

Fractionally Integrated Moving Average Stable Processes With Long-Range Dependence

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Abstract. Long memory processes driven by Lévy noise with finite second-order moments have been well studied in the literature. They form a very rich class of processes presenting an autocovariance function that decays like a power function. Here, we study a class of Lévy processes whose second-order moments are infinite, the so-called α -stable processes. Based on [Samorodnitsky and Taqqu \(1994\)](#), we construct an isometry that allows us to define stochastic integrals concerning the linear fractional stable motion using Riemann-Liouville fractional integrals. With this construction, an integration by parts formula follows naturally. We then present a family of stationary $S\alpha S$ processes with the property of long-range dependence, using a generalized measure to investigate its dependence structure. At the end, the law of large number's result for a time's sample of the process is shown as an application of the isometry and integration by parts formula.

1. Introduction

In this work, we are interested in continuous-time fractionally integrated stable processes and its property of long-range dependence (LRD) or long memory. There are many interesting applications of such processes throughout the literature (see [Samorodnitsky, 2006](#) for an intuitive and historical introduction). This property presents a phenomenon in which some proper measure of the process' dependence structure - usually the autocovariance function when well defined - behaves like a power function, having a slow decay. Slow in the sense that the corresponding sum will diverge (power function with exponent in $(-1, 0)$). Linear fractional stable motion (LFSM) is a generalization of the well-known fractional Brownian motion (fBm) to the α -stable case. Accordingly to [Samorodnitsky and Taqqu \(1994\)](#), it has the following moving average representation. For $0 < \alpha \leq 2$, $0 < H < 1$, $H \neq \frac{1}{\alpha}$ and $a, b \in \mathbb{R}$ such that $|a| + |b| > 0$,

Received by the editors December 15th, 2020; accepted February 3rd, 2022.

2010 *Mathematics Subject Classification.* 62H20, 60G10, 62M10, 62E10, 54E99.

Key words and phrases. Fractionally integrated moving average stable processes, long-range dependence, linear fractional stable motion.

G. de L. Feltes has been supported by CAPES-Brazil 441802/2029-01. S. R. C. Lopes has been supported by CNPq-Brazil grant 303453/2018-4.

$$M_{\alpha,H}(t) = \int_{-\infty}^{\infty} a \left[(t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}} \right] + b \left[(t-x)_-^{H-\frac{1}{\alpha}} - (-x)_-^{H-\frac{1}{\alpha}} \right] dL_{\alpha}(x), \quad (1.1)$$

for $t \in \mathbb{R}$, where $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$ are the positive and negative part of u , respectively, and $\{L_{\alpha}(t)\}_{t \in \mathbb{R}}$ is a $S\alpha S$ process with Lebesgue control measure. The process $\{M_{\alpha,H}(t)\}_{t \in \mathbb{R}}$ is self-similar with Hurst parameter H and it has stationary increments. Defining the increment process $Y_{\alpha,H}(t) := M_{\alpha,H}(t+1) - M_{\alpha,H}(t)$, $t \in \mathbb{R}$ which is also stationary, consider as a measure of its dependence structure the difference between the joint characteristic function of the pair $(Y_{\alpha,H}(t), Y_{\alpha,H}(0))$ and the product of the marginal characteristic functions of $Y_{\alpha,H}(t)$ and $Y_{\alpha,H}(0)$, given by

$$r(t) := \mathbb{E} \exp\{i(\theta_1 Y_d(t) + \theta_2 Y_d(0))\} - \mathbb{E} \exp\{i\theta_1 Y_d(t)\} \mathbb{E} \exp\{i\theta_2 Y_d(0)\}.$$

Astrauskas et al. (1991) show that, for a positive constant K , as $t \rightarrow \infty$,

$$|r(t)| \sim K|t|^{\alpha H - \alpha},$$

and when $1 < \alpha < 2$ and $\frac{1}{\alpha} < H < 1$ we have $-1 < \alpha H - \alpha < 0$. Therefore, one can say that the increment process of the LFSM has the property of long-range dependence. This fact suggests that we may be able to define a continuous-time α -stable process with long memory if we succeed in building stochastic integrals with respect to the LFSM.

This paper is organized as follows. Section 2 starts with some basic facts and introductory notation. Using Riemann-Liouville fractional integrals we build in Section 3 an isometry which allows us to define the stochastic integral with respect to an LFSM and prove an interesting formula for representing the fractionally integrated α -stable processes. In Section 4 we define a long memory fractionally integrated moving average α -stable process and relate it to the LFSM using previous results. As an application of the isometry and integration by parts formula, we show a law of large number's results at the end of this section. Section 5 concludes this work.

2. Definitions and Preliminary Results

Let $\{L_{\alpha}(t)\}_{t \in \mathbb{R}}$ be a $S\alpha S$ process with index of stability $0 < \alpha \leq 2$ and Lebesgue control measure $dm(x) = dx$. Accordingly to Samorodnitsky and Taqqu (1994), for a measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \in L^{\alpha}(\mathbb{R})$ the stable stochastic integral

$$I(f) = \int_{\mathbb{R}} f(x) dL_{\alpha}(x) \quad (2.1)$$

is well defined and has the following properties.

Proposition 2.1. *The distribution of $I(f)$ is given by*

$$I(f) \sim S_{\alpha}(\sigma_f, 0, 0),$$

where \sim denotes equality in distribution and

$$\sigma_f = \left(\int_{\mathbb{R}} |f(x)|^{\alpha} dx \right)^{\frac{1}{\alpha}} = \|f\|_{\alpha}, \quad (2.2)$$

which means that

$$\mathbb{E} [\exp \{i\theta I(f)\}] = \exp \left\{ -|\theta|^{\alpha} \int_{\mathbb{R}} |f(x)|^{\alpha} dx \right\}.$$

Proof. See proposition 3.4.1 in Samorodnitsky and Taqqu (1994). \square

In order to define the fractionally integrated stable processes, we need to introduce the fractional integrals and derivatives of Riemann-Liouville (see Samko et al., 1993).

Definition 2.2. Let $d \in (0, 1)$. For $f \in L^p(\mathbb{R})$, where $1 \leq p < \frac{1}{d}$, we define

$$(I_-^d f)(x) = \frac{1}{\Gamma(d)} \int_x^\infty f(t)(t-x)^{d-1} dt \quad \text{and} \quad (I_+^d f)(x) = \frac{1}{\Gamma(d)} \int_{-\infty}^x f(t)(x-t)^{d-1} dt, \quad (2.3)$$

respectively, the *right and left-sided Riemann-Liouville fractional integrals of order d* , where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Gamma function.

On the other hand, fractional differentiation was introduced as the inverse of fractional integration. Consider $d \in (0, 1)$, $1 \leq p < \frac{1}{d}$ and denote by $I_\pm^d(L^p)$ the class of functions $\phi \in L^p$ that can be represented as a fractional integral $I_\pm^d f$ of some function $f \in L^p$.

Definition 2.3. Let $\phi \in I_\pm^d(L^p)$ be of the form $\phi = I_\pm^d f$, for a function $f \in L^p$. Then f is precisely the *right or left-sided Riemann-Liouville fractional derivative of order d* , respectively, given by

$$(\mathcal{D}_-^d \phi)(x) = -\frac{1}{\Gamma(1-d)} \frac{d}{dx} \int_x^\infty \phi(t)(t-x)^{-d} dt \quad (2.4)$$

and

$$(\mathcal{D}_+^d \phi)(x) = \frac{1}{\Gamma(1-d)} \frac{d}{dx} \int_{-\infty}^x \phi(t)(x-t)^{-d} dt, \quad (2.5)$$

where $\frac{d}{dx}$ is the usual operation of differentiation with respect to x and the convergence of the integrals at the singularity $t = x$ holds pointwise with probability 1, if $p = 1$ and in the L^p sense, if $p > 1$.

Remark 2.4. In fractional processes as the LFSM defined in (1.1), for $0 < \alpha \leq 2$, one has an interesting relationship between the fractional parameter d and the Hurst parameter H of self-similarity which is given by

$$H = d + \frac{1}{\alpha}. \quad (2.6)$$

As we are more interested in the fractional integration point of view, we will often mention the parameter $d = H - \frac{1}{\alpha}$.

3. Main Results I

This section is based on Pipiras and Taqqu (2000) and Marquardt (2006) in which we define stochastic integrals with respect to the linear fractional stable motion (LFSM) and prove a formula that can be seen as integration by parts' formula, analogous to proposition 5.5 on Marquardt (2006) for the fractional Brownian motion. We restrict ourselves to the case $1 < \alpha < 2$ since we are interested in processes with the property of long-range dependence. While the cited authors deal with Lévy processes with a finite second moment, we handle the infinite second-order moments of $S\alpha S$ processes by using their finite lower-order moments in a suitable space.

3.1. *Fractionally Integrated Processes and the $B_{\alpha,p}$ Space.* Let $\{X(t)\}_{t \in \mathbb{R}}$ be a $S\alpha S$ process, with index of stability $1 < \alpha < 2$, given in its integral representation, accordingly to the family $\{f_t\}_{t \in \mathbb{R}}$ of functions such that, for every $t \in \mathbb{R}$, $f_t : \mathbb{R} \rightarrow \mathbb{R}$, $f_t \in L^\alpha(\mathbb{R})$ and $\{L_\alpha(t)\}_{t \in \mathbb{R}}$ is a $S\alpha S$ process with Lebesgue control measure $dm(x) = dx$. Then, according to Proposition 2.1, we know that

$$X(t) = \int_{\mathbb{R}} f_t(x) dL_\alpha(x) \sim S_\alpha(\sigma_t, 0, 0) \quad (3.1)$$

defined and its distribution is totally determined by

$$\sigma_t = \left(\int_{\mathbb{R}} |f_t(x)|^\alpha dx \right)^{\frac{1}{\alpha}} = \|f_t\|_\alpha. \quad (3.2)$$

It would be important to have an isometry property for the $S\alpha S$ processes as the Itô's isometry for processes with finite second-order moment. Since the α -stable distributions only have finite moments of order $p < \alpha$, we shall consider the following known result.

Proposition 3.1. *Let $\{X(t)\}_{t \in \mathbb{R}}$ be the process defined in (3.1) and let $1 \leq p < \alpha$. Then*

$$\mathbb{E}[|X(t)|^p] = \mathbb{E}[|L_\alpha(1)|^p] \|f_t\|_\alpha^p. \tag{3.3}$$

Proof. From expression (3.1) and since for any positive a , $S_\alpha(a\sigma, 0, 0) \sim aS_\alpha(\sigma, 0, 0)$, where \sim denotes equality in distribution, we have

$$X(t) \sim S_\alpha(\sigma t, 0, 0) \sim \sigma t S_\alpha(1, 0, 0) \sim \sigma t L_\alpha(1).$$

Therefore

$$\mathbb{E}[|X(t)|^p] = \mathbb{E}[|\sigma t L_\alpha(1)|^p] = \mathbb{E}[|L_\alpha(1)|^p] \|f_t\|_\alpha^p.$$

□

Now, for measurable functions $g : \mathbb{R} \rightarrow \mathbb{R}$, $g \in L^1(\mathbb{R})$ and $d \in (0, 1 - \frac{1}{\alpha})$, consider the right-sided Riemann-Liouville fractional integral of order d , $I_-^d g = \frac{1}{\Gamma(d)} \int_x^\infty g(t)(t-x)^{d-1} dt$ and denote by \tilde{B}_α the set

$$\tilde{B}_\alpha = \left\{ g \in L^1(\mathbb{R}) : \int_{-\infty}^\infty |(I_-^d g)(x)|^\alpha dx < \infty \right\}. \tag{3.4}$$

The set \tilde{B}_α can be seen as the set of kernel functions $g \in L^1(\mathbb{R})$ for which the stable stochastic integral of its respective fractional integral $\int_{\mathbb{R}} (I_-^d g)(x) dL_\alpha(x)$ is well defined. For the next proposition, we will need the following lemma (Leach, 1956).

Lemma 3.2. *Let $1 < p < \infty$ and $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For a measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f \in L^p(\mathbb{R})$ if and only if there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}} |f(x)h(x)| dx \leq C \|h\|_q, \tag{3.5}$$

for every function $h \in L^q(\mathbb{R})$.

Note that since α lies in the interval $(1, 2)$, $L^\alpha(\mathbb{R})$ is a Banach space. Using the previous lemma, we have the following.

Proposition 3.3. *If $g \in L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$, then $g \in \tilde{B}_\alpha$.*

Proof. For every $h \in L^\beta(\mathbb{R})$, such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have

$$\begin{aligned} \int_{\mathbb{R}} |h(x)(I_-^d g)(x)| dx &\leq \int_{-\infty}^\infty \int_x^\infty \left| h(x)g(t) \frac{(t-x)^{d-1}}{\Gamma(d)} \right| dt dx \\ &= \frac{1}{\Gamma(d)} \int_{-\infty}^\infty \int_0^\infty |h(x)g(s+x)s^{d-1}| ds dx. \end{aligned} \tag{3.6}$$

Rewriting the last expression, we get

$$\int_{-\infty}^\infty \int_0^\infty |h(x)g(s+x)s^{d-1}| ds dx = I_1 + I_2, \tag{3.7}$$

where

$$I_1 = \int_{-\infty}^\infty \int_1^\infty |h(x)g(s+x)s^{d-1}| ds dx \quad \text{and} \quad I_2 = \int_{-\infty}^\infty \int_0^1 |h(x)g(s+x)s^{d-1}| ds dx.$$

By using Fubini's theorem and Holder's inequality we get

$$I_2 = \int_0^1 s^{d-1} \int_{-\infty}^\infty |h(x)g(s+x)| dx ds \leq \int_0^1 s^{d-1} \|h\|_\beta \|g\|_\alpha ds = \frac{1}{d} \|h\|_\beta \|g\|_\alpha. \tag{3.8}$$

For I_1 in (3.1), set $t = s + x$ and apply Holder's inequality to get

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} |g(t)| \int_1^{\infty} |h(t-s)s^{d-1}| ds dt \leq \int_{-\infty}^{\infty} |g(t)| \|h\|_{\beta} \left(\int_1^{\infty} s^{\alpha(d-1)} ds \right)^{\frac{1}{\alpha}} dt \\ &= \int_{-\infty}^{\infty} |g(t)| \|h\|_{\beta} \frac{1}{(\alpha(1-d) - 1)^{\frac{1}{\alpha}}} dt = \frac{1}{(\alpha(1-d) - 1)^{\frac{1}{\alpha}}} \|g\|_1 \|h\|_{\beta}. \end{aligned} \tag{3.9}$$

Combine (3.6), (3.7), (3.8) and (3.9) to finally get

$$\int_{\mathbb{R}} |h(x)(I_{-}^d g)(x)| dx \leq \left[\frac{1}{\Gamma(d)} \left(\frac{\|g\|_{\alpha}}{d} + \frac{\|g\|_1}{(\alpha(1-d) - 1)^{\frac{1}{\alpha}}} \right) \right] \|h\|_{\beta}, \tag{3.10}$$

where

$$C = \frac{1}{\Gamma(d)} \left(\frac{\|g\|_{\alpha}}{d} + \frac{\|g\|_1}{(\alpha(1-d) - 1)^{\frac{1}{\alpha}}} \right) > 0.$$

□

Remark 3.4. As well as for the right-sided Riemann-Liouville fractional integral $I_{-}^d g$, the same is true for the left-sided Riemann-Liouville fractional integral $I_{+}^d g$, which means that if $g \in L^1(\mathbb{R}) \cap L^{\alpha}(\mathbb{R})$, then $\int_{\mathbb{R}} |(I_{+}^d g)(x)|^{\alpha} dx < \infty$ and its stable stochastic integral $\int_{\mathbb{R}} (I_{+}^d g)(x) dL_{\alpha}(x)$ is well defined. The proof is completely analogous and will be omitted.

As we have seen, $L^1(\mathbb{R}) \cap L^{\alpha}(\mathbb{R}) \subset \tilde{B}_{\alpha}$, so we would like to define a norm which could allow us to work with isometries for processes of the form $\int_{\mathbb{R}} (I_{+}^d g)(x) dL_{\alpha}(x)$. The following norm is based on theorem 3.2 of Pipiras and Taqqu (2000), which in that case it comes from an inner product. Now, inspired by Proposition 3.1 and using the previous remark, we give the following definition.

Definition 3.5. For a fixed $1 < p < \alpha$ and $g \in L^1(\mathbb{R}) \cap L^{\alpha}(\mathbb{R})$, let the norm $\|\cdot\|_{\alpha,p}$ be defined as

$$\|g\|_{\alpha,p} = \mathbb{E}[|L_{\alpha}(1)|^p]^{\frac{1}{p}} \left(\int_{\mathbb{R}} |(I_{-}^d g)(x)|^{\alpha} dx \right)^{\frac{1}{\alpha}} = \mathbb{E}[|L_{\alpha}(1)|^p]^{\frac{1}{p}} \|I_{-}^d g\|_{\alpha}.$$

Proposition 3.6. The expression given in Definition 3.5 defines a norm. Moreover, for $g \in L^1(\mathbb{R}) \cap L^{\alpha}(\mathbb{R})$, there exist $M, N > 0$ such that

$$\|g\|_{\alpha,p} \leq M \|g\|_1 + N \|g\|_{\alpha}. \tag{3.11}$$

Proof. First, note that $\|g\|_{\alpha,p} = D \|I_{-}^d g\|_{\alpha}$, where $D = (\mathbb{E}[|L_{\alpha}(1)|^p])^{\frac{1}{p}}$ does not depend on g and $\|\cdot\|_{\alpha}$ is the usual $L^{\alpha}(\mathbb{R})$ norm.

- (i) since $D > 0$, $\|g\|_{\alpha,p} = 0 \iff \|I_{-}^d g\|_{\alpha} = 0 \iff I_{-}^d g = 0 \iff g = \mathcal{D}_{-}^d(I_{-}^d g) = 0$;
- (ii) for $a \in \mathbb{R}$, $\|ag\|_{\alpha,p} = D \|I_{-}^d ag\|_{\alpha} = D \|aI_{-}^d g\|_{\alpha} = |a| \|g\|_{\alpha,p}$;
- (iii) for $f, g \in L^1(\mathbb{R}) \cap L^{\alpha}(\mathbb{R})$, $\|f + g\|_{\alpha,p} = D \|I_{-}^d(f + g)\|_{\alpha} = D \|I_{-}^d f + I_{-}^d g\|_{\alpha} \leq D \|I_{-}^d f\|_{\alpha} + D \|I_{-}^d g\|_{\alpha} = \|f\|_{\alpha,p} + \|g\|_{\alpha,p}$.

Finally, (3.11) can be seen from the proof of Proposition 3.3: by (3.10) we get that $G(h) = \int_{\mathbb{R}} h(x)(I_{-}^d g)(x) dx$ is a bounded linear functional in $L^{\beta}(\mathbb{R})$, where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Moreover,

$$\|(I_{-}^d g)\|_{\alpha} = \sup_{\|h\|_{\beta}=1} |G(h)| \leq \left[\frac{1}{\Gamma(d)} \left(\frac{\|g\|_{\alpha}}{d} + \frac{\|g\|_1}{(\alpha(1-d) - 1)^{\frac{1}{\alpha}}} \right) \right].$$

And (3.11) follows by setting

$$M = \frac{(\mathbb{E}[|L_{\alpha}(1)|^p])^{\frac{1}{p}}}{\Gamma(d)} \frac{1}{(\alpha(1-d) - 1)^{\frac{1}{\alpha}}} > 0 \text{ and } N = \frac{(\mathbb{E}[|L_{\alpha}(1)|^p])^{\frac{1}{p}}}{\Gamma(d)} \frac{1}{d} > 0.$$

□

We are now able to define the space which will allow us to build the desired stochastic integrals.

Definition 3.7. Let $B_{\alpha,p}$ be the space $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$ under the equivalence relation $\sim_{\alpha,p}$ in which for $f, g \in L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$,

$$f \sim_{\alpha,p} g \iff \|f - g\|_{\alpha,p} = 0. \tag{3.12}$$

3.2. Stochastic Integral and Integration by Parts Formula. We proceed with the construction of the stochastic integral concerning the linear fractional stable motion. Details on the stochastic integration concerning the fractional Brownian motion can be found in [Pipiras and Taqqu \(2000\)](#), [Nualart \(2006\)](#), and [Carmona et al. \(2003\)](#) and concerning fractional Lévy processes as a generalization of the fBm using Malliavin Calculus in [He \(2014\)](#).

Firstly, let us recall the LFSM defined in (1.1): let $\{L_\alpha(t)\}_{t \in \mathbb{R}}$ be a $S\alpha S$ process with index of stability $1 < \alpha < 2$ and Lebesgue control measure $dm(x) = dx$ and let $0 < d < 1 - \frac{1}{\alpha}$ be the fractional integration parameter. Then, taking $a = (\Gamma(d + 1))^{-1}$ and $b = 0$ and using the notation of Remark 2.4, we define

$$M_d(t) := \frac{1}{\Gamma(d + 1)} \int_{-\infty}^{\infty} [(t - x)_+^d - (-x)_+^d] dL_\alpha(x), \quad t \in \mathbb{R}. \tag{3.13}$$

Equipped with the $B_{\alpha,p}$ space given in Definition 3.7 and the norm $\|\cdot\|_{\alpha,p}$ given in Definition 3.5, we want to define, for functions $g \in B_{\alpha,p}$, the stochastic integral

$$I_{M_d}(g) = \int_{\mathbb{R}} g(x) dM_d(x). \tag{3.14}$$

As usual, we start by defining the integral for simple functions. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be of the form

$$\phi(x) = \sum_{i=1}^{n-1} a_i \mathbb{I}_{(t_i, t_{i+1}]}(x),$$

where $a_i \in \mathbb{R}$ and $-\infty < t_1 < t_2 < \dots < t_n < \infty$. It is easy to see that $\phi \in B_{\alpha,p}$. So we define the integral

$$I_{M_d}(\phi) = \int_{\mathbb{R}} \phi(x) dM_d(x) = \sum_{i=1}^{n-1} a_i [M_d(t_{i+1}) - M_d(t_i)].$$

Note that the integral $I_{M_d}(\cdot)$ is linear for simple functions. Moreover, we have the proposition below.

Proposition 3.8. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a simple function. Then*

$$\int_{\mathbb{R}} \phi(x) dM_d(x) = \int_{\mathbb{R}} (I_-^d \phi)(x) dL_\alpha(x) \tag{3.15}$$

and $\phi \rightarrow I_{M_d}(\phi)$ is an isometry between $B_{\alpha,p}$ and $L^p(\Omega, \mathbb{P})$, in the sense that $\|I_{M_d}(\phi)\|_{L^p(\Omega, \mathbb{P})}^p = \|\phi\|_{\alpha,p}^p$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space.

Proof. It suffices to show (3.15) for indicator functions $\phi(x) = \mathbb{I}_{(0,t]}(x)$, $t > 0$, because every other simple function is a linear combination of such functions. In fact,

$$\int_{\mathbb{R}} \phi(x) dM_d(x) = \int_{\mathbb{R}} \mathbb{I}_{(0,t]}(x) dM_d(x) = M_d(t)$$

and for the right-hand side of (3.15), we have

$$\begin{aligned} \int_{\mathbb{R}} (I_-^d \phi)(x) dL_\alpha(x) &= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_x^\infty (s - x)^{d-1} \mathbb{I}_{(0,t]}(s) ds dL_\alpha(x) \\ &= \frac{1}{\Gamma(d + 1)} \int_{\mathbb{R}} [(t - x)_+^d - (-x)_+^d] dL_\alpha(x) = M_d(t). \end{aligned} \tag{3.16}$$

Furthermore, for all simple functions ϕ it follows from Proposition 3.1 that

$$\begin{aligned} \|I_{M_d}(\phi)\|_{L^p(\Omega, \mathbb{P})}^p &= \mathbb{E} \left[\left| \int_{\mathbb{R}} (I_-^d \phi)(x) dL_\alpha(x) \right|^p \right] = \mathbb{E}[|L_\alpha(1)|^p] \left(\int_{\mathbb{R}} |(I_-^d \phi)(x)|^\alpha dx \right)^{\frac{p}{\alpha}} \\ &= \|\phi\|_{\alpha, p}^p. \end{aligned} \tag{3.17}$$

□

Having the integral $I_{M_d}(\cdot)$ well defined for simple functions, we are ready to prove the next theorem and to define the integral for any $g \in B_{\alpha, p}$.

Theorem 3.9. *Let $M_d = \{M_d(t)\}_{t \in \mathbb{R}}$ be the linear fractional stable motion defined in (3.13) and let the function $g : \mathbb{R} \rightarrow \mathbb{R}$ be in $B_{\alpha, p}$. Then there exists a sequence of simple functions $\phi_n : \mathbb{R} \rightarrow \mathbb{R}$, $n \in \mathbb{Z}_+$, satisfying $\|\phi_n - g\|_{\alpha, p} \rightarrow 0$ as $n \rightarrow \infty$ such that $I_{M_d}(\phi_n)$ converges in $L^p(\Omega, \mathbb{P})$ to a limit denoted by $I_{M_d}(g) = \int_{\mathbb{R}} g(x) dM_d(x)$ and $I_{M_d}(g)$ is independent of the approximating sequence ϕ_n . Moreover, the isometry property holds*

$$\|I_{M_d}(g)\|_{L^p(\Omega, \mathbb{P})} = \|g\|_{\alpha, p}. \tag{3.18}$$

Proof. As we know, simple functions are dense in $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$ and $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$ is dense in $B_{\alpha, p}$ by construction of the space $B_{\alpha, p}$. Also (3.11) holds, so whenever a sequence converges in $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$ it will converge in $B_{\alpha, p}$ as well. Therefore, the simple functions are dense in $B_{\alpha, p}$. Hence, there exists a sequence ϕ_n of simple functions such that $\|\phi_n - g\|_{\alpha, p} \rightarrow 0$, as $n \rightarrow \infty$. It follows from the isometry property (3.17) that $\int_{\mathbb{R}} \phi_n(x) dM_d(x)$ converges in $L^p(\Omega, \mathbb{P})$ towards a limit denoted by $\int_{\mathbb{R}} g(x) dM_d(x)$ and the isometry property is preserved in this procedure, so (3.18) holds. Finally, (3.18) implies that the limit denoted by the integral $\int_{\mathbb{R}} g(x) dM_d(x)$ is the same for all approximating sequences ϕ_n converging to g . □

Note that convergence in the L^p sense implies convergence in probability, which is enough to define the integrals. Usually, they are defined with equality in probability or even in the L^p sense, most commonly in L^2 (see Kuo, 2006). The next theorem states that we can interchange the fractional integration between the integrand function and the process concerning which we are integrating. This formula reminds the well-known integration by parts formula from traditional calculus.

Theorem 3.10. *Let $g \in B_{\alpha, p}$. Then*

$$\int_{\mathbb{R}} g(x) dM_d(x) = \int_{\mathbb{R}} (I_-^d g)(x) dL_\alpha(x). \tag{3.19}$$

Proof. It follows from Proposition 3.8 and Theorem 3.9. □

Now, as a direct application of the previous results, we state a convergence theorem that focuses on using the isometry property (3.18) and the norm $\|\cdot\|_{\alpha, p}$ to make sense the analysis of the convergence in $L^p(\Omega, \mathbb{P})$ of $I_{M_d}(g_n) \rightarrow I_{M_d}(g)$, as $n \rightarrow \infty$, for a suitable choice of g_n and g . We will study convergence on $L^p(\Omega, \mathbb{P})$ by dealing with convergence in $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$.

Consider g, g_n real measurable functions in $B_{\alpha, p}$ such that the stochastic integrals $I_{M_d}(g_n)$ and $I_{M_d}(g)$ are all well defined according to Theorem 3.9. We say that the sequence of processes $I_{M_d}(g_n)$ converges to the limit process $I_{M_d}(g)$ if

$$\|I_{M_d}(g_n) - I_{M_d}(g)\|_{L^p(\Omega, \mathbb{P})} \rightarrow 0 \tag{3.20}$$

and we denote

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} g_n(x) dM_d(x) = \int_{\mathbb{R}} g(x) dM_d(x), \tag{3.21}$$

with equality in the $L^p(\Omega, \mathbb{P})$ sense. Of course, it also implies convergence in probability for \mathbb{P} . Equality has the same meaning as in Theorem 3.9.

Theorem 3.11. *If $g_n \rightarrow g$ in both $L^1(\mathbb{R})$ and $L^\alpha(\mathbb{R})$, then $I_{M_d}(g_n) \rightarrow I_{M_d}(g)$ in $L^p(\Omega, \mathbb{P})$ and*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} g_n(x) dM_d(x) = \int_{\mathbb{R}} g(x) dM_d(x). \quad (3.22)$$

Proof. By (3.18) and (3.11) we have

$$\|I_{M_d}(g_n) - I_{M_d}(g)\|_{L^p(\Omega, \mathbb{P})} = \|g_n - g\|_{\alpha, p} \leq M \|g_n - g\|_1 + N \|g_n - g\|_\alpha \quad (3.23)$$

and it follows that $\|I_{M_d}(g_n) - I_{M_d}(g)\|_{L^p(\Omega, \mathbb{P})} \rightarrow 0$, as $n \rightarrow \infty$. \square

4. Main Results II

In this section, we are interested in defining a class of continuous-time moving average stable processes that present the property of long-range dependence. Let $\{L_\alpha(t)\}_{t \in \mathbb{R}}$ be a $S\alpha S$ process with index of stability $1 < \alpha < 2$ and Lebesgue control measure. Accordingly to Samorodnitsky and Taqqu (1994), if g is a real measurable function defined on \mathbb{R} satisfying $\int_{\mathbb{R}} |g(x)|^\alpha dx < \infty$, and define

$$X(t) = \int_{-\infty}^{\infty} g(t-x) dL_\alpha(x), \quad t \in \mathbb{R}, \quad (4.1)$$

then $\{X(t)\}_{t \in \mathbb{R}}$ is well defined and stationary. A process of this form is called $S\alpha S$ moving average process. The function g is often called the kernel function.

It's known in the literature that one can obtain a long-memory process by considering moving average processes whose kernel function is the fractional integral of a short-memory function (see Samorodnitsky, 2006 and Marquardt, 2006). Henceforth, we are considering functions with short memory and with positive support, which means that we will assume the following conditions:

- (C1) $g(t) = 0$, for all $t < 0$;
- (C2) $|g(t)| \leq C e^{-ct}$, for some constants $C > 0$ and $c > 0$.

Note that every such function g satisfying (C1) and (C2) belongs to $B_{\alpha, p}$.

Let us recall the left-sided Riemann-Liouville fractional integral of order $d \in (0, 1)$ of a function $g \in L^1(\mathbb{R})$, as defined in (2.3):

$$(I_+^d g)(x) = \frac{1}{\Gamma(d)} \int_{-\infty}^x g(t)(x-t)^{d-1} dt = \frac{1}{\Gamma(d)} \int_0^\infty g(x-s)(s)^{d-1} ds.$$

So for $1 < \alpha < 2$ and $1 < p < \alpha$ the idea is to define the moving average stable process using the left-sided Riemann-Liouville fractional integral of a kernel function g satisfying conditions (C1) and (C2).

Definition 4.1 (FIMA Stable process). Let $\{L_\alpha(t)\}_{t \in \mathbb{R}}$ be a $S\alpha S$ process with index of stability $1 < \alpha < 2$ and Lebesgue control measure. For $d \in (0, 1 - \frac{1}{\alpha})$, define

$$Y_d(t) = \int_{-\infty}^t (I_+^d g)(t-x) dL_\alpha(x), \quad t \in \mathbb{R}. \quad (4.2)$$

Theorem 4.2. *The FIMA Stable process $\{Y_d(t)\}_{t \in \mathbb{R}}$ in (4.2) is well defined and stationary.*

Proof. We know that $g \in B_{\alpha, p}$ implies $g \in L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$, so by Remark 3.4 it follows that $\int_{\mathbb{R}} |(I_+^d g)(x)|^\alpha dx < \infty$ and the process is well defined. Now, for $t_1, \dots, t_d, h, \theta_1, \dots, \theta_d \in \mathbb{R}$, setting

$y = x - h$, we have

$$\begin{aligned} \left\| \sum_{i=1}^d \theta_i Y_d(t_i + h) \right\|_\alpha^\alpha &= \int_{-\infty}^\infty \left| \sum_{i=1}^d \theta_i \mathbb{I}_{(-\infty, t_i+h]}(x) (I_+^d g)(t_i + h - x) \right|^\alpha dx \\ &= \int_{-\infty}^\infty \left| \sum_{i=1}^d \theta_i \mathbb{I}_{(-\infty, t_i]}(y) (I_+^d g)(t_i - y) \right|^\alpha dy = \left\| \sum_{i=1}^d \theta_i Y_d(t_i) \right\|_\alpha^\alpha. \end{aligned} \tag{4.3}$$

□

Now, using Theorem 3.10, we can represent the FIMA Stable process with the linear fractional stable motion.

Theorem 4.3. *Let $\{Y_d(t)\}_{t \in \mathbb{R}}$ be the FIMA Stable process defined in (4.2) and $\{M_d(t)\}_{t \in \mathbb{R}}$ be the linear fractional stable motion defined in (3.13). Then, Y_d can be represented as*

$$Y_d(t) = \int_{-\infty}^t g(t-x) dM_d(x), \quad t \in \mathbb{R}. \tag{4.4}$$

Proof. For each $t \in \mathbb{R}$, consider $g_t(x) := g(t-x)$, so from Theorem 3.10, we have

$$\int_{-\infty}^t g(t-x) dM_d(x) = \int_{-\infty}^t g_t(x) dM_d(x) = \int_{-\infty}^t (I_-^d g_t)(x) dL_\alpha(x).$$

By using the definition of the right-sided fractional integral $(I_-^d g_t)(x)$ and setting $w = u - x$ we get

$$\begin{aligned} \int_{-\infty}^t (I_-^d g_t)(x) dL_\alpha(x) &= \frac{1}{\Gamma(d)} \int_{-\infty}^t \int_x^\infty (u-x)^{d-1} g_t(u) du dL_\alpha(x) \\ &= \frac{1}{\Gamma(d)} \int_{-\infty}^t \int_x^\infty (u-x)^{d-1} g(t-u) du dL_\alpha(x) \\ &= \frac{1}{\Gamma(d)} \int_{-\infty}^t \int_0^\infty w^{d-1} g(t-x-w) dw dL_\alpha(x). \end{aligned} \tag{4.5}$$

Finally, using the definition of the left-sided fractional integral $(I_+^d g)(t-x)$ we get the result

$$\frac{1}{\Gamma(d)} \int_{-\infty}^t \int_0^\infty w^{d-1} g(t-x-w) dw dL_\alpha(x) = \int_{-\infty}^t (I_+^d g)(t-x) dL_\alpha(x) = Y_d(t). \tag{4.6}$$

□

Remark 4.4. Note that this representation allows us to work with a rapid decay kernel function g instead of a slow decay kernel function $(I_+^d g)$, which can be useful for more efficient simulation algorithms. It also can be used to show a convergence of sequences of such processes by dealing with the associated moving average kernel function. See Theorems 3.11 and 4.11.

Now we turn our attention to the long-range dependence, or long-memory property of the process $\{Y_d(t)\}_{t \in \mathbb{R}}$. As we know, for stationary Gaussian processes one can describe the long-range dependence property as the slow decaying of its autocovariance function. Since we cannot use such a function in our setting, as Maejima and Yamamoto (2003) we are going to use the following and proceed with similar calculations as the cited authors. For $\theta_1, \theta_2 \in \mathbb{R}$, let

$$\begin{aligned} r(t) := r(\theta_1, \theta_2; t) &= \mathbb{E} \left[\exp\{i(\theta_1 Y_d(t) + \theta_2 Y_d(0))\} \right] \\ &\quad - \mathbb{E} \left[\exp\{i\theta_1 Y_d(t)\} \right] \mathbb{E} \left[\exp\{i\theta_2 Y_d(0)\} \right] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} I(t) := I(\theta_1, \theta_2; t) &= -\log \left(\mathbb{E} \left[\exp\{i(\theta_1 Y_d(t) + \theta_2 Y_d(0))\} \right] \right) \\ &+ \log \left(\mathbb{E} \left[\exp\{i\theta_1 Y_d(t)\} \right] \right) + \log \left(\mathbb{E} \left[\exp\{i\theta_2 Y_d(0)\} \right] \right). \end{aligned} \quad (4.8)$$

Setting

$$\begin{aligned} K(\theta_1, \theta_2; t) &= \mathbb{E} \left[\exp\{i\theta_1 Y_d(t)\} \right] \mathbb{E} \left[\exp\{i\theta_2 Y_d(0)\} \right] \\ &= \mathbb{E} \left[\exp\{i\theta_1 Y_d(0)\} \right] \mathbb{E} \left[\exp\{i\theta_2 Y_d(0)\} \right] =: K \end{aligned}$$

which does not depend on t since $\{Y_d(t)\}_{t \in \mathbb{R}}$ is stationary. There is a relationship among r , I and K given by

$$r(\theta_1, \theta_2; t) = K(\theta_1, \theta_2; t) (e^{-I(\theta_1, \theta_2; t)} - 1). \quad (4.9)$$

Furthermore, if $I(t) \rightarrow 0$ as $t \rightarrow \infty$, then $r(t) \sim -KI(t)$ as $t \rightarrow \infty$, which means that the quantities are asymptotically equal. Note that the quantity $r(\theta_1, \theta_2; t)$ is the difference between the joint characteristic function of the pair $(Y_d(t), Y_d(0))$ and the product of the marginal characteristic functions of $Y_d(t)$ and $Y_d(0)$. So if the process is independent, $r(\theta_1, \theta_2; t) = 0$, and if the process is Gaussian or has finite second-order moments, the quantity $-I(1, -1; t)$ coincides with the autocovariance function. The quantity $r(t)$ has been proved to be a proper tool for describing the dependence structure of stable processes and has been used by several authors such as [Astrauskas et al. \(1991\)](#); [Levy and Taqqu \(1991\)](#); [Magdziarz and Weron \(2007\)](#); [Magdziarz \(2007\)](#). Besides, it is used in [Maruyama \(1970\)](#) relating it to the mixing property for stationary infinitely divisible processes. Also, [Samorodnitsky and Taqqu \(1994\)](#) point out that the quantity $I(\cdot)$ can be used to show that two processes are different by showing that their respective quantities are asymptotically different. Let us then define what we mean by the long-range dependence or long-memory property in our setting.

Definition 4.5. Let $\{X(t)\}_{t \in \mathbb{R}}$ be a stationary $S\alpha S$ process and $r(t)$ as in (4.7). We say the process $\{X(t)\}_{t \in \mathbb{R}}$ has *long-range dependence* or *long-memory property* if there exists $\theta \in (0, 1)$ and a constant $c_r > 0$ such that $r(t)$ satisfies

$$\lim_{t \rightarrow \infty} |r(t)| \frac{t^\theta}{c_r} = 1, \quad (4.10)$$

or, equivalently,

$$|r(t)| \sim c_r t^{-\theta}.$$

Note that since the process $\{Y_d(t)\}_{t \in \mathbb{R}}$ is stationary,

$$(Y_d(t), Y_d(0)) \stackrel{d}{=} (Y_d(0), Y_d(-t)),$$

thus $r(\theta_1, \theta_2; t) = r(\theta_2, \theta_1; -t)$. Therefore, the asymptotic behavior of $r(\theta_1, \theta_2; t)$ as $t \rightarrow \infty$ is essentially the same as $r(\theta_1, \theta_2; t)$ as $t \rightarrow -\infty$. Since the computations for $t < 0$ as $t \rightarrow -\infty$ are a bit easier in our setting, as in [Maejima and Yamamoto \(2003\)](#), we will analyze the asymptotic behavior of $r(\theta_1, \theta_2; t)$ for $t < 0$, as $t \rightarrow -\infty$.

Before we give the proof of the asymptotic behavior of $r(\theta_1, \theta_2; t)$, for $t < 0$, as $t \rightarrow -\infty$, for a FIMA process (see Theorem 4.3 below), we shall give three auxiliary lemmas.

Lemma 4.6. For $1 < \alpha < 2$ and for $r, s \in \mathbb{R}$, it is true that

$$\left| |r + s|^\alpha - |r|^\alpha - |s|^\alpha \right| \leq \alpha |r| |s|^{\alpha-1} + (\alpha + 1) |r|^\alpha. \quad (4.11)$$

Proof. See lemma 5.2 in [Maejima and Yamamoto \(2003\)](#). □

We define the following functions $\phi(\cdot, x)$ and $\psi(\cdot, x)$, for $x > 1$,

$$\phi(t, x) = \frac{1}{\Gamma(d)} \int_1^x \theta_1 g(t(1-u))(x-u)^{d-1} du \quad \text{and} \quad \psi(t, x) = \frac{1}{\Gamma(d)} \int_0^x \theta_2 g(-tu)(x-u)^{d-1} du. \tag{4.12}$$

Lemma 4.7. *Let the functions $\phi(\cdot, x)$ and $\psi(\cdot, x)$ be defined in (4.12). Then, there exist constants $K_1, K_2 > 0$ such that, for any $t < 0$ and $x > 1$ we have*

$$|t\phi(t, x)| \leq K_1(x-1)^{d-1} \quad \text{and} \quad |t\psi(t, x)| \leq K_2x^{d-1}. \tag{4.13}$$

Proof. First, note that setting $w = u + 1$, we have

$$\begin{aligned} \theta_2^{-1}\psi(t, x-1) &= \frac{1}{\Gamma(d)} \int_0^{x-1} g(-tu)(x-1-u)^{d-1} du \\ &= \frac{1}{\Gamma(d)} \int_1^x g(t(1-w))(x-w)^{d-1} dw = \theta_1^{-1}\phi(t, x). \end{aligned} \tag{4.14}$$

Thus it suffices to show (4.13) for $\phi(\cdot, x)$. In fact, we have

$$\begin{aligned} |t\phi(t, x)| &= \left| \frac{1}{\Gamma(d)} \int_1^x t\theta_1 g(t(1-u))(x-u)^{d-1} du \right| \\ &\leq \left| \frac{1}{\Gamma(d)} \int_1^{\frac{x+1}{2}} t\theta_1 g(t(1-u))(x-u)^{d-1} du \right| + \left| \frac{1}{\Gamma(d)} \int_{\frac{x+1}{2}}^x t\theta_1 g(t(1-u))(x-u)^{d-1} du \right| \\ &= I_1 + I_2. \end{aligned} \tag{4.15}$$

Recall that $t < 0$. For I_1 , it follows that

$$I_1 \leq \frac{|\theta_1 C|}{\Gamma(d)} \left(\frac{x-1}{2}\right)^{d-1} \int_1^{\frac{x+1}{2}} e^{ct(u-1)} du \leq \frac{|\theta_1 C|}{\Gamma(d)c} \left(\frac{x-1}{2}\right)^{d-1} \tag{4.16}$$

and

$$\begin{aligned} I_2 &\leq \frac{|\theta_1 C|}{\Gamma(d)} e^{\frac{ct(x-1)}{2}} \int_{\frac{x+1}{2}}^x (x-u)^{d-1} du = \frac{|\theta_1 C|}{\Gamma(d)} \left(\frac{x-1}{2}\right)^d |t| e^{\frac{ct(x-1)}{2}} \\ &\leq \frac{|\theta_1 C|}{\Gamma(d)ce} \left(\frac{x-1}{2}\right)^{d-1}, \end{aligned} \tag{4.17}$$

since $z(t) := |t|e^{\frac{ct(x-1)}{2}}$ takes its maximum value $\frac{2}{c(x-1)e}$ at $t = \frac{-2}{c(x-1)}$. Combining (4.16) and (4.17), we have

$$|t\phi(t, x)| \leq I_1 + I_2 \leq \frac{|\theta_1 C|}{\Gamma(d)c} \left(\frac{x-1}{2}\right)^{d-1} + \frac{|\theta_1 C|}{\Gamma(d)ce} \left(\frac{x-1}{2}\right)^{d-1} \leq K_1(x-1)^{d-1}, \tag{4.18}$$

which is (4.13). □

Lemma 4.8. *Let the functions $\phi(\cdot, x)$ and $\psi(\cdot, x)$ be defined in (4.12). There exists a constant $L \in \mathbb{R}$ such that*

$$\lim_{t \rightarrow -\infty} t\phi(t, x) = L\theta_1(x-1)^{d-1} \quad \text{and} \quad \lim_{t \rightarrow -\infty} t\psi(t, x) = L\theta_2x^{d-1}. \tag{4.19}$$

Proof. For (4.19) it suffices to show for the function $\phi(\cdot, x)$. It follows that

$$\begin{aligned} \lim_{t \rightarrow -\infty} t\phi(t, x) &= \lim_{t \rightarrow -\infty} \frac{1}{\Gamma(d)} \int_1^{\frac{x+1}{2}} t\theta_1 g(t(1-u))(x-u)^{d-1} du \\ &+ \lim_{t \rightarrow -\infty} \frac{1}{\Gamma(d)} \int_{\frac{x+1}{2}}^x t\theta_1 g(t(1-u))(x-u)^{d-1} du = J_1 + J_2. \end{aligned} \quad (4.20)$$

So, set $w = t(u-1)$ to get

$$\begin{aligned} J_1 &= \lim_{t \rightarrow -\infty} \frac{1}{\Gamma(d)} \int_0^{\frac{t(x-1)}{2}} \theta_1 g(-w) \left(x - \frac{w}{t} - 1\right)^{d-1} dw \\ &= - \lim_{t \rightarrow -\infty} \frac{\theta_1}{\Gamma(d)} \int_{-\infty}^0 \mathbb{I}_{\left[\frac{t(x-1)}{2}, 0\right]}(w) g(-w) \left(x - \frac{w}{t} - 1\right)^{d-1} dw \end{aligned} \quad (4.21)$$

but by condition (C2),

$$\begin{aligned} \left| \mathbb{I}_{\left[\frac{t(x-1)}{2}, 0\right]}(w) g(-w) \left(x - \frac{w}{t} - 1\right)^{d-1} \right| &\leq C \left| \mathbb{I}_{\left[\frac{t(x-1)}{2}, 0\right]}(w) e^{cw} \left(x - \frac{w}{t} - 1\right)^{d-1} \right| \\ &\leq C e^{cw} \left(\frac{x-1}{2}\right)^{d-1}, \end{aligned} \quad (4.22)$$

which is w integrable in $(-\infty, 0)$. Thus, by the Dominated Convergence theorem and setting $u = -w$, we get

$$J_1 = - \frac{\theta_1}{\Gamma(d)} \int_{-\infty}^0 g(-w) (x-1)^{d-1} dw = \left[\frac{1}{\Gamma(d)} \int_0^\infty g(u) du \right] \theta_1 (x-1)^{d-1}. \quad (4.23)$$

And for J_2 ,

$$\begin{aligned} |J_2| &\leq \lim_{t \rightarrow -\infty} \frac{|\theta_1|}{\Gamma(d)} \int_{\frac{x+1}{2}}^x |tg(t(1-u))| (x-u)^{d-1} du \\ &\leq \lim_{t \rightarrow -\infty} \frac{|t\theta_1|}{\Gamma(d)} \int_{\frac{x+1}{2}}^x e^{ct(u-1)} (x-u)^{d-1} du \leq \lim_{t \rightarrow -\infty} \frac{|t\theta_1|}{\Gamma(d)} e^{\frac{ct(x-1)}{2}} \int_{\frac{x+1}{2}}^x (x-u)^{d-1} du = 0. \end{aligned} \quad (4.24)$$

Therefore, from equations (4.17)-(4.20), it follows that

$$\lim_{t \rightarrow -\infty} t\phi(t, x) = \left[\frac{1}{\Gamma(d)} \int_0^\infty g(u) du \right] \theta_1 (x-1)^{d-1} \quad (4.25)$$

and we get (4.19) by setting

$$L = \frac{1}{\Gamma(d)} \int_0^\infty g(u) du.$$

□

Now we are ready for Theorem 4.9 below. Recall that for a $S\alpha S$ process $\{X(t)\}_{t \in \mathbb{R}}$ given by (3.1), the Proposition 2.1 states that

$$\mathbb{E} \left[\exp \left\{ i\theta \int_{\mathbb{R}} f_t(x) dL_\alpha(x) \right\} \right] = \exp \left\{ -|\theta|^\alpha \int_{\mathbb{R}} |f_t(x)|^\alpha dx \right\}. \quad (4.26)$$

Theorem 4.9. *Let $\{Y_d(t)\}_{t \in \mathbb{R}}$ be the FIMA Stable process defined in (4.2). If $1 < \alpha < 2$ and $0 < d < 1 - \frac{1}{\alpha}$, then, as $t \rightarrow -\infty$,*

$$r(t) = r(\theta_1, \theta_2; t) \sim -KC|t|^{\alpha(d-1)+1}, \quad (4.27)$$

where

$$K = \exp \left\{ - \frac{\theta_1^\alpha + \theta_2^\alpha}{\Gamma(d)^\alpha} \int_{-\infty}^0 \left| \int_x^0 g(-u)(u-x)^{d-1} du \right|^\alpha dx \right\}, \tag{4.28}$$

and

$$C = \left(\frac{1}{\Gamma(d)} \int_0^\infty g(u)du \right)^\alpha \int_1^\infty \left(\left| \theta_1(x-1)^{d-1} + \theta_2 x^{d-1} \right|^\alpha - \left| \theta_1(x-1)^{d-1} \right|^\alpha - \left| \theta_2 x^{d-1} \right|^\alpha \right) dx. \tag{4.29}$$

Proof. We start by calculating $K = \mathbb{E}[\exp\{i\theta_1 Y_d(0)\}] \mathbb{E}[\exp\{i\theta_2 Y_d(0)\}]$. By (4.26), (4.5) and (4.6)

$$\begin{aligned} K &= \exp \left\{ - \frac{1}{\Gamma(d)^\alpha} \int_{-\infty}^0 \left(\left| \int_x^0 \theta_1 g(-u)(u-x)^{d-1} du \right|^\alpha - \left| \int_x^0 \theta_2 g(-u)(u-x)^{d-1} du \right|^\alpha \right) dx \right\} \\ &= \exp \left\{ - \frac{\theta_1^\alpha + \theta_2^\alpha}{\Gamma(d)^\alpha} \int_{-\infty}^0 \left| \int_x^0 g(-u)(u-x)^{d-1} du \right|^\alpha dx \right\}. \end{aligned} \tag{4.30}$$

Now, let us compute $I(\cdot)$. Again by (4.26), (4.5) and (4.6), we have

$$\begin{aligned} I(t) &= -\log \left(\mathbb{E} \left[\exp\{i(\theta_1 Y_d(t) + \theta_2 Y_d(0))\} \right] \right) + \log \left(\mathbb{E} \left[\exp\{i\theta_1 Y_d(t)\} \right] \right) \\ &\quad + \log \left(\mathbb{E} \left[\exp\{i\theta_2 Y_d(0)\} \right] \right) \\ &= \frac{1}{\Gamma(d)^\alpha} \int_{-\infty}^t \left(\left| \int_x^t \theta_1 g(t-u)(u-x)^{d-1} du + \int_x^0 \theta_2 g(-u)(u-x)^{d-1} du \right|^\alpha \right. \\ &\quad \left. - \left| \int_x^t \theta_1 g(t-u)(u-x)^{d-1} du \right|^\alpha - \left| \int_x^0 \theta_2 g(-u)(u-x)^{d-1} du \right|^\alpha \right) dx. \end{aligned} \tag{4.31}$$

Set $x = xt$ and $u = ut$ and recall that $t < 0$ to get

$$\begin{aligned} I(t) &= \frac{1}{\Gamma(d)^\alpha} \int_1^\infty \left(\left| \int_1^x \theta_1 g(t(1-u))(x-u)^{d-1} |t|^{d-1} |t| du \right. \right. \\ &\quad \left. \left. + \int_0^x \theta_2 g(-tu)(x-u)^{d-1} |t|^{d-1} |t| du \right|^\alpha - \left| \int_1^x \theta_1 g(t(1-u))(x-u)^{d-1} |t|^{d-1} |t| du \right|^\alpha \right. \\ &\quad \left. - \left| \int_0^x \theta_2 g(-tu)(x-u)^{d-1} |t|^{d-1} |t| du \right|^\alpha \right) |t| dx. \end{aligned} \tag{4.32}$$

Since $(|t|^{d-1}|t|)^\alpha |t| = |t|^{\alpha d+1}$, it follows that

$$\begin{aligned} I(t) &= \frac{|t|^{\alpha d+1}}{\Gamma(d)^\alpha} \int_1^\infty \left(\left| \int_1^x \theta_1 g(t(1-u))(x-u)^{d-1} du + \int_0^x \theta_2 g(-tu)(x-u)^{d-1} du \right|^\alpha \right. \\ &\quad \left. - \left| \int_1^x \theta_1 g(t(1-u))(x-u)^{d-1} du \right|^\alpha - \left| \int_0^x \theta_2 g(-tu)(x-u)^{d-1} du \right|^\alpha \right) dx \\ &=: |t|^{\alpha d+1} \int_1^\infty |\phi(t, x) + \psi(t, x)|^\alpha - |\phi(t, x)|^\alpha - |\psi(t, x)|^\alpha dx, \end{aligned} \tag{4.33}$$

where $\phi(\cdot, x)$ and $\psi(\cdot, x)$, for $x > 1$, are given in (4.12).

Therefore,

$$\lim_{t \rightarrow -\infty} |t|^{\alpha-\alpha d-1} I(t) = \lim_{t \rightarrow -\infty} \int_1^\infty (|\phi(t, x) + t\psi(t, x)|^\alpha - |\phi(t, x)|^\alpha - |t\psi(t, x)|^\alpha) dx. \tag{4.34}$$

We want to use the Dominated Convergence theorem to deal with the above limit. From Lemmas 4.6, with $r = t\psi(t, x)$ and $s = t\phi(t, x)$, and Lemma 4.7, it follows that

$$\begin{aligned} |s + r|^\alpha - |s|^\alpha - |r|^\alpha &\leq \alpha|r||s|^{\alpha-1} + (\alpha + 1)|r|^\alpha \\ &\leq \alpha K_2 x^{d-1} (K_1(x-1)^{d-1})^{\alpha-1} + (\alpha + 1)(K_2 x^{d-1})^\alpha \\ &= \alpha K_1^{(\alpha-1)(d-1)} K_2 x^{d-1} (x-1)^{(\alpha-1)(d-1)} + (\alpha + 1) K_2^\alpha x^{\alpha(d-1)}. \end{aligned} \tag{4.35}$$

Note that $1 < \alpha < 2$ implies $0 < \alpha - 1 < 1$ and $0 < d < 1 - \frac{1}{\alpha}$ implies $-1 < d - 1 < -\frac{1}{\alpha}$, so $-1 < (\alpha - 1)(d - 1)$ and $\alpha(d - 1) < -1$. Thus, $x^{\alpha(d-1)}$ clearly is in $L^1(1, \infty)$. As for $x^{d-1}(x-1)^{(\alpha-1)(d-1)}$, the integral may diverge when $x \rightarrow 1$ and $x \rightarrow \infty$. But when $x \rightarrow 1$, $x^{d-1}(x-1)^{(\alpha-1)(d-1)}$ behaves like $(x-1)^{(\alpha-1)(d-1)}$, which is integrable since $-1 < (\alpha - 1)(d - 1)$. And when $x \rightarrow \infty$, $x^{d-1}(x-1)^{(\alpha-1)(d-1)}$ behaves like $x^{\alpha(d-1)}$. Hence,

$$\int_1^\infty \left| \alpha K_1^{(\alpha-1)(d-1)} K_2 x^{d-1} (x-1)^{(\alpha-1)(d-1)} + (\alpha + 1) K_2^\alpha x^{\alpha(d-1)} \right| dx < \infty. \tag{4.36}$$

Now by applying the Dominated Convergence theorem in (4.34) altogether with Lemma 4.8 we obtain

$$\begin{aligned} \lim_{t \rightarrow -\infty} |t|^{\alpha-\alpha d-1} I(t) &= \int_1^\infty \lim_{t \rightarrow -\infty} |t\phi(t, x) + t\psi(t, x)|^\alpha - |t\phi(t, x)|^\alpha - |t\psi(t, x)|^\alpha dx \\ &= L^\alpha \int_1^\infty \left| \theta_1(x-1)^{d-1} + \theta_2 x^{d-1} \right|^\alpha - \left| \theta_1(x-1)^{d-1} \right|^\alpha - \left| \theta_2 x^{d-1} \right|^\alpha dx. \end{aligned} \tag{4.37}$$

The last above integral in expression (4.37) is the constant C in (4.29) (see the proof of Lemma 4.8 for the value of L). Thus, $I(t) \rightarrow 0$, as $t \rightarrow \infty$. Therefore,

$$r(t) \sim -KI(t) \sim -KC|t|^{\alpha(d-1)+1}, \text{ as } t \rightarrow \infty.$$

□

The following theorem proves the long-range dependence property for the FIMA Stable process.

Theorem 4.10. *Let $\{Y_d(t)\}_{t \in \mathbb{R}}$ be the FIMA Stable process defined in (4.2). If $1 < \alpha < 2$ and $0 < d < 1 - \frac{1}{\alpha}$, then it has the property of long-range dependence or long-memory, in the sense of Definition 4.5.*

Proof Since $1 < \alpha < 2$ and $-1 < d - 1 < -\frac{1}{\alpha}$ imply that $-2 < \alpha(d - 1) < -1$, we have by the Theorem 4.9 that $r(t)$ satisfies

$$|r(t)| = |r(\theta_1, \theta_2; t)| \sim KCt^{\alpha(d-1)+1}, \tag{4.38}$$

with $-1 < \alpha(d - 1) + 1 < 0$.

□

Now, as an application of the representation Theorem 4.3 and the convergence Theorem 3.11, we will prove a result which is basically a LLN result for our setting. Suppose $\{t_j\}_{j \in \mathbb{Z}_+}$ is a sequence of times such that $t_j \in \mathbb{R}$ and for every $j \in \mathbb{Z}_+$ we consider $Y_d(t_j)$ with representation

$$Y_d(t_j) = \int_{-\infty}^{t_j} g(t_j - x) dM_d(x) = \int_{\mathbb{R}} \mathbb{I}_{\{x \leq t_j\}} g(t_j - x) dM_d(x) =: \int_{\mathbb{R}} g_j(x) dM_d(x). \tag{4.39}$$

Define the sequence of partial sums $\{S_n\}_{n \in \mathbb{Z}_+}$ by

$$S_n := \sum_{j=1}^n Y_d(t_j) = \sum_{j=1}^n I_{M_d}(g_j) = I_{M_d} \left(\sum_{j=1}^n g_j \right) = \int_{\mathbb{R}} \left[\sum_{j=1}^n g_j(x) \right] dM_d(x) \tag{4.40}$$

from the linearity of the integral I_{M_d} . So we state the following theorem.

Theorem 4.11. *If the sequence of times $\{t_j\}_{j \in \mathbb{Z}_+}$ is a sequence of natural times, $t_j = j$, then*

$$\frac{S_n}{n} \rightarrow 0, \text{ as } n \rightarrow \infty \text{ in the } L^p(\Omega, \mathbb{P}) \text{ sense.} \tag{4.41}$$

Proof. We start with some definitions. Let

$$G_n(x) := \sum_{j=1}^n g_j(x).$$

So the sequence in (4.40) becomes

$$\frac{S_n}{n} = \int_{\mathbb{R}} \left[\frac{G_n(x)}{n} \right] dM_d(x).$$

Thus by Theorem 3.11 it suffices to show that $\|G_n(x)/n\|_1 \rightarrow 0$ and $\|G_n(x)/n\|_\alpha \rightarrow 0$, as $n \rightarrow \infty$, to conclude that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left[\frac{G_n(x)}{n} \right] dM_d(x) = \int_{\mathbb{R}} 0 dM_d(x) = 0.$$

We first note that as g satisfies $|g(x)| \leq C e^{-cx}$, for $x \geq 0$ and some constants $C, c > 0$,

$$\left| \frac{G_n(x)}{n} \right| \leq \frac{1}{n} \sum_{j=1}^n \mathbb{I}_{\{x \leq j\}} |g(j-x)| \leq \frac{C}{n} \sum_{j=1}^n \mathbb{I}_{\{x \leq j\}} e^{-c(j-x)}. \tag{4.42}$$

For every fixed $x \in \mathbb{R}$, the series on the right-hand side of the last equality in equation (4.42) converges to some positive constant limit D_x ,

$$\sum_{j=1}^{\infty} \mathbb{I}_{\{x \leq j\}} e^{-c(j-x)} = D_x.$$

We get that for all $x \in \mathbb{R}$,

$$\left| \frac{G_n(x)}{n} \right| \leq \frac{C D_x}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Furthermore, since $s \leq t \Rightarrow \mathbb{I}_{\{x \leq t\}} e^{-c(t-x)} \leq \mathbb{I}_{\{x \leq s\}} e^{-c(s-x)} \Rightarrow \sup_j \mathbb{I}_{\{x \leq j\}} e^{-c(j-x)} = \mathbb{I}_{\{x \leq 1\}} e^{-c(1-x)}$ then, for every fixed $n \in \mathbb{Z}_+$, by (4.42) we get that

$$\left| \frac{G_n(x)}{n} \right| \leq \frac{C}{n} \sum_{j=1}^n \mathbb{I}_{\{x \leq 1\}} e^{-c(1-x)} = C \mathbb{I}_{\{x \leq 1\}} e^{-c(1-x)}, \tag{4.43}$$

which clearly belongs to $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$. We obtain the desired result by the Dominated Convergence theorem. \square

The following corollary is a simple extension of the last result.

Corollary 4.12. *If the sequence of times $\{t_j\}_{j \in \mathbb{Z}_+}$ is bounded below ($t_j \in [a, \infty)$) such that $\lim_{j \rightarrow \infty} j^{-\beta} t_j \geq K$ for some constants $\beta, K > 0$ then*

$$\frac{S_n}{n} \rightarrow 0, \text{ as } n \rightarrow \infty, \text{ in the } L^p(\Omega, \mathbb{P}) \text{ sense.} \tag{4.44}$$

Proof. It suffices to note that since $\sup_j \mathbb{I}_{\{x \leq t_j\}} e^{-c(t_j-x)} = \mathbb{I}_{\{x \leq a\}} e^{-c(a-x)}$, for every $x \in \mathbb{R}$,

$$\left| \frac{G_n(x)}{n} \right| \leq \mathbb{I}_{\{x \leq a\}} e^{-c(a-x)},$$

which also clearly belongs to $L^1(\mathbb{R}) \cap L^\alpha(\mathbb{R})$. Since $\lim_{j \rightarrow \infty} j^{-\beta} t_j \geq K$, we get that for all $x \in \mathbb{R}$,

$$\left| \frac{G_n(x)}{n} \right| \leq \frac{C}{n} \sum_{j=1}^n \mathbb{I}_{\{x \leq Kj^\beta\}} e^{-c(Kj^\beta - x)} \leq \frac{Ce^{cx}}{n} \sum_{j=1}^n e^{-cKj^\beta} \leq \frac{Ce^{cx} \tilde{D}}{n} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

for

$$\tilde{D} := \sum_{j=1}^{\infty} e^{-cKj^\beta}.$$

□

5. Conclusions

We have successfully defined a continuous-time fractionally integrated moving average α -stable process with the long-range dependence property. The results presented here can be applied in models that capture both high variability behavior and long-range dependence structure. We have shown an isometry allowing us to define stochastic integrals concerning the linear fractional stable motion by using the Riemann-Liouville fractional integrals. From this, a by-product was obtained as integration by parts formula. Theorem 4.3 reveals a core relationship between a FIMA Stable process and the LFSM, while Theorem 4.10 showed the long-range dependence property for the FIMA Stable processes. As an application of the isometry and integration by parts formula, law of large number's results follow immediately.

Acknowledgements

The authors would like to express their sincere thanks to an anonymous referee that read carefully the original manuscript pointing out several mistakes on it.

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