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CVaR optimization of high dimensional portfolios using dynamic factor copulas

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Resumo

Modelos de cópulas tornaram-se um método popular para a otimização de portfólios via Valor-em-Risco Condicional (CVaR). A abordagem de estimação normalmente é composta por dois passos: no primeiro, modelos ARMA-GARCH univariados são utilizados para ajustar cada retorno dos ativos, enquanto que em um segundo passo, a estrutura de dependência do retorno dos ativos é modelada utilizando funções de cópulas. Com o aumento do número de ativos compondo um portfólio, a estimação de modelos tradicionais de cópulas dinâmicas torna-se computacionalmente onerosa. Neste trabalho, nossa contribuição principal é de utilizarmos modelos de cópulas fatoriais dinâmicas para encontrarmos um portfólio de alta dimensão ótimo no sentido de minimizar o seu CVaR. Cópulas fatoriais são capazes de lidar com a "maldição da dimensionalidade" enquanto ainda oferecem um alto nível de complexidade e flexibilidade em seus modelos. Para introduzir variação temporal nos parâmetros de dependência das cópulas, utilizamos o modelo Generalizado de Scores Autoregressivos (GAS). Ainda, consideramos duas estruturas distintas de dependência: dependência homogênea e dependência em blocos. Utilizando dados de ações do Ibovespa de Janeiro de 2013 a Dezembro de 2020, aplicamos uma janela móvel de um dia para estimar ambos os modelos univariados e as funções de cópulas e também achar os pesos ótimos do portfólio para o dia seguinte. Os resultados empíricos sugerem que os modelos de cópulas fatoriais têm medidas de risco e retorno similares ou superiores em relação a um portfólio de uma cópula Gaussiana tradicional, sendo também consideravelmente superiores a dois portfólios de Markowitz de média-variância diferentes, um portfólio com pesos iguais para cada ativo e o índice IBRX50.

Abstract

Copula models have become a popular Conditional Value-at-Risk (CVaR) portfolio optimization method. The estimation approach normally is composed by two steps: in the first, univariate ARMA-GARCH models are commonly fit to each asset return; whereas in a second step, the returns dependence structure is modeled using copula functions. As the number of assets in a portfolio increases, the estimation of traditional dynamic copulas becomes computationally burdensome. In this work, our novel contribution is to employ dynamic factor copula models to find an optimal high dimensional portfolio in the sense of minimizing its CVaR. Factor copulas are able to address the ”curse of dimensionality” while offering a high level of complexity and flexibility to the models. We introduce time variation into the copula dependence parameters using a Generalized Autoregressive Scores (GAS) model. Two distinct dependence structures are considered: homogeneous dependence and block dependence. Using data consisting of Ibovespa Brazilian stocks from January 2013 to December 2020, we apply a one-day rolling window to estimate both univariate models and copula functions and also to find optimal portfolio weights for the following day. Empirical results suggest that our min-CVaR-factor-copula proposed strategy has superior or similar risk and return measures with respect to a traditional Gaussian copula while being considerably superior to two different Markowitz mean-variance portfolios, an Equal Weights portfolio and the IBRX50 index.

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Chapter 1

Introduction

This dissertation studies a dynamic factor copula method to optimize portfolios consisting of many assets with regard to its Conditional-Value-at-Risk (CVaR). The study is fully presented in chapter 2 of the present work. In addition, the current chapter discloses the main concepts employed throughout the study as well as our findings.

1.1 Portfolio optimization using CVaR

Over recent years, CVaR has become a popular risk measure in portfolio optimization problems due to its helpful properties. As opposed to the famous Markowitz mean-variance optimization, CVaR considers only the downside risk of a portfolio in order to find optimal asset allocation. Specifically, it measures the risk of potential extreme losses an investor may face while not limiting the upside risk as variance minimization problems do.

Following Rockafellar and Uryasev (2000), we define CVaR at a confidence level β as

$$CVaR_\beta(\boldsymbol{w}) = \frac{1}{1-\beta} \int_{f(\boldsymbol{w},\boldsymbol{r}) \geqslant VaR_\beta(\boldsymbol{w})} f(\boldsymbol{w},\boldsymbol{r}) p(\boldsymbol{r})\, d\boldsymbol{r}. \tag{1.1}$$

In the equation, $f(\boldsymbol{w},\boldsymbol{r})$ is a loss function depending upon a decision vector $\boldsymbol{w}$ that belongs a set $W \in \mathbb{R}^n$ of feasible portfolios and a random vector $\boldsymbol{r} \in \mathbb{R}^m$. The vector $\boldsymbol{r}$ stands for the uncertainties that can affect the loss. Then for each $\boldsymbol{w}$, the loss $f(\boldsymbol{w},\boldsymbol{r})$ is a random variable having a distribution in $\mathbb{R}$ conditioned by that of $\boldsymbol{r}$. It is assumed that $\boldsymbol{r}$ has a probability density function denoted by $p(\boldsymbol{r})$.

Given equation 1.1, it is clear that CVaR uses VaR in its definition. As VaR renders non-convex portfolio optimization problems, the use of the equation above can be troublesome. The main contribution of Rockafellar and Uryasev (2000) is the proposal of an auxiliary function that can be used to calculate CVaR without the need to compute VaR risk first:

$$F_\beta(\boldsymbol{w},\alpha) = \alpha + \frac{1}{1-\beta} \int_{\boldsymbol{r}\in\mathbb{R}^m} (f(\boldsymbol{w},\boldsymbol{r}) - \alpha)^+ \, p(\boldsymbol{r})\, d\boldsymbol{r}, \tag{1.2}$$

where $(h)^+ = max(h, 0)$.

Then, by minimizing the auxiliary function, we end up obtaining the CVaR, i.e.,

$$CVaR_\beta(\boldsymbol{w}) = \min_{\alpha \in \mathbb{R}} F_\beta(\boldsymbol{w}, \alpha). \tag{1.3}$$

An important feature of $F_\beta(\boldsymbol{w}, \alpha)$ is that it is convex with respect to α and continuously differentiable. This is especially useful from a numerical point of view since continuously differentiable convex problems are straightforward to solve.

Furthermore, the integral in equation 1.2 can be approximated by sampling $\boldsymbol{r_j}$ observations from the probability density function $p(\boldsymbol{r})$, where $j = 1, ..., J$. Then, the approximation of $F_\beta(\boldsymbol{w}, \alpha)$ is given as

$$F^d_\beta(\boldsymbol{w}, \alpha) = \alpha + \frac{1}{(1-\beta)J} \sum_{j=1}^{J} (f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha)^+, \tag{1.4}$$

which is convex and linear regarding α. Although not differentiable with respect to α, we can obtain its minimum with a linear programming problem.

Finally, if $f(\boldsymbol{w}, \boldsymbol{r})$ is convex with respect to $\boldsymbol{w}$, then $CVaR_\beta(\boldsymbol{w})$ is convex regarding $\boldsymbol{w}$ and $F^d_\beta(\boldsymbol{w}, \alpha)$ is convex regarding $(\boldsymbol{w}, \boldsymbol{r})$. Additionally, if the set W of feasible portfolios including constraints is a convex set, minimizing the CVaR associated with the loss function is equivalent to minimize $F^d_\beta(\boldsymbol{w}, \alpha)$ over all $(\boldsymbol{w}, \alpha) \in W \times \mathbb{R}$, i.e,

$$\min_{\boldsymbol{w} \in W} CVaR_\beta(\boldsymbol{w}) = \min_{\boldsymbol{w} \in W,\, \alpha \in \mathbb{R}} F^d_\beta(\boldsymbol{w}, \alpha). \tag{1.5}$$

By solving (1.5) we can simultaneously obtain the optimal portfolio vector of weights, $\boldsymbol{w}^*$, the portfolio's corresponding VaR, α^*, and the minimum CVaR associated to the loss, which equals to $F^d_\beta(\boldsymbol{w}^*, \alpha)$.

In portfolio optimization problems, a common loss function is the negative of the portfolio return. We interpret $\boldsymbol{r}$ as the random vector that constitutes the joint distribution of the returns of the assets in the portfolio, with density $p(\boldsymbol{r})$ and independent of $\boldsymbol{w}$. Then, $f(\boldsymbol{w}, \boldsymbol{r}) = -\boldsymbol{w}'\boldsymbol{r}$, which is convex and linear regarding $\boldsymbol{w}$.

Using equation 1.4, the minimization of the approximation $F^d_\beta(\boldsymbol{w}, \alpha)$ with the defined loss function can be solved via convex programming. With the use of auxiliary variables, $b_j \in \mathbb{R}$, $j = 1, ..., J$, minimizing the approximation is equivalent to minimizing the linear expression

$$\begin{aligned} \min_{\boldsymbol{w} \in W,\, b \in \mathbb{R}^J,\, \alpha \in \mathbb{R}} \quad & \alpha + \frac{1}{(1-\beta)J} \sum_{j=1}^{J} b_j \\ \text{s.t} \quad & b_j \geqslant f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha, \quad j = 1, ..., J, \\ & b_j \geqslant 0, \quad j = 1, ..., J, \end{aligned} \tag{1.6}$$

subject to additional linear constraints. It is then possible to solve the minimization problem using efficient linear programming algorithms such as *simplex*.

1.2 Copulas

Often, it is not desired to work with $p(\boldsymbol{r})$ itself. Methods to obtain the joint return distribution can be computationally burdensome in a portfolio of many assets since there are many parameters to estimate. A common approach is to assume that the joint probability of the asset returns $\boldsymbol{r}$ follows a multivariate Normal distribution. However, asset returns can possess tail dependence, heavy tails and asymmetry. This behavior is usually referred to as financial stylized facts, and while not addressed by the multivariate Normal assumption, still an essential feature in the financial analysis of a portfolio.

Consider the copula function C(.),

$$C(u_1, ..., u_n) = P(U_1 \leqslant u_i, ..., U_n \leqslant u_n), \tag{1.7}$$

in which u_i are realizations of U_i for $i = 1, ..., n$, and $U_i \sim U[0, 1]$. Copulas have become a popular method in portfolio optimization problems. Following Kakouris and Rustem (2014), it is possible to represent the density function of an n-copula as

$$c(u_1, ..., u_n) = \frac{\partial^n C(u_1, ..., u_n)}{\partial u_1, ..., \partial u_n} = \frac{f(r_1, ..., r_n)}{\prod_{i=1}^{n} f_i(r_i)}. \tag{1.8}$$

This useful representation allows us to replace the estimation of the complex joint distribution of the asset returns by separately modeling the univariate distribution of the assets and their dependence structure captured by the copula.

Joe and Xu (1996) show that a two-step procedure is a consistent method to estimate the dependence of the asset returns. In a first step, the conditional marginal distribution of each asset's time series is estimated using univariate models. In a second step, one estimates the multivariate dependence parameters associated with the copula functions. Then, the optimal portfolio is obtained by generating return scenarios derived from the univariate models and the dependence between the asset returns, as in equation 1.4.

Different copula functions and estimation methods have surfaced in the last decades. Common methods include Vine Copulas and mixtures of copula distributions. A mixture, as in Pfaff (2012), aims to use a linear combination of different copula functions to capture different asset return dependence structures, such as lower and upper tail dependence. Vine copulas, proposed by Joe (1994) and Joe (1996), aim for a graphical construction of a tree of bivariate copulas, which together build on to a desired multivariate distribution. Due to good performance when dealing with a moderate number of assets, they have become prevalent in financial analysis and have many variations, such as Tófoli et al. (2016), who introduced a dynamic vine copula approach.

Nevertheless, dynamic vine copulas can also become an infeasible method when dealing with portfolios of high dimensions due to the large number of parameters to estimate.

1.3 Dynamic Factor Copulas

A factor copula model, introduced by Oh and Patton (2017), aims to build a copula model with fewer parameters to estimate based on data dimension reduction methods. A factor copula model with one factor for N asset returns time series is given by

$$X_{i_t} = \lambda_{i_t} Z_t + \epsilon_{i_t}, \qquad i = 1, 2, ..., N, \tag{1.9}$$

where $Z_t \sim F_Z(\gamma_Z)$, $\epsilon_{i_t} \sim F_\epsilon(\gamma_\epsilon)$, $Z \perp \epsilon_i \ \forall i$, λ_{i_t} is the ith factor loading at time t, γ_Z and γ_ϵ are vectors of the parameters from the distribution of Z and ϵ respectively and $\boldsymbol{X}_t = (X_{1_t}, ..., X_{N_t})$ is an underlying vector of variables whose copula is the same one as the copula from the observable variables $\boldsymbol{Y}_t = (Y_{1_t}, ..., Y_{N_t})$.

In order to model Z and ϵ, Oh and Patton (2017) propose the use of a skew-t distribution. This way, important stylized facts of financial random variables, such as asymmetry, tail dependence and heavy tails, can be addressed.

Two essential aspects of factor copula models are how the dependence structure is built and how it varies over time. Regarding the former, we consider two different dependence structures: Equidependence (Homogeneous dependence), where all the asset returns equally depend on each other, and block dependence, which divides the assets among groups with equal intra-group return dependence. Regarding the latter, we use the GAS model developed by Creal et al. (2013) to deal with the time variation of the factor copula loadings, as applied in Oh and Patton (2018).

Usually, factor copulas do not have a closed form and their estimation can be compromised. When dealing with time-varying factor copulas, the most common estimation method is based on maximizing a numerical approximation of the likelihood function.

Let $c_t(u_{1_t}, ..., u_{N_t})$ be the density of the factor copula $\boldsymbol{X}_t$. Then,

$$c_t(u_{1_t}, ..., u_{N_t}) = \frac{h_{X_t}[F^{-1}_{X_{1_t}}(u_{1_t}), ..., F^{-1}_{X_{N_t}}(u_{N_t})]}{f_{1_t}(F^{-1}_{X_{1_t}}(u_{1_t})) \times ... \times f_{N_t}(F^{-1}_{X_{N_t}}(u_{N_t}))}. \tag{1.10}$$

We define

$$f_{i_t}(x_{i_t}) = \int_0^1 f_\epsilon(x_{i_t} - \lambda_{i_t} F_Z^{-1}(m))dm, \tag{1.11}$$

$$F_{X_{i_t}}(x_{i_t}) = \int_0^1 F_\epsilon(x_{i_t} - \lambda_{i_t} F_Z^{-1}(m))dm, \tag{1.12}$$

and

$$h_{X_t}(x_{1_t}, ..., x_{N_t}) = \int_0^1 \prod_{i=1}^{N} f_\epsilon(x_{i_t} - \lambda_{i_t} F_Z^{-1}(m))dm. \tag{1.13}$$

The log of the copula likelihood function is then given by

$$l_c(\theta_c|\hat{\theta}_1, ..., \hat{\theta}_N, \boldsymbol{w}) = \sum_{t=1}^{T} \ln c[F_1(y_{1_t}|\hat{\theta}_1, \boldsymbol{w}), ..., F_n(y_{N_t}|\hat{\theta}_N, \boldsymbol{w})|\boldsymbol{w}, \theta_c], \tag{1.14}$$

where $\theta_c = [\gamma_Z, \gamma_\epsilon, \gamma_\lambda]'$ is the set of copula parameters to be estimated.

1.4 Proposal of this work

We propose using an ARMA-GARCH GAS factor copula model to model and optimize a minimal-CVaR high-dimensional portfolio. We apply the framework for stocks composing the Ibovespa index and compare the method to a traditional Gaussian copula model with fixed parameters, a Markowitz mean-variance portfolio, a second Markowitz portfolio with a smaller estimation window, an Equally Weighted portfolio and the index of the 50 more tradeable stocks in Brazilian market, IBRX50. We find that the performance of our proposed strategy is superior to both Markowitz portfolios, the Equally Weighted portfolio and the IBRX50 index and similar or superior to the Gaussian one regarding different risk and return measures. The proposal is presented in an article format in chapter 2.

Chapter 2

Article

The attached research article, *CVaR optimization of high dimensional portfolios using dynamic factor copulas* (Alovisi and Ziegelmann, 2022), comprises the main contribution of the present Thesis.

CVaR optimization of high dimensional portfolios using dynamic factor copulas

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Abstract

Copula models have become a popular Conditional Value-at-Risk (CVaR) portfolio optimization method. The estimation approach normally is composed by two steps: in the first, univariate ARMA-GARCH models are commonly fit to each asset return; whereas in a second step, the returns dependence structure is modeled using copula functions. As the number of assets in a portfolio increases, the estimation of traditional dynamic copulas becomes computationally burdensome. In this work, our novel contribution is to employ dynamic factor copula models to find an optimal high dimensional portfolio in the sense of minimizing its CVaR. Factor copulas are able to address the "curse of dimensionality" while offering a high level of complexity and flexibility to the models. We introduce time variation into the copula dependence parameters using a Generalized Autoregressive Scores (GAS) model. Two distinct dependence structures are considered: homogeneous dependence and block dependence. Using data consisting of Ibovespa Brazilian stocks from January 2013 to December 2020, we apply a one-day rolling window to estimate both univariate models and copula functions and also to find optimal portfolio weights for the following day. Empirical results suggest that our min-CVaR-factor-copula proposed strategy has superior or similar risk and return measures with respect to a traditional Gaussian copula while being considerably superior to two different Markowitz mean-variance portfolios, an Equal Weights portfolio and the IBRX50 index.

Keywords: Portfolio Optimization, Factor Copulas, CVaR, Financial Econometrics, Financial Risk

1. Introduction

Portfolio optimization is the practice of distributing resources among different investments. This may be, for example, among different asset classes, such as stocks, funds, bonds and real estate, or among different stocks in an equity portfolio. Generally, in this kind of problem, the asset's returns are described as random variables, and the selection of an optimal risk-return portfolio depends on the underlying assumptions about the behavior of the returns and the choice of an adequate risk measure.

Markowitz (1952) in his seminal paper introduced the mean-variance optimal portfolio, one of the earliest portfolio optimization frameworks. In a traditional mean-variance optimization, the portfolio's risk and mean-return estimation is obtained by the asset returns covariance matrix and the sample average of the returns, respectively. Markowitz identified that by diversifying a portfolio among different assets and return patterns, it is possible to build an efficient portfolio with either (i) minimum risk for a specified target of portfolio mean-returns or (ii) maximum expected return for a specified risk target.

An important feature of the traditional mean-variance portfolio optimization is the assumption of normality to represent the behavior of financial returns. However, returns often exhibit characteristics that set them apart from normality, such as asymmetry, fat tails and tail dependence. These characteristics, known as financial stylized facts, are not addressed by the normality, but remain of great importance in portfolio optimization. Therefore, newer optimization frameworks and risk measures have emerged over the last decades.

As Pfaff (2012) highlighted, through 1990s and 2000s the focus of investors shifted due to the need to measure losses during financial crises of the period more accurately. Downside risk, as defined in Sortino and van der Meer (1991), is the risk of an asset's actual return being lower than the expected return and the uncertainty of the magnitude of this difference. Given the need for updated risk measures, Value-at-Risk (VaR) has become very popular in financial risk management over the last decades as a measure of downside risk.

Nevertheless, the use of VaR as a risk measure for portfolio optimization has been criticized over recent years. Firstly, using VaR in a portfolio optimization does not result in a convex problem, meaning that the optimization can be hard to solve and have many local extrema. Also, it does not have sub-additivity property as in Artzner et al. (1999), which means that the portfolio risk is greater than the sum of the single risk measures of the assets included in the portfolio. Lastly, it gives a percentile of the loss distribution of the portfolio that does not provide the whole picture of the entire possible losses of the tail.

Under general conditions, the Conditional Value-at-Risk (CVaR), presented by Rockafellar and Uryasev (2000), can be defined as the expectation of the values of the tail distribution, that is, of those exceeding VaR. As a measure of risk, CVaR exhibits more desirable properties than VaR. The authors showed that finding the portfolio that minimizes CVaR can be achieved by minimizing an auxiliary and more tractable function that computes CVaR without the need to integrate over VaR values (numerical optimization of CVaR is difficult due to this dependence). It is also shown that using CVaR as a risk measure for a portfolio yields convex optimization problems that can be turned to linear programming, in which efficient algorithms to optimize portfolios with large dimensions exist.

In order to compute CVaR (and VaR) for a desired portfolio, one has to make assumptions about the underlying probability density of the loss function, which depends on the joint distribution of the asset returns. However, this is not an easy task and tends to become even more difficult as the number of assets increases. A popular way to deal with this issue is to assume that the asset returns follow a multivariate Normal distribution, although, as mentioned above, normality can seriously fail to represent the behavior of financial time series. To handle some of these issues, copulas have recently become a popular alternative - see Cherubini et al. (2004). They are functions that connect marginals to their multivariate distribution, thus having all pertinent information concerning the dependence structure among random variables.

The advantage of using copulas is that they allow splitting the estimation of the marginals distribution from the estimation of the multivariate dependence structure. Thus, complex characteristics presented in the marginal distributions are modeled in a first stage using univariate models such as ARMA-GARCH for each asset, while copula functions focus only on the dependence relationship among the asset returns. In finance, this originated the popular Copula-ARMA-GARCH modeling framework that has become a strong option for such problems.

In order to capture all possible dependence structures between the asset returns, common methods include Pair Copulas (for example Vine Copulas) and mixtures of different copula distributions. The latter aims to use a linear combination of copula functions that captures different dependence structures, such as lower and upper tail dependence. An application of the mixture method can be found in Pfaff (2012). However, as the number of assets of the portfolio increases, mixture copulas can become difficult to estimate as there are many different parameters, including those associated with the linear combination.

Vine copulas proposed by Joe (1994) and later refined in Joe (1996) aim for a graphical construction of a tree of bivariate copulas, which together build on to a multivariate distribution that the modeler seeks for. Vine copulas are extensively reviewed in Czado (2010) and tend to perform well when dealing with a moderate number of assets compared to traditional multivariate models. Thus, they became very popular in financial analysis and had many variations, such as Tófoli et al. (2016), who introduced a dynamic vine copula approach.

Nevertheless, dynamic vine copulas also become infeasible to estimate when dealing with portfolios of high dimensions due to a large number of parameters. In order to deal with the "curse of dimensionality" of a portfolio with many assets, copula models based on data reduction have been recently proposed. Oh and Patton (2017) and Krupskii and Joe (2013) independently developed two different methods of factor copulas, both with the same purpose of reducing the dimension of the data assuming that common factors can adequately represent the original group of dependence associations. In this work, we use the proposal of Oh and Patton (2017).

Two important elements of factor copulas are how the dependence structure is built and how it varies over time. Regarding the former, we consider two different dependence structures: Equidependence (Homogeneous dependence), where all the asset returns equally depend on each other, and block dependence, which divides the assets among groups with equal intra-group return dependence. Regarding the latter, we use the GAS model developed by Creal et al. (2013) to deal with the time variation of the factor copula loadings, as applied in Oh and Patton (2018).

Therefore, we propose using an ARMA-GARCH GAS factor copula model to model and optimize a minimal-CVaR high dimensional portfolio. We apply the framework for stocks composing the Ibovespa index and compare the method to a traditional Gaussian copula model, two distinct Markowitz mean-variance portfolios, an Equally Weighted portfolio and the IBRX50 index. We find that the performance of our proposed strategy is superior to the IBRX50 index, Markowitz and Equally Weighted portfolios and similar or superior to the Gaussian one regarding risk and return measures.

The paper is structured as follows: Section 2 describes the CVaR and the (factor) copula portfolio optimization methodology. Section 3 presents the empirical optimization strategy for the portfolios. Section 4 shows the empirical results for the optimization strategies, whereas section 5 concludes the study.

2. Methodology

Consider the CVaR portfolio optimization problem where there is a need to model the joint distribution of the asset returns. We follow Joe and Xu (1996) Inference From Margins approach, which shows that a two-step estimation procedure is consistent. In a first step, the conditional marginal distribution of each asset's time series is estimated using univariate ARMA-GARCH models. In a second step, one estimates the multivariate dependence parameters associated with the (factor) copula functions. Then, the optimal portfolio is obtained by generating return scenarios derived from the univariate models and the dependence between the asset returns. This section describes the CVaR optimization of a portfolio and copula methods in what follows.

2.1. Conditional Value-at-Risk

For a thorough definition and understanding of CVaR, first, we need to define Value-at-Risk (VaR). Let $f(\boldsymbol{w}, \boldsymbol{r})$ be defined as a loss function depending upon a decision vector $\boldsymbol{w}$ that belongs to any arbitrarily chosen subset $W \in \mathbb{R}^n$ and a random vector $\boldsymbol{r} \in \mathbb{R}^m$. The vector $\boldsymbol{w}$ can be interpreted as representing a portfolio that belongs to the set W of feasible portfolios, including constraints. The vector $\boldsymbol{r}$ stands for the uncertainties that can affect the loss. Then for each $\boldsymbol{w}$, the loss $f(\boldsymbol{w}, \boldsymbol{r})$ is a random variable having a distribution in $\mathbb{R}$ conditioned by that of $\boldsymbol{r}$. We interpret $\boldsymbol{r}$ as a random vector that constitutes the joint distribution of the returns of the assets in the portfolio, with density $p(\boldsymbol{r})$ and independent of $\boldsymbol{w}$. However, an analytical representation of $p(\boldsymbol{r})$ is not needed to implement a CVaR optimization, as will be shown later.

For a fixed $\boldsymbol{w}$, we denote the cumulative distribution function of the loss associated with $\boldsymbol{w}$ as

$$\Psi(\boldsymbol{w}, \alpha) = \int_{f(\boldsymbol{w},\boldsymbol{r}) \geq \alpha} p(\boldsymbol{r}) d\boldsymbol{r}, \tag{1}$$

i.e., the probability of $f(\boldsymbol{w}, \boldsymbol{r})$ not exceeding a threshold α. It is worth noting that $\Psi(\boldsymbol{w}, \alpha)$ is non-decreasing with respect to α and continuous from the right.

Then, for a given confidence level β, the Value-at-Risk associated with $\boldsymbol{w}$ is given as

$$VaR_{\beta}(\boldsymbol{w}) = min\{\alpha \in \mathbb{R} \ : \ \Psi(\boldsymbol{w}, \alpha) \geq \beta\}. \tag{2}$$

Furthermore, the CVaR, at a confidence level β, can be defined as

$$CVaR_{\beta}(\boldsymbol{w}) = \frac{1}{1-\beta} \int_{f(\boldsymbol{w},\boldsymbol{r}) \geq VaR_{\beta}(\boldsymbol{w})} f(\boldsymbol{w}, \boldsymbol{r}) p(\boldsymbol{r}) \, d\boldsymbol{r}. \tag{3}$$

An important feature of CVaR is its coherence as a risk measure, in the sense of Artzner et al. (1999). A *coherent* risk measure satisfies four properties desirable in the context of portfolio optimization. For a thorough presentation of coherence and its properties, see Appendix A. Acerbi and Tasche (2002) and Rockafellar and Uryasev (2002) give a formal proof of the coherence of CVaR as a risk measure. Furthermore, it is worth noting that VaR is not coherent since it does not satisfy the property of Subadditivity - see Dańıelsson et al. (2005).

In addition, as we will further explore below, the use of the CVaR in the context of portfolio optimization generates a convex optimization problem. This is a desirable outcome since most convex optimization problems nowadays are straightforward to solve. However, using VaR in the same context does not render this useful characteristic. Thus, because of the important properties above, CVaR has become popular in applications of portfolio optimization.

2.2. CVaR Optimization Problem

Following Wuertz et al. (2010), an intuitive minimal-CVaR optimization problem can be written as

$$\begin{aligned} \text{min} \quad & CVaR_{\beta}(\boldsymbol{w}) \\ \text{s.t} \quad & \boldsymbol{w} \in W, \end{aligned} \tag{4}$$

where W is a set of feasible portfolio's optimal solutions, $\boldsymbol{w}$ is an asset weights vector where $\boldsymbol{w} \in W$ an β the desired CVaR significance. Additional linear constraints can be included in the feasible set W.

Given equation 3, it is clear that CVaR optimization uses VaR in its definition. As stated before, VaR's portfolio optimization problems are neither convex nor linear. Rockafellar and Uryasev (2000) in their main contribution, define a simpler and convenient auxiliary function that can be used to calculate CVaR without any need to compute VaR first:

$$F_\beta(\boldsymbol{w}, \alpha) = \alpha + \frac{1}{1-\beta} \int_{\boldsymbol{r} \in \mathbb{R}^m} (f(\boldsymbol{w}, \boldsymbol{r}) - \alpha)^+ \, p(\boldsymbol{r}) \, d\boldsymbol{r}, \tag{5}$$

where $(h)^+ = max(h, 0)$.

It is shown that $F_\beta(\boldsymbol{w}, \alpha)$ is convex with respect to α and continuously differentiable. By minimizing the auxiliary function, we end up obtaining the CVaR, i.e.,

$$CVaR_\beta(\boldsymbol{w}) = \min_{\alpha \in \mathbb{R}} F_\beta(\boldsymbol{w}, \alpha). \tag{6}$$

They also developed a method to approximate $F_\beta(\boldsymbol{w}, \alpha)$ by sampling $\boldsymbol{r}_j, \; j = 1, ..., J$ observations from the density function $p(\boldsymbol{r})$ using J scenarios, supposing that the analytical characterization for the density $p(\boldsymbol{r})$ is not available. Then, it's possible to approximate equation 5 by its discrete version:

$$F_\beta^d(\boldsymbol{w}, \alpha) = \alpha + \frac{1}{(1-\beta)J} \sum_{j=1}^{J} (f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha)^+, \tag{7}$$

which is convex and linear with respect to α. Although not differentiable with respect to α, we can obtain its minimum with a linear programming problem.

Furthermore, if $f(\boldsymbol{w}, \boldsymbol{r})$ is convex with respect to $\boldsymbol{w}$, then $CVaR_\beta(\boldsymbol{w})$ is convex regarding $\boldsymbol{w}$ and $F_\beta^d(\boldsymbol{w}, \alpha)$ is convex regarding $(\boldsymbol{w}, \boldsymbol{r})$. Additionally, if the set W of feasible portfolios including constraints is a convex set, minimizing the CVaR associated with the loss function is equivalent to minimize $F_\beta^d(\boldsymbol{w}, \alpha)$ over all $(\boldsymbol{w}, \alpha) \in W \times \mathbb{R}$, i.e,

$$\min_{\boldsymbol{w} \in W} CVaR_\beta(\boldsymbol{w}) = \min_{\boldsymbol{w} \in W, \, \alpha \in \mathbb{R}} F_\beta^d(\boldsymbol{w}, \alpha). \tag{8}$$

By solving (8) we can simultaneously obtain the optimal portfolio vector of weights, $\boldsymbol{w}^*$, the portfolio's corresponding VaR, α^*, and the optimal CVaR, which equals to $F^d_\beta(\boldsymbol{w}^*, \alpha)$.

Finally, if the loss function $f(\boldsymbol{w}, \boldsymbol{r}_j)$ is convex and linear with respect to $\boldsymbol{w}$ we can reduce (8) to a desired linear programming problem. The final linear problem investigated is then given by

$$\begin{aligned} \min_{\boldsymbol{w}\in W,\, b\in\mathbb{R}^J,\, \alpha\in\mathbb{R}} \quad & \alpha + \frac{1}{(1-\beta)J}\sum_{j=1}^{J} b_j \\ \text{s.t} \quad & b_j \geq f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha, \quad j = 1, ..., J, \\ & b_j \geq 0, \quad j = 1, ..., J, \\ & \boldsymbol{w} \in W, \\ & \boldsymbol{w}^{'}\boldsymbol{1} = 1, \\ & \boldsymbol{w}^{'}\hat{\boldsymbol{\mu}} \geq R, \\ & w_i \geq 0, \quad \forall\, w_i \in \boldsymbol{w}. \end{aligned} \tag{9}$$

The reduction to a linear programming problem is achieved by adding auxiliary variables b_j to replace $(f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha)^+$, imposing the linear constraints $b_j \geq f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha$ and $b_j \geq 0$.

Note that the constraints $b_j \geq f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha$ and $b_j \geq 0$ alone cannot ensure that $b_j = (f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha)^+$, since b_j can be larger than both of the right-hand terms while still being feasible. Nevertheless, as we are minimizing the objective function which involves a positive multiple of b_j, it will never be optimal to assign b_j a value larger than the maximum of the two quantities $f(\boldsymbol{w}, \boldsymbol{r}_j) - \alpha$ and 0, and therefore, in an optimal solution b_j will be precisely $(f(\boldsymbol{w}, \boldsymbol{r}_j)^+ - \alpha)^+$.

The loss function for a minimal-CVaR portfolio is defined as $f(\boldsymbol{w}, \boldsymbol{r}) = -\boldsymbol{w}'\boldsymbol{r}$, which is clearly convex and linear concerning $\boldsymbol{w}$. Additional feasible linear constraints of no short-selling, full investment and target portfolio returns are included in the problem, where R is a target portfolio return and $\hat{\boldsymbol{\mu}}$ is the asset mean return vector. Thus, for the final optimization, we have that no portfolio weights must be negative, the weights must sum to one and the optimal portfolio must produce the desired target return. Then, CVaR portfolio optimization can be carried out using efficient linear programming algorithms (for instance, *simplex*).

2.3. Copulas

The theory of copula functions was first presented by Sklar (1959). By the mid-1990s, copulas became a popular tool to model dependence between asset returns in empirical finance, as seen in Pfaff (2012). Copulas are multivariate distribution functions with standard uniform distributed margins. A copula $C(.)$ is a function where

$$C(u_1, ..., u_n) = P(U_1 \leq u_i, ..., U_n \leq u_n), \tag{10}$$

in which u_i are realizations of U_i for $i = 1, ..., n$, and $U_i \sim U[0, 1]$.

The major advantage of using copula functions is that the estimation of complicated multivariate distribution functions can be replaced by the estimation of univariate models and the copula functions. In a first step, all possible complexities of the marginal behavior of the asset returns are captured by the univariate model, while the use of the copula functions captures the multivariate dependence structure of these returns.

A famous theorem that originated this modelling method is called the Sklar's Theorem:

Theorem 2.1 (Sklar's Theorem). *Let F be an n-dimensional distribution function with margins $F_1, ...F_n$. Then, there exists an n-copula C, such that for all $r \in \mathbb{R}^n$,*

$$F(r_1, ..., r_n) = C(F_1(r_1), ..., F_n(r_n)). \tag{11}$$

Furthermore, if $F_1, ...F_n$ are continuous, then C is unique.

The margins $F_1, ..., F_n$ and the multivariate distribution function F are as defined above. The margins u_i can be replaced by $F_i(r_i)$ as they both belong to the domain $\mathbb{I}$ and are uniformly distributed, i.e, let $u \sim U(0, 1)$, then $P(F(r) \leq u) = P(r \leq F^{-1}(u)) = F(F^{-1}(u)) = u$. With the use of the theorem above, Kakouris and Rustem (2014) derive a relation between the probability density functions and copula functions. They define the copula density of a copula function with dimension n as:

Definition 2.1. *Let f be the multivariate probability density function of the probability distribution F and $f_1, ..., f_n$ the univariate probability density functions of the margins $F_1, ..., F_n$. The copula's density function of an n-copula C is the function c: $U[0, 1]^n \mapsto [0, \infty)$ such that*

$$c(u_1, ..., u_n) = \frac{\partial^n C(u_1, ..., u_n)}{\partial u_1, ..., \partial u_n} = \frac{f(r_1, ..., r_n)}{\prod_{i=1}^n f_i(r_i)}. \tag{12}$$

The definition allows us to separate modeling the marginals $F_i(r_i)$ from the dependence structure captured by C. I.e., copulas decompose the asset joint p.d.f from its margins, allowing a simpler estimation of the marginals and a copula dependence function instead of having to estimate and infer the complex multivariate distribution of the asset returns.

As in Hofert et al. (2018b), we can replace the estimation of the multivariate distribution with this simpler method in two parts: (i) finding the marginal distribution for each r_i (ii) finding the dependency between the filtered data from (i). As there are many different univariate and copula models and copula functions, this method provides high flexibility concerning common joint distribution models. An extensive review of copula modelling in finance can be found on Fan and Patton (2014) and Patton (2008).

2.4. Dynamic Factor Copulas

High-dimensional data, expressed here as a portfolio of many assets, often correlates to data dimension reduction methods. These methods include popular techniques such as principal component analysis (PCA) and factor analysis. The aim of introducing factor copulas in such financial problems is to have lesser parameters to estimate concerning traditional copulas while still capturing important dependence structures of the asset return.

Oh and Patton (2017) propose a factor copula approach in the context of modeling the joint distribution of a set of time series, allowing for a certain degree of flexibility and not a very high number of parameters to estimate. A factor copula model with one factor for N asset returns time series is given by

$$X_{i_t} = \lambda_{i_t} Z_t + \epsilon_{i_t}, \qquad i = 1, 2, ..., N, \tag{13}$$

where $Z_t \sim F_Z(\gamma_Z)$, $\epsilon_{i_t} \sim F_\epsilon(\gamma_\epsilon)$, $Z \perp \epsilon_i \ \forall i$, λ_{i_t} is the ith factor loading at time t, γ_Z and γ_ϵ are vectors of the parameters from the distribution of Z and ϵ respectively and $\boldsymbol{X}_t = (X_{1_t}, ..., X_{N_t})$ is an underlying vector of variables whose copula is the same one as the copula from the observable variables $\boldsymbol{Y}_t = (Y_{1_t}, ..., Y_{N_t})$. Then, we have that

$$Y_t \sim H_Y(y_{1_t}, ..., y_{N_t}|\boldsymbol{w}) = C(F_{1_t}(y_{1_t}|\boldsymbol{w}), ..., F_{N_t}(y_{N_t}|\boldsymbol{w})|\boldsymbol{w}) \tag{14}$$

and

$$X_t \sim H_X(x_{1_t}, ..., x_{N_t}) = C(F_{X_{1_t}}(x_{1_t}), ..., F_{X_{N_t}}(x_{N_t})). \tag{15}$$

In order to model Z and ϵ, we use a skew-t distribution and a t distribution, respectively, as Oh and Patton (2017). This way, we address important stylized facts of financial random variables, such as asymmetry, tail dependence and heavy tails.

2.5. Dependence structure

Even in a factor copula framework, the number of parameters to be estimated tends to grow as the number of time series grows. To deal with the increasing computation complexity implied, Oh and Patton (2017) suggest three different levels of time series dependence structures to ponder between optimization complexity and factor copula flexibility.

The first and least flexible structure is the homogeneous (equidependent) dependence, which assumes all the asset returns depend on each other in the same strength, i.e., a unique factor loading for every series. Oppositely, the most flexible structure, heterogeneous dependence, assumes different factor loadings for each series. Although having a high degree of flexibility, the latter has many parameters to estimate, presenting a complex optimization for the purpose of this work. Finally, on an intermediate level, the structured denominated block dependence splits the set of assets into homogeneous groups defined ad-hoc based on the market similarities between the assets. This way, asset returns contained in the same group have the same dependence strength and factor loading.

A factor copula model can then be defined in terms of its dependence structure, i.e.,

$$X_{i_t} = \lambda_{g(i),t}(\gamma_\lambda) Z_t + \epsilon_{i_t}, \qquad i = 1, 2, ..., N, \tag{16}$$

where $g(i) \in \{1, ..., G\}$ is the group to which the ith time series belongs, G is the number of homogeneous groups and γ_λ is the set of parameters linked to λ. Thus, $G = N$ is the particular case of heterogeneous dependence, $G = 1$ is the case of homogeneous dependence and $1 < G < N$ is the block dependence one.

2.5.1. Time-varying parameters

A common way to model dependence using copulas considers fixed parameters over time. Nevertheless, it makes sense to address a variation in dependence parameters as market conditions and asset behavior tend to change due to macroeconomic shocks. Patton (2006) proposes using a nonlinear restricted ARMA model to address time-varying copula dependence parameters. da Silva Filho et al. (2012) allow the ARMA dynamics of the parameters to be conducted by a Hidden Markov Chain.

Nonetheless, a recent popular method to address time variation in higher dimensions is the Generalized Autoregressive Score model (GAS), proposed by Creal et al. (2013). The method assumes a time-varying parameter f_t and a conditional observation density $p(y_t|f_t)$ for observation y_t. Then, the parameter f_t follows the recursion $f_{t+1} = \omega + \beta f_t + \alpha S(f_t)\frac{\partial \log p(y_t|f_t)}{\partial f_t}$, where $S(f_t)$ is a scaling function for the score of the log observation density. The method uses the scaled score to drive the time variation of the parameter f_t, linking the shape of the conditional observation density to the dynamics of f_t.

Oh and Patton (2018) use a GAS model to describe dynamics for factor copula parameters. This method will be used in this work and adapted to the context of portfolio optimization using factor copulas. The model structure can be described as

$$\ln \lambda_{g,t} = \omega_g + \beta \ln \lambda_{g,t-1} + \alpha \frac{\partial \ln c(\boldsymbol{u}_{t-1}, \lambda_{t-1}, \gamma_Z, \gamma_\epsilon)}{\partial \lambda_{g,t-1}}, \quad g = 1, ..., G, \tag{17}$$

where w_g have dimension G, $\lambda_t = \left[\lambda_{1,t}, ..., \lambda_{G,t}\right]'$ and $\gamma_\lambda = \left[\omega_g, \beta, \alpha\right]'$. There are $G + 2$ parameters to estimate; thus, in the heterogeneous case where $G = N$, it can become very burdensome to estimate the parameters as the number of dimensions (assets) increases. As this work focuses on offering computationally feasible estimations of time-varying factor copulas in higher dimensions, only the other two aforementioned dependence structures will be explored - block dependence and homogeneous dependence.

2.5.2. Factor Copulas Estimation

Usually, factor copulas do not have a closed form and their estimation can be compromised. For static copulas, Oh and Patton (2017) present a Method of Moments based estimation centered on comparing the observed data moments with simulated data moments. However, when dealing with time-varying copulas, the most common estimation method is based on maximizing a numerical approximation of the likelihood function.

Let $c_t(u_{1_t}, ..., u_{N_t})$ be the density of the factor copula $\boldsymbol{X}_t$. Then,

$$c_t(u_{1_t}, ..., u_{N_t}) = \frac{h_{X_t}[F^{-1}_{X_{1_t}}(u_{1_t}), ..., F^{-1}_{X_{N_t}}(u_{N_t})]}{f_{1_t}(F^{-1}_{X_{1_t}}(u_{1_t})) \times ... \times f_{N_t}(F^{-1}_{X_{N_t}}(u_{N_t}))}. \tag{18}$$

We define

$$f_{i_t}(x_{i_t}) = \int_0^1 f_\epsilon(x_{i_t} - \lambda_{i_t} F_Z^{-1}(m))dm, \tag{19}$$

$$F_{X_{i_t}}(x_{i_t}) = \int_0^1 F_\epsilon(x_{i_t} - \lambda_{i_t} F_Z^{-1}(m))dm, \tag{20}$$

and

$$h_{X_t}(x_{1_t}, ..., x_{N_t}) = \int_0^1 \prod_{i=1}^{N} f_\epsilon(x_{i_t} - \lambda_{i_t} F_Z^{-1}(m))dm. \tag{21}$$

The log of the copula likelihood function is then given by

$$l_c(\theta_c|\hat{\theta}_1, ..., \hat{\theta}_N, \boldsymbol{w}) = \sum_{t=1}^{T} \ln c[F_1(y_{1_t}|\hat{\theta}_1, \boldsymbol{w}), ..., F_n(y_{N_t}|\hat{\theta}_N, \boldsymbol{w})|\boldsymbol{w}, \theta_c], \tag{22}$$

where $\theta_c = [\gamma_Z, \gamma_\epsilon, \gamma_\lambda]'$ is the set of copula parameters to be estimated.

2.6. Marginal Modelling

Due to the high dimensional nature of the portfolio optimization problem, as there are many optimizations to run, we opt for an ARMA(1,1)-GARCH(1,1) model to estimate the conditional mean and the conditional variance of each of the asset returns. ARMA-GARCH models are known to be parsimonious while still offering satisfactory performance. The model is defined as follows:

$$\begin{aligned} R_t &= \mu + \phi R_{t-1} + \theta\epsilon_{t-1} + \epsilon_t \\ \epsilon_t &= \sigma_t Z_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \end{aligned} \tag{23}$$

where $Z_t \sim iid(0,1)$ are the standardized errors, which are independent of $\epsilon_{t-l},\ l \geq 1$. The standardized errors Z_t can follow any distribution with 0 mean and variance 1. Common distributions include the standard normal, GED and Student's t-distribution.

However, in order to correctly capture known asset's stylized facts of heavy tails and skewness, we use the Generalized Hyperbolic Skew Student's t-distribution, popularized by Aas and Haff (2006) and parameterized by Ghalanos (2014). The probability density function of the distribution is given by

$$f_Z(z) = \frac{2^{(1-\nu)/2}\delta^\nu|\beta|^{(\nu+1)/2}K_{(\nu+1)/2}\left(\sqrt{\beta^2(\delta^2 + (z-\mu)^2)}\right)e^{\beta(z-\mu)}}{\Gamma(\nu/2)\sqrt{\pi}\left(\sqrt{\delta^2 + (z-\mu)^2}\right)^{(\nu+1)/2}}, \tag{24}$$

where $\beta \in \mathbb{R}$ and $\nu > 0$. For the unconditional variance to be finite we need that $\nu > 4$ and for the existence of skewness and kurtosis, $\nu > 6$ and $\nu > 8$, respectively.

2.7. Copula CVaR Optimization

Now that the CVaR optimization problem for a random vector of distributions has been defined, as well as theorems to associate copulas and these distributions, we are able to define CVaR optimization with copula functions. This is accomplished following Kakouris and Rustem (2014) and Zhu and Fukushima (2009).

Let $\boldsymbol{w} \in W$ be a decision vector belonging to a feasible set W, $\boldsymbol{u} \in U[0,1]^n$ a random vector that follows a continuous distribution with copula density function $c(.)$ and $\boldsymbol{F}(\boldsymbol{r}) = (F_1(r_1), ..., F_n(r_n))$ a set of marginal distributions where $\boldsymbol{u} = \boldsymbol{F}(\boldsymbol{r})$. Also, $\tilde{g}(\boldsymbol{w}, \boldsymbol{u}) = g(\boldsymbol{w}, \boldsymbol{F}^{-1}(\boldsymbol{u})) = g(\boldsymbol{w}, \boldsymbol{r})$ maps the domain of the cost function from $\mathbb{R}^n$ to $\mathbb{I}^n$, as implied by the transformation $u_i = F_i(r_i)$. The copula definition of CVaR with respect to VaR similar as equation (3) is as follows:

$$\begin{aligned} CVaR_\beta(\boldsymbol{w}) &= \frac{1}{1-\beta} \int_{g(\boldsymbol{w},\boldsymbol{r}) \geq VaR_\beta(\boldsymbol{w})} g(\boldsymbol{w}, \boldsymbol{r}) p(\boldsymbol{r}) d\boldsymbol{r} \\ &= \frac{1}{1-\beta} \int_{g(\boldsymbol{w},\boldsymbol{r}) \geq VaR_\beta(\boldsymbol{w})} g(\boldsymbol{w}, \boldsymbol{r}) c(\boldsymbol{F}(\boldsymbol{r})) \prod_{i=i}^{n} f_i(r_i) d\boldsymbol{r} \\ &= \frac{1}{1-\beta} \int_{\tilde{g}(\boldsymbol{w},\boldsymbol{u}) \geq VaR_\beta(\boldsymbol{w})} \tilde{g}(\boldsymbol{w}, \boldsymbol{u}) c(\boldsymbol{u}) d\boldsymbol{u}. \end{aligned} \tag{25}$$

Then, the auxiliary copula corresponding equation of (5) is

$$\begin{aligned} G_\beta(\boldsymbol{w}, \alpha) &= \alpha + \frac{1}{1-\beta} \int_{\boldsymbol{r} \in \mathbb{R}^n} (g(\boldsymbol{w}, \boldsymbol{r}) - \alpha)^+ \, p(\boldsymbol{r}) \, d\boldsymbol{r} \\ &= \alpha + \frac{1}{1-\beta} \int_{\boldsymbol{u} \in U[0,1]^n} (\tilde{g}(\boldsymbol{w}, \boldsymbol{u}) - \alpha)^+ \, c(\boldsymbol{u}) \, d\boldsymbol{u}. \end{aligned} \tag{26}$$

In order for the above equations to be computed, exact knowledge of the distribution of $p(\boldsymbol{r})$ or copula density $c(\boldsymbol{u})$ and the margins $\boldsymbol{F}(\boldsymbol{r})$ are needed. Since we introduce copulas in our models not to deal with $p(\boldsymbol{r})$ directly, the latter will be used. Knowledge of the copula $C(\boldsymbol{u})$ and its margins

$u_i = F_i(r_i)$, $i = 1, ..., n$ implies knowledge of $p(\boldsymbol{r})$ and $c(\boldsymbol{u})$. The discrete version sampling K scenarios, similar to (7), is represented as

$$G^d_\beta(\boldsymbol{w}, \alpha) = \alpha + \frac{1}{(1-\beta)K} \sum_{k=1}^{K} (\tilde{g}(\boldsymbol{w}, \boldsymbol{u}_k) - \alpha)^+, \tag{27}$$

where we evaluate the function using Monte Carlo simulations. This is done by sampling realizations from the copula $C(.)$ using as inputs the filtered uniform margins, where $\boldsymbol{u}_k$ is the k-th, sample drawn from the copula $C(.)$, $k = 1, ..., K$.

Following the assumptions of convexity and linearity of the loss function $\tilde{g}(\boldsymbol{w}, \boldsymbol{u})$ with respect to $\boldsymbol{w}$, the optimization problem,

$$\min_{\boldsymbol{w} \in W} CVaR_\beta(\boldsymbol{w}) = \min_{\boldsymbol{w} \in W,\, \alpha \in \mathbb{R}} G^d_\beta(\boldsymbol{w}, \alpha), \tag{28}$$

can be modelled following Rockafellar and Uryasev (2002) minimal-CVaR approach as

$$\begin{aligned} \min_{\boldsymbol{w} \in \mathbb{R}^n,\, b \in \mathbb{R}^K,\, \alpha \in \mathbb{R}} \quad & \alpha + \frac{1}{(1-\beta)K} \sum_{k=1}^{K} b_k, \\ \text{s.t} \quad & b_k \geq \tilde{g}(\boldsymbol{w}, \boldsymbol{u}_k) - \alpha, \quad k = 1, ..., K, \\ & b_k \geq 0, \quad k = 1, ..., K, \\ & \boldsymbol{w} \in W, \\ & \boldsymbol{w}' \boldsymbol{1} = 1, \\ & \boldsymbol{w}' \hat{\boldsymbol{\mu}} \geq R, \\ & w_i \geq 0, \quad \forall\, w_i \in \boldsymbol{w}, \end{aligned} \tag{29}$$

where the loss function for a minimal-CVaR optimization is defined as $\tilde{g}(\boldsymbol{w}, \boldsymbol{u}) = -\boldsymbol{w}' \boldsymbol{F}^{-1}(\boldsymbol{u})$.

3. Optimization Strategy

The empirical study of the portfolio optimization methods described above considers a dataset of 30 Brazilian stocks downloaded from *Yahoo! Finance* from January 3, 2013 to November 13, 2020. The dataset encompasses 1957 trading days, where 1956 daily price log returns are calculated.

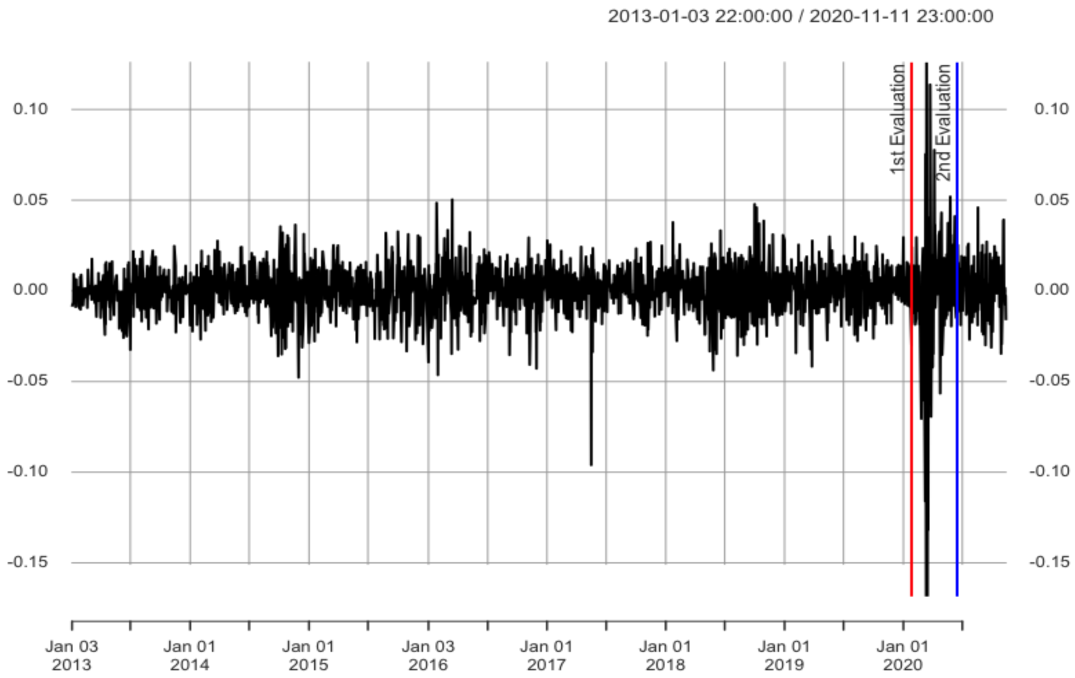

Figure 1: Portfolio log returns

To estimate each model, we consider a period of $T = 1755$ days. Then, a rolling window optimization approach, similarly to Xi (2014), is applied as follows:

- Optimization 1: Use return 1 to return 1755 to estimate the Copula-CVaR models and determine portfolio weights for day 1756.
- Optimization 2: Use return 2 to return 1756 to estimate the Copula-CVaR models and determine portfolio weights for day 1757.
- ...
- Optimization 200: Use return 200 to return 1955 to estimate the Copula-CVaR models and determine portfolio weights for day 1956.

Figure 1 shows the portfolio log returns for the whole period. It can be seen that the portfolio has a big volatility cluster in the year 2020 due to the COVID pandemic's shock on Brazilian financial markets. We purposely chose this time frame when assessing portfolio performance to analyze CVaR optimization during stressful market periods. Observations on the right side of the red line on January 27, 2020, are used to assess the optimization performance of the entire evaluation window. Observations on the right side of the blue line on June 12, 2020, are used to assess the optimization performance in a calmer market scenario, where the initial COVID shock has already passed.

Table 1 shows five different blocks of stocks, tickers and names. Each block has stocks from similar markets used to model the Block Dependence factor copula. The categorization of the assets into the blocks is arbitrary, although being an ad-hoc choice based on the assumption that firms that operate in the same segment have similar dependence structures. The stock simple return distributions' descriptive statistics are shown in Table 2. Most stocks have negative or positive skewness and a positive excess kurtosis, supporting our choice of using a Skewed-t distribution to model univariate log returns instead of traditional Normal or t distributions.

3.1. Empirical Strategy

To apply the Factor Copula-CVaR portfolio optimization, we follow mainly the steps presented in Pfaff (2012), Hofert et al. (2018a), Hofert et al. (2018b), Xi (2014), Bartels and Ziegelmann (2016) and Oh and Patton (2018). Below, the steps are presented and repeated for all the 200 optimizations that we previously mentioned. In addition, we consider as benchmarks the following portfolios: an Equal Weight, a minimum CVaR Gaussian Copula, a traditional Markowitz mean-variance portfolio, a second Markowitz portfolio using a smaller estimation window of 300 observations and the IBRX50 index.

(1) Fit an ARMA(1,1)-GARCH(1,1) model with skewed t-distributed innovations to each univariate log-return time series.

(2) Using the estimated parametric model, construct the standardized residuals vector for each i asset, $i = 1, ..., 30$, as

$$\frac{\hat{\epsilon}_{i,t}}{\hat{\sigma}_{i,t}}, \quad t = 1, ..., 1755 \quad \text{and} \quad i = 1, ..., 30. \tag{30}$$

(3) Calculate pseudo-uniform variables (the residuals cdf inverse transforms) from the standardized residuals parametrically using the Skewed-t

Code	Asset	Block	Block Code
B3SA3.SA	B3 S.A. - Brasil, Bolsa, Balcão	Finance	1
BBAS3.SA	Banco do Brasil S.A.	Finance	1
BBDC4.SA	Banco Bradesco S.A.	Finance	1
ITUB4.SA	Itaú Unibanco Holding S.A.	Finance	1
SANB11.SA	Banco Santander (Brasil) S.A.	Finance	1
CIEL3.SA	Cielo S.A.	Finance	1
SULA11.SA	Sul América S.A.	Finance	1
ITSA4.SA	Itaúsa - Investimentos Itaú SA	Finance	1
ABEV3.SA	Ambev S.A.	Retail	2
MGLU3.SA	Magazine Luiza S.A.	Retail	2
LREN3.SA	Lojas Renner S.A.	Retail	2
AMER3.SA	Americanas S.A.	Retail	2
JBSS3.SA	JBS S.A.	Retail	2
RENT3.SA	Localiza Rent a Car S.A.	Retail	2
QUAL3.SA	Qualicorp S.A.	Retail	2
PETR4.SA	Petróleo Brasileiro S.A. - Petrobras	Energy	3
PRIO3.SA	Petro Rio S.A.	Energy	3
CMIG4.SA	Companhia Energética de Minas Gerais	Energy	3
UGPA3.SA	Ultrapar Participações S.A.	Energy	3
CSNA3.SA	Companhia Siderúrgica Nacional	Energy	3
TAEE11.SA	Transmissora Aliança de Energia Elétrica S.A.	Energy	3
EQTL3.SA	Equatorial Energia S.A.	Energy	3
ENBR3.SA	EDP - Energias do Brasil S.A.	Energy	3
MULT3.SA	Multiplan Empreendimentos Imobiliários S.A.	Construction	4
CCRO3.SA	CCR S.A.	Construction	4
CYRE3.SA	Cyrela Brazil Realty S.A.	Construction	4
MRVE3.SA	MRV Engenharia e Participações S.A.	Construction	4
GGBR4.SA	Gerdau S.A.	Construction	4
GOLL4.SA	Gol Linhas Aéreas Inteligentes S.A.	Aviation	5
EMBR3.SA	Embraer S.A.	Aviation	5

Table 1: Assets and respective blocks

distribution of the GARCH error process. This can also be done semi-parametrically using the empirical distribution functions of the standardized residual vectors; see Pfaff (2012) or Hofert et al. (2018a) for an example.

(4) Estimate Homogeneous Dependence and Block Dependence factor

copula models to the data that has been transformed to uniform [0,1] margins.

(5) Use the dependence structure estimated by the factor copula models to generate $K = 3000$ scenarios of random variates for the pseudo-uniformly distributed variables.

Series	Min	Mean	Sd	Max	Skewness	Kurtosis
ABEV3.SA	-0.1578	0.0001	0.0167	0.0987	-0.3694	11.6527
AMER3.SA	-0.1697	0.0016	0.0390	0.4161	1.2898	13.4375
B3SA3.SA	-0.1612	0.0010	0.0237	0.1924	0.1487	8.1349
BBAS3.SA	-0.2117	0.0005	0.0289	0.1713	-0.0544	9.5692
BBDC4.SA	-0.1427	0.0004	0.0233	0.1687	0.1213	9.1317
CCRO3.SA	-0.1793	0.0001	0.0254	0.2216	0.2777	12.8619
CIEL3.SA	-0.2118	-0.0004	0.0259	0.2346	0.9124	15.9014
CMIG4.SA	-0.2105	0.0003	0.0285	0.1780	-0.2726	9.2928
CSNA3.SA	-0.2529	0.0010	0.0387	0.2082	0.3477	7.2138
CYRE3.SA	-0.2465	0.0005	0.0268	0.1806	-0.5253	14.2854
EMBR3.SA	-0.2644	-0.0001	0.0255	0.2250	-0.1465	18.8457
ENBR3.SA	-0.1277	0.0004	0.0206	0.1556	0.1242	7.3696
EQTL3.SA	-0.1084	0.0010	0.0169	0.0815	-0.3002	6.8250
GGBR4.SA	-0.1796	0.0005	0.0299	0.1745	0.1173	6.4267
GOLL4.SA	-0.3629	0.0013	0.0477	0.5033	1.2066	18.5140
ITSA4.SA	-0.1087	0.0005	0.0199	0.1027	-0.0338	5.7802
ITUB4.SA	-0.1205	0.0005	0.0209	0.1177	0.1995	6.3375
JBSS3.SA	-0.3134	0.0011	0.0319	0.2460	0.3568	14.3872
LREN3.SA	-0.2112	0.0009	0.0228	0.1500	-0.1201	11.7651
MGLU3.SA	-0.2108	0.0029	0.0396	0.3729	1.3519	15.2224
MRVE3.SA	-0.2015	0.0007	0.0278	0.2114	-0.0357	9.7884
MULT3.SA	-0.2239	0.0003	0.0221	0.1730	-0.0800	15.6592
PETR4.SA	-0.2970	0.0006	0.0327	0.2222	-0.2797	10.6809
PRIO3.SA	-0.3654	0.0014	0.0520	0.8367	2.6899	45.1696
QUAL3.SA	-0.2937	0.0006	0.0278	0.3664	0.2539	26.7847
RENT3.SA	-0.2361	0.0012	0.0258	0.2682	0.3447	16.9381
SANB11.SA	-0.1347	0.0007	0.0234	0.1578	0.1639	8.0727
SULA11.SA	-0.1753	0.0008	0.0227	0.1760	-0.1270	9.8877
TAEE11.SA	-0.0831	0.0003	0.0159	0.0873	-0.2172	4.8489
UGPA3.SA	-0.2136	0.0002	0.0233	0.2338	0.1360	21.1323

Table 2: Asset returns descriptive statistics

(6) Calculate Skewed-t quantiles for these Monte Carlo draws, $z_{i,k},\ k = 1, ..., K$.

(7) Determine the K scenarios of simulated daily log returns for the out-of-sample following day we are forecasting for each asset i,

$$r_{i,k} = \hat{R}^{sim}_{i,k} + \hat{\epsilon}_{i,k}, \tag{31}$$

in which $\hat{R}^{sim}_{i,k}$ is provided by the ARMA(1,1)-GARCH(1,1) model,

$$\hat{R}^{sim}_{i,k} = \hat{\epsilon}_{i,k} + \sum_{l=0}^{1} \hat{\phi}_{i,l}\ R^{sim}_{i,k-1} + \sum_{l=0}^{1} \hat{\theta}_{i,l}\ \hat{\epsilon}_{i,k-1} \tag{32}$$

where $\hat{\epsilon}_{i,k}$ is given as

$$\begin{aligned} \hat{\epsilon}_{i,k} &= \hat{\sigma}_{i,k}\, z_{i,k} \\ \hat{\sigma}^2_{i,k} &= \hat{\alpha}_{i,0} + \hat{\alpha}_{i,1}\, \hat{\epsilon}^2_{i,k-1} + \hat{\beta}_{i,1}\, \sigma^2_{i,k-1}. \end{aligned} \tag{33}$$

(8) Finally, use the simulated returns data as input to optimize the portfolio weights, finding the minimal CVaR for a confidence level of 5% and a given portfolio target return. This is done using the results from Wuertz et al. (2010), in which the method by Rockafellar and Uryasev (2002) is applied for optimizing CVaR, reducing the optimization to a linear problem. The output is a vector of optimal portfolio weights, $\boldsymbol{W^*} = [w^*_1, ...w^*_{30}]$. To assess optimization performance, we run the optimization for target daily returns equal to the average of the portfolio returns.

Similar steps are applied when optimizing the Gaussian copula portfolio for every period. Nevertheless, as there is no need to estimate a factor copula model, step (4) of the optimization fits a Gaussian multivariate copula to the pseudo-uniform data following Hofert et al. (2018b) method.

For each of the 200 optimizations for the compared portfolios, the estimated optimal asset weight and the observed simple returns of the data-set, r^{obs}_i, are used to calculate the following day's out-of-sample portfolio returns, as

$$R^{port}_n = \sum_{i=1}^{30} w^*_{i_n}\, r^{obs}_{i_n}, \quad n = 1, ..., 200. \tag{34}$$

For the Equal Weight portfolio, the weight of each asset is simply 1/30. As we repeat this procedure for every optimization, the compared portfolios are automatically rebalanced daily by their weights. After running the optimizations, we obtain a vector of portfolio returns, $\boldsymbol{R^{port}} = [R^{port}_1, ..., R^{port}_{200}]$, that will be used to assess the portfolio's performance.

3.2. Performance Measures

In order to assess the portfolio's risk and return measures, we propose several indicators as in Peterson and Carl (2019) and Bacon (2008): Annualized Returns, Annualized Standard Deviation, VaR, CVaR, Semi-Deviation, Conditional Drawdown-at-Risk (CDaR), Average Drawdown, Annualized Sharpe Ratio, Sortino Ratio, Upside Potential, Downside Frequency, Calmar Ratio and Drawdown Deviation. The choice of the performance measures was made in order to capture different aspects of the portfolio's performance: cumulative returns, variability of the returns, downside risk, tail downside risk and risk-return measures.

Annualized Returns are a convenient way to compare a standardized period of returns. It is calculated as

$$An.Return = \left(\prod_{i=1}^{n}(1 + R_i^{port})\right)^{f/n}, \tag{35}$$

where R_i^{port} is the daily simple return observations of the portfolio, n is the number of periods under analysis and f is the number of periods within the year. On average, we have over 252 trading days yearly, such that $f = 252$. As we run 200 optimizations, $n = 200$.

In order to measure the variability of the returns from the mean return of the portfolio, we compute the Annualized Standard Deviation,

$$An.StdDev = \sigma\sqrt{f}, \tag{36}$$

where σ is the portfolio's standard deviation and $f = 252$.

From an investor's perspective considering absolute returns and wishing to avoid losses, a continuous losing period, or drawdown, is an intuitive and well-known risk measure. In this sense, the Worst Drawdown is defined as the largest individual uninterrupted loss in a return series and it is calculated as

$$WorstDD = |\max D_j|, \quad j = 0, ..., d \tag{37}$$

where D_j is the j-th Drawdown over the entire period and d is the total number of Drawdowns in the entire period.

Drawdown Deviation ($DDDev.$) calculates a statistic similar to standard deviation, considering individual drawdowns:

$$DDDev. = \sqrt{\sum_{j=1}^{j=d} \frac{D_j^2}{n}}. \tag{38}$$

Risk measures that use standard deviation consider a portfolio's upside and downside risk. Nonetheless, investors can have a bigger aversion to downside risk than to upside risk, as variability in positive returns is not viewed with the same concern as variability in negative returns. Therefore, the analysis of risk measures regarding only the negative returns in a portfolio is called Downside Risk Assessment.

In this sense, Downside Deviation measures under-performance variability below a Minimum Acceptable Return - MAR. It is defined as

$$\sigma_D = \sqrt{\sum_{i=1}^{n} \frac{\min[R_i - MAR, 0]^2}{n}}, \tag{39}$$

where n is the total number of returns. Semi-Deviation ($Semi.Dev$) is a particular case of σ_D where the Minimum Acceptable Return is the mean of $\boldsymbol{R^{port}}$.

To calculate Downside Frequency, ($DownsideFreq.$), we take the subset of returns that are less than the MAR, n_d, and divide the length of this subset by the total number of returns:

$$DownsideFreq = \frac{n_d}{n}. \tag{40}$$

In order to evaluate portfolios with different levels of risk and different returns, distinct risk-reward measures can be calculated. A traditional risk-return measure is Sharpe Ratio, which measures the return per unit of risk, expressed as variability. It represents the additional amount of return an investor receives for an additional unit of risk, considering a Risk Free rate. It is calculated as

$$SharpeRatio = \frac{\frac{\sum_{i=1}^{n}(R_i^{port} - R_f)}{n}}{\sigma}. \tag{41}$$

The Sortino Ratio, similar to the Sharpe Ratio, is a risk-adjusted evaluation of the return of an investment. The difference is that Sortino Ratio only factors in downside risk. It is defined as

$$SortinoRatio = \frac{\frac{\sum_{i=1}^{n}(R_i^{port} - MAR)}{n}}{\sigma_D}. \tag{42}$$

The Upside Potential Ratio builds on Sortino Ratio, considering only Upside Risk (the opposite of Downside Risk) on the numerator.

It is also possible to define a risk measure similar to the Sharpe Ratio using the worst computed Drawdown rather than the standard deviation to reflect the investor's risk. The Calmar Ratio, ($CalmarRatio$), is calculated as

$$CalmarRatio = \frac{An.Return}{W_d}, \tag{43}$$

where W_d is the worst drawdown observed in the portfolio.

Lastly, we also calculate downside risk measures concerning the tail of the returns' distributions: VaR, CVaR and Conditional Drawdown-at-Risk, CDaR. VaR and CVaR calculation is already explained. For a thorough definition of CDaR see Chekhlov et al. (2004). We use a significance level of 5% to compute the tail downside risk statistics.

4. Empirical Results

As mentioned previously, to capture financial stylized facts of skewness and kurtosis of the asset returns distributions as well as variance clustering, we used an ARMA(1,1)-GARCH(1,1) following a Skewed-t distribution for each series. As we run many estimations for every asset, the ARMA-GARCH parameter estimation results are not shown, although we chose the model by its capability of addressing fat tails and variance clustering while still being parsimonious.

To validate subsequent copula analysis, we perform Kolmogorov-Smirnov (KS) tests for the probability integral transforms of the standardized residuals. In order to uphold correct copula modeling, it is expected that the transformation provides margins with Uniform [0,1] distributions. In every case, p-values of the test were above 0.1, meaning they do not statistically differ from the Uniform [0,1] distribution.

Regarding estimation of factor copula with time-varying parameters, Figure 2 shows the estimated out-of-sample Homogeneous factor copula loadings. We note that the loadings vary between 0.35 and 1.20, with a mean of around 0.84, with the smallest value occurring on March 10, 2020 and the highest value on April 15, 2020 - the most volatile period during the COVID market crisis. By the sharp decline in asset dependence on the most critical early days of the crisis in March 2020, followed by a sharp elevation afterward, it is possible to conclude that during the initial stages of COVID crisis, the various assets behaved differently while gaining a more dependent behavior in the following days of the crisis. As the most volatile period passed, the asset dependence gradually decayed over time.

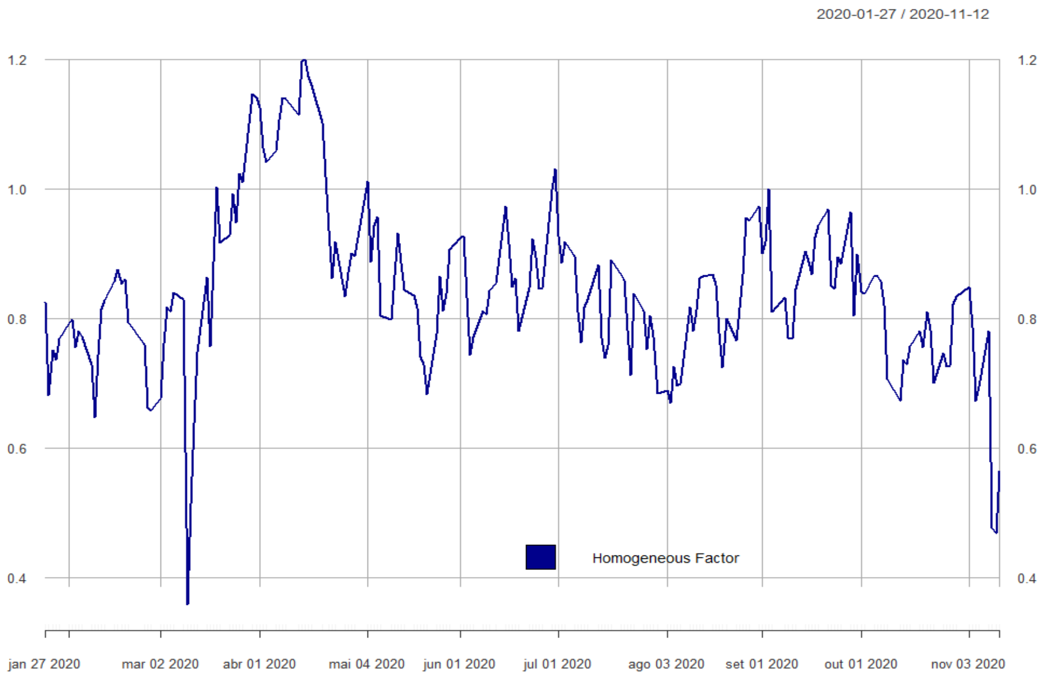

Figure 2: Homogeneous factor copula loadings

Figure 3 shows factor loadings evolution through time for the Block Dependence case. We see that asset returns from the Finance block are the ones that have the highest dependence, reaching a maximum of 1.44 during COVID crises and slowly decaying afterward, as well as a minimum of 0.57 before COVID. On the other hand, the Aviation block has the lowest dependence variability, increasing from around 0.62 to 0.76 after the crisis. For the Retail block, the most heterogeneous block of assets, it is possible to see the increase of dependence during the initial stages of the crisis but a fast decay as the volatile period passes. Finally, the Energy and Construction ones behaved similarly to the Finance block but with a different magnitude. Thus, given the different behavior of the blocks, it seems to make sense to separate the dependence estimation of the asset returns.

Table 4 computes, for the whole evaluation window, out-of-sample An-

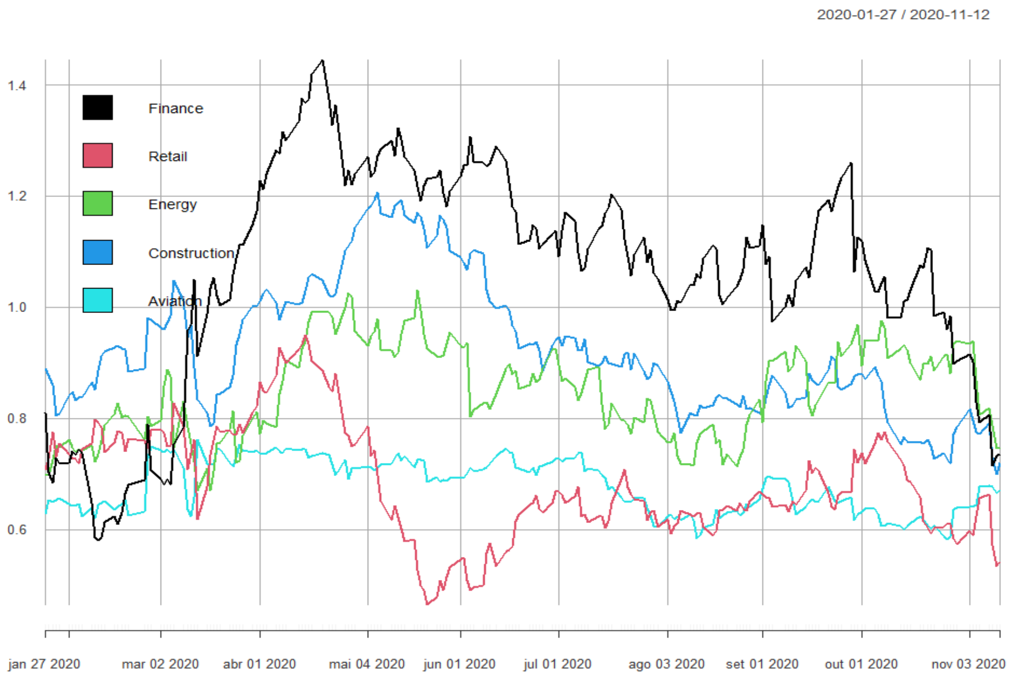

Figure 3: Block Dependent factor copula loadings

nualized Mean Return, Annualized Standard Deviation, $VaR_{0.95}$, $CVaR_{0.95}$, Semi-Deviation, $CDaR_{0.95}$, Average Drawdown, Annualized Sharpe Ratio, Sortino Ratio, Upside Potential, Downside Frequency and Drawdown Deviation, using a daily target return equal to the mean return of the assets. Due to the hard task of optimizing portfolio returns in the context of extreme market losses during COVID crisis, Table 4 computes the same risk metrics for a subset of the evaluation window after the initial shock of COVID crisis. We chose this particular subset window in order to have risk and return metrics for a stable market condition in addition to the extreme volatile period mentioned above. For the calculation of risk measures, due to the extreme negative market condition, we consider a Risk-Free Rate of 0% and a Minimum Accepted Return of 0%.

By analyzing the seven portfolios of the whole out-of-sample optimization window, we find that factor and Gaussian copula models present considerably better risk and return measures with respect to the Equal Weight portfolio, the Markowitz portfolios and the IBRX50 index. Copula portfolios have over twice the Annualized Return regarding the Equal Weight and the Markowitz portfolio with the same estimation window as the copula ones, with lower or similar downside risk measures - VaR, CVaR, CDaR, Semi-Deviation and Drawdown Deviation. An increase in portfolio return is also followed by a lower increase in risk measures, as we can see by better Sharpe Ratio, Calmar Ratio, Sortino Ratio and Upside potential. The three portfolios also exhibited lesser Worst Drawdown and Annualized Standard Deviation, even though the Markowitz mean-variance portfolio optimization aims to minimize the asset's covariance. The Markowitz portfolio with a smaller estimation window has a good performance regarding risk statistics, however the low Annualized Return severely hinders its use, as can be seen in the bad performance of risk/return statistics such as Sharpe Ratio.

Risk/Return Measure	Hom. Factor	Block Dep. Factor	Equal Weight Portfolio	Markowitz	Markowitz Smaller	Gaussian Copula	IBRX-50
Annualized Returns	-0.132	**-0.088**	-0.220	-0.236	-0.203	-0.108	-0.264
Annualized S.D.	0.363	0.369	0.530	0.390	**0.312**	0.366	0.511
VaR	-0.034	-0.035	-0.042	-0.036	**-0.027**	-0.032	-0.045
CVaR	-0.068	-0.067	-0.095	-0.072	**-0.063**	-0.068	-0.093
CDaR	**0.388**	0.416	0.499	0.411	**0.388**	0.408	0.507
Semi-Deviation	0.018	0.018	0.025	0.019	**0.016**	0.018	0.025
Drawdown Deviation	**0.027**	0.029	0.035	0.029	**0.027**	0.029	0.036
Worst Drawdown	0.388	0.416	0.499	0.411	**0.372**	0.408	0.507
Sharpe Ratio	-0.363	**-0.238**	-0.416	-0.604	-0.650	-0.294	-0.517
Sortino Ratio	-0.016	**-0.005**	-0.016	-0.039	-0.044	-0.010	-0.027
Upside Potential	0.534	**0.591**	0.533	0.579	0.530	0.515	0.531
Downside Frequency	0.49	0.505	0.47	**0.515**	0.5	0.465	0.485

Table 3: Risk and return measures for whole evaluation window

A comparison between the Homogeneous factor copula, Block Dependent factor copula and the Gaussian copula model requires more attention, as the observed statistics are closer to each other. Block Dependent factor copula presented the best risk-return measures: Sharpe Ratio, Calmar Ratio, Sortino Ratio and Upside Potential. This means that an increase in portfolio returns is accompanied by a lower increase in overall risk and downside risk concerning the other portfolios. The Block Dependent portfolio also exhibited better Annualized Returns and CVaR, the optimization target. Concerning drawdown statistics, the Homogeneous Dependence factor portfolio had the best measures, shown with lower CDaR, Drawdown Deviation and Average Drawdown. The Gaussian copula portfolio had good

Risk/Return Measure	Hom. Factor	Block Dep. Factor	Equal Weight Portfolio	Markowitz	Markowitz Smaller	Gaussian Copula	IBRX-50
Annualized Returns	0.450	**0.470**	0.210	0.076	0.167	0.417	0.188
Annualized S.D	0.188	0.202	0.245	0.194	**0.152**	0.195	0.242
VaR	-0.016	-0.017	-0.025	-0.019	**-0.013**	-0.018	-0.023
CVaR	-0.020	-0.021	-0.029	-0.023	**-0.016**	-0.021	-0.029
CDaR	0.050	0.047	0.058	**0.036**	0.055	0.056	0.037
Semi-Deviation	0.008	0.008	0.011	0.008	**0.006**	0.008	0.011
Drawdown Deviation	0.009	0.009	0.011	0.014	0.009	0.010	0.013
Worst Drawdown	**0.067**	0.078	0.101	0.136	0.069	0.069	0.120
Sharpe Ratio	**2.397**	2.331	0.856	0.390	1.100	2.142	0.777
Sortino Ratio	**0.226**	0.225	0.083	0.045	0.109	0.201	0.076
Upside Potential	0.990	**1.105**	0.782	0.885	0.859	0.920	0.813
Downside Frequency	0.467	**0.505**	0.458	**0.505**	0.467	0.458	0.467

Table 4: Risk and return measures for a stable market subsample

all-around behavior, with the lowest Downside Frequency and an Annualized Return between Block Dependent and Homogeneous dependence portfolio measures.

Lastly, Figure 4 shows the cumulative return time-series of each model. It is possible to see that during the COVID market crash of March 2020, the Equal Weight Portfolio and the IBRX50 index suffered significantly bigger losses with respect to the other portfolios, even though they ended with a cumulative return similar to the Markowitz ones. CVaR-targeted portfolios exhibited comparable initial losses regarding the Markowitz portfolios. However, they could better capture the market recovery dynamics, as all three exhibited better cumulative returns than the Markowitz benchmark. The Block Dependent factor copula had the best cumulative return at the end of the testing period.

In the smaller subsample results analysis, all of the considered portfolios showed positive Annualized Returns. However, as well as in the whole evaluation window, CVaR-targeted copula portfolios exhibited considerably better returns concerning Equal Weight, Markowitz and IBRX50 benchmarks. Compared to Equal Weight, the traditional Markowitz and IBRX50, the increase in portfolio returns of the copula models does not follow an increase in risk: copula portfolios have less SD, VaR, CVaR, Drawdown Deviation and Average Drawdown, while having better risk-return metrics such as Sharpe Ratio. Thus, copula portfolios can consolidate higher returns with less or similar risk metrics concerning the other portfolios. Even though the smaller estimation window Markowitz portfolio produced better risk measures, the

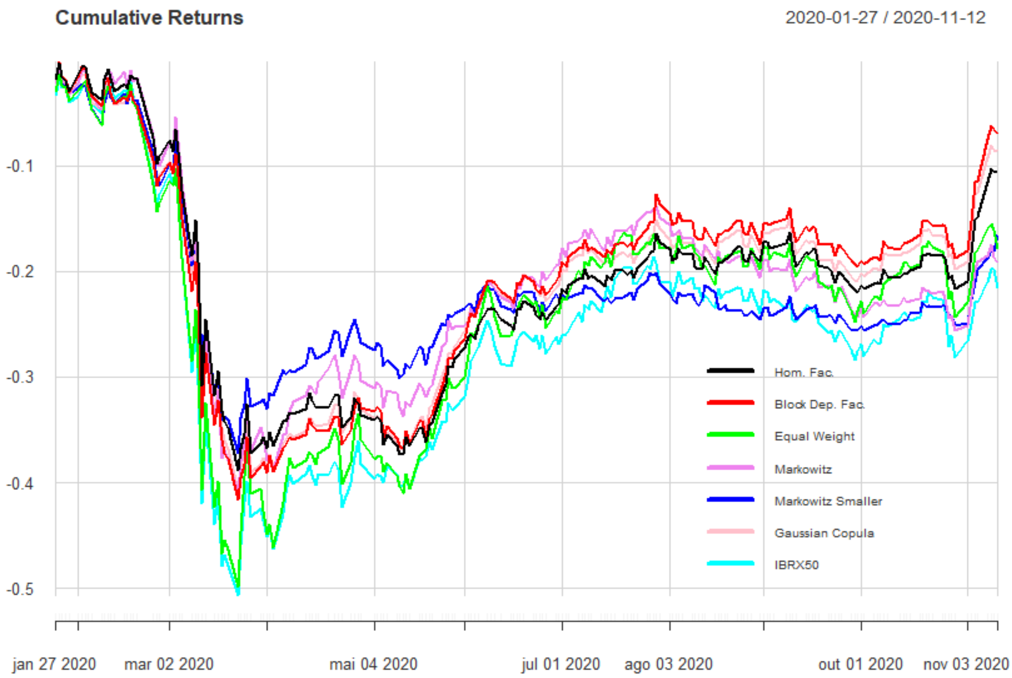

Figure 4: Cumulative returns of the considered models for whole evaluation window

poor Annualized Return performance hinders its use.

With respect to the three copula portfolios, the factor copula ones delivered higher Annualized Returns and lower SD, VaR, CDaR, Drawdown Deviation and Average Drawdown, while having similar Semi Deviation and CVaR. Risk-return measures are also better than the Gaussian counterpart. Thus, although factor copulas reduce the data dimension, they can still capture significant asset dependence behavior with time-varying loadings. By solely comparing the Homogeneous and Block Dependent factor copula portfolios, we conclude that the Block Dependent optimization generated better Annualized Returns at the expense of slightly worse risk and risk-return measures.

Figure 5 shows cumulative returns for the smaller subsample. While the traditional Markowitz portfolio had a good initial performance during the market upturn, it failed to deliver consistent performance through the considered period. The Equal Weight portfolio and the IBRX50 index performed better in the more stable market condition, an expected behavior considering the stylized fact that the asset returns tend to have a similar (panicking) dynamic in the event of extreme market losses. The Gaussian copula portfolio showed slightly worse overall performance concerning the factor copula portfolios. The Markowitz portfolio with a smaller estimation window exhibited a sideways behavior, not being able to capture the positive returns in the market upturn as the rest of the portfolios. While the Block Dependent factor copula and the Homogeneous factor copula portfolios behaved similarly, the former had a better performance at the end of the evaluation period, with the highest cumulative return of all the considered portfolios.

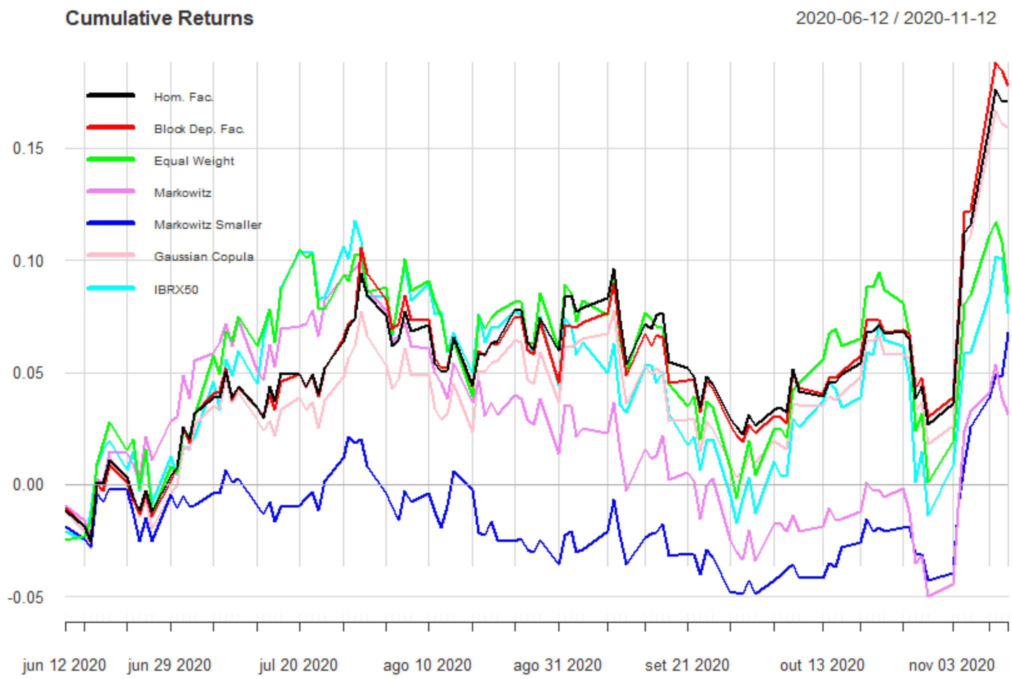

Figure 5: Cumulative returns of the considered models for a stable market subsample

5. Final discussion and future work

This work aims to investigate and offer an alternative to traditional computationally expensive Archimedian and Elliptical copula modeling in the context of minimal-CVaR portfolio optimization with many assets. We employ two time-varying factor copula models with distinct dependence structures in order to reduce the number of copula parameters to estimate while still capturing significant asset dependence behavior.

The empirical portfolio optimization analysis suggests that both factor models can deliver similar or better risk/return measures concerning traditional Gaussian copula methods, as well as a Markowitz mean-variance portfolios and an Equal Weight portfolio. Both factor copula models have also superior risk/return measures with respect to the IBRX50 index. Regarding the two distinct factor copulas, the Block Dependent portfolio optimization exhibited higher Annualized Returns and risk-reward measures at the expense of marginally higher downside risk measures considering the larger optimization window.

In summary, although not popularly employed in CVaR optimization of portfolios, factor copulas can be a promising alternative to the "curse of dimensionality" implied by the increase in the number of assets composing the portfolio. Similar to Bartels and Ziegelmann (2016), future work could expand on different analyzed factor copulas models and dependence structures of the asset returns.

Appendix A. Coherence of a risk measure

In this appendix, we aim to give a simple yet complete explanation of the coherence of a risk measure. In the sense of portfolio optimization, a coherent risk measure ensures that the portfolio holds important characteristics regarding broader aspects such as regulations. Artzner et al. (1999) defines a risk measure as coherent if it satisfies four axioms. Let ρ denote a risk measure and $\rho(L)$ the risk value of a portfolio, where the loss $L \in G$ is a random variable.

Axiom Appendix A.1 (Translation Invariance). *Let $l \in \mathbb{R}$. Then, $\rho(L + l) = \rho(L) - l$.*

The first axiom states that adding (subtracting) an initial amount l to an initial position and investing it in a reference instrument decreases (increases) the measure of risk by l. I.e., it ensures that the losses and the risk measure are defined in the same units.

Axiom Appendix A.2 (Subadditivity). *Let L_1 and $L_2 \in G$. Then, $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$.*

A simple interpretation of axiom Appendix A.2 is that the portfolio's risk is lesser or equal to the sum of the risk of the assets that create the portfolio. I.e., by holding different assets, we obtain a less risky portfolio.

Axiom Appendix A.3 (Positive homogeneity). *Let $c \geq 0$ and $L \in G$. Then, $\rho(cL) = c\rho(L)$.*

The axiom above ensures the scaling of the risk measure regarding the size of the position. By having this property, one ensures that the size of the portfolio does not directly influence its riskiness.

Axiom Appendix A.4 (Monotonicity). *Let L_1 and $L_2 \in G$ with $L_1 \leq L_2$. Then, $\rho(L_1) \leq \rho(L_2)$.*

A straightforward interpretation of axiom Appendix A.4 is that the relation between the losses also reflects when risk measures are calculated regarding each loss. Traditional risk measures are ruled out as a coherent measure when checking for this property, including Standard Deviation and Semi-Deviation.

VaR is not coherent because it does not satisfy axiom Appendix A.2 - see Daníelsson et al. (2005) for a complete study. Even though CVaR includes VaR, a non-coherent risk measure in its definition, Rockafellar and Uryasev (2002) and Acerbi and Tasche (2002) give a complete proof of the coherence of CVaR, i.e., satisfying all of the axioms above.

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