\}$, $\sigma_B \in \{0, 1\}$. We are interested in the Lagrangian structure of (7). We summarize what (7) models depending on $\sigma_E$ and $\sigma_B$:

- **Relativistic Vlasov-Poisson equations:** charged particles under a self-consistent electric field or particles under a self-consistent electric and gravitational fields with particle charge $q > q_c$ if $\sigma_E = 1$, $\sigma_B = 0$; motion of galaxy clusters under a gravitational field or particles under a self-consistent electric and gravitational fields with particle charge $q < q_c$ if $\sigma_E = -1$, $\sigma_B = 0$ (see, for instance, [13, Chapter 5] and references therein);

- **Relativistic Vlasov-Biot-Savart equations[2]:** charged particles under a self-consistent magnetic field; particles under a self-consistent

[2] This terminology, albeit not standard, is in analogy to the Vlasov-Poisson system, since the magnetic field obeys the Biot-Savart law.

quasi-electrostatic (QES) electromagnetic and gravitational fields with particle charge $q = q_c$ if $\sigma_E = 0$ and $\sigma_B = 1$;

- **QES relativistic Vlasov-Maxwell equations:** charged particles under a self-consistent QES electromagnetic field; particles under a self-consistent QES electromagnetic and gravitational fields with particle charge $q > q_c$ if $\sigma_E = \sigma_B = 1$;

- **Relativistic gravitational Vlasov-Biot-Savart equations:**

  charged particles under a self-consistent magnetic and gravitational fields; particles under a self-consistent quasi-magnetostatic (QMS) electromagnetic and gravitational fields with particle charge $q < q_c$ if $\sigma_E = -1$ and $\sigma_B = 1$.

Note we allow $\sigma_B = \sigma_E = 0$, that is, (7) to be the linear transport equation, but its theory is classical and we shall not consider it. Moreover, the fact that the critical charge evolution system coincides with the Vlasov-Biot-Savart system suggests that the displacement current $\partial_t E_t$ behaves like a lower order term; see (5). This is well-known in Electrodynamics [22]; Maxwell predicted theoretically as a correction of Ampère's law. Nonetheless, we show that it behaves like a lower order term in the magnetic potential energy; see Lemma 2.19 and Remark 4.

Concerning the existence of classical solutions of (7), we refer to [6, 21, 23], where the existence of local solutions for the relativistic Vlasov-Poisson system is established. As mentioned in [13, Chapter 5, Section 1.5], very little is known regarding the existence of global solutions for general initial data. However, existence results can be found, for instance, for spherically and axially symmetric initial data; see [20, 19]. In the aforementioned results, it is required higher integrability assumptions and moment conditions on the initial data. To be more physically relevant, it is desired to avoid such hypotheses even though classical solutions may fail to exist. We thus consider renormalized and generalized solutions, which allow us to establish a Lagrangian structure for the system, global existence results, and (under suitable energy bounds) a global in time maximal regular flow, as we explain in 2.

The main goal of the second part of the thesis is to study the Lagrangian structure of (7) under suitable hypothesis. In the seminal paper of DiPerna and Lions, they introduced the concept of renormalized solution (see Definition 2.1) in order to overcome the ill-posedness of (7) in the distributional sense if $f$ is merely $L^1$. Notice that, by the transport structure of the Vlasov equation, one expect that, in suitable sense, the initial condition transported by a flow associated to $(\hat{v}, E_t + \hat{v} \times B_t)$ is a solution of (7). But since we are dealing with renormalized solution, one might lose the relation between Lagrangian and Eulerian pictures. The first main result of Appendix 2 states that there exists a flow associated to (7) which transports the initial condition for renormalized and/or distributional solutions (see Theorem 2.2 and Corollary 2.10):

**Theorem** (Rough version)**.** *Assume that $f$ is a distributional or a renormalized solution of (7). Then $f$ is a Lagrangian solution transported by the flow (in a suitable sense) $\boldsymbol{X}(t,\cdot)$ associated to $(\hat{v}, E_t + \hat{v} \times B_t)$. Moreover, if the relativistic and the electromagnetic energy are integrable in time, that is,*

$$\int_0^T \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} |E_t|^2 + |B_t|^2\,\mathrm{d}x\,\mathrm{d}t < \infty,$$

*Then the flow is globally defined for $t \in [0,T]$ and $f_t = \boldsymbol{X}(t,\cdot)_{\#} f_0$.*

The second main result states that, if we only consider the "effective" particle density and current density $\rho^{\mathrm{eff}}$, $J^{\mathrm{eff}}$, respectively, in the electromagnetic field, that is, if we now consider

$$E_t^{\mathrm{eff}}(x) = \sigma_E \int_{\mathbb{R}^3} \rho_t^{\mathrm{eff}}(y) K(x-y)\,\mathrm{d}y, \quad B_t^{\mathrm{eff}}(x) = \sigma_B \int_{\mathbb{R}^3} J_t^{\mathrm{eff}}(y) \times K(x-y)\,\mathrm{d}y,$$

instead of $E_t$, $B_t$ in (7), we still have a existence of a Lagrangian solution result. A renormalized solution of this "effective" version of (7) combined with suitable hypothesis of $\rho^{\mathrm{eff}}$, $J^{\mathrm{eff}}$ will be called "generalized solution". The results reads (see Theorem 2.3):

**Theorem** (Rough version)**.** *If $f_0 \in L^1(\mathbb{R}^6)$ is nonnegative, then there exists a generalized Lagrangian solution of (7) transported by the flow (in a suitable sense) associated to $(\hat{v}, E_t^{\mathrm{eff}} + \hat{v} \times B_t^{\mathrm{eff}})$.*

Finally, the third main result states that if the initial condition has finite energy (in a suitable sense), then the distribution of particles is continuous in time, and it is transported by a globally defined flow (see Theorem 2.4):

**Theorem** (Rough version)**.** *If $f_0$ has every energy bounded, then there exists a global Lagrangian solution $f \in C([0,\infty); L^1(\mathbb{R}^6))$, and its flow is globally defined on $[0,\infty)$. Moreover, $f$ has every energy bounded, and the electromagnetic field $E_t$, $B_t$ is strongly continuous in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$.*

# Chapter 1

# Regularity of solutions of general obstacle problem

We now make a more precise hypotheses on the system (3), namely, we consider continuous viscosity solutions of

$$\begin{cases} \min\{\partial_t u + (-\Delta)^s u - b\cdot\nabla u - \mathcal{I}u - ru,\ u-\psi\} = 0 & \text{in } (0,T]\times\mathbb{R}^n, \\ u(0,x) = \psi(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where

(*i*) the obstacle $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ is assumed to be a function of class $W^{2,\infty}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$;

(*ii*) $b \in \mathbb{R}^n$ is a constant vector;

(*iii*) $r \in \mathbb{R}$ is a constant[1];

(*iv*) $\mathcal{I}$ is a non-local, convex[2], translation-invariant, uniformly elliptic operator with respect to $\mathscr{L}_0$. The latter means that for all $v$, $w \in C^{2\sigma+0^+}(x)$

[1] Although the condition $r \geq 0$ might seem natural (see [7, Section 5]), our main result holds even for $r < 0$.

[2] The convexity of $\mathcal{I}$ is only used in Lemma 1.17. Thus, if $u$ is a semiconvex solution of (1.1) and $\mathcal{I}$ is not convex, the results of this paper still hold.

which satisfy[3]

$$\int_{\mathbb{R}^n} \frac{|v(y)|+|w(y)|}{1+|y|^{n+2\sigma}}\,\mathrm{d}y < \infty,$$

we have that $\mathcal{I}v(x)$ and $\mathcal{I}w(x)$ are well defined and

$$M^{-}_{\mathscr{L}_0}(v-w)(x) \le \mathcal{I}v(x) - \mathcal{I}w(x) \le M^{+}_{\mathscr{L}_0}(v-w)(x), \tag{1.2}$$

where $\mathscr{L}_0$ is the set of operators $L$ such that

$$\begin{aligned} Lu(x) &:= \int_{\mathbb{R}^n} \delta u(x,y) K(y)\mathrm{d}y, \\ \frac{\lambda}{|y|^{n+2\sigma}} &\le K(y) \le \frac{\Lambda}{|y|^{n+2\sigma}}, \text{ and } \quad K(y) = K(-y). \end{aligned} \tag{1.3}$$

The extremal operators $M^{+}_{\mathscr{L}_0}$ and $M^{-}_{\mathscr{L}_0}$ are analogous to Pucci operators

$$M^{+}_{\mathscr{L}_0}u(x) := \sup_{L\in\mathscr{L}_0} Lu(x) \equiv \int_{\mathbb{R}^n} \frac{\Lambda(\delta u(x,y))^{+} - \lambda(\delta u(x,y))^{-}}{|y|^{n+2\sigma}}\mathrm{d}y,$$

$$M^{-}_{\mathscr{L}_0}u(x) := \inf_{L\in\mathscr{L}_0} Lu(x) \equiv \int_{\mathbb{R}^n} \frac{\lambda(\delta u(x,y))^{+} - \Lambda(\delta u(x,y))^{-}}{|y|^{n+2\sigma}}\mathrm{d}y,$$

where $\delta u(x,y) := u(x+y) + u(x-y) - 2u(x)$. For simplicity, we assume $\mathcal{I}(0) = 0$. We assume that $\mathcal{I}$ is a lower order diffusion operator when compared to the fractional Laplacian $(-\Delta)^s$ in the sense that $s > \sigma > 0$.

Notice that since $Lu \in C^{\alpha}(\mathbb{R}^n)$ whenever $u \in C^{2\sigma+\alpha}(\mathbb{R}^n)$ for some $\alpha > 0$, thus $\mathcal{I}u \in C^{\alpha}(\mathbb{R}^n)$.

Quintessential examples of $\mathcal{I}$ are $-(-\Delta)^{\sigma}$ and $L$, and by [10], more sophisticated examples arise, such as

$$\mathcal{I}u = \sup_{\beta} L_{\beta}u, \quad \mathcal{I}u(x) = \int_{\mathbb{R}^n} \frac{G(u(x+y)-u(x))}{|y|^{n+2\sigma}}\mathrm{d}x,$$

where $K_{\beta}$ satisfy (1.3) uniformly with respect to $\beta$ and $G$ is a convex monotone Lipschitz function and $G(0) = 0$.

[3] We recall that $\phi$ is said to be $C^{2\sigma+0^+}$ punctually at $x$ if there exists $v \in \mathbb{R}^n$ and $M > 0$ such that $|\phi(x+y) - \phi(x)| \le M|y|^{2\sigma+0^+}$ if $2\sigma + 0^+ \le 1$ and $|\phi(x+y) - \phi(x) - v\cdot y| \le M|y|^{2\sigma-1+0^+}$ if $2\sigma + 0^+ > 1$ for small $y$.

Throughout this chapter, by $C^{\alpha\pm 0^+}$ we mean $C^{\alpha\pm\epsilon}$ for all $\epsilon > 0$.

In the light of [30], we remark that since we need the well posedness of the inverse of the fractional Laplacian and we assume that $s > 1/2$, we consider throughout the paper that the dimension satisfies $n \geq 2$.

## 1.1 Comparison results and first regularity estimates

We first recall the general definition of a viscosity solution for a nonlocal problem $\mathcal{L}u = f$, where $f$ is a bounded continuous function and $\mathcal{L}u = \partial_t u + (-\Delta)^s u - b \cdot \nabla u - \mathcal{I}u - ru$:

**Definition 1.1.** An upper semicontinuous $u$ on $(0,T] \times \mathbb{R}^n$ is a subsolution of $\mathcal{L}u = f$ at $(t_0, x_0)$ if for all functions $\phi \in C^{1,2}(\overline{B_R(t_0,x_0)})$ such that $0 = (u-\phi)(t_0,x_0) > (u-\phi)(t,x)$ for all $(t,x) \in B_R(t_0,x_0) \setminus \{(t_0,x_0)\}$ for some $R > 0$, the function

$$v(t,x) \coloneqq \begin{cases} \phi(t,x) \text{ in } B_R(t_0,x_0); \\ u(t,x) \text{ at } (0,T] \times \mathbb{R}^n \setminus B_R(t_0,x_0) \end{cases} \tag{1.4}$$

satisfies $\mathcal{L}v(t_0,x_0) \leq f(t_0,x_0)$.

Analogously, a lower semicontinuous $u$ on $(0,T] \times \mathbb{R}^n$ is a supersolution of $\mathcal{L}u = f$ at $(t_0,x_0)$ if for all functions $\varphi \in C^{1,2}(\overline{B_R(t_0,x_0)})$ such that $0 = (u-\varphi)(t_0,x_0) < (u-\varphi)(t,x)$ for all $(t,x) \in B_R(t_0,x_0) \setminus \{(t_0,x_0)\}$ for some $R > 0$, the function

$$v(t,x) \coloneqq \begin{cases} \varphi(t,x) \text{ in } B_R(t_0,x_0); \\ u(t,x) \text{ at } (0,T] \times \mathbb{R}^n \setminus B_R(t_0,x_0) \end{cases} \tag{1.5}$$

satisfies $\mathcal{L}v(t_0,x_0) \geq f(t_0,x_0)$.

A solution of $\mathcal{L}u = f$ is a continuous function that is both a subsolution and a supersolution for all $(t,x) \in (0,T] \times \mathbb{R}^n$.

We remark that the definition of the auxiliar function $v$ is necessary due to the nonlocal operators $(-\Delta)^s$ and $-\mathcal{I}$.

We assume throughout the paper that $u$ is a solution of (1.1) in the following sense:

**Definition 1.2.** An upper semicontinuous, bounded $u$ on $(0,T]\times\mathbb{R}^n$ is a subsolution of (1.1) if $\mathcal{L}u(t,x)\le 0$ in viscosity sense for all $(t,x)\in(0,T]\times\mathbb{R}^n$ such that $u(t,x)>\psi(x)$, and $u(0,\cdot)\le\psi$.

Analogously, a lower semicontinuous, bounded $u$ on $(0,T]\times\mathbb{R}^n$ is a supersolution of (1.1) if $u(t,\cdot)\ge\psi$ for all $t\in(0,T]$, $\mathcal{L}u(t,x)\ge 0$ in viscosity sense for all $(t,x)\in(0,T]\times\mathbb{R}^n$, and $u(0,\cdot)\ge\psi$.

A solution of (1.1) is a bounded continuous function that is both a subsolution and a supersolution.

We remark that the definition of subsolution and supersolution are not symmetric. Moreover, we can relax Definition 1.2 by dropping the hypothesis $u(t,\cdot)\ge\psi$ (but assuming that a supersolution also has a empty semi-jet set, see [1, Definition 2] for the classical case).

Since the fractional Laplacian is the leading term, we now define the lower order operator $\mathcal{R}$ as

$$\mathcal{R}u(t,x) \coloneqq (\mathcal{I}+b\cdot\nabla+r)u(t,x).$$

We now prove the existence, uniqueness and regularity of solutions of (1.6) (see Lemma 1.3, Lemma 1.4, and Lemma 1.5). These tools will be needed in order to prove existence, uniqueness and regularity of solutions for the penalized equation (1.13). The techniques are fairly standard and are presented for the sake of completeness.

**Remark 1.** Notice that operators $(\partial_t+(-\Delta)^s-b\cdot\nabla-M^{\pm}_{\mathscr{L}_0}-r+\gamma)$ satisfy the weak maximum principle for $\gamma$ big enough (namely, $\gamma=r$). Indeed, since $[(-\Delta)^s-M^{\pm}_{\mathscr{L}_0}]\varphi(t_0,x_0)\ge 0$ if $\varphi$ attains its maximum at $(t_0,x_0)$, we have that

- if $(\partial_t+(-\Delta)^s-b\cdot\nabla-M^{\pm}_{\mathscr{L}_0}-r+\gamma)u\le 0$ in $(0,T]\times\mathbb{R}^n$, then

$$\max_{(0,T]\times\mathbb{R}^n} u \le \max_{\{t=0\}\times\mathbb{R}^n} u^+;$$

- if $(\partial_t+(-\Delta)^s-b\cdot\nabla-M^{\pm}_{\mathscr{L}_0}-r+\gamma)u\ge 0$ in $(0,T]\times\mathbb{R}^n$, then

$$\min_{(0,T]\times\mathbb{R}^n} u \ge -\max_{\{t=0\}\times\mathbb{R}^n} u^-,$$

and its proof is a straightforward adaptation of [16, Theorem 9, Section 7.1].

**Lemma 1.3** (Uniqueness)**.** *Assume $\psi$, $b$, $r$, and $\mathcal{I}$ satisfy (i), (ii), (iii), and (iv), respectively. For continuous functions $f \in L^\infty((0,T] \times \mathbb{R}^n)$ and $u \in L^\infty((0,T] \times \mathbb{R}^n) \cap C^{1,2}_{t,x}((0,T] \times \mathbb{R}^n)$, which satisfy*

$$\begin{aligned} \partial_t u + (-\Delta)^s u - \mathcal{R}u = f \quad &\textit{in} \quad (0,T] \times \mathbb{R}^n; \\ u(0,\cdot) = \psi \quad &\textit{on}\ \mathbb{R}^n, \end{aligned} \tag{1.6}$$

*we have*

$$\|u\|_{L^\infty((0,T]\times\mathbb{R}^n)} \le C\Big(\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}\Big),$$

*where $C = C(r,T)$. Moreover, the solution is unique.*

*Proof.* For the first claim, it suffices to prove for $u$ which satisfies

$$\Big(\partial_t + (-\Delta)^s - M^+_{\mathscr{L}_0} - b\cdot\nabla - r\Big)u \le f \le \big(\partial_t + (-\Delta)^s - M^-_{\mathscr{L}_0} - b\cdot\nabla - r\big)\, u.$$

The result follows by noticing that

$$\begin{aligned} \big(\partial_t + (-\Delta)^s - M^-_{\mathscr{L}_0} - b\cdot\nabla + \gamma - r\big)\,&\big(\pm e^{-\gamma t}u + \|f\|_{L^\infty(\mathbb{R}^n)}\big) \\ &\ge \pm e^{-\gamma t} f + (\gamma - r)\|f\|_{L^\infty(\mathbb{R}^n)} \\ &\ge e^{-\gamma t}\big(\pm f + e^{\gamma t}\|f\|_{L^\infty(\mathbb{R}^n)}\big) \ge 0 \end{aligned}$$

where $\gamma \coloneqq r+1$. Hence, by the minimum principle

$$\begin{aligned} \pm e^{-\gamma t}u + \|f\|_{L^\infty(\mathbb{R}^n)} &\ge -\max_{\{t=0\}\times\mathbb{R}^n}(\pm e^{-\gamma t}u + \|f\|_{L^\infty(\mathbb{R}^n)})^- \\ &= -\max_{\mathbb{R}^n}(\pm\psi + \|f\|_{L^\infty(\mathbb{R}^n)})^-, \end{aligned}$$

which gives

$$\begin{aligned} \|u\|_{L^\infty((0,T]\times\mathbb{R}^n)} &\le e^{\gamma T}(\|f\|_{L^\infty(\mathbb{R}^n)} + \max_{\mathbb{R}^n}(\pm\psi + \|f\|_{L^\infty(\mathbb{R}^n)})^-) \\ &\le 2e^{\gamma T}(\|f\|_{L^\infty(\mathbb{R}^n)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}) \end{aligned}$$

Now, if $u$ and $v$ satisfy (1.6), then

$$\begin{aligned} &(\partial_t + (-\Delta)^s - b\cdot\nabla - r)(u-v) + \mathcal{I}v - \mathcal{I}u = 0; \\ &(u-v)(0,\cdot) = 0. \end{aligned} \tag{1.7}$$

By the ellipticity of $\mathcal{I}$ (see (1.2)), we have

$$\begin{aligned} &(\partial_t + (-\Delta)^s - M^-_{\mathscr{L}_0} - b\cdot\nabla + \gamma - r)e^{-\gamma t}(u-v) \ge 0 \\ &(\partial_t + (-\Delta)^s - M^+_{\mathscr{L}_0} - b\cdot\nabla + \gamma - r)e^{-\gamma t}(u-v) \le 0 \\ &(u-v)(0,\cdot) = 0. \end{aligned}$$

By minimum and maximum principle, respectively, we conclude $u \equiv v$. □

Next, we need a regularity result for the fractional heat equation. Namely, by [26, Theorems 2.3 and 3.1], if $v$ satisfies

$$\partial_t v + (-\Delta)^s v = f$$

with $f \in C^{\alpha,\beta}_{t,x}((0,T];\mathbb{R}^n)$, $\alpha, \beta \in (0,1)$, then

$$\|\partial_t v\|_{C^{\alpha,\beta}_{t,x}((0,T];\mathbb{R}^n)} + \|(-\Delta)^s v\|_{C^{\alpha,\beta}_{t,x}((0,T];\mathbb{R}^n)} \le C(1 + \|f\|_{C^{\alpha,\beta}_{t,x}((0,T];\mathbb{R}^n)}). \quad (1.8)$$

Moreover, since $\mathcal{R}$ is a lower order operator with respect to $(-\Delta)^s$, we shall perform an interpolation inequality, in the sense that given a bounded function $u$, we have by classical Hölder interpolation inequalities (see, for instance, [18, Lemma 6.32]) and [30, Propositions 2.1.8 and 2.1.9] that

$$\begin{aligned} \|\mathcal{R}u\|_{C^\alpha(\mathbb{R}^n)} \le C\|u\|_{C^{\alpha+\max\{1,2\sigma\}}(\mathbb{R}^n)} &\le \epsilon\|u\|_{C^{\alpha+2s}(\mathbb{R}^n)} + C_\epsilon\|u\|_{L^\infty(\mathbb{R}^n)} \\ &\le C\epsilon\|(-\Delta)^s u\|_{C^\alpha(\mathbb{R}^n)} + C_\epsilon\|u\|_{L^\infty(\mathbb{R}^n)}; \\ \|\mathcal{R}u\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{C^{\max\{1,2\sigma+0^+\}}(\mathbb{R}^n)} &\le \epsilon\|u\|_{C^{2s-0^+}(\mathbb{R}^n)} + C_\epsilon\|u\|_{L^\infty(\mathbb{R}^n)} \end{aligned} \quad (1.9)$$

for all $\epsilon > 0$ (recall that $\max\{1, 2\sigma\} < 2s$) and $\alpha \in (0,1)$.

We now prove a priori estimates for classical solutions of (1.6) (see [29, Lemma 2.6] for a proof in the elliptic case).

**Lemma 1.4** (A priori Schauder estimates)**.** *Assume $\psi$, $b$, $r$, and $\mathcal{I}$ as in (i), (ii), (iii), and (iv), respectively. Then, there exists a constant*

$$C(n, s, \lambda, \Lambda, \sigma, T, r, b)$$

*such that for any $f \in C^{\alpha,\beta}_{t,x}$ and $u \in C^{1+\alpha,2}_{t,x}$ bounded functions which satisfy*

$$\begin{aligned} \partial_t u + (-\Delta)^s u - \mathcal{R}u &= f \quad \text{on } (0,T] \times \mathbb{R}^n, \\ u(0,\cdot) &= \psi \quad \text{on } \mathbb{R}^n, \end{aligned}$$

*we have the estimate*

$$\|\partial_t u\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + \|(-\Delta)^s u\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \le C(\|\psi\|_{C^2(\mathbb{R}^n)} + \|f\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}).$$

*In particular, we have $u(t,\cdot) \in C^{2s+\beta}$.*

*Proof.* We first notice that $v := u - \psi \in C^{1+\alpha,2}_{t,x}$ satisfies

$$(\partial_t + (-\Delta)^s - M^+_{\mathscr{L}_0} - b\cdot\nabla - r)v \le \tilde{f} \le (\partial_t + (-\Delta)^s - M^-_{\mathscr{L}_0} - b\cdot\nabla - r)v;$$
$$v(0,\cdot) = 0.$$

where $\tilde{f} := f - ((-\Delta)^s - \mathcal{R})\psi$. By (1.8), there exists a constant $C$ such that

$$\begin{aligned}\|\partial_t v\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} &+ \|(-\Delta)^s v\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}\\ &\le C(1 + \|\partial_t v + (-\Delta)^s v\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}).\end{aligned}$$

By Lemma 1.3, we have $\|v\|_{L^\infty((0,T]\times\mathbb{R}^n)} \le C\|\tilde{f}\|_{L^\infty((0,T]\times\mathbb{R}^n)}$ for a constant $C = C(T,r)$. Moreover, by interpolation inequality (as in (1.9)), for all $\epsilon > 0$ and $C_\epsilon = C(\epsilon, n, s, \lambda, \Lambda, \sigma, T, r, b)$ such that

$$(1-\epsilon)\left(\|\partial_t v\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + \|(-\Delta)^s v\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}\right) \le C_\epsilon \|\tilde{f}\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)},$$

hence the estimative follows by choosing $\epsilon = 1/2$. By the regularity of the fractional Laplacian, see [30], we have $u(t,\cdot) \in C^{2s+\beta}$. □

We use the previous results to prove the existence and uniqueness of solutions of (1.6).

**Lemma 1.5.** *In the same setting, there is a unique bounded solution* $u \in C^{1+\alpha,2s+\beta}_{t,x}$ *of* (1.6) *for* $\alpha \in (0,1)$ *and* $\beta \in (2-2s,1)$*, with the bound*

$$\|\partial_t u\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + \|(-\Delta)^s u\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \le C(\|f\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + \|\psi\|_{C^2(\mathbb{R}^n)}). \tag{1.10}$$

*Proof.* Uniqueness and boundedness follow from Lemma 1.3. We first assume that $\psi \in C^\infty_c$ and $f \in C^\infty$, with compact support in space. We define the operator $\mathcal{L}_0$ as the fractional heat operator, that is, $\mathcal{L}_0 := \partial_t + (-\Delta)^s$. We claim that a solution of

$$\begin{aligned}&\mathcal{L}_0 u = f \text{ on } (0,T]\times\mathbb{R}^n,\\ &u(0,\cdot) = \psi,\end{aligned} \tag{1.11}$$

is smooth, vanishing as $|x| \longrightarrow \infty$. Indeed, denoting $\mathscr{F}u$ the Fourier transform of $u$ in space, we have

$$u(t,x) := \mathscr{F}^{-1}\left(e^{-|\xi|^{2s}t}\mathscr{F}\psi\right) + \mathscr{F}^{-1}\left(\int_0^t e^{-|\xi|^{2s}(t-s)}\mathscr{F}f(s,\xi)\,\mathrm{d}s\right).$$

By the regularity of $\psi$ and $f$, we obtain $u \in C^\infty$ and it vanishes as $|x| \longrightarrow \infty$, concluding the claim. By Lemma 1.3 and Lemma 1.4, we obtain (1.10) for $\mathcal{L}_0$.

Now, note that functions $f \in C^{\alpha,\beta}_{t,x}$ and $\psi \in C^2$ can be approximated by $\{f_k\}_{k\geq 0} \subset C^\infty$, with compact support in space, and $\{\psi_k\}_{k\geq 0} \subset C^\infty_c$, respectively. More precisely, we have $f_k \longrightarrow f$ and $\psi_k \longrightarrow \psi$ pointwise, and the sequences are uniformly bounded. Let $u_k \in C^{1+\alpha,2s+\beta}_{t,x}$ (vanishing as $|x| \longrightarrow \infty$) be a solution of

$$\begin{aligned} &\mathcal{L}_0 u_k = f_k \text{ on } (0,T] \times \mathbb{R}^n, \\ &u_k(0,\cdot) = \psi_k. \end{aligned}$$

By the Arzelá-Ascoli Theorem, we obtain a subsequence

$$u_{k_j} \longrightarrow u$$

in $C^{1+\alpha,2s+\beta}_{t,x}$, thus $u$ is a solution of (1.11). By assumptions (ii), (iii), and (iv), the operator $\mathcal{L} : C^{1+\alpha,2s+\beta}_{t,x} \longrightarrow C^{\alpha,\beta}_{t,x}$ is well defined. Now, we proceed by continuity method: we write $\mathcal{L}_t = \mathcal{L}_0 - t\mathcal{R}$. By Lemma 1.3, $\mathcal{L}$ is an injective operator. Since we have proven that $\mathcal{L}_0$ is a surjective operator, we conclude that $\mathcal{L}_0$ is a bijective operator, and the inverse $\mathcal{L}_0^{-1}$ is well-defined. Hence,

$$\mathcal{L}_t u = f \iff u = \mathcal{L}_0^{-1}(f + t\mathcal{R}u) =: \mathcal{S}_0 u.$$

If we show that $\mathcal{S}_0$ is a contraction map, we have that $\mathcal{L}$ is bijective, hence the claim will be proven. Indeed, by Lemma 1.3 and Lemma 1.4, for any $u,\, v \in C^{1+\alpha,2s+\beta}_{t,x}$ such that $(u-v)(0,\cdot) \equiv 0$, we conclude

$$\|\mathcal{S}_0 u - \mathcal{S}_0 v\|_{C^{1+\alpha,2s+\beta}((0,T]\times\mathbb{R}^n)} \leq t\, C\, \|\mathcal{R}u - \mathcal{R}v\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}$$

where $C$ is a universal constant. Now, by the regularity of $u$ and $v$, we have

$$t\, C\, \|\mathcal{R}u - \mathcal{R}v\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \leq t\, C_0\, \|u - v\|_{C^{\alpha,\beta+\max\{1,2\sigma\}}((0,T]\times\mathbb{R}^n))}, \qquad (1.12)$$

where $C_0$ does not depend on $t$. Since $\max\{1, 2\sigma\} < 2s$, we obtain $\mathcal{S}_0$ is a contraction map for $t_0 < C_0^{-1}$. Hence, $\mathcal{L}_{t_0}$ is bijective, and the lemma follows by iterating the same argument for the map $\mathcal{S}_{t_0} u := \mathcal{L}_{t_0}^{-1}(f + (t-t_0)\mathcal{R}u)$. □

Once the Hölder regularity of solutions of (1.6) is established, we can prove the existence of solutions to the penalized equation

$$\begin{cases} u^\epsilon_t + (-\Delta)^s u^\epsilon - b\cdot\nabla u^\epsilon - \mathcal{I}u^\epsilon - ru^\epsilon = \beta_\epsilon(u^\epsilon - \psi^\epsilon) & \text{in } (0,T]\times\mathbb{R}^n; \\ u^\epsilon(0,x) = \psi^\epsilon(x) & \text{in } \mathbb{R}^n, \end{cases} \qquad (1.13)$$

where $\psi^\epsilon \coloneqq \eta_\epsilon * \psi$, $\eta_\epsilon$ being the standard mollifier, $\beta_\epsilon(z) \coloneqq e^{-z/\epsilon}$, and $\epsilon > 0$.

**Lemma 1.6.** *Assume that $\psi$, $b$, $r$, and $\mathcal{I}$ as in (i), (ii), (iii), and (iv), respectively. Then, there exists a solution $u^\epsilon \in C^{1+\alpha,2s+\beta}_{t,x}$ to the penalized problem (1.13), where $\alpha \in (0,1)$ and $\beta \in (2-2s,2)$.*

*Proof.* We construct $u_k \in C^{1+\alpha,2s+\beta}_{t,x}$ iteratively as the unique solution to the equation

$$\begin{aligned} &\mathcal{L}u_k = \beta_\epsilon(u_{k-1} - \psi^\epsilon) \quad \text{ on } (0,T] \times \mathbb{R}^n, \\ &u_k(0,\cdot) = \psi^\epsilon, \end{aligned} \tag{1.14}$$

where $u_0 \equiv 0$. Indeed, for $k = 1$, $u_1 \in C^{1+\alpha,2s+\beta}_{t,x}$ since $\beta_\epsilon(-\psi^\epsilon) \in C^\infty$. Assuming the regularity holds for $u_{k-1}$, $u_k \in C^{1+\alpha,2s+\beta}_{t,x}$ since $\beta_\epsilon(u_{k-1} - \psi^\epsilon) \in C^{1+\alpha,2s+\beta}_{t,x}$ (by the regularity of $u_{k-1}$ and $\psi^\epsilon$).

By (1.10) and the regularity above, we have for $k \geq 1$

$$\begin{aligned} \|\partial_t u_k\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + &\|(-\Delta)^s u_k\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \\ &\leq C(\|\beta_\epsilon(u_{k-1} - \psi^\epsilon)\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + \|\psi^\epsilon\|_{C^2(\mathbb{R}^n)}). \end{aligned} \tag{1.15}$$

We now claim that for $k \geq 1$

$$\begin{aligned} &\|\beta_\epsilon(u_{k-1} - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq C_\epsilon, \\ &\|\beta_\epsilon(u_{k-1} - \psi^\epsilon)\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \leq C_\epsilon(1 + \|u_{k-1}\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}), \end{aligned} \tag{1.16}$$

where $C_\epsilon$ depends on $\epsilon$ (but does not depend on $k$). Indeed, for $k = 1$,

$$\begin{aligned} &\|\beta_\epsilon(u_0 - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq e^{\epsilon^{-1}\|\psi^\epsilon\|_{L^\infty(\mathbb{R}^n)}} \leq C_\epsilon; \\ &\|\beta_\epsilon(u_0 - \psi^\epsilon)\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \leq \frac{1}{\epsilon}\|\beta_\epsilon(u_0 - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)}\|\psi^\epsilon\|_{C^\beta(\mathbb{R}^n)} \leq C_\epsilon. \end{aligned}$$

Now, suppose that (1.16) holds for $k \geq 2$. Then by Lemma 1.3, we have

$$\begin{aligned} &\|\beta_\epsilon(u_k - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq e^{\epsilon^{-1}(\|\psi^\epsilon\|_{L^\infty(\mathbb{R}^n)} + \|\beta_\epsilon(u_{k-1}-\psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)})} \leq C_\epsilon; \\ &\|\beta_\epsilon(u_k - \psi^\epsilon)\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \\ &\qquad\qquad \leq \frac{1}{\epsilon}\|\beta_\epsilon(u_k - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)}\|u_k - \psi^\epsilon\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \\ &\qquad\qquad \leq C_\epsilon(1 + \|u_k\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}). \end{aligned}$$

Hence, the claim follows. Combining (1.15) and (1.16), we have

$$\|\partial_t u_k\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + \|(-\Delta)^s u_k\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \le C_\epsilon(1 + \|u_{k-1}\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}).$$

We finally claim that

$$\|u_{k-1}\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} \le C_\epsilon. \tag{1.17}$$

Once proven the claim, we will have a uniform bound (with respect to $k$) of

$$\|\partial_t u_k\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)} + \|(-\Delta)^s u_k\|_{C^{\alpha,\beta}((0,T]\times\mathbb{R}^n)}.$$

The claim follows by the regularity of the fractional heat equation with bounded source (see (1.54) in Section 1.A), we have

$$\begin{aligned} &\|u_{k-1}\|_{C^{1-0^+}((0,T];L^\infty(\mathbb{R}^n))} + \|u_{k-1}\|_{L^\infty((0,T];C^{2s-0^+}(\mathbb{R}^n))} \\ &\le C(\|(\partial_t + (-\Delta)^s)u_{k-1}\|_{L^\infty((0,T]\times\mathbb{R}^n)} + \|u_{k-1}\|_{L^\infty((0,T]\times\mathbb{R}^n)}). \end{aligned}$$

By interpolation inequality (see (1.9)) and Lemma 1.3, there exists a constant

$$C(n, s, \lambda, \Lambda, \sigma, T, r, b, \|\psi\|_{C^2(\mathbb{R}^n)}) > 0$$

such that

$$\begin{aligned} &\|u_{k-1}\|_{C^{1-0^+}((0,T];L^\infty(\mathbb{R}^n))} + \|u_{k-1}\|_{L^\infty((0,T];C^{2s-0^+}(\mathbb{R}^n))} \\ &\qquad \le C(1 + \|\beta_\epsilon(u_{k-1} - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)}). \end{aligned}$$

Hence, by (1.16), we conclude the claim.

Now, by the uniform bound of $\{u_k\}_{k\ge 0}$ in $C^{1+\alpha,2s+\beta}_{t,x}$, we have a subsequence convergent in compact subsets of $(0,T]\times\mathbb{R}^n$ in $C^{1+\alpha,2s+\beta}_{t,x}$ to a function $u^\epsilon \in C^{1+\alpha,2s+\beta}_{t,x}((0,T]\times\mathbb{R}^n)$. Moreover, we have

$$\mathcal{L}u_k \to \mathcal{L}u_\epsilon, \quad \beta_\epsilon(u_k - \psi^\epsilon) \to \beta_\epsilon(u^\epsilon - \psi^\epsilon) \quad \text{as } k \to \infty.$$

Hence, we conclude

$$\begin{aligned} &\mathcal{L}u^\epsilon = \beta_\epsilon(u^\epsilon - \psi^\epsilon) \quad \text{on } (0,T]\times\mathbb{R}^n, \\ &u^\epsilon(0,\cdot) = \psi^\epsilon. \end{aligned}$$

□

We now prove a uniform bound of $\beta_\epsilon(u^\epsilon - \psi^\epsilon)$ with respect to $\epsilon$, which combined with Lemma 1.3 gives a uniform bound of $u^\epsilon$.

**Lemma 1.7.** *Assume that $\psi$, $b$, $r$, and $\mathcal{I}$ as in (i), (ii), (iii), and (iv), respectively. Then, there is a constant $C(n, s, \sigma, \lambda, \Lambda, T, \|\psi\|_{C^2(\mathbb{R}^n)}, b, r) > 0$ such that*

$$\begin{aligned} &\|\beta_\epsilon(u^\epsilon - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq C \\ &\|u^\epsilon\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq C. \end{aligned} \tag{1.18}$$

*Proof.* We remark that we only need an upper bound, since $\beta_\epsilon(u^\epsilon - \psi^\epsilon) \geq 0$. We assume for $\gamma \geq 0$

$$\inf_{(0,T]\times\mathbb{R}^n} (e^{-\gamma t} u^\epsilon - \psi^\epsilon) < 0,$$

for otherwise $u^\epsilon \geq e^{\gamma t}\psi^\epsilon \geq \psi^\epsilon$, hence $\beta_\epsilon(u^\epsilon - \psi^\epsilon) \leq 1$. Take $\varphi$ a nonnegative smooth function that grows as $|x|^\sigma$ at infinity. Now, we claim that for $\delta > 0$ sufficiently small, we have

$$\min_{(0,T]\times\mathbb{R}^n} \left( e^{-\gamma t} u^\epsilon - \psi^\epsilon + \frac{\delta}{T-t} + \delta\varphi \right) < 0, \tag{1.19}$$

and the minimum is a interior point of $(0,T] \times \mathbb{R}^n$. Indeed, since $u^\epsilon$ is bounded[4], we may take $\delta$ small enough so that we can consider $(t^\epsilon_\delta, x^\epsilon_\delta)$ the minimizer of $e^{-\gamma t}u^\epsilon - \psi^\epsilon + \frac{\delta}{T-t} + \delta\varphi$. Now, since $\inf_{(0,T]\times\mathbb{R}^n}(e^{-\gamma t}u^\epsilon - \psi^\epsilon) < 0$, we may assume (1.19) (taking $\delta$ smaller if necessary). To prove that the minimizer is at the interior, we remark that the function blows up as $|x| \to \infty$ and $t \to T^-$. Moreover, if the minimum were at $t = 0$, then

$$0 < \delta \left( \frac{1}{T} + \inf_{\mathbb{R}^n} \varphi \right) = \inf_{(0,T]\times\mathbb{R}^n} \left( e^{-\gamma t} u^\epsilon - \psi^\epsilon + \frac{\delta}{T-t} + \delta\varphi \right) < 0.$$

Hence, the claim is proven. Thus,

$$\begin{aligned} \partial_t u^\epsilon(t^\epsilon_\delta, x^\epsilon_\delta) - \gamma u^\epsilon(t^\epsilon_\delta, x^\epsilon_\delta) + \frac{\delta e^{\gamma t^\epsilon_\delta}}{(T - t^\epsilon_\delta)^2} = 0, \\ \nabla u^\epsilon(t^\epsilon_\delta, x^\epsilon_\delta) - e^{\gamma t^\epsilon_\delta} \nabla\psi^\epsilon(x^\epsilon_\delta) + e^{\gamma t^\epsilon_\delta} \delta \nabla\varphi(x^\epsilon_\delta) = 0, \end{aligned}$$

$$\text{and} \quad (-\Delta)^s u^\epsilon(t^\epsilon_\delta, x^\epsilon_\delta) - e^{\gamma t^\epsilon_\delta}(-\Delta)^s \psi^\epsilon(x^\epsilon_\delta) + \delta\, e^{\gamma t^\epsilon_\delta}(-\Delta)^s\varphi(x^\epsilon_\delta) \leq 0.$$

Furthermore, by (1.2), we have

$$-\mathcal{I}u^\epsilon(t^\epsilon_\delta, x^\epsilon_\delta) \leq -M^-_{\mathscr{L}_0} u^\epsilon(t^\epsilon_\delta, x^\epsilon_\delta) \leq e^{\gamma t^\epsilon_\delta}(\delta M^+_{\mathscr{L}_0}\varphi(x^\epsilon_\delta) - M^-_{\mathscr{L}_0}\psi^\epsilon(x^\epsilon_\delta)).$$

[4]By Lemma 1.3 and taking the limit $k \to \infty$ at (1.16), we conclude $\|u^\epsilon\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq C_\epsilon$. However, we do not have a uniform bound with respect to $\epsilon$.

Hence, choosing $\gamma \coloneqq r$, we have

$$\beta_\epsilon(u^\epsilon - \psi^\epsilon)(t_\delta^\epsilon, x_\delta^\epsilon) \le e^{\gamma t_\delta^\epsilon}\Bigg( -\frac{\delta}{(T-t_\delta^\epsilon)^2} - b\cdot\nabla\psi^\epsilon(x_\delta^\epsilon) + \delta\, b\cdot\nabla\varphi(x_\delta^\epsilon)$$
$$+(-\Delta)^s\psi^\epsilon(x_\delta^\epsilon) - \delta\,(-\Delta)^s\varphi(x_\delta^\epsilon) + \delta\, M^+_{\mathscr{L}_0}\varphi(x_\delta^\epsilon) - M^-_{\mathscr{L}_0}\psi^\epsilon(x_\delta^\epsilon) + (\gamma - r)\psi^\epsilon(x_\delta^\epsilon)$$
$$-(\gamma - r)\frac{\delta}{(T-t_\delta^\epsilon)^2} - \delta(\gamma - r)\varphi(x_\delta^\epsilon)\Bigg) \le C + O(\delta),$$

where $C(n, s, \sigma, \lambda, \Lambda, T, b, r, \|\psi\|_{C^2(\mathbb{R}^n)}) > 0$. Since

$$(u^\epsilon - \psi^\epsilon)(t_\delta^\epsilon, x_\delta^\epsilon) \longrightarrow \inf_{(0,T]\times\mathbb{R}^n}(u^\epsilon - \psi^\epsilon)$$

as $\delta \to 0$ and $\beta_\epsilon$ is decreasing, we obtain

$$\sup_{(0,T]\times\mathbb{R}^n}\beta_\epsilon(u^\epsilon - \psi^\epsilon) = \lim_{\delta\to 0}\beta_\epsilon(u^\epsilon - \psi^\epsilon)(t_\delta^\epsilon, x_\delta^\epsilon) \le C\|\psi\|_{C^2(\mathbb{R}^n)}.$$

For the second inequality in (1.18), by Lemma 1.3 and the previous result, we conclude

$$\|u^\epsilon\|_{L^\infty((0,T]\times\mathbb{R}^n)} \le C(\|\beta_\epsilon(u^\epsilon - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)} + \|\psi\|_{L^\infty(\mathbb{R}^n)}) \le C\|\psi\|_{C^2(\mathbb{R}^n)}.$$
□

We finally prove that $u^\epsilon$ being a solution of (1.13) converges to $u$ being a solution of (1.1).

**Theorem 1.8** (Approximation by Penalization Method)**.** *Assume that $\psi$, $b$, $r$, and $\mathcal{I}$ as in (i), (ii), (iii), and (iv), respectively. Then, there exists a viscosity solution of (1.1) which is an approximation of a solution $u^\epsilon$ of (1.13), i.e., $u^\epsilon \longrightarrow u$ as $\epsilon \longrightarrow 0^+$, and $u \in C^{1-0^+}((0,T]; L^\infty(\mathbb{R}^n)) \cap L^\infty((0,T]; C^{2s-0^+}(\mathbb{R}^n))$.*

*Proof.* We know by Lemma 1.6 that for each $\epsilon > 0$, $u^\epsilon$ is a $C^{1+\alpha, 2s+\beta}_{t,x}$ function. By (1.54) (see Section 1.A) and interpolation inequality (as in (1.9)), we have

$$\|u^\epsilon\|_{C^{1-0^+}((0,T];L^\infty(\mathbb{R}^n))} + \|u^\epsilon\|_{L^\infty((0,T];C^{2s-0^+}(\mathbb{R}^n))}$$
$$\le C(\|\beta_\epsilon(u^\epsilon - \psi^\epsilon)\|_{L^\infty((0,T]\times\mathbb{R}^n)} + \|u^\epsilon\|_{L^\infty((0,T]\times\mathbb{R}^n)}).$$

By Lemma 1.7, we conclude that $u^\epsilon \longrightarrow u$ in both $C_t^{1-0^+}; L_x^\infty$ and $L_t^\infty; C_x^{2s-0^+}$ norms as $\epsilon \longrightarrow 0^+$, and $u \in C^{1-0^+}((0,T]; L^\infty(\mathbb{R}^n)) \cap L^\infty((0,T]; C^{2s-0^+}(\mathbb{R}^n))$.

Moreover, $\psi^\epsilon \longrightarrow \psi$ in $C^2$ norm. To show that in fact $u$ is a viscosity solution of (1.1), let $\phi \in C^{1,2}_{t,x}$ such that $u-\phi$ has a strict local maximum at $(t_0,x_0)$. Choose $R>0$ such that $0=(u-\phi)(t_0,x_0)>(u-\phi)(t,x)$ for $(t,x)\in B_R(t_0,x_0)\setminus\{(t_0,x_0)\}$ and consider $(t^\epsilon,x^\epsilon)$ the maximum of $u^\epsilon-\phi$ at $K\coloneqq B_{R/2}(t_0,x_0)$. By the compactness of $K$, we have (up to a subsequence) $(t^\epsilon,x^\epsilon)\longrightarrow(s,y)$ as $\epsilon\longrightarrow 0^+$. By the definition of $(t^\epsilon,x^\epsilon)$ and its limit, we have

$$(u-\phi)(t_0,x_0)\le(u-\phi)(s,y).$$

Since $(t_0,x_0)$ is a strict local maximum, $(s,y)=(t_0,x_0)$. Now, since $u^\epsilon$ solves (1.13) classically and the definition of $(t^\epsilon,x^\epsilon)$, we have

$$(\partial_t v^\epsilon+(-\Delta)^s v^\epsilon-\mathcal{I}v^\epsilon-b\cdot\nabla v^\epsilon-ru^\epsilon)(t^\epsilon,x^\epsilon)\le\beta_\epsilon(u^\epsilon-\psi^\epsilon)(t^\epsilon,x^\epsilon),$$

where $v^\epsilon$ as in (1.4), replacing $u$ by $u^\epsilon$. By the uniform bound (1.18), we have that $u(t,\cdot)\ge\psi$ for all $t\in(0,T]$. By letting $\epsilon\longrightarrow 0^+$ at the above inequality, we conclude

$$(\partial_t v+(-\Delta)^s v-\mathcal{I}v-b\cdot\nabla v-rv)(t_0,x_0)\le\begin{cases}1, & \text{if } v(t_0,x_0)=\psi(x_0);\\ 0, & \text{if } v(t_0,x_0)>\psi(x_0).\end{cases}$$

Hence,

$$\begin{cases}(\partial_t v+(-\Delta)^s v-\mathcal{I}v-b\cdot\nabla v-rv)(t_0,x_0)\le 0 & \text{if } u(t_0,x_0)>\psi(x_0);\\ u(0,x)=\psi(x) & \forall\, x\in\mathbb{R}^n.\end{cases}$$

Since $(t_0,x_0)$ is arbitrary, we conclude that $u$ is a viscosity subsolution. To show that $u$ is also a viscosity supersolution, we remark that by the same ideas as above that for $0=(u-\varphi)(t_0,x_0)<(u-\varphi)(t,x)$ for $(t,x)\in B_R(t_0,x_0)\setminus\{(t_0,x_0)\}$, we obtain that

$$(\partial_t v+(-\Delta)^s v-\mathcal{I}v-b\cdot\nabla v-rv)(t_0,x_0)\ge\begin{cases}1, & \text{if } v(t_0,x_0)=\psi(x_0);\\ 0, & \text{if } v(t_0,x_0)>\psi(x_0),\end{cases}$$

where $v$ as in (1.5). Thus,

$$\begin{cases}(\partial_t v+(-\Delta)^s v-\mathcal{I}v-b\cdot\nabla v-rv)(t_0,x_0)\ge 0;\\ u(0,\cdot)=\psi,\end{cases}$$

and it follows that $u$ is a viscosity supersolution, and hence $u$ is a viscosity solution of (1.1). □

We now want to establish the uniqueness of solutions of (1.1). In order to do so, we proceed as in [10, Section 5] and define sup-convolution and inf-convolution and Γ-convergence.

**Definition 1.9.** Given an upper semicontinuous function $u$, we define the sup-convolution approximation $u^\epsilon$ by

$$u^\epsilon(t,x) = \sup_{(s,y)\in(0,T]\times\mathbb{R}^n} u(t+s,x+y) - \frac{|(s,y)|^2}{\epsilon}.$$

On the other hand, if $u$ is a lower semicontinuous, the inf-convolution $u_\epsilon$ is given by

$$u_\epsilon(t,x) = \inf_{(s,y)\in(0,T]\times\mathbb{R}^n} u(t+s,x+y) + \frac{|(s,y)|^2}{\epsilon}.$$

We remark that if $u$ is bounded, then $u^\epsilon$ and $u_\epsilon$ are also bounded.

**Definition 1.10.** A sequence of lower semicontinuous functions $u_k$ is said to Γ-converge to in $(0,T]\times\mathbb{R}^n$ if the two following conditions hold:

- For every sequence $(t_k,x_k) \longrightarrow (t,x)$, $\liminf_{k\to\infty} u_k(t_k,x_k) \geq u(t,x)$.
- For every $(t,x)$, there is a sequence $(t_k,x_k) \longrightarrow (t,x)$ such that

$$\limsup_{k\to\infty} u_k(t_k,x_k) = u(t,x).$$

We now prove that one can change the test functions $\phi$ at Definition 1.1 by functions that touches from above (below) that are punctually $C^{1;1,1}_{t,x}$ (see the proof below).

**Proposition 1.11.** *Let $u$ be an upper semicontinuous function such that $\mathcal{L}u \leq f$ in the viscosity sense. Let $\phi$ be a bounded function such that $\phi \in C^{1;1,1}_{t,x}$ punctually at $(t,x)$. Assume that $\phi$ touches $u$ from above at $(t,x)$. Then $\mathcal{L}\phi(t,x)$ is defined in the classical sense and $\mathcal{L}\phi(t,x) \leq f(t,x)$.*

*Proof.* By the regularity of $\phi(t,x)$, $\mathcal{L}\phi(t,x)$ is defined classically. Moreover, there exists a quadratic polynomial in space and linear in time such that $q$ touches from above $\phi$ at $(t,x)$. Let

$$v_r := \begin{cases} \phi & \text{in } B_r(t,x), \\ u & \text{in } (0,T]\times\mathbb{R}^n \setminus B_r(t,x), \end{cases}$$

Since $\mathcal{L}u \leq f$ in the viscosity sense, $\mathcal{L}v_r(t,x) \leq f(t,x)$, and $\mathcal{L}v_r(t,x)$ is well-defined. Let

$$u_r := \begin{cases} q & \text{in } B_r(t,x), \\ \phi & \text{in } (0,T] \times \mathbb{R}^n \setminus B_r(t,x). \end{cases}$$

Thus, we have

$$\begin{aligned} \mathcal{L}\phi(t,x) &\leq \mathcal{L}u_r(t,x) + (M^+_{\mathscr{L}_0} - (-\Delta)^s)(u_r - \phi)(t,x) \leq \mathcal{L}u_r(t,x) \\ &\leq \mathcal{L}v_r(t,x) + (M^+_{\mathscr{L}_0} - (-\Delta)^s)(v_r - u_r)(t,x) \\ &\leq f(t,x) + \Lambda \int_{B_r(x)} \frac{(\delta(\phi - q)(x,y,t))^+}{|y|^{n+2\sigma}} \mathrm{d}y \\ &+ \int_{B_r(x)} \frac{(\delta(\phi - q)(x,y,t))}{|y|^{n+2s}} \mathrm{d}y \leq f(t,x) + \epsilon, \end{aligned}$$

for any $\epsilon > 0$, since both integrands are bounded by $|y|^{2-2\sigma-n}$. Thus, the proposition follows. ☐

Of course, Proposition 1.11 holds for supersolutions, and its proof is similar. Analogously to [10, Propositions 5.4 and 5.5], we have the following proposition:

**Proposition 1.12.** *If $u$ is bounded and lower-semicontinuous in $(0,T] \times \mathbb{R}^n$, then $u_\epsilon$ $\Gamma$-converges to $u$. Likewise, if $u$ is bounded and upper-semicontinuous in $(0,T] \times \mathbb{R}^n$, then $-u^\epsilon$ $\Gamma$-converges to $-u$. If $u$ satisfies $\mathcal{L}u \leq f$ in the viscosity sense, then $\mathcal{L}u^\epsilon \leq f + d_\epsilon$ in the viscosity sense; if $v$ satisfies $\mathcal{L}v \geq f$ in the viscosity sense, then $\mathcal{L}v_\epsilon \geq f - d_\epsilon$ in the viscosity sense, where $d_\epsilon \longrightarrow 0$ as $\epsilon \longrightarrow 0$ and depends on the modulus of continuity.*

*Proof.* The first claim is just a generalization $u^\epsilon \longrightarrow u$ locally uniformly if $u$ is continuous. For the second claim, suppose that the $f$ has modulus of continuity $\omega$. Let $(t_0,x_0)$ be such that $u^\epsilon(t,x) - \phi(t,x) < u^\epsilon(t_0,x_0) - \phi(t_0,x_0) = 0$ for all $(t,x) \in B_R(t_0,x_0) \setminus \{(t_0,x_0)\}$, $\phi \in C^{1,2}_{t,x}$. Define $\eta(s,y) := \phi(s - s_0 + t_0, y - y_0 + x_0) + \epsilon^{-1}|(s_0 - t_0, y_0 - x_0)|^2$, where $(s_0,y_0)$ is such that

$$u^\epsilon(t_0,x_0) = u(s_0,y_0) - \frac{|(s_0 - t_0, y_0 - x_0)|^2}{\epsilon}.$$

Then $u(s,y) - \eta(s,y) < u(s_0,y_0) - \eta(s_0,y_0) = 0$ for all $(s,y) \in B_R(s_0,y_0) \setminus \{(s_0,y_0)\}$, and so

$$\mathcal{L}v^\epsilon(t_0,x_0) = \mathcal{L}v(s_0,y_0) \leq f(s_0,y_0),$$

where $v$ as in(1.4), and $v^\epsilon$ as in (1.4), replacing $u$ by $u^\epsilon$. Since $f$ has a modulus of continuity $\omega$, we have $f(s_0, y_0) \leq f(t_0, x_0) + \omega(|(t_0 - s_0, x_0 - y_0)|)$. Noticing that $u^\epsilon \geq u$, one has

$$\frac{|(s_0 - t_0, y_0 - x_0)|^2}{\epsilon} \leq u(s_0, y_0) - u(t_0, x_0) \leq 2\|u\|_{L^\infty((0,T]\times\mathbb{R}^n)},$$

we conclude

$$\mathcal{L}v^\epsilon(t_0, x_0) \leq f(t_0, x_0) + d_\epsilon,$$

where $d_\epsilon := \omega(2\|u\|_{L^\infty((0,T]\times\mathbb{R}^n)}\epsilon^{1/2})$. The proof for supersolutions is analogous. □

The next lemma is a straightforward adaptation of [10, Lemma 5.8], since the main difficulty of the operator $\mathcal{L}$ is the nonlocal part $(-\Delta)^s - \mathcal{I}$.

**Lemma 1.13.** *Let $v$ be a lower-semicontinuous and $\mathcal{L}v \geq g$ in the viscosity sense, and $u$ is upper-semicontinuous and $\mathcal{L}u \leq f$ in the viscosity sense. Moreover, assume that $u$ and $v$ are bounded functions. Then*

$$\mathcal{L}^+(u-v) := (\partial_t + (-\Delta)^s - b\cdot\nabla - r - M^+_{\mathscr{L}_0})(u-v) \leq f - g \text{ in the viscosity sense.}$$

*Proof.* By Proposition 1.12 and the stability of viscosity solutions under $\Gamma$-limits (see [10, Lemma 4.5]), it is enough to show that $\mathcal{L}^+(u^\epsilon - v_\epsilon) \leq f - g + 2d_\epsilon$ in the viscosity sense for every $\epsilon > 0$. Let $\phi \in C^{1,2}_{t,x}$ touching from above $u^\epsilon - v_\epsilon$ at $(t, x)$. Since $u$ and $v$ are bounded, then $u^\epsilon$ and $v_\epsilon$ are also bounded. Since $u^\epsilon - v_\epsilon$ is touched by above at $(t, x)$ by a $C^{1,2}_{t,x}$, then both $u^\epsilon$ and $-v_\epsilon$ must be $C^{1,2}_{t,x}$ punctually at $(t, x)$. Moreover, by (1.11), we can evaluate $\mathcal{L}u^\epsilon$ and $\mathcal{L}v_\epsilon$ at $(t, x)$ in the classical sense. Thus, by Proposition 1.12,

$$\mathcal{L}^+(u^\epsilon - v_\epsilon)(t, x) \leq \mathcal{L}u^\epsilon(t, x) - \mathcal{L}v_\epsilon(t, x) \leq f(t, x) - g(t, x) + 2d_\epsilon.$$

Hence, $\mathcal{L}^+\phi(t, x) \leq f(t, x) - g(t, x) + 2d_\epsilon$ since $\phi$ touches $v_\epsilon - u^\epsilon$ by above. Thus, $\mathcal{L}^+(u^\epsilon - v_\epsilon) \leq f - g + 2d_\epsilon$ in the viscosity sense. □

We now prove an analogous of maximum principle for $\mathcal{L}^+u \leq f$. This is the key result for the comparison principle Theorem 1.15.

**Lemma 1.14.** *Let $u$ is a bounded function defined in $(0, T] \times \mathbb{R}^n$, upper-semicontinuous such that $\mathcal{L}^+u \leq f$ at $\Omega \subset (0, T] \times \mathbb{R}^n$ in the viscosity sense, where $\Omega$ is a open set. Then there exists a constant $C(T) > 0$ such that*

$$\sup_\Omega u \leq C(T)\left(\|f^+\|_{L^\infty((0,T]\times\mathbb{R}^n)} + \sup_{\Omega^c} u\right).$$

*Proof.* Let

$$\phi_M(t) := e^{\gamma t}(M + \epsilon + \|f^+\|_{L^\infty((0,T]\times\mathbb{R}^n)}) \quad \text{for all } t \in (0,T],$$

where $\gamma := r + 1$, and $\epsilon > 0$. Note that $\mathcal{L}^+\phi_M(t,x) = (\gamma - r(x))\phi_M(t) > \|f^+\|_{L^\infty((0,T]\times\mathbb{R}^n)}$. Let $M_0$ be the smallest value of $M$ for which $\phi_M \geq u$ in $(0,T] \times \mathbb{R}^n$. We assume by contradiction that $M_0 > \sup_{\Omega^c} u$. Then, there exists $(t_0, x_0) \in \Omega$ such that $u(t_0, x_0) = \phi_{M_0}(t_0)$ (by the minimality of $M_0$), hence $\phi$ touches from above $u$ at $(t_0, x_0)$. Since $u$ is a viscosity subsolution at $\Omega$, we would have $\mathcal{L}^+\phi_{M_0}(t_0, x_0) \leq f(t_0, x_0)$, a contradiction. Therefore, for $(t,x) \in (0,T] \times \mathbb{R}^n$, we have

$$\begin{aligned} u(t,x) \leq \phi_{M_0}(t) &\leq e^{\gamma T}(M_0 + \epsilon + \|f^+\|_{L^\infty((0,T]\times\mathbb{R}^n)}) \\ &\leq e^{\gamma T}\left(\sup_{\Omega^c} u + \epsilon + \|f^+\|_{L^\infty((0,T]\times\mathbb{R}^n)}\right). \end{aligned}$$

Letting $\epsilon \longrightarrow 0$, we conclude the proof. □

We now prove the comparison principle for (1.1):

**Theorem 1.15** (Comparison Principle)**.** *Let $u, v$ be bounded viscosity subsolution and supersolution of* (1.1)*, respectively. Then $u \leq v$ in $(0,T] \times \mathbb{R}^n$.*

*Proof.* We first notice that $u(0,\cdot) \leq \psi \leq v(0,\cdot)$. For $t > 0$, if $u(t,x) \leq \psi(x)$, then $v(t,x) \geq \psi(x) \geq u(t,x)$. Moreover, by Lemma 1.13 we have $\mathcal{L}^+(u-v) \leq 0$ in the viscosity sense at $\{u > \psi\}$. By Lemma 1.14, we conclude $u \leq v$. □

As a direct consequence (combined with Theorem 1.8), we have the following corollary.

**Corollary 1.16** (Existence, Uniqueness and Regularity)**.** *There exists a unique bounded viscosity solution $u$ of* (1.1)*. Moreover, $u \in C^{1-0^+, 2s-0^+}_{t,x}$ and $u$ is approximated by a solution of the penalized equation* (1.13)*.*

*Proof.* The uniqueness follows from Theorem 1.15. The existence, regularity and approximation follows from Theorem 1.8. □

Once the Comparison Principle is established, we are able to adapt preliminary regularity properties of solutions analogous to [7, Lemma 3.2]. We implicitly use that $\mathcal{L}$ is translation invariant, since $b$ and $r$ are fixed and $\mathcal{I}$ is assumed to be translation invariant (see (iv)).

**Lemma 1.17.** *Let $u$ be a solution of* (1.1)*. Then, for any fixed $t > 0$, the $u(t, \cdot)$ is globally Lipschitz and uniformly semiconvex. Moreover, for any fixed $x \in \mathbb{R}^n$, the function $t \longmapsto u(t, x)$ non-decreasing.*

*Proof.* Fix $v \in \mathbb{R}^n$ and define $\tilde{u}(t, x) \coloneqq u(t, x + v) + C|v|$. Hence, $\tilde{u}$ solves

$$\begin{cases} \min\{\mathcal{L}\tilde{u} + rC|v|,\ \tilde{u} - \tilde{\psi}\} = 0 & \text{in } (0, T] \times \mathbb{R}^n, \\ \tilde{u}(0, x) = \tilde{\psi} & \text{in } \mathbb{R}^n, \end{cases}$$

where $\tilde{\psi}(x) \coloneqq \psi(x+v)+C|v|$. Choosing $C \coloneqq \|\nabla\psi\|_{L^\infty(\mathbb{R}^n)}$, if $u(t, x) = \psi(x)$, then $u(t, x) \le \tilde{\psi}(x) \le \tilde{u}(t, x)$. If $u(t, x) > \psi(x)$, then by Lemma 1.14, we have $\mathcal{L}^+(u - \tilde{u})(t, x) \le C|v|$, and by Theorem 1.15, $(u - \tilde{u})(t, x) \le Cre^{\gamma T}|v|$, hence $u(t, \cdot)$ is globally Lipschitz[5].

Moreover, for any fixed $\eta \ge 0$, the function $\tilde{u}(t, x) \coloneqq u(t + \eta, x)$ solves

$$\begin{cases} \min\{\mathcal{L}\tilde{u},\ \tilde{u} - \psi\} = 0 & \text{in } (-\eta, T - \eta] \times \mathbb{R}^n, \\ \tilde{u}(0, x) = u(\eta, x) & \text{in } \mathbb{R}^n. \end{cases}$$

We know $u(t, x) \ge \psi(x)$ so that, in particular, $u(\eta, x) \ge \psi(x)$; therefore, by Theorem 1.15,

$$u(t + \eta, x) \ge u(t, x) \ \text{ for every } \eta, t \ge 0.$$

Finally, denoting $C \coloneqq 2\,\|D^2\psi\|_{L^\infty(\mathbb{R}^n)}$, for a fixed $v \in \mathbb{R}^n$, we have

$$\begin{aligned} \tilde{u}(t, x) \coloneqq \frac{u(t, x + v) + u(t, x - v) + C|v|^2}{2} &\ge \frac{\psi(x + v) + \psi(x - v) + C|v|^2}{2} \\ &\ge \psi(x). \end{aligned}$$

If $u(t, x) = \psi(x)$, then $u$ is semiconvex. Moreover, since $\mathcal{I}$ is convex, $\tilde{u}$ satisfies

$$\mathcal{L}\tilde{u}(t, x) \ge \frac{1}{2}(\mathcal{L}u(t, x + v) + \mathcal{L}u(t, x - v) - rC|v|^2) \ge -\frac{rC}{2}|v|^2.$$

Hence, if $u(t, x) > \psi(x)$, then by Lemma 1.14, $\mathcal{L}^+(u - \tilde{u})(t, x) \le rC/2|v|^2$, and by Theorem 1.15, $(u - \tilde{u})(t, x) \le Cre^{\gamma T}/2|v|^2$. Since $x, v$ are arbitrary, the $C_0$-semiconvexity of $u(t, \cdot)$ follows, where $C_0 \coloneqq \|D^2\psi\|_{L^\infty(\mathbb{R}^n)}(1 + re^{\gamma T})$. □

[5] Notice that by Theorem 1.8 we already have that $u(t, \cdot)$ is globally Lipschitz, but we have improved its Lipschitz constant from Lemma 1.17.

Our next lemma deals with basic estimates of our parabolic operator, which gives a Lipschitz regularity in spacetime (see Corollary 1.18) and a comparison between $(-\Delta)^s u$ and $\mathcal{R}u$ at the contact set $\{u(t,\cdot)=\psi\}$ and the open set $\{u(t,\cdot)>\psi\}$ (see Lemma 1.19).

As a direct consequence, we have the following corollary.

**Corollary 1.18.** *If $u$ solves (1.1), then $u$ is Lipschitz in space-time, with*

$$\|\partial_t u\|_{L^\infty((0,T]\times\mathbb{R}^n)} + \|\nabla u\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq C(s,n,\lambda,\Lambda,r,b,T,\|\psi\|_{C^2(\mathbb{R}^n)}).$$

*Proof.* The Lipschitz regularity in space is just a restatement of Lemma 1.17. Now, since $u$ solves (1.1), then by Corollary 1.16 $u$ is the limit of $u^\epsilon$ which solves (1.13). We denote by $\partial_t^h$ the differential quotient with respect to time variable. Hence $w^\epsilon(t,x) := e^{-rt}\partial_t^h u^\epsilon(t,x)$ solves

$$\begin{cases} \partial_t w^\epsilon + (-\Delta)^s w^\epsilon - b\cdot\nabla w^\epsilon - M^+_{\mathscr{L}_0} w^\epsilon + \epsilon^{-1}\beta_\epsilon(\xi) w^\epsilon \leq 0 & \text{in } (0,T]\times\mathbb{R}^n, \\ |w^\epsilon(0,\cdot)| \leq |(b\cdot\nabla + \mathcal{I} + r - (-\Delta)^s)\ \psi^\epsilon| + \delta & \text{in } \mathbb{R}^n, \end{cases}$$

where $\xi \in L^\infty((0,T]\times\mathbb{R}^n)$ is nonnegative, and $|\partial_t^h u^\epsilon(0,\cdot) - \partial_t u^\epsilon(0,\cdot)| \leq \delta$ for $h$ small for $\delta>0$. Hence, by maximum principle we have

$$\|\partial_t^h u^\epsilon\|_{L^\infty((0,T]\times\mathbb{R}^n)} \leq e^{rT}(C\|\psi\|_{C^2(\mathbb{R}^n)} + \delta).$$

Letting $\epsilon \longrightarrow 0^+$ and $h \longrightarrow 0^+$, we conclude the proof. □

We notice that since $u \in C^{1-0^+}((0,T];L^\infty(\mathbb{R}^n)) \cap L^\infty((0,T];C^{2s-0^+}(\mathbb{R}^n))$ (see Corollary 1.16), we have

$$\mathcal{R}u \in L^\infty((0,T];C^\gamma(\mathbb{R}^n)), \quad \gamma := s - \max\{\sigma, 1/2\}. \tag{1.20}$$

Here, we chose the exponent for simplicity, but one has the general regularity

$$\mathcal{R}u \in L^\infty((0,T];C^{2\gamma-0^+}(\mathbb{R}^n)).$$

**Lemma 1.19.** *For a solution $u$ of (1.1) and a fixed $t_0>0$, we have*

$$0 \leq (-\Delta)^s u(t_0,\cdot) - \mathcal{R}u(t_0,\cdot) < +\infty \quad \textit{a.e. in} \quad \{u(t_0,\cdot)=\psi\}; \tag{1.21}$$

$$(-\Delta)^s u(t_0,\cdot) - \mathcal{R}u(t_0,\cdot) \leq 0 \quad \textit{in} \quad \{u(t_0,\cdot)>\psi\}. \tag{1.22}$$

*Proof.* Combining the Corollary 1.18 and Lemma 1.17, we have that $\partial_t u \geq 0$ a.e. Also, $\partial_t u = 0$ almost everywhere on the contact set $\{u = \psi\}$ so that

$$\partial_t u + (-\Delta)^s u - \mathcal{R}u = 0 \text{ in } \{u > \psi\} \text{ and } \partial_t u = 0 \text{ a.e. on } \{u = \psi\}.$$

This can be rewritten as

$$\partial_t u + (-\Delta)^s u - \mathcal{R}u = ((-\Delta)^s u - \mathcal{R}u)\, \chi_{\{u=\psi\}}. \tag{1.23}$$

This can be understood not only in the almost everywhere sense, but also in the distributional sense; incidentally, the right hand side is well defined by Lemma 1.7 and Corollary 1.18 implies that $(-\Delta)^s u - \mathcal{R}u$ is a bounded function.

Notice that (1.23) implies $\partial_t u + (-\Delta)^s u - \mathcal{R}u$ is globally bounded and vanishes in the open set $\{u > \psi\}$, so that we infer $u$ is smooth inside $\{u > \psi\}$. Hence, we are then allowed to write, for a fixed $t_0 > 0$,

$$(-\Delta)^s u(t_0, \cdot) - \mathcal{R}u(t_0, \cdot) = -\partial_t u(t_0, \cdot) \leq 0 \text{ in } \{u(t_0, \cdot) > \psi\},$$

which is (1.22).

Next, since $\partial_t u = 0$ a.e. on the contact set $\{u = \psi\}$, we have (see Lemma 1.7 and Corollary 1.18) that

$$0 \leq (-\Delta)^s u - \mathcal{R}u < \infty. \tag{1.24}$$

for almost every $(t, x) \in \{u = \psi\}$.

However, we need the same bound to hold for a.e. $x \in \mathbb{R}^n$, for every $t_0 \in (0, T]$. Note that Lipschitz continuity of $u$, see Corollary 1.18, implies that the map $t \longmapsto u(t, \cdot) \in L^2_{\text{loc}}(\mathbb{R}^n)$ is uniformly continuous. In turn, by (1.21), this implies weak continuity of the map

$$t \longmapsto (-\Delta)^s u(t, \cdot) - \mathcal{R}u(t, \cdot) \in L^2_{\text{loc}}(\mathbb{R}^n). \tag{1.25}$$

Now, consider $\epsilon > 0$ and a bounded Borel set $A \subset \{u(t_0, \cdot) = \psi\}$, multiply (1.24) by $\chi_{[t_0-\epsilon, t_0]}\chi_A$, and integrate to obtain

$$0 \leq \int_{[t_0-\epsilon, t_0]\times A} \Big[(-\Delta)^s u - \mathcal{R}u\Big] \leq C\, |A|\, \epsilon,$$

because, by Lemma 1.17, $\{u(t, \cdot) = \psi\}$ is decreasing in time and thus so is $[t_0 - \epsilon, t_0] \times A \subset \{u = \psi\}$. Since the map 1.25 is weakly continuous, we obtain as $\epsilon \to 0$:

$$0 \leq \int_A \Big[(-\Delta)^s u(t_0, \cdot) - \mathcal{R}u(t_0, \cdot)\Big] \leq C\, |A|$$

for all bounded Borel set $A \subset \{u(t_0, \cdot) = \psi\}$. This concludes the proof. □

## 1.2 Hölder-space decay of fractional Laplacian

For a given fixed $t > 0$, we assume, without loss of generality, that $0 \in \partial\{u(t,\cdot) = \psi\}$ and consider the $L_a$-harmonic function $v : \mathbb{R}^n \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ given by[6]

$$v(x,y) \coloneqq u(t,x,y) + \frac{\mathcal{R}u(t,0)}{1-a} y^{1-a},$$

where $u(t,x,y)$ denotes the harmonic extension of $u(t,x)$ to the upper half space, that is,

$$\begin{cases} L_a u(t,x,y) \coloneqq \operatorname{div}_{x,y}\left[y^a \nabla_{x,y} u(t,x,y)\right] = 0 \ \text{ for } x \in \mathbb{R}^n \ \text{ and } \ y > 0, \\ u(t,x,0) = u(t,x). \end{cases}$$

See, for instance, Caffarelli-Silvestre [9] where the authors characterize the fractional Laplacian as

$$\lim_{y \to 0^+} y^a u_y(t,x,y) = -(-\Delta)^s u(t,x) \ \text{ with } \ a = 1-2s.^{7} \tag{1.26}$$

By Lemma 1.17, we have

$$u(t,x+h,0) + u(t,x-h,0) - 2u(t,x,0) \leq -2C_0|h|^2 \ \text{ for every } \ h \in \mathbb{R}^n,$$

so that the maximum principle implies

$$u(t,x+h,y) + u(t,x-h,y) - 2u(t,x,y) \leq -2C_0|h|^2$$

for every $h \in \mathbb{R}^n$ and $y > 0$. This means that $u(t,x,y)$ is $C_0$-semiconvex with respect to $x$ for all $y \geq 0$ and, in particular,

$$\partial_y\big(y^a u_y(t,x,y)\big) \leq nC_0 y^a.$$

Now, consider the function

$$\tilde{v}(x,y) \coloneqq v(x,y) - \psi(x)$$

and set $\Lambda \coloneqq \{\tilde{v}(x,0) = 0\} = \{v(x,0) = \psi(x)\}$.

[6]By $\mathcal{R}u(t,0)$ we mean the evaluation of the function $\mathcal{R}u(t,\cdot)$ at the point $x = 0$.

[7]We remark that we actually have $\lim_{y\to 0^+} y^a \partial_y u(t,x,y) = -c_{n,a}(-\Delta)^s u(t,x)$, and so we are taking for simplicity the normalization constant as $c_{n,a} = 1$.

**Lemma 1.20.** *The following properties hold.*

(*a*) *We have* $\tilde{v} \geq 0$ *in the set* $\mathbb{R}^n \times \mathbb{R}^+ \setminus \Lambda \times \{0\}$*;*

(*b*) *The function* $\tilde{v}$ *is* $2C_0$*-semiconvex with respect to* $x$ *for all* $y \geq 0$ *and*

$$\partial_y\big(y^a \tilde{v}_y(x,y)\big) \leq 2nC_0 y^a;$$

(*c*) *For a.e.* $x \in \Lambda$*,*

$$\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) \leq C_1|x|^\gamma$$

*and, for all* $x \in \mathbb{R}^n \setminus \Lambda$*,*

$$\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) \geq -C_1|x|^\gamma;$$

(*d*) *For all* $x \in \Lambda$*,*

$$\tilde{v}(x,y) - \tilde{v}(x,0) \leq \frac{nC_0}{1+a} y^2 + \frac{C|x|^\gamma}{1-a} y^{1-a};$$

*Proof.* The first item only restates that $u(t,x) \geq \psi(x)$. Next, (b) follows from the semiconvexity of $v$ and $\psi$: we obtain that $\tilde{v}$ is $2C_0$-semiconvex with respect to $x$ for all $y \geq 0$, which implies

$$\partial_y(y^a \tilde{v}_y(x,y)) \leq 2nC_0 y^a.$$

In order to show (c), we first use (1.21) and (1.20) to conclude that, for a.e. $x \in \Lambda$,

$$\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) = -(-\Delta)^s u(t,x) + \mathcal{R}u(t,0) \leq |\mathcal{R}u(t,x) - \mathcal{R}u(t,0)| \leq C_1|x|^\gamma.$$

Then, we use (1.22) (and again (1.20)) to obtain that, for every $x \in \mathbb{R}^n \setminus \Lambda$,

$$\lim_{y\to 0^+} y^a \tilde{v}_y(x,y) = -(-\Delta)^s u(t,x) + \mathcal{R}u(t,0) \geq \mathcal{R}u(t,0) - \mathcal{R}u(t,x) \geq -C_1|x|^\gamma.$$

Now we prove (d). For a.e. $x \in \Lambda$, we have

$$\begin{aligned}
\tilde{v}(x,y) - \tilde{v}(x,0) &= \int_0^y \frac{s^a \tilde{v}_y(x,s)}{s^a}\,\mathrm{d}s \\
&= \int_0^y \frac{1}{s^a}\left(\int_0^s \partial_y(\tau^a \tilde{v}_y(x,\tau))\,\mathrm{d}\tau + \lim_{z\to 0^+} z^a \tilde{v}_y(x,z)\right)\mathrm{d}s \\
&\leq \int_0^y \frac{1}{s^a}\left(\frac{2nC_0 s^{a+1}}{a+1} + C|x|^\gamma\right)\mathrm{d}s = \frac{nC_0}{1+a} y^2 + \frac{C|x|^\gamma}{1-a} y^{1-a},
\end{aligned}$$

where the inequality relies in (b) and (c) above. Moreover, by continuity, the estimate holds for every $x \in \Lambda$. $\square$

Now, let us analyze a first decay property of $y^a \tilde{v}_y$.

**Proposition 1.21.** *There exists $c > 0$ and $\mu \in (0,1)$ for which*

$$\inf_{\Gamma_{4^{-k}}} y^a \tilde{v}_y(x,y) \geq -c\mu^k, \tag{1.27}$$

*where $\Gamma_r := B_r \times [0, \eta r]$ and $\eta := \sqrt{\frac{1+a}{2n}}$.*

*Proof.* The result follows by induction. To obtain the case $k = 0$, we note

$$\begin{aligned} L_{-a}(y^a \tilde{v}_y) &= \operatorname{div}_{x,y}\left(y^{-a}\nabla_{x,y}(y^a \tilde{v}_y)\right) = \operatorname{div}_{x,y}\left(y^{-a}\nabla_{x,y}(y^a u_y)\right) \\ &= \Delta_x(u_y) + \partial_y(y^{-a}\partial_y(y^a u_y)) = \partial_y\left(\Delta_x u + y^{-a}\partial_y(y^a u_y)\right) \\ &= \partial_y(y^{-a} L_a u(t,x,y)) = 0. \end{aligned}$$

Then, since

$$\lim_{y \to 0^+} y^a \tilde{v}_y(x,y) = -(-\Delta)^s u(t,x) + \mathcal{R}u(t,0)$$

is bounded, we obtain that $y^a \tilde{v}_y(x,y)$ remains bounded, for $y > 0$, by the maximum principle. This is enough for the case $k = 0$.

Now, assume that (1.27) holds for some $k \in \mathbb{N}$, where $c$ and $\mu$ are to be chosen later. Set

$$\tilde{V}(x,y) := \frac{4^{2sk}}{c\mu^k}\, \tilde{v}\left(\frac{x}{4^k}, \frac{y}{4^k}\right).$$

The induction hypothesis (recall $a = 1-2s$ and $v_y = \tilde{v}_y$) reads

$$\begin{aligned} \inf_{\Gamma_1}(y^a \tilde{V}_y) = \inf_{\Gamma_1}(y^a \bar{V}_y) = \frac{1}{c\mu^k} \inf_{\Gamma_1}\left[\frac{y^a}{4^{ka}}\, \tilde{v}_y\left(\frac{x}{4^k}, \frac{y}{4^k}\right)\right] &= \frac{1}{c\mu^k} \inf_{\Gamma_{4^{-k}}} y^a \tilde{v}_y(x,y) \\ &\geq -1. \end{aligned} \tag{1.28}$$

So, in this renormalized notation, it is enough to show that

$$\inf_{\Gamma_{1/4}} y^a \tilde{V}(x,y) > -\mu.$$

In order to do that, consider the auxiliary function

$$\bar{V}(x,y) := \frac{4^{2sk}}{c\mu^k}\, \bar{v}\left(\frac{x}{4^k}, \frac{y}{4^k}\right). \tag{1.29}$$

where $\bar{v}(x,y) \coloneqq v(x,y) - \Big(\psi(0) + \nabla\psi(0)\cdot x\Big)$. Both $\bar{v}$ and $\bar{V}$ are $L_a$-harmonic functions. Also, by using the $C_0$-semiconvexity of $v$, we obtain

$$
\begin{aligned}
|\tilde{V}(x,y) - \bar{V}(x,y)| &\le \frac{4^{2sk}}{c\mu^k}\left|\tilde{v}\left(\frac{x}{4^k},\frac{y}{4^k}\right) - \bar{v}\left(\frac{x}{4^k},\frac{y}{4^k}\right)\right| \\
&\le \frac{4^{2sk}}{c\mu^k}\left|\psi(0) - \psi(4^{-k}x) + \nabla\psi(0)\cdot(4^{-k}x)\right| \\
&\le \frac{C_0 4^{2(s-1)k}}{2c\mu^k}|x|^2.
\end{aligned}
\tag{1.30}
$$

Moreover, Lemma 1.20(b) yields

$$
\partial_y(y^a\tilde{V}_y) = \partial_y(y^a\bar{V}_y) \le \frac{2nC_0}{c4^{2(1-s)k}\mu^k}\, y^a. \tag{1.31}
$$

Furthermore, both $\tilde{V}$ and $\bar{V}$ are semiconvex in the set $\Gamma_1$ with constant $\frac{2C_0}{c4^{2(1-s)k}\mu^k}$.

Let us fix $L \gg C_0$ yet to be chosen. As can be checked below, we can assume this constant depends only $n$, $a$, and $C_0$. Set

$$
\bar{W}(x,y) \coloneqq \bar{V}(x,y) + \frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{c4^{\gamma k}\mu^k(1-a)}\, y^{1-a} - \frac{L}{c4^{2(1-s)k}\mu^k}\left(|x|^2 - \frac{n}{1+a}y^2\right).
$$

We have the following properties:

$(i)$ By a straightforward computation, $\bar{W}$ is an $L_a$-harmonic function.

$(ii)$ The semiconvexity of $\psi$ implies that, for every $x \in \Lambda \setminus \{0\}$,

$$
\begin{aligned}
\bar{W}(x,0) &= \frac{4^{2sk}}{K_1\mu^k}\left[v\left(\frac{x}{4^k},0\right) - \psi(0) - \nabla\psi(0)\cdot\frac{x}{4^k} - L\left|\frac{x}{4^k}\right|^2\right] \\
&\le \frac{4^{2sk}}{K_1\mu^k}\left[v\left(\frac{x}{4^k},0\right) - \psi\left(\frac{x}{4^k}\right) - (L-C_0)\left|\frac{x}{4^k}\right|^2\right] < 0.
\end{aligned}
$$

$(iii)$ By the continuity of $v$,

$$
\lim_{(x,y)\to(0,0)} \bar{W}(x,y) = \frac{4^{2sk}}{c\mu^k}\Big[v(0,0) - \psi(0)\Big] = 0.
$$

$(iv)$ By Lemma 1.20(c), we have, for $|x| < B_{1/8} \setminus \Lambda$,

$$\lim_{y\to 0^+} y^a \bar{W}_y(x,y) > 0.$$

Indeed, we have

$$\begin{aligned} y^a \bar{W}_y &= y^a \bar{V}_y + \frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{c4^{\gamma k}\mu^k} + \frac{2nL}{c4^{2(1-s)k}\mu^k(1+a)}\, y^{1+a} \\ &= \frac{1}{c\mu^k}\Bigg[\left(\frac{y}{4^k}\right)^a \tilde{v}_y\left(\frac{x}{4^k},\frac{y}{4^k}\right) + \frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{4^{\gamma k}} \\ &\quad + \frac{2nL}{4^{2(1-s)k}(1+a)}\, y^{1+a}\Bigg]. \end{aligned}$$

Let $y \to 0^+$ and recall that $C_1$ is a Hölder constant for $\mathcal{R}u(t,\cdot)$ to infer that

$$\lim_{y\to 0^+} y^a \bar{W}_y \geq \frac{1}{c4^{\gamma k}\mu^k}\left[-C_1|x|^\gamma + \|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}\right] > 0.$$

In particular, (iv) implies that for a fixed $x \in B_{1/8}\setminus\Lambda$, $\bar{W}(x,y) > \bar{W}(x,0)$ for all $(x,y) \in (B_{1/8}\setminus\Lambda)\times(0,\delta)$, once $\delta > 0$ is small enough so that $\bar{W}_y(x,y) > 0$ for all $y \in [0,\delta)$.

These properties and Hopf's Lemma (see, for instance, [18, Theorem 3.5]) imply that the maximum of $\bar{W}$ is non-negative and attained on $\partial\Gamma_{1/8}\setminus\{y = 0\}$. Hence, this maximum is achieved either at a point on the top $\partial\Gamma_{1/8}\cap\{y = \eta/8\}$ of the cylinder or at a point on the side $\partial B_{1/8}\times(0,\eta/8)$. In what follows, we analyze each case separately.

If the maximum is attained on $\partial\Gamma_{1/8}\cap\{y = \eta/8\}$, there exists $x_0 \in B_{1/8}$ for which $\bar{W}(x_0,\eta/8) \geq 0$. Thus, we have

$$\bar{V}(x_0,\eta/8) + A\,\frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{c4^{\gamma k}\mu^k} \geq -B\,\frac{L}{c4^{2(1-s)k}\mu^k},$$

where $A := \frac{\eta^{1-a}}{(1-a)8^{1-a}}$ and $B := \frac{n\eta^2}{64(a+1)}$. Since $\eta$ depends only of $n$ and $a$, so do the positive constants $A$ and $B$. By the semiconvexity of $\bar{V}$, see (1.31), we can write

$$\bar{V}(x,\eta/8) \geq \bar{V}(x_0,\eta/8) + \langle\nabla_x\bar{V}(x_0,\eta/8), x - x_0\rangle - \frac{2C_0}{c4^{2(1-s)k}\mu^k}\,|x - x_0|^2,$$

so that, in the half-ball

$$\mathrm{HB}_{1/2}(x_0,\eta/8) := \{z \in B_{1/2}(x_0);\ \langle \nabla_x \bar{V}(x_0,\eta/8), z-x_0\rangle \geq 0\},$$

there holds

$$\bar{V}(x,\eta/8) + A\,\frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{c4^{\gamma k}\mu^k} \geq -\frac{BL+C_0}{c4^{2(1-s)k}\mu^k} \geq -\frac{2BL}{c4^{2(1-s)k}\mu^k}. \tag{1.32}$$

In the last inequality is used the fact that $L$ is choosen much larger than $C_0$. Now, recall $\bar{V}_y = \tilde{V}_y$; hence, Lemma 1.20(c) gives

$$\begin{aligned} \lim_{y\to 0^+} y^a \bar{V}_y(x,y) &\leq \frac{C_1}{c4^{\gamma k}\mu^k}\,|x|^\gamma \quad \text{if} \quad \tilde{V}(x,0)=0 \quad \text{and} \\ \lim_{y\to 0^+} y^a \bar{V}_y(x,y) &\geq -\frac{C_1}{c4^{\gamma k}\mu^k}\,|x|^\gamma \quad \text{if} \quad \tilde{V}(x,0)>0. \end{aligned} \tag{1.33}$$

Integrate (1.31) with respect to $y$ in the interval $[0,y]$, with $y<\eta/8$ to obtain

$$\lim_{y\to 0^+} y^a \bar{V}_y(x,y) + \frac{2nC_0\eta^{a+1}}{c4^{2(1-s)k}\mu^k(a+1)8^{a+1}} \geq \frac{\eta^a \bar{V}_y(x,y)}{8^a}.$$

Integrating the inequality above with respect to $y$ in the interval $[0,\eta/8]$ combined with (1.32) and (1.33) yield, for all $x \in \mathrm{HB}_{1/2}(x_0,\eta/8)$,

$$\lim_{y\to 0^+} y^a \bar{V}_y(x,y) + A\frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{c4^{\gamma k}\mu^k} \geq -B'\frac{L}{c(4^{2(1-s)}\mu)^k}$$

where $A' = \frac{\eta^{a-1}A}{8^{a-1}}$ and $B' = \frac{2B\eta^{a-1}}{8^{a-1}} + \frac{2n\eta^{a+1}}{(a+1)8^{a+1}} + \frac{1}{4}$ are positives constants that depend only on $a$ and $n$. This is again possible because of the choice $L \gg C_0$.

On the other hand, suppose the non-negative maximum of $\bar{W}$ is attained on a point $(x_0,y_0) \in \partial B_{1/8} \times (0,\eta/8)$. The definition of $\eta$ implies $0 \leq y_0^2 \leq \frac{1+a}{2n8^2} = \frac{1+a}{2n}|x_0|^2$. Thus, since $\bar{W}(x_0,y_0) \geq 0$,

$$\bar{V}(x_0,y_0) + D'\,\frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{c4^{\gamma k}\mu^k} \geq \frac{L}{2^7 c4^{2(1-s)k}\mu^k},$$

where $D' = \frac{\eta^{2(1-a)}}{(1-a)8^{2(1-a)}}$. We can repeat the argument of the previous case to obtain that

$$\lim_{y\to 0^+} y^a \bar{V}_y(x,y) + D''\,\frac{\|\mathcal{R}u(t,\cdot)\|_{C^\gamma(B_1)}}{c4^{\gamma k}\mu^k} \geq -B''\frac{L}{c(4^{2(1-s)}\mu)^k},$$

for all $x \in \mathrm{HB}_{1/2}(x_0, y_0)$, where $D'' = D' + \frac{\eta^{a-1}}{8^{a-1}}$, and $B'' = \frac{2B\eta^{a-1}}{8^{a-1}} + \frac{1}{4}$.

In any case, there exist $C > 0$, $D > 0$, $\bar{y} \in [0, \eta/8]$, and $\bar{x} \in B_{1/8}$ such that, for all $x \in \mathrm{HB}_{1/2}(\bar{x}, \bar{y})$,

$$\lim_{y \to 0^+} y^a \bar{V}_y(x, y) \geq -D \frac{\|\mathcal{R}u(t, \cdot)\|_{C^\gamma(B_1)}}{c 4^{\gamma k} \mu^k} - \frac{C}{c 4^{2(1-s)k} \mu^k}.$$

We observe the constants above depend only on $n$, $a$, and $C_0$. The choices

$$\max\{4^{-\gamma}, 4^{-2+2s}\} \leq \mu < 1 \quad \text{and} \quad c > 2\big(C + D\|\mathcal{R}u(t, \cdot)\|_{C^\gamma(B_1)}\big)$$

then provides us with

$$\lim_{y \to 0^+} y^a \bar{V}_y(x, y) > -\frac{1}{2}. \tag{1.34}$$

As in case $k = 1$, we have that $y^a \bar{V}_y(x, y)$ solves $L_{-a}(y^a \bar{V}_y(x, y)) = 0$ in $\mathbb{R}^n \times \mathbb{R}^+$. From this, we now show that (1.34) and (1.28) imply that there exists $\theta < 1$ such that, for every $x \in B_{1/4}$,

$$\left(\frac{\eta}{4}\right)^a \bar{V}_y(x, \eta/4) \geq -\theta. \tag{1.35}$$

Indeed, by the maximum (actually, minimum) principle, we have

$$\inf_{x \in B_{5/8}} \lim_{y \to 0^+} y^a \bar{V}_y(x, y) \leq \inf_{(x,y) \in \Gamma_{5/8}} y^a \bar{V}_y(x, y) \leq \left(\frac{\eta}{4}\right)^a \bar{V}_y(x, \eta/4) \tag{1.36}$$

for all $x \in B_{1/4}$. Then, (1.28) and Harnack's inequality yield

$$1 + \sup_{x \in B_{5/8}} \lim_{y \to 0^+} y^a \bar{V}_y(x, y) \leq C \left( \inf_{x \in B_{5/8}} \lim_{y \to 0^+} y^a \bar{V}_y(x, y) + 1 \right),$$

for some constant $C > 0$ depending only on $s$ and $n$. Since $\mathrm{HB}_{1/2}(\bar{x}, \bar{y}) \subset B_{5/8}$, by (1.34), we have

$$\inf_{x \in B_{5/8}} \lim_{y \to 0^+} y^a \bar{V}_y(x, y) \geq \frac{1}{C} \sup_{x \in \mathrm{HB}_{1/2}(\bar{x}, \bar{y})} \lim_{y \to 0^+} y^a \bar{V}_y(x, y) - 1 + \frac{1}{C} \geq \frac{1}{2C} - 1 =: -\theta.$$

By the above and (1.36), we conclude (1.35).

Next, integrate (1.31) with respect to $y$ in the interval $[y, \eta/4]$ to obtain

$$\left(\frac{\eta}{4}\right)^a \bar{V}_y(x, \eta/4) - y^a \bar{V}_y(x, y) \leq \frac{2nC_0}{c 4^{2(1-s)k} \mu^k (a+1)} \left[(\eta/4)^{a+1} - y^{a+1}\right] \leq \frac{\hat{C}}{c},$$

where $\hat{C} = \frac{2nC_0\eta^{a+1}}{(a+1)4^{a+1}}$ is a positive constant that depends only on $n$, $a$, and $C_0$. We thus have

$$y^a \bar{V}_y(x,y) \geq -\theta - \frac{\hat{C}}{c}.$$

First enlarge, if necessary, $c$ so that $\theta + \hat{C}/K_1 < 1$; then, enlarge $\mu$ (if necessary) so that $\theta + \hat{C}/K_1 < \mu < 1$. Therefore,

$$y^a \tilde{V}_y(x,y) = y^a \bar{V}_y(x,y) > -\mu,$$

for every $x \in B_{1/4}$ and every $y \in [0, \eta/4]$, which is what we wanted. □

Once Proposition 1.21 is established, we show in a standard manner (see, for instance, [7, Lemma 4.4]) how a bound from below of the form $\inf_{\Gamma_r} y^a \tilde{v}_y \geq -Cr^\alpha$ provides control of the $l^\infty$-norm of $\tilde{v}$ in a smaller cylinder.

**Lemma 1.22.** *For $C > 0$, $\alpha \in (0,1)$, and $r \in (0,1]$ such that $\inf_{\Gamma_r} y^a \tilde{v}_y \geq -Cr^\alpha$, there exists $M > 0$ for which*

$$\sup_{\Gamma_{r/8}} |\tilde{v}| \leq M r^{\alpha+2s}.$$

*Moreover, the constant $M$ is independent of $r$ and depends only on $C, \alpha, a$, and $C_0$.*

*Proof.* We consider only the case where $r > 0$ is small, for $\tilde{v}$ is globally bounded. By Lemma 1.20(a) and by our assumption, we have, for every $(x,y) \in \Gamma_r$,

$$\tilde{v}(x,y) \geq \tilde{v}(x,0) - Cr^\alpha \int_0^y \tau^{-a} \, \mathrm{d}\tau \geq -\frac{C\eta^{1-a}}{1-a} r^{\alpha+2s}.$$

This provides a lower bound on $\tilde{v}$.

Let us assume, by contradiction, that the upper bound does not hold, that is, for any $M > 0$, there exists $(x_0, y_0) \in \Gamma_{r/8}$ such that $\tilde{v}(x_0, y_0) \geq M r^{\alpha+2s}$. Our assumption, by integration, yields

$$\begin{aligned} \tilde{v}(x_0, \eta r/2) &\geq \tilde{v}(x_0, y_0) - \frac{C\eta^{2s}}{(1-a)2^{2s}} r^{\alpha+2s} + \frac{Cy_0^{2s}}{1-a} r^\alpha \\ &\geq \left( M - \frac{C\eta^{2s}}{(1-a)2^{2s}} \right) r^{\alpha+2s}. \end{aligned}$$

In particular, for sufficiently large $M > 0$, namely $M \geq \frac{4C\eta^{2s}}{3(1-a)2^{2s}}$, we can write

$$\tilde{v}(x_0, \eta r/2) \geq \frac{M}{4} r^{\alpha+2s}.$$

Next, denote $\bar{v}$ as in (1.29) and observe that the semiconvexity of $\psi$ implies $|\bar{v} - \tilde{v}| \leq C_0 r^2$ in $\Gamma_r$. Then, the lower bound above gives

$$\bar{v}(x, y) + \frac{Cr^{\alpha+2s}}{1-a} + C_0 r^2 \geq 0 \quad \text{for every} \quad (x, y) \in \Gamma_r.$$

Now, $B_{\eta r/2}(x_0, \eta r/2) \subset \Gamma_r$ and $(0, \eta r/2) \in B_{\eta r/4}(x_0, \eta r/2)$, so that Harnack inequality, applied in $B_{\eta r/2}(x_0, \eta r/2)$, gives

$$\frac{M}{4} r^{\alpha+2s} \leq \sup_{B_{\eta r/4}} \left[ \bar{v} + \frac{Cr^{\alpha+2s}}{1-a} + C_0 r^2 \right] \leq c \left( \bar{v}(0, \eta r/2) + \frac{Cr^{\alpha+2s}}{1-a} + C_0 r^2 \right).$$

Hence, there exists $c_0 > 0$ such that

$$\tilde{v}(0, \eta r/2) + C_0 r^2 \geq \bar{v}(0, \eta r/2) \geq c_0 M r^{\alpha+2s} - \frac{Cr^{\alpha+2s}}{1-a} - C_0 r^2.$$

Recall $0 \in \Lambda$; then, by Lemma 1.20(d),

$$\begin{aligned} 0 = \tilde{v}(0,0) &\geq \tilde{v}(0, \eta r/2) - \frac{nC_0\eta^2}{4(1+a)} r^2 \\ &\geq c_0 M r^{\alpha+2s} - \frac{Cr^{\alpha+2s}}{1-a} - \frac{nC_0\eta^2}{4(1+a)} r^2 - 2C_0 r^2. \end{aligned}$$

In particular, we have a bound for $M$:

$$M \leq \frac{1}{c_0 r^{\alpha+2s}} \left( \frac{Cr^{\alpha+2s}}{1-a} + \frac{nC_0\eta^2}{4(1+a)} r^2 + 2C_0 r^2 \right).$$

This is in contradiction to our assumption because the constant $M > 0$ is should be arbitrary. □

We are now in a position to prove a first regularity estimate at a free boundary point.

**Theorem 1.23.** *Let $u$ be a solution of (1.1), and $\psi$, $b$, $r$, and $\mathcal{I}$ as in (i), (ii), (iii), and (iv), respectively. Then, there exist $\bar{C} > 0$ and $\alpha \in (0, \gamma)$ such that, for every $r \in (0, 1)$ and every $x_0 \in \partial\{u(t, \cdot) = \psi\}$,*

$$\sup_{B_r(x_0)} |u(t, \cdot) - \psi| \leq \bar{C}\, r^{\alpha+2s} \quad \textit{and} \tag{1.37}$$

$$\sup_{B_r(x_0)} \left| \left[(-\Delta)^s u(t, \cdot) - \mathcal{R}u(t, \cdot)\right] \chi_{\{u(t,\cdot)=\psi\}} \right| \leq \bar{C}\, r^{\alpha}. \tag{1.38}$$

*Proof.* The estimate in (1.37) is a direct consequence of Lemma 1.22. In order to prove (1.38), we assume, as before, $x_0 = 0$. Recall that, by the definition of $\tilde{v}$,

$$(-\Delta)^s u(t, x') - \mathcal{R}u(t, 0) = -\lim_{y\to 0^+} y^a \tilde{v}_y(x', y) = -\lim_{y\to 0^+} y^a v_y(x', y)$$

and so

$$(-\Delta)^s u(t, x') - \mathcal{R}u(t, x') = -\lim_{y\to 0^+} y^a v_y(x', y) + \mathcal{R}u(t, 0) - \mathcal{R}u(t, x').$$

By (1.21) and (1.20) we have that

$$\sup_{B_r} \left| \left[(-\Delta)^s u(t, \cdot) - \mathcal{R}u(t, \cdot)\right] \chi_{\{u(t,\cdot)=\psi\}} \right| \leq -\inf_{B_r} \lim_{y\to 0^+} y^a v_y(\cdot, y) + C_1 r^{\gamma}.$$

Now, if $1/4 < r < 1$, then by Proposition 1.21 for $k = 0$ we have that

$$-\inf_{B_r} \lim_{y\to 0^+} y^a v_y(x', y) \leq c \leq 4cr;$$

on the other hand, if $r \leq 1/4$, by taking $\beta$ such that $\beta \leq \log_4 \mu^{-1}$ combined with Proposition 1.21, we obtain

$$-\inf_{B_r} \lim_{y\to 0^+} y^a v_y(x', y) \leq -\inf_{B_{1/4}} \lim_{y\to 0^+} y^a v_y(x', y) \leq c\mu \leq c\mu^k \leq c4^{-k\beta}.$$

Hence, choosing $k$ large enough so that $4^{-k} < r$ gives (1.38) for

$$\alpha = \min\{\beta, \gamma\}. \qquad \square$$

**Corollary 1.24.** *In the same setting of Theorem 1.23, there exist $\bar{C}' > 0$ and $\alpha \in (0, \gamma)$ such that*

$$\Big\| \left[(-\Delta)^s u(t,\cdot) - \mathcal{R}u(t,\cdot)\right] \chi_{\{u(t,\cdot)=\psi\}} \Big\|_{C^\alpha(\mathbb{R}^n)} \le \bar{C}',$$

*that is,*

$$\left[(-\Delta)^s u(t,\cdot) - \mathcal{R}u(t,\cdot)\right] \chi_{\{u(t,\cdot)=\psi\}} \in C^\alpha(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

*Proof.* Let $\alpha$ obtained in Theorem 1.23. Over the set $\Lambda = \{u(t,\cdot) = \psi\}$, recall that the function $(-\Delta)^s u(t,\cdot) - \mathcal{R}u(t,\cdot)$ is bounded, by (1.21). It is then enough to show that, for $|x_1 - x_2| \le 1/4$ with $x_1, x_2 \in \Lambda$,

$$\Big|(-\Delta)^s u(t,x_1) - (-\Delta)^s u(t,x_2) - \mathcal{R}u(t,x_1) + \mathcal{R}u(t,x_2)\Big| \le C|x_1 - x_2|^\alpha.$$

Given $x \in \Lambda$, let $d(x, \partial\Lambda)$ denote the distance from $x$ to $\partial\Lambda$. We then analyze two possible situations.

- Suppose first that

$$|x_1 - x_2| \le \frac{1}{4} \max\big\{d(x_1, \partial\Lambda), d(x_2, \partial\Lambda)\big\}.$$

By Theorem 1.23, we have, for any $r \in (0,1)$,

$$\sup_{B_r(x_i)} |u(t,\cdot) - \psi| \le \bar{C} r^{\alpha+2s}.$$

In particular, $u(t,\cdot) = \psi$ in the set $S \coloneqq B_{4|x_1-x_2|}(x_1) \cap B_{4|x_1-x_2|}(x_2)$. Also, we trivially have

$$|u(t,\cdot) - \psi| \le M \coloneqq \|u(t,\cdot) - \psi\|_{L^\infty(\mathbb{R}^n)}$$

outside the set $B_1(x_1) \supset B_{1/2}(x_2)$, and then

$$\begin{aligned} |(-\Delta)^s f(x_1) - (-\Delta)^s f(x_2) - \mathcal{R}f(x_1) + \mathcal{R}f(x_2)| &\le C_1|x_1 - x_2|^\gamma \\ &+ \int_{\mathbb{R}^n \setminus S} |f(x')| \left| \frac{1}{|x' - x_1|^{n+2s}} - \frac{1}{|x' - x_2|^{n+2s}} \right| \mathrm{d}x' \\ \le C_1|x_1 - x_2|^\gamma + C\left[\bar{C}\int_{|x_1-x_2|}^1 \tau^{\alpha'-2}\,\mathrm{d}\tau + M\right]|x_1 - x_2| &\le C|x_1 - x_2|^\alpha, \end{aligned}$$

where $f \coloneqq u(t,\cdot) - \psi$. Because $\|(-\Delta)^s\psi\|_{C_x^{1-s}(\mathbb{R}^n)}$ is bounded, this gives the result.

- If, on the other hand,

$$|x_1 - x_2| \geq \frac{1}{4} \max \big\{ d(x_1, \partial\Lambda), d(x_2, \partial\Lambda) \big\},$$

we take $\bar{x}_1, \bar{x}_2 \in \partial\Lambda$ for which $|x_1 - \bar{x_1}| = d(x_1, \partial\Lambda)$ and $|x_2 - \bar{x_2}| = d(x_2, \partial\Lambda)$. Therefore, by Theorem 1.23, we have

$$\begin{aligned} |(-\Delta)^s f(x_1) - (-\Delta)^s f(x_2) - \mathcal{R}f(x_1) + \mathcal{R}f(x_2)| \leq C_1 |x_1 - x_2|^\gamma \\ + \sup_{B_{4|x_1-x_2|}(\bar{x}_1)} |(-\Delta)^s f| + \sup_{B_{4|x_1-x_2|}(\bar{x}_2)} |(-\Delta)^s f| \leq \bar{C}' |x_1 - x_2|^\alpha. \end{aligned}$$

□

## 1.3 Monotonicity formula and optimal regularity in space

We recall a regularity property provided by the fractional heat operator (see, for instance, [7, Appendix A]); namely, that if $v$ satisfies

$$\partial_t v + (-\Delta)^s v = f$$

with $f \in L^\infty((0,T]; C^\beta(\mathbb{R}^n))$ and $\beta \in (0,1)$, then

$$\begin{aligned} \|\partial_t v\|_{L^\infty((0,T];C^{\beta-0^+}(\mathbb{R}^n))} + \|(-\Delta)^s v\|_{L^\infty((0,T];C^{\beta-0^+}(\mathbb{R}^n))} \\ \leq C \left(1 + \|f\|_{L^\infty((0,T];C^\beta(\mathbb{R}^n))}\right). \end{aligned} \tag{1.39}$$

Incidentally, we have shown in Corollary 1.24 and (1.20) that

$$\partial_t u + (-\Delta)^s u = \big[(-\Delta)^s u - \mathcal{R}u\big] \chi_{\{u=\psi\}} + \mathcal{R}u \in L^\infty\big((0,T]; C^\alpha(\mathbb{R}^n)\big)$$

so that (1.39) holds for our solution $u$. Hence, $(-\Delta)^s u \in C^{\alpha-0^+}$, and since $u$ is bounded Lemma 1.7, by [30, Proposition 2.1.8], we have

$$u \in L^\infty\big((0,T]; C^{2s+\alpha-0^+}(\mathbb{R}^n)\big).$$

Moreover, the lower term has the regularity

$$\mathcal{R}u \in L^\infty\big((0,T]; C^{\alpha+\gamma}(\mathbb{R}^n)\big). \tag{1.40}$$

Next, we consider $0 \in \partial\{u(t,\cdot)=\psi\}$. Moreover, let $w : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ be the function which solves, with fixed $t > 0$,

$$\begin{cases} L_{-a} w = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ w(x,0) = \big[(-\Delta)^s u(t,x) - \mathcal{R}u(t,0)\big]\chi_{\{u(t,\cdot)=\psi\}}(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{1.41}$$

Hence, by the boundedness obtained in Lemma 1.19, the maximum principle for $w$, and the regularity $C^\alpha(\mathbb{R}^n)$ of $w(x,0)$ (given by Corollary 1.24), we have

$$\sup_{|x|^2+y^2 \le r^2} w(x,y) \le C r^\alpha,$$

for a uniform constant $C > 0$. The goal is to obtain the estimative above with $1-s$ replacing $\alpha$. Hence, without loss of generality, from now on we assume that $\alpha < 1-s$.

We begin with the following lemma, which is the analogous of [7, Lemma 4.5].

**Lemma 1.25.** *Let $\bar{C} > 0$ and $\alpha$ be as in Theorem 1.23 and set*

$$\delta = \delta(\alpha, s) = \frac{1}{4}\left(\frac{\alpha}{\alpha+2s} - \frac{\alpha}{2}\right).$$

*Then, there exists $r_0 > 0$, depending on $\alpha, s, \bar{C}$, and $C_0$, such that $\operatorname{co}(\Omega \cap B_r)$ does not contain the origin for any $r \in (0, r_0)$, where*

$$\Omega \coloneqq \{x \in \mathbb{R}^n;\ w(x,0) \ge r^{\alpha+\delta}\}$$

*and $\operatorname{co} A$ stands for the convex hull of the set $A$.*

*Proof.* Let $x \in \Omega$ and assume, by contradiction, that $0 \in \operatorname{co}(\Omega \cap B_r)$. By the definition of $w$, we must have $u(t,x) = \psi(x)$, or equivalently, $\tilde{v}(x,0) = 0$. Note that, for $x \in \Omega$,

$$\lim_{y \to 0^+} y^a \tilde{v}_y(x,y) = -(-\Delta)^s u(t,x) + \mathcal{R}u(t,0) = -w(x,0) \le -r^{\alpha+\delta}.$$

Moreover, by Lemma 1.20(b),

$$\begin{aligned}\tilde{v}(x,y) &= \int_0^y \tilde{v}_y(x,\tau)\,\mathrm{d}\tau = \int_0^y \frac{1}{\tau^a}\left(\lim_{\rho\to 0^+}\rho^a\tilde{v}_y(x,\rho) + \int_0^\tau \rho^a\tilde{v}_y(x,\rho)\,\mathrm{d}\rho\right)\mathrm{d}\tau \\ &\leq -\frac{r^{\alpha+\delta}y^{2s}}{2s} + \int_0^y \frac{2nC_0}{\tau^a}\int_0^\tau \rho^a\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &= -\frac{r^{\alpha+\delta}y^{2s}}{2s} + \frac{2nC_0}{1+a}\int_0^y \tau\,\mathrm{d}\tau \\ &= -\frac{r^{\alpha+\delta}y^{2s}}{2s} + \frac{nC_0y^2}{1+a}.\end{aligned}$$

Now, by Theorem 1.23, we know $\tilde{v}(0,y) \geq -\bar{C}y^{\alpha+2s}$. Also, by the semiconvexity of $\tilde{v}$, given by Lemma 1.20(b), we have

$$\tilde{v}(0,y) + \nabla_x\tilde{v}(0,y)\cdot x \leq \tilde{v}(x,y) + C_0r^2.$$

Thus, since $0 \in \mathrm{co}(\Omega\cap B_r)$,

$$\sup_{x\in\mathrm{co}(\Omega\cap B_r)} \nabla_x\tilde{v}(0,y)\cdot x \geq -|\nabla_x\tilde{v}(0,y)| \inf_{\mathrm{co}(\Omega\cap B_r)} |x| = 0$$

and we have

$$\tilde{v}(0,y) \leq \sup_{x\in\mathrm{co}(\Omega\cap B_r)} \tilde{v}(x,y) + C_0r^2.$$

Putting all these together, we have, for any $r,y\in(0,1)$,

$$\bar{C}y^{\alpha+2s} + \frac{nC_0}{1+a}y^2 + C_0r^2 \geq \frac{r^{\alpha+\delta}y^{2s}}{2s}. \tag{1.42}$$

In order to get a contradiction, we relate $y$ and $r$ by the formula $y^\alpha = r^{\alpha+2\delta}$, so that (1.42) implies

$$\bar{C}r^\delta + \frac{nC_0}{1+a}r^{4\delta\alpha^{-1}+\delta+\gamma} + C_0r^{\delta+\gamma} \geq \frac{1}{2s},$$

where $\gamma \coloneqq 2-\alpha^{-1}(\alpha+2s)(\alpha+2\delta)$ which is positive by the definition of $\delta$. Now, the left hand side goes to zero as $r\to 0$ and we have a contradiction for small values of $r$. □

We remark that $\delta < \gamma$, since $2s > 1$ and $\alpha < \gamma$. The next two technical lemmas are key ingredients to prove the monotonicity formula Lemma 1.28.

**Lemma 1.26.** *There exists $C > 0$ such that, for every $r \geq 0$,*

$$\limsup_{y \to 0^+} \int_{B_r} \frac{y^{-a}\, \partial_y \big(w(x,y)^2\big)}{(|x|^2 + y^2)^{(n-1-a)/2}} \, \mathrm{d}x \geq -C r^{\alpha+1+a}.$$

*Moreover,*

$$\lim_{y \to 0^+} \int_{B_r} \partial_y \left( \left( |x|^2 + y^2 \right)^{-(n-1-a)/2} \right) y^{-a} w(x,y)^2 \, \mathrm{d}x = 0.$$

*Proof.* To show the first estimate, we begin by noticing the following properties:

$(i)$ From Lemma 1.19, we have $w(x,0) = 0$ for $x \in \mathbb{R}^n \setminus \Lambda$ and $w(x,0) \geq 0$ for $x \in \Lambda$. Hence, by the maximum principle $w(x,y) \geq 0$, that is, $w(x,y) \geq w(x,0)$ for all $x \in \mathbb{R}^n \setminus \Lambda$ and $y > 0$.

$(ii)$ From Lemma 1.20, we have

$$y^a v_y(x,y) \leq \lim_{\tau \to 0^+} \tau^a v_y(x,\tau) + \frac{nC_0}{1+a} y^{1+a},$$

with the limit well-defined since $-(-\Delta)^s u(t,x) + \mathcal{R}u(t,0)$ is Hölder continuous on $\Lambda$ and smooth outside (by Lemma 1.19).

$(iii)$ The function $y^a v_y$ is a solution of

$$\begin{cases} L_{-a}(y^a v_y) = 0; \\ \lim_{y \to 0^+} y^a v_y(x,y) = -(-\Delta)^s u(t,x) + \mathcal{R}u(t,0). \end{cases}$$

Moreover, from Lemma 1.19, we have that $w(x,0) \geq (-\Delta)^s u(t,x) - \mathcal{R}u(t,0)$ and then, by the maximum principle, $w(x,y) \geq -y^a v_y(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^+$. Since

$$w(x,0) = -\lim_{y \to 0^+} y^a v_y(x,y) \quad \text{in} \quad \Lambda,$$

the previous item implies that, for all $x \in \Lambda$ and $y > 0$,

$$w(x,y) \geq w(x,0) - \frac{nC_0}{1+a} y^{1+a}. \tag{1.43}$$

From $(i)$ and $(iii)$, we have that (1.43) actually holds for all $x \in \mathbb{R}^n$ and $y > 0$. Furthermore, since $w$ is non-negative and $C_x^\alpha$, we conclude

$$w(x,y)^2 - w(x,0)^2 \geq -\frac{nC_0}{1+a} y^{1+a} [w(x,y) + w(x,0)] \geq -Ky^{1+a}(r+y)^\alpha,$$

for all $x \in B_r$, $y > 0$, and a uniform constant $K > 0$.

We now use the change of variable $\tau(y) := \left(\frac{y}{1+a}\right)^{1+a}$ and define $\tilde{w}(x,\tau) := w(x,y)$. Then, the above inequality can be rewritten as

$$\tilde{w}(x,y)^2 - \tilde{w}(x,0)^2 \geq -K'\tau(r + \tau^{1/(1+a)})^\alpha, \tag{1.44}$$

for all $x \in B_r$, $y > 0$, and a uniform constant $K' > 0$. Using that

$$y^{-a}\partial_y\big(w(x,y)^2\big) = \partial_\tau\big(\tilde{w}(x,\tau)^2\big),$$

we have that

$$\begin{aligned}
&\limsup_{y \to 0^+} \int_{B_r} \frac{y^{-a}\partial_y\big(w(x,y)^2\big)}{(|x|^2 + y^2)^{(n-1-a)/2}} \, \mathrm{d}x \\
&= \limsup_{s \to 0^+} \int_{B_r} \frac{\partial_\tau\big(\tilde{w}(x,\tau)^2\big)}{\big(|x|^2 + (1+a)^2\tau^{2/(1+a)}\big)^{(n-1-a)/2}} \, \mathrm{d}x.
\end{aligned}$$

To estimate the right hand side above, we consider the average with respect to $\tau \in [0,\epsilon]$ and we use Fubini's Theorem to obtain

$$\begin{aligned}
I_\epsilon &:= \frac{1}{\epsilon} \int_0^\epsilon \int_{B_r} \frac{\partial_\tau\big(\tilde{w}(x,\tau)^2\big)}{\big(|x|^2 + (1+a)^2\tau^{2/(1+a)}\big)^{(n-1-a)/2}} \, \mathrm{d}x \, \mathrm{d}\tau \\
&= \frac{1}{\epsilon} \int_{B_r} \left( \frac{\tilde{w}(x,\epsilon)^2}{\big(|x|^2 + (1+a)^2\epsilon^{2/(1+a)}\big)^{(n-1-a)/2}} - \frac{\tilde{w}(x,0)^2}{|x|^{n-1-a}} \right) \mathrm{d}x \\
&\quad - \frac{1}{\epsilon} \int_0^\epsilon \int_{B_r} \tilde{w}(x,\tau)^2 \frac{\mathrm{d}}{\mathrm{d}\tau} \left(|x|^2 + (1+a)^2\tau^{2/(1+a)}\right)^{-(n-1-a)/2} \mathrm{d}x \, \mathrm{d}\tau.
\end{aligned}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(|x|^2 + (1+a)^2\tau^{2/(1+a)}\right)^{-(n-1-a)/2} \leq 0.$$

Hence, by (1.44) and the fact that $w(\cdot,0)=\tilde{w}(\cdot,0)\in C^\alpha_x$, we have

$$
\begin{aligned}
I_\epsilon &\geq \frac{1}{\epsilon}\int_{B_r}\left(\frac{\tilde{w}(x,0)^2-K'\epsilon(r+\epsilon^{1/(1+a)})^\alpha}{(|x|^2+(1+a)^2\epsilon^{2/(1+a)})^{(n-1-a)/2}}-\frac{\tilde{w}(x,0)^2}{|x|^{n-1-a}}\right)\mathrm{d}x\\
&\geq \int_{B_r}\frac{-K'(r+\epsilon^{1/(1+a)})^\alpha}{|x|^{n-1-a}}\,\mathrm{d}x\\
&+\frac{C}{\epsilon}\int_{B_r}\left[\frac{|x|^{2\alpha}}{(|x|^2+(1+a)^2\epsilon^{2/(1+a)})^{(n-1-a)/2}}-\frac{|x|^{2\alpha}}{|x|^{n-1-a}}\right]\mathrm{d}x\\
&=: I_{1\epsilon}+I_{2\epsilon}.
\end{aligned}
$$

We have

$$
\lim_{\epsilon\to 0} I_{1\epsilon}=-\frac{K'n\omega_n}{a+1}r^{\alpha+1+a}=-K'C_{n,a}r^{\alpha+1+a}.
$$

For the second term $I_{2\epsilon}$, we split the integral over $B_{\epsilon^\beta}$ and over $B_r\setminus B_{\epsilon^\beta}$, denoting these by $I^1_{2\epsilon}$ and $I^2_{2\epsilon}$, respectively, and the exponent $\beta>0$ is yet to be chosen. On the one hand, to estimate $I^1_{2\epsilon}$, we choose $\beta\in\left(\frac{1}{2\alpha+a+1},\frac{1}{a+1}\right)$, and we have that

$$
\lim_{\epsilon\to 0} I^1_{2\epsilon}\geq -\lim_{\epsilon\to 0}\frac{C}{\epsilon}\int_{B_{\epsilon^\beta}}\frac{|x|^{2\alpha}}{|x|^{n-1-a}}\,\mathrm{d}x=-\frac{Cn\omega_n}{2\alpha+a+1}\lim_{\epsilon\to 0}\epsilon^{\beta(2\alpha+a+1)-1}=0.
$$

On the other hand, for all $|x|\geq\epsilon^\beta$, we have $\epsilon^{2/(1+a)}\leq|x|^2$, so that

$$
\left(|x|^2+(1+a)^2\epsilon^{2/(1+a)}\right)^{(n-1-a)/2}\leq C\left(|x|^{n-1-a}+C\epsilon^{2/(1+a)}|x|^{n-3-a}\right)
$$

and the term $I^2_{2\epsilon}$ can be estimated as

$$
\begin{aligned}
I^2_{2\epsilon} &\geq \frac{C}{\epsilon}\int_{\epsilon^\beta}^r\left[\frac{\rho^{n-1+2\alpha}}{\rho^{n-1-a}+C\epsilon^{2/(1+a)}\rho^{n-3-a}}-\frac{\rho^{n-1+2\alpha}}{\rho^{n-1-a}}\right]\mathrm{d}\rho\\
&= -\frac{C}{\epsilon}\int_{\epsilon^\beta}^r\rho^{2\alpha+a}\frac{\epsilon^{2/(1+a)}}{\rho^2+C\epsilon^{2/(1+a)}}\,\mathrm{d}\rho\\
&\geq -\frac{C\epsilon^{2/(1+a)}}{\epsilon}\int_{\epsilon^\beta}^r\rho^{2\alpha+a-2}\,\mathrm{d}\rho\\
&\geq -C_r\epsilon^{2/(1+a)-1}\left[1+\epsilon^{\beta(2\alpha+a-1)}\right].
\end{aligned}
$$

Recall that $2>1+a$, and so we only need to consider the case $2\alpha+a-1<0$, since otherwise we clearly have $\lim_{\epsilon\to 0} I^2_{2\epsilon}\geq 0$. Moreover, since $\beta<1/(1+a)$, we have

$$
\frac{2}{1+a}-1+\beta(2\alpha+a-1)\geq\frac{2\alpha}{1+a}>0,
$$

which also gives $\lim_{\epsilon\to 0} I_{2\epsilon}^2 \geq 0$, so that $\lim_{\epsilon\to 0} I_{2\epsilon} \geq 0$. Hence, we conclude that

$$\liminf_{\epsilon\to 0} I_\epsilon \geq -K' C_{n,a} r^{\alpha+1+a}.$$

From this, we deduce

$$\begin{aligned}\limsup_{\epsilon\to 0} \int_{B_r} \frac{\partial_\tau\big(\tilde{w}(x,\epsilon)^2\big)}{\big(|x|^2+(1+a)^2\epsilon^{2/(1+a)}\big)^{(n-1-a)/2}}\,\mathrm{d}x &\geq \liminf_{\epsilon\to 0} I_\epsilon \\ &\geq -K' C_{n,a} r^{\alpha+1+a},\end{aligned}$$

which is what we wanted.

To show the second claim of the lemma, we observe that, by the $C_x^\alpha$-regularity of $w$, we have

$$\begin{aligned}\left|\int_{B_r} \partial_y \left(\big(|x|^2+y^2\big)^{-(n-1-a)/2}\right) y^{-a} w(x,y)^2\,\mathrm{d}x\right| \qquad&\\ \leq \int_{B_r} \frac{y^{1-a}}{(|x|^2+y^2)^{(n+1-a)/2-\alpha}}\,\mathrm{d}x&\\ \leq C y^{1-a}\int_0^r \frac{\rho^{n-1}}{(\rho^2+y^2)^{(n+1-a)/2-\alpha}}\,\mathrm{d}\rho&\\ \leq C y^{1-a}\int_0^r \frac{\rho^{n-1}}{(\rho+y)^{n+1-a-2\alpha}}\,\mathrm{d}\rho&\\ \leq C_r \frac{y^{1-a}}{y^{1-a-2\alpha}} = C_r y^{2\alpha},&\end{aligned}$$

which gives

$$\lim_{y\to 0^+} \int_{B_r} \partial_y \left(\big(|x|^2+y^2\big)^{-(n-1-a)/2}\right) y^{-a} w(x,y)^2\,\mathrm{d}x = 0. \qquad \square$$

The next lemma is the result [7, Lemma 4.7] on the first eigenvalue of a weighted Laplacian on the half-sphere. The result applies to our modified function $w$ as proved below.

Let us denote by $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ the $n$-dimensional sphere, and set

$$\mathbb{S}^n_+ := \mathbb{S}^n \cap \{x_{n+1} \geq 0\}.$$

Let us also denote

$$\mathscr{H}_0^{1/2} := \Big\{h \in H^{1/2}\big(\partial(\mathbb{S}^n_+)\big);\ h = 0 \text{ on } \partial(\mathbb{S}^n_+) \cap \{x_{n+1}=0\} \cap \{x_n \geq 0\}\Big\}.$$

In other words, $h \in \mathscr{H}_0^{1/2}$ when it is Sobolev in the boundary $\partial\mathbb{S}^n_+ \simeq \mathbb{S}^{n-1}$ of the upper sphere, and it vanishes on the upper part of the $(n-1)$-dimensional sphere $\partial(\mathbb{S}^n_+) \cap \{x_{n+1} = 0\}$.

**Lemma 1.27.** *[7, Lemma 4.7] We have*

$$\inf_{h\in\mathscr{H}_0^{1/2}} \frac{\displaystyle\int_{\mathbb{S}^n_+} |\nabla_\theta h|^2 y^{-a}\,\mathrm{d}\sigma}{\displaystyle\int_{\mathbb{S}^n_+} h^2 y^{-a}\,\mathrm{d}\sigma} = (1-s)(n-1+s),$$

*where $\nabla_\theta$ is the derivative with respect to the angular variables.*

*Proof.* For convenience of the reader, we reproduce the proof by Caffarelli and Figalli. Let

$$\bar{H}(x,y) \coloneqq (\sqrt{x_n^2+y^2} - x_n)^{1-s}$$

and denote by $\bar{h}(\theta)$ its restriction to $\mathbb{S}^n_+$, which gives $\bar{H} = r^{1-s}\bar{h}(\theta)$. As shown in [8, Proposition 5.4], $\bar{h}$ is the first eingenfunction related to the minimization problem above. If $\lambda_1$ is the correspoding eigenvalue, our goal is to show that $\lambda_1 = -(1-s)(n-1+s)$.

First, we claim that $\bar{H}$ satisfies $L_{-a}\bar{H} = 0$ for $y > 0$. Indeed, the function

$$G(x_n, y) \coloneqq (\sqrt{x_n^2+y^2} - x_n)^{1/2}$$

is harmonic in $y > 0$ as the imaginary part of $z \longmapsto z^{1/2}$. Since $\bar{H} = G^{1+a}$, direct computation yields

$$\begin{aligned} L_{-a}\bar{H} = L_{-a}G^{1+a} =&(1+a)y^{-a}G^a\Delta_{x,y}G \\ &+ (1+a)ay^{-a}G^{a-1}\left(|\nabla_{x,y}G|^2 - \frac{GG_y}{y}\right) = 0. \end{aligned}$$

Next, since $\bar{h}$ is an eigenfunction, we have $\mathrm{div}_\theta(y^{-a}\nabla_\theta\bar{h}) = \lambda_1\bar{h}$. In particular,

$$\Delta_\theta\bar{h}(0,1) = \lambda_1\bar{h}(0,1).$$

Moreover, by spherical coordinates,

$$0 = L_{-a}\bar{H} = \Delta_r\bar{H} + \frac{n}{r}\bar{H}_r + \frac{1}{r^2}\Delta_\theta\bar{H} - \frac{a}{y}\bar{H}_y,$$

and we obtain

$$\begin{aligned}0 &= \Delta_r \bar{H}(0,1) + n\bar{H}_r(0,1) + \Delta_\theta \bar{H}(0,1) - a\bar{H}_y(0,1)\\ &= -(1-s)s\bar{h}(0,1) + (1-s)(n-a)\bar{h}(0,1) + \Delta_\theta \bar{h}(0,1).\end{aligned}$$

Therefore,

$$\lambda_1 \bar{h}(0,1) = \Delta_\theta \bar{h}(0,1) = -(1-s)(n-1+s)\bar{h}(0,1). \qquad \square$$

We now prove the monotonicity formula: the result and its proof are found in [7, Lemma 4.8]. For the convenience of the reader, we reproduce the proof.

**Lemma 1.28** (Monotonicity Formula)**.** *Let $w$ be given by (1.41) and denote*

$$B_r^+ := \{z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^+;\ |z| < r\}.$$

*For $r \in (0,1]$, define*

$$\varphi(r) := \frac{1}{r^{2(1-s)}} \int_{B_r^+} \frac{|\nabla_z w(z)|^2 y^{-a}}{|z|^{n-1-a}}\,\mathrm{d}z.$$

*Then, there exists $C > 0$ such that, for all $r \in (0,1]$,*

$$\varphi(r) \le C\left(1 + r^{2\alpha+\delta-a-1}\right).$$

*Proof.* Set

$$\varphi_\epsilon(r) := \frac{1}{r^{2(1-s)}} \int_{B_r^+ \cap \{y>\epsilon\}} \frac{|\nabla_z w(z)|^2 y^{-a}}{|z|^{n-1-a}}\,\mathrm{d}z.$$

By the Monotone Convergence Theorem, we can bound $\varphi$ by $\liminf_{\epsilon\to 0} \varphi_\epsilon$. Moreover, we note that $\varphi(r)$ is bounded by $\varphi(1)$. Hence, we only need to bound $\liminf_{\epsilon\to 0} \varphi_\epsilon(1)$. Let $\chi : \mathbb{R}^n \to [0,1]$ be a smooth compactly supported function with $\chi \equiv 1$ in $B_1 \subset \mathbb{R}^n$. Thus,

$$\varphi_\epsilon(r) \le \int_\epsilon^1 \int_{\mathbb{R}^n} \frac{|\nabla_z w(z)|^2 y^{-a}}{|z|^{n-1-a}} \chi(x)\,\mathrm{d}x\,\mathrm{d}y.$$

The definition of $w$ in (1.41) gives $L_{-a} w = 0$ and so we have $L_{-a}(w^2) = 2|\nabla_z w|^2 y^{-a}$. Then, integration by parts gives

$$\begin{aligned}\varphi_\epsilon(r) \le &-\int_\epsilon^1 \int_{\mathbb{R}^n} \nabla_z(w^2) \cdot \nabla_z \left(\frac{1}{2|z|^{n-1-a}}\right) y^{-a} \chi(x)\,\mathrm{d}x\,\mathrm{d}y\\ &- \int_\epsilon^1 \int_{\mathbb{R}^n} \nabla_x(w^2) \cdot \nabla_x \chi(x) \frac{y^{-a}}{2|z|^{n-1-a}}\,\mathrm{d}x\,\mathrm{d}y\\ &+ \int_{\mathbb{R}^n} \partial_y(w^2) \frac{y^{-a}}{2|z|^{n-1-a}}\,\mathrm{d}x \Big|_{y=\epsilon}^{y=1}.\end{aligned}$$

Using that $L_{-a}|z|^{-n+1+a} = C\delta_{(0,0)}$, we can integrate by parts once more to obtain

$$
\begin{aligned}
\varphi_\epsilon(r) \leq & \int_\epsilon^1 \int_{\mathbb{R}^n} w^2 \Delta_x \chi(x) \frac{y^{-a}}{2|z|^{n-1-a}} \,\mathrm{d}x\,\mathrm{d}y \\
& + \int_\epsilon^1 \int_{\mathbb{R}^n} w^2 \nabla_x \chi(x) \cdot \nabla_x \left( \frac{1}{|z|^{n-1-a}} \right) y^{-a} \,\mathrm{d}x\,\mathrm{d}y \\
& - \int_{\mathbb{R}^n} w^2 \partial_y \left( \frac{1}{2|z|^{n-1-a}} \right) y^{-a} \chi(x) \,\mathrm{d}x \bigg|_{y=\epsilon}^{y=1} \\
& + \int_{\mathbb{R}^n} \partial_y (w^2) \frac{y^{-a}\chi(x)}{2|z|^{n-1-a}} \,\mathrm{d}x \bigg|_{y=\epsilon}^{y=1},
\end{aligned}
$$

since $(0,0) \notin [\epsilon, 1] \times \mathbb{R}^n$. Recall that $\chi \equiv 1$ in $B_1$, that $w$ is of class $C^\alpha_x$, and that $\operatorname{supp} \chi \subseteq B_R$ for some $R > 0$, so that

$$
\begin{aligned}
& \int_\epsilon^1 \int_{\mathbb{R}^n} \left( w^2 \Delta_x \chi(x) \frac{y^{-a}}{2|z|^{n-1-a}} + w^2 \nabla_x \chi(x) \cdot \nabla_x \left( \frac{1}{|z|^{n-1-a}} \right) y^{-a} \right) \mathrm{d}x\,\mathrm{d}y \\
& \qquad \leq C \int_\epsilon^1 y^{-a} \int_1^R \left( |r^2 + y^2|^{\alpha + a/2} + |r^2 + y^2|^{\alpha + (1+a)/2} \right) \mathrm{d}r\,\mathrm{d}y < +\infty.
\end{aligned}
$$

Moreover, since $w$ is smooth for $y > 0$, we obtain

$$
\begin{aligned}
& - \int_{\mathbb{R}^n} w^2 \partial_y \left( \frac{1}{2|z|^{n-1-a}} \right) y^{-a} \chi(x) \,\mathrm{d}x \bigg|_{y=1} \\
& + \int_{\mathbb{R}^n} \partial_y (w^2) \frac{y^{-a}\chi(x)}{2|z|^{n-1-a}} \,\mathrm{d}x \bigg|_{y=1} < +\infty.
\end{aligned}
$$

Using Lemma 1.26, we conclude that

$$
\varphi(r) \leq \liminf_{\epsilon \to 0} \varphi_\epsilon(1) \leq C.
$$

Hence, we have that $\varphi_\epsilon(r) \longrightarrow \varphi(r)$ locally uniformly in $(0, 1]$. This shows, in particular, that $\varphi(r)$ is well-defined. Now, take $\epsilon < r$ and use again that

$L_{-a}|z|^{-n+1+a} = C\delta_{(0,0)}$ to obtain

$$\begin{aligned}
\varphi_\epsilon'(r) &= -\frac{1-s}{r^{3-2s}} \int_{B_r^+\cap\{y>\epsilon\}} \frac{L_{-a}(w^2)}{|z|^{n-1-a}} \,\mathrm{d}z + \frac{1}{r^n} \int_{\partial B_r^+\cap\{y>\epsilon\}} |\nabla_z w(z)|^2 y^{-a} \,\mathrm{d}\sigma \\
&= -\frac{2(1-s)}{r^{1+2(1-s)}} \int_{\partial(B_r^+\cap\{y>\epsilon\})} w\nabla_z w\cdot\nu \frac{y^{-a}}{|z|^{n-1-a}} \,\mathrm{d}\sigma \\
&\quad + \frac{1-s}{r^{1+2(1-s)}} \int_{B_r^+\cap\{y>\epsilon\}} \nabla_z(w^2)\cdot\nabla_z\left(\frac{1}{|z|^{n-1-a}}\right) y^{-a} \,\mathrm{d}z \\
&\quad + \frac{1}{r^n} \int_{\partial B_r^+\cap\{y>\epsilon\}} |\nabla_z w(z)|^2 y^{-a} \,\mathrm{d}\sigma =: A_\epsilon + B_\epsilon + C_\epsilon.
\end{aligned}$$

We estimate each of these three terms. By Lemma 1.26 and the Cauchy-Schwarz's inequality,

$$\begin{aligned}
\lim_{\epsilon\to0^+} A_\epsilon &= -\frac{1-s}{r^{n+1}} \lim_{\epsilon\to0^+} \int_{\partial B_r^+\cap\{y>\epsilon\}} (w^2)_r y^{-a} \,\mathrm{d}\sigma \\
&\quad + \frac{1-s}{r^{1+2(1-s)}} \lim_{\epsilon\to0^+} \int_{B_r^+\cap\{y=\epsilon\}} (w^2)_y \frac{y^{-a}}{|z|^{n-1-a}} \,\mathrm{d}\sigma \\
&\geq -\frac{(1-s)^2}{r^{n+2}} \int_{\partial B_{r,+}} w^2 y^{-a} \,\mathrm{d}\sigma - \frac{1}{r^n} \int_{\partial B_{r,+}} (w_r)^2 y^{-a} \,\mathrm{d}\sigma - Cr^{\alpha-1} \\
&\geq -\frac{(1-s)^2}{r^{n+2}} \int_{\partial B_{r,+}} w^2 y^{-a} \,\mathrm{d}\sigma - \frac{1}{r^n} \int_{\partial B_{r,+}} |\nabla_z w(z)|^2 y^{-a} \,\mathrm{d}\sigma \\
&\quad + \frac{1}{r^{n+2}} \int_{\partial B_{r,+}} |\nabla_\theta w|^2 y^{-a} \,\mathrm{d}\sigma - Cr^{\alpha-1}.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_{\epsilon\to0^+} B_\epsilon &= -\frac{(1-s)(n-1-a)}{r^{n+2}} \lim_{\epsilon\to0^+} \int_{\partial B_r^+\cap\{y>\epsilon\}} w^2 y^{-a} \,\mathrm{d}\sigma \\
&\quad - \frac{1-s}{r^{1+2(1-s)}} \lim_{\epsilon\to0^+} \int_{B_r^+\cap\{y=\epsilon\}} w^2 \partial_y \left(\frac{1}{|z|^{n-1-a}}\right) y^{-a} \,\mathrm{d}\sigma \\
&= -\frac{(1-s)(n-1-a)}{r^{n+2}} \int_{\partial B_{r,+}} w^2 y^{-a} \,\mathrm{d}\sigma.
\end{aligned}$$

Hence, using that $\varphi_\epsilon$ converges uniformly as $\epsilon\to0^+$, we have that the distributional derivative $D_r\varphi$ satisfies

$$D_r\varphi \geq \frac{1}{r^{n+2}} \int_{\partial B_{r,+}} |\nabla_\theta w|^2 y^{-a} \,\mathrm{d}\sigma - Cr^{\alpha-1} + \frac{\lambda_1}{r^{n+2}} \int_{\partial B_{r,+}} w^2 y^{-a} \,\mathrm{d}\sigma,$$

where $\lambda_1$ as in (the proof of) Lemma 1.27. Consider $\bar{W} \coloneqq (w - r^{\alpha+\delta})^-$. By Lemma 1.25, $\bar{W}$ is admissible for the eigenvalue problem in Lemma 1.27. We compute

$$|\nabla_\theta \bar{W}|^2 \leq |\nabla_\theta w|^2 \quad \text{and} \quad (w - \bar{W})^2 + 2\bar{W}(w - \bar{W}) + \bar{W}^2 = w^2,$$

to conclude that

$$\begin{aligned} D_r\varphi &\geq \frac{\lambda_1}{r^{n+2}} \int_{\partial B_{r,+}} \left[(w - \bar{W})^2 + 2\bar{W}(w - \bar{W})\right] y^{-a}\, d\sigma - Cr^{\alpha-1} \\ &\geq -Cr^{2\alpha+\delta-a-2} - Cr^{\alpha-1}, \end{aligned}$$

since $|\bar{W}| \leq |w| \leq Cr^\alpha$ and $|w - \bar{W}| \leq r^{\alpha+\delta}$. Therefore, integration in the interval $[r, 1]$ yields

$$\varphi(r) \leq \varphi(1) + Cr^{2\alpha+\delta-a-1} + C \leq C(1 + r^{2\alpha+\delta-a-1})$$

for all $r \in (0, 1]$, since $1 + a > 0$ and $\varphi(1)$ is universally bounded. □

Now, we we are able to obtain the optimal modulus of continuity of $w$. In particular, we obtain an improved regularity and the optimal regularity of the lower order and free boundary term, respectively.

**Proposition 1.29.** *Let $u$ be the solution of (1.1). Then*

$$[(-\Delta)^s u - \mathcal{R}u]\chi_{\{u=\psi\}} \in L^\infty((0, T]; C^{1-s}(\mathbb{R}^n)),\ \mathcal{R}u \in L^\infty((0, T]; C^{1-s+\gamma}(\mathbb{R}^n)).$$

*Proof.* Let $\eta_\epsilon$ be a mollifier, define $w_\epsilon \coloneqq \eta_\epsilon * w$, and observe $L_{-a}w_\epsilon = (L_{-a}w) * \eta_\epsilon = 0$. Moreover,

$$w_\epsilon(x, y) - w_\epsilon(x, 0) \geq -\frac{nC_0}{1 + a} y^{1+a}.$$

Set $\bar{W}_\epsilon \coloneqq (w_\epsilon - r^{\alpha+\delta})^+$, which satisfies $L_{-a}\bar{W}_\epsilon \leq 0$ in the set $\{y > 0\}$ and

$$\bar{W}_\epsilon(x, y) - \bar{W}_\epsilon(x, 0) \geq -\frac{nC_0}{1 + a} y^{1+a}.$$

We now consider, for $(x, y) \in \mathbb{R}^n \times \mathbb{R}$,

$$\tilde{w}_\epsilon(x, y) \coloneqq \bar{W}_\epsilon(x, |y|) + \left(1 + \frac{nC_0}{1 + a}\right) |y|^{1+a}.$$

Note that $L_{-a}\tilde{w}_\epsilon \le 0$ in the set $\{y \neq 0\}$, and

$$\tilde{w}_\epsilon(x,y) - \tilde{w}_\epsilon(x,0) \ge |y|^{1+a}.$$

Since $\tilde{w}_\epsilon$ is smooth in $x$, we conclude $\tilde{w}_\epsilon$ is a subsolution for $L_{-a}$ in the whole $\mathbb{R}^n \times \mathbb{R}$. Then, let $\epsilon \to 0$ so that

$$\tilde{w}(x,y) := (w(x,|y|) - r^{\alpha+\delta})^+ + \left(1 + \frac{nC_0}{1+a}\right)|y|^{1+a}$$

is a subsolution globally. By Lemma 1.25, the convex hull of the set where $\tilde{w}(\cdot,0) \ge 0$ does not contain the origin and it is thus contained in "some half" of $B_r$. In particular, $\tilde{w}(\cdot,0) \equiv 0$ in a set which is bigger than the other half of $B_r$. So, by a weighted Poincaré inequality (see [17, Theorem 1.5]) and the definition of $\varphi$ (see Lemma 1.28), we obtain, for all $r \in (0,1]$,

$$\begin{aligned}
\int_{B_r^+} \tilde{w}(z)^2 y^{-a}\,\mathrm{d}z &\le Cr^2 \int_{B_r^+} |\nabla_z \tilde{w}(z)|^2 y^{-a}\,\mathrm{d}z \\
&\le Cr^2 \left[\int_{B_r^+} |\nabla_z w(z)|^2 y^{-a}\,\mathrm{d}z + r^{n+1+a}\right] \\
&\le Cr^{n+2}\big(\varphi(r) + r^{1+a}\big) \\
&\le Cr^{n+2}\big(1 + \varphi(r)\big),
\end{aligned}$$

since $|\nabla_z \tilde{w}|^2 \le |\nabla_z w|^2 + C|y|^{2a}$. Then, since $\tilde{w}$ is $L_{-a}$-subharmonic and Lemma 1.28, we get

$$\sup_{B_{r/2}^+} \tilde{w}^2 \le \frac{C}{r^{n+1-a}} \int_{B_r^+} \tilde{w}(z)^2 y^{-a}\,\mathrm{d}z \le C\big(r^{1+a} + r^{2\alpha+\delta}\big).$$

Hence, for all $r \in (0,1]$,

$$\sup_{B_r} w \le C\Big(\sup_{B_{r/2}^+} \tilde{w} + r^{\alpha+\delta} + r^{1+a}\Big) \le C\big(r^{1-s} + r^{\alpha+\delta/2}\big). \tag{1.45}$$

We conclude by the same argument as Corollary 1.24 that

$$\|w\|_{C_x^{\beta_\alpha}(\mathbb{R}^n)} \le C, \qquad \text{where} \qquad \beta_\alpha := \min\{1-s, \alpha+\delta/2\}.$$

As remarked previously, $\delta < \gamma$, thus $\beta_\alpha < \gamma$. Hence, by (1.40), we have

$$\big[(-\Delta)^s u(t,\cdot) - \mathcal{R}u(t,\cdot)\big]\chi_{\{u(t,\cdot)=\psi\}} \in C^{\beta_\alpha}(\mathbb{R}^n).$$

Therefore, we have

$$\partial_t u + (-\Delta)^s u = \big[(-\Delta)^s u - \mathcal{R}u\big]\chi_{\{u=\psi\}} + \mathcal{R}u \in L^\infty((0,T];C^{\beta_\alpha}(\mathbb{R}^n)).$$

Hence, by (1.39), $(-\Delta)^s u(t,\cdot) \in C^{\beta_\alpha - 0^+}(\mathbb{R}^n)$, and by [30, Proposition 2.1.8], we have $u(t,\cdot) \in C^{\beta_\alpha+2s-0^+}(\mathbb{R}^n)$, thus

$$\mathcal{R}u \in L^\infty((0,T];C^{\beta_\alpha+\gamma}(\mathbb{R}^n)). \tag{1.46}$$

By the definition of $\delta$ (see Lemma 1.25), given $\alpha_0 > 0$, there exists $\delta_0 > 0$ such that $\delta \geq \delta_0 > 0$ for all $\alpha' \in [\alpha_0, 1-s]$. If $\beta_\alpha = 1-s$, the proposition follows. Otherwise, we apply the monotonicity formula (as in (1.45)) $k$ times and the argument above to obtain

$$\sup_{B_r} w \leq C\big(r^{1-s} + r^{\alpha+k\,\delta_0/2}\big), \quad \mathcal{R}u \in L^\infty((0,T];C^{\min\{1-s,\alpha+k\,\delta_0/2\}+\gamma}(\mathbb{R}^n)).$$

Choosing $k$ large enough and using the same argument as Corollary 1.24, the proposition follows. □

## 1.4 Almost optimal regularity in time

We note that Proposition 1.29 implies that

$$\partial_t u + (-\Delta)^s u = \big[(-\Delta)^s u - \mathcal{R}u\big]\chi_{\{u=\psi\}} + \mathcal{R}u \in L^\infty((0,T];C^{1-s}(\mathbb{R}^n)),$$

which gives (see (1.39))

$$\partial_t u \ \text{ and } \ (-\Delta)^s u \ \in \ L^\infty((0,T];C^{1-s-0^+}(\mathbb{R}^n)). \tag{1.47}$$

From this, we are able to show the first step of the iteration procedure that eventually grants us the optimal regularity of the solution. We remark that by (1.20) and Proposition 1.29, we have

$$\mathcal{R}u \in L^\infty((0,T];C^{1-s+\gamma}(\mathbb{R}^n)).$$

**Lemma 1.30.** *We have* $\big[(-\Delta)^s u - \mathcal{R}u\big]\chi_{\{u=\psi\}} \in C_{t,x}^{\frac{1-s}{1+s}-0^+,1-s}((0,T]\times\mathbb{R}^n)$.

*Proof.* We need to estimate

$$\big[(-\Delta)^s u - \mathcal{R}u\big](t,x)\chi_{\{u(t,\cdot)=\psi\}} - \big[(-\Delta)^s u - \mathcal{R}u\big](s,x)\chi_{\{u(s,\cdot)=\psi\}}.$$

We notice that we only need to consider $x \in \{u(\tau,\cdot) = \psi\}$ for some $\tau$ (otherwise the expression vanishes). Let $0 < s < t \leq T$. By Lemma 1.17, $\{u(t,\cdot) = \psi\} \subseteq \{u(s,\cdot) = \psi\}$, we can assume, without loss of generality, that $x \in \{u(s,\cdot) = \psi\}$. If $x \in \{u(s,\cdot) = \psi\} \setminus \{u(t,\cdot) = \psi\}$, by Lemma 1.19, the left hand side below vanishes and we can find $\tau \in (s,t)$ such that $x \in \partial\{u(\tau,\cdot) = \psi\}$

$$\big[(-\Delta)^s u - \mathcal{R}u\big](\tau,x)\chi_{\{u(\tau,\cdot)=\psi\}} = \big[(-\Delta)^s u - \mathcal{R}u\big](t,x)\chi_{\{u(t,\cdot)=\psi\}}.$$

Then, we can estimate the free boundary part replacing $t$ with $\tau$. Hence, we need only consider $x \in \{u(t,\cdot) = \psi\}$. In other words, we only need to estimate both terms

$$\big|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)\big|, \tag{1.48}$$

$$\big|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)\big|. \tag{1.49}$$

By the same strategy as in [7, Lemma 4.12], we bound the (1.48) by

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \big[(-\Delta)^s u(t,x) - (-\Delta)^s u(t,z)\big]\phi_r(x-z)\,\mathrm{d}z\right| \\
&+ \left|\int_{\mathbb{R}^n} \big[(-\Delta)^s u(t,z) - (-\Delta)^s u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right| \\
&+ \left|\int_{\mathbb{R}^n} \big[(-\Delta)^s u(s,x) - (-\Delta)^s u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right|,
\end{aligned}$$

where $\phi$ is a normalized smooth cutoff function supported in $B_1$, and $\phi_r(x) \coloneqq r^{-n}\phi(x/r)$. Then, the first and third terms can be controlled by $Cr^{1-s-0^+}$. For the second term, we integrate by parts $(-\Delta)^s$ and recall the Lipschitz-in-time regularity of $u$ (see Corollary 1.18), so that

$$\big|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)\big| \leq C\left(r^{1-s-0^+} + \frac{(t-s)}{r^{2s}}\right).$$

The choice $r \coloneqq (t-s)^{1/(1+s)}$ thus yields

$$\big|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)\big| \leq C(t-s)^{\frac{1-s}{1+s}-0^+}.$$

Analogously, we bound (1.49) by

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\big[\mathcal{R}u(t,x)-\mathcal{R}u(t,z)\big]\phi_r(x-z)\,\mathrm{d}z\right|\\&+\left|\int_{\mathbb{R}^n}\big[\mathcal{R}u(t,z)-\mathcal{R}u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right|\\&+\left|\int_{\mathbb{R}^n}\big[\mathcal{R}u(s,x)-\mathcal{R}u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z.\right|,\end{aligned}$$

By the space regularity of $\mathcal{R}u$ (see (1.46)), the first and third terms can be controlled by $Cr^{1-s+\gamma}$. By Corollary 1.18 and performing an integration by parts, we can bound the second term by

$$C\left((t-s)+\frac{(t-s)}{r}+\left|\int_{\mathbb{R}^n}M^+_{\mathscr{L}_0}(u(t,z)-u(s,z))\phi_r(x-z)\,\mathrm{d}z\right|\right).$$

Recalling that $M^+_{\mathscr{L}_0}v \coloneqq \sup_{L\in\mathscr{L}_0} Lv$, we have that for all $\epsilon>0$, $M^+_{\mathscr{L}_0}v-\epsilon\leq Lv$. Since $\|\phi_r\|_{L^1(B_r)}=1$ and the fact that $L$ is an integrable by parts operator, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}M^+_{\mathscr{L}_0}(u(t,z)-u(s,z))\phi_r(x-z)\,\mathrm{d}z\right|\\&\leq\int_{\mathbb{R}^n}|u(t,z)-u(s,z)||M^+_{\mathscr{L}_0}\phi_r(x-z)|\,\mathrm{d}z+\epsilon.\end{aligned}$$

Now, by the explicit form of Pucci operator (see (iv)) and the Lipschitz in time regularity of $u$, we can bound the integral above by $\frac{(t-s)}{r^{2\sigma}}$ after letting $\epsilon\longrightarrow 0^+$. Thus,

$$\big|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)\big|\leq C\left(r^{1-s+\gamma}+(t-s)+\frac{(t-s)}{r}+\frac{(t-s)}{r^{2\sigma}}\right).$$

Assume without loss of generality that $r\leq 1$ (otherwise, we have Lipschitz regularity in time). Now, if $2\sigma\leq 1$, we have $\gamma=s-1/2$, and so

$$\big|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)\big|\leq C\left(r^{1/2}+(t-s)+\frac{(t-s)}{r}\right),$$

and so the choice $r\coloneqq(t-s)^{2/3}$ gives

$$\big|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)\big|\leq C(t-s)^{1/3}.$$

Now, if $2\sigma > 1$, we have $\gamma = s - \sigma$ and so

$$\left|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)\right| \le C\left(r^{1-\sigma} + (t-s) + \frac{(t-s)}{r^{2\sigma}}\right),$$

and so the choice $r \coloneqq (t-s)^{1/(1+\sigma)}$ gives

$$\left|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)\right| \le C(t-s)^{\frac{1-\sigma}{1+\sigma}}.$$

Finally, observe that since $s > 1/2$,

$$\frac{1}{3} > \frac{1-s}{1+s} \quad \text{and} \quad \frac{1-\sigma}{1+\sigma} > \frac{1-s}{1+s}. \tag{1.50}$$

□

By (1.8), Lemma 1.30 and interpolation inequality (see (1.9)), we obtain

$$\partial_t u \ \text{ and } \ (-\Delta)^s u \ \in \ C_{t,x}^{\frac{1-s}{1+s}-0^+,1-s}((0,T]\times\mathbb{R}^n). \tag{1.51}$$

The next lemma is the key ingredient to create a bootstrap in Theorem 1.32.

**Lemma 1.31.** *Let $\alpha \in (0, \frac{1-s}{2s})$ and assume*

$$\partial_t u \in C_{t,x}^{\alpha,1-s}((0,T]\times\mathbb{R}^n) \quad \textit{and} \quad (-\Delta)^s u \in L^\infty((0,T];C^{1-s}(\mathbb{R}^n)).$$

*Then, with a uniform bound,*

$$\left[(-\Delta)^s u - \mathcal{R}u\right]\chi_{\{u=\psi\}} \ \in \ C_{t,x}^{(1+\alpha)\frac{1-s}{1+s},1-s}((0,T]\times\mathbb{R}^n).$$

*Proof.* As in the proof of the previous lemma, we first consider $x \in \{u(\tau,\cdot) = \psi\}$ for some $\tau$, thus without loss of generality, $x \in \{u(t,\cdot)=\psi\} \subseteq \{u(s,\cdot) = \psi\}$, where $0 < s < t < T$, and estimate

$$\begin{aligned}
&\left|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)\right| \\
&\le \left|\int_{\mathbb{R}^n} \left[(-\Delta)^s u(t,x) - (-\Delta)^s u(t,z)\right]\phi_r(x-z)\,\mathrm{d}z\right| \\
&+ \left|\int_{\mathbb{R}^n} \left[(-\Delta)^s u(t,z) - (-\Delta)^s u(s,z)\right]\phi_r(x-z)\,\mathrm{d}z\right| \\
&+ \left|\int_{\mathbb{R}^n} \left[(-\Delta)^s u(s,x) - (-\Delta)^s u(s,z)\right]\phi_r(x-z)\,\mathrm{d}z\right|.
\end{aligned}$$

Again, the first and third terms can be controlled by $Cr^{1-s}$. The second term we integrate by parts to obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\big[(-\Delta)^s u(t,z)-(-\Delta)^s u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right|\\&\le\left(\int_{\mathbb{R}^n}\big|\partial_t u(s,z)\big|\,\big|(-\Delta)^s\phi_r(x-z)\big|\,\mathrm{d}z\right)(t-s)\\&+\left|\int_{\mathbb{R}^n}\big[u(t,z)-u(s,z)-\partial_t u(s,z)(t-s)\big]\,(-\Delta)^s\phi_r(x-z)\,\mathrm{d}z\right|.\end{aligned}$$

Since $\partial_t u(\cdot,z)$ is of class $C^\alpha$ in time and $\|(-\Delta)^s\phi_r\|_{L^1(B_r)}\le C/r^{2s}$, the last term on the right hand side above is bounded by $C(t-s)^{1+\alpha}/r^{2s}$. For the integral in the first term, we use that $\partial_t u$ is of class $C^{1-s}$ in space and that $\partial_t u$ vanishes at $(t,x)\in\{u=\psi\}$ to show it is bounded by

$$C\int_{\mathbb{R}^n}\min\{|x-z|^{1-s},1\}\,|(-\Delta)^s\phi_r(x-z)|\,\mathrm{d}z. \tag{1.52}$$

Since $\phi$ is compactly supported, $|(-\Delta)^s\phi(w)|\le C|w|^{-n-2s}$ when $|w|$ is large enough. Hence, scaling yields, for all $w\in\mathbb{R}^n$,

$$|(-\Delta)^s\phi_r(w)|\le\frac{C}{r^{n+2s}+|w|^{n+2s}}.$$

Thus, (1.52) can be controled, up to a constant, by

$$\begin{aligned}\int_{B_1}\frac{|w|^{1-s}}{r^{n+2s}+|w|^{n+2s}}\,\mathrm{d}w+\int_{\mathbb{R}^n\setminus B_1}\frac{1}{|w|^{n+2s}}\,\mathrm{d}w&\\ \le\frac{C}{r^{n+2s}}\int_{B_r}|w|^{1-s}\,\mathrm{d}w&\\ +C\int_{B_1\setminus B_r}|w|^{1-3s-n}\,\mathrm{d}w+C.&\end{aligned}$$

This implies

$$\int_{\mathbb{R}^n}\min\{|x-z|^{1-s},1\}\,|(-\Delta)^s\phi_r(x-z)|\,\mathrm{d}z\le C(1+r^{1-3s}).$$

Finally, we obtain

$$\big|(-\Delta)^s(u(t,x)-u(s,x))\big|\le C\left[r^{1-s}+\frac{(t-s)^{1+\alpha}}{r^{2s}}+C(t-s)(1+r^{1-3s})\right].$$

Also, since $\alpha < (1-s)/2s$, we have

$$\alpha \le \frac{(1-s)(1+\alpha)}{1+s} \le 1 + \frac{(1-3s)(1+\alpha)}{1+s}$$

Therefore, the choice $r \coloneqq (t-s)^{(1+\alpha)/(1+s)}$ ensures

$$|(-\Delta)^s u(t,x) - (-\Delta)^s u(s,x)| \le C(t-s)^{(1+\alpha)\frac{1-s}{1+s}}.$$

Analogously, we estimate

$$\begin{aligned}|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)| \le &\left|\int_{\mathbb{R}^n} \big[\mathcal{R}u(t,x) - \mathcal{R}u(t,z)\big]\phi_r(x-z)\,\mathrm{d}z\right| \\ &+ \left|\int_{\mathbb{R}^n} \big[\mathcal{R}u(t,z) - \mathcal{R}u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right| \\ &+ \left|\int_{\mathbb{R}^n} \big[\mathcal{R}u(s,x) - \mathcal{R}u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z.\right|,\end{aligned}$$

Once again, the first and third integrals are bounded by $Cr^{1-s+\gamma}$. For the second integral, we split the integral into

$$\begin{aligned}C\Bigg(&\left|\int_{\mathbb{R}^n} \big[\mathcal{I}u(t,z) - \mathcal{I}u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right| \\ &+ \left|\int_{\mathbb{R}^n} \big[\nabla u(t,z) - \nabla u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right| \\ &+ \left|\int_{\mathbb{R}^n} \big[u(t,z) - u(s,z)\big]\phi_r(x-z)\,\mathrm{d}z\right|\Bigg).\end{aligned} \tag{1.53}$$

Notice that the third term in (1.53) is Lipschitz-in-time, thus there is nothing to prove. For the remaining terms, we proceed analogously and bound them by

$$\begin{aligned}C(t-s)\Bigg(1 + \frac{(t-s)^\alpha}{r} + \frac{(t-s)^\alpha}{r^{2\sigma}} + \frac{1}{r^{n+1}}\int_{B_r}|w|^{1-s}\,\mathrm{d}w + \int_{B_1\setminus B_r}|w|^{-s-n}\,\mathrm{d}w \\ + \frac{1}{r^{n+2\sigma}}\int_{B_r}|w|^{1-s}\,\mathrm{d}w + \int_{B_1\setminus B_r}|w|^{1-2\sigma-s-n}\,\mathrm{d}w\Bigg).\end{aligned}$$

Now, we estimate the integrals above:

$$\frac{1}{r^{n+1}}\int_{B_r}|w|^{1-s}\,\mathrm{d}w+\int_{B_1\setminus B_r}|w|^{-s-n}\,\mathrm{d}w\le C\left(1+\frac{1}{r^s}\right),$$

$$\frac{1}{r^{n+2\sigma}}\int_{B_r}|w|^{1-s}\,\mathrm{d}w+\int_{B_1\setminus B_r}|w|^{1-2\sigma-s-n}\,\mathrm{d}w\le C\Bigg(\frac{1}{r^{2\nu+s-1}}$$
$$+\begin{cases}1 & \text{if } \sigma<(1-s)/2;\\ 1+|\log(r)| & \text{if } \sigma=(1-s)/2;\\ 1+r^{1-2\sigma-s} & \text{if } \sigma>(1-s)/2.\end{cases}\Bigg)$$

Hence, we conclude that, for $\sigma\ge 0$,

$$|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)|\le Cr^{1-s+\gamma}+C(t-s)\Bigg(\frac{(t-s)^\alpha}{r}+\frac{(t-s)^\alpha}{r^{2\sigma}}+\frac{1}{r^s}$$
$$+\frac{1}{r^{2\sigma+s-1}}+\begin{cases}1 & \text{if } \sigma<(1-s)/2;\\ 1+|\log(r)| & \text{if } \sigma=(1-s)/2;\\ 1+r^{1-2\sigma-s} & \text{if } \sigma>(1-s)/2.\end{cases}\Bigg)$$

If $0\le\sigma<(1-s)/2$, then (recall that $\gamma=s-1/2$)

$$|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)|\le C\left(r^{1/2}+(t-s)+\frac{(t-s)^{1+\alpha}}{r}+\frac{(t-s)}{r^s}\right).$$

Then, by choosing $r:=(t-s)^{2(1+\alpha)/3}$, we have

$$|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)|\le C\left((t-s)^{\frac{1}{3}(1+\alpha)}+(t-s)^{1-\frac{2s}{3}(1+\alpha)}\right).$$

We claim that $\frac{1}{3}(1+\alpha)\le 1-\frac{2s}{3}(1+\alpha)$. Indeed, the claim holds if, and only if $(1+\alpha)(1+2s)\le 3$. Now, since $\alpha<(1-s)/2s$, we have $(1+\alpha)(1+2s)\le\frac{3}{2}+s+\frac{1}{2s}$. Moreover, $s+\frac{1}{2s}\le\frac{3}{2}\iff 2s^2+1-3s<0$, and the latter holds since $1/2<s<1$. Hence, we conclude

$$|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)|\le C(t-s)^{(1+\alpha)/3}.$$

If $(1-s)/2\le\sigma\le 1/2$, then

$$|\mathcal{R}u(t,x)-\mathcal{R}u(s,x)|\le C\Bigg(r^{1/2}+(t-s)\left(1+|\log(r)|\right)$$
$$+\frac{(t-s)^{1+\alpha}}{r}+\frac{(t-s)}{r^s}\Bigg).$$

Once again, by choosing $r \coloneqq (t-s)^{2(1+\alpha)/3}$, we obtain

$$|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)| \le C(t-s)^{(1+\alpha)/3}.$$

Finally, if $\sigma > 1/2$, then

$$|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)| \le C\left(r^{1-\nu} + (t-s) + \frac{(t-s)^{1+\alpha}}{r^{2\sigma}} + \frac{(t-s)}{r^{2\sigma+s-1}}\right).$$

Choosing $r \coloneqq (t-s)^{(1+\alpha)/(1+\sigma)}$, we obtain

$$|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)| \le C\left((t-s)^{(1+\alpha)\frac{1-\sigma}{1+\sigma}} + (t-s)^{1-(1+\alpha)\frac{2\sigma+s-1}{1+\sigma}}\right).$$

We claim that $(1+\alpha)\frac{1-\sigma}{1+\sigma} \le 1-(1+\alpha)\frac{2\sigma+s-1}{1+\sigma}$. Indeed, the claim holds if, and only if $(1+\alpha)(\sigma+s) \le 1+\sigma$. Once again, since $\alpha < (1-s)/2s$, we have $(1+\alpha)(\sigma+2s-1) \le \frac{1+\sigma}{2} + \frac{s}{2} + \frac{\sigma}{2s}$. Moreover, $\frac{s}{2} + \frac{\sigma}{2s} \le \frac{1+\sigma}{2} \iff s^2 - (1+\sigma)s + \sigma \le 0$, and the latter holds since $1/2 < \nu < s$. Hence, we conclude

$$|\mathcal{R}u(t,x) - \mathcal{R}u(s,x)| \le C(t-s)^{(1+\alpha)\frac{1-\sigma}{1+\sigma}}.$$

Thus, by (1.50), we conclude the lemma. □

We are now ready to prove our main regularity result.

**Theorem 1.32.** *Assume that $\psi$, $b$, $r$, and $\mathcal{I}$ as in* (i), (ii), (iii), *and* (iv), *respectively, and let $u$ be the unique (continuous) viscosity solution of* (1.1). *Then $u$ is globally Lipschitz in space-time on $(0,T] \times \mathbb{R}^n$, and satisfies*

$$\begin{cases} \partial_t u \in C_{t,x}^{\frac{1-s}{2s}-0^+,1-s}((0,T]\times\mathbb{R}^n); \\ (-\Delta)^s u \in C_{t,x}^{\frac{1-s}{2s},1-s}((0,T]\times\mathbb{R}^n). \end{cases}$$

*Proof.* The global Lipschitz regularity follows from Corollary 1.18. Given $\alpha \in \left(0, \frac{1-s}{2s}\right)$, denote by $\Phi$ the affine function

$$\Phi(\alpha) \coloneqq (1+\alpha)\frac{1-s}{1+s}$$

which is strictly increasing and satisfies $\Phi(\frac{1-s}{2s}) = \frac{1-s}{2s}$. By (1.51), we can apply Lemma 1.31, which gives

$$\big[(-\Delta)^s u - \mathcal{R}u\big]\chi_{\{u=\psi\}} \in C_{t,x}^{\Phi\left(\frac{1-s}{1+s}-0^+\right),1-s}((0,T]\times\mathbb{R}^n).$$

Then, by (1.8) and interpolation inequality (1.9), we have

$$\partial_t u,\ (-\Delta)^s u \ \in\ C_{t,x}^{\Phi\left(\frac{1-s}{1+s}-0^+\right),1-s}((0,T]\times\mathbb{R}^n),$$

since $\Phi(\alpha) < \frac{1-s}{2s}$ for $\alpha < \frac{1-s}{2s}$. Next, we apply Lemma 1.31 and (1.8) iteratively to obtain

$$\partial_t u,\ (-\Delta)^s u \ \in\ C_{t,x}^{\Phi^n\left(\frac{1-s}{1+s}-0^+\right),1-s}((0,T]\times\mathbb{R}^n),$$

which combined with [7, Estimate A.5] gives

$$(-\Delta)^s u \in C_{t,x}^{\frac{1-s}{2s},1-s}((0,T]\times\mathbb{R}^n).$$

Since $\Phi^n\left(\frac{1-s}{1+s}-0^+\right) \longrightarrow \frac{1-s}{2s}$ as $n\to\infty$, we also conclude

$$\partial_t u \in C_{t,x}^{\frac{1-s}{2s}-0^+,1-s}((0,T]\times\mathbb{R}^n). \qquad \square$$

## 1.A Regularity results for $\partial_t + (-\Delta)^s$ for $f \in L^\infty$

We now adress the approximation of a solution of (1.1) by a solution of (1.13). The main ideas are found in [29]. We already used the regularities (1.39) and (1.8) of the fractional heat equation. However, in both cases the source $f$ is Hölder continuous. Nonetheless, we need a regularity result when $f$ is merely a bounded function in spacetime. Namely, for $\partial_t v + (-\Delta)^s v = f$, we have

$$\|v\|_{C^{1-0^+}((0,T];L^\infty(\mathbb{R}^n))} + \|v\|_{L^\infty((0,T];C^{2s-0^+}(\mathbb{R}^n))} \le C(1+\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)}). \tag{1.54}$$

In order to show (1.54), we proceed as in [7]: we notice that

$$v(t,x) = \Gamma_s(t) * v(0) + \int_0^t \Gamma_s(t-\tau) * f(\tau)\,\mathrm{d}\tau,$$

where $\Gamma_s(t,y)$ is fundamental solution of the fractional heat equation and it behaves as

$$|\Gamma_s(t,y)| \sim \frac{t}{t^{\frac{n+2s}{2s}} + |y|^{n+2s}}, \quad |\partial_t \Gamma_s(t,y)| \le C\frac{1}{t^{\frac{n+2s}{2s}} + |y|^{n+2s}},$$
$$|\nabla_y \Gamma_s(t,y)| \le C\frac{1}{|y|}\frac{t}{t^{\frac{n+2s}{2s}} + |y|^{n+2s}}, \quad |D_y^2 \Gamma_s(t,y)| \le C\frac{1}{|y|^2}\frac{t}{t^{\frac{n+2s}{2s}} + |y|^{n+2s}}. \tag{1.55}$$

Since the initial condition of (1.1) is well-behaved (namely, it satisfies (i)), the first term is smooth. Thus, we only need to estimate the source term. In order to do it, we need the following estimates:

- there exists a constant $C > 0$ such that, for all $h > 0$,

$$\int_{\mathbb{R}^n} \frac{h}{h^{\frac{n+2s}{2s}} + |z|^{n+2s}} \, \mathrm{d}z \leq C(1+h); \tag{1.56}$$

- there exists a constant $C > 0$ such that, for all $h > 0$,

$$\int_0^t \frac{t-\tau}{(t-\tau)^{\frac{n+2s}{2s}} + h^{n+2s}} \, \mathrm{d}\tau \leq C \min\{h^{-n-2s}, h^{-n+2s}\}. \tag{1.57}$$

The proof of both are very simple: for (1.56), one splits the integral into $B_{h^{1/2s}}$ and $\mathbb{R}^n \setminus B_{h^{1/2s}}$, thus

$$\int_{\mathbb{R}^n} \frac{h}{h^{\frac{n+2s}{2s}} + |z|^{n+2s}} \, \mathrm{d}z \leq \frac{1}{h^{n/2s}} |B_{h^{1/2s}}| + h \int_{\mathbb{R}^n \setminus B_{h^{1/2s}}} \frac{1}{|z|^{n+2s}} \, \mathrm{d}z \leq C(1+h).$$

For (1.57), if $h \geq 1$, the bound is trivial, since the integrand is bounded by $h^{-n-2s}$; otherwise, if $h \in (0,1]$, we split the integral into $[0, t-h^{2s}]$ and $[t-h^{2s}, t]$, thus for $n \geq 2$ we have

$$\begin{aligned}\int_0^t \frac{t-\tau}{(t-\tau)^{\frac{n+2s}{2s}} + h^{n+2s}} \, \mathrm{d}\tau &\leq \int_0^{t-h^{2s}} (t-\tau)^{-n/2s} \, \mathrm{d}\tau + \frac{1}{h^{n+2s}} \int_{t-h^{2s}}^t (t-\tau) \, \mathrm{d}\tau \\ &\leq Ch^{-n+2s}.\end{aligned}$$

For the time regularity of (1.54), notice that for $u < t$, we have

$$\begin{aligned}|v(t) - v(u)| \leq C \Bigg( &\int_u^t |\Gamma_s(t-\tau) * f(\tau)| \, \mathrm{d}\tau \\ &+ \int_0^u |(\Gamma_s(t-\tau) - \Gamma_s(u-\tau)) * f(\tau)| \, \mathrm{d}\tau \Bigg).\end{aligned}$$

By (1.55) and (1.56) with $h = t - \tau$, the first term can be bounded by

$$C\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)} \int_u^t [1 + (t-\tau)] \, \mathrm{d}\tau \leq C\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)} (t-u),$$

while the second term can be bounded by

$$
\begin{aligned}
&C\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)}\int_0^u \min\{t-u,u-\tau\}(1+(u-\tau)^{-1})\,\mathrm{d}\tau\\
&\le C\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)}\Bigg((t-u)\int_0^{u-(t-u)}(1+(u-\tau)^{-1})\,\mathrm{d}\tau\\
&+\int_{u-(t-u)}^u (1+(u-\tau))\,\mathrm{d}\tau\Bigg)\\
&\le C\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)}(t-u)|\log(t-u)|.
\end{aligned}
$$

For the space regularity of (1.54), since $2s>1$, we evaluate

$$
\begin{aligned}
&|\nabla v(x)-\nabla v(z)|\\
&\le C\|f\|_{L^\infty((0,T]\times\mathbb{R}^n)}\int_0^t\int_{\mathbb{R}^n}|\nabla\Gamma_s(t-\tau,x-y)-\nabla\Gamma_s(t-\tau,z-y)|\,\mathrm{d}y\,\mathrm{d}\tau.
\end{aligned}
$$

We split the space integral into $\{|x-z|\le|x-y|/2\}$ and $\{|x-z|\ge|x-y|/2\}$. For the first region, by (1.55) and (1.57) with $h=|x-y|$, we can bound the integral by

$$
\begin{aligned}
&C|x-z|\int_{\{|x-z|\le|x-y|/2\}}|x-y|^{-2}\min\{|x-y|^{-n-2s},|x-y|^{-n+2s}\}\,\mathrm{d}y\\
&\le C|x-z|^{1+2s-2}\int_{\{|x-z|\le|x-y|/2\le 1\}}|x-y|^{-n}\,\mathrm{d}y\\
&+C|x-z|\int_{\{|x-y|/2\ge 1\}}|x-y|^{-2-n-2s}\,\mathrm{d}y\\
&\le C|x-z|^{2s-1}\left|\log|x-z|\right|,
\end{aligned}
$$

while For the second region, by (1.55), (1.57) with $h=|x-y|$ and noticing that $\{|x-z|\ge|x-y|/2\}\subset B_{3|x-z|}(x)\cap B_{3|x-z|}(z)$, we can bound the integral by

$$
\int_{B_{3|x-z|}(x)}\frac{1}{|x-y|}|x-y|^{-n+2s}\,\mathrm{d}y\le|x-z|^{2s-1}.
$$

# Chapter 2

# Lagrangian structure of relativistic Vlasov systems

As already mentioned in the Introduction, we are interested in the Lagrangian structure of solutions of (7), the existence of generalized solutions under minimal assumptions, and the existence of renormalized solutions if the initial condition has bounded initial energy and a higher integrability.

For our purposes, a crucial observation is that (7) can be written as

$$\partial_t f_t + \mathbf{b}_t \cdot \nabla_{x,y} f_t = 0, \tag{2.1}$$

where, for each fixed $t > 0$, the vector field $\mathbf{b}_t : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ is given by $\mathbf{b}_t(x,v) = (\hat{v}, E_t + \hat{v} \times B_t)$. Moreover, the vector field is divergence-free, since

$$\nabla_{x,v} \cdot \mathbf{b}_t = \nabla_v \cdot (\hat{v} \times B_t) = (\nabla_v \times \hat{v}) \cdot B_t - \hat{v} \cdot (\nabla_v \times B_t) = 0.$$

By the transport nature of (2.1), it is expected that solutions have a Lagrangian structure, meaning that the initial condition $f_0$ is transported to $f_t$ by an associated flow. In the weak regularity regime, however, the existence of such flow is not guaranteed by the classical Cauchy-Lipschitz theory. Indeed, since $K$ is locally integrable, we have $E_t$, $B_t \in L^1_{\text{loc}}(\mathbb{R}^3;\mathbb{R}^3)$ whenever $f_t \in L^1(\mathbb{R}^6)$, so that $\mathbf{b}_t$ is only in $L^1_{\text{loc}}(\mathbb{R}^6;\mathbb{R}^6)$.

Since $\mathbf{b}_t$ is divergence-free, (2.1) can be rewritten as

$$\partial_t f_t + \nabla_{x,y} \cdot (\mathbf{b}_t f_t) = 0.$$

The latter can be interpreted in the distributional sense provided $\mathbf{b}_t f_t$ is locally integrable which, however, does not follow only from the assumption

$f_t \in L^1(\mathbb{R}^6)$. To treat this problem, we introduce a function $\beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\partial_t \beta(f_t) + \nabla_{x,v} \cdot (\mathbf{b}_t \beta(f_t)) = 0 \tag{2.2}$$

whenever $f_t$ is a smooth solution of (2.1). Hence, $\mathbf{b}_t \beta(f_t) \in L^1_{\text{loc}}$, which leads to the concept of a renormalized solution; see [14].

**Definition 2.1** (Renormalized solution)**.** For a Borel vector field

$$\mathbf{b} \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^6; \mathbb{R}^6),$$

we say that a Borel function $f \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^6)$ is a renormalized solution of (2.1) starting from $f_0$ if (2.2) holds in the sense of distributions, that is,

$$\begin{aligned} &\int_{\mathbb{R}^6} \phi_0(x,v) \beta(f_0(x,v))\, \mathrm{d}x\, \mathrm{d}v \\ &+ \int_0^T \int_{\mathbb{R}^6} \Big[\partial_t \phi_t(x,v) + \nabla_{x,v} \phi_t(x,v) \cdot \mathbf{b}_t(x,v)\Big] \beta(f_t(x,v))\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t = 0 \end{aligned} \tag{2.3}$$

for all $\phi \in C^1_c([0,T) \times \mathbb{R}^6)$ and $\beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Moreover, $f \in L^\infty((0,T); L^1(\mathbb{R}^6))$ is called a renormalized solution of (7) starting from $f_0$ if, by setting

$$\rho_t(x) \coloneqq \int_{\mathbb{R}^3} f_t(x,v)\, \mathrm{d}v, \quad E_t(x) \coloneqq \sigma_E \int_{\mathbb{R}^3} \rho_t(y)\, K(x-y)\, \mathrm{d}y,$$

$$J_t(x) \coloneqq \int_{\mathbb{R}^3} \hat{v} f_t(x,v)\, \mathrm{d}v, \quad B_t(x) \coloneqq \sigma_B \int_{\mathbb{R}^3} J_t(y) \times K(x-y)\, \mathrm{d}y, \tag{2.4}$$

$$\text{and} \quad \mathbf{b}_t(x,v) \coloneqq (\hat{v}, E_t(x) + \hat{v} \times B_t(x)), \tag{2.5}$$

we have that $f_t$ satisfies (2.3), for every $\phi \in C^1_c([0,T) \times \mathbb{R}^6)$, with $\mathbf{b}_t$ as in (2.5).

Observe that the integrability assumption $f_t \in L^1(\mathbb{R}^6)$ is used so that $\rho_t$, $J_t$, $E_t$, and $B_t$ are well defined. From now on, we refer to $E_t$ and $B_t$ as the electric and the magnetic fields, respectively, even though $E_t$ may represent a gravitational field as well.

Our first main result shows that distributional or renormalized solutions of (7) are in fact Lagrangian solutions. This gives a characterization of solutions of (7), since Lagrangian solutions are generally stronger than renormalized or distributional solutions.

**Theorem 2.2.** *Let $T > 0$ and $f$ be a nonnegative function. Assume $f \in L^\infty([0,T); L^1(\mathbb{R}^6))$ is weakly continuous in the sense that*

$$t \longmapsto \int_{\mathbb{R}^6} f_t \, \varphi \, \mathrm{d}x \, \mathrm{d}v \quad \text{is continuous for any } \varphi \in C_c(\mathbb{R}^6).$$

*Assume further that:*

- $(i)$ *either $f \in L^\infty((0,T); L^\infty(\mathbb{R}^6))$ and $f$ is a distributional solution of (7) starting from $f_0$; or*

- $(ii)$ *$f$ is a renormalized solution of (7) starting from $f_0$.*

*Then, $f$ is a Lagrangian solution transported by the Maximal Regular Flow $\boldsymbol{X}(t,x)$ associated to $\boldsymbol{b}_t(x,v) = (\hat{v}, E_t(x) + \hat{v} \times B_t(x))$ (see Definition 2.9), starting from $0$. In particular, $f_t$ is renormalized.*

Next, in Definition 2.11, we introduce the concept of generalized solutions, which allows the electromagnetic field to be generated by effective densities $\rho^{\text{eff}}$ and $J^{\text{eff}}$. This may be interpreted as particles vanishing from the phase space but still contributing in the electromagnetic field in the physical space. In fact, generalized solutions are renormalized if the number of particles is conserved in time, as follows from Lemma 2.12. This indicates that, should renormalized solutions fail to exist, there must be a loss of mass/charge as $\hat{v}$ approaches the speed of light (we recall that the speed of light $c = 1$).

Our second main theorem provides, under minimal assumptions on the initial datum, the global existence of generalized solutions.

**Theorem 2.3** (Existence of generalized solutions)**.** *Let $f_0 \in L^1(\mathbb{R}^6)$ be a nonnegative function. Then there exists a generalized solution $(f_t, \rho_t^{\text{eff}}, J_t^{\text{eff}})$ of (7) starting from $f_0$ (see Definition 2.11)). Moreover, the map*

$$t \in [0,\infty) \longmapsto f_t \in L^1_{\text{loc}}(\mathbb{R}^6)$$

*is continuous and the solution $f_t$ is transported by the Maximal Regular Flow associated to field $\boldsymbol{b}_t^{\text{eff}}(x,v) = (\hat{v}, E_t^{\text{eff}} + \hat{v} \times B_t^{\text{eff}})$.*

In view of Theorem 2.3, if we assume higher integrability on the initial datum and bounded initial energy, we can prove the existence of a global Lagrangian solution. Moreover, we show strong continuity of densities and fields and that each energy remains bounded in later times. Furthermore, we emphasize that our result holds even in the gravitational case $\sigma_E = -1$.

**Theorem 2.4** (Existence of global Lagrangian solution)**.** *Let $f_0$ be a nonnegative function with every energy bounded (see Definition 2.15). Then there exists a global Lagrangian (hence renormalized) solution*

$$f_t \in C([0, \infty); L^1(\mathbb{R}^6))$$

*of (7) with initial datum $f_0$, and the flow is globally defined on $[0, \infty)$ for $f_0$-almost every $(x, v) \in \mathbb{R}^6$, with $f_t$ being the image of $f_0$ through the incompressible flow.*

*Moreover, the following properties hold:*

*(i) the densities $\rho_t$, $J_t$ and the fields $E_t$, $B_t$ are strongly continuous in $L^1_{\mathrm{loc}}(\mathbb{R}^6)$;*

*(ii) for every $t \geq 0$, $f_t$ has every energy bounded independently of time.*

The chapter is organized as follows. In Section 2.1, we prove Theorem 2.2. More explicitly, we rely on the machinery for nonsmooth vector fields developed in [3] to prove the equivalence of renormalized and Lagrangian solutions. Moreover, in Corollary 2.10, we show that if the electromagnetic and relativistic energies are integrable in $[0, T]$, then its associated flow is globally defined in time. In Section 2.2, we extend the notion of generalized solutions from [4, Definition 2.6] to our setting (see Definition 2.11) in order to allow an "effective" density current of particles (along with the corresponding "effective" density of particles) and we prove the existence of a Lagrangian solution with the "effective" acceleration (Theorem 2.3). Finally, in Section 2.3, we prove Theorem 2.4 under the condition of each bounded energy (see Definition 2.15), obtaining a globally defined flow and a solution of (7) for all range of $\sigma_E$ and $\sigma_B$.

## 2.1 Lagrangian solution and associated flow

In this section, we prove Theorem 2.2 which says that Lagrangian and renormalized solutions of (7) are equivalent. For this, we use the machinery developed in [4, Sections 4 and 5] combined with a version of [4, Theorem 4.4] that we show holds for our vector field $\mathbf{b}$ as well. From now on, we denote by $\mathscr{M}$ the space of measures with finite total mass, by $\mathscr{M}_+$ the space of nonnegative measures in $\mathbb{R}^3$ with finite total mass, by $\mathrm{AC}(I; \mathbb{R}^6)$ the space of absolutely continuous curves on the interval $I$ with values in $\mathbb{R}^6$, and by

$\mathscr{L}^6$ the Lebesgue measure in $\mathbb{R}^6$. We begin with the preliminary definitions of renormalized solutions, and of regular and maximum regular flows:

**Definition 2.5** (Regular flow)**.** Fix $\tau_1 < \tau_2$ and $B \subseteq \mathbb{R}^6$ a Borel set. For a Borel vector field $\mathbf{b} : (\tau_1, \tau_2) \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$, we say that $\mathbf{X} : [\tau_1, \tau_2] \times B \longrightarrow \mathbb{R}^6$ is a regular flow with vector $\mathbf{b}$ when

$(i)$ for a.e. $x \in B$, we have that $\mathbf{X}(\cdot, x) \in \mathrm{AC}([\tau_1, \tau_2]; \mathbb{R}^6)$ and that it solves the equation $\dot{x}(t) = \mathbf{b}_t(x(t))$ a.e. in $(\tau_1, \tau_2)$ with initial condition $\mathbf{X}(x, \tau_1) = x$;

$(ii)$ there exists $C > 0$ such that $\mathbf{X}(t, \cdot)_{\#}(\mathscr{L}^6 \llcorner B) \leq C\mathscr{L}^6$ for all $t \in [\tau_1, \tau_2]$. Note that $C$ can depend on the particular flow $\mathbf{X}$.

Here, we denote $\boldsymbol{X}_{\#}\mu$ as the pushforward of a measure $\mu$ by $\boldsymbol{X}$ and $\nu \llcorner B$ as the measure $\nu$ restricted to the set $B$.

**Definition 2.6** (Maximum regular flow)**.** For every $s \in (0, T)$, a Borel map $\mathbf{X}(\cdot, s, \cdot) : (T^-_{s,\mathbf{X}}, T^+_{s,\mathbf{X}}) \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ is said to be a maximum regular flow (starting at $s$) if there exist two Borel maps $T^+_{s,\mathbf{X}} : \mathbb{R}^6 \longrightarrow (s, T]$, $T^-_{s,\mathbf{X}} : \mathbb{R}^6 \longrightarrow [0, s)$ such that $\mathbf{X}(\cdot, s, x)$ is defined in $(T^-_{s,\mathbf{X}}(x), T^+_{s,\mathbf{X}}(x))$ and

$(i)$ for a.e. $x \in \mathbb{R}^6$, we have that $\mathbf{X}(\cdot, s, x) \in \mathrm{AC}((T^-_{s,\mathbf{X}}, T^+_{s,\mathbf{X}}); \mathbb{R}^6)$ and that it solves the equation $\dot{x}(t) = \mathbf{b}_t(x(t))$ a.e. in $(T^-_{s,\mathbf{X}}, T^+_{s,\mathbf{X}})$ with $\mathbf{X}(s, s, x) = x$;

$(ii)$ there exists a constant $C > 0$ such that $\mathbf{X}(t, s, \cdot)_{\#}(\mathscr{L}^6 \llcorner \{T^-_{s,\mathbf{X}} < t < T^+_{s,\mathbf{X}}\}) \leq C\mathscr{L}^6$ for all $t \in [0, T]$. As berfore, this constant $C$ can depend on $\mathbf{X}$ and $s$;

$(iii)$ for a.e. $x \in \mathbb{R}^6$, either $T^+_{s,\mathbf{X}} = T$ and $\mathbf{X}(\cdot, s, x) \in C([s, T]; \mathbb{R}^6)$, or $\lim_{t \uparrow T^+_{s,\mathbf{X}}} |\mathbf{X}(t, s, x)| = \infty$. Analogousy, either $T^-_{s,\mathbf{X}} = 0$ and $\mathbf{X}(\cdot, s, x) \in C([0, s]; \mathbb{R}^6)$, or $\lim_{t \downarrow T^-_{s,\mathbf{X}}} |\mathbf{X}(t, s, x)| = \infty$.

The following lemma (compare with [4, Theorem 4.4]), combined with the facts that $\mathbf{b}_t$ is divergence-free in the sense of distribution a.e. in time and that $\boldsymbol{b} \in L^\infty((0, T); L^1_{\mathrm{loc}}(\mathbb{R}^6; \mathbb{R}^6))$, provides a sufficient condition to the existence and the uniqueness of a maximum regular flow for the continuity equation. From now on, a convolution $f * \mu$, where $f$ is a function and $\mu$ is a measure should be understood in the sense that $f * \mu(x) := \int f(x-y)\,\mathrm{d}\mu(y)$. We remark that if $\mu(\mathbb{R}^3) < \infty$, then (recall that $K(x) = \frac{x}{4\pi|x|}$)

$$\int_{B_R} |K * \mu(x)|\,\mathrm{d}x < \infty \quad \forall R > 0.$$

**Lemma 2.7.** *Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ be given by*

$$\boldsymbol{b}_t(x,v) = (\boldsymbol{b}_{1t}(v), \boldsymbol{b}_{2t}(x,v)),$$

*where*

$$\begin{aligned} \boldsymbol{b}_1 &\in L^\infty((0,T); W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)), \\ \boldsymbol{b}_{2t}(x,v) &= K * \rho_t(x) + \boldsymbol{b}_{1t}(v) \times \int_{\mathbb{R}^3} K(y-x) \times \mathrm{d}J_t(y) \\ &=: K * \rho_t(x) + \boldsymbol{b}_{1t}(v) \times \tilde{\boldsymbol{b}}_{2t}(x), \end{aligned}$$

*with $\rho \in L^\infty((0,T); \mathscr{M}_+(\mathbb{R}^3))$ and $|J| \in L^\infty((0,T); \mathscr{M}_+(\mathbb{R}^3))$. Then, $\boldsymbol{b}$ satisfies the following: for any nonnegative $\bar{\rho} \in L^\infty(\mathbb{R}^3)$ with compact support and any closed interval $[a,b] \subset [0,T]$, both continuity equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t \pm \nabla_{x,v} \cdot (\boldsymbol{b}_t \rho_t) = 0 \quad \text{in}\,(a,b) \times \mathbb{R}^6$$

*have at most one solution in the class of all weakly* nonnegative continuous functions $[a,b] \ni t \longrightarrow \tilde{\rho}_t$ with $\rho_a = \bar{\rho}$ and $\cup_{t \in [a,b]} \operatorname{supp} \rho_t \Subset \mathbb{R}^6$.*

*Proof.* Let $\mathscr{P}(X)$ be the set of probability measures on $X$ and

$$e_t : C([0,T]; \mathbb{R}^6) \longrightarrow \mathbb{R}^6$$

the evaluation map at time $t$, which means $e_t(\eta) := \eta(t)$. By the same argument as in [4, Theorem 4.4], it is enough to show that given $\boldsymbol{\eta} \in \mathscr{P}(C([0,T]; B_R \times B_R))$ for some $R > 0$ concentrated on integral curves of $\mathbf{b}$ such that $(e_t)_\# \boldsymbol{\eta} \le C_0 \mathscr{L}^6$ for all $t \in [0,T]$, the disintegration $\boldsymbol{\eta}_x$ of $\boldsymbol{\eta}$ with respect to $e_0$ is a Dirac delta for $\bar{\rho}$-a.e. $x$. Recall that the disintegration of $\boldsymbol{\eta}$ with respect to $e_0$ is a family of measures $\boldsymbol{\eta}_x$ such that, for all $E \in C([0,T]; B_R \times B_R)$,

$$\boldsymbol{\eta}(E) = \int_{\mathbb{R}^6} \boldsymbol{\eta}_x(E \cap e_0^{-1}(x))\,\mathrm{d}x.$$

For this purpose, the authors of [4] consider the function

$$\Phi_{\delta,\zeta}(t) \coloneqq \iiint \log\left(1 + \frac{|\gamma^1(t) - \eta^1(t)|}{\zeta\delta} + \frac{|\gamma^2(t) - \eta^2(t)|}{\delta}\right) \mathrm{d}\mu(x, \eta, \gamma),$$

where $\delta$, $\zeta \in (0, 1)$ are small constants to be chosen later, $t \in [0, T]$, $\bar{\rho} \coloneqq (e_0)_{\#}\mathrm{d}\boldsymbol{\eta}$, $\mathrm{d}\mu(x, \eta, \gamma) \coloneqq \mathrm{d}\boldsymbol{\eta}_x(\gamma)\mathrm{d}\boldsymbol{\eta}_x(\eta)\mathrm{d}\bar{\rho}(x)$[1], with notation

$$\eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

and assume by contradiction that $\boldsymbol{\eta}_x$ is not a Dirac delta for $\bar{\rho}$-a.e. $x$, which means that there exists a constant $a > 0$ such that

$$\iiint \left(\int_0^T \min\{|\gamma(t) - \eta(t)|, 1\}\mathrm{d}t\right) \mathrm{d}\mu(x, \eta, \gamma) \geq a.$$

Indeed, if $\boldsymbol{\eta}_x$ is a Dirac delta for $\bar{\rho}$-a.e. $x$, the integrand above would vanish.

Moreover, they show that, without loss of generality, by assuming $a \leq 2T$, it holds

$$\Phi_{\delta,\zeta}(t_0) \geq \frac{a}{2T} \log\left(1 + \frac{a}{2\delta T}\right). \tag{2.6}$$

Now, computing the time derivative of $\Phi_{\delta,\zeta}$, we have that

$$\begin{aligned} \frac{\mathrm{d}\Phi_{\delta,\zeta}}{\mathrm{d}t}(t) \leq \iiint \Bigg( & \frac{|\mathbf{b}_{1t}(\gamma^2(t)) - \mathbf{b}_{1t}(\eta^2(t))|}{\zeta(\delta + |\gamma^2(t) - \eta^2(t)|)} \\ & + \frac{\zeta|\mathbf{b}_{1t}(\gamma^2(t)) \times (\tilde{\mathbf{b}}_{2t}(\gamma^1(t)) - \tilde{\mathbf{b}}_{2t}(\eta^1(t)))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \\ & + \frac{\zeta|(\mathbf{b}_{1t}(\gamma^2(t)) - \mathbf{b}_{1t}(\eta^2(t))) \times \tilde{\mathbf{b}}_{2t}(\eta^1(t))|}{\zeta\delta + |\gamma^2(t) - \eta^2(t)|} \\ & + \frac{\eta|K * \rho_t(\gamma^1(t)) - K * \rho_t(\eta^1(t))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|)}\Bigg)\mathrm{d}\mu(x, \eta, \gamma). \end{aligned} \tag{2.7}$$

By our assumption on $\mathbf{b}_{1t}$, the first summand is easily estimated using the Lipschitz regularity of $\mathbf{b}_{1t}$ in $B_R$:

$$\iiint \frac{|\mathbf{b}_{1t}(\gamma^2(t)) - \mathbf{b}_{1t}(\eta^2(t))|}{\zeta(\delta + |\gamma^2(t) - \eta^2(t)|)} \mathrm{d}\mu(x, \eta, \gamma) \leq \frac{\|\nabla \mathbf{b}_1\|_{L^\infty((0,T)\times B_R)}}{\zeta}. \tag{2.8}$$

[1] Note that $\mu \in \mathscr{P}(\mathbb{R}^3 \times C([0, T]; \mathbb{R}^3)^2)$ and $\Phi_{\delta,\zeta}(0) = 0$.

Analogously, the third summand is estimated using that $\tilde{\mathbf{b}}_2$ is locally integrable and the Lipschitz regularity of $\mathbf{b}_1$ in $B_R$:

$$\begin{aligned}\iiint \frac{\zeta|(\mathbf{b}_{1t}(\gamma^2(t)) - \mathbf{b}_{1t}(\eta^2(t))) \times \tilde{\mathbf{b}}_{2t}(\eta^1(t))|}{\zeta\delta + |\gamma^2(t) - \eta^2(t)|} \mathrm{d}\mu(x,\eta,\gamma) \\ \leq \zeta\|\nabla \mathbf{b}_1\|_{L^\infty((0,T)\times B_R)}\|\tilde{\mathbf{b}}_2\|_{L^\infty((0,T);L^1(B_R))}.\end{aligned} \tag{2.9}$$

For the second term, we have

$$\begin{aligned}&\iiint \frac{\zeta|\mathbf{b}_{1t}(\gamma^2(t)) \times (\tilde{\mathbf{b}}_{2t}(\gamma^1(t)) - \tilde{\mathbf{b}}_{2t}(\eta^1(t)))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(x,\eta,\gamma) \\ &\qquad \leq C\|\mathbf{b}_1\|_{L^\infty((0,T)\times B_R)} \iiint \frac{\zeta|K * \tilde{\rho}_t(\gamma^1(t)) - K * \tilde{\rho}_t(\eta^1(t))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(x,\eta,\gamma),\end{aligned}$$

where $\tilde{\rho}_t(y) \coloneqq \sup_i |J_i|_t(y)$. Since $J_i \in L^\infty((0,T);\mathscr{M}(\mathbb{R}^3))$, its total variation is well-defined and has finite measure, thus

$$\tilde{\rho} \in L^\infty((0,T);\mathscr{M}_+(\mathbb{R}^3)).$$

By [4, Theorem 4.4, estimate (4.13)], we have that[2]

$$\iiint \frac{\zeta|K * \bar{\rho}_t(\gamma^1(t)) - K * \bar{\rho}_t(\eta^1(t))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(x,\eta,\gamma) \leq C\zeta\left(1 + \log\left(\frac{C}{\zeta\delta}\right)\right), \tag{2.10}$$

where $\bar{\rho} \in L^\infty((0,\infty);\mathscr{M}_+(\mathbb{R}^3))$ and $C$ depends only on $\sup_{t\in(0,T)} \bar{\rho}_t(\mathbb{R}^3)$ and $R$. Hence, the second and fourth terms can be estimated by (2.10).

Then, using (2.8), (2.9), and (2.10), one can integrate (2.7) with respect to time in $[0,t_0]$ to obtain

$$\frac{d\Phi_{\delta,\zeta}}{dt}(t_0) \leq Ct_0\left(\frac{1}{\zeta} + \zeta + \zeta\log\left(\frac{C}{\zeta}\right) + \zeta\log\left(\frac{1}{\delta}\right)\right),$$

where $C$ is a constant depending only on $R$, $\sup_{t\in(0,T)} \rho_t(\mathbb{R}^3)$, $\sup_{t\in(0,T)} \tilde{\rho}_t(\mathbb{R}^3)$, and $\|\mathbf{b}_1\|_{L^\infty((0,T);W^{1,\infty}(B_R))}$. Choosing first $\zeta > 0$ small enough in order to have $Ct_0\zeta < a/(2T)$ and then letting $\delta \longrightarrow 0$, we find a contradiction with (2.6), concluding the proof. □

[2]As mentioned by the authors, although their proof is in an autonomous setting, the result also holds for $\bar{\rho}_t$.

As mentioned before, by [3, Theorems 5.7, 6.1, 7.1], we obtain existence, uniqueness, and a semigroup property for the maximum regular flow (for a concise statement, see [4, Theorem 4.3]). We now define generalized flow (analogous to Definition 2.5) and Lagrangian solutions. For this, we define $\bar{\mathbb{R}}^6 = \mathbb{R}^6 \cup \{\infty\}$, and given a open set $A \subset [0, \infty)$, $\mathrm{AC}_{\mathrm{loc}}(A; \mathbb{R}^6)$ the set of continuous curves $\gamma : A \longrightarrow \mathbb{R}$ that are absolutely continuous when restricted to any closed interval in $A$.

**Definition 2.8** (Generalized flow)**.** For a Borel vector field $\boldsymbol{b} : (0, T) \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$, the measure $\boldsymbol{\eta} \in \mathscr{M}_+(C[0, T]; \bar{\mathbb{R}}^6)$ is said to be a generalized flow of $\boldsymbol{b}$ if $\boldsymbol{\eta}$ is concentrated on the (Borel) set

$$\begin{aligned} \Gamma \coloneqq \{\eta \in C([0, T]; \bar{\mathbb{R}}^6) : \eta \in \mathrm{AC}_{\mathrm{loc}}(\{\eta \neq \infty\}; \mathbb{R}^6) \\ \text{and } \dot{\eta}(t) = \boldsymbol{b}_t(\eta(t)) \text{ for a.e. } t \in \{\eta(t) \neq \infty\}\}. \end{aligned}$$

The generalized flow is regular if there exists $C \geq 0$ such that

$$(e_t)_{\#}\boldsymbol{\eta} \llcorner \mathbb{R}^6 \leq C \mathscr{L}^6 \quad \forall t \in [0, T].$$

**Definition 2.9** (Transported measures and Lagrangian solutions)**.** Let $\boldsymbol{b} : (0, T) \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ be a Borel vector field having a maximal regular flow $\boldsymbol{X}$, and $\boldsymbol{\eta} \in \mathscr{M}_+(C[0, T]; \bar{\mathbb{R}}^6)$ with $(e_t)_{\#}\boldsymbol{\eta} \ll \mathscr{L}^6$ for all $t \in [0, T]$. We say that $\boldsymbol{\eta}$ is transported by $\boldsymbol{X}$ if, for all $s \in [0, T]$, $\boldsymbol{\eta}$ is concentrated on

$$\begin{aligned} \{\eta \in C([0, T]; \bar{\mathbb{R}}^6) : \eta(s) = \infty \text{ or } \dot{\eta}(\cdot) = \boldsymbol{X}(\cdot, s, \eta(s)) \\ \text{in } (T^-_{s,\mathbf{X}}(\eta(s)), T^+_{s,\mathbf{X}}(\eta(s)))\}. \end{aligned}$$

Moreover, let $\rho \in L^\infty((0, T); L^1(\mathbb{R}^6))$ be a nonnegative distributional solution of the continuity equation, weakly continuous on $[0, T]$ in the sense that for all $\varphi \in C_c(\mathbb{R}^6)$, the map $t \mapsto \int \varphi \rho_t$ is weakly continuous. We say that $\rho_t$ is a Lagrangian solution if there exists $\eta \in \mathscr{M}_+(C([0, T]; \bar{\mathbb{R}}^6))$ transported by $\boldsymbol{X}$ with $(e_t)_{\#}\boldsymbol{\eta} = \rho_t \mathscr{L}^6$ for every $t \in [0, T]$.

By [4, Theorem 4.7], we have that for $\boldsymbol{b}$ as in Lemma 2.7, regular generalized flows are transported by its maximal regular flow $\boldsymbol{X}$. We are now ready to prove Theorem 2.2.

*Proof of Theorem 2.2.* We have already established that the vector field $\boldsymbol{b}$ satisfies

$$\boldsymbol{b} \in L^\infty((0, T); L^1_{\mathrm{loc}}(\mathbb{R}^6; \mathbb{R}^6)),$$

(see the introduction of Chapter 2) is divergence-free, and satisfies the uniqueness of bounded compactly supported nonnegative distributional solutions of the continuity equation (see Lemma 2.7). Therefore by [4, Theorem 5.1], we deduce that: if (i) holds, then $f$ is a Lagrangian solution; if (ii) holds and it is not bounded, then $\beta(f_t)$ is a Lagrangian solution, where we choose $\beta(s) := \arctan(s)$. In particular, by [4, Theorem 4.10] we have that $f_t$ is a renormalized solution. □

We have a direct corollary that provides conditions to obtain a globally defined flow, that is, to avoid a finite-time blow up.

**Corollary 2.10.** *Fix $T > 0$ and let $f \in L^\infty((0,T); L^1(\mathbb{R}^6))$ be a nonnegative renormalized solution of (7) (as in Definition 2.1). Assume that*

$$\int_0^T \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{2}|E_t|^2 + \tfrac{1}{2}|B_t|^2\,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{2.11}$$

*that is, the relativistic energy and the electromagnetic energy (7) are integrable in time.*

*Then*

*(i) The maximal regular flow $\boldsymbol{X}(t,\cdot)$ associated to $\boldsymbol{b}_t = (\hat{v}, E_t + \hat{v} \times B_t)$ and starting from 0 is globally defined on $[0,T]$ for $f_0$-a.e. $(x,v)$;*

*(ii) $f_t$ is the image of $f_0$ through this flow, that is, $f_t = \boldsymbol{X}(t,\cdot)_\# f_0 = f_0 \circ \boldsymbol{X}^{-1}(t,\cdot)$ for all $t \in [0,T]$:*

$$\int_{\mathbb{R}^6} \phi(x,v) f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^6} \phi\left(\boldsymbol{X}(t,x,v)\right) f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v$$

*for all $\phi \geq 0$, $t \in [0,T]$;*

*(iii) the map*

$$[0,T] \ni t \longmapsto \int_{\mathbb{R}^6} \psi(f_t(x,v))\,\mathrm{d}x\,\mathrm{d}v$$

*is constant in time for all Borel $\psi : [0,\infty) \longrightarrow [0,\infty)$.*

*Proof.* Thanks to Theorem 2.2, the solution is transported by the maximal regular flow associated to $\boldsymbol{b}_t = (v, v \times B_t)$. Moreover, since $f_t$ is renormalized,

$g_t \coloneqq \frac{2}{\pi}\arctan f_t : (0,T)\times\mathbb{R}^3 \longrightarrow [0,1]$ is a solution of the continuity equation with vector field $\boldsymbol{b}$. Since $g_t^2 \le g_t \le f_t$ and $|\hat{v}| < 1$, we have

$$
\begin{aligned}
I &\coloneqq \int_0^T \int_{\mathbb{R}^6} \frac{|\boldsymbol{b}_t(x,v)| g_t(x,v)}{(1+(|x|^2+|v|^2)^{1/2})\log(2+(|x|^2+|v|^2)^{1/2})} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&\le C\int_0^T \int_{\mathbb{R}^6} f_t \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^6} \frac{(|E_t|+|B_t|)g_t}{(1+|v|)\log(2+|v|)} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\
&\le \left(\int_{\mathbb{R}^3} \frac{1}{(1+|v|)^3\log^2(2+|v|)} \mathrm{d}v\right)\left(\int_0^T \int_{\mathbb{R}^3} |E_t|^2 + |B_t|^2 \,\mathrm{d}x\,\mathrm{d}t\right) \\
&+ C\int_0^T \int_{\mathbb{R}^6} (1+|v|) f_t \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.
\end{aligned}
$$

By (2.11) and $(1+|v|) \le \sqrt{2(1+|v|^2)}$, we conclude $I$ is bounded.

Now, by the no blow-up criterion in [4, Proposition 4.11] we obtain that the maximal regular flow $\boldsymbol{X}$ of $\boldsymbol{b}$ is globally defined on $[0,T]$ (hence, it follows (i)). Moreover, the trajectories $\boldsymbol{X}(\cdot,x,v)$ belong to $\mathrm{AC}([0,T];\mathbb{R}^6)$ for $g_0$-a.e. $(x,v) \in \mathbb{R}^6$, and $g_t = \boldsymbol{X}(t,\cdot)_{\#} g_0 = g_0 \circ \boldsymbol{X}^{-1}(t,\cdot)$. Since $f_t = \tan\left(\frac{\pi}{2} g_t\right)$ and the map $[0,1) \ni s \longrightarrow \tan\left(\frac{\pi}{2}s\right) \in [0,\infty)$ is a diffeomorphism, we obtain that $f_t = \boldsymbol{X}(t,\cdot)_{\#} f_0 = f_0 \circ \boldsymbol{X}^{-1}(t,\cdot)$ (hence, it follows (ii)). In particular, for all Borel functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ we have

$$
\int_{\mathbb{R}^6} \psi(f_t)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^6} \psi(f_0) \circ \boldsymbol{X}^{-1}(t,\cdot)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^6} \psi(f_0)\,\mathrm{d}x\,\mathrm{d}v,
$$

where the second equality follows by the incompressibility of the flow, which gives (iii). □

**Remark 2.** As in [4, Remark 2.4], given $0 \le s \le t \le T$, it is possible to reconstruct $f_t$ from $f_s$ by using the flow, that is, $f_t = \boldsymbol{X}(t,s,\cdot)_{\#}(f_s)$.

## 2.2 Existence of generalized solution

We now introduce the concept of a generalized solution, which allows the electromagnetic field to be generated by effective densities $\rho^{\mathrm{eff}}$ and $J^{\mathrm{eff}}$. We may interpret it as particles vanishing from the phase space but still contributing in the electromagnetic field in the physical space. Thus, it is natural to assume that $\rho_t^{\mathrm{eff}}$ may be larger than $\rho_t$, but it is bounded by the initial particle density $\rho_0$. Moreover, we assume that the particle current density

$J_t^{\text{eff}}$ is relativistic and compatible with $\rho_t^{\text{eff}}$, that is, $|J_t^{\text{eff}}| < \rho_t^{\text{eff}}$ and satisfies the continuity equation (see (2.13a), (2.13b), and (2.13c) below).

**Definition 2.11** (Generalized solution)**.** Given $\bar{f} \in L^1(\mathbb{R}^6)$, let

$$f \in L^\infty((0,\infty); L^1(\mathbb{R}^6))$$

be a nonnegative function,

$$\rho_t^{\text{eff}} \in L^\infty((0,\infty); \mathscr{M}_+(\mathbb{R}^3))\text{and } (J_t^{\text{eff}})_i \in L^\infty((0,\infty); \mathscr{M}(\mathbb{R}^3))$$

for each component $i \in \{1, 2, 3\}$. We say that the triplet $(f_t, \rho_t^{\text{eff}}, J_t^{\text{eff}})$ is a (global in time) generalized solution of (7) starting from $\bar{f}$ if, setting

$$\begin{aligned} \rho_t(x) &\coloneqq \int_{\mathbb{R}^3} f_t(x,v)\,\mathrm{d}v, \quad E_t^{\text{eff}}(x) \coloneqq \sigma_E \int_{\mathbb{R}^3} K(x-y)\,\mathrm{d}\rho_t^{\text{eff}}(y), \\ J_t(x) &\coloneqq \int_{\mathbb{R}^3} \hat{v} f_t(x,v)\,\mathrm{d}v, \quad B_t^{\text{eff}}(x) \coloneqq \sigma_B \int_{\mathbb{R}^3} K(y-x) \times \mathrm{d}J_t^{\text{eff}}(y), \quad \text{and} \\ \mathbf{b}_t^{\text{eff}}(x,v) &\coloneqq (\hat{v}, E_t^{\text{eff}}(x) + \hat{v} \times B_t^{\text{eff}}(x)), \end{aligned} \tag{2.12}$$

the following hold: $f_t$ is a renormalized solution of the continuity equation with vector field $\boldsymbol{b}_t^{\text{eff}}$ starting from $\bar{f}$,

$$\rho_t \le \rho_t^{\text{eff}}, \quad |J_t^{\text{eff}}| < \rho_t^{\text{eff}} \quad \text{as measures for a.e. } t \in (0,\infty), \tag{2.13a}$$

$$\rho_t^{\text{eff}}(\mathbb{R}^3) \le \|f_0\|_{L^1(\mathbb{R}^6)} \quad \text{for a.e. } t \in (0,\infty), \text{ and} \tag{2.13b}$$

$$\partial_t \rho_t^{\text{eff}} + \nabla \cdot J_t^{\text{eff}} = 0 \quad \text{with initial condition } \bar{\rho} = \int_{\mathbb{R}^3} \bar{f}\,\mathrm{d}v, \text{ i.e.,} \tag{2.13c}$$

$$\int_{\mathbb{R}^3} \phi_0\,\mathrm{d}\bar{\rho} + \int_0^\infty \int_{\mathbb{R}^3} (\partial_t \phi_t\,\mathrm{d}\rho_t^{\text{eff}} + \nabla \phi_t \cdot \mathrm{d}J_t^{\text{eff}})\,\mathrm{d}t = 0 \quad \forall\, \phi \in C_c^1([0,\infty) \times \mathbb{R}^3).$$

Notice that by the Radon-Nikodym's Theorem, combined with (2.13a), there exists a vector field $V^{\text{eff}} \in L^\infty((0,\infty); L^1(\rho^{\text{eff}}; \mathbb{R}^3))$ such that $\mathrm{d}J_t^{\text{eff}} = V_t^{\text{eff}}\,\mathrm{d}\rho_t^{\text{eff}}$ and $|V_t^{\text{eff}}(x)| < 1$ for a.e. $(t,x) \in (0,\infty) \times \mathbb{R}^3$. This is analogous to the continuity equation associated to (7) with initial condition $\rho_0$, which is obtained by integrating (7) with respect to $v$ over the whole domain $\mathbb{R}^3$:

$$\int_{\mathbb{R}^3} \phi_0\,\mathrm{d}\rho_0 + \int_0^\infty \int_{\mathbb{R}^3} (\partial_t \phi_t + \nabla \phi_t \cdot V_t)\,\mathrm{d}\rho_t\,\mathrm{d}t = 0 \quad \forall \phi \in C_c^1([0,\infty) \times \mathbb{R}^3), \tag{2.14}$$

where $V \coloneqq J/\rho \in L^\infty((0,\infty); L^1(\rho;\mathbb{R}^3))$ satisfies $\mathrm{d}J_t = V_t\,\mathrm{d}\rho_t$ and $|V_t(x)| < 1$ for a.e. $(t,x) \in (0,\infty)\times\mathbb{R}^3$. Notice that the second condition in (2.13a) and (2.13c) are not present in the definition of generalized solutions of the non-relativistic Vlasov-Poisson system [4, Definition 2.6], and are imposed in order to allow an effective magnetic field which reduces to the self-consistent one if the number of particles is conserved.

To see that Definition 2.11 is in fact a generalization of Definition 2.1, we remark that $\|\rho_t\|_{L^1(\mathbb{R}^3)} = \|f_t\|_{L^1(\mathbb{R}^6)}$, hence it follows by (2.13a) and (2.13b) that, if the number of particles is conserved a.e. in time, i.e., if $\|f_t\|_{L^1(\mathbb{R}^6)} = \|f_0\|_{L^1(\mathbb{R}^6)}$ for a.e. $t$, then $\rho_t^{\mathrm{eff}} = \rho_t$. Indeed, if the conservation holds, we have

$$\rho_t(\mathbb{R}^3) \le \rho_t^{\mathrm{eff}}(\mathbb{R}^3) \le \|f_0\|_{L^1(\mathbb{R}^6)} = \rho_0(\mathbb{R}^3) = \rho_t(\mathbb{R}^3),$$

thus $\rho_t(\mathbb{R}^3) = \rho_t^{\mathrm{eff}}(\mathbb{R}^3)$. Now, if there exists $E \subset \mathbb{R}^3$ and $t > 0$ such that $\rho_t(E) < \rho_t^{\mathrm{eff}}(E)$, then

$$(\rho_t^{\mathrm{eff}} - \rho_t)(\mathbb{R}^3) \ge (\rho_t^{\mathrm{eff}} - \rho_t)(E) > 0,$$

a contradiction.

Notice that by (2.13c) and (2.14), we have that $\rho_t$ satisfy the continuity equation with both velocities $V_t$ and $V_t^{\mathrm{eff}}$ with initial condition $\rho_0$. The following lemma gives that $V = V^{\mathrm{eff}}$, whence $J = J^{\mathrm{eff}}$.

**Lemma 2.12.** *Assume $\rho_t$ satisfies the continuity equation with the same initial condition and both vector fields $V$, $V^{\mathrm{eff}}$. Assume further that $V$, $V^{\mathrm{eff}}$ satisfy*

$$\int_0^T \int_{\mathbb{R}^3} \frac{|V(x)| + |V^{\mathrm{eff}}(x)|}{1+|x|}\,\mathrm{d}\rho_t(x)\,\mathrm{d}t < \infty.$$

*Then, $V = V^{\mathrm{eff}}$.*

*Proof.* Consider a class $\mathscr{L}_{\boldsymbol{b}}$ of measured-value solutions $\mu_t \in \mathscr{M}_+(\mathbb{R}^3)$ of continuity equation with vector field $\boldsymbol{b}_t$ satisfying

$$0 \le \partial_t\mu_t \le \mu_t \quad\Longrightarrow\quad \partial_t\mu_t \in \mathscr{L}_{\boldsymbol{b}}$$

whenever $\partial_t\mu_t$ still solves the continuity equation with vector field $\boldsymbol{b}_t$, and the integrability condition

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\boldsymbol{b}_t(x)|}{1+|x|}\,\mathrm{d}\mu_t(x)\,\mathrm{d}t < \infty.$$

Notice that $\rho_t \in \mathscr{L}_V \cap \mathscr{L}_{V^{\text{eff}}}$ for all $T > 0$, hence by [15], we have

$$\rho_t = \boldsymbol{X}(t,\cdot)_{\#}\rho_0 = \boldsymbol{X}^{\text{eff}}(t,\cdot)_{\#}\rho_0 \quad \forall\, t \in [0,T], \tag{2.15}$$

where $\boldsymbol{X}$ and $\boldsymbol{X}^{\text{eff}}$ are $\mathscr{L}_V$ and $\mathscr{L}_{V^{\text{eff}}}$ Lagragian flows, respectively, that is, $\boldsymbol{X}(t,\cdot)$ and $\boldsymbol{X}^{\text{eff}}(t,\cdot)$ are (unique) absolutely continuous functions in $[0,T]$ starting from $\rho_0$ (at time 0) such that

$$\begin{aligned} &\dot{\boldsymbol{X}}(t,\cdot) = V_t(\boldsymbol{X}(t,\cdot)), \quad \dot{\boldsymbol{X}}^{\text{eff}}(t,\cdot) = V_t^{\text{eff}}(\boldsymbol{X}^{\text{eff}}(t,\cdot)), \\ &\boldsymbol{X}(0,\cdot) = \boldsymbol{X}^{\text{eff}}(0,\cdot) = \mathrm{Id} \end{aligned}$$

for $\rho_0$-almost everywhere. By (2.15) and the uniqueness of $\boldsymbol{X}$ and $\boldsymbol{X}^{\text{eff}}$, we conclude that $V_t = V_t^{\text{eff}}$. □

It follows that, if the number of particles is conserved in time, then generalized solutions are renormalized ones. This observation indicates that a generalized solution which is not renormalized must lose mass/charge as the velocity approaches the speed of light.

Next, our goal is to prove the global existence of generalized solutions $f_t$ for any nonnegative $f_0 \in L^1(\mathbb{R}^6)$ (Theorem 2.3). In order to do so, we need to establish the existence of a (unique) distributional solution with smooth kernel and initial data. More precisely, we show that by smoothing the kernel $K$ and with nonnegative initial condition in $C_c^\infty(\mathbb{R}^6)$, we obtain a classical solution of (7). To avoid any confusion with the notation of Theorem 2.3 and Theorem 2.4, we denote by $\mathcal{K} \coloneqq \eta * K$ and by $g$ the smoothed kernel and the initial condition, respectively.

**Proposition 2.13.** *Let $g \in C_c^\infty(\mathbb{R}^6)$ be a nonnegative function. Then, there exists a unique nonnegative Lagrangian solution $f \in C^\infty([0,\infty) \times \mathbb{R}^6)$ of the smoothed system (7):*

$$\begin{cases} \partial_t f_t + \hat{v}\cdot\nabla_x f_t + (E_t + \hat{v}\times B_t)\cdot\nabla_v f_t = 0 & \text{in} \quad (0,\infty)\times\mathbb{R}^6; \\ \rho_t(x) = \int_{\mathbb{R}^3} f_t(x,v)\,\mathrm{d}v, \quad J_t(x) = \int_{\mathbb{R}^3} \hat{v} f_t(x,v)\,\mathrm{d}v & \text{in} \quad (0,\infty)\times\mathbb{R}^3; \\ E_t(x) = \sigma_E \int_{\mathbb{R}^3} \rho_t(y)\mathcal{K}(x-y)\,\mathrm{d}y & \text{in} \quad (0,\infty)\times\mathbb{R}^3; \\ B_t(x) = \sigma_B \int_{\mathbb{R}^3} J_t(y)\times\mathcal{K}(x-y)\,\mathrm{d}y & \text{in} \quad (0,\infty)\times\mathbb{R}^3; \\ f_0(x,v) = g(x,v) & \text{in} \quad \mathbb{R}^3\times\mathbb{R}^3. \end{cases} \tag{2.16}$$

*Proof.* In this proof, we adapt ideas and techniques from [13, Chapter 5]. We construct by induction a sequence of smooth functions $f_t^n$ with initial condition $g$ which converges to a solution of (2.16). For $n = 1$, let $f^1$ be a solution of the linear transport equation

$$\begin{cases} \partial_t f_t^1(x, v) + \nabla_x \cdot (\hat{v} f_t^1)(x, v) = 0, \\ f_0^1(x, v) = g(x, v) \end{cases}$$

which gives that

$$f_t^1(x, v) = g(x - t\hat{v}, v) \in C_c^\infty([0, \infty) \times \mathbb{R}^6).$$

Moreover, we have that $f^1$ is a Lagrangian solution, since there exists a unique solution $\boldsymbol{Z}^0(t, \cdot) \coloneqq (\boldsymbol{X}^0, \boldsymbol{V}^0)(t, \cdot)$ of

$$\begin{cases} \dot{\boldsymbol{Z}}(t, \cdot) = \boldsymbol{b}_t^0(\boldsymbol{Z}(t, \cdot)); \\ \boldsymbol{Z}(0, \cdot) = \mathrm{Id}, \end{cases}$$

where $\boldsymbol{b}_t^0(x, v) \coloneqq (\hat{v}, 0)$. Hence,

$$f_t^1 = g \circ \boldsymbol{Z}^0(t), \quad \|f_t^1\|_{L^1(\mathbb{R}^6)} = \|g\|_{L^1(\mathbb{R}^6)}, \quad \text{and} \quad \|f_t^1\|_{L^\infty(\mathbb{R}^6)} = \|g\|_{L^\infty(\mathbb{R}^6)}.$$

Now, for $n \geq 2$, assume that there exists a smooth Lagrangian function

$$f^n \in L^\infty([0, \infty) \times \mathbb{R}^6) \cap L^\infty([0, \infty); L^1(\mathbb{R}^6))$$

which satisfies

$$\begin{cases} \partial_t f_t^n(x, v) + \nabla_{x,v} \cdot (\boldsymbol{b}^{n-1} f_t^n)(x, v) = 0, \\ f_0^n(x, v) = g(x, v), \end{cases} \tag{2.17}$$

where

$$\begin{aligned} E_t^n(x) &= \sigma_E \int_{\mathbb{R}^3} \rho_t^n(y) \mathcal{K}(x - y) \,\mathrm{d}y, \\ B_t^n(x) &= \sigma_B \int_{\mathbb{R}^3} J_t^n(y) \times \mathcal{K}(x - y) \,\mathrm{d}y, \\ \boldsymbol{b}_t^n(x, v) &= (\hat{v},\, E_t^n + \hat{v} \times B_t^n)(x, v), \end{aligned}$$

and define $f^{n+1}$ as a solution of (2.17) with vector field $\boldsymbol{b}_t^n$. Notice that $\boldsymbol{b}_t^n$ is divergence-free, and since $f^n$ and $\mathcal{K}$ are smooth, we obtain that $\boldsymbol{b}_t^n$ is also smooth. Moreover, we have $\boldsymbol{b}^n \in L^\infty([0,\infty); W^{k,\infty}(\mathbb{R}^6;\mathbb{R}^6))$ for all $k \in \mathbb{N}$, since by Young's inequality (recall that $|J^n| < \rho^n$ a.e.)

$$\begin{aligned} &\|D^k_{x,v}\boldsymbol{b}^n_t\|_{L^\infty([0,\infty);L^\infty(\mathbb{R}^6;\mathbb{R}^6))} \\ \leq & C\Big(1 + \|K\|_{L^1(B_1;\mathbb{R}^3)}\|D^k\eta\|_{L^\infty(\mathbb{R}^3)}\|\rho^n\|_{L^\infty([0,\infty);L^1(\mathbb{R}^3))} \\ &+ \|K\|_{L^\infty(\mathbb{R}^3\setminus B_1;\mathbb{R}^3)}\|D^k\eta\|_{L^1(\mathbb{R}^3)}\|\rho^n\|_{L^\infty([0,\infty);L^1(\mathbb{R}^3))}\Big). \end{aligned} \tag{2.18}$$

Thus, we have for all $t \geq 0$ a smooth incompressible flow

$$\boldsymbol{Z}^n(t) = (\boldsymbol{X}^n, \boldsymbol{V}^n)(t)$$

which satisfies

$$\begin{cases} \dot{\boldsymbol{Z}}(t,\cdot) = \boldsymbol{b}^n_t(\boldsymbol{Z}(t,\cdot)); \\ \boldsymbol{Z}(0,\cdot) = \mathrm{Id}, \end{cases} \tag{2.19}$$

and the following properties hold:

$$\begin{aligned} f^{n+1}_t = g \circ \boldsymbol{Z}^n(t), \quad \|f^{n+1}_t\|_{L^1(\mathbb{R}^6)} &= \|g\|_{L^1(\mathbb{R}^6)}, \\ \text{and} \quad \|f^{n+1}_t\|_{L^\infty(\mathbb{R}^6)} &= \|g\|_{L^\infty(\mathbb{R}^6)}. \end{aligned} \tag{2.20}$$

Now, we want to exploit the fact that (recall that $g \in C^\infty_c$)

$$|f^{n+1}_t - f^n_t| \leq C|\boldsymbol{Z}^n(t) - \boldsymbol{Z}^{n-1}(t)| \tag{2.21}$$

to show that $f^n$ is a Cauchy sequence in $C([0,T]\times\mathbb{R}^6)$. For this purpose, notice that (we omit the $t$ and $(x,v)$ arguments for a cleaner presentation)

$$\begin{aligned} |\boldsymbol{X}^n(s) - \boldsymbol{X}^{n-1}(s)| \leq & \int_s^t |\boldsymbol{V}^n(\tau) - \boldsymbol{V}^{n-1}(\tau)| \\ & + |\boldsymbol{V}^n(\tau)|\left|\frac{1}{\sqrt{1+|\boldsymbol{V}^n(\tau)|^2}} - \frac{1}{\sqrt{1+|\boldsymbol{V}^{n-1}(\tau)|^2}}\right| \mathrm{d}\tau \\ \leq & \int_s^t |\boldsymbol{V}^n(\tau) - \boldsymbol{V}^{n-1}(\tau)| \\ & + \left|\sqrt{1+|\boldsymbol{V}^n(\tau)|^2} - \sqrt{1+|\boldsymbol{V}^{n-1}(\tau)|^2}\right| \mathrm{d}\tau. \end{aligned}$$

Thus, by mean value theorem, we conclude that

$$|\boldsymbol{X}^n(s) - \boldsymbol{X}^{n-1}(s)| \leq 2\int_s^t |\boldsymbol{V}^n(\tau) - \boldsymbol{V}^{n-1}(\tau)|\,\mathrm{d}\tau.$$

Moreover, define $E^n$ and $B^n$ as in (2.16) with densities $\rho^n$ and $J^n$, respectively. Now, by the same procedure as before combined with the uniform boundedness of $B^n$ (by (2.18) and (2.20)), we have

$$\begin{aligned}|\boldsymbol{V}^n(s)-\boldsymbol{V}^{n-1}(s)| \leq & C\int_s^t |E^n_\tau(\boldsymbol{X}^n(\tau))-E^{n-1}_\tau(\boldsymbol{X}^{n-1}(\tau))| \\ & + |B^n_\tau(\boldsymbol{X}^n(\tau))-B^{n-1}_\tau(\boldsymbol{X}^{n-1}(\tau))| \\ & + |\boldsymbol{V}^n(\tau)-\boldsymbol{V}^{n-1}(\tau)|\,\mathrm{d}\tau.\end{aligned}$$

By (2.18) and (2.20), $E^n$ and $B^n$ are uniformly bounded with respect to $n$ and $t$, thus

$$\begin{aligned}|E^n_\tau(\boldsymbol{X}^n(\tau))-E^{n-1}_\tau(\boldsymbol{X}^{n-1}(\tau))| \leq & |(E^n_\tau-E^{n-1}_\tau)(\boldsymbol{X}^n(\tau))| \\ & + |E^{n-1}_\tau(\boldsymbol{X}^n(\tau))-E^{n-1}_\tau(\boldsymbol{X}^{n-1}(\tau))| \\ \leq & \|E^n_\tau-E^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3)} \\ & + C|\boldsymbol{X}^n(\tau)-\boldsymbol{X}^{n-1}(\tau)|,\end{aligned}$$

and, analogously,

$$|B^n_\tau(\boldsymbol{X}^n(\tau))-B^{n-1}_\tau(\boldsymbol{X}^{n-1}(\tau))| \leq \|B^n_\tau-B^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3)}+C|\boldsymbol{X}^n(\tau)-\boldsymbol{X}^{n-1}(\tau)|.$$

Hence, we obtain that

$$\begin{aligned}|\boldsymbol{Z}^n(s)-\boldsymbol{Z}^{n-1}(s)| \leq C\int_s^t & \|E^n_\tau-E^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3)}+\|B^n_\tau-B^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3)} \\ & + |\boldsymbol{Z}^n(\tau)-\boldsymbol{Z}^{n-1}(\tau)|\,\mathrm{d}\tau.\end{aligned}$$

Thus, by Gronwall's inequality, we conclude that

$$|\boldsymbol{Z}^n(t)-\boldsymbol{Z}^{n-1}(t)| \leq C\int_0^t \|E^n_\tau-E^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3)}+\|B^n_\tau-B^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}\tau.$$

Now, by (2.20), we have that $f^n \in C_c^\infty$, which combined with (2.21) and Young's inequality gives that

$$\begin{aligned}\|f^{n+1}_t-f^n_t\|_{L^\infty(\mathbb{R}^6)} & \leq C\int_0^t \|\rho^n_\tau-\rho^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3)}+\|J^n_\tau-J^{n-1}_\tau\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\,\mathrm{d}\tau \\ & \leq C\int_0^t \|f^n_\tau-f^{n-1}_\tau\|_{L^\infty(\mathbb{R}^6)}\,\mathrm{d}\tau.\end{aligned} \tag{2.22}$$

Therefore, by induction, we have that for all $T > 0$,

$$\|f_t^{n+1} - f_t^n\|_{L^\infty(\mathbb{R}^6)} \leq C\frac{T^n}{n!}, \quad t \in [0, T],$$

and we conclude that $f^n$ converges uniformly to a function $f \in C([0, \infty) \times \mathbb{R}^6)$. Moreover, by (2.20), we have that $f_t = g \circ \boldsymbol{Z}(t)$, and

$$f \in L^\infty([0, \infty); L^1(\mathbb{R}^6)) \cap L^\infty([0, \infty) \times \mathbb{R}^6,$$

where

$$\boldsymbol{Z}(t, \cdot) \coloneqq \lim_{n\to\infty} \boldsymbol{Z}^n(t, \cdot)$$

Notice that $f_t$ has compact support (since $g \in C_c^\infty$), thus $\rho^n$ and $J^n$ converge to $\rho$ and $J$ in $C([0, \infty) \times \mathbb{R}^6)$, respectively. Therefore, $E^n$ and $B^n$ converge to $E$ and $B$, thus $\boldsymbol{b}^n$ converges to $\boldsymbol{b}$ in $C([0, \infty) \times \mathbb{R}^6)$. By the same computation as (2.18), we have in fact that $\boldsymbol{b} \in C([0, \infty); W^{k,\infty}(\mathbb{R}^6))$ for all $k \in \mathbb{N}$, and we conclude by passing the limit in (2.19) that $\boldsymbol{Z} \in C^1([0, \infty); C^\infty(\mathbb{R}^6))$, and we have $f \in C^1([0, \infty); C^\infty(\mathbb{R}^6))$. By iteration, we conclude that $f$ is a smooth nonnegative Lagrangian solution of (2.16), where $\boldsymbol{Z} \in C^\infty([0, \infty) \times C^\infty(\mathbb{R}^6))$ solves

$$\begin{cases} \dot{\boldsymbol{Z}}(t, \cdot) = \boldsymbol{b}_t(\boldsymbol{Z}(t, \cdot)); \\ \boldsymbol{Z}(0, \cdot) = \operatorname{Id}. \end{cases} \tag{2.23}$$

In particular, we have that $f \in C_c^\infty([0, \infty) \times \mathbb{R}^6)$.

To prove the uniqueness, assume that there exists Lagrangian solutions $f$, $\tilde{f}$ of (2.16). Thus,

$$f_t \coloneqq g \circ \boldsymbol{Z}(t), \quad \tilde{f}_t \coloneqq g \circ \tilde{\boldsymbol{Z}}(t)$$

where both $\boldsymbol{Z}$, $\tilde{\boldsymbol{Z}}$ solve (2.23). Thus, we may repeat the proof of (2.22) for $f_t - \tilde{f}_t$ to obtain

$$\|f_t - \tilde{f}_t\|_{L^\infty(\mathbb{R}^6)} \leq C \int_0^t \|f_\tau - \tilde{f}_\tau\|_{L^\infty(\mathbb{R}^6)} \,\mathrm{d}\tau,$$

and we conclude by Gronwall's inequality that $f \equiv \tilde{f}$. □

We are now able to prove our second main result.

*Proof of Theorem 2.3.* Our proof follows the same general structure of the proof of [4, Theorem 2.7]: we begin by approximating $f$ as a $L^1$ limit of $f^n$ (Steps 1 and 2), which was already shown in [4]; then, we approximate $(\rho_t^{\text{eff}},\, J_t^{\text{eff}})$ and show that the electromagnetic field of the approximation converges to the effective field $(E_t^{\text{eff}},\, B_t^{\text{eff}})$ (Steps 3 and 4); finally, in Step 5, we combine stability results for the continuity equation obtained in [4, Section 5] to take limits in the approximated system and conclude that the limiting solution is transported by the limit of the incompressible flow.

**Step 1: Approximating solutions.** Consider $K^n \coloneqq K * \eta^n$, where $\eta^n(x) \coloneqq n^3\eta(nx)$, and $\eta$ is a standard convolution kernel in $\mathbb{R}^3$. Let $f_0^n \in C_c^\infty(\mathbb{R}^6)$ be a sequence such that

$$f_0^n \longrightarrow f_0 \text{ in } L^1(\mathbb{R}^6). \tag{2.24}$$

Moreover, denote $f_t^n$ the smooth solution of (7) with initial condition $f_0^n$ and kernel $K^n$ (see Proposition 2.13), and its respective charge density, electric field, density current, and magnetic field defined by

$$\rho_t^n(x) \coloneqq \int_{\mathbb{R}^3} f_t^n(x,v)\,\mathrm{d}v, \quad E_t^n(x) \coloneqq \sigma_E \int_{\mathbb{R}^3} \rho_t^n(y) K^n(x-y)\,\mathrm{d}y,$$
$$J_t^n(x) \coloneqq \int_{\mathbb{R}^3} \hat{v}\, f_t^n(x,v)\,\mathrm{d}v, \quad \text{and} \quad B_t^n(x) \coloneqq \sigma_B \int_{\mathbb{R}^3} J_t^n(y) \times K^n(x-y)\,\mathrm{d}y.$$

Since $K^n$ is smooth and vanishes at infinity, we have

$$E^n,\, B^n \in L^\infty([0,\infty); W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)),$$

but without a uniform bound with respect to $n$, nonetheless (see the proof of (2.18)). Hence, $\boldsymbol{b}_t^n \coloneqq (\hat{v}, E_t^n + \hat{v} \times B_t^n)$ is a Lipschitz divergence-free vector field, and its flow $\boldsymbol{X}^n(t) : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ is well defined and incompressible, hence by theory for the transport equation, for all $t \in [0,\infty)$ and each component $i \in \{1,\, 2,\, 3\}$,

$$f_t^n = f_0^n \circ \boldsymbol{X}^n(t)^{-1} \quad \text{and}$$
$$\|J_t^n\|_{L^1(\mathbb{R}^3,\mathbb{R}^3)} \le \||\hat{v}| f_t^n\|_{L^1(\mathbb{R}^6)} < \|\rho_t^n\|_{L^1(\mathbb{R}^3)} = \|f_t^n\|_{L^1(\mathbb{R}^6)} = \|f_0^n\|_{L^1(\mathbb{R}^6)}. \tag{2.25}$$

Assume without loss of generality that $\mathscr{L}^6(\{f_0 = k\}) = 0$ for every $k \in \mathbb{N}$ (otherwise, consider $\mathscr{L}^6(\{f_0 = k + \tau\}) = 0$ for $\tau \in (0,1)$), we deduce that for all $k$

$$f_0^{n,k} \coloneqq \mathbf{1}_{\{k \le f_0^n < k+1\}} f_0^n \longrightarrow f_0^k \coloneqq \mathbf{1}_{\{k \le f_0 < k+1\}} f_0 \quad \text{in } L^1(\mathbb{R}^6). \tag{2.26}$$

Thus, by defining $f_t^{n,k} := \mathbf{1}_{\{k \le f_t^n < k+1\}} f_t^n$, we have that $f_t^{n,k}$ is a distributional solution of the continuity equation (with vector field $\boldsymbol{b}_t^n$) and $f_0^n$ initial datum. Moreover, we have for all $t \in [0, \infty)$

$$f_t^{n,k} = \mathbf{1}_{\{k \le f_0^n \circ \boldsymbol{X}^n(t)^{-1} < k+1\}} f_0^n \circ \boldsymbol{X}^n(t)^{-1}, \quad \|f_t^{n,k}\|_{L^1(\mathbb{R}^6)} = \|f_0^{n,k}\|_{L^1(\mathbb{R}^6)}. \tag{2.27}$$

**Step 2: Limit in phase space.** By construction, $(f^{n,k})_{n \in \mathbb{N}}$ is a nonnegative uniformly bounded sequence. Hence, there exists $f^k \in L^\infty((0,\infty) \times \mathbb{R}^6)$ such that

$$f^{n,k} \rightharpoonup f^k \quad \text{weakly* in } L^\infty((0,\infty) \times \mathbb{R}^6) \quad \text{as } n \longrightarrow \infty \quad \forall\, k \in \mathbb{N}. \tag{2.28}$$

Moreover, for any $K \subset\subset \mathbb{R}^6$, and any bounded function $\phi : (0,\infty) \longrightarrow (0,\infty)$ with compact support, we use test function $\phi(t)\mathbf{1}_K(x,v)\,\mathrm{sign}(f_t^k)(x,v)$ for the previous two weak convergence combined with Fatou's lemma, the convergence of $(f_t^{n,k})_{n \in \mathbb{N}}$, and (2.27) to obtain

$$\begin{aligned}\int_0^\infty \phi(t)\|f_t^k\|_{L^1(K)} \mathrm{d}t &\le \left(\int_0^\infty \phi(t)\,\mathrm{d}t\right) \liminf_{n\to\infty} \|f_0^{n,k}\|_{L^1(\mathbb{R}^6)} \\ &= \left(\int_0^\infty \phi(t)\,\mathrm{d}t\right) \|f_0^k\|_{L^1(\mathbb{R}^6)},\end{aligned}$$

Since $\phi$ was arbitrary the supremum among all compact subset $K \subset \mathbb{R}^6$ we obtain

$$\|f_t^k\|_{L^1(\mathbb{R}^6)} \le \|f_0^k\|_{L^1(\mathbb{R}^6)} \quad \text{for a.e. } t \in (0,\infty), \tag{2.29}$$

so, in particular, $f^k \in L^\infty((0,\infty); L^1(\mathbb{R}^6))$. Moreover, by defining $f = \sum_{k=0}^\infty f^k$, we have

$$\|f_t\|_{L^1(\mathbb{R}^6)} \le \|f_0\|_{L^1(\mathbb{R}^6)} \quad \text{for a.e. } t \in [0,\infty). \tag{2.30}$$

Noticing that $f^n = \sum_{k=0}^\infty f^{n,k}$, by fixing $\varphi \in L^\infty((0,T) \times \mathbb{R}^6)$, (2.27), and (2.29), we have for all $k_0 \ge 1$,

$$\begin{aligned}\left|\int_0^T \int_{\mathbb{R}^6} \varphi(f^n - f)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\right| &\le \left|\sum_{k=0}^{k_0-1} \int_0^T \int_{\mathbb{R}^6} \varphi(f^{n,k} - f^k)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\right| \\ &\quad + T\|\varphi\|_{L^\infty((0,T)\times\mathbb{R}^6)} \sum_{k=k_0}^\infty \int_{\mathbb{R}^6} (|f_0^{n,k}| + |f_0^k|)\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

Now, by the convergence (2.28) the first term vanishes as $n \longrightarrow \infty$. Thus, by convergences (2.24) and (2.26), we have

$$\limsup_{n\to\infty} \left| \int_0^T \int_{\mathbb{R}^6} \varphi(f^n - f)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t \right| \leq 2T \|\varphi\|_{L^\infty((0,T)\times\mathbb{R}^6)} \|f_0 \mathbf{1}_{\{f_0 \geq k_0\}}\|_{L^1(\mathbb{R}^6)}.$$

Letting $k_0 \longrightarrow \infty$ and since $\varphi \in L^\infty$ was arbitrary, we conclude

$$f^n \rightharpoonup f \quad \text{weakly in } L^1((0,T)\times\mathbb{R}^6). \tag{2.31}$$

**Step 3: Limit in physical densities.** Since $(\rho^n)_{n\in\mathbb{N}}$ and $(J_i^n)_{n\in\mathbb{N}}$ are bounded sequences in $L^\infty((0,\infty); \mathscr{M}_+(\mathbb{R}^3))$ and $L^\infty((0,\infty); \mathscr{M}(\mathbb{R}^3))$, respectively, for each component $i \in \{1, 2, 3\}$ (see (2.25)), and the fact that $L^\infty((0,\infty); \mathscr{M}(\mathbb{R}^3)) = [L^1((0,\infty); C_0(\mathbb{R}^3))]^*$, there exist

$$\rho^{\mathrm{eff}} \in L^\infty((0,\infty); \mathscr{M}_+(\mathbb{R}^3)) \text{ and } J_i^{\mathrm{eff}} \in L^\infty((0,\infty); \mathscr{M}(\mathbb{R}^3))$$

such that

$$\begin{aligned} \rho^n &\rightharpoonup \rho^{\mathrm{eff}} \quad \text{weakly* in } L^\infty((0,\infty); \mathscr{M}_+(\mathbb{R}^3)); \\ J_i^n &\rightharpoonup J_i^{\mathrm{eff}} \quad \text{weakly* in } L^\infty((0,\infty); \mathscr{M}(\mathbb{R}^3)). \end{aligned} \tag{2.32}$$

for each component $i \in \{1, 2, 3\}$. Hence, by the lower semicontinuity of the norm under weak* convergence, we have

$$\begin{aligned} \operatorname{ess\,sup}_{t\in(0,\infty)} |\rho_t^{\mathrm{eff}}|(\mathbb{R}^3) \leq \lim_{n\to\infty} \left( \sup_{t\in(0,\infty)} \|\rho_t^n\|_{L^1(\mathbb{R}^3)} \right) &= \lim_{n\to\infty} \|\rho_0^n\|_{L^1(\mathbb{R}^3)} \\ &= \|f_0\|_{L^1(\mathbb{R}^6)}. \end{aligned} \tag{2.33}$$

Now, fixing a nonnegative function $\varphi \in C_c((0,\infty)\times\mathbb{R}^3)$, by (2.31) and (2.32), we obtain that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^3} \varphi_t(x)\, \mathrm{d}\rho_t^{\mathrm{eff}}(x)\, \mathrm{d}t &\geq \lim_{R\to\infty} \liminf_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^3\times B_R} f_t^n(x,v)\varphi_t(x)\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t \\ &= \int_0^\infty \int_{\mathbb{R}^6} f_t(x,v)\varphi_t(x)\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t \\ &= \int_0^\infty \int_{\mathbb{R}^3} \varphi_t(x)\, \mathrm{d}\rho_t(x)\, \mathrm{d}t. \end{aligned}$$

Moreover, by recalling that $|\hat{v}| < 1$, we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^3} \varphi_t(x)\,\mathrm{d}\rho_t^{\mathrm{eff}}(x)\,\mathrm{d}t &= \lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^6} f_t^n(x,v)\varphi_t(x)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &> \lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^6} |\hat{v}|\, f_t^n(x,v)\varphi_t(x)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int_0^\infty \int_{\mathbb{R}^3} \varphi_t(x)\,\mathrm{d}|J_t^{\mathrm{eff}}|(x)\,\mathrm{d}t.\end{aligned}$$

Thus,

$$\rho_t \leq \rho_t^{\mathrm{eff}}, \quad |J_t^{\mathrm{eff}}| < \rho_t^{\mathrm{eff}} \quad \text{as measures for a.e. } t \in (0,\infty). \tag{2.34}$$

Finally, by the same argument to show (2.14), we notice that

$$\int_{\mathbb{R}^3} \phi_0\,\mathrm{d}\rho_0^n + \int_0^\infty \int_{\mathbb{R}^3} (\partial_t\phi_t\,\mathrm{d}\rho_t^n + \nabla\phi_t\cdot\,\mathrm{d}J_t^n)\,\mathrm{d}t = 0 \quad \forall\phi \in C_c^1([0,\infty)\times\mathbb{R}^3).$$

Hence, by (2.24) and (2.32), we conclude by taking the limit $n \longrightarrow \infty$ that

$$\int_{\mathbb{R}^3} \phi_0\,\mathrm{d}\rho_0 + \int_0^\infty \int_{\mathbb{R}^3} (\partial_t\phi_t\,\mathrm{d}\rho_t^{\mathrm{eff}} + \nabla\phi_t\cdot\mathrm{d}J_t^{\mathrm{eff}})\,\mathrm{d}t = 0 \quad \forall\,\phi \in C_c^1([0,\infty)\times\mathbb{R}^3),$$

i.e.,

$$\partial_t\rho_t^{\mathrm{eff}} + \nabla\cdot J_t^{\mathrm{eff}} = 0 \quad \text{as measures with initial condition } \rho_0. \tag{2.35}$$

**Step 4: Limit of vector fields.** Using the definition (2.12), we claim that

$$\boldsymbol{b}^n \rightharpoonup \boldsymbol{b}^{\mathrm{eff}} \quad \text{weakly in } L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^6;\mathbb{R}^6) \tag{2.36}$$

and that, for every ball $B_R \subset \mathbb{R}^3$,

$$[E^n + \hat{v}\times B^n](x+h) \longrightarrow [E^n + \hat{v}\times B^n](x) \text{ as } |h|\to 0 \text{ in } L^1_{\mathrm{loc}}((0,\infty);L^1(B_R)) \tag{2.37}$$

uniformly in $n$.

For this purpose, we first prove that the sequence $(\boldsymbol{b}^n)_{n\in\mathbb{N}}$ is bounded in $L^p_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^6;\mathbb{R}^6)$ for every $p \in [1,3/2)$. Indeed, by using Young's inequality, for every $t \geq 0$, $n \in \mathbb{N}$, and $r > 0$,

$$\begin{aligned}\|B_t^n\|_{L^p(B_r;\mathbb{R}^3)} + \|E_t^n\|_{L^p(B_r;\mathbb{R}^3)} \leq{}& \|(|J_t^n| * \eta^n) * |K|\|_{L^p(B_r;\mathbb{R}^3)} \\ &+ \|(\rho_t^n * \eta^n) * K\|_{L^p(B_r;\mathbb{R}^3)}\end{aligned}$$

The first term can be bounded by

$$
\begin{aligned}
&\|(|J_t^n| * \eta^n) * (|K|\mathbf{1}_{B_1})\|_{L^p(B_r;\mathbb{R}^3)} + \|(|J_t^n| * \eta^n) * (|K|\mathbf{1}_{\mathbb{R}^3\setminus B_1})\|_{L^p(B_r;\mathbb{R}^3)} \\
&\le \||J_t^n|\|_{L^1(\mathbb{R}^3)}\|\eta^n\|_{L^1(\mathbb{R}^3)}\|K\|_{L^p(B_1;\mathbb{R}^3)} \\
&+ \mathscr{L}^3(B_r)^{1/p}\||J_t^n\|_{L^1(\mathbb{R}^3)}\|\eta^n\|_{L^1(\mathbb{R}^3)}\|K\|_{L^\infty(\mathbb{R}^3\setminus B_1;\mathbb{R}^3)}.
\end{aligned}
$$

Likewise, the second term can be bounded by

$$
\begin{aligned}
&\|\rho_t^n\|_{L^1(\mathbb{R}^3)}\|\eta^n\|_{L^1(\mathbb{R}^3)}\|K\|_{L^p(B_1;\mathbb{R}^3)} \\
&\qquad\qquad + \mathscr{L}^3(B_r)^{1/p}\|\rho_t^n\|_{L^1(\mathbb{R}^3)}\|\eta^n\|_{L^1(\mathbb{R}^3)}\|K\|_{L^\infty(\mathbb{R}^3\setminus B_1;\mathbb{R}^3)}.
\end{aligned}
$$

Thus, up to subsequences, the sequence $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ converges weakly in $L^p_{\mathrm{loc}}$. We now claim that for every $\varphi \in C_c((0,\infty)\times\mathbb{R}^3)$,

$$
\lim_{n\to\infty}\int_0^\infty\int_{\mathbb{R}^3}(E_t^n + \hat{v}\times B_t^n)\,\varphi_t\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty\int_{\mathbb{R}^3}(E_t^{\mathrm{eff}} + \hat{v}\times B_t^{\mathrm{eff}})\,\varphi_t\,\mathrm{d}x\,\mathrm{d}t.
$$

Indeed, denoting $T_\varphi$ the upper time support of $\varphi$, we have

$$
\begin{aligned}
&\left|\int_0^\infty\int_{\mathbb{R}^3}(E_t^n + \hat{v}\times B_t^n)\,\varphi_t\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}^3}(E_t^{\mathrm{eff}} + \hat{v}\times B_t^{\mathrm{eff}})\,\varphi_t\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\le \left|\int_0^\infty\int_{\mathbb{R}^3}(\rho_t^n - \rho_t^{\mathrm{eff}})\varphi_t * K\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\quad + \left|\int_0^\infty\int_{\mathbb{R}^3}\rho_t^n(\varphi_t * K - \varphi_t * K * \eta^n)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\quad + \left|\int_0^\infty\int_{\mathbb{R}^3}(J_t^n - J_t^{\mathrm{eff}})\times\varphi_t * K\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\quad + \left|\int_0^\infty\int_{\mathbb{R}^3}J_t^n\times(\varphi_t * K - \varphi_t * K * \eta^n)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\le \left|\int_0^\infty\int_{\mathbb{R}^3}(\rho_t^n - \rho_t^{\mathrm{eff}})\varphi_t * K\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\quad + \left|\int_0^\infty\int_{\mathbb{R}^3}(J_t^n - J_t^{\mathrm{eff}})\times\varphi_t * K\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\quad + T_\varphi(\|\rho^n\|_{L^\infty((0,\infty);L^1(\mathbb{R}^3))} \\
&\qquad\quad + \|J^n\|_{L^\infty((0,\infty);L^1(\mathbb{R}^3;\mathbb{R}^3))})\|\varphi * K - \varphi * K * \eta^n\|_{L^\infty((0,\infty)\times\mathbb{R}^3;\mathbb{R}^3)}.
\end{aligned}
$$

By the weak convergence (2.32) and the fact that $\varphi * K$ is a bounded continuous function, the first and second terms vanish as $n \longrightarrow \infty$. Moreover, the last term also vanishes, since the first factor is bounded by $C\|f_0\|_{L^1(\mathbb{R}^6)}$, where $C > 0$ is a universal constant and $\varphi * K * \eta^n$ convergences uniformly to $\varphi * K$ in $(0,\infty)\times\mathbb{R}^3$. Thus, we have proven (2.36).

We now prove (2.37). For this purpose, we combine the fact that $K \in W^{\alpha,p}(\mathbb{R}^3;\mathbb{R}^3)$ for every $\alpha < 1$ and $p < 3/(2+\alpha)$, and Young's inequality to obtain

$$\|E_t^n + \hat{v}\times B_t^n\|_{W^{\alpha,p}(B_R;\mathbb{R}^3)} \le C(R)\|(\rho_t^n + |J_t^n|) * \eta^n\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}.$$

Combining $\|\eta^n\|_{L^1(\mathbb{R}^3)} = 1$ with (2.25), we can bound the right term independently of $n$ and $t$, which combined with the embedding of fractional Sobolev spaces and Nikolsky spaces [24] gives

$$\|\boldsymbol{b}_t^n(\cdot + h) - \boldsymbol{b}_t^n(\cdot)\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)} \le C\left(p,\alpha,R,\|\boldsymbol{b}_t^n\|_{W^{\alpha,p}(B_{2R};\mathbb{R}^3)}\right)|h|^\alpha \quad \forall |h| \le R,$$

and (2.37) follows.

**Step 5: Conclusion.** By (2.36) and (2.37), we can apply the stability result from [15] to deduce that $f^k$ is a weakly continuous distributional solution of the continuity equation with vector field $\boldsymbol{b}^{\text{eff}}$ and starting from $f_0^k$ for every $k \in \mathbb{N}$. We now exploit the linearity of the continuity equation to show that $F^m := \sum_{k=1}^m f^k$ is also a bounded distributional solution for every $m \in \mathbb{N}$. Using the same arguments as in the proof of Theorem 2.2, we obtain that $F^m$ is a renormalized solution for every $m \in \mathbb{N}$. Since $F^m \longrightarrow f$ strongly in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^6)$ as $m \longrightarrow \infty$, we obtain that $f$ is a renormalized solution of the continuity equation with vector field $\boldsymbol{b}^{\text{eff}}$ and starting from $f_0$, which combined with (2.30) (2.33), (2.34), and (2.35) proves that the trio $(f_t, \rho_t^{\text{eff}}, J_t^{\text{eff}})$ is a generalized solution starting from $f_0$ according to Definition 2.11.

To show that $f$ is transported by the maximum regular flow associated to $\boldsymbol{b}^{\text{eff}}$, we simply use that each $f^k$ is transported (once again with the same argument as in Theorem 2.2) combined with the definition of $f$ and (2.30). Finally, by [4, Theorem 4.10], we conclude that the map

$$[0,\infty) \ni t \longmapsto f_t \in L^1_{\text{loc}}(\mathbb{R}^6) \quad \text{is continuous.} \qquad \square$$

## 2.3 Finite energy solutions

Up to now, we have established the existence of a generalized solution (see Theorem 2.3) and that renormalized and generalized solutions coincide in case the mass/charge is conserved in time. In this section, we investigate whether the existence of renormalized solutions can be shown under the more natural condition that the initial total energy is bounded, that is,

$$\mathscr{E}_0 := \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{\sigma_E}{2}\int_{\mathbb{R}^3} (H * \rho_0)\rho_0\,\mathrm{d}x \\ + \frac{\sigma_B}{2}\int_{\mathbb{R}^3} (H * J_0)\cdot J_0\,\mathrm{d}x < \infty, \tag{2.38}$$

where the first term is the relativistc (initial) total energy and the second and third are the electric and magnetic potential (initial) energies, respectively, and $H(x) := (4\pi|x|)^{-1}$. For this purpose, we recall that by integrating the first equation of (7) with respect to $(x, v)$ on the whole domain $\mathbb{R}^6$ gives that the relativistic energy (formally) satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v &= \int_{\mathbb{R}^6} \hat{v}\cdot(E_t + \hat{v}\times B_t) f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^3} E_t\cdot J_t\,\mathrm{d}x. \end{aligned}$$

Now, Poynting's Theorem gives that the relativistic Vlasov-Maxwell equation has its electromagnetic total energy (formally) conserved, i.e.,

$$\begin{aligned} &\int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{1}{2}\int_{\mathbb{R}^3} |E_t|^2 + |B_t|^2\,\mathrm{d}x \\ &= \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &+ \frac{1}{2}\int_{\mathbb{R}^3} |E_0|^2 + |B_0|^2\,\mathrm{d}x, \end{aligned}$$

while for the system (7) we obtain a similar expression (see (2.41) below):

$$\begin{aligned} &\int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{\sigma_E}{2}\int_{\mathbb{R}^3} (H * \rho_t)\rho_t\,\mathrm{d}x \\ &= \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{\sigma_E}{2}\int_{\mathbb{R}^3} (H * \rho_0)\rho_0\,\mathrm{d}x. \end{aligned} \tag{2.39}$$

Although (2.39) is formal, we shall exploit a semicontinuity argument to show it in the form of an inequality (see the proof of Theorem 2.4):

$$\begin{aligned}&\int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{\sigma_E}{2}\int_{\mathbb{R}^3}(H*\rho_t)\rho_t\,\mathrm{d}x\\ &\le \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{\sigma_E}{2}\int_{\mathbb{R}^3}(H*\rho_0)\rho_0\,\mathrm{d}x.\end{aligned} \tag{2.40}$$

Notice that the magnetic potential energy does not appear in the conservation above. On the other hand, one can (formally) integrate by parts the electric and magnetic energy to obtain the relations

$$\begin{aligned}&\int_{\mathbb{R}^3} |E_t|^2\,\mathrm{d}x = \int_{\mathbb{R}^3}(H*\rho_t)\rho_t\,\mathrm{d}x;\\ &\int_{\mathbb{R}^3} |B_t|^2\,\mathrm{d}x = \int_{\mathbb{R}^3}(H*J_t)\cdot J_t\,\mathrm{d}x - \int_{\mathbb{R}^3}\left(\nabla\cdot(H*J_t)\right)^2\,\mathrm{d}x.\end{aligned} \tag{2.41}$$

We can interpret $H*\rho_t$ and $H*J_t$ as the electric potential and magnetic vector potential, respectively (see [22]). Notice that, on one hand, the electric potential energy is fully converted into the electric energy. On the other hand, the magnetic potential energy is converted into the magnetic energy and the displacement current $\partial_t E_t$, since

$$-\int_{\mathbb{R}^3}\left(\nabla\cdot(H*J_t)\right)^2\,\mathrm{d}x = \int_{\mathbb{R}^3}\nabla\cdot(H*J_t)\,\partial_t(H*\rho_t)\,\mathrm{d}x = \int_{\mathbb{R}^3}(H*J_t)\cdot\partial_t E_t\,\mathrm{d}x. \tag{2.42}$$

Moreover, we obtain (formally) that the magnetic potential energy is nonnegative for a.e. $t\in[0,\infty)$. We observe that (2.39) and (2.41) do not have any magnetic energy terms, so that it is unclear whether the classical energy estimate $\mathscr{E}_t < \mathscr{E}_0$ holds, where the total energy of the system at time $t$ is given by

$$\mathscr{E}_t := \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v + \frac{\sigma_E}{2}\int_{\mathbb{R}^3}(H*\rho_t)\rho_t\,\mathrm{d}x + \frac{\sigma_B}{2}\int_{\mathbb{R}^3}(H*J_t)\cdot J_t\,\mathrm{d}x.$$

**Remark 3.** Although the formal argument that leads to (2.41) suggests the magnetic potential energy is nonnegative, we rigorously justify it in the proof of Lemma 2.19. Hence, (2.38) implies that the right-hand side of (2.40) is bounded.

**Remark 4.** By (2.39) and (2.42), we (formally) have

$$\int_{\mathbb{R}^3} |B_t|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} A_t \cdot (J_t + \partial_t E_t) \,\mathrm{d}x, \tag{2.43}$$

where $A_t \coloneqq H * J_t$ is the magnetic vector potential. Since we can interpret $\partial_t E_t$ as a density current, one might define the magnetic vector potential as $H * (J_t + \partial_t E)$, and therefore (2.42) does not provide a relation between magnetic energy and magnetic potential energy. We claim that (2.43) still holds if $A_t = H * (J_t + \partial_t E)$; thus, we may interpret $\partial_t E_t$ as a lower order term. Indeed, define a magnetic field with density current $\tilde{J}_t \coloneqq J_t + \partial_t E_t$, that is, $\tilde{B} = \nabla \times (H * \tilde{J}_t)$, and a calculation analogous to (2.39) gives that

$$\int_{\mathbb{R}^3} |\tilde{B}_t|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} (H * \tilde{J}_t) \cdot \tilde{J}_t \,\mathrm{d}x - \int_{\mathbb{R}^3} \Big(\nabla \cdot (H * \tilde{J})\Big)^2 \,\mathrm{d}x. \tag{2.44}$$

Notice that $\nabla \cdot (H * \tilde{J}) = H * (\nabla \cdot J + \partial_t \rho_t) = 0$, hence the last term vanishes. Moreover, since $E_t$ is irrotational, $B_t = \tilde{B}_t$; thus, combining (2.43) and (2.44), we conclude that

$$\int_{\mathbb{R}^3} (H * \partial_t E_t) \cdot (J_t + \partial_t E_t) \,\mathrm{d}x = 0.$$

Therefore, had we defined the magnetic vector potential as $H * \tilde{J}$, (2.43) would be unaltered.

Notice that if $\sigma_E = 1$, a bound as (2.40) gives that each energy term of $\mathscr{E}_t$ is bounded, since $|J| < \rho$ a.e. in space-time. However, it does not provide, in general, control of relativistic energy, electric and magnetic potential energies if $\sigma_E = -1$ or $\sigma_E = 0$. If we also assume a higher integrability of $f_0$ and a suitable smallness condition on its norm, the next lemma can be used to bound each energy.

**Lemma 2.14.** *Let $f \in L^1(\mathbb{R}^6) \cap L^q(\mathbb{R}^6)$ be a nonnegative function for some $q \geq 1$ and $\sqrt{1 + |v|^2} f \in L^1(\mathbb{R}^6)$. Set $p \coloneqq \frac{4q-3}{3q-2}$. Then $\rho = \int_{\mathbb{R}^3} f(\cdot, v) \,\mathrm{d}v \in L^p(\mathbb{R}^3)$ and there exists a constant $C > 0$, depending only on $q$ such that*

$$\|\rho\|_{L^p(\mathbb{R}^3)} \leq C \|\sqrt{1 + |v|^2} f\|_{L^1(\mathbb{R}^6)}^{\theta} \|f\|_{L^q(\mathbb{R}^6)}^{1-\theta},$$

*where $\theta \coloneqq \frac{3(q-1)}{4q-3}$.*

*Proof.* We begin choosing $R > 0$ splitting the integral of $\rho$ on the sets $\{|v| < R\}$ and $\{|v| \geq R\}$. Hence, for each $x \in \mathbb{R}^3$,

$$\rho(x) \leq R^{3(q-1)/q} \|f(x,\cdot)\|_{L^q(\mathbb{R}^3)} + R^{-1} \|\sqrt{1+|v|^2} f(x,\cdot)\|_{L^1(\mathbb{R}^3)}.$$

By minimizing the right-hand side with respect to $R$, we have

$$\rho(x) \leq C \|\sqrt{1+|v|^2} f(x,\cdot)\|_{L^1(\mathbb{R}^3)}^{3(q-1)/(4q-3)} \|f(x,\cdot)\|_{L^q(\mathbb{R}^3)}^{q/(4q-3)}.$$

Taking the $L^p$-norm on $\rho$ and using Hölder's inequality, the result follows. $\square$

As anticipated, if $f_0$ satisfies

$$f_0 \in \begin{cases} L^1(\mathbb{R}^6) & \text{if } \sigma_E = 1; \\ L^1(\mathbb{R}^6) \cap L^{3/2}(\mathbb{R}^6) & \text{if } \sigma_E = 0; \\ L^1(\mathbb{R}^6) \cap L^{3/2}(\mathbb{R}^6) \text{ and } \|f_0\|_{L^{3/2}(\mathbb{R}^6)} \leq \epsilon & \text{if } \sigma_E = -1 \end{cases} \tag{2.45}$$

for some suitable $\epsilon > 0$, the previous lemma allows us to bound each relativistic energy, electric and magnetic potential energies. Indeed, by Calderón-Zygmund estimates and the Sobolev embedding, we have that

$$\|H * \rho_t\|_{L^6(\mathbb{R}^3)} \leq C \|D^2(H * \rho_t)\|_{L^{6/5}(\mathbb{R}^3)} \leq C \|\rho_t\|_{L^{6/5}(\mathbb{R}^3)} \tag{2.46}$$

for some universal constant $C > 0$. Combining (2.46) with Hölder's inequality and Lemma 2.14 with $p = 6/5$ and $q = 3/2$ gives

$$\begin{aligned} \int_{\mathbb{R}^3} (H * \rho_t)\rho_t \, \mathrm{d}x \leq \|H * \rho_t\|_{L^6(\mathbb{R}^3)} \|\rho_t\|_{L^{6/5}(\mathbb{R}^3)} \leq C \|\rho_t\|_{L^{6/5}(\mathbb{R}^3)}^2 \\ \leq C \|\sqrt{1+|v|^2} f_t\|_{L^1(\mathbb{R}^6)} \|f_t\|_{L^{3/2}(\mathbb{R}^6)}. \end{aligned} \tag{2.47}$$

Notice that $\|f\|_{L^\infty([0,\infty);L^{3/2}(\mathbb{R}^6))} \leq \|f_0\|_{L^{3/2}(\mathbb{R}^6)}$ when the solution is built by approximation (see (2.30)). Hence, if (2.40) holds, we already have a bound of the relativistic energy in the pure magnetic case $\sigma_E = 0$, and by the previous bound, we obtain the following boundedness of the magnetic and electric potential energies (recall that $|J| < \rho$ a.e. in space-time):

$$\begin{aligned} \int_{\mathbb{R}^3} (H * J_t) \cdot J_t \, \mathrm{d}x &\leq \int_{\mathbb{R}^3} (H * \rho_t)\rho_t \, \mathrm{d}x \\ &\leq C \|f_0\|_{L^{3/2}(\mathbb{R}^6)} \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v) \, \mathrm{d}x \, \mathrm{d}v. \end{aligned}$$

Now, in the repulsive case $\sigma_E = -1$, we obtain by (2.40) and (2.47) that

$$\begin{aligned}&\big(1 - C\|f\|_{L^\infty([0,\infty);L^{3/2}(\mathbb{R}^6))}\big) \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &\leq \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v - \int_{\mathbb{R}^3} (H * \rho_0)\rho_0\,\mathrm{d}x.\end{aligned}$$

Assuming that $f$ is built by approximation as before and that $\|f_0\|_{L^{3/2}(\mathbb{R}^6)} < 1/C =: \epsilon$, we have a bound of the relativistic energy; therefore, by (2.47), the electric and magnetic potential energies are bounded as well. This motivates the following:

**Definition 2.15.** We say that $f_0$ has every energy bounded if (2.38) and (2.45) hold. Moreover, if $f_t$ also satisfies (2.40) for almost every $t \in [0,\infty)$, then we say that $f_t$ has every energy bounded.

**Remark 5.** Notice that we need stronger assumptions on the initial data compared to the nonrelativistic Vlasov-Poisson case for $\sigma_E = -1$, where it is only needed that $f_0 \in L^{9/7}(\mathbb{R}^3)$, with no smallness assumption (see [11]). This is due to the fact that classical kinetic energy grows as $|v|^2$, whereas the relativistic energy as $|v|$.

We now prove that if $f_0$ has every energy bounded, then we have a smooth sequence $(f_0^n)_{n\in\mathbb{N}}$ and a mollified sequence of kernels $(H * \eta^{k_n})_{n\in\mathbb{N}}$ with uniform bounded energy. We denote by $L_c^\infty$ the space of bounded measurable functions with compact support.

**Lemma 2.16.** *Let $\eta^k(x) \coloneqq k^3\eta(kx)$, where $\eta$ is a standard convolution kernel in $\mathbb{R}^3$. Let $f_0$ be a nonnegative function with every energy bounded. Then there exists a sequence $(f_0^n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^6)$ and a sequence $(k_n)_{n\in\mathbb{N}}$ such that $k_n \longrightarrow \infty$ and, by setting $\rho_0^n = \int_{\mathbb{R}^3} f_0^n(\cdot,v)\,\mathrm{d}v$ and $J_0^n = \int_{\mathbb{R}^3} \hat{v} f_0^n(\cdot,v)\,\mathrm{d}v$,*

$$\begin{aligned}\lim_{n\to\infty} \Bigg( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0^n(x,v)\,\mathrm{d}x\,\mathrm{d}v &+ \frac{\sigma_E}{2}\int_{\mathbb{R}^3} (H * \eta^{k_n} * \rho_0^n)\rho_0^n\,\mathrm{d}x \\ &+ \frac{\sigma_B}{2}\int_{\mathbb{R}^3} (H * \eta^{k_n} * J_0^n)\cdot J_0^n\,\mathrm{d}x \Bigg) \\ = \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v &+ \frac{\sigma_E}{2}\int_{\mathbb{R}^3} (H * \rho_0)\rho_0\,\mathrm{d}x \\ &+ \frac{\sigma_B}{2}\int_{\mathbb{R}^3} (H * J_0)\cdot J_0\,\mathrm{d}x.\end{aligned}$$

*Proof.* We split the proof in three steps: in Step 1, we assume that $f_0 \in L^\infty_c(\mathbb{R}^6)$ and approximate it by a sequence of smooth functions with compact support; in Step 2, we obtain the desired limit without the mollification of $H$; in Step 3, we introduce the mollification of the kernel $\eta^k * H$, and conclude that the limit holds if we extract a subsequence of $k$ which depends on $n$.

**Step 1: $\boldsymbol{f_0 \in L^\infty_c(\mathbb{R}^6)}$.** Consider smooth functions $f_0^n$ which converge pointwise such that $\|f_0^n\|_{L^\infty(\mathbb{R}^6)} \leq \|f_0\|_{L^\infty(\mathbb{R}^6)}$ and $\operatorname{supp} f_0^n \subset B_R$ for all $n$ for some $R > 0$. Thus, $\|J_0^n\|_{L^\infty(\mathbb{R}^3,\mathbb{R}^3)} < \|\rho_0^n\|_{L^\infty(\mathbb{R}^3)} \leq \|\rho_0\|_{L^\infty(\mathbb{R}^3)}$, and $\operatorname{supp}|J_0^n| \subset \operatorname{supp}\rho_0^n \subseteq B_R$. Moreover, $|H * J_0^n| < H * \rho_0^n < \infty$ and $H * \rho_0^n \longrightarrow H * \rho_0$ and $H * J_0^n \longrightarrow H * J_0$ in $L^p_{\mathrm{loc}}$ for every $p$, and we conclude by dominated convergence that

$$
\begin{aligned}
\lim_{n\to\infty} \Bigg( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0^n(x,v)\,\mathrm{d}x\,\mathrm{d}v &+ \frac{\sigma_E}{2}\int_{\mathbb{R}^3} (H * \rho_0^n)\rho_0^n\,\mathrm{d}x \\
&+ \frac{\sigma_B}{2}\int_{\mathbb{R}^3} (H * J_0^n)\cdot J_0^n\,\mathrm{d}x \Bigg) \\
= \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v &+ \frac{\sigma_E}{2}\int_{\mathbb{R}^3} (H * \rho_0)\rho_0\,\mathrm{d}x \\
&+ \frac{\sigma_B}{2}\int_{\mathbb{R}^3} (H * J_0)\cdot J_0\,\mathrm{d}x.
\end{aligned}
\tag{2.48}
$$

**Step 2: $\boldsymbol{f_0 \in L^1(\mathbb{R}^6)}$ without mollification of $\boldsymbol{H}$.** By Step 1, it is enough to approximate $f_0$ by $(f_0^n)_{n\in\mathbb{N}} \subset L^\infty_c(\mathbb{R}^6)$ with converging energies to obtain (2.48). For this purpose, define

$$
f_0^n(x,v) := \min\{n, \mathbf{1}_{B_n}(x,v) f_0(x,v)\}, \quad (x,v) \in \mathbb{R}^6.
$$

Since $H \geq 0$, the first two integrands on the left-hand side of (2.48) converges monotonically, and we conclude by monotone convergence. Since $|(H * J_0^n)\cdot J_0^n| < (H * \rho_0)\rho_0$ a.e., and $(H * \rho_0)\rho_0$ is integrable (since $f_0$ has every energy bounded), we conclude that the last integral on the left-hand side converges by the dominated convergence.

**Step 3: Approximation of the kernel.** Given $(f_0^n)_{n\in\mathbb{N}} \in C^\infty_c(\mathbb{R}^6)$ provided by the previous two steps, we have

$$
\begin{aligned}
\lim_{k\to\infty} &\left( \int_{\mathbb{R}^3} (H * \eta^k * \rho_0^n)\rho_0^n\,\mathrm{d}x + \int_{\mathbb{R}^3} (H * \eta^k * J_0^n)\cdot J_0^n\,\mathrm{d}x \right) \\
&= \int_{\mathbb{R}^3} (H * \rho_0^n)\rho_0^n\,\mathrm{d}x + \int_{\mathbb{R}^3} (H * J_0^n)\cdot J_0^n\,\mathrm{d}x
\end{aligned}
$$

for every fixed $n$. Hence, there exists $k_n$ sufficiently large such that

$$\left| \int_{\mathbb{R}^3} (H * \eta^{k_n} * \rho_0^n)\rho_0^n \,\mathrm{d}x + \int_{\mathbb{R}^3} (H * \eta^{k_n} * J_0^n) \cdot J_0^n \,\mathrm{d}x \right.$$
$$\left. - \int_{\mathbb{R}^3} (H * \rho_0^n)\rho_0^n \,\mathrm{d}x - \int_{\mathbb{R}^3} (H * J_0^n) \cdot J_0^n \,\mathrm{d}x \right| \leq \frac{1}{n},$$

and the lemma follows. □

In what follows, we need the following result from [4, Lemma 3.3] that we state for convenience of the reader.

**Lemma 2.17.** *Let $T > 0$ and $\phi \in C_c((0,T))$ be a nonnegative function. Then, for every sequence $(\rho^n)_{n\in\mathbb{N}} \subset C([0,T]; \mathscr{M}_+(\mathbb{R}^3))$ such that*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \rho_t^n(\mathbb{R}^3) < \infty$$

*and*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^3} \varphi \,\mathrm{d}(\rho_t^n - \rho_t) \right| = 0 \quad \text{for every } \varphi \in C_c^\infty(\mathbb{R}^3). \tag{2.49}$$

*we have*

$$\int_0^T \phi(t) \int_{\mathbb{R}^3} H * \rho_t(x) \,\mathrm{d}\rho_t(x) \,\mathrm{d}t \leq \liminf_{n\to\infty} \int_0^T \phi(t) \int_{\mathbb{R}^3} H * \eta^n * \rho_t^n(x) \,\mathrm{d}\rho_t^n(x) \,\mathrm{d}t, \tag{2.50}$$

Although the previous lemma is enough for $\sigma_E \in \{0,1\}$, we need a slight higher integrability assumption in the gravitational case $\sigma_E = -1$. This is due to the fact that we obtain (2.40) by a lower semicontinuity argument, and (2.50) is not sufficient if the electric potential energy is nonpositive. Nonetheless, if $\rho \in L^{6/5}$, we obtain (2.50) with a limit and an equality, and we prove it in the next lemma.

**Lemma 2.18.** *Let $\rho^n$, $\rho \in L^\infty([0,T]; L^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3))$ in the same setting as Lemma 2.17. Moreover, assume that*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|\rho_t^n\|_{L^{6/5}(\mathbb{R}^3)} < \infty. \tag{2.51}$$

*Then*

$$\lim_{n\to\infty}\int_0^T \phi(t)\int_{\mathbb{R}^3} H*\eta^n*\rho_t^n(x)\,\mathrm{d}\rho_t^n(x)\,\mathrm{d}t = \int_0^T \phi(t)\int_{\mathbb{R}^3} H*\rho_t(x)\,\mathrm{d}\rho_t(x)\,\mathrm{d}t. \tag{2.52}$$

*Proof.* Notice that

$$\begin{aligned}
&\int_{\mathbb{R}^3} H*\eta^n*\rho_t^n(x)\rho_t^n(x) - H*\rho_t(x)\rho_t(x)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^3} H*\eta^n*(\rho_t^n(x)-\rho_t(x))\rho_t^n(x)\,\mathrm{d}x\\
&+ \int_{\mathbb{R}^3} H*(\eta^n*\rho_t(x)-\rho_t(x))\rho_t^n(x)\,\mathrm{d}x\\
&+ \int_{\mathbb{R}^3} H*\rho_t(x)(\rho_t^n(x)-\rho_t(x))\,\mathrm{d}x =: I_1+I_2+I_3.
\end{aligned}$$

Now, by (2.46) and Hölder inequality, we obtain that

$$|I_2| \le C\|\eta^n*\rho_t-\rho_t\|_{L^{6/5}(\mathbb{R}^3)}\sup_{n\in\mathbb{N}}\|\rho_t^n\|_{L^{6/5}(\mathbb{R}^3)}.$$

Letting $n \longrightarrow \infty$, we obtain that $I_2$ vanishes. We now define $\zeta_k \in C_c^\infty(\mathbb{R}^3)$ as a cutoff function in the annular set $B_k \setminus B_{1/k}$, namely,

$$\begin{cases}
\zeta_k = 1 & \text{in} \quad B_k\setminus B_{1/k};\\
\zeta_k = 0 & \text{in} \quad B_{k+1}^c \cup B_{1/(k+1)};\\
0\le\zeta_k\le 1 & \text{in} \quad \mathbb{R}^3.
\end{cases}$$

We write $I_3$ as

$$\begin{aligned}
|I_3| \le &\left|\int_{\mathbb{R}^3} H*\rho_t(x)(\rho_t^n(x)-\rho_t(x))\zeta_k(x)\,\mathrm{d}x\right|\\
&+\left|\int_{\mathbb{R}^3} H*\rho_t(x)(\rho_t^n(x)-\rho_t(x))(1-\zeta_k(x))\,\mathrm{d}x\right|
\end{aligned}$$

We want to take first the limit $n \longrightarrow \infty$ and after $k \longrightarrow \infty$ to be able to use

(2.49). Now, by (2.46), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} H * \rho_t(x)(\rho_t^n(x) - \rho_t(x))(1-\zeta_k(x))\,\mathrm{d}x\right| \\
&\leq \left|\int_{B_{1/k}\cup B_k^c} H * \rho_t(x)(\rho_t^n(x) - \rho_t(x))\,\mathrm{d}x\right| \\
&\leq C\|\rho_t\|_{L^{6/5}(\mathbb{R}^3)} \sup_{n\in\mathbb{N}} \|\rho_t^n - \rho_t\|_{L^{6/5}(B_{1/k}\cup B_k^c)}.
\end{aligned}$$

Defining measures $\mathrm{d}\mu_t^n \coloneqq (\rho_t^n - \rho_t)^{6/5}\mathrm{d}x$ and $\mu \coloneqq \sup_{n\in\mathbb{N}} \mu^n$, by (2.51) and the continuity from below for measures gives that

$$\begin{aligned}
\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \int_{B_{1/k}\cup B_k^c} (\rho_t^n - \rho_t)^{6/5}\,\mathrm{d}x &= \lim_{k\to\infty} \mu_t(B_{1/k}\cup B_k^c) \\
&= \mu_t\left(\cap_{k=1}^{\infty} B_{1/k}\right) + \mu_t\left(\cap_{k=1}^{\infty} B_k^c\right) = 0,
\end{aligned}$$

and we conclude that second term vanishes as $k \longrightarrow \infty$. Now, we bound the first term by

$$\|H * \rho_t\|_{L^\infty(B_{k+1}\setminus B_{1/(k+1)})} \left|\int_{\mathbb{R}^3} \zeta_k(\rho_t^n(x) - \rho_t(x))\,\mathrm{d}x\right|$$

By Young's inequality, we have

$$\|H * \rho_t\|_{L^\infty(B_{k+1}\setminus B_{1/(k+1)})} \leq \|H\|_{L^\infty(B_{k+1}\setminus B_{1/(k+1)})}\|\rho_t\|_{L^1(\mathbb{R}^3)} < \infty.$$

Hence, by (2.49), $I_3$ vanishes as $n \longrightarrow \infty$ and $k \longrightarrow \infty$. Analogously, we have

$$\begin{aligned}
|I_1| &= \left|\int_{\mathbb{R}^3} H * \eta^n * \rho_t^n(x)(\rho_t^n(x) - \rho_t(x))\,\mathrm{d}x\right| \\
&\leq \|H * \rho_t^n\|_{L^\infty(B_{k+1}\setminus B_{1/(k+1)})} \left|\int_{\mathbb{R}^3} \zeta_k(\rho_t^n(x) - \rho_t(x))\,\mathrm{d}x\right| \\
&+ C \sup_{n\in\mathbb{N}} \|\rho_t^n\|_{L^{6/5}(\mathbb{R}^3)} \sup_{n\in\mathbb{N}} \|\rho_t^n - \rho_t\|_{L^{6/5}(B_{1/k}\cup B_k^c)},
\end{aligned}$$

and by the same argument as before, $I_1$ vanishes as $n \longrightarrow \infty$ and $k \longrightarrow \infty$, and the lemma follows. $\square$

We now want to rigorously justify (2.41) for $|J| < \rho \in L^1(\mathbb{R}^3)$. Actually, the same argument yields the result for $|J| < \rho \in \mathscr{M}_+(\mathbb{R}^3)$. The following lemma gives (2.41) with an inequality; in particular, the magnetic potential energy is nonnegative.

**Lemma 2.19.** *For every $|J| < \rho \in L^1(\mathbb{R}^3)$ nonnegative,*

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla(H * \rho)|^2 \,\mathrm{d}x &\leq \int_{\mathbb{R}^3} (H * \rho)\rho \,\mathrm{d}x; \\ \int_{\mathbb{R}^3} |\nabla \times (H * J)|^2 \,\mathrm{d}x &\leq \int_{\mathbb{R}^3} (H * J) \cdot J \,\mathrm{d}x - \int_{\mathbb{R}^3} (\nabla \cdot (H * J))^2 \,\mathrm{d}x. \end{aligned} \tag{2.53}$$

*In particular, we obtain that the magnetic potential energy is nonnegative.*

*Proof.* We split the proof similarly to Lemma 2.16:

**Step 1: $J_i$, $\rho \in L_c^\infty(\mathbb{R}^3)$.** Consider first $\rho$, $J$ smooth compactly supported functions, and perform an integration by parts to obtain

$$\begin{aligned} \int_{B_R} |\nabla(H * \rho)|^2 \,\mathrm{d}x &= \int_{B_R} (H * \rho)\rho \,\mathrm{d}x + \int_{\partial B_R} H * \rho \, \nabla(H * \rho) \cdot \nu_{B_R} \,\mathrm{d}\mathscr{H}^2; \\ \int_{B_R} |\nabla \times (H * J)|^2 \,\mathrm{d}x &= \int_{B_R} (H * J) \cdot J \,\mathrm{d}x - \int_{\mathbb{R}^3} (\nabla \cdot (H * J))^2 \,\mathrm{d}x \\ &\quad - \int_{\partial B_R} [(H * J) \times (\nabla \times (H * J))] \cdot \nu_{B_R} \,\mathrm{d}\mathscr{H}^2 \\ &\quad + \int_{\partial B_R} \nabla \cdot (H * J) H * J \cdot \nu_{B_R} \,\mathrm{d}\mathscr{H}^2. \end{aligned}$$

The same identity holds for $J_i$, $\rho \in L_c^\infty(\mathbb{R}^3)$ by approximation for each component $i \in \{1, 2, 3\}$. Since $H * \mu$ and $\nabla(H * \mu)$ decay as $R^{-1}$ and $R^{-2}$ when evaluated at $\partial B_R$ for all $\mu \in L_c^\infty(\mathbb{R}^3)$, the boundary terms vanish as $R \longrightarrow \infty$, and we obtain that (2.53) holds with an equality.

**Step 2: $J_i$, $\rho \in L^1(\mathbb{R}^3)$.** We consider the truncations

$$\rho^n \coloneqq \min\{n, \mathbf{1}_{B_n}(x, v)\rho\}, \quad J_i^n \coloneqq \min\{n, \mathbf{1}_{B_n}(x, v)J_i\}.$$

Since $H \geq 0$, by monotone convergence and Step 1 we obtain that

$$\int_{\mathbb{R}^3} (H * \rho)\rho \,\mathrm{d}x = \lim_{n \to \infty} \int_{\mathbb{R}^3} (H * \rho^n)\rho^n \,\mathrm{d}x = \lim_{n \to \infty} \int_{\mathbb{R}^3} |\nabla(H * \rho^n)|^2 \,\mathrm{d}x.$$

Moreover, since $|J| < \rho$, by dominated convergence and Step 1 we obtain that

$$\begin{aligned}\int_{\mathbb{R}^3} (H * J) \cdot J \, \mathrm{d}x &= \lim_{n\to\infty} \int_{\mathbb{R}^3} (H * J^n) \cdot J^n \, \mathrm{d}x \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla \times (H * J^n)|^2 \, \mathrm{d}x \\ &+ \lim_{n\to\infty} \int_{\mathbb{R}^3} (\nabla \cdot (H * J^n))^2 \, \mathrm{d}x.\end{aligned}$$

Assuming without loss of generality that $(H * \rho)\rho \in L^1(\mathbb{R}^3)$, we get bounded sequences $(\nabla(H*\rho^n))_{n\in\mathbb{N}}$, $(\nabla\cdot(H*J^n))_{n\in\mathbb{N}}$, and $(\nabla\times(H*J^n))_{n\in\mathbb{N}}$ in $L^2$. Since each sequence converges in the sense of distributions to $\nabla(H*\rho)$, $\nabla\cdot(H*J)$, and $\nabla \times (H * J)$, respectively, and the lower semicontinuity of the $L^2$-norm with respect to the weak convergence, we conclude (2.53). $\square$

Finally, we prove our third main result.

*Proof of Theorem 2.4.* The proof of existence of renormalized solutions begins similarly to the proof of Theorem 2.3: let $(f_0^n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^6)$ and $(k_n)_{n\in\mathbb{N}}$ given by Lemma 2.16. By Steps 1-3 in the proof of Theorem 2.3 we get a sequence of smooth functions $f^n$ satisfying (7) with initial condition $f_0^n$ and kernel $K^n$ (see Proposition 2.13) such that

$$\begin{aligned} f^n &\rightharpoonup f \quad \text{weakly in } L^1([0,T]\times\mathbb{R}^6) \text{ for any } T>0; \\ \rho^n &\rightharpoonup \rho^{\mathrm{eff}} \quad \text{weakly* in } L^\infty((0,\infty);\mathscr{M}_+(\mathbb{R}^3)); \\ |J^{\mathrm{eff}}| &< \rho^{\mathrm{eff}} \quad \text{as measures;} \\ \partial_t\rho^{\mathrm{eff}} + \nabla\cdot J^{\mathrm{eff}} &= 0 \quad \text{as measures with initial condition } \rho_0, \end{aligned} \tag{2.54}$$

where $\rho_t^n(x) := \int_{\mathbb{R}^3} f_t^n(x,v)\,\mathrm{d}v$. Analogously to (2.30), we have that for $\sigma_E \in \{-1, 0\}$,

$$\|f_t^n\|_{L^{3/2}(\mathbb{R}^6)} \le \|f_0\|_{L^{3/2}(\mathbb{R}^6)}, \quad \|f_t\|_{L^{3/2}(\mathbb{R}^6)} \le \|f_0\|_{L^{3/2}(\mathbb{R}^6)} \quad \text{for a.e. } t\in[0,\infty). \tag{2.55}$$

Moreover, since (2.39) holds for classical solutions and $f_0$ has every energy bounded, we obtain that

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,\infty)} \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t^n \,\mathrm{d}x\,\mathrm{d}v \le C, \tag{2.56}$$

and by the lower semicontinuity of the relativistic energy we deduce that, for every $T > 0$,

$$\int_0^T \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^T \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t^n \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \le CT. \tag{2.57}$$

We now claim that $\rho^{\text{eff}} = \rho$ and, consequently, $J = J^{\text{eff}}$, where $|J| < \rho \in L^\infty((0,T); L^1(\mathbb{R}^6))$ as in (7). For this, consider $\zeta_k : \mathbb{R}^6 \longrightarrow [0,1]$ a nonnegative function which equals 1 inside $B_k$ and 0 in $B_{k+1}^c$ and compute

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^3} (\rho_t^n - \rho_t)\varphi_t \,\mathrm{d}x\,\mathrm{d}t &= \int_0^\infty \int_{\mathbb{R}^6} (f_t^n(x,v) - f_t(x,v))\varphi_t(x)\zeta_k(v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\infty \int_{\mathbb{R}^6} f_t^n(x,v)\varphi_t(x)(1-\zeta_k(v))\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\infty \int_{\mathbb{R}^6} f_t(x,v)\varphi_t(x)(\zeta_k(v)-1)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By the weak convergence in $L^1$ in (2.54), the first term vanishes as $n \longrightarrow \infty$. The second and third terms can be estimated using (2.56) and (2.57):

$$\begin{aligned}&\Bigg| \int_0^\infty \int_{\mathbb{R}^6} f_t^n(x,v)\varphi_t(x)(1-\zeta_k(v))\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^\infty \int_{\mathbb{R}^6} f_t(x,v)\varphi_t(x)(\zeta_k(v)-1)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \Bigg| \\ &+ \frac{\|\varphi\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}}{k} \int_0^{T_\varphi} \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t^n \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &+ \frac{\|\varphi\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}}{k} \int_0^{T_\varphi} \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &\le \frac{CT_\varphi \|\varphi\|_{L^\infty((0,\infty)\times\mathbb{R}^3)}}{k},\end{aligned}$$

where $T_\varphi$ is the time support of $\varphi$. Letting $k \longrightarrow \infty$, we conclude that $\rho^n$ converges to $\rho$ weakly* in $L^\infty((0,\infty); \mathscr{M}_+(\mathbb{R}^3))$, which combined with (2.54) gives that $\rho = \rho^{\text{eff}}$. Hence, by (2.54) and Lemma 2.12, we conclude that $J = J^{\text{eff}}$, and in Steps 4 and 5 in the proof of Theorem 2.3, we obtain a global Lagrangian (hence renormalized) solution $f_t \in C([0,\infty); L^1_{\text{loc}}(\mathbb{R}^6))$ of (7) with initial datum $f_0$.

We now prove properties by a lower semicontinuous argument on the energy of $f^n$.

**Step 1: Bound on the total energy for $\mathscr{L}^1$-almost every time.** We use the weak convergence of $f^n$ (see (2.54)) with $\phi(t)\sqrt{1+|v|^2}\chi_r(x,v)$ as a test function, where $\phi \in C_c^\infty((0,\infty))$ and $\chi_r \in C_c^\infty(\mathbb{R}^6)$ are nonnegative functions, with $\chi_r$ being a cutoff between $B_r$ and $B_{r+1}$, we obtain

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^6} f_t(x,v)\sqrt{1+|v|^2}\phi(t)\chi_r(x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\le \liminf_{n\to\infty}\int_0^\infty \phi(t)\int_{\mathbb{R}^6}\sqrt{1+|v|^2}f_t^n(x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Taking the supremum with respect to $r$, we deduce that

$$\begin{aligned}&\int_0^\infty \phi(t)\int_{\mathbb{R}^6}\sqrt{1+|v|^2}f_t(x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\le \liminf_{n\to\infty}\int_0^\infty \phi(t)\int_{\mathbb{R}^6}\sqrt{1+|v|^2}f_t^n(x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.58}$$

Since $\phi$ is arbitrary, we have that $\sqrt{1+|v|^2}f_t \in L^1_{\mathrm{loc}}(\mathbb{R}^6)$ for almost every $t$. Moreover, since we can decompose the density current as $J = V\rho$ (see remark after Definition 2.11), where $|V| < 1$ a.e. in spacetime, we have that

$$\sup_{t\in[0,\infty)}\int_{\mathbb{R}^3}|V_t(x)|\,\mathrm{d}\rho_t(x) < \infty,$$

hence by [2, Theorem 8.1.2], we have that $\rho_t$ has a weakly* continuous representative. Furthermore, since $\rho^n$ satisfies a similar continuity equation, by the proof of [2, Theorem 8.1.2], we have that

$$\left|\int_{\mathbb{R}^3}(\rho_t^n-\rho_s^n)\varphi\,\mathrm{d}x\right| \le \|\varphi\|_{C^1(\mathbb{R}^3)}\int_s^t\int_{\mathbb{R}^3}|V_r^n|\rho_r^n\,\mathrm{d}x\,\mathrm{d}r \le C|t-s|$$

for all $\varphi \in C_c^\infty(\mathbb{R}^3)$, which gives that the map $t \longmapsto \int_{\mathbb{R}^3}\varphi\,\mathrm{d}\rho_t^n$ is equicontinuous. By the weak* convergence of $\rho^n$ to $\rho$ in $L^\infty((0,\infty);\mathscr{M}_+(\mathbb{R}^3))$, we have a uniform boundedness, thus Arzelà-Ascoli theorem implies that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\int_{\mathbb{R}^3}\varphi\,\mathrm{d}(\rho_t^n-\rho_t)\right| = 0 \quad \text{for every } \varphi\in C_c^\infty(\mathbb{R}^3). \tag{2.59}$$

Combining the above with the fact that $\rho_t^n$ is uniformly bounded with respect to $n$ and $t$, by Lemma 2.17 we obtain

$$\int_0^\infty \phi(t) \int_{\mathbb{R}^3} H*\rho_t(x)\,\mathrm{d}\rho_t(x)\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^\infty \phi(t) \int_{\mathbb{R}^3} H*\eta^{k_n}*\rho_t^n(x)\,\mathrm{d}\rho_t^n(x)\,\mathrm{d}t. \tag{2.60}$$

Combining (2.58), (2.60), and (2.39), we conclude that for $\sigma_E \in \{0, 1\}$

$$\begin{aligned}
&\int_0^\infty \phi(t) \left( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}v\,\mathrm{d}x + \frac{\sigma_E}{2} \int_{\mathbb{R}^3} H * \rho_t(x)\rho_t(x)\,\mathrm{d}x \right) \mathrm{d}t \\
&\le \liminf_{n\to\infty} \int_0^\infty \phi(t) \Bigg( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0^n(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&+ \frac{\sigma_E}{2} \int_{\mathbb{R}^3} H * \eta^{k_n} * \rho_0^n(x)\rho_0^n(x)\,\mathrm{d}x \Bigg) \mathrm{d}t \\
&= \left( \int_0^\infty \phi(t)\,\mathrm{d}t \right) \Bigg( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&+ \frac{\sigma_E}{2} \int_{\mathbb{R}^3} H * \rho_0(x)\rho_0(x)\,\mathrm{d}x \Bigg).
\end{aligned}$$

The case $\sigma_E = -1$ is subtler: by (2.58) and (2.39) we have that

$$\begin{aligned}
&\int_0^\infty \phi(t) \left( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} H * \rho_t(x)\rho_t(x)\,\mathrm{d}x \right) \mathrm{d}t \\
&\le \liminf_{n\to\infty} \int_0^\infty \phi(t) \Bigg( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t^n(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&- \frac{1}{2} \int_{\mathbb{R}^3} H * \rho_t(x)\rho_t(x)\,\mathrm{d}x \Bigg) \mathrm{d}t \\
&\le \left( \int_0^\infty \phi(t)\,\mathrm{d}t \right) \left( \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_0(x,v)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^3} H * \rho_0(x)\rho_0(x)\,\mathrm{d}x \right) \\
&+ \frac{1}{2} \limsup_{n\to\infty} \int_0^\infty \phi(t) \left( \int_{\mathbb{R}^3} H * \eta^{k_n} * \rho_t^n(x)\rho_t^n(x) - H * \rho_t(x)\rho_t(x)\,\mathrm{d}x \right) \mathrm{d}t
\end{aligned}$$

Notice that by (2.57), (2.55) and (2.46), we have for every $T > 0$,

$$\sup_{t\in[0,T]} \|\rho_t\|_{L^{6/5}(\mathbb{R}^3)} + \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|\rho_t^n\|_{L^{6/5}(\mathbb{R}^3)} < \infty.$$

Thus, by Lemma 2.18, we obtain that the last term equals 0. Since $\phi$ was arbitrary and since $f_0$ has every energy bounded, we conclude that $f_t$ has every energy bounded for $\mathscr{L}^1$-almost every $t \in (0, \infty)$.

**Step 2: Bound on the total energy for every time.** Notice that the relativistic and electric potential energy is lower semicontinuous with respect to the strong $L^1_{\text{loc}}$ and weak* $\mathscr{M}_+$ convergences, respectively. Hence, by the continuity of $t \longmapsto f_t \in L^1(\mathbb{R}^6)$ and $t \longmapsto \rho_t \in \mathscr{M}_+(\mathbb{R}^3)$ for the $L^1_{\text{loc}}$ and weak* $\mathscr{M}_+$ convergences, respectively, combined with Step 1, we have that for $t_n \longrightarrow \bar{t} \in [0, \infty)$ such that (2.40) holds for all $t_n$, we may pass the limit and obtain (2.40) for $t = \bar{t}$.

**Step 3: Strong $\boldsymbol{L^1_{\text{loc}}}$-continuity of the $\boldsymbol{\rho}$, $\boldsymbol{J}$, $\boldsymbol{E}$, $\boldsymbol{B}$.** Given $t \in [0, \infty)$, let $t_n \longrightarrow t$. Fix $r > 0$, and for any $R > 0$

$$\int_{B_r} \int_{\mathbb{R}^3} |f_{t_n} - f_t| \, \mathrm{d}v \, \mathrm{d}x \leq \int_{B_r} \int_{B_R} |f_{t_n} - f_t| \, \mathrm{d}v \, \mathrm{d}x \\ + R^{-1} \int_{B_r} \int_{\mathbb{R}^3} \sqrt{1 + |v|^2} (f_{t_n} + f_t) \, \mathrm{d}v \, \mathrm{d}x.$$

By the uniform boundedness of the relativistic energy with respect to time and the $L^1_{\text{loc}}$ continuity of $f_t$, by taking the limit in $n$ and then in $R$, we conclude that $\rho_{t_n} \longrightarrow \rho_t$ in $L^1_{\text{loc}}$. Moreover, since $|\hat{v}| < 1$, we have

$$\int_{B_r} |J_{t_n} - J_t| \, \mathrm{d}x < \int_{B_r} \int_{\mathbb{R}^3} |f_{t_n} - f_t| \, \mathrm{d}v \, \mathrm{d}x \longrightarrow 0,$$

thus $J_{t_n} \longrightarrow J_t$ in $L^1_{\text{loc}}$. Finally, since $K \in L^1_{\text{loc}}$ and $|J|(\mathbb{R}^3) < \rho(\mathbb{R}^3) < \infty$, we conclude that $E_t$, $B_t$ are also strongly continuous in $L^1_{\text{loc}}(\mathbb{R}^3)$.

**Step 4: Globally defined flow.** We can combine the fact that $f_t$ has every energy bounded and Lemma 2.19 to obtain that

$$E_t, \, B_t \in L^\infty([0, \infty); L^2(\mathbb{R}^3)),$$

thus by Corollary 2.10 we conclude that the trajectories of the maximal regular flow starting at any given $t$ do not blow up for $f_t$-almost every $(x, v) \in \mathbb{R}^6$.

**Step 5: Strong $\boldsymbol{L^1}$-continuity of $\boldsymbol{f}$.** By Theorem 2.2 and $L^1_{\text{loc}}$-continuity of $f_t$, we deduce that finite energy solutions conserve mass, i.e., $\rho_t(\mathbb{R}^3) = \rho_0(\mathbb{R}^3)$ for every $t \in [0, \infty)$. In particular, solutions are strongly continuous in $L^1(\mathbb{R}^6)$ and not only $L^1_{\text{loc}}(\mathbb{R}^6)$ (see [4, Theorem 4.10]). □

# Future problems

We now discuss further desirable results concerning the regularity of the free boundary associated to (3) and the Lagrangian structure obtained in Appendix 2 for the relativistic Vlasov-Maxwell system. Once again, due to the different nature of problems, we split the section in two parts.

## Free boundary regularity

Since existence, uniqueness and (almost) optimal regularity of solutions of (3) are achieved in chapter 1 (see Theorem 1.32), a natural problem arises: does the free boundary $\partial\{u = \psi\}$ have some regularity? The answer is affirmative for the fractional heat equation, that is, when $b \equiv 0$, $r \equiv 0$, $\mathcal{I} \equiv 0$, and for a obstacle $\psi$ with regularity $\psi \in C^4$, with $\|D^k\psi\|_{L^\infty(\mathbb{R}^n)} < \infty$ for $1 \leq k \leq 4$, and it was proven in [5]. Their main result states that there exists $\alpha > 0$ such that free boundary is $C^{1,\alpha}$ in space-time near regular points, i.e, points $(t_0, x_0)$ such that $u - \psi$ have subquadratic growth in the fractional parabolic cylinder $\mathcal{Q}_r(t_0, x_0)$. We remark that although that the regularity result of [7] comprehends the full range $0 < s < 1$, the free boundary regularity obtained in [5] is restricted to $s > 1/2$ due to the space scaling slower than time in this setting. Moreover, its proof heavily uses the scale invariance of the fractional heat equation, thus its adaptation for the full problem (3) is not straightforward. Nonetheless, it is expected that a similar result holds, since the regularity is given by the fractional heat operator and $\mathcal{R}$ is a lower order operator with respect to $(-\Delta)^s$.

We remark that one can investigate if the regularity of solutions of (3) is optimal, that is, if one can improve the time derivative regularity to $C_{t,x}^{\frac{1-s}{2s},1-s}$, however this is an open problem even for the fractional heat equation.

# Lagrangian structure for relativistic Vlasov-Maxwell system

We now recall that the system (7) is in fact a approximation of the relativistic Vlasov-Maxwell system

$$\begin{cases} \partial_t f_t + \hat{v} \cdot \nabla_x f_t + (E_t + \hat{v} \times B_t) \cdot \nabla_v f_t = 0; \\ \rho_t(x) = \int_{\mathbb{R}^3} f_t(x,v)\,\mathrm{d}v, \quad J_t(x) = \int_{\mathbb{R}^3} \hat{v} f_t(x,v)\,\mathrm{d}v; \\ \nabla \cdot E_t(x) = \rho_t(x), \quad \partial_t B_t(x) + \nabla \times E_t(x) = 0; \\ \nabla \cdot B_t(x) = 0, \quad \partial_t E_t(x) - \nabla \times B_t(x) = -J_t(x); \\ f_{t=0} = f_0. \end{cases} \tag{2.61}$$

Thus, it is natural to ask if the results of Appendix 2 (namely, Theorem 2.2, Theorem 2.3 and Theorem 2.4) can be extended to (2.61). The main difficulty at adapting the results is the lack of explicit representation of the electromagnetic field. In particular, the finite initial energy assumption in Theorem 2.4 probably does not work for a general electromagnetic field, since the potential energy and the electromagnetic energy may not have a simple relation as (2.41). One alternative is to use the Jefimenko equations [22] in order to recover an explicit form for the electromagnetic field, but additional difficulties arise, such as desynchronization of fields and sources, besides time derivatives of sources in the formula for such equations. Moreover, most of tools developed in [4] and its adaptations in Appendix 2 heavily use the explicit form of fields, thus preventing a straightforward generalization.

# Bibliography


[1] A. L. Amadori. Obstacle problem for nonlinear integro-differential equations arising in option pricing. *Ric. Mat.*, 56(1):1–17, 2007.

[2] L. Ambrosio. Gradient flows in metric spaces and in the Wasserstein space of probability measures. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 29:15–33, 2005.

[3] L. Ambrosio, M. Colombo, and A. Figalli. Existence and uniqueness of maximal regular flows for non-smooth vector fields. *Arch. Ration. Mech. Anal.*, 218(2):1043–1081, 2015.

[4] L. Ambrosio, M. Colombo, and A. Figalli. On the Lagrangian structure of transport equations: the Vlasov-Poisson system. *Duke Math. J.*, 166(18):3505–3568, 2017.

[5] B. Barrios, A. Figalli, and X. Ros-Oton. Free boundary regularity in the parabolic fractional obstacle problem. *Comm. Pure Appl. Math.*, 71(10):2129–2159, 2018.

[6] F. Bouchut, F. Golse, and C. Pallard. Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system. *Arch. Ration. Mech. Anal.*, 170(1):1–15, 2003.

[7] L. Caffarelli and A. Figalli. Regularity of solutions to the parabolic fractional obstacle problem. *J. Reine Angew. Math.*, 680:191–233, 2013.

[8] L. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.*, 171(2):425–461, 2008.

[9] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[10] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[11] M. Colombo. *Flows of non-smooth vector fields and degenerate elliptic equations*, volume 22 of *Tesi. Scuola Normale Superiore di Pisa (Nuova Series) [Theses of Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2017. With applications to the Vlasov-Poisson and semigeostrophic systems.

[12] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[13] C. M. Dafermos and E. Feireisl, editors. *Handbook of differential equations: evolutionary equations. Vol. III*. Handbook of Differential Equations. Elsevier/North-Holland, Amsterdam, 2007.

[14] R. DiPerna and P.-L. Lions. Solutions globales d'équations du type Vlasov-Poisson. *C. R. Acad. Sci. Paris Sér. I Math.*, 307(12):655–658, 1988.

[15] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[16] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[17] E. B. Fabes, C. E. Kenig, and R. P. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.

[18] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[19] R. T. Glassey and J. Schaeffer. On global symmetric solutions to the relativistic Vlasov-Poisson equation in three space dimensions. *Math. Methods Appl. Sci.*, 24(3):143–157, 2001.

[20] R. T. Glassey and Jack Schaeffer. On symmetric solutions of the relativistic Vlasov-Poisson system. *Comm. Math. Phys.*, 101(4):459–473, 1985.

[21] R. T. Glassey and W. A. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Rational Mech. Anal.*, 92(1):59–90, 1986.

[22] J. D. Jackson. *Classical electrodynamics*. John Wiley & Sons, Inc., New York-London-Sydney, second edition, 1975.

[23] S. Klainerman and G. Staffilani. A new approach to study the Vlasov-Maxwell system. *Comm. Pure Appl. Anal.*, 1(1):103–125, 2002.

[24] J. Kristensen and G. Mingione. The singular set of minima of integral functionals. *Arch. Ration. Mech. Anal.*, 180(3):331–398, 2006.

[25] P. Laurence and S. Salsa. Regularity of the free boundary of an american option on several assets. *Comm. Pure and Appl. Math.*, 62(7):969–994, 2009.

[26] W. R. Madych and N. M. Rivière. Multipliers of the Hölder classes. *J. Functional Analysis*, 21(4):369–379, 1976.

[27] G. Manfredi. Non-relativistic limits of Maxwell's equations. *European J. of Phys.*, 34(4):859–871, apr 2013.

[28] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *J. of Financial Econ.*, 3(1):125 – 144, 1976.

[29] A. Petrosyan and C. A. Pop. Optimal regularity of solutions to the obstacle problem for the fractional Laplacian with drift. *J. Funct. Anal.*, 268(2):417–472, 2015.

[30] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.