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Counting orientations of graphs with no strongly connected tournaments[☆]

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Abstract

Let $S_k(n)$ be the maximum number of orientations of an n-vertex graph G in which *no copy* of K_k is strongly connected. For all integers $n, k \geq 4$ where $n \geq 5$ or $k \geq 5$, we prove that $S_k(n) = 2^{t_{k-1}(n)}$, where $t_{k-1}(n)$ is the number of edges of the n-vertex $(k-1)$-partite Turán graph $T_{k-1}(n)$. Moreover, we prove that $T_{k-1}(n)$ is the only graph having $2^{t_{k-1}(n)}$ orientations with no strongly connected copies of K_k.

Keywords: orientations; tournaments; complete graphs

1. Introduction

Let G be a graph and $\vec{F}$ be an oriented graph. An orientation $\vec{G}$ of G is $\vec{F}$*-free* if $\vec{G}$ contains no copy of $\vec{F}$. Given a family $\vec{\mathcal{F}}$ of oriented graphs, denote by $\mathcal{D}(G, \vec{\mathcal{F}})$ the family of orientations of G that are $\vec{F}$-free for every $\vec{F} \in \vec{\mathcal{F}}$ and

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write $D(G,\vec{\mathcal{F}}) = |\mathcal{D}(G,\vec{\mathcal{F}})|$. Finally, let

$$D(n,\vec{\mathcal{F}}) = \max\{D(G,\vec{\mathcal{F}}) : |V(G)| = n\}. \tag{1}$$

An n-vertex graph G that achieves equality in (1) is called $\vec{\mathcal{F}}$*-extremal* or, alternatively, *extremal for* $\vec{\mathcal{F}}$. Note that if G is a graph without copies of the underlying graphs of the orientated graphs in $\vec{\mathcal{F}}$, then $D(G,\vec{\mathcal{F}}) = 2^{|E(G)|}$. Therefore, if $k = \min\{\chi(H) : H \in \vec{\mathcal{F}}\}$, then $D(n,\vec{\mathcal{F}}) \geq 2^{t_{k-1}}$, where $t_{k-1}(n)$ denotes the number of edges of the *Turán graph* $T_{k-1}(n)$, the n-vertex graph without copies of K_k and maximum number of edges possible, which is the balanced complete $(k-1)$-partite graph.

The problem of determining $D(n,\vec{\mathcal{F}})$ has been solved by Alon and Yuster [1] for large n when $\vec{\mathcal{F}}$ consists of a single *tournament*. Recently, Araújo and the first and last authors [2] extended this result to every n in the case where the forbidden tournament is the strongly connected triangle, here denoted by $C_3^{\circlearrowright}$.

Theorem 1.1. *[2] For $n \geq 1$, we have $D(n,\{C_3^{\circlearrowright}\}) = \max\{2^{t_2(n)}, n!\}$. Furthermore, among all graphs G with $n \geq 8$ vertices, $D(G,\{C_3^{\circlearrowright}\}) = 2^{t_2(n)}$ if and only if G is the Turán graph $T_2(n)$.*

Recently, Bucić and Sudakov [4] proved that for sufficiently large n, we have $D(n,\{C_{2k+1}^{\circlearrowright}\}) = 2^{t_2(n)}$, where $C_{2k+1}^{\circlearrowright}$ is the strongly connected cycle of length $2k+1$,

Note that the result of Alon and Yuster determining $D(n,\{\vec{K}_k\})$ for any fixed tournament $\vec{K}_k$ on $k \geq 3$ vertices and sufficiently large n immediately implies the following: for any nonempty family $\vec{\mathcal{F}}$ of orientations of K_k, we also have $D(n,\vec{\mathcal{F}}) = 2^{t_{k-1}(n)}$ for large n. More generally, if $\vec{\mathcal{F}}$ is a family of tournaments (not necessarily with the same size) and k is the minimum size of a tournament in $\vec{\mathcal{F}}$, we must also have $D(n,\vec{\mathcal{F}}) = 2^{t_{k-1}(n)}$ for large n. However, the problem of determining $D(n,\vec{\mathcal{F}})$ for all values of $n \geq 1$ is still open for nontrivial families $\vec{\mathcal{F}}$ of tournaments other than $\vec{\mathcal{F}} = \{C_3^{\circlearrowright}\}$.

In Section 2 we prove general results for orientations that avoid graphs from a family of tournaments (see Lemmas 2.5 and 2.6). These results imply that if G is an $\vec{\mathcal{F}}$-extremal graph that is not complete multipartite, then one can construct $\vec{\mathcal{F}}$-extremal complete multipartite graphs G_1 and G_2 such that $|E(G_2)| < |E(G)| \leq |E(G_1)|$. In Section 3 we use these tools to extend Theorem 1.1 to the family $\vec{\mathcal{S}}_k$ of all strongly connected tournaments on $k \geq 4$ vertices. More precisely, we prove in Theorem 3.1 that for all integers $n, k \geq 4$ where $n \geq 5$ or $k \geq 5$, we have $S_k(n) = 2^{t_{k-1}(n)}$ and we show that $T_{k-1}(n)$ is the only $\vec{\mathcal{S}}_k$-extremal graph. For $n = k = 4$, we have $S_4(4) = 40$ and K_4 is the only $\vec{\mathcal{S}}_4$-extremal graph.

We remark that this result implies a similar one for any family $\vec{\mathcal{F}}$ of orientations of complete graphs with at least k vertices for which $\vec{\mathcal{S}}_k \subset \vec{\mathcal{F}}$. More precisely, one can conclude that for $n \geq 5$ or $k \geq 5$ we have $D(n,\vec{\mathcal{F}}) = 2^{t_{k-1}(n)}$ and that $T_{k-1}(n)$ is the only $\vec{\mathcal{F}}$-extremal graph. For example, this implies that the number of orientations in which every tournament with k vertices is transitively oriented is at most $2^{t_{k-1}(n)}$.

2. Complete multipartite extremal graphs

In this section we obtain some results derived with the approach in [3], which in turn was influenced by the proof of Turán's Theorem by Zykov Symmetrization. To highlight the similarities, we use the notation of [3] whenever it is possible. Fix a family of tournaments (not necessarily with the same number of vertices) on at least three vertices and fix a positive integer n. We prove that there is a complete multipartite n-vertex graph that is extremal for this family (see Lemmas 2.5 and 2.6).

Let $\vec{x}$ be a vector whose coordinates are indexed by a set T. Given $t \in T$, we denote by $x(t)$ the value of x at coordinate t. Given $p \in (0,\infty)$, the ℓ_p*-norm* of $\vec{x}$, denoted by $\|\vec{x}\|_p$, is given by

$$\|x\|_p = \left(\sum_{t\in T} |x(t)|^p\right)^{1/p}.$$

Moreover, for a sequence of vectors $\vec{x}_1,\ldots,\vec{x}_s$, each indexed by T, their pointwise product $\vec{y}$, i.e., the vector in which $y(t) = \prod_{k=1}^{s} x_k(t)$ for $t \in T$, is denoted by $\prod_{k=1}^{s} \vec{x}_k$.

Let G be a graph and let $\vec{\mathcal{F}}$ be a family of oriented graphs. Given a subgraph H of G and an $\vec{\mathcal{F}}$-free orientation $\vec{H}$ of H, we denote by $c_{\vec{\mathcal{F}}}(G \mid \vec{H})$ the number of ways to orient the edges in $E(G) \setminus E(H)$ in order to extend $\vec{H}$ to an $\vec{\mathcal{F}}$-free orientation of G. For simplicity, given $v \in V(G) \setminus V(H)$, if H is an induced subgraph of G, then we write $c_{\vec{\mathcal{F}}}(v, \vec{H})$ for $c_{\vec{\mathcal{F}}}(G[V(H) \cup \{v\}], \vec{H})$. Similarly, given an edge $\{u, v\}$ with $u, v \in V(G) \setminus V(H)$, we use $c_{\vec{\mathcal{F}}}(\{u, v\}, \vec{H})$ for the number of ways to orient the edge $\{u, v\}$ and the edges between $V(H)$ and the vertices u and v (again avoiding $\vec{\mathcal{F}}$).

We also define $\vec{v}_{H,\vec{\mathcal{F}}}$ as the vector indexed by the set $\mathcal{D}(H, \vec{\mathcal{F}})$ of all $\vec{\mathcal{F}}$-free orientations of H, whose coordinate corresponding to an orientation $\vec{H}$ is given by $\vec{v}_{H,\vec{\mathcal{F}}}(\vec{H}) = c_{\vec{\mathcal{F}}}(v, \vec{H})$. When $\vec{\mathcal{F}}$ is a family of tournaments, we obtain the following simple observation.

Fact 2.1. *Let $\vec{\mathcal{F}}$ be a family of tournaments. If H is an induced subgraph of G such that $S = V(G) \setminus V(H)$ is an independent set in G, and $\vec{H}$ is an $\vec{\mathcal{F}}$-free orientation of H, then*

$$c_{\vec{\mathcal{F}}}(G \mid \vec{H}) = \prod_{v \in S} c_{\vec{\mathcal{F}}}(v, \vec{H}).$$

We shall also use the following consequence of Hölder's inequality.

Lemma 2.2. *Let $\vec{x}_1, \ldots, \vec{x}_s$ be complex-valued vectors indexed by a set T. Then,*

$$\left\| \prod_{k=1}^{s} \vec{x}_k \right\|_1 \le \prod_{k=1}^{s} \left\| \vec{x}_k \right\|_s .$$

Furthermore, equality holds if and only if at least one of the following two conditions holds: $\vec{x}_i$ is the zero vector for some $i \in [s]$ or, for every $i, j \in [s]$, there exists $\alpha_{i,j}$ with the property that $x_i(t) = \alpha_{i,j} x_j(t)$ for all $t \in T$.

We say that two non-adjacent vertices of a graph are *twins* if they have the same neighborhood. For the next lemma we consider the following operation: given an independent set S of a graph G and a particular vertex $v \in S$, delete all vertices in $S \setminus \{v\}$ and add $|S| - 1$ new twins of v. We show that there is a vertex $v \in S$ for which the graph generated by this operation contains at least as many $\vec{\mathcal{F}}$-free orientations as G.

Lemma 2.3. *Let $\vec{\mathcal{F}}$ be a family of tournaments, and let G be a graph on n vertices. If $S \subset V(G)$ is a non-empty independent set, then the following holds.*

(a) if v is a vertex in S that maximizes $\left\| \vec{v}_{G-S,\vec{\mathcal{F}}} \right\|_{|S|}$ among all vertices of S, then the graph $\widetilde{G}$ obtained from G by replacing the vertices in $S \setminus \{v\}$ with $|S| - 1$ twins of v is such that $D(\widetilde{G}, \vec{\mathcal{F}}) \ge D(G, \vec{\mathcal{F}})$; and

(b) if G is $\vec{\mathcal{F}}$-extremal, then $\vec{u}_{G-S,\vec{\mathcal{F}}} = \vec{w}_{G-S,\vec{\mathcal{F}}}$ for any vertices $u, w \in S$.

Proof. Let $\vec{\mathcal{F}}$, G, and S be as in the statement, and put $H = G - S$ and $s = |S|$. By Fact 2.1, the total number of $\vec{\mathcal{F}}$-free orientations of G is given by

$$D(G, \vec{\mathcal{F}}) = \sum_{\vec{H} \in D(H, \vec{\mathcal{F}})} c_{\vec{\mathcal{F}}}(G \mid \vec{H}) = \sum_{\vec{H} \in D(H, \vec{\mathcal{F}})} \prod_{u \in S} c_{\vec{\mathcal{F}}}(u, \vec{H}) = \left\| \prod_{u \in S} \vec{u}_{H,\vec{\mathcal{F}}} \right\|_1 , \tag{2}$$

where, in the last equality, we used the fact that every coordinate of $\vec{u}_{H,\vec{\mathcal{F}}}$ is non-negative.

Let v be a vertex in S for which $\left\| \vec{v}_{H,\vec{\mathcal{F}}} \right\|_s$ is maximum. By Lemma 2.2, we have

$$\left\| \prod_{u \in S} \vec{u}_{H,\vec{\mathcal{F}}} \right\|_1 \le \prod_{u \in S} \left\| \vec{u}_{H,\vec{\mathcal{F}}} \right\|_s \le \left\| \vec{v}_{H,\vec{\mathcal{F}}} \right\|_s^s . \tag{3}$$

On the other hand, let $\widetilde{G}$ be the graph obtained from G by replacing the vertices in $S \setminus \{v\}$ by $s-1$ twins of v. Then, we have:

$$D(\widetilde{G}, \vec{\mathcal{F}}) = \sum_{\vec{H} \in D(H, \vec{\mathcal{F}})} c_{\vec{\mathcal{F}}}(v, \vec{H})^s = \left\| \vec{v}_{H,\vec{\mathcal{F}}} \right\|_s^s. \tag{4}$$

Therefore, combining (2), (3) and (4), we have $D(\widetilde{G}, \vec{\mathcal{F}}) \geq D(G, \vec{\mathcal{F}})$. This proves (a).

Now, assume G is $\vec{\mathcal{F}}$-extremal. Since $D(\widetilde{G}, \vec{\mathcal{F}}) \geq D(G, \vec{\mathcal{F}})$, we have $D(\widetilde{G}, \vec{\mathcal{F}}) = D(G, \vec{\mathcal{F}})$. This implies that both inequalities in (3) hold with equality, and hence for every vertex $u \in S$, we must have $\left\| \vec{u}_{H,\vec{\mathcal{F}}} \right\|_s = \left\| \vec{v}_{H,\vec{\mathcal{F}}} \right\|_s$. By the equality conditions of Lemma 2.2, together with the fact that all coordinates are nonnegative, we have $\vec{u}_{H,\vec{\mathcal{F}}} = \vec{v}_{H,\vec{\mathcal{F}}}$. This proves (b). □

One can easily obtain the following corollary from Lemma 2.3.

Corollary 2.4. *Let $\vec{\mathcal{F}}$ be a family of tournaments, and let G be an $\vec{\mathcal{F}}$-extremal graph. If $u, v \in V(G)$ are non-adjacent, then the graph obtained from G by replacing v with a twin of u is also $\vec{\mathcal{F}}$-extremal.*

Note that a graph G is a complete multipartite graph if and only if, for any $u, v, w \in V(G)$ such that $\{u, v\}, \{u, w\} \notin E(G)$, we have $\{v, w\} \notin E(G)$. In other words, G is a complete multipartite graph if every two non-adjacent vertices are twins. By repeatedly applying Corollary 2.4, we show that *there exists* a complete multipartite graph on n vertices that is $\vec{\mathcal{F}}$-extremal.

Lemma 2.5. *Let $\vec{\mathcal{F}}$ be a family of tournaments and let G be an n-vertex $\vec{\mathcal{F}}$-extremal graph. Then there exists an n-vertex complete multipartite graph G^* that is $\vec{\mathcal{F}}$-extremal and satisfies $|E(G^*)| \geq |E(G)|$.*

Proof. Let G be an n-vertex $\vec{\mathcal{F}}$-extremal graph. A vertex u is *eccentric* if there is a non-neighbor of u that is not its twin. Put $G_0 = G$ and let $G_0, \ldots, G_t$ be a maximal sequence of graphs in which, for $i = 0, \ldots, t-1$, the graph G_{i+1} is obtained from G_i by picking an eccentric vertex u of maximum degree in G_i and replacing *every* non-neighbor of u with a twin of u. Thus, every vertex of G_{i+1} is either a neighbor or a twin of u. Clearly, u is not eccentric in G_{i+1}. Moreover, if a vertex v is eccentric in G_{i+1}, then v is also eccentric in G_i, and hence G_{i+1} contains fewer eccentric vertices than G_i, from which we conclude that the sequence $G_0, \ldots, G_t$ is finite. By construction, since we always pick an eccentric vertex u with maximum degree, $|E(G_t)| \geq |E(G)|$. Then, by Corollary 2.4, G_t is $\vec{\mathcal{F}}$-extremal. Moreover, every pair of non-adjacent vertices u and v in G_t are twins. Therefore, G_t is a complete multipartite graph. □

The following result implies that any $\vec{\mathcal{F}}$-extremal graph may also be turned into an $\vec{\mathcal{F}}$-extremal multipartite graph by removing edges.

Lemma 2.6. *Let $\vec{\mathcal{F}}$ be a family of tournaments, let G be an $\vec{\mathcal{F}}$-extremal graph, and let u, v, w be distinct vertices of G such that $\{u, v\}, \{u, w\} \notin E(G)$ and $\{v, w\} \in E(G)$. Then, the graph obtained from G by deleting the edge $\{v, w\}$ is $\vec{\mathcal{F}}$-extremal. Furthermore, for $H = G - \{u, v, w\}$ we have $\vec{u}_{H,\vec{\mathcal{F}}} = \vec{w}_{H,\vec{\mathcal{F}}} = \vec{v}_{H,\vec{\mathcal{F}}}$.*

Proof. Let $\vec{\mathcal{F}}, G, u, v$, and w be as in the statement. Let $H = G - \{u, v, w\}$. For $x \in \{u, v, w\}$ we write $H^x = G[V(H) \cup x]$ and let G^x_{twins} be the graph obtained from H^x by adding twins x_1 and x_2 of x. Applying Fact 2.1 to G^u_{twins} with $S = \{u, u_1, u_2\}$, we have

$$D(G^u_{\text{twins}}, \vec{\mathcal{F}}) = \sum_{\vec{H} \in D(H, \vec{\mathcal{F}})} c_{\vec{\mathcal{F}}}(G^u_{\text{twins}} \mid \vec{H}) = \sum_{\vec{H} \in D(H, \vec{\mathcal{F}})} c_{\vec{\mathcal{F}}}(u, \vec{H})^3 = \left\| \vec{u}_{H,\vec{\mathcal{F}}} \right\|_3^3.$$

By an analogous computation, we have $D(G^w_{\text{twins}}, \vec{\mathcal{F}}) = \left\| \vec{w}_{H,\vec{\mathcal{F}}} \right\|_3^3$.

By Corollary 2.4 applied twice, the graph G^u_{twins} is also $\vec{\mathcal{F}}$-extremal. Therefore,

$$D(G, \vec{\mathcal{F}}) = D(G^u_{\text{twins}}, \vec{\mathcal{F}}). \tag{5}$$

However, since w is a neighbor of v, Corollary 2.4 cannot be applied to the graph G^w_{twins}, so we cannot conclude that G^w_{twins} is $\vec{\mathcal{F}}$-extremal. But since G^u_{twins} is $\vec{\mathcal{F}}$-extremal, we have

$$\left\|\vec{w}_{H,\vec{\mathcal{F}}}\right\|_3^3 = D(G^w_{\text{twins}}, \vec{\mathcal{F}}) \leq D(G^u_{\text{twins}}, \vec{\mathcal{F}}) = \left\|\vec{u}_{H,\vec{\mathcal{F}}}\right\|_3^3.$$

Since there are no edges between u and $\{v, w\}$, we can compute $D(G, \vec{\mathcal{F}})$ as follows:

$$D(G, \vec{\mathcal{F}}) = \sum_{\vec{H} \in D(H,\vec{\mathcal{F}})} c_{\vec{\mathcal{F}}}(u, \vec{H}) c_{\vec{\mathcal{F}}}(G - u \mid \vec{H}) = \sum_{\vec{H} \in D(H,\vec{\mathcal{F}})} \left(c_{\vec{\mathcal{F}}}(u, \vec{H}) \sum_{\vec{H^w} \mid \vec{H}} c_{\vec{\mathcal{F}}}(v, \vec{H^w}) \right),$$

where the inner sum is taken over the $\vec{\mathcal{F}}$-free orientations of H^w that extend a given $\vec{\mathcal{F}}$-free orientation of H, that is, over the orientations of the edges between w and H, for which the resulting orientation is $\vec{\mathcal{F}}$-free. By Lemma 2.3(b), since G is $\vec{\mathcal{F}}$-extremal and $\{u, v\} \notin E(G)$, we have $\vec{v}_{H^w,\vec{\mathcal{F}}} = \vec{u}_{H^w,\vec{\mathcal{F}}}$, i.e., $c_{\vec{\mathcal{F}}}(v, \vec{H^w}) = c_{\vec{\mathcal{F}}}(u, \vec{H^w})$ for every $\vec{H^w}$. Finally, note that since $\vec{\mathcal{F}}$ is a family of tournaments and u and w are not adjacent, $c_{\vec{\mathcal{F}}}(u, \vec{H^w})$ does not depend on the orientation of the edges between w and H, so $c_{\vec{\mathcal{F}}}(u, \vec{H^w}) = c_{\vec{\mathcal{F}}}(u, \vec{H})$. Therefore,

$$\begin{aligned}
D(G, \vec{\mathcal{F}}) &= \sum_{\vec{H} \in D(H,\vec{\mathcal{F}})} \left(c_{\vec{\mathcal{F}}}(u, \vec{H}) \sum_{\vec{H^w} \mid \vec{H}} c_{\vec{\mathcal{F}}}(u, \vec{H}) \right) = \sum_{\vec{H} \in D(H,\vec{\mathcal{F}})} \left(c_{\vec{\mathcal{F}}}(u, \vec{H}) c_{\vec{\mathcal{F}}}(u, \vec{H}) \sum_{\vec{H^w} \mid \vec{H}} 1 \right) \\
&= \sum_{\vec{H} \in D(H,\vec{\mathcal{F}})} c_{\vec{\mathcal{F}}}(u, \vec{H})^2 c_{\vec{\mathcal{F}}}(w, \vec{H}) \\
&\leq \left\|\vec{u}_{H,\vec{\mathcal{F}}}\right\|_3 \left\|\vec{u}_{H,\vec{\mathcal{F}}}\right\|_3 \left\|\vec{w}_{H,\vec{\mathcal{F}}}\right\|_3 \qquad (6) \\
&\leq \left\|\vec{u}_{H,\vec{\mathcal{F}}}\right\|_3^3, \qquad (7)
\end{aligned}$$

where (6) follows from Lemma 2.2, and (7) follows from (6). Note that, by (5) we have $D(G, \vec{\mathcal{F}}) = D(G^u_{\text{twins}}, \vec{\mathcal{F}}) = \left\|\vec{u}_{H,\vec{\mathcal{F}}}\right\|_3^3$. Therefore, we must have equality in both (6) and (7), which in turn leads to $\left\|\vec{u}_{H,\vec{\mathcal{F}}}\right\|_3 = \left\|\vec{w}_{H,\vec{\mathcal{F}}}\right\|_3$. The equality condition in Lemma 2.2 implies that $\vec{u}_{H,\vec{\mathcal{F}}} = \vec{w}_{H,\vec{\mathcal{F}}}$. Analogously, $\vec{u}_{H,\vec{\mathcal{F}}} = \vec{v}_{H,\vec{\mathcal{F}}}$.

Finally, let G^- be the graph obtained from G by deleting the edge $\{v, w\}$. It follows that

$$D(G^u_{\text{twins}}, \vec{\mathcal{F}}) = \sum_{\vec{H}} c_{\vec{\mathcal{F}}}(u, \vec{H})^3 = \sum_{\vec{H}} c_{\vec{\mathcal{F}}}(u, \vec{H}) c_{\vec{\mathcal{F}}}(v, \vec{H}) c_{\vec{\mathcal{F}}}(w, \vec{H}) = D(G^-, \vec{\mathcal{F}}).$$

Therefore, G^- is $\vec{\mathcal{F}}$-extremal. This concludes the proof. □

An interesting consequence of the main results of this section (Lemmas 2.5 and 2.6) is that if G is an $\vec{\mathcal{F}}$-extremal graph that is not complete multipartite, then one can construct $\vec{\mathcal{F}}$-extremal complete multipartite graphs G_1 and G_2 such that $|E(G_2)| < |E(G)| \leq |E(G_1)|$. We conclude this section showing that for some families $\vec{\mathcal{F}}$ of forbidden tournaments *every* $\vec{\mathcal{F}}$-extremal graph is a complete multipartite graph.

Lemma 2.7. *Let $\vec{\mathcal{F}}$ be a family of tournaments with no source and let $n \geq 4$ be a positive integer. Then, every n-vertex $\vec{\mathcal{F}}$-extremal graph is complete multipartite.*

Proof. Let n and $\vec{\mathcal{F}}$ be as in the statement. Let G be an n-vertex $\vec{\mathcal{F}}$-extremal graph and assume for a contradiction that G is not complete multipartite. Fix vertices u, v, w such that $\{u, v\}, \{u, w\} \notin E(G)$ and $\{v, w\} \in E(G)$.

Let $H = G - \{u, v, w\}$, and $H^x = G[V(H) \cup x]$ for $x \in \{u, v, w\}$. From Lemma 2.6, we have $\vec{u}_{H,\vec{\mathcal{F}}} = \vec{w}_{H,\vec{\mathcal{F}}} = \vec{v}_{H,\vec{\mathcal{F}}}$, so for every orientation $\vec{H}$ of H we have $c_{\vec{\mathcal{F}}}(u, \vec{H}) = c_{\vec{\mathcal{F}}}(w, \vec{H}) = c_{\vec{\mathcal{F}}}(v, \vec{H})$. Note that since $\vec{\mathcal{F}}$ is a family of tournaments and u and w are not adjacent, for every extension of $\vec{H}$ to an orientation $\vec{H^w}$, we must have $c_{\vec{\mathcal{F}}}(u, \vec{H^w}) = c_{\vec{\mathcal{F}}}(u, \vec{H})$.

Finally, since u and v are not adjacent, by Lemma 2.3 (b), we have $\vec{u}_{H^w,\vec{\mathcal{F}}} = \vec{v}_{H^w,\vec{\mathcal{F}}}$, that is, $c_{\vec{\mathcal{F}}}(u, \vec{H^w}) = c_{\vec{\mathcal{F}}}(v, \vec{H^w})$ for every orientation $\vec{H^w}$ of H^w. It follows that, for every $\vec{\mathcal{F}}$-free extension $\vec{H^w}$ of $\vec{H}$, we must have

$$c_{\vec{\mathcal{F}}}(v, \vec{H^w}) = c_{\vec{\mathcal{F}}}(v, \vec{H}). \tag{8}$$

We get a contradiction from this fact and hence such a graph cannot exist. For that, we find an orientation of $\vec{H}$ and an extension of it to H^w that is $\vec{\mathcal{F}}$-free and such that (8) does not hold. By the definition of $\vec{\mathcal{F}}$, we may start with a transitive orientation $\vec{H}$ of H, which can be extended to an $\vec{\mathcal{F}}$-free orientation $\vec{H^w}$ of H^w by orienting all edges $\{x, w\}$ between w and $x \in V(H)$ as $\vec{wx}$. Let $\mathcal{H}(v)$ and $\mathcal{H}^w(v)$ be the classes of all $\vec{\mathcal{F}}$-free orientations that extend $\vec{H}$ to H^v and $\vec{H^w}$ to $G - u$, respectively. We show that there is an injective mapping $\phi\colon \mathcal{H}(v) \to \mathcal{H}^w(v)$ that is not surjective.

Given an orientation $\vec{H^v} \in \mathcal{H}(v)$, let $\phi(\vec{H^v})$ be the orientation of $G - u$ that extends $\vec{H^w}$ by orienting any edge $e = \{v, x\}$ between v and $x \in V(H)$ with the same orientation as e in $\vec{H^v}$ and by assigning the orientation $\vec{wv}$ to $\{v, w\}$. The function ϕ is clearly injective. We claim that $\phi(\mathcal{H}(v)) \in \mathcal{H}^w(v)$. Indeed, suppose that $\phi(\mathcal{H}(v))$ contains a tournament $\vec{T}$ with no source. Clearly, this copy involves both v and w, otherwise it would also occur in $\vec{H^v}$, a contradiction. However, w is a source in $\phi(\mathcal{H}(v))$, so it cannot lie in $\vec{T}$.

On the other hand, any transitive orientation of $G - u$ where $\{v, w\}$ is oriented $\vec{vw}$ and all edges $\{x, v\}$ and $\{y, w\}$ with $x, y \in V(H)$ are oriented $\vec{vx}$ and $\vec{wy}$, respectively, must lie in $\mathcal{H}^w(v)$. However, it does not lie in $\phi(\mathcal{H}(v))$, as ϕ always orients $\{v, w\}$ as $\vec{wv}$. So ϕ is not surjective, as desired. □

3. Avoiding strongly connected tournaments

Recall that $\vec{\mathcal{S}}_k$ is the family of all strongly connected tournaments with k vertices. Given a graph G, we denote by $\vec{\mathcal{S}}_k(G)$ the family of $\vec{\mathcal{S}}_k$-free orientations of G. We write $S_k(G) = |\vec{\mathcal{S}}_k(G)|$ and define

$$S_k(n) = \max\{S_k(G) \colon G \text{ is an } n\text{-vertex graph}\}.$$

In this section we prove the following theorem, which is the main result of this paper.

Theorem 3.1. *Let $n \geq k \geq 4$ Then, $S_k(n) = 40$ if $n = k = 4$, and $S_k(n) = 2^{t_{k-1}(n)}$ if $n \geq 5$ or $k \geq 5$. Furthermore, if $n \geq 5$ or $k \geq 5$, then the Turán graph $T_{k-1}(n)$ is the only $\vec{\mathcal{S}}_k$-extremal graph.*

We refer to a directed Hamilton cycle (resp. directed Hamilton path) in an oriented graph simply as Hamilton cycle (resp. Hamilton path). We need the following basic result about Hamilton paths and cycles in graphs.

Fact 3.2. *Every tournament contains a Hamilton path and every strongly connected tournament contains a Hamilton cycle.*

By using Fact 3.2 in an induction argument, we can guarantee the existence of strongly connected subtournaments of any length in strongly connected tournaments.

Lemma 3.3. *Every strongly connected tournament $\vec{K}$ contains a strongly connected tournament of order ℓ for every $3 \leq \ell \leq |V(\vec{K})|$.*

In the next two lemmas we estimate in how many ways one can extend orientations of complete graphs on r vertices, with $r \in \{k-1, k, k+1\}$, to larger complete graphs avoiding a strongly connected copy of K_k.

Lemma 3.4. *Let K be a complete subgraph of a graph G with $r \in \{k-1, k, k+1\}$ vertices and let u be a vertex in $V(G) \setminus V(K)$ that is adjacent to all vertices of K. Then, for any orientation $\vec{K}$ of K, we have $c_{\vec{\mathcal{S}}_k}(u, \vec{K}) \leq (r-k+4) \cdot 2^{k-3}$.*

Proof. We assume $r = k+1$ as the proof for the other values of r is analogous. Let K, G and u be as in the statement. Let $\vec{K}$ be an orientation of K and note that by Fact 3.2 there is a Hamilton path $(v_1, \ldots, v_{k+1})$ in $\vec{K}$. By Lemma 3.3, any orientation of the edges between u and $\vec{K}$ that extends $\vec{K}$ to an $\vec{\mathcal{S}}_k$-free orientation cannot form a directed cycle of length at least k. If $\{u, v_1\}$ is oriented towards v_1, then the edges between any w in $\{v_{k-1}, v_k, v_{k+1}\}$ and u must be oriented towards w. Since the edges $\{u, v_j\}$ with $j \in \{2, \ldots, k-2\}$ can be oriented in two ways, there are at most 2^{k-3} possible orientations. If $\{u, v_1\}$ is oriented towards u and $\{u, v_2\}$ is oriented towards v_2, then the edges between any w in $\{v_k, v_{k+1}\}$ and u are oriented towards w. Again, since the edges $\{u, v_j\}$ with $j \in \{3, \ldots, k-1\}$ can be oriented in two ways, there are at most 2^{k-3} possible orientations. Analogously, if $\{u, v_1\}$ and $\{u, v_2\}$ are oriented towards u and $\{u, v_3\}$ is oriented towards v_3, then there are at most 2^{k-3} possible orientations. Finally, there are at most 2^{k-2} possible orientations for which $\{u, v_1\}$, $\{u, v_2\}$ and $\{u, v_3\}$ are oriented towards u. Therefore, $c_{\vec{S}_k}(u, \vec{K}_r) \leq 2^{k-3} + 2^{k-3} + 2^{k-3} + 2^{k-2} = 5 \cdot 2^{k-3}$, as desired. □

Lemma 3.5. *Let K be a complete subgraph of a graph G with $k-1$ vertices and let u, v be adjacent vertices in $V(G) \setminus V(K)$ that are adjacent to all vertices of K. Then, $c_{\vec{S}_k}(\{u, v\}, \vec{K}) < 2 \cdot 3 \cdot 2^{2k-5}$.*

Proof. Let K, G and u be as in the statement. Let $\vec{K}$ be an orientation of K and note that by Fact 3.2 there is a Hamilton path $(v_1, \ldots, v_{k-1})$ in $\vec{K}$. There are two possible orientations for the edge $\{u, v\}$. Let us estimate in how many ways one can orient the edges between $\vec{K}$ and $\{u, v\}$ without creating a strongly connected K_k. Suppose without loss of generality that $\{u, v\}$ is oriented towards u.

Note that, by Lemma 3.3, any orientation of the edges between u and $\vec{K}$ that extends $\vec{K}$ to an $\vec{S}_k$-free orientation cannot form a directed cycle of length at least k. Analogously to the proof of Lemma 3.4, if $\{u, v_1\}$ is oriented towards v_1, then the edge between any v_{k-1} and u must be oriented towards v_{k-1}, and the edges between any w in $\{v_{k-2}, v_{k-1}\}$ and v must be oriented towards w. Since the remaining edges from u or v to K can be oriented in two ways, there are at most 2^{2k-6} possible orientations. Similarly, there are at most 2^{2k-5} possible orientations in which $\{u, v_1\}$ is oriented towards u and $\{u, v_2\}$ is oriented towards v_2, and there are at most 2^{2k-6} possible orientations in which $\{u, v_1\}$ and $\{u, v_2\}$ are oriented towards u, and $\{v, v_1\}$ is oriented towards v_1. Finally, there are at most 2^{2k-5} possible orientations in which $\{u, v_1\}$ and $\{u, v_2\}$ are oriented towards u, and $\{v, v_1\}$ is oriented towards v. Therefore, $c_{\vec{S}_k}(uv, \vec{K}_x) \leq 2 \cdot (2^{2k-6} + 2^{2k-5} + 2^{2k-6} + 2^{2k-5}) = 2 \cdot 3 \cdot 2^{2k-5}$. □

The next result states that for $k \geq 5$, at least half of the orientations of K_k are strongly connected. Let $\mathcal{SC}(G)$ denote the number of strongly connected orientations of G.

Lemma 3.6. *Let $k \geq 5$ be a positive integer. Then $\mathcal{SC}(K_k) > 2^{\binom{k}{2}-1}$.*

Proof. The proof is by induction on k. Let $\{u_1, \ldots, u_k\}$ be the vertex set of $G = K_k$ and consider the complete graph $G' = G - u_k$ on $k-1$ vertices. First suppose $k = 5$. We show that $\mathcal{SC}(K_5) \geq 512$. It is easy to see that out of the 40 orientations of G' that are not strongly connected, there are 24 transitive orientations and 16 non-transitive orientations. Note that since each of these non-transitive orientations contains a copy of $C_3^{\circlearrowright}$ and a sink or source, there are 7 ways to extend such orientation to obtain a strongly connected orientation of G. Transitive and strongly connected orientations of G' may be extended in 4 and $2^4 - 2$ ways, respectively. Noting that there are 24 strongly connected orientations of G', we have $\mathcal{SC}(K_5) = 24 \cdot 4 + 24 \cdot (2^4 - 2) + 16 \cdot 7 = 544 > 2^{\binom{5}{2}-1}$.

Thus, we may assume $k \geq 6$ and we suppose that $\mathcal{SC}(K_{k-1}) > 2^{\binom{k-1}{2}-1}$. Let $\vec{G'}$ be an orientation of G' and suppose that $\vec{G'}$ is strongly connected. If u_k has indegree and outdegree at least 1, then this gives a strongly connected orientation of G. Therefore, $\vec{G'}$ can be extended in precisely $2^{k-1} - 2$ ways. Now, suppose that $\vec{G'}$ is not strongly connected. By Fact 3.2, G' contains a Hamilton path P. We may assume, without loss of generality, that $P = u_1, \ldots, u_{k-1}$. By orienting $\{u_1, u_k\}$ towards u_1 and $\{u_{k-1}, u_k\}$ towards u_k, and orienting the edges $\{u_i, u_k\}$ in any direction, for $i = 1, \ldots, k-2$, we obtain a strongly connected orientation of G. Thus, there are at least 2^{k-3} strongly connected orientations of G from $\vec{G'}$. Therefore, we have

$$\mathcal{SC}(K_k) \geq (2^{k-1} - 2) \cdot \mathcal{SC}(K_{k-1}) + 2^{k-3} \cdot (2^{\binom{k-1}{2}} - \mathcal{SC}(K_{k-1})) > 2^{\binom{k}{2}-1}.$$

□

Combining some of the previous results, we prove that cliques of size k and $k+1$ are not $\vec{S}_k$-extremal.

Corollary 3.7. *For $k \geq 5$ we have $\vec{S}_k(K_k) < 2^{t_{k-1}(k)}$ and for $k \geq 4$ we have $\vec{S}_k(K_{k+1}) < 2^{t_{k-1}(k+1)}$.*

Proof. We start by showing that $\vec{S}_4(K_5) < 2^{t_3(5)} = 2^8$. Every non-transitive orientation of K_5 contains a $C_3^{\circlearrowright}$. We claim that there are precisely $\binom{5}{2} \cdot 2 \cdot 6$ such orientations without a strongly connected K_4. In fact, there are $\binom{5}{3} = \binom{5}{2}$ triangles in a K_5 and two strongly connected orientations of each triangle. It is not hard to see that there are at most six ways to orient the edges outside the triangle to obtain an orientation of K_5 with no strongly connected K_4. Since there are 120 transitive orientations of K_5, we obtain $\vec{\mathcal{S}}_4(K_5) \le 120 + \binom{5}{2} \cdot 2 \cdot 6 = 240 < 2^8$.

Assume $k \ge 5$. Note that a strongly connected orientation of K_{k-1} can be extended in only two (resp. six) ways to a orientation of K_k (resp. K_{k+1}) that does not contain a strongly connected K_k. By Lemma 3.4, every orientation of K_{k-1} can be extended in at most $3 \cdot 2^{k-3}$ ways to an $\vec{\mathcal{S}}_k$-free orientation of K_k, which gives

$$\vec{\mathcal{S}}_k(K_k) \le 2 \cdot \mathcal{SC}(K_{k-1}) + 3 \cdot 2^{k-3}(2^{\binom{k-1}{2}} - \mathcal{SC}(K_{k-1})). \tag{9}$$

To obtain an estimate for $\vec{\mathcal{S}}_k(K_{k+1})$ we use Lemma 3.5, which implies that every orientation of K_{k-1} can be extended in at most $2 \cdot 3 \cdot 2^{2k-5}$ ways to an $\vec{\mathcal{S}}_k$-free orientation of K_{k+1}. Then,

$$\vec{\mathcal{S}}_k(K_{k+1}) \le 6 \cdot \mathcal{SC}(K_{k-1}) + 2 \cdot 3 \cdot 2^{2k-5}(2^{\binom{k-1}{2}} - \mathcal{SC}(K_{k-1})). \tag{10}$$

If $k = 5$, then by using Lemma 3.6, we have $S_5(K_5) = 2^{\binom{5}{2}} - \mathcal{SC}(K_5) < 2^9 = 2^{t_4(5)}$. From (10), since $\mathcal{SC}(K_4) = 24$, we obtain $\vec{\mathcal{S}}_k(K_{k+1}) \le 6 \cdot 24 + 2 \cdot 3 \cdot 2^5(2^6 - 24) = 7824 < 2^{13} = 2^{t_4(6)}$.

Now we assume that $k \ge 6$. From (9), since $t_{k-1}(k) = t_{k-1}(k-1) + k - 2 = \binom{k-1}{2} + k - 2$ we have

$$\begin{aligned}\vec{\mathcal{S}}_k(K_k) &\le 2 \cdot \mathcal{SC}(K_{k-1}) + 3 \cdot 2^{k-3}(2^{\binom{k-1}{2}} - \mathcal{SC}(K_{k-1}))\\ &\le 2 \cdot 2^{\binom{k-1}{2}} + 3 \cdot 2^{k-4} 2^{\binom{k-1}{2}} \le 2^{k-2} 2^{\binom{k-1}{2}} = 2^{t_{r-1}(k)}.\end{aligned}$$

Analogously, from (10), since $t_{k-1}(k+1) = t_{k-1}(k-1) + 2k - 3 = \binom{k-1}{2} + 2k - 3$ we have

$$\begin{aligned}\vec{\mathcal{S}}_k(K_{k+1}) &\le 6 \cdot \mathcal{SC}(K_{k-1}) + 2 \cdot 3 \cdot 2^{2k-5}(2^{\binom{k-1}{2}} - \mathcal{SC}(K_{k-1}))\\ &\le 6 \cdot 2^{\binom{k-1}{2}} + 3 \cdot 2^{2k-5} 2^{\binom{k-1}{2}} \le 2^{2k-3} 2^{\binom{k-1}{2}} = 2^{t_{r-1}(k+1)},\end{aligned}$$

which concludes the proof. □

Since Lemma 2.7 implies that every $\vec{\mathcal{F}}$-extremal graph is complete and multipartite for the family $\vec{\mathcal{F}}$ of k-vertex tournaments with no source, to prove Theorem 3.1 it is enough to show that every $\vec{\mathcal{F}}$-extremal graph is $(k-1)$-partite. In what follows we prove the main result of this paper.

Proof of Theorem 3.1. Let G be an $\vec{\mathcal{S}}_k$-extremal r-partite graph with $r \ge k$. The proof is by induction on n. For $k \ge 5$ and $n < k$ the result is trivial. For $k = 4$ and $5 \le n \le 8$, we verified the result by exhaustion using the computer programs NAUTY [5] and SageMath [6]. We now proceed with the induction step.

Let G be an n-vertex graph and assume that the statement holds for graphs with fewer vertices. First, suppose that G contains a clique K of size $k + 1$. From Corollary 3.7, we have for $k \ge 4$ that $\vec{\mathcal{S}}_k(K) < 2^{t_{k-1}(k+1)}$. Note that since G is a complete multipartite graph, if $u \notin V(K)$, then u is adjacent to either k or $k + 1$ vertices of K. Thus, by Lemma 3.4, we have $c_{\vec{\mathcal{S}}_k}(u, \vec{K}) \le 5 \cdot 2^{k-3}$. Therefore, we have

$$\vec{\mathcal{S}}_k(G) \le \vec{\mathcal{S}}_k(K)(5 \cdot 2^{k-3})^{n-k-1} \vec{\mathcal{S}}_k(G \setminus K) < 2^{t_{k-1}(k+1)+(\log 5+k-3)(n-k-1)} \vec{\mathcal{S}}_k(G \setminus K). \tag{11}$$

If $k = 4$ and $n - k - 1 > 4$, or $k \ge 5$ then, by the induction hypothesis, we have $\vec{\mathcal{S}}_k(G \setminus K) \le 2^{t_{k-1}(n-k-1)}$. Therefore, from (11) and some calculations one can conclude that

$$\vec{\mathcal{S}}_k(G) < 2^{t_{k-1}(k+1)+(\log 5+k-3)(n-k-1)} \cdot \vec{\mathcal{S}}_k(G \setminus K) < 2^{t_{k-1}(n)}.$$

Now, suppose $k = 4$ and $n - k - 1 = 4$. In this case, $\vec{\mathcal{S}}_k(G \setminus K) \le 40$, and hence, from (11) we have $\vec{\mathcal{S}}_4(G) < 2^{t_3(5)+4(\log 5+1)} \cdot \vec{\mathcal{S}}_k(G \setminus K)$, which implies $\vec{\mathcal{S}}_4(G) < 2^{8+4(\log 5+1)+\log 40} < 2^{27} = 2^{t_3(9)}$. Therefore we can assume that G does not contain a clique of size $k + 1$, and hence G is k-partite. We first deal with the case $k \ge 5$. If $n = k$, then $G \simeq K_k$, and hence, by Corollary 3.7, we have $\vec{\mathcal{S}}_k(G) < 2^{t_{k-1}(k)}$. Thus, we may assume that $n \ge k + 1$. Let K be a clique of size k in G. Since G is a complete k-partite graph, if $u \notin V(K)$, then u is adjacent to precisely $k - 1$ vertices of K. Thus, by Lemma 3.4, we have $c_{\vec{\mathcal{S}}_k}(u, \vec{K}) \le 3 \cdot 2^{k-3}$. Therefore, we have

$$\vec{\mathcal{S}}_k(G) \le \vec{\mathcal{S}}_k(K)(3 \cdot 2^{k-3})^{n-k} \cdot \vec{\mathcal{S}}_k(G \setminus K)$$

Clearly, if $G \setminus K$ is $(k-1)$-partite, then $\vec{\mathcal{S}}_k(G \setminus K) \leq 2^{t_{k-1}(n-k)}$; and if $G \setminus K$ is not $(k-1)$-partite, then, by the induction hypothesis, we have $\vec{\mathcal{S}}_k(G \setminus K) \leq 2^{t_{k-1}(n-k)}$. Moreover, from Corollary 3.7, we have $\vec{\mathcal{S}}_k(K) < 2^{t_{k-1}(k)}$. Therefore, with a few calculations, from (11) one can conclude that

$$\vec{\mathcal{S}}_k(G) < 2^{t_{k-1}(k)+(\log 3+k-3)(n-k)} \cdot \vec{\mathcal{S}}_k(G \setminus K) \leq 2^{t_{k-1}(k)+(\log 3+k-3)(n-k)+t_{k-1}(n-k)} < 2^{t_{k-1}(n)}$$

If $n \geq 9$, then removing a copy of K_4 from G results in a graph which is not a copy of K_4. Then, by arguments analogous to the ones above (for $k \geq 5$), we get $\vec{\mathcal{S}}_4(G) < 2^{t_3(n)}$. □

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