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MAS-COLELL AND RAZIN MODEL OF INTERSECTORAL MIGRATION AND GROWTH WITH DISTINCT POPULATION GROWTH RATES

Porto Alegre
2021

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Work presented in partial fulfillment of the requirements for the degree of Bachelor in Economics.

Advisor: Prof. Dr. João Plínio Juchem Neto

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RESUMO

Neste trabalho propomos a generalização do modelo de dois setores de crescimento econômico e migração intersetorial de Mas-Colell e Razin a partir da introdução de taxas de crescimento populacional distintas para cada setor. A dinâmica do nosso modelo generalizado é definida por um sistema de duas Equações Diferenciais Ordinárias, uma para o capital agregado *per capita* e a outra para a proporção de população no setor industrial. Através da análise de estabilidade, mostra-se que nosso modelo possui sempre um estado estacionário estável. Simulações numéricas foram feitas considerando três cenários, um com taxas de crescimento populacionais iguais e dois com taxas de crescimento populacionais distintas, com três diferentes políticas fiscais para cada cenário. Comparando os cenários sob as mesmas políticas fiscais, observamos que para o cenário com uma maior taxa de crescimento populacional no setor industrial houve uma maior acúmulo de capital per capita agregado e mais pessoas vivendo no setor industrial em relação ao cenário com taxas de crescimento populacional iguais, para qualquer política fiscal. Para o cenário com uma maior taxa de crescimento populacional no setor agrícola, o único caso com um acúmulo de capital per capita agregado menor do que no cenário com taxas de crescimento populacional iguais é aquele com um subsídio; além disso, houve menos pessoas vivendo no setor industrial em relação ao cenário com taxas de crescimento populacional iguais, para as três políticas fiscais.

Palavras-chave: Modelo de Crescimento Econômico de Dois Setores. Migração. Crescimento Populacional Distinto. Política Fiscal.

ABSTRACT

In this work we propose a generalization of the Mas-Colell and Razin model of intersectoral migration and growth through the introduction of distinct population growth rates for each sector. The dynamics of our generalized model is given by a system of two Ordinary Differential Equations, one for the aggregate per capita capital and another for the proportion of the population in the industrial sector. Furthermore, through a stability analysis, we show that our model will always have a stable steady-state. Numerical simulations were performed given three scenarios, one with equal population growth rates and two with distinct population growth rates, with three different tax policies for each scenario. Comparing the scenarios under the same tax policies, we observed that for the scenario with a bigger population growth rate in the industrial sector, there was a bigger aggregate per capita capital accumulation and more people living in the industrial sector in relation to the scenario with equal population growth rates, for any tax policy. For the scenario with a bigger population growth rate in the agricultural sector, the only case with an aggregate per capita capital accumulation smaller than in the scenario with equal population growth rates was the one with a subsidy; besides, there was less people living in the industrial sector relative to the scenario with equal population growth rates, for all three tax policies.

Keywords: Two-sector Growth Model. Migration. Distinct Population Growth. Tax Policy.

LIST OF FIGURES

LIST OF TABLES

CONTENTS

1 INTRODUCTION

The interest for economic growth is certainly not new. Since the dawn of Economics with the Physiocrats - when economic growth was believed to be governed exclusively by natural laws, going through the early stages of industrial development with the first industrial revolution, economists had already had their attention focused on investigating the national wealth and how to increase it. From the 1930s, in the shadow of the Great Depression, and essentially after the Second World War, in the 1950s and 1960s, there was a significant increase in the production of works in theory of economic growth, precisely because it was an economically chaotic period. According to Kregel (1972), “for while an economy is growing there may be no need to wonder at the cause, but when it is not there is a pressing necessity to consider why not.” (KREGEL, 1972, p. 9).

In this context, the Neoclassical growth models stand out from the contestation of the assumptions and results of post-Keynesian works like the Harrod-Domar model. Solow (1956) and Swan (1956) formulated the first neoclassical model that gained prominence and motivated far-reaching models, such as Uzawa (1961). These early models that began to investigate the mechanisms of growth bothered some authors like Jorgenson (1961), who argued that they embodied too much the advanced and industrial economies, leaving aside the developing economies. Because of that, Jorgenson (1961) developed a two-sector model whose structure most faithfully represented those of developing countries. However, Mas-Colell and Razin (1973) showed that there was no need for Jorgenson to formulate a new model, because his model’s growth patterns could be obtained and explained within a neoclassical framework.

The Mas-Colell and Razin model is a neoclassical two-sector growth model with intersectoral migration. It consists of two sectors working under perfect competition, an industrial and an agricultural sector, and two goods, one for consumption and investment produced in the former and one only for consumption produced in the latter. It contemplates only two inputs, capital and labor, which are fully employed and move freely between the sectors. Production functions for both sectors are linear homogeneous of Cobb-Douglas type. The Mas-Colell and Razin model introduces a function that models the migration of labor from one sector to another, adding a migratory dynamic which is one of the main differences between this model and the others considered.

The objective of this work is to generalize the Mas-Colell and Razin model of intersectoral migration and growth, which considers the same population growth rate in both sectors, in order to consider two distinct population growth rates, one for the agricultural sector and another for the industrial one. This generalization allows us to incorporate in our model a factor that expresses demographic changes more plausibly, since population growth usually diverges between agricultural and urban (industrial) regions. To formulate our generalized model, we first derived the Mas-Colell and Razin model. After that, we introduced the distinct sectoral population growth rates, which caused the redefinition of the main variables. To finish the formulation of our generalized model, we carried out a stability analysis of its steady-states. The final step was to investigate the impacts of different population growth rates and tax policies on the endogenous variables of the model, which was done performing numerical simulations and through the analysis of the phase diagrams of our model, using the software MATLAB® and Maple®.

This work is structured as follows: we begin with a literature review focusing on Neoclassical growth models, especially the ones with two sectors. The third chapter presents and derives mathematically the Mas-Colell and Razin model, including the introduction of a tax policy object. The fourth chapter presents our generalized model and the stability analysis with distinct population growth rates. The fifth chapter shows the numerical simulations and phase diagrams for three scenarios - one where the population growth rates are equal and two where they are distinct -, given three different tax policies for each scenario, and discuss the results. We close out with the conclusions and perspectives of future research in the sixth chapter.

2 LITERATURE REVIEW

One topic that has intrigued many economists, notably Marx and Keynes, is the relation between capital accumulation and employment (DOMAR, 1946). Since the first industrial revolution, economists have been concerned that in the new industrialized nations capital would replace labor, causing massive unemployment and consequently social and political disorder.

The second half of the XX century was pivotal in the theory of economic growth. The seminal works of Harrod (1939) and Domar (1946) formulated a post-keynesian model of economic growth that had a huge influence on the field of neoclassical growth theory, mainly for provoking a series of studies since the 1950s contesting their results. One of the most influential one was conducted by Solow (1956), who made a critical analysis of the Harrod-Domar model employing a neoclassical framework.

Among the assumptions of the Harrod-Domar model, the most problematic one in the view of the neoclassical models was the one that assumes fixed proportions between inputs - capital and labor - and output, making the capital-labor ratio and capital-output ratio fixed, and therefore inserting rigidity into the model (WAN, 1971; HAHN; MATTHEWS, 1964) . According to Hahn and Matthews (1964), "the amounts of capital and of labour needed to produce a unit of output are both uniquely given; for the moment this may be thought of as the result of technological considerations - fixed coefficients in production" (HAHN; MATTHEWS, 1964, p. 783).

Solow noted that the equilibrium of the economic system considered by Harrod and Domar "[...] boils down to a comparison between the natural rate of growth which depends [...] on the increase of the labor force, and the warranted rate of growth which depends on the saving and investing habits of households and firms" (SOLOW, 1956, p.65). In the Harrod-Domar model, the natural rate of growth is a constant rate at which the labor force grows, and the warranted rate of growth is the model's equilibrium rate of growth, defined by the quotient between the saving-income ratio and the capital-output ratio. It can only have a steady growth with full employment if the natural rate equals the warranted rate (HAHN; MATTHEWS, 1964). This creates an environment in which the growth equilibrium is always at the so called "knife-edge", meaning that the equilibrium is unstable, mostly because the natural and the warranted rates of growth are independently determined (HAHN; MATTHEWS, 1964). As stated by Solow, such opposition between the natural and warranted rates is built upon the assumption of fixed proportions in production. Without this strong assumption, it looks like the unstable equilibrium on the "knife-edge" vanishes with it (SOLOW, 1956).

2.1 SOLOW-SWAN MODEL

Accepting all the assumptions of the Harrod-Domar model but that of fixed proportions, Solow (1956) built a model of long-run growth under a neoclassical framework. The economy is represented by one sector that produces a single homogeneous good, whose rate of production is given by a global production function $Y(t) = F(L, K)$, which uses two factors of production, labor and capital, and shows constant returns to scale; hence, it's homogeneous of degree one. The output produced is either consumed or saved with the rate of saving being given by the exogenous constant $s \in (0, 1)$. The rate of capital depreciation is given by $\delta \in (0, 1)$, which is also constant. Last, but not least, there is full employment of capital and labor. The population growth - and hence, the labor force, according to the assumption of full employment - is given by the constant exogenous rate $\frac{\dot{L}}{L} = n$. Therefore, the net increase of the stock of capital is expressed as:

$$\dot{K} = sY - \delta K,$$

where sY is the gross investment and δK the depreciation of capital. Thus, the dynamics of the model is given by the following system of Ordinary Differential Equations (ODEs):

$$\begin{cases} \dot{K} = sY - \delta K \\ \dot{L} = Ln, \end{cases} \tag{2.1}$$

given the initial conditions $K(0) = K_0 > 0$ and $L(0) = L_0 > 0$. Defining $k = \frac{K}{L}$ as the capital-labor ratio, after some algebraic manipulations the system (2.1) can be expressed as:

$$\dot{k} = s\, f(k) - (n + \delta)\, k, \tag{2.2}$$

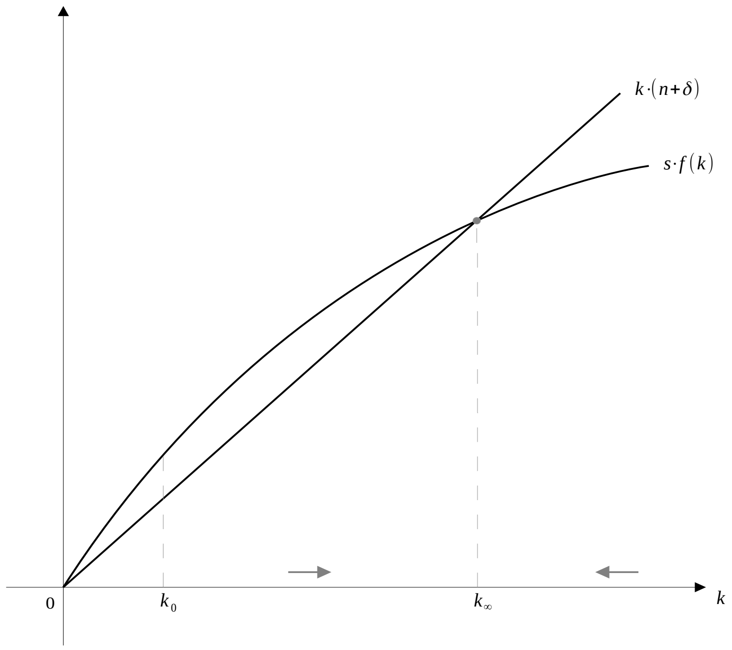
given the initial condition $k(0) = \frac{K_0}{L_0} = k_0 > 0$. Equation (2.2) gives the dynamic of the per capita capital accumulation in the economy. In line with Barro and Martin (2004), the production function $f(k)$ in (2.2) follows the neoclassical conditions, like constant returns to scale, positive and diminishing returns to inputs, the Inada conditions - $f(k)$ presents a strictly concave behaviour -, and essentiality of the inputs[1]. The term $(n + \delta)k$ is the effective depreciation of per capita capital; if the saving rate were zero, then k would decline either by the depreciation of per capita capital at the rate δ or by the population increase at the rate n (BARRO; MARTIN, 2004).

Figure 1 shows the dynamics of k over time given by (2.2). Let the aggregate capital-labor ratio be k_∞ at the intersection of both curves. At this point, $\dot{k} = 0 \iff sf(k_\infty) = (n + \delta)k_\infty$, means that the gross investment is equal to the effective depreciation of the per capita capital, and so there is no per capita capital accumulation. Point k_∞ is the steady-state level of per capita capital. Considering some initial aggregate capital-labor ratio $k_0 > k_\infty$, it is easily seen that $sf(k) < (n + \delta)k$, that is, the effective depreciation of per capita capital is bigger than the gross investment; consequently, k will decrease towards k_∞. On the other hand, if $k_0 < k_\infty$ then $sf(k) > (n + \delta)k$, and so the gross investment outpaces the depreciation; thus, there is per capita capital accumulation and k will increase towards k_∞. This analysis indicates that the steady-state k_∞ is a stable equilibrium (SOLOW, 1956, p. 70).

In this way, Solow proved that Harrod and Domar could have achieved a stable equilibrium with no unemployment if they had abandoned the fixed proportions assumption. Working on the same problem, although independently, Swan (1956) confirmed the results obtained by Solow. Their work on the matter came to be known as the Solow-Swan model of economic growth, a landmark in the neoclassical growth theory. The welcomed results had an immediate impact, catching the attention of intellectuals that began to investigate if the model was solid enough to maintain the main results under different conditions.

[1] The essentiality says that each input is essential for production, meaning that there cannot be any production if the level of one of the factors of production is zero. For more details on the neoclassical conditions, see Barro and Martin (2004), p.28.

Figure 1 – The Solow-Swan model.



Source: Adapted from Barro and Martin (2004).

2.2 SHINKAI'S TWO-SECTOR MODEL

Shinkai (1960) came to notice the differences between the Harrod-Domar and the Solow-Swan models. At the former, he says that "is impossible to sustain full employment unless the warranted rate of growth coincides with the natural rate, and the initial stock of capital just equal the initial supply of labor multiplied by 'the' capital-labor ratio" (SHINKAI, 1960, p. 107)[2]. Meanwhile, the latter assumes flexible technology assumptions - production works under neoclassical conditions -, making the capital-labor ratio adjustable. Thus, Shinkai developed a two-sector model that could express either the unstable equilibrium of Harrod and Domar or the stable equilibrium of Solow and Swan, depending on the capital-labor ratio considered (SHINKAI, 1960).

Shinkai's model was built upon an economy with two sectors, one based on a capital-goods industry and the other on a consumption-goods industry. There are two kinds of goods, with capital - investment - goods only demanded for production and consumption goods only demanded by consumers. In regard to factors of production, both sectors use solely capital and labor. Each industry uses the same technology, and so all technological coefficients are constant. Depreciation of the capital stock is

[2]The emphasis on the capital-labor ratio is a result of the fixed proportions assumption, which makes the capital-labor ratio constant.

neglected, which makes the output of the capital-goods sector equal to the rate of net investment. Additionally, there is free movement of capital between sectors. The supply of labor N grows at an exogenous constant rate n; labor is homogeneous, meaning that the real wage rate W is the same for all workers (SHINKAI, 1960, p. 108). Under these assumptions, Shinkai (1960) shows that growth equilibrium will be stable only if the consumption-goods industry is more capital-intensive than the capital-goods industry, corroborating with the results found by Solow. Otherwise, if the capital-goods industry is more capital-intensive than the consumption-goods one, then the equilibrium is unstable, reproducing the results found by Harrod and Domar (SHINKAI, 1960).

2.3 UZAWA'S TWO-SECTOR MODEL

Looking at investigating further the two-sector scheme, Uzawa (1961) developed a neoclassical version of Shinkai's model. As it was, he assumes that production follows the neoclassical conditions, the same ones considered by the Solow-Swan model. To begin establishing the model's short-run equilibrium conditions, Uzawa defines the production function of sector i as $Y_i = F_i(K_i, L_i), i = 1, 2$, where K_i and L_i are the stocks of capital and labor of sector i, respectively. The subscript 1 refers to the investment-goods sector and the subscript 2 refers to the consumption-goods sector. Considering P_1 and P_2 as the prices of the respective goods, then the marginal productivity conditions can be defined as:

$$P_i \frac{\partial F_i}{\partial K_i} = r, P_i \frac{\partial F_i}{\partial L_i} = w, i = 1, 2, \tag{2.3}$$

with r being the return to capital and w the wage rate. Considering that there is free movement of factors, r is equal in sectors 1 and 2, as well as w. Also, the hypothesis of full employment of capital and labor results in $K_1 + K_2 = K$ and $L_1 + L_2 = L$. The assumptions that labor does not save, meaning that wages are spent entirely on consumption goods, and that capital does not consume, meaning that profits are spent entirely on capital goods, are defined respectively by:

$$P_2 Y_2 = wL, \tag{2.4}$$

$$P_1 Y_1 = rK. \tag{2.5}$$

Equations (2.4) and (2.5) close the model's short-run equilibrium conditions. Now, letting $\rho_i = \frac{L_i}{L}, i = 1, 2$, be the labor allocation and $k_i, i = 1, 2$, the per capita capital in each sector, then the full employment hypothesis can be rewritten as:

$$\rho_1 k_1 + \rho_2 k_2 = k, \tag{2.6}$$

$$\rho_1 + \rho_2 = 1. \tag{2.7}$$

Taking the production function $Y_i = F_i(K_i, L_i)$ and dividing by L_i we have $\frac{Y_i}{L_i} = F_i(k_i, 1) = f_i(k_i)$. Thus, the output per capita is defines as:

$$y_i = \frac{Y_i}{L_i} = f_i(k_i), i = 1, 2. \tag{2.8}$$

Uzawa (1961) sets the wage-rental ratio as $\omega = \frac{w}{r}, \omega > 0$, which is nothing more than the price ratio of labor and capital; consequently, ω controls the optimum quantity of k in each sector. Applying (2.3)[3] into $\omega = \frac{w}{r}$ we have:

$$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i, i = 1, 2. \tag{2.9}$$

Equation (2.9) gives the optimum quantity of per capita capital in each sector. Also, the labor allocation to the capital-goods sector ρ_1 is determined by:

$$\rho_1 f_1(k_1) = f_1'(k_1) k, \tag{2.10}$$

The set of equations (2.6-2.10) are the reduced short-run equilibrium conditions of the model (UZAWA, 1961).

[3]Given that $\frac{\partial F_i}{\partial K_i} = f_i'(k_i)$ and $\frac{\partial F_i}{\partial L_i} = f_i(k_i) - k_i f_i'(k_i)$ (BARRO; MARTIN, 2004, p. 28).

From (2.9), $\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i \iff \omega + k_1 = \frac{f_1(k_1)}{f_1'(k_1)}$. Coming out of (2.10), $\rho_1 f_1(k_1) = f_1'(k_1)k \iff \frac{k}{\rho_1} = \frac{f_1(k_1)}{f_1'(k_1)}$. Thus, ρ_1 can be rewritten as:

$$\frac{k}{\rho_1} = \omega + k_1 \implies \rho_1 = \frac{k}{\omega + k_1}. \tag{2.11}$$

Isolating ρ_1 in (2.7), together with (2.11), it is shown that:

$$\frac{k}{\omega + k_1} = 1 - \rho_2 \implies \rho_2 = 1 - \frac{k}{\omega + k_1} = \frac{\omega + k_1 - k}{\omega + k_1}.$$

Replacing ρ_2 in (2.6) together with (2.11):

$$\left(\frac{k}{\omega + k_1}\right) k_1 + \left(\frac{\omega + k_1 - k}{\omega + k_1}\right) k_2 = k \implies \frac{kk_1 + \omega k_2 + k_1 k_2 - kk_2}{\omega + k_1} = k$$

$$\implies kk_1 + \omega k_2 + k_1 k_2 - kk_2 = \omega k + kk_1 \implies \omega k_2 + k_1 k_2 = \omega k + kk_2$$

$$\therefore k(\omega + k_2) = (\omega + k_1)k_2.$$

Hence, Uzawa (1961) expresses k as:

$$k = \left(\frac{\omega + k_1}{\omega + k_2}\right) k_2, \tag{2.12}$$

where $k_i = k_i(\omega), i = 1, 2$. Then, solving the implicit equation (2.12) for ω gives its equilibrium value (UZAWA, 1961).

Finally, considering the capital depreciation rate as δ and the population growth rate as n, the growth dynamics of Uzawa's two-sector model is given by the following system of ODEs:

$$\begin{cases} \dot{K} = Y_1 - \delta K \\ \dot{L} = Ln, \end{cases} \tag{2.13}$$

given the initial conditions $K(0) = K_0 > 0$ and $L(0) = L_0 > 0$. The hypothesis that capital does not consume defines $P_1Y_1 = rK \Rightarrow Y_1 = K\frac{r}{P_1}$. Taking the system (2.13), together with (2.3), the per capita capital accumulation takes the form:

$$\dot{k} = f_1'(k_1)k - (\delta + n)k, \tag{2.14}$$

given the initial condition $k(0) = k_0 > 0$, and where $f_1'(k_1)$ is the marginal product of capital in the investment-goods sector. The non trivial steady-state of (2.14) is then given by $\dot{k} = 0$, that is:

$$f_1'(k_{1_\infty}) = \delta + n,$$

where $k_{1_\infty} = k_1(\omega_\infty)$, with ω_∞ and k_∞ being the steady-state levels of the wage-rental ratio and the aggregate per capita capital accumulation, respectively. Under the hypothesis that the consumption-goods sector is more capital-intensive than the investment-goods sector (capital-intensity hypothesis), that is, $k_2(\omega) > k_1(\omega)$, as proposed by Shinkai (1960), Uzawa shows that k_∞ is unique and stable. To that matter, he says "the uniqueness of the balanced capital-labor ratio and its stability crucially hinge on the hypothesis that the consumption-goods sector is more capital intensive than the investment-good sector" (UZAWA, 1961, p. 45)[4].

The importance of the assumption that says that the uniqueness and stability of the equilibrium rely on the capital-intensity hypothesis was analyzed by Solow (1961), on a pathway of showing that k and k_1 always move together in Uzawa's model. It is clear that the per capita capital from both sectors increases when, and only when, the wage-rentals ratio increases. The only way this association can be broken, according to Solow (1961), is "if, while the separate machine/labor ratios should be rising, the less machine-intensive industry should gain enough at the expense of the more machine-intensive one to permit a fall in the overall machine/labor ratio" (SOLOW, 1961, p. 49). Hence, capital-intensity hypothesis, it is true for any ω that:

$$\frac{K_2}{L_2} > \frac{K_1}{L_1} \iff \frac{r}{w}\frac{K_2}{L_2} > \frac{r}{w}\frac{K_1}{L_1}$$

[4]Uzawa uses the term "balanced capital-labor ratio" to refer to k_∞.

When ω increases, the price ratio $\frac{P_2}{P_1}$ is going to increase or decrease depending on the amount of labor in sector 2 in relation to that of sector 1. If the consumption-goods sector - sector 2 - is more capital-intensive, it employs more capital than labor. On the other hand, the investment-goods sector - sector 1 - is therefore more labor-intensive, and so employs more labor than capital. Now, the increase of $\omega = \frac{w}{r}$ will make P_1 increase relatively more than P_2, because the wage rate is increasing relatively more than the rentals, making the labor-intensive product, the capital good, relatively more expensive than the consumption good. Thus, an increase in ω leads to a decrease in the price ratio $\frac{P_2}{P_1}$, the same as an increase in $\frac{P_1}{P_2}$ (SOLOW, 1961).

In this matter, considering the hypothesis given by (2.4) and (2.5), Solow (1961) shows that:

$$P_2Y_2 = wL \Longrightarrow L = \frac{P_2Y_2}{w},$$

$$P_1Y_1 = rK \Longrightarrow K = \frac{P_1Y_1}{r},$$

$$\therefore \frac{K}{L} = \frac{w}{r}\frac{P_1}{P_2}\frac{Y_1}{Y_2}.$$

So, if $\frac{w}{r}$ rises, $\frac{P_1}{P_2}$ must rise as well. For that, $\frac{K}{L}$ shall increase except if $\frac{Y_1}{Y_2}$ decreases, but that cannot happen[5]; therefore, $\frac{K}{L}$ must grow and k and k_1 must move on the same direction (SOLOW, 1961).

2.4 UZAWA'S TWO-SECTOR MODEL REVISITED

In his note on Uzawa's model, Solow indicates that the assumption that one sector should be more capital-intensive than the other for the equilibrium to be unique and stable is at best peculiar. Besides, he shows that the assumption that guarantees the stability of the equilibrium path is the one that says that wages only consume and rentals only save, which is the exact proposition that Uzawa (1963) replaces, introducing the parameter propensity to save[6] into the model. Solow's note made Uzawa revisit his

[5] For a detailed explanation, see Solow (1961), p. 49.

[6] The way Uzawa (1963) refers to the rate of saving.

model, changing assumptions and including a new parameter, but it did not make him drop the capital-intensity hypothesis (SOLOW, 1961; UZAWA, 1963).

The basic assumptions and structure used by Uzawa (1963) are exactly the same ones used by Uzawa (1961), as presented before. There are two sectors, one producing only industrial goods and the other producing only consumption goods, labelled respectively 1 and 2. There are only two homogeneous factors, labor L and capital K. The first one grows at a constant exogenous rate n, the second one depreciates at a constant rate δ. Considering neoclassical production conditions, the production functions of both sectors are given by

$$Y_1(t) = F_1(K_1, L_1) \tag{2.15}$$

and

$$Y_2(t) = F_2(K_2, L_2). \tag{2.16}$$

The assumption that factors are fully employed is expressed through

$$K = K_1 + K_2 \tag{2.17}$$

and

$$L = L_1 + L_2. \tag{2.18}$$

Also, the gross national product in terms of consumption goods is represented by:

$$Y(t) = Y_2(t) + p(t)Y_1(t), \tag{2.19}$$

where $p = p(t)$ is the price ratio of the new industrial good in terms of the consumption good, which is essentially the supply price of new investment goods. Considering that the production factor's market operates under perfect competition, and that there is free movement of factors between sectors, then the wage w equals the marginal product of labor and the rentals r equals the marginal product of capital:

$$w = \frac{\partial F_2}{\partial L_2} = p\frac{\partial F_1}{\partial L_1}, \tag{2.20}$$

$$r = \frac{\partial F_2}{\partial K_2} = p\frac{\partial F_1}{\partial K_1}. \tag{2.21}$$

Wrapping up the model, the quantity of new investment goods is given by the following equation:

$$pY_1 = sY, \tag{2.22}$$

in which the constant $s \in (0, 1)$ is the average propensity to save[7]. The set of equations (2.15-2.22) represents the short-run equilibrium conditions. Therefore, the growth dynamics is given by the following system of ODEs:

$$\begin{cases} \dot{K} = Y_1 - \delta K \\ \dot{L} = Ln, \end{cases} \tag{2.23}$$

given the initial conditions $K(0) = K_0 > 0$ and $L(0) = L_0 > 0$. Equation (2.22) defines $Y_1 = \frac{sY}{p}$ in (2.23) (UZAWA, 1963).

Following the same procedure as in his previous work, Uzawa introduces the per capita variables k_i, ρ_i and $y_i, i = 1, 2$, as well as the wage-rentals ratio ω. Therefore, equations (2.19) and (2.22) can be rewritten respectively as:

$$y = y_2 + py_1, \tag{2.24}$$

and

$$py_1 = sy, \tag{2.25}$$

[7] Uzawa (1963) dedicates a section of his work to handle a variable propensity to save. As this goes beyond the purpose of this work, it will be purposely omitted.

where the sectoral outputs per capita $y_i, i = 1, 2$ are

$$y_1 = f_1\left(\frac{k_2 - k}{k_2 - k_1}\right) \tag{2.26}$$

and

$$y_2 = f_2\left(\frac{k - k_1}{k_2 - k_1}\right), \tag{2.27}$$

alongside with the relative price of the new capital good

$$p = \frac{f_2'}{f_1'}. \tag{2.28}$$

The optimum per capita capital $k_i, i = 1, 2$ is determined by the same equation (2.9), which is derived applying the marginal productivity conditions (2.20) and (2.21) into ω. This equation, together with (2.24) - (2.28), close the model's reduced short-run equilibrium conditions. Taking (2.26) - (2.28), (2.24) turns out to be:

$$y = f_2\left(\frac{k - k_1}{k_2 - k_1}\right) + \frac{f_2'}{f_1'} f_1\left(\frac{k_2 - k}{k_2 - k_1}\right) \Longrightarrow y = \frac{f_2 f_1'(k - k_1) + f_1 f_2'(k_2 - k)}{f_1'(k_2 - k_1)}.$$

Substituting y above together with (2.26) and (2.28) into (2.25):

$$\frac{f_2'}{f_1'} f_1\left(\frac{k_2 - k}{k_2 - k_1}\right) = s\left[\frac{f_2 f_1'(k - k_1) + f_1 f_2'(k_2 - k)}{f_1'(k_2 - k_1)}\right]$$

$$\Longrightarrow \frac{f_2' f_1(k_2 - k)}{f_1'(k_2 - k_1)} = \frac{s}{f_1'(k_2 - k_1)}\left[f_2 f_1'(k - k_1) + f_1 f_2'(k_2 - k)\right]$$

$$\Longrightarrow f_2' f_1(k_2 - k) = s\left[f_2 f_1'(k - k_1) + f_1 f_2'(k_2 - k)\right].$$

From (2.9), $k_i = \frac{f_i}{f_i'} - \omega, i = 1, 2$. Then:

$$f_2' f_1\left(\frac{f_2}{f_2'} - \omega - k\right) = s\left\{f_2 f_1'\left[k - \left(\frac{f_1}{f_1'} - \omega\right)\right] + f_1 f_2'\left[\left(\frac{f_2}{f_2'} - \omega\right) - k\right]\right\}$$

$$\implies f_1 f_2 - f_2' f_1 \omega - f_2' f_1 k = s(f_2 f_1' k - f_1 f_2 + f_2 f_1' \omega + f_1 f_2 - f_2' f_1 \omega - f_2' f_1 k)$$

$$\implies f_1 f_2 - f_2' f_1 (\omega + k) = s[f_2 f_1' (\omega + k) - f_2' f_1 (\omega + k)]$$

$$\implies f_1 f_2 = s f_2 f_1' (\omega + k) - s f_2' f_1 (\omega + k) + f_2' f_1 (\omega + k)$$

$$\implies f_1 f_2 = (\omega + k)[s f_2 f_1' + (1 - s) f_2' f_1]$$

Dividing both sides by $f_1' f_2'$:

$$\frac{f_1}{f_1'} \frac{f_2}{f_2'} = (\omega + k) \left[s \frac{f_2}{f_2'} + (1 - s) \frac{f_1}{f_1'} \right] \implies (\omega + k) = \frac{\frac{f_1}{f_1'} \frac{f_2}{f_2'}}{\left[s \frac{f_2}{f_2'} + (1 - s) \frac{f_1}{f_1'} \right]}$$

Given that $k_i = \frac{f_i}{f_i'} - \omega \iff \frac{f_i}{f_i'} = k_i + \omega, i = 1, 2$, then:

$$(\omega + k) = \frac{(k_1 + \omega)(k_2 + \omega)}{s(k_2 + \omega) + (1 - s)(k_1 + \omega)}. \tag{2.29}$$

Equation (2.29) determines the equilibrium wage-rentals ratio. As stated by Uzawa (1963), holding the capital-intensity hypothesis[8], it is shown that the supply price of the new capital good p is positively related to the wage-rentals ratio $\omega(k)$, as already observed by Solow (1961) (SOLOW, 1961; UZAWA, 1963).

The gross national product y is uniquely determined at each given value of $\omega(k)$. Taking (2.24) together with (2.26), (2.27) and (2.28):

$$y = f_2 \left(\frac{k - k_1}{k_2 - k_1} \right) + \frac{f_2'}{f_1'} f_1 \left(\frac{k_2 - k}{k_2 - k_1} \right)$$

$$\implies y = \frac{f_2 f_1' (k - k_1) + f_1 f_2' (k_2 - k)}{f_1' (k_2 - k_1)}$$

[8]Just as Uzawa said, the capital-intensity assumption "is required mainly for reasons of a mathematical nature and for which it seems to be difficult to give any economic justification" (UZAWA, 1963, p. 109). In fact, this justification is another contribution of Solow (1961).

$$\Longrightarrow y = \frac{f_2 f_1' k - f_2 f_1' k_1 + f_1 f_2' k_2 - f_1 f_2' k}{f_1' k_2 - f_1' k_1}$$

Using $k_i = \frac{f_i}{f_i'} - \omega, i = 1, 2$ again:

$$y = \frac{f_2 f_1' k - f_2 f_1' \left(\frac{f_1}{f_1'} - \omega\right) + f_1 f_2' \left(\frac{f_2}{f_2'} - \omega\right) - f_1 f_2' k}{f_1' \left(\frac{f_2}{f_2'} - \omega\right) - f_1' \left(\frac{f_1}{f_1'} - \omega\right)}$$

$$\Longrightarrow y = \frac{f_2 f_1' k - f_2 f_1 + f_2 f_1' \omega + f_1 f_2 - f_2' f_1 \omega - f_1 f_2' k}{\frac{f_1' f_2}{f_2'} - f_1' \omega - f_1 + f_1' \omega}$$

$$\Longrightarrow y = \frac{f_2 f_1' k + f_2 f_1' \omega - f_2' f_1 \omega - f_1 f_2' k}{\frac{f_1' f_2}{f_2'} - f_1}$$

$$\Longrightarrow y = \frac{f_2 f_1' (\omega + k) - f_2' f_1 (\omega + k)}{\frac{f_1' f_2 - f_1 f_2'}{f_2'}}$$

$$\Longrightarrow y = (\omega + k)(f_2 f_1' - f_2' f_1) \frac{f_2'}{f_1' f_2 - f_1 f_2'}$$

$$\therefore y = f_2'(k_2)(\omega + k). \tag{2.30}$$

According to Uzawa (1963), “the gross national product per capita is an increasing function of the wage-rentals ratio if and only if the capital-intensity hypothesis is satisfied” (UZAWA, 1963, p. 110). Finally, if the capital-intensity assumption holds, it can be shown that the higher the average propensity to save s, the higher the wage-rentals ratio ω, and the higher the quantity of new investment goods y_1[9] (UZAWA, 1963).

In per capita terms, the system (2.23) can be reduced to:

$$\dot{k} = y_1 - (\delta + n)k.$$

[9]For a mathematical explanation, see (UZAWA, 1963, p. 109-110).

From (2.25), the quantity of new capital goods can be expressed as $y_1 = \frac{sy}{p}$. Thus, $\dot{k}$ can be written as:

$$\dot{k} = \frac{sy}{p} - (\delta + n)k.$$

Yet, using (2.30) alongside with $p = \frac{f_2'}{f_1'}$:

$$\dot{k} = sf_1'(k_1)(\omega + k) - (\delta + n)k, \tag{2.31}$$

given the initial condition $k(0) = k_0 > 0$. Equation (2.31) is the per capita capital accumulation of Uzawa's revisited two-sector model. The steady-state of (2.31) is obtained by making $\dot{k} = 0$:

$$\dot{k} = 0 \Longrightarrow sf_1'(k_{1_\infty})\frac{(\omega_\infty + k_\infty)}{k_\infty} = (\delta + n),$$

where $k_{1_\infty} = k_1(\omega_\infty)$, with the steady-state level of the wage-rental ratio being ω_∞, related to the respective steady-state level of per capita capital k_∞[10]. In Uzawa's words, "if the capital-intensity hypothesis is satisfied, there always exists a uniquely determined balanced capital-labor ratio k_∞, corresponding to each level of the average propensity to save s" (UZAWA, 1963, p. 111)[11].

Comparing (2.31) with (2.14), it shows more than anything how the parameter s assumes an important role in the dynamic of the per capita accumulation. In fact, this revisited model works as an expansion of the Solow-Swan model to a two-sector economy. The comparison between Solow's and Uzawa's models started the moment Solow published his note on Uzawa's work. The discussion referred, among other things, to the difference between the two models. According to Inada (1963), "in Solow's one-sector model, a time path may be possible in which the capital-labor ratio increases without limit or decreases to zero. Contrary to this, such a path is impossible in Uzawa's model" (INADA, 1963, p. 119). Solow (1961) blamed the assumptions expressed by (2.4) and (2.5) for this, and even said that if Uzawa had considered saving as a fraction

[10]The model's functioning is similar to the one seen from (UZAWA, 1961).

[11]The original numeration referring to the capital-intensity hypothesis was omitted, and the symbol used to identify the steady-state per capita capital was changed in the citation.

of the aggregate income, then stability would not hold and the two-sector model would behave qualitatively like his one-sector model. Inada (1963) cleared up the discussion, showing that the difference was, in fact, in another set of assumptions that Uzawa made on the production functions[12](SOLOW, 1961; INADA, 1963).

2.5 ALTERNATIVE TWO-SECTOR MODELS

All the models mentioned prior to this point belong to the theory of growth, applied to advanced economies. There were just a few works on the theory of development, for underdeveloped economies, whose "emphasis is laid on the balance between capital accumulation and the growth of population, each adjusting to the other." (JORGENSON, 1961, p. 310). Looking to fill this gap, Jorgenson (1961) presented a theory of development of a dual economy, a two-sector model where one sector represents an advanced - or modern - sector, and the other represents a traditional - or agricultural - sector. In Jorgenson's model, the population growth depends on a balance between the per capita food supply and mortality; if there is enough food supply to maintain the population growth, then the model presents what Jorgenson called agricultural surplus. In this case, some labor force working on the agricultural sector can be relocated to the advanced sector. The model of a dual economy was developed under the hypothesis that if an agricultural surplus arose, it would persist. This is the critical condition for an economy to exhibit sustained growth[13]. Otherwise, if somehow the agricultural surplus started to decrease - because of a change in the net rate of reproduction, for example -, then the economy would be caught in a low-level equilibrium trap, which is stable for any initial condition (JORGENSON, 1961).

Later on, Jorgenson (1967) presented two alternative approaches to his theory of development of a dual economy, one classical and one neoclassical, developing both under the same framework to make comparisons possible. The differences between both approaches are basically "assumptions made about the technology of the agricultural sector and about conditions governing the supply of labour" (JORGENSON,

[12]Inada (1963) derived a generalization of Uzawa's model, and showed that Solow's one-sector model is actually a particular case of Uzawa's generalized model. For a better understanding, see (INADA, 1963).

[13]According to Jorgenson (1961), "The characteristics of an economy which experiences steady growth depend not only on the existence of an agricultural surplus but also on technical conditions in the advanced sector" (JORGENSON, 1961, p. 334).

1967, p. 308). Dixit (1970) pointed out and analyzed the main differences between the patterns of growth of the two approaches (DIXIT, 1970; JORGENSON, 1967). Subsequently, Mas-Colell and Razin (1973) showed that the growth patterns exhibited by the Jorgenson's model, such as "a decreasing rate of migration from rural to urban sector; a stage of accelerated accumulation of capital; etc." (MAS-COLELL; RAZIN, 1973, p. 72), could be explained by a neoclassical growth model. To accomplish this, Mas-Colell and Razin introduced migration into a two-sector growth model (MAS-COLELL; RAZIN, 1973), which will be discussed in the next chapter.

3 MAS-COLELL AND RAZIN MODEL

Beginning with the mathematical derivation of Mas-Colell and Razin (1973), let us examine the model's basic assumptions. The stocks of labor and capital, which are fully employed, are given respectively by $L = L_A + L_I$ - for simplicity, it is assumed that $L_I + L_A = 1$ - and $K = K_A + K_I$, where I and A stands for the industrial and agricultural sectors, respectively. About the production functions, they are linear homogeneous of Cobb-Douglas type:

$$Y_I = K_I^{\beta} L_I^{1-\beta}, \tag{3.1}$$

$$Y_A = K_A^{\alpha} L_A^{1-\alpha}. \tag{3.2}$$

Both Y_I and Y_A can be rewritten as:

$$Y_I = K_I^{\beta} L_I^{1} L_I^{-\beta} = L_I \frac{K_I^{\beta}}{L_I^{\beta}} = L_I \left(\frac{K_I}{L_I}\right)^{\beta},$$

$$Y_A = K_A^{\alpha} L_A L_A^{-\alpha} = L_A \frac{K_A^{\alpha}}{L_A^{\alpha}} = L_A \left(\frac{K_A}{L_A}\right)^{\alpha}.$$

Given that the per capita capital in each sector is expressed by $k_i = \frac{K_i}{L_i}, i = I, A$, then:

$$Y_I = L_I k_I^{\beta},$$

$$Y_A \; = \; L_A \, k_A^{\alpha}.$$

Defining $\rho = \frac{L_I}{L} = \frac{L_I}{L_I + L_A}$ as the proportion of total labour force employed in the industrial sector, and $(1-\rho) = \frac{L_A}{L} = \frac{L_A}{L_I + L_A}$ as the proportion of total labour force employed in the agricultural sector, and dividing both functions by L, we obtain:

$$y_I = \frac{Y_I}{L} = \frac{L_I}{L} k_I^{\beta} = \rho k_A^{\beta},$$

$$y_A = \frac{Y_A}{L} = \frac{L_A}{L} k_A^{\alpha} = (1-\rho) k_A^{\alpha}.$$

Therefore, equations (3.1) and (3.2) in per capita terms can be rewritten as:

$$y_I = \rho k_I^{\beta}, \tag{3.3}$$

$$y_A = (1-\rho) k_A^{\alpha}. \tag{3.4}$$

The stock of capital[1] is also represented in per capita form:

$$K_A + K_I = K \Leftrightarrow \rho k_I + (1-\rho) k_A = k. \tag{3.5}$$

Taking (3.1) and (3.2) and differentiating with respect to the sectoral stocks of capital, we get the marginal productivities of capital on both sectors:

$$\frac{\partial Y_I}{\partial K_I} = \beta K_I^{\beta-1} L_I^{1-\beta} = \beta \frac{K_I^{\beta-1}}{L_I^{\beta-1}} = \beta \left(\frac{K_I}{L_I}\right)^{\beta-1} = \beta k_I^{\beta-1},$$

$$\frac{\partial Y_A}{\partial K_A} = \alpha K_A^{\alpha-1} L_A^{1-\alpha} = \alpha \frac{K_A^{\alpha-1}}{L_A^{\alpha-1}} = \alpha \left(\frac{K_A}{L_A}\right)^{\alpha-1} = \alpha k_A^{\alpha-1}.$$

The value of the marginal productivity of capital in the industrial sector is $VPMgK_I = p\beta k_I^{\beta-1}$, with p being the price of the industrial good in terms of the agricultural good[2], while the one in the agricultural sector is $VPMgK_A = \alpha k_A^{\alpha-1}$. Considering that there is perfect movement of capital between the sectors, the equilibrium condition is given by the equalization of the marginal productivities:

$$p\beta k_I^{\beta-1} = \alpha k_A^{\alpha-1}. \tag{3.6}$$

[1]Given the full employment of capital and labor, it is true that

$$k = \frac{K}{L} = \frac{K}{\rho + (1-\rho)} = \frac{K}{1} \Rightarrow k = K.$$

[2]The agricultural good serves as the *numéraire* of the economy.

Likewise, differentiating (3.1) and (3.2) with respect to the labor stocks from each sector the marginal productivities of labor is:

$$\frac{\partial Y_I}{\partial L_I} = (1-\beta)K_I^{\beta}L_I^{1-\beta-1} = (1-\beta)\frac{K_I^{\beta}}{L_I^{\beta}} = (1-\beta)\left(\frac{K_I}{L_I}\right)^{\beta} = (1-\beta)k_I^{\beta},$$

$$\frac{\partial Y_A}{\partial L_A} = (1-\alpha)K_A^{\alpha}L_A^{1-\alpha-1} = (1-\alpha)\frac{K_A^{\alpha}}{L_A^{\alpha}} = (1-\alpha)\left(\frac{K_A}{L_A}\right)^{\alpha} = (1-\alpha)k_A^{\alpha}.$$

Then, the labor market in each sector will be in equilibrium when:

$$w_I = p(1-\beta)k_I^{\beta}, w_A = (1-\alpha)k_A^{\alpha}, \quad (3.7)$$

where w_I, w_A are the wage rates in both sectors. The adjust in the labor market is not instantaneous, allowing w_I and w_A to differ momentarily according to ρ. That is, if ρ is small, meaning that the proportion of labor in the industrial sector is small, there will be more workers on the agricultural sector, implying in a stronger competition in the labor market in that region, which drives w_A down. On the other hand, there will be less competition in the labor market in the industrial region, driving w_I up. Hence, this creates a scenario where $w_I > w_A$. The logic for a high level of ρ is analogous; in this case, $w_I < w_A$ due to the stronger competition in the industrial labor market relatively to the labor market in the agricultural region, which drives w_I down.

Defining s as the rate of saving and δ as the proportion of income spent on industrial goods for consumption - both constants -, with $py_I + y_A$ being the per capita national income, it is possible to determine the demand for industrial product as $s(py_I + y_A) + \delta(py_I + y_A) = (s+\delta)(py_I + y_A)$, with the first term of the equation being the share of income spent on industrial good for investment purposes and the second one representing the share spent on industrial good for consumption purposes[3]. Meanwhile, the industrial supply is expressed by py_I; hence, the market equilibrium for the industrial goods is given by:

$$(s+\delta)(py_I + y_A) = py_I. \quad (3.8)$$

[3]It is important to notice that the first term represents the income spent on industrial output for investment because the model considers implicitly that savings equals investment.

Replacing (3.3) and (3.4) in (3.8) and isolating k_A^α we have:

$$(s+\delta)(p\rho k_I^\beta + (1-\rho)k_A^\alpha) = p\rho k_I^\beta$$

$$\Longrightarrow k_A^\alpha = \frac{p\rho k_I^\beta \left(\frac{1}{(s+\delta)} - 1\right)}{(1-\rho)}.$$

Dividing the last equation by (3.6), we obtain:

$$k_A = \frac{\rho\alpha k_I \left(\frac{1}{(s+\delta)} - 1\right)}{(1-\rho)\beta}.$$

Now, going back to equation (3.5), k_A can also be isolated:

$$\rho k_I + (1-\rho)k_A = k \Longrightarrow k_A = \frac{k-\rho k_I}{(1-\rho)}.$$

Finally, we obtain an equation relying just on k_I:

$$\frac{k-\rho k_I}{(1-\rho)} = \frac{\rho\alpha k_I \left(\frac{1}{(s+\delta)} - 1\right)}{(1-\rho)\beta}$$

$$\Longrightarrow k_I = \frac{\beta}{\left(\frac{\alpha}{(s+\delta)} - \alpha + \beta\right)} \left(\frac{k}{\rho}\right).$$

Therefore, multiplying both the numerator and the denominator by $\left(\frac{s}{s}\right)$, the final form of the per capita capital of the industrial sector is given:

$$k_I = \frac{\beta s}{\alpha\frac{s}{(s+\delta)} - \alpha s + \beta s} \left(\frac{k}{\rho}\right) \Longrightarrow k_I = \frac{\beta s}{\alpha(\lambda - s) + \beta s} \left(\frac{k}{\rho}\right),$$

where $\lambda = \frac{s}{(s+\delta)}$ is defined as the proportion of total industrial output in the form of industrial good for investment purposes, i.e., the rate of investment. Defining the parameter $\theta = \frac{\beta s}{\alpha(\lambda - s) + \beta s}$ for simplicity, Mas-Colell and Razin (1973) got the final equation for k_I:

$$k_I = \theta \frac{k}{\rho}. \tag{3.9}$$

To derive the per capita capital of the agricultural sector, let us replace (3.9) on $k_A = \frac{\rho \alpha k_I \left(\frac{1}{(s+\delta)} - 1 \right)}{(1-\rho)\beta}$:

$$k_A = \frac{\alpha}{\beta} \frac{\rho}{(1-\rho)} \left(\frac{1-s-\delta}{s+\delta} \right) \theta \frac{k}{\rho}.$$

Given that θ is defined as $\theta = \frac{(s+\delta)\beta}{(s+\delta)\beta + (1-s-\delta)\alpha}$, after some simplifications k_A is denoted by:

$$k_A = (1-\theta) \frac{k}{(1-\rho)}. \tag{3.10}$$

Moving forward, let us start looking into the equations that give the model its dynamics. Knowing that the aggregate per capita capital is $k = \frac{K(t)}{L(t)}$, it can be differentiated with respect to time[4]:

$$\frac{dk}{dt} = \frac{d}{dt} \left(\frac{K}{L} \right) = \frac{\dot{K}L - K\dot{L}}{L^2} \Longrightarrow \dot{k} = \frac{\dot{K}L}{L^2} - \frac{K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - kn$$

$$\Longrightarrow \dot{k} = \frac{\lambda L_I \left(\frac{K_I}{L_I} \right)^{\beta}}{L} - kn \Longrightarrow \dot{k} = \lambda k_I^{\beta} \left(\frac{L_I}{L} \right) - kn = \lambda k_I^{\beta} \rho - kn$$

$$\therefore \dot{k} = \lambda y_I - kn.$$

[4]The capital stock accumulation and the population growth are given by the following differential equations:

$$\dot{K} = \lambda Y_I = \lambda K_I^{\beta} L_I^{1-\beta} = \lambda K_I^{\beta} L_I^{1} L_I^{-\beta} = \lambda L_I \left(\frac{K_I}{L_I} \right)^{\beta}, \tag{3.11}$$

$$\frac{\dot{L}}{L} = n \longrightarrow \dot{L} = Ln, \tag{3.12}$$

where n is a constant rate at which the population grows.

Taking (3.3) and (3.9) into account, and dividing both sides by k, it gives the dynamic of the aggregate per capita capital, which is the per capita capital accumulation:

$$\frac{\dot{k}}{k} = \lambda\theta^{\beta}\left(\frac{k}{\rho}\right)^{\beta-1} - n. \tag{3.13}$$

The Mas-Colell and Razin model introduces a function that models the migration of labor from one sector to the other, which is a positive function of the wage differential from both sectors and is given by the following general form (MAS-COLELL; RAZIN, 1973):

$$\frac{\dot{\rho}}{\rho} = f(w_I, w_A). \tag{3.14}$$

To express the behavior of the relative rate of growth of migration $\frac{\dot{\rho}}{\rho}$, Mas-Colell and Razin (1973) use a more specific form, in order to meet some requirements[5]:

$$\frac{\dot{\rho}}{\rho} = \gamma\left[\frac{w_I - w_A}{w_A}\right], \tag{3.15}$$

where $\gamma > 0$. Replacing (3.7) in (3.15) gives:

$$\frac{\dot{\rho}}{\rho} = \gamma\left[\frac{p(1-\beta)k_I^{\beta} - (1-\alpha)k_A^{\alpha}}{(1-\alpha)k_A^{\alpha}}\right] \Longrightarrow \frac{\dot{\rho}}{\rho} = \gamma\left[\frac{p(1-\beta)k_I^{\beta}}{(1-\alpha)k_A^{\alpha}} - 1\right].$$

Given that $p\beta k_I^{\beta-1} = \alpha k_A^{\alpha-1}$, multiplying the right side of the equality by $\frac{\alpha k_A^{\alpha-1}}{p\beta k_I^{\beta-1}} = 1$ yields:

$$\frac{\dot{\rho}}{\rho} = \gamma\frac{\alpha k_A^{\alpha-1}}{p\beta k_I^{\beta-1}}\left[\frac{p(1-\beta)k_I^{\beta}}{(1-\alpha)k_A^{\alpha}} - 1\right] \Longrightarrow \frac{\dot{\rho}}{\rho} = \gamma\left[\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{k_A^{\alpha-1}}{k_A^{\alpha}}\frac{k_I^{\beta}}{k_I^{\beta-1}} - 1\right]$$

$$\Longrightarrow \frac{\dot{\rho}}{\rho} = \gamma\left[\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{k_I}{k_A} - 1\right].$$

Substituting (3.9) and (3.10), we get that the dynamics of migration is expressed by[6]

$$\frac{\dot{\rho}}{\rho} = \gamma\left[\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{\theta}{(1-\theta)}\frac{(1-\rho)}{\rho} - 1\right]. \tag{3.16}$$

[5] If $\frac{w_I}{w_A} \to \infty$, then $\frac{\dot{\rho}}{\rho} \to \infty$.

[6] Through our derivation of the original model, we found out that Mas-Colell and Razin (1973) made a mistake defining $\frac{\dot{\rho}}{\rho}$, forgetting the term $\frac{\theta}{(1-\theta)}$. See Mas-Colell and Razin (1973), p. 76.

The direction of the migratory flow respects the wage differential in both sectors, that is, labor flows into the sector paying the highest wage. If the wages are equal, there is no migration. Thus, the equilibrium is the point where there will be no migration, meaning that at this point the wages defined by (3.7) are equal:

$$f(w_I, w_A) = 0 \iff w_I = w_A \implies p(1-\beta)k_I^{\beta} = (1-\alpha)k_A^{\alpha}.$$

Dividing by (3.6):

$$\frac{p(1-\beta)k_I^{\beta}}{p\beta k_I^{\beta-1}} = \frac{(1-\alpha)k_A^{\alpha}}{\alpha k_A^{\alpha-1}} \implies \frac{(1-\beta)}{\beta}k_I^{\beta-(1-\beta)} = \frac{(1-\alpha)}{\alpha}k_A^{\alpha-(1-\alpha)}$$

$$\implies \frac{(1-\beta)}{\beta}k_I = \frac{(1-\alpha)}{\alpha}k_A.$$

Replacing by (3.9) and (3.10):

$$\frac{(1-\beta)}{\beta}\theta\frac{k}{\rho} = \frac{(1-\alpha)}{\alpha}(1-\theta)\frac{k}{(1-\rho)}.$$

Making some algebraic manipulations:

$$\frac{(1-\beta)\theta\alpha}{\beta\rho} = \frac{(1-\alpha)(1-\theta)}{(1-\rho)} \implies \frac{(1-\beta)\theta\alpha}{\beta(1-\alpha)(1-\theta)} = \frac{\rho}{(1-\rho)}$$

$$\implies \rho = \frac{(1-\beta)\theta\alpha}{\beta(1-\alpha)(1-\theta)\left[1+\frac{(1-\beta)\theta\alpha}{\beta(1-\alpha)(1-\theta)}\right]}$$

Consequently, the steady-state level of migration is defined by:

$$\rho_{\infty} = \frac{(1-\beta)\theta\alpha}{\beta(1-\alpha)(1-\theta)+(1-\beta)\theta\alpha}. \tag{3.17}$$

Now, for the analysis of the per capita capital on the steady-state, let us rewrite equation (3.13) as $\frac{\dot{k}}{k} = \frac{\lambda y_I}{k} - n = \lambda\theta^{\beta}\left(\frac{k}{\rho}\right)^{\beta-1} - n$. Thus:

$$\dot{k} = 0 \Longleftrightarrow 0 = \lambda\theta^{\beta}\left(\frac{k}{\rho}\right)^{\beta-1} - n \Longrightarrow n = \lambda\theta^{\beta}\left(\frac{k}{\rho}\right)^{\beta-1} \Longrightarrow n = \lambda\theta^{\beta}\frac{\rho^{1-\beta}}{k^{1-\beta}}$$

$$\therefore k = \left(\frac{\lambda}{n}\right)^{\frac{1}{1-\beta}}\theta^{\frac{\beta}{1-\beta}}\rho.$$

Defining the parameter $c = \left(\frac{\lambda}{n}\right)^{\frac{1}{1-\beta}}\theta^{\frac{\beta}{1-\beta}}$ for matters of simplification, the following equation gives the steady-state level of the per capita capital:

$$k_{\infty} = c\rho_{\infty}. \tag{3.18}$$

Summarizing the Mas-Colell and Razin two-sector model of intersectoral migration and growth, the dynamics of the model is given by the following system of ODEs:

$$\begin{cases} \dot{k} = k\lambda\theta^{\beta}\left(\dfrac{k}{\rho}\right)^{\beta-1} - kn, \\ \dot{\rho} = \rho\gamma\left[\dfrac{(1-\beta)}{\beta}\dfrac{\alpha}{(1-\alpha)}\dfrac{\theta}{(1-\theta)}\dfrac{(1-\rho)}{\rho} - 1\right], \end{cases} \tag{3.19}$$

for the initial conditions $k(0) = k_0 > 0$ and $\rho(0) = \rho_0 > 0$ (MAS-COLELL; RAZIN, 1973). The stability of the steady-state given by (3.17) and (3.18) can be verified by the phase diagram in Figure 2.

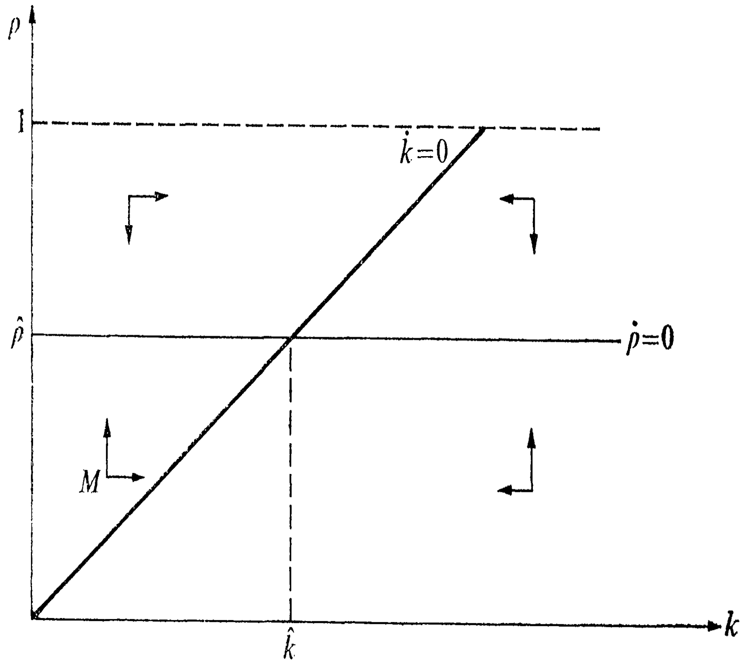
3.1 TAX POLICY

Seeking to introduce a tax policy into the model, Mas-Colell and Razin (1973) included an *ad valorem* subsidy (tax) given to the agricultural sector[7]. The rate τ represents a change in the price of the agricultural sector, i.e., $p_A = 1.(1+\tau)$, and therefore will affect the marginal productivity of capital and the wage rate of the agricultural sector, changing equations (3.6) and (3.7) as shown below (MAS-COLELL; RAZIN, 1973):

$$p\beta k_I^{\beta-1} = (1+\tau)\alpha k_A^{\alpha-1}, \tag{3.20}$$

[7] It is supposed that the government raises - or gives - these funds from an income tax (MAS-COLELL; RAZIN, 1973).

Figure 2 – Phase diagram for the Mas-Colell and Razin model.



Source: Mas-Colell and Razin (1973).

$$w_A = (1+\tau)(1-\alpha)k_A^{\alpha}, \tag{3.21}$$

where $\tau > -1$, with $-1 < \tau < 0$ being a tax and $\tau > 0$ being a subsidy. From (3.20) and (3.21), the Mas-Colell and Razin model can be derived again. Dividing $k_A^{\alpha} = \frac{p\rho k_I^{\beta}\left(\frac{1}{(s+\delta)}-1\right)}{(1-\rho)}$ by (3.20) gives:

$$\frac{k_A^{\alpha}}{(1+\tau)\alpha k_A^{\alpha-1}} = \frac{\frac{p\rho k_I^{\beta}\left(\frac{1}{(s+\delta)}-1\right)}{(1-\rho)}}{p\beta k_I^{\beta-1}} \longrightarrow \frac{k_A^{\alpha-(\alpha-1)}}{(1+\tau)\alpha} = \frac{\rho k_I^{\beta-(\beta-1)}\left(\frac{1}{(s+\delta)}-1\right)}{(1-\rho)\beta}$$

$$\Longrightarrow k_A = \frac{(1+\tau)\alpha\rho k_I\left(\frac{1}{(s+\delta)}-1\right)}{(1-\rho)\beta}.$$

Isolating k_A in (3.5) and equalizing with the equation above:

$$k_I = \frac{\beta s}{(1+\tau)\alpha(\lambda-s)+\beta s}\left(\frac{k}{\rho}\right).$$

Defining $\theta_\tau = \frac{\beta s}{(1+\tau)\alpha(\lambda - s)+\beta s}$, we obtained the new equation of the per capita capital of the industrial sector:

$$k_I = \theta_\tau \frac{k}{\rho}. \tag{3.22}$$

Now, replacing (3.22) on $k_A = \frac{(1+\tau)\alpha\rho k_I\left(\frac{1}{(s+\delta)}-1\right)}{(1-\rho)\beta}$ gives:

$$k_A = \frac{(1+\tau)\alpha}{\beta}\frac{\rho}{(1-\rho)}\left(\frac{1-s-\delta}{s+\delta}\right)\theta_\tau \frac{k}{\rho}.$$

The parameter θ_τ can be rewritten as $\theta_\tau = \frac{(s+\delta)\beta}{(s+\delta)\beta+(1-s-\delta)(1+\tau)\alpha}$. After some simplifications, we get:

$$k_A = \left[1 - \frac{(s+\delta)\beta}{(s+\delta)\beta + (1-s-\delta)(1+\tau)\alpha}\right]\frac{k}{(1-\rho)}.$$

Thus, the new per capita capital of the agricultural sector is given by:

$$k_A = (1-\theta_\tau)\frac{k}{(1-\rho)}. \tag{3.23}$$

The parameter θ_τ plays a pivotal role in the Mas-Colell and Razin model with tax policy. As we can see, the transmission of the policy to the per capita capital in both sectors takes place through θ_τ, and the same happens for the rest of the variables. The new dynamics of the model is given by the following system of ODEs:

$$\begin{cases} \dot{k} = k\lambda\theta_\tau^\beta \left(\dfrac{k}{\rho}\right)^{\beta-1} - kn, \\ \dot{\rho} = \rho\gamma \left[\dfrac{(1-\beta)}{\beta}\dfrac{\alpha}{(1-\alpha)}\dfrac{\theta_\tau}{(1-\theta_\tau)}\dfrac{(1-\rho)}{\rho} - 1\right]. \end{cases} \tag{3.24}$$

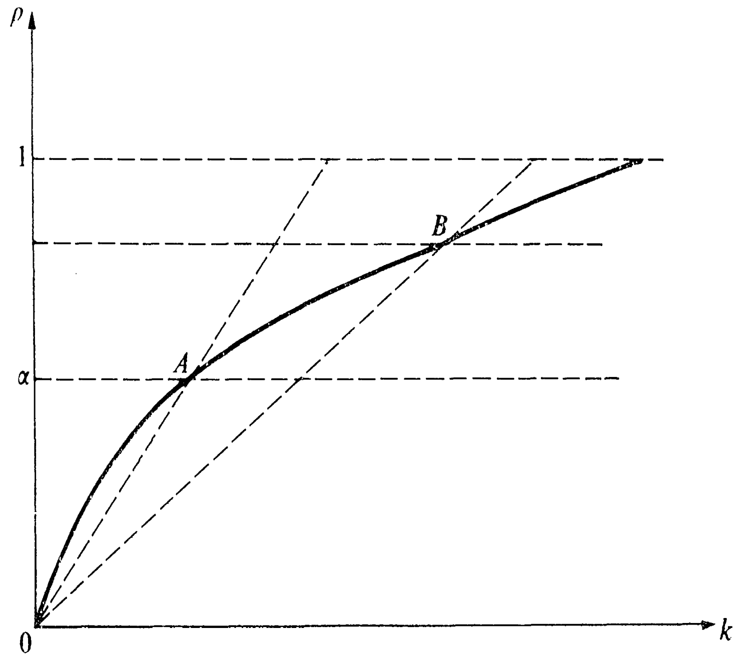
for the initial conditions $k(0) = k_0 > 0$ and $\rho(0) = \rho_0 > 0$. The new form of the steady-states levels of the per capita capital and migration are given by:

$$k_\infty = c\,\rho_\infty, \tag{3.25}$$

$$\rho_\infty = \frac{(1-\beta)\theta_\tau\alpha}{\beta(1-\alpha)(1-\theta_\tau)+(1-\beta)\theta_\tau\alpha}. \tag{3.26}$$

where $c = \left(\frac{\lambda}{n}\right)^{\frac{1}{1-\beta}} \theta_\tau^{\frac{\beta}{1-\beta}}$. It is easy to see that if $\tau = 0$ then $\theta_\tau = \frac{(s+\delta)\beta}{(s+\delta)\beta+(1-s-\delta)\alpha} = \frac{\beta s}{\alpha(\lambda-s)+\beta s}$, recovering the parameter θ of the model without the tax policy. The following phase diagram shows how changes in the tax policy affect the steady-state levels given by (3.25) and (3.26) (MAS-COLELL; RAZIN, 1973):

Figure 3 – Phase diagram for different tax policies.



Source: Mas-Colell and Razin (1973).

Point A in Figure 3 is a steady-state in a case with a subsidy. As τ decreases, meaning a decrease in the subsidy or even the change for a tax, the slope of the isocline $\dot{k} = 0$ decreases, while the horizontal isocline $\dot{\rho} = 0$ goes up. Therefore, for any decrease of τ, at the steady-state we will have bigger values for both ρ and k, as shown by point B.

4 GENERALIZED MODEL WITH DISTINCT POPULATION GROWTH RATES

The objective of this section is to generalize the Mas-Colell and Razin model of intersectoral migration and growth presented in the previous chapter, which considers a single population growth rate, in order to consider two population growth rates, one for each sector. Considering n_A as the population growth rate of the agricultural sector and n_I as the population growth rate of the industrial sector, the population growth in each sector is given by the following differential equations:

$$\dot{L_I} = L_I n_I, \tag{4.1}$$

$$\dot{L_A} = L_A n_A. \tag{4.2}$$

Hence, the new labor dynamic is now given by $\dot{L} = \dot{L_I} + \dot{L_A}$. Dividing $\dot{L}$ by L and using (4.1) and (4.2) we obtain:

$$\frac{\dot{L}}{L} = \frac{\dot{L_I}}{L} + \frac{\dot{L_A}}{L} = \frac{L_I n_I}{L} + \frac{L_A n_A}{L}$$

$$\Longrightarrow \frac{\dot{L}}{L} = n_I \rho + n_A(1-\rho). \tag{4.3}$$

Equation (4.3) determines the dynamic of the aggregate population growth considering distinct population growth rates. If we consider $n_A = n_B = n$ in (4.3), we get the population growth of the original model given by (3.12). Next, knowing that the aggregate per capita capita is $k = \frac{K(t)}{L(t)}$, we can differentiate it with respect to time:

$$\dot{k} = \frac{dk}{dt} = \frac{d}{dt}\left(\frac{K}{L}\right) = \frac{\dot{K}L - K\dot{L}}{L^2} \Longrightarrow \dot{k} = \frac{\dot{K}L}{L^2} - \frac{K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} = \frac{\dot{K}}{L} - k[n_I\rho + n_A(1-\rho)]$$

$$\Longrightarrow \dot{k} = \frac{\dot{K}}{L} - k[n_I\rho + n_A(1-\rho)].$$

Replacing (3.11) in the equation above:

$$\dot{k} = \frac{\left(\lambda K_I^{\beta} L_I^{1-\beta}\right)}{L} - k\left[n_I\rho + n_A(1-\rho)\right] = \frac{\left[\lambda L_I\left(\frac{K_I}{L_I}\right)^{\beta}\right]}{L} - k[n_I\rho + n_A(1-\rho)]$$

$$\Longrightarrow \dot{k} = \lambda k_I^{\beta}\frac{L_I}{L} - k\left[n_I\rho + n_A(1-\rho)\right] \Longrightarrow \dot{k} = \lambda k_I^{\beta}\rho - k[n_I\rho + n_A(1-\rho)].$$

Dividing both sides by k, we get:

$$\frac{\dot{k}}{k} = \frac{\lambda k_I^{\beta}\rho}{k} - \left[n_I\rho + n_A(1-\rho)\right],$$

Replacing (3.22):

$$\frac{\dot{k}}{k} = \frac{\lambda\left(\theta_\tau\frac{k}{\rho}\right)^{\beta}\rho}{k} - \left[n_I\rho + n_A(1-\rho)\right] = \lambda\theta_\tau^{\beta}\frac{k^{\beta}}{k}\frac{\rho}{\rho^{\beta}} - \left[n_I\rho + n_A(1-\rho)\right]$$

$$\Longrightarrow \frac{\dot{k}}{k} = \lambda\theta_\tau^{\beta}k^{\beta-1}\rho^{1-\beta} - \left[n_I\rho + n_A(1-\rho)\right] = \lambda\theta_\tau^{\beta}\frac{k^{\beta-1}}{\rho^{\beta-1}} - \left[n_I\rho + n_A(1-\rho)\right]$$

Therefore, the dynamic of the aggregate per capita capital, given distinct population growth rates, is defined as:

$$\frac{\dot{k}}{k} = \lambda\theta_\tau^{\beta}\left(\frac{k}{\rho}\right)^{\beta-1} - \left[n_I\rho + n_A(1-\rho)\right]. \tag{4.4}$$

As we can see, making $n_I = n_A$, we recover the dynamic of k as defined in the original model by (3.13). In order to obtain the aggregate per capita capital at the steady-state, we proceed as follows:

$$\dot{k} = 0 \Longleftrightarrow \lambda\theta_\tau^{\beta}\left(\frac{k}{\rho}\right)^{\beta-1} - \left[n_I\rho + n_A(1-\rho)\right] = 0$$

$$\therefore k_\infty = \theta_\tau^{\frac{\beta}{1-\beta}}\rho_\infty\left[\frac{\lambda}{n_I\rho_\infty + n_A(1-\rho_\infty)}\right]^{\frac{1}{1-\beta}}. \tag{4.5}$$

Once again, it is easily seen that if we take $n_I = n_A$, we recover k_∞ of the original model as defined in (3.18).

The final step in defining our generalized model is to derive an equation for the dynamics of ρ:

$$\frac{\dot{\rho}}{\rho} = \frac{\left(\frac{L_I}{L}\right)'}{\left(\frac{L_I}{L}\right)} = \frac{L}{L_I}\left(\frac{L_I}{L}\right)' = \frac{L}{L_I}\left(\frac{\dot{L}_I L - L_I \dot{L}}{L^2}\right) = \frac{\dot{L}_I L - L_I \dot{L}}{L_I L}$$

$$\longrightarrow \frac{\dot{\rho}}{\rho} = \frac{\dot{L}_I L}{L_I L} - \frac{L_I \dot{L}}{L_I L} = \frac{\dot{L}_I}{L_I} - \frac{\dot{L}}{L}.$$

Replacing (4.3):

$$\frac{\dot{\rho}}{\rho} = \frac{\dot{L}_I}{L_I} - n_I \rho + n_A(1-\rho).$$

Knowing that the Mas-Colell and Razin model defines migration as $M = \dot{L}_I - n_I L_I$, we get:

$$\frac{\dot{\rho}}{\rho} = \frac{M + n_I L_I}{L_I} - [n_I \rho + n_A(1-\rho)] = \frac{M}{L_I} + n_I - n_I \rho - n_A(1-\rho)$$

$$\Longrightarrow \frac{\dot{\rho}}{\rho} = \frac{M}{L_I} + n_I(1-\rho) - n_A(1-\rho)$$

$$\Longrightarrow \frac{\dot{\rho}}{\rho} = \frac{M}{L_I} + (1-\rho)(n_I - n_A).$$

Given that the rate of migration towards the industrial sector in the original model is defined by (3.16), we end up with:

$$\frac{\dot{\rho}}{\rho} = \gamma \left[\frac{(1-\beta)}{\beta} \frac{\alpha}{(1-\alpha)} \frac{\theta_\tau}{(1-\theta_\tau)} \frac{(1-\rho)}{\rho} - 1\right] + (1-\rho)(n_I - n_A). \tag{4.6}$$

As we can see, taking $n_I = n_A$ we recover (3.16). Equation (4.6) shows that in our generalized model the population dynamic in the industrial sector is determined not only by the rate of migration towards the industrial sector, but also by the differential of the sectoral population growth rates. In order to obtain the fraction of the population in the industrial sector at the steady-state, we proceed as such:

$$\dot{\rho} = 0 \iff \gamma\rho\left[\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{\theta_\tau}{(1-\theta_\tau)}\frac{(1-\rho)}{\rho} - 1\right] + \rho(1-\rho)(n_I - n_A) = 0$$

$$\iff \rho^2(n_I - n_A) + \rho\left[n_A - n_I + \gamma + \gamma\left(\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{\theta_\tau}{(1-\theta_\tau)}\right)\right] - \gamma\left[\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{\theta_\tau}{(1-\theta_\tau)}\right] = 0.$$

Defining $\sigma = \gamma\left[\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{\theta_\tau}{(1-\theta_\tau)}\right]$ and replacing it in the equation above:

$$\rho^2(n_I - n_A) + \rho(n_A - n_I + \gamma + \sigma) - \sigma = 0.$$

Now, replacing $\Delta n = n_I - n_A$, we get a quadratic equation for ρ:

$$\Delta n \rho^2 + (\gamma + \sigma - \Delta n)\rho - \sigma = 0. \tag{4.7}$$

This means that mathematically our model may have two steady-states for ρ[1]. We can verify that making $\Delta n = 0$, we recover ρ_∞ as defined in the Mas-Colell and Razin model[2], which has only one steady-state. Now, dividing (4.7) by Δn, we get:

$$\rho^2 + \left(\frac{\gamma + \sigma - \Delta n}{\Delta n}\right)\rho - \frac{\sigma}{\Delta n} = 0.$$

Applying the quadratic formula to the equation above, we can see that the steady-states for ρ are:

$$\rho_\infty^{1,2} = \frac{1}{2}\left[-\left(\frac{\gamma + \sigma - \Delta n}{\Delta n}\right) \pm \sqrt{\left(\frac{\gamma + \sigma - \Delta n}{\Delta n}\right)^2 + \frac{4\sigma}{\Delta n}}\right]. \tag{4.8}$$

Summarizing, the dynamics is then defined by the following system of ODEs:

$$\begin{cases} \dot{k} = k\lambda\theta_\tau^\beta\left(\dfrac{k}{\rho}\right)^{\beta-1} - k[n_I\rho + n_A(1-\rho)] \\ \dot{\rho} = \rho\gamma\left[\dfrac{(1-\beta)}{\beta}\dfrac{\alpha}{(1-\alpha)}\dfrac{\theta_\tau}{(1-\theta_\tau)}\dfrac{(1-\rho)}{\rho} - 1\right] + \rho(1-\rho)(n_I - n_A), \end{cases} \tag{4.9}$$

[1]Although ρ has two steady-states, we note that only one makes economic sense, i.e., $0 < \rho_\infty < 1$.

[2]Taking (4.7) and making $\Delta n = 0$ gives us equation (3.26):

$$(\gamma + \sigma)\rho - \sigma = 0 \Longrightarrow \rho_\infty = \frac{\sigma}{\gamma + \sigma} \Longrightarrow \rho_\infty = \frac{(1-\beta)\alpha\theta_\tau}{(1-\beta)\alpha\theta_\tau + \beta(1-\alpha)(1-\theta_\tau)}.$$

for the initial conditions $k(0) = k_0 > 0$ and $\rho(0) = \rho_0 > 0$. Besides that, it is worth mentioning that the per capita capital, the per capita output and the wage in each sector are determined by the following already known equations:

$$k_I = \theta_\tau \frac{k}{\rho}, k_A = (1-\theta_\tau)\frac{k}{(1-\rho)},$$

$$y_I = \rho k_I^\beta, y_A = (1-\rho)k_A^\alpha,$$

$$w_I = p(1-\beta)k_I^\beta, w_A = (1+\tau)(1-\alpha)k_A^\alpha.$$

With that we conclude the derivation of our generalized model.

4.1 EQUILIBRIA STABILITY ANALYSIS

4.1.1 Stability of ρ_∞

First, let us define the function $p(\rho)$ as the equation (4.7). In section 4, we had determined the steady-states for ρ in (4.8), which in the notation adopted here are the roots for $p(\rho)$. Rewriting (4.8), we get:

$$\rho_{1,2} = \frac{1}{2\Delta n}\left[-(\sigma+\gamma-\Delta n) \pm \sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma}\right], \tag{4.10}$$

where

$$\rho_1 = \frac{1}{2\Delta n}\left[-(\sigma+\gamma-\Delta n) + \sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma}\right], \tag{4.11}$$

$$\rho_2 = \frac{1}{2\Delta n}\left[-(\sigma+\gamma-\Delta n) - \sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma}\right]. \tag{4.12}$$

We are interested in the signals of (4.11) and (4.12), and want to show that only one has a feasible value, satisfying the condition $\rho \in (0,1)$[3]. Now, we know that $\gamma > 0$, so

[3]Which is the interval of values that makes economic sense, once ρ is the proportion of people living in the industrial sector.

$\rho_{1,2}$ depends on the signals of σ and Δn. Let's take a closer look at σ for a moment. We defined $\sigma = \gamma\left[\frac{(1-\beta)}{\beta}\frac{\alpha}{(1-\alpha)}\frac{\theta_\tau}{(1-\theta_\tau)}\right]$, where $\alpha, \beta \in (0,1)$ and $\tau > -1$. Therefore, it all relies upon θ_τ. Now, we already know that $\theta_\tau = \frac{(s+\delta)\beta}{(s+\delta)\beta+(1-s-\delta)(1+\tau)\alpha}$, where $\delta, s \in [0,1]$. Rewriting $1-s-\delta = 1-(s+\delta)$ and dividing both the numerator and the denominator of θ_τ by $(s+\delta)\beta$, we get:

$$\theta_\tau = \frac{1}{\frac{\alpha(1+\tau)}{\beta}\left[\frac{1}{(s+\delta)}-1\right]+1}.$$

Once $(s+\delta)$ is the proportion of income spent on industrial goods (for consumption and investment purposes), $1-(s+\delta)$ is the proportion of income spent on agricultural goods, and so $1-(s+\delta) > 0$. Then:

$$1-(s+\delta) > 0 \Longrightarrow (s+\delta) < 1 \Longrightarrow \frac{1}{(s+\delta)} > 1.$$

If $\frac{1}{(s+\delta)} > 1$, then $\left[\frac{1}{(s+\delta)}-1\right] > 0$, which means that $\frac{\alpha(1+\tau)}{\beta}\left[\frac{1}{(s+\delta)}-1\right]+1 > 1$, due to $\frac{\alpha(1+\tau)}{\beta} > 0, \forall \alpha, \beta, \tau$. Thus:

$$\theta_\tau = \frac{1}{\frac{\alpha(1+\tau)}{\beta}\left[\frac{1}{(s+\delta)}-1\right]+1} < 1 \Longrightarrow (1-\theta_\tau) > 0.$$

The fact that $(1-\theta_\tau) > 0$ guarantees that σ is strictly positive. Now that we know that $\sigma > 0$, there is only Δn to analyse. We express our results in the following proposition:

Proposition 4.1.1 *Considering any scenario with distinct population growth rates, there will always be one, and only one, feasible ρ.*

Proof: we have two possible cases, depending on the signal of Δn. Let us start with the case where the population growth rate is bigger in the industrial sector than in the agricultural sector.

Case 1: $\Delta n > 0$

In this case, $-(\sigma+\gamma-\Delta n) \lesseqgtr 0$. Starting with $-(\sigma+\gamma-\Delta n) = 0$, we have $\Delta n = \sigma+\gamma$, implying that from (4.10) we get:

$$\rho_{1,2} = \pm \frac{1}{2\Delta n}\sqrt{4\Delta n \sigma} = \pm \sqrt{\frac{\sigma}{\Delta n}}$$

$$\implies \rho_1 = \sqrt{\frac{\sigma}{\Delta n}}, \rho_2 = -\sqrt{\frac{\sigma}{\Delta n}}$$

$$\implies \rho_1 > 0, \rho_2 < 0, \forall \sigma, \gamma > 0$$

Observing that $\rho_1 = \sqrt{\frac{\sigma}{\Delta n}} = \sqrt{\frac{\sigma}{\sigma+\gamma}}$, it is clear that $\rho_1 < 1$. Therefore:

$$\rho_1 \in (0,1), \rho_2 < 0.$$

Now, for $-(\sigma + \gamma - \Delta n) > 0$, we get $\Delta n > \sigma + \gamma$. Thus:

$$\sqrt{(\sigma + \gamma - \Delta n)^2 + 4\Delta n \sigma} > -(\sigma + \gamma - \Delta n), \forall \sigma, \gamma$$

Therefore, $\rho_1 > 0$ and $\rho_2 < 0$. Taking ρ_1, we can rewrite it as:

$$\rho_1 = \frac{1}{2}\left[\frac{-(\sigma + \gamma - \Delta n)}{\Delta n} + \sqrt{\left(\frac{\sigma + \gamma - \Delta n}{\Delta n}\right)^2 + \frac{4\sigma}{\Delta n}}\right].$$

Let us show that $\rho_1 < 1$ by contradiction. Assuming that $\rho_1 \geq 1$, we have:

$$\frac{1}{2}\left[\frac{-(\sigma + \gamma - \Delta n)}{\Delta n} + \sqrt{\left(\frac{\sigma + \gamma - \Delta n}{\Delta n}\right)^2 + \frac{4\sigma}{\Delta n}}\right] \geq 1$$

$$\iff \frac{-(\sigma + \gamma - \Delta n)}{\Delta n} + \sqrt{\left(\frac{\sigma + \gamma - \Delta n}{\Delta n}\right)^2 + \frac{4\sigma}{\Delta n}} \geq 2$$

$$\iff -\left(\frac{\sigma + \gamma}{\Delta n} - 1\right)\sqrt{\left(\frac{\sigma + \gamma - \Delta n}{\Delta n}\right)^2 + \frac{4\sigma}{\Delta n}} \geq 2$$

$$\iff \sqrt{\left(\frac{\sigma+\gamma-\Delta n}{\Delta n}\right)^2+\frac{4\sigma}{\Delta n}} \geq \frac{\sigma+\gamma}{\Delta n}+1$$

$$\iff \left(\sqrt{\left(\frac{\sigma+\gamma-\Delta n}{\Delta n}\right)^2+\frac{4\sigma}{\Delta n}}\right)^2 \geq \left(\frac{\sigma+\gamma}{\Delta n}+1\right)^2$$

$$\iff \left(\frac{\sigma+\gamma}{\Delta n}-1\right)^2+\frac{4\sigma}{\Delta n} \geq \frac{(\sigma+\gamma)^2}{(\Delta n)^2}+1+2\left(\frac{\sigma+\gamma}{\Delta n}\right)$$

$$\iff -2\left(\frac{\sigma+\gamma}{\Delta n}\right)+\frac{4\sigma}{\Delta n} \geq 2\left(\frac{\sigma+\gamma}{\Delta n}\right)$$

$$\iff \frac{4\sigma}{\Delta n} \geq 4\left(\frac{\sigma+\gamma}{\Delta n}\right)$$

$$\iff \sigma \geq \sigma+\gamma \Leftrightarrow 0 \geq \gamma,$$

what is an absurd, because $\gamma > 0$. Hence, we conclude that $\rho_1 < 1$, and so:

$$\rho_1 \in (0,1), \rho_2 < 0.$$

For last, we have $-(\sigma+\gamma-\Delta n) < 0$, for what we get $\Delta n < \sigma+\gamma$. For this condition, we also have $\rho_1 > 0$ and $\rho_2 < 0$, because:

$$\sqrt{(\sigma+\gamma-\Delta n)^2+4\Delta n\sigma} > -(\sigma+\gamma-\Delta n), \forall \sigma, \gamma.$$

Using the same reasoning as the previous occasion, we can show that $\rho_1 < 1$.

$$\therefore \Delta n > 0 \Longrightarrow \rho_1 \in (0,1), \rho_2 < 0.$$

Case 2: $\Delta n < 0$

In a case where the population growth rate is bigger in the agricultural sector than in the industrial sector, we always have $-(\sigma+\gamma-\Delta n)<0$, which makes $-(\sigma+\gamma-\Delta n)<\sqrt{(\sigma+\gamma-\Delta n)^2+4\Delta n\sigma}$, considering that $4\Delta n\sigma<0$. Hence, from (4.10) is easy to see that:

$$\rho_{1,2}>0.$$

As before, we have to prove that only one between (4.11) and (4.12) belongs to the interval $(0,1)$, that is, we have to show that one of them is smaller than 1 while the other one is greater than 1. In that line, let us prove first that $\rho_1<1$ by contradiction. Considering $\rho_1\geq 1$:

$$\frac{1}{2\Delta n}\left[-(\sigma+\gamma-\Delta n)+\sqrt{(\sigma+\gamma-\Delta n)^2+4\Delta n\sigma}\right]\geq 1$$

$$\Longleftrightarrow -(\sigma+\gamma-\Delta n)+\sqrt{(\sigma+\gamma-\Delta n)^2+4\Delta n\sigma}\leq 2\Delta n$$

$$\Longleftrightarrow \sqrt{(\sigma+\gamma-\Delta n)^2+4\Delta n\sigma}\leq \sigma+\gamma+\Delta n,$$

where $\sigma+\gamma+\Delta n \gtreqless 0$, due to $\Delta n<0$. If $\sigma+\gamma+\Delta n\leq 0$, then we have an absurd (following the last inequality), proving that $\rho_1<1$. On the contrary, if $\sigma+\gamma+\Delta n>0$, we shall continue the analysis:

$$\Longleftrightarrow \left(\sqrt{(\sigma+\gamma-\Delta n)^2+4\Delta n\sigma}\right)^2\leq(\sigma+\gamma+\Delta n)^2$$

$$\Longleftrightarrow (\sigma+\gamma-\Delta n)^2+4\Delta n\sigma\leq(\sigma+\gamma+\Delta n)^2$$

$$\Longleftrightarrow (\sigma+\gamma)^2+\Delta n^2-2\Delta n(\sigma+\gamma)+4\Delta n\sigma\leq(\sigma+\gamma)^2+\Delta n^2+2\Delta n(\sigma+\gamma)$$

$$\iff 4\Delta n\sigma \leq 4\Delta n\sigma + 4\Delta n\gamma \Leftrightarrow 4\Delta n\gamma \geq 0,$$

which is an absurd, because $\Delta n < 0$. Therefore, $\rho_1 < 1 \Longrightarrow \rho_1 \in (0,1)$. Last, but not least, let us show - also by contradiction - that $\rho_2 > 1$. Assuming that $\rho_2 \leq 1$:

$$\frac{1}{2\Delta n}\left[-(\sigma+\gamma-\Delta n) - \sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma}\right] \leq 1$$

$$\iff -(\sigma+\gamma-\Delta n) - \sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma} \geq 2\Delta n$$

$$\iff -\sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma} \geq \sigma+\gamma+\Delta n.$$

Once again, $\sigma+\gamma+\Delta n \gtreqless 0$. If $\sigma+\gamma+\Delta n \geq 0$, then we have an absurd, proving that $\rho_2 > 1$. On the contrary, if $\sigma+\gamma+\Delta n < 0$, we may continue multiplying the last inequality by (-1):

$$\iff \left(\sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma}\right)^2 \leq (-(\sigma+\gamma+\Delta n))^2,$$

$$\iff (\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma \leq (\sigma+\gamma+\Delta n)^2$$

$$\iff (\sigma+\gamma)^2 + \Delta n^2 - 2\Delta n(\sigma+\gamma) + 4\Delta n\sigma \leq (\sigma+\gamma)^2 + \Delta n^2 + 2\Delta n(\sigma+\gamma)$$

$$\iff 4\Delta n\sigma \leq 4\Delta n\sigma + 4\Delta n\gamma \iff 4\Delta n\gamma \geq 0,$$

which is also an absurd, proving that $\rho_2 > 1$. With that, we conclude the following:

$$\therefore \Delta n < 0 \Longrightarrow \rho_1 \in (0,1), \rho_2 > 1.$$

□

Summarizing, proposition 4.1.1 has shown that for any scenario with distinct population growth rates only ρ_1 makes economic sense. Then, we may conclude the following about the steady-state of ρ:

$$\rho_\infty = \begin{cases} \frac{1}{2\Delta n}\left[-(\sigma+\gamma-\Delta n)+\sqrt{(\sigma+\gamma-\Delta n)^2+4\Delta n\sigma}\right], & \text{for } \Delta n \neq 0 \\ \frac{(1-\beta)\theta_\tau\alpha}{\beta(1-\alpha)(1-\theta_\tau)+(1-\beta)\theta_\tau\alpha}, & \text{for } \Delta n = 0. \end{cases} \tag{4.13}$$

The effort of obtaining (4.13) is a big deal for our generalized model. Back in section 4, when we derived the steady-states for ρ in (4.8), we had the uncertainty of having two possible steady-states, which obscured our results. Now we know only one of them is appropriate, and we know exactly how to calculate it.

Finally, we can carry out a stability analysis on the steady-state ρ_∞. From (4.9), we get that $\frac{\dot\rho}{\rho} = \sigma\left(\frac{1-\rho}{\rho}\right) - \gamma + (1-\rho)\Delta n$. Hence:

$$\dot\rho \lesseqqgtr 0 \iff \sigma\left(\frac{1-\rho}{\rho}\right) - \gamma + (1-\rho)\Delta n \lesseqqgtr 0$$

$$\dot\rho \lesseqqgtr 0 \iff \sigma(1-\rho) - \gamma\rho + \rho\Delta n - \rho^2\Delta n \lesseqqgtr 0$$

$$\dot\rho \lesseqqgtr 0 \iff -\Delta n\rho^2 + (\Delta n - \sigma - \gamma)\rho + \sigma \lesseqqgtr 0$$

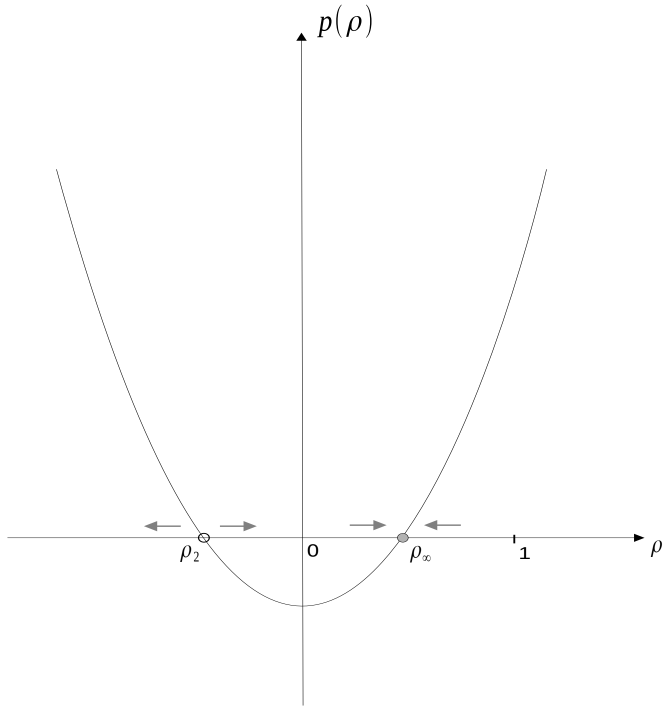
$$\dot\rho \lesseqqgtr 0 \iff \Delta n\rho^2 + (\gamma + \sigma - \Delta n)\rho - \sigma \gtreqqless 0$$

$$\therefore \dot\rho \lesseqqgtr 0 \iff p(\rho) \gtreqqless 0. \tag{4.14}$$

Relation (4.14) shows that ρ_∞ is stable for $\Delta n \lesseqqgtr 0$. If $\Delta n = 0$, then $p(\rho) = (\sigma+\gamma)\rho - \sigma \lesseqqgtr 0 \iff \rho \lesseqqgtr \frac{\sigma}{\sigma+\gamma} = \rho_\infty$. Therefore, $\dot\rho \gtreqqless 0 \iff \rho \lesseqqgtr \rho_\infty$, where the stability of ρ_∞ can be easily verified. For $\Delta n \leqslant 0$, the stability can be verified graphically through Figures 4 and

5[4]. Figure 4 is a sketch of $p(\rho)$ for $\Delta n > 0$. As expressed in (4.14), for $p(\rho) > 0$ we have $\dot{\rho} < 0$, and so for any positive value of $p(\rho)$ we observe a decrease in ρ, represented by the grey arrows pointing to the left. Now, if $p(\rho) < 0$ then we have $\dot{\rho} > 0$, showing that for any negative value of $p(\rho)$ we observe an increase in ρ, movement represented by the grey arrows pointing to the right. The arrows pointing to ρ_∞ indicate its stability, showing that no matter the value of the $p(\rho)$, it tends to ρ_∞. Analogously, Figure 5 shows the stability of ρ_∞ in a scenario with $\Delta n < 0$, where the same signal reasoning is used.

Figure 4 – Sketch of $p(\rho)$ for $\Delta n > 0$.



Source: Elaborated by the author (2021).

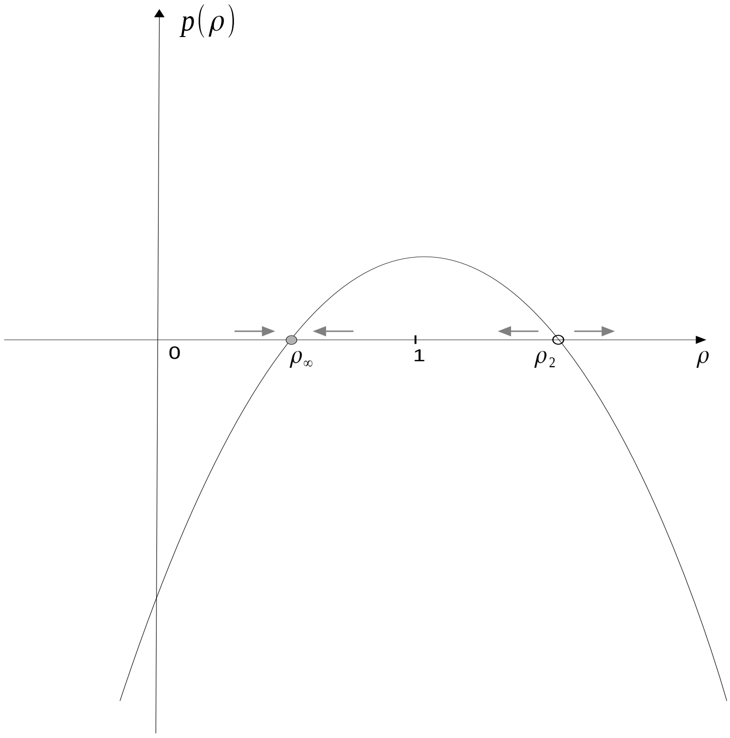
4.1.2 Stability of k_∞

The stability analysis of k_∞ is similar to that of ρ_∞. From (4.9), we get that $\frac{\dot{k}}{k} = \lambda\theta_\tau^\beta \left(\frac{k}{\rho}\right)^{\beta-1} - [n_I\rho + n_A(1-\rho)]$. Thus:

$$\dot{k} \lesseqgtr 0 \Longleftrightarrow \lambda\theta_\tau^\beta \left(\frac{k}{\rho}\right)^{\beta-1} - [n_I\rho + n_A(1-\rho)] \lesseqgtr 0$$

[4]For our purposes we can ignore $\rho \notin (0,1)$ as we have shown that it makes no economic sense. In fact, ρ_2 was included in our figures only for formal reasons.

Figure 5 – Sketch of $p(\rho)$ for $\Delta n < 0$.



Source: Elaborated by the author (2021).

$$\dot{k} \lesseqqgtr 0 \iff \lambda\theta_\tau^\beta \left(\frac{k}{\rho}\right)^{\beta-1} \lesseqqgtr [n_I\rho + n_A(1-\rho)]$$

$$\dot{k} \lesseqqgtr 0 \iff \lambda\theta_\tau^\beta \frac{\rho^{1-\beta}}{k^{\beta-1}} \lesseqqgtr [n_I\rho + n_A(1-\rho)]$$

$$\dot{k} \lesseqqgtr 0 \iff \frac{1}{k^{\beta-1}} \lesseqqgtr \frac{\lambda\theta_\tau^\beta \rho^{1-\beta}}{[n_I\rho + n_A(1-\rho)]}$$

$$\dot{k} \lesseqqgtr 0 \iff k^{\beta-1} \gtreqqless \frac{\lambda\theta_\tau^\beta \rho^{1-\beta}}{[n_I\rho + n_A(1-\rho)]}$$

$$\dot{k} \lesseqqgtr 0 \iff k \gtreqqless \theta_\tau^{\frac{\beta}{1-\beta}} \rho_\infty \left[\frac{\lambda}{n_I\rho_\infty + n_A(1-\rho_\infty)}\right]^{\frac{1}{1-\beta}}$$

$$\therefore \dot{k} \lesseqqgtr 0 \iff s(\rho) \gtreqqless k, \tag{4.15}$$

for $s(\rho) = \theta_\tau^{\frac{\beta}{1-\beta}} \rho \left[\frac{\lambda}{n_I\rho + n_A(1-\rho)}\right]^{\frac{1}{1-\beta}}$. At the steady-state, $k_\infty = s(\rho_\infty)$, which is constant. Thus, $k \lesseqgtr k_\infty \Longrightarrow \dot{k} \gtreqless 0$; this is shown by relation (4.15).When $s(\rho)$ is positive - at any point to the right of k_∞ - $\dot{k} < 0$, and so k decreases towards k_∞. On the other hand, if $s(\rho)$ is negative - at any point to the left of k_∞ - $\dot{k} > 0$, k increases towards k_∞. This proves the stability of k_∞.

Summarizing, the equilibrium of our generalized model is composed of the following steady-states:

$$\rho_\infty = \begin{cases} \dfrac{1}{2\Delta n}\left[-(\sigma+\gamma-\Delta n) + \sqrt{(\sigma+\gamma-\Delta n)^2 + 4\Delta n\sigma}\right], & \text{for } \Delta n \neq 0 \\ \dfrac{(1-\beta)\theta_\tau\alpha}{\beta(1-\alpha)(1-\theta_\tau) + (1-\beta)\theta_\tau\alpha}, & \text{for } \Delta n = 0, \end{cases}$$

$$k_\infty = \theta_\tau^{\frac{\beta}{1-\beta}} \rho \left[\frac{\lambda}{n_I\rho_\infty + n_A(1-\rho)}\right]^{\frac{1}{1-\beta}}, \text{for } \Delta n \lesseqgtr 0.$$

5 RESULTS AND DISCUSSION

5.1 NUMERICAL SIMULATIONS

In this section we are going to perform numerical simulations to analyze the behavior of the endogenous variables and present the phase diagrams of our generalized model. Our problem boils down to solve the system of ODEs given by (4.9) that defines the generalized model. We have chosen to solve it numerically using the Euler's method[1] through the implementation of an algorithm using the software MATLAB®; all the results were validated by the MATLAB command ode45. The script developed - available in Appendix B - carries out the numerical simulations and makes the graphs of the temporal evolution of our variables. The phase diagrams were generated by the software Maple®.

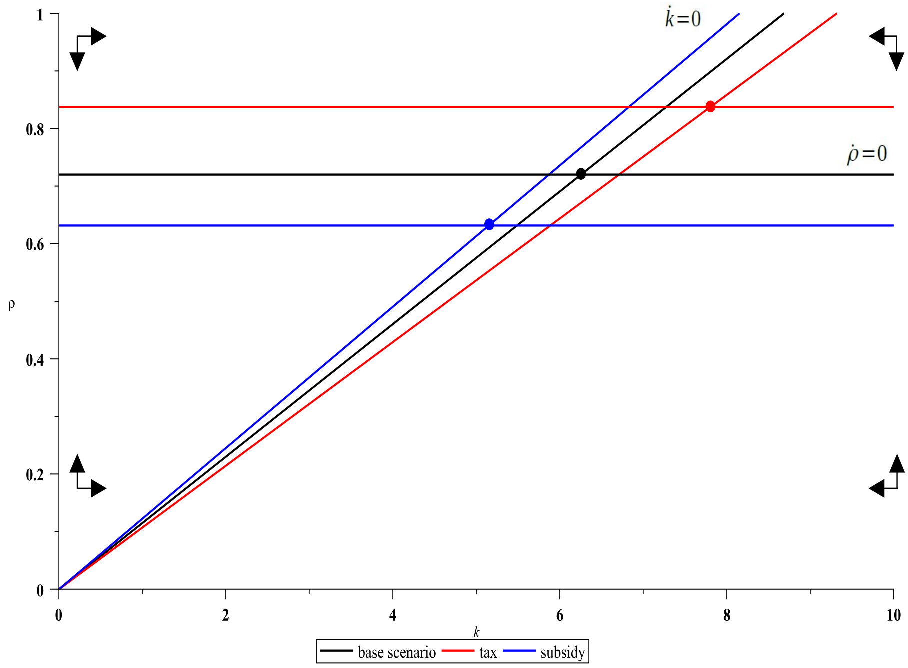
Our simulations and graphical analysis consider three scenarios concerning the sectoral population growth rates, one with equal rates, where $n_I = n_A = 0.05$, which will be referred to as scenario 1, and two scenarios with distinct rates, one with the rate in the industrial sector bigger than in the agricultural sector, where we take $n_I = 0.05$ and $n_A = 0.01$, which will be referred to as scenario 2, and the other with the rate in the agricultural sector bigger than in the industrial sector, where we take $n_I = 0.01$ and $n_A = 0.05$, which will be referred to as scenario 3. We also contemplate three tax policies, a case where no tax or subsidy is applied, and so $\tau = 0$, a case with a tax, where we take $\tau = -0.5$, and a case with a subsidy, where $\tau = 0.5$. At this point, it is prudent to remember that the tax or subsidy is applied to the agricultural sector, and so it affects the marginal productivity of capital and the wage rate of that region. The rest of our parameters and its values are $\beta = 0.4$, $\alpha = 0.3$, $s = 0.15$, $\gamma = 0.01$ and $\delta = 0.6$, which are the same values used by Mas-Colell and Razin (1973). The variables we are interested in analyzing are the aggregate per capita capital (k), the share of the population in the industrial sector (ρ), and the sectoral per capita capital (k_I and k_A) and output (y_I and y_A); it is also worth looking at the wage rates (w_I and w_A) because they explain the dynamics of the intersectoral migration.

We begin by presenting the phase diagrams in Figures 6 to 8. The diagrams are composed by isoclines $\dot{k} = 0$ and $\dot{\rho} = 0$, where the last generates a horizontal curve,

[1] See Appendix A.

because ρ_∞ is always constant. We also opted for using the same color to identify the isoclines belonging to the same tax policy, in order to facilitate the identification of the steady-state and seeking to differentiate well the different policies. Last, the black arrows on the edges indicate the dynamics.

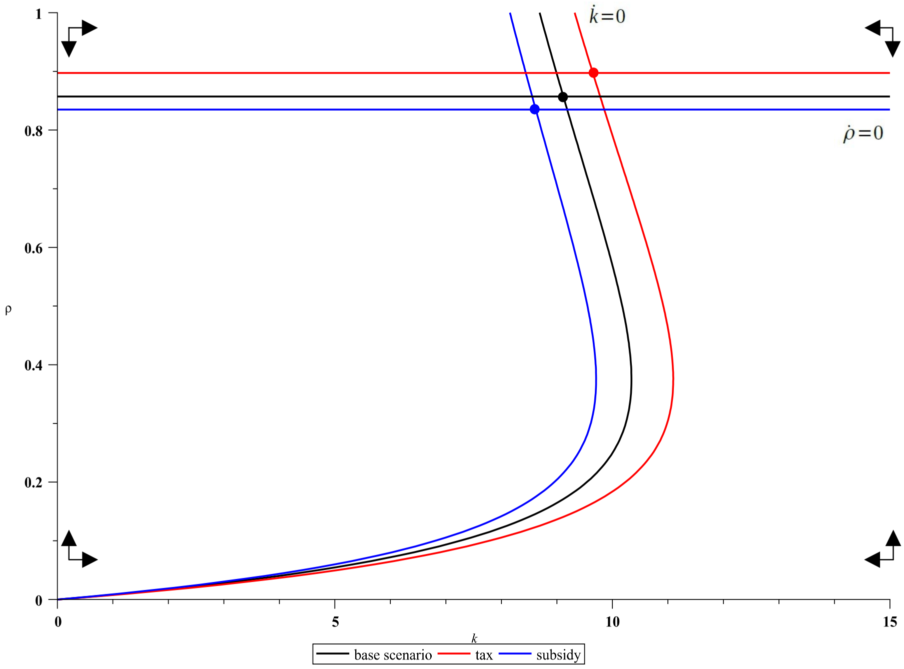
Figure 6 – Isoclines $\dot{k} = 0$ and $\dot{\rho} = 0$ for $n_I = n_A$.



Source: Elaborated by the author (2021).

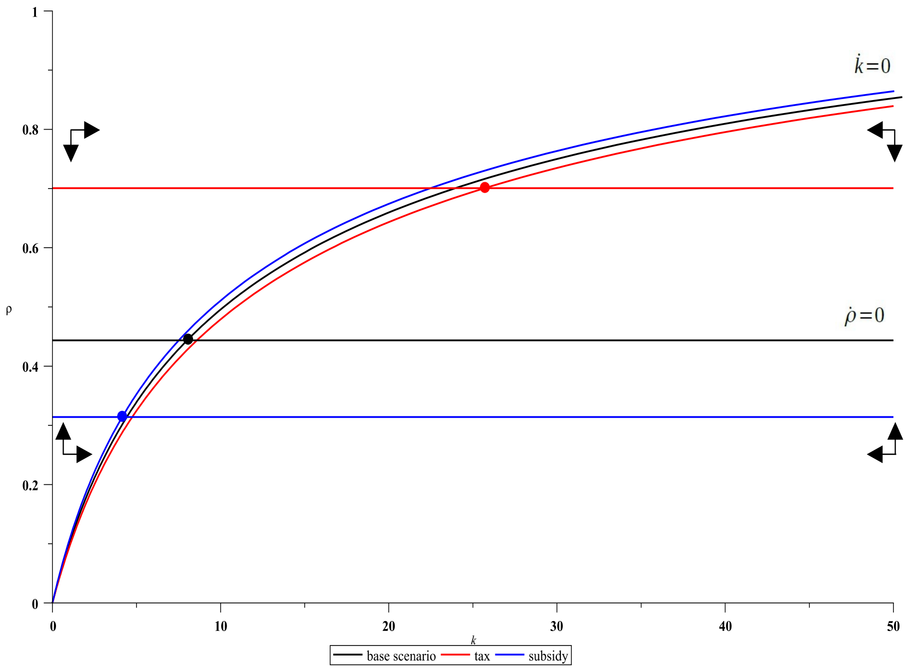
Starting with Figure 6, it shows the phase diagrams of scenario 1. The black isoclines form the same diagram as Figure 2, while the blue and red isoclines express the same changes that Figure 3 as discussed back in section 3.1. Thus, a subsidy policy leads to an equilibrium with smaller k and ρ than the cases with no tax policy or with a tax on the agricultural sector. As we can see, the black arrows show that the steady-states are stable. Next, Figure 7 shows the phase diagrams of scenario 2; the biggest difference from this scenario to the previous is the formats of $\dot{k} = 0$. Aside from that, the conclusion is pretty much the same, that is, a subsidy policy leads to an equilibrium with smaller k and ρ than the other two tax policies, and the steady-states are also stable. Closing the phase diagrams analysis, Figure 8 shows the diagrams of scenario 3. Again, the formats of $\dot{k} = 0$ differ from the other scenarios. Once again, the pattern for the equilibrium showed in the other two scenarios repeat, but in this context with more intensity, showed by a bigger distance between steady-states given each tax policy (compare with scenario 2, for example). As before, the steady-states are stable, as shown by the direction of the black arrows.

Figure 7 – Isoclines $\dot{k} = 0$ and $\dot{\rho} = 0$ for $n_I > n_A$.



Source: Elaborated by the author (2021).

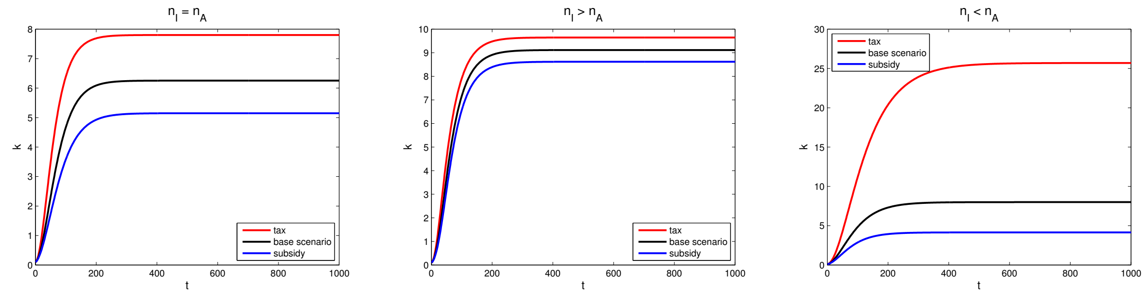
Figure 8 – Isoclines $\dot{k} = 0$ and $\dot{\rho} = 0$ for $n_I < n_A$.



Source: Elaborated by the author (2021).

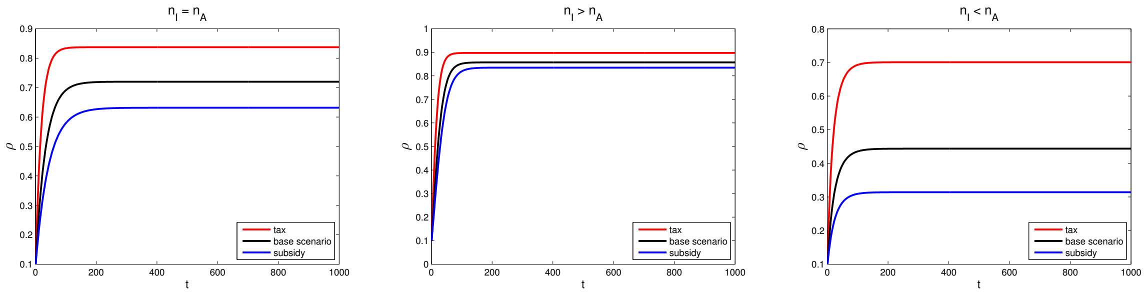
The next step is to present the temporal evolution until the steady-state is reached of our endogenous variables in all the scenarios considered. Each figure contains three graphs, where each graph represents one of the scenarios involving the population growth rates and each curve expresses a tax policy. The numerical simulations complement the analysis of the phase diagrams, corroborating our results. The discussion is left for the next subsection.

Figure 9 – Temporal evolution of k



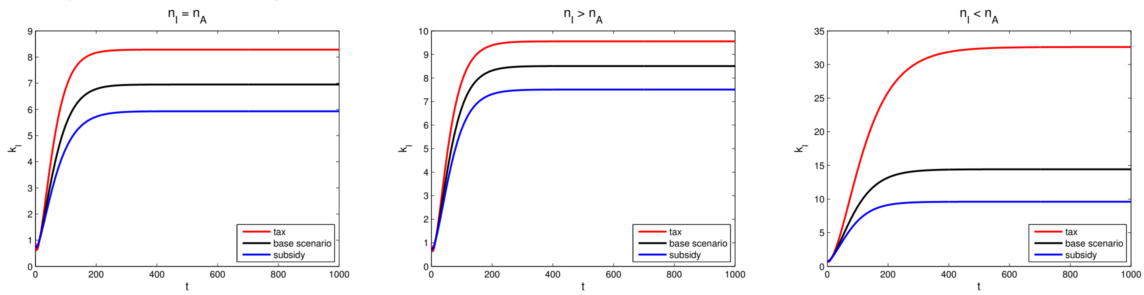
Source: Elaborated by the author (2021).

Figure 10 – Temporal evolution of ρ



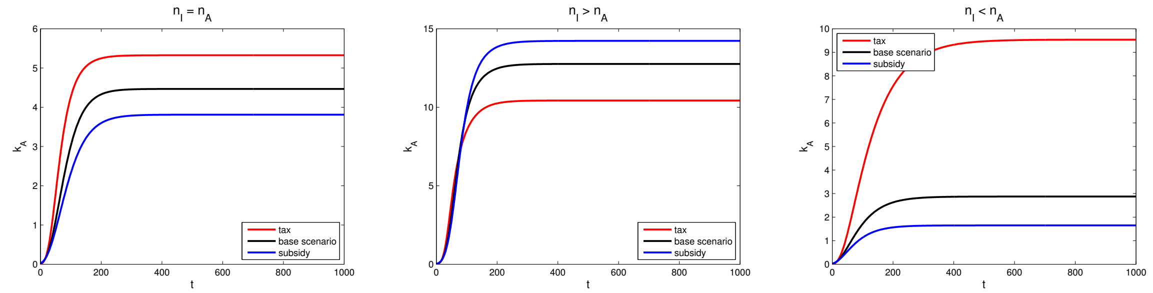
Source: Elaborated by the author (2021).

Figure 11 – Temporal evolution of k_I



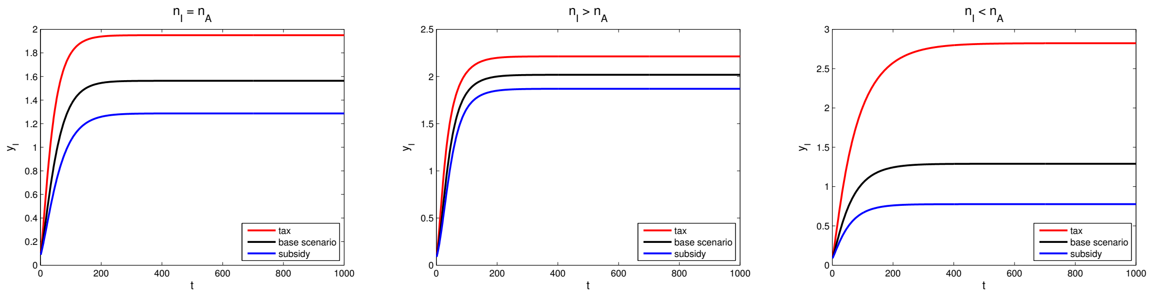
Source: Elaborated by the author (2021).

Figure 12 – Temporal evolution of k_A



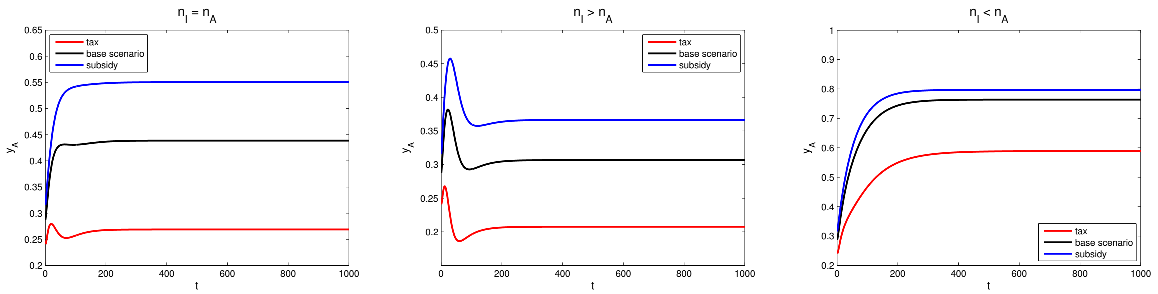
Source: Elaborated by the author (2021).

Figure 13 – Temporal evolution of y_I



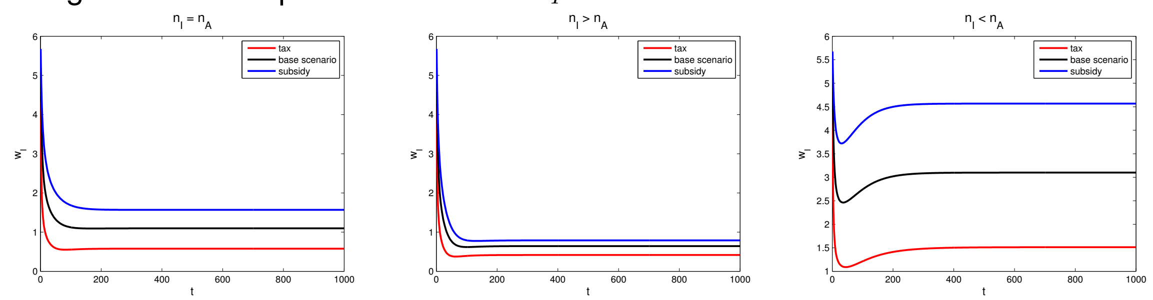
Source: Elaborated by the author (2021).

Figure 14 – Temporal evolution of y_A



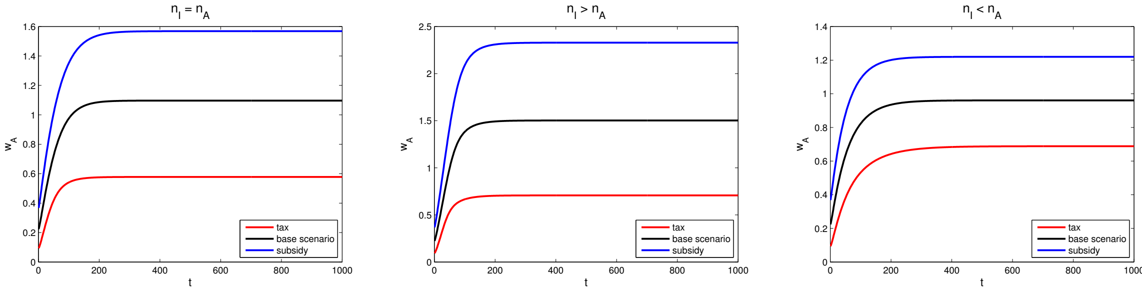
Source: Elaborated by the author (2021).

Figure 15 – Temporal evolution of w_I



Source: Elaborated by the author (2021).

Figure 16 – Temporal evolution of w_A



Source: Elaborated by the author (2021).

5.2 DISCUSSION

In this subsection we are going to discuss the results of our numerical simulations. Starting with the aggregate per capita capital, as seen in Figure 9, the biggest values from all three scenarios are presented in the case with a tax, while the smallest ones are from the case with a subsidy. Besides that, the biggest k appears in scenario 3, and it is by a large margin. Also, it is curious to note that the smallest value in scenario 2 is still bigger than the biggest one in scenario 1. About the share of the population in the industrial sector, shown in Figure 10, the biggest values from all the three scenarios are from the case with a tax, while the smallest values are from the case with a subsidy, just as in the analysis of the aggregate per capita capital. The meaning of this is that no matter the scenario, an agricultural product tax apparently induces the population to move towards the industrial sector, while a subsidy serves as a stimulus for more people to stay in the agricultural sector, relatively. It is easier to see this in scenarios 1 and 3, especially in the last one, where the difference between ρ from the cases with a tax and a subsidy is the biggest possible. The steady-state values of k and ρ can be seen in Table 1:

Table 1 – Steady-state values of k and ρ

	$n_I = n_A$		$n_I > n_A$		$n_I < n_A$	
	k_∞	ρ_∞	k_∞	ρ_∞	k_∞	ρ_∞
base scenario	6.254	0.72	9.1143	0.8571	7.9987	0.4436
tax	7.8013	0.8372	9.6457	0.8972	25.697	0.7006
subsidy	5.1482	0.6316	8.6181	0.8348	4.1471	0.3141

Source: Elaborated by the author (2021).

Dealing with the sectoral per capita capitals in Figures 11 and 12, k_I and k_A seems to respect the pattern of k, with the exception of k_A in scenario 2, where its value in the case with a subsidy is bigger than the one in the case with a tax. For some reason, the introduction of a subsidy in the agricultural sector, in a scenario with a bigger population growth rate in the industrial sector, makes the per capita capital in the agricultural region grow atypically. One possible explanation might be the high proportion of people living in the industrial sector in this scenario with a subsidy; in this case, $\rho = 0.8348$ exactly, which is the biggest value from all the three scenarios with a subsidy. Because there is few people living in the agricultural region, the capital per capita will be substantial; in fact, $k_A > k_I$ for any tax policy in scenario 2, and that might be because of ρ. Now, considering the sectoral per capita outputs in Figures 13 and 14, y_I is bigger than y_A for any tax policy in scenarios 1 and 2; as a matter of fact, the only case where $y_A > y_I$ is in scenario 3 with a subsidy, where $y_A = 0.7969$ and $y_I = 0.7763$, only slightly bigger. Especially in scenarios 1 and 2, the difference between the per capita output in both sectors is considerable, showing that somehow the level of development in the industrial region is higher than in the agricultural region. This development of the industrial region can be explained by two things; first, we can easily see from Figure 10 that the share of the population living in the industrial sector is bigger than the one living in the agricultural sector for any tax policy - notably in scenario 2, where they are much bigger -, meaning that the industrial region is succeeding in attracting people; a second reason might be in the analysis of k_I and k_A. Despite the per capita capital in the agricultural sector being bigger than the one in the industrial sector for any tax policy, the difference between their values is not so big, meaning that although there are much more people living in the industrial region, the per capita capital there is proportionally high, which is also a sign of development. The steady-state values of k_I, k_A, y_I and y_A can be seen in Tables 2 and 3:

Table 2 – Steady-state values of k_I and k_A

	$n_I = n_A$		$n_I > n_A$		$n_I < n_A$	
	k_{I_∞}	k_{A_∞}	k_{I_∞}	k_{A_∞}	k_{I_∞}	k_{A_∞}
base scenario	6.9489	4.4672	8.5067	12.76	14.4262	2.875
tax	8.2828	5.3247	9.5558	10.43	32.6018	9.5374
subsidy	5.9283	3.811	7.508	14.23	9.6026	1.649

Source: Elaborated by the author (2021).

Last, but not least, we take a look at the wages in Figures 15 and 16. More than their values, we are interested in their behaviors. Starting with scenario 1, which is

Table 3 – Steady-state values of y_I and y_A

	$n_I = n_A$		$n_I > n_A$		$n_I < n_A$	
	y_{I_∞}	y_{A_∞}	y_{I_∞}	y_{A_∞}	y_{I_∞}	y_{A_∞}
base scenario	1.5635	0.4387	2.0182	0.3067	1.2901	0.7638
tax	1.9503	0.2689	2.2132	0.2076	2.8235	0.5889
subsidy	1.2871	0.5504	1.8698	0.3664	0.7763	0.7969

Source: Elaborated by the author (2021).

the scenario that simulates the same conditions as in the original Mas-Colell and Razin model, we can see that w_I and w_A converge on the same value in all cases, and that is what we expect them to do, once the dynamics of ρ in the original model is equal to the migration rate, which is a function of the wage differential. So, there will be no migration when the wages are equal, and that is why in the original model ρ converges. The migration rate is zero at the steady-state, which causes population stability. Now, in scenarios 2 and 3, where there are distinct population growth rates, wages do not converge to the same value in neither case, and they should not indeed. The dynamics of ρ for scenarios with distinct population growth rates is determined not only by the rate of migration towards the industrial sector, but also by the differential of the sectoral population growth rates[2]. In these scenarios migration is not null at the steady-state, and it is exactly the existence of that migration that serves as counterweight to the larger population growth in a region, causing populations in the sectors to stabilize. For example, if we take scenario 2 - where the population growth rate is bigger in the industrial sector - regarding any tax policy, at $t = 0$ w_I is much bigger than w_A because the population in the industrial sector is tiny (remember that $\rho(0) = 0.1$)[3]; this wage difference indulges the industrial region, making it attractive to workers, what drives ρ up. Over time, as more workers are attracted to the industrial region, wages in this region begin to fall, while the one in the agricultural region rises. At some point, it w_A surpasses w_I, which changes the direction of migration towards the agricultural region. This condition is maintained at the steady state, and it is exactly the migration towards the agricultural sector that serves as counterweight to the larger population growth in the industrial region, stabilizing the population size in both sectors. The analysis for scenario 3 is analogous, but considering that the population growth rate is bigger in the agricultural sector, and so w_I is bigger than w_A the hole time, causing a continuous

[2]As mentioned before, this is easily seen by the equation that defines $\dot{\rho}$ in (4.9).
[3]The lack of competition in the labor market makes the wage in that sector initially bigger.

migratory flow towards the industrial region, which serves as a counterweight to the larger population growth in the agricultural sector, stabilizing the population at the steady state.

6 CONCLUSION

In this work, we generalized the Mas-Colell and Razin two-sector model of intersectoral migration and growth by introducing two population growth rates, one for each sector. The introduction of sectoral population growth rates allowed us to determine a new dynamic of the aggregate population growth, which came to be determined by the rate of migration towards the industrial sector, as in the original model, and by the differential of the sectoral population growth rates. Likewise, the per capita capital accumulation of our generalized model also came to depend on the differential of the sectoral population growth rates. The stability analysis showed that our generalized model, determined by a system of two ODEs, has an unique economically feasible and stable equilibrium given by ρ_∞ and k_∞.

To investigate the behavior of our endogenous variables, numerical simulations were performed. With the different population growth rates it was possible to consider three scenarios in our discussions, one with equal population growth rates, in which we brought back the original model, and two with distinct population growth rates, where we could observe the effects on the different rates. Besides that, we also considered three tax policies, that is, a case with no tax or subsidy, a case with a tax and a case with a subsidy. Overall, we made simulations for eight variables in three scenarios with three tax policies each.

Considering the aggregate per capita capital (k) and comparing scenarios under the same tax policies, any scenario with distinct population growth rates leads to a bigger per capita capital accumulation relative to the scenario of the original model (with equal population growth rates), except for the case of a subsidy in the scenario with a bigger population growth rate in the agricultural sector; for this case, the scenario of the original model leads to a bigger per capita capital accumulation. For the proportion of the population in the industrial sector (ρ), for a population growth rate in the industrial sector bigger than in the agricultural sector, the steady-state value of ρ is bigger than in the scenario with equal population growth rates for any tax policy. For a population growth rate in the agricultural sector bigger than in the industrial sector, the value of ρ at the steady-state is smaller than in the scenario of the original model for all the three tax policies. Thus, comparing cases with the same tax policies, the scenario with a bigger population growth rate in the industrial sector results in more people living in the industrial sector in relation to the scenario with equal population growth rates, no matter the tax policy; while the scenario with a bigger population growth rate in the agricultural sector results in less people living in the industrial sector in relation to the scenario of the original model, for all three tax policies.

As for the capital per capita in the industrial sector (k_I), comparing cases with the same tax policies, any scenario with distinct population growth rates leads to a bigger steady-state value relative to the scenario with equal population growth rate. For the capital per capita in the agricultural sector (k_A), we observe that for the case with a tax, any scenario with distinct population growth rates leads to a bigger steady-state value relative to the scenario of the original model; now, for the cases with no tax policy and a subsidy, the scenario with a bigger population growth rate in the industrial sector has bigger steady-state values than the scenario with equal population growth rates, but the latter has bigger steady-states than the scenario with a bigger population growth rate in the agricultural sector.

Finally, considering the output per capita in the industrial sector (y_I) under the same tax policy, we notice that for the case with a tax, any scenario with distinct population growth rates leads to a bigger steady-state value relative to the scenario of the original model; on the other hand, for the cases without a tax policy and a subsidy, the scenario with a bigger population growth rate in the industrial sector has bigger steady-state values than the scenario with equal population growth rates, but the latter has bigger steady-state values than the scenario with a bigger population growth rate in the agricultural sector. For the output per capita in the agricultural sector (y_A), all steady-state values for the scenario of the original model are bigger than those in the scenario with a bigger population growth rate in the industrial sector, but are smaller than those of the scenario with a bigger population growth rate in the agricultural sector.

Future research may consider formulating the model with different production functions. In this sense, Christiaans (2017) shows very interesting ideas, like using an agricultural production function linear in labor. Other possibilities are considering logistic population growth rather than Malthusian, imperfect capital mobility between the sectors, and technological progress.

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APPENDIX A – EULER'S METHOD

Our generalized model is represented by a system of ODEs defined in (4.9). The method chosen to solve it was the Euler's method, which is a numerical method for solving initial value problems of ODEs, which allows one's to calculate approximations at certain points. Following Chapra (2018), in general terms the method works like this. Given any function that depends on the time $y = f(t)$, the derivative of y in relation to t is a function of both y and t,

$$\frac{dy}{dt} = f(t, y).$$

We can define $f(t_i, y_i)$ as being the ODE in time i, where $i = 0, 1, 2, \cdots$. Therefore, the Euler's method is defined by the following equation:

$$y_{i+1} = y_i + f(t_i, y_i)h,$$

where h represents the difference between t_{i+1} e t_i. Here, $\dot{k}$ and $\dot{\rho}$ in (4.9) represent $f(t_i, y_i)$ (CHAPRA, 2018). The algorithm that implements the method can be seen in our MATLAB® script, lines 36-44, in Appendix B.

APPENDIX B – MATLAB SCRIPT FOR THE NUMERICAL SIMULATIONS

```
clear, clc;

% parameters
beta = 0.4; % parameter of the production function from the industrial sector
alpha = 0.3; % parameter of the production function from the agricultural sector
n_I = 0.05; % industrial population growth rate
n_A = 0.05; % agricultural population growth rate
delta_n = n_I - n_A;
s = 0.15; % rate of saving
delta = 0.6; % income spent on industrial goods
lambda = s/(s+delta);
tau = -0.5; %tax/subsidy given to the agricultural sector
gamma = 0.01;

while tau <= 0.5
    theta = (s*beta)/((s*beta)+((lambda-s)*alpha*(1+tau)));
    sigma = gamma*(((1-beta)/beta)*(alpha/(1-alpha))*(theta/(1-theta)));
    % Initial conditions for rho and k
    rho(1) = 0.1;
    k(1) = 0.1;

    % steady-states
    if n_I == n_A
        rho_ss = (1-beta)*alpha*theta/((beta*(1-alpha)*(1-theta))+((1-beta)*alpha*theta))
    else
```

```
        rho1_ss = (-((gamma+sigma-delta_n)/delta_n) + sqrt(((gamma+sigma-delta_n)/delta_n)^2 + (4*sigma/delta_n)))/2;
        rho2_ss = (-((gamma+sigma-delta_n)/delta_n) - sqrt(((gamma+sigma-delta_n)/delta_n)^2 + ((4*sigma)/delta_n)))/2;
        if rho1_ss >= 0 & rho1_ss <= 1
            rho_ss = rho1_ss
        else
            rho_ss = rho2_ss
        end
    end
    k_ss = theta^(beta/(1-beta))*rho_ss*((lambda/((n_I*rho_ss)+(n_A*(1-rho_ss))))^(1/(1-beta)))

    % Euler's method
    T = 850; % total time
    N = 1000; % points
    dt = T/N;
    t = 0:dt:T;
    for i = 1:N
        rho(i+1) = rho(i) + (gamma*rho(i)*((1-beta)/beta*alpha/(1-alpha)*theta/(1-theta)*(1-rho(i))/rho(i)-1)+rho(i)*(1-rho(i))*(n_I-n_A))*dt;
        k(i+1) = k(i) + (k(i)*lambda*theta^beta*(k(i)/rho(i))^(beta-1)-k(i)*(n_I*rho(i)+n_A*(1-rho(i))))*dt;
    end

    % per-capita capital, per-capita product, wage rates and migration, for both
    % sectors
```

```

     for i = 1:N
        k_I(i) = (theta*k(i))/rho(i);
        k_A(i) = ((1-theta)*k(i))/(1-rho(i));
        y_I(i) = rho(i)*(k_I(i)^beta);
        y_A(i) = (1-rho(i))*(k_A(i)^alpha);
        p(i) = (y_A(i)/y_I(i))*((s+delta)/(1-s-delta)); % relative price of the industrial good
        w_I(i) = p(i)*(1-beta)*(k_I(i)^beta);
        w_A(i) = (1+tau)*(1-alpha)*(k_A(i)^alpha);
        PMGK_I(i) = p(i)*beta*(k_I(i))^(beta-1);
        PMGK_A(i) = (1+tau)*alpha*(k_A(i))^(alpha-1);
     end

    % convergence graphs
    if tau == -0.5
       plot(k, 'r', 'LineWidth', 2);
       xlabel('t');
       ylabel('k');
    %   title('k vs. time');
       grid off;
       hold on
    elseif tau == 0
       plot(k, 'k', 'LineWidth', 2);
       xlabel('t');
       ylabel('k');
    %   title('k vs. time');
       grid off;
       hold on
    else
       plot(k, 'b', 'LineWidth', 2);
       xlabel('t');
```

```
        ylabel('k');
    %    title('k vs. time');
        grid off;
        hold on
    end
    tau = tau + 0.5;
end

legend('tax', 'base scenario', 'subsidy', 'location', '
   northwest')
% ylabel('$$\rho$$','Interpreter','latex','FontSize',16)
```