

## Research Article

# Skew Polynomial Extensions over Zip Rings

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In this article, we study the relationship between left (right) zip property of  $R$  and skew polynomial extension over  $R$ , using the skew versions of Armendariz rings.

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## 1. Introduction

Throughout this paper  $R$  denotes an associative ring with identity and  $\sigma : R \rightarrow R$  an automorphism of  $R$ , otherwise unless stated. We denote  $R[[x; \sigma]]$  ( $R[[x, x^{-1}; \sigma]]$ ) the skew series rings (skew Laurent series rings) whose elements are the series  $\sum_{i \geq 0} a_i x^i$  ( $\sum_{j=p}^{\infty} b_j x^j$ ), where the addition is defined as usual and the multiplication is defined by the rule,  $xa = \sigma(a)x$  ( $xa = \sigma(a)x$  and  $x^{-1}a = \sigma^{-1}(a)x$ ), for any  $a \in R$ . Note that the skew polynomial rings of automorphism type  $R[x; \sigma]$  (skew Laurent of polynomial  $R[x, x^{-1}; \sigma]$ ) are subrings of  $R[[x; \sigma]]$  ( $R[[x, x^{-1}; \sigma]]$ ) whose elements are  $\sum_{i=0}^n a_i x^i$  ( $\sum_{j=q}^m b_j x^j$ ) where the sum and multiplication are defined as before.

Rege and Chhawchharia in [1] introduced the notion of an Armendariz ring. A ring  $R$  is called Armendariz if whenever polynomials  $\sum_{i=0}^n a_i x^i, \sum_{j=0}^m b_j x^j \in R[x]$  satisfy  $f(x)g(x) = 0$ , then  $a_i b_j = 0$  for each  $0 \leq i \leq n$  and  $0 \leq j \leq m$ . The name Armendariz ring was chosen because Armendariz [2] had shown that a reduced ring (i.e., ring without nonzero nilpotent elements) satisfies this condition. Some properties of Armendariz rings have been studied by Rege and Chhawchharia [1], Armendariz [2], Anderson and Camillo [3], and Kim and Lee [4].

Faith in [5] called a ring  $R$  right zip if the right annihilator  $r_R(X)$  of a subset  $X$  of  $R$  is zero, then  $r_R(Y) = 0$  for a finite subset  $Y \subseteq X$ ; equivalently, for a left ideal  $L$  of  $R$  with  $r_R(L) = 0$ , there exists a finitely generated left ideal  $L_1 \subseteq L$  such that  $r_R(L_1) = 0$ .  $R$  is zip if it is right and left zip. The concept of zip rings was initiated by Zelmanowitz [6] and appeared in various papers [5, 7–12], and references therein. Zelmanowitz stated that any ring satisfying

the descending chain condition on right annihilators is a right zip ring (although not so-called at that time), but the converse does not hold. Extensions of zip rings were studied by several authors. Beachy and Blair [7] showed that if  $R$  is a commutative zip ring, then the polynomial ring  $R[x]$  over  $R$  is zip. The authors in [13] proved that  $R$  is a right (left) zip ring if and only if  $R[x]$  is a right (left) zip ring when  $R$  is an Armendariz ring.

In this paper, we study skew polynomial extensions over zip rings by using skew versions of Armendariz rings and we generalized the results of [13]. Our skew versions of Armendariz rings follow the ideas of [14, Definition]. Moreover, we provide some examples to display some of the phenomenas of Section 2.

## 2. Skew polynomial extensions over zip rings

Throughout this paper  $\sigma$  is an automorphism of  $R$  unless otherwise stated and  $S$  will denote one of the following rings:  $R[x; \sigma]$ ,  $R[[x; \sigma]]$ ,  $R[x, x^{-1}; \sigma]$ , and  $R[[x, x^{-1}; \sigma]]$ . A left (right) annihilator of a subset  $U$  of  $R$  is defined by  $l_R(U) = \{a \in R : aU = 0\}$  ( $r_R(U) = \{a \in R : Ua = 0\}$ ). For a ring  $R$ , put  $r\text{Ann}_R(2^R) = \{r_R(U) : U \subseteq R\}$  and  $l\text{Ann}_R(2^R) = \{l_R(U) : U \subseteq R\}$ .

We begin with the following lemma and use it without further mention.

**Lemma 2.1.** *Let  $S$  be one of the rings above and  $U$  a subset of  $R$ . The following statements hold:*

- (i)  $l_S(U) = Sl_R(U)$ ,
- (ii)  $r_S(U) = r_R(U)S$ .

*Proof.* (i) We only prove for the case  $S = R[x; \sigma]$  because the other cases are similar. Let  $f(x) = \sum_{i=0}^n a_i x^i \in R[x; \sigma]$  such that  $f(x)U = 0$ . Then  $\sigma^{-i}(a_i)U = 0$  for all  $0 \leq i \leq n$  and it follows that  $\sigma^{-i}(a_i) \in l_R(U)$  for all  $0 \leq i \leq n$ . Hence  $f(x) = \sum_{i=0}^n x^i \sigma^{-i}(a_i) \in R[x; \sigma]l_R(U)$ . So  $l_{R[x; \sigma]}(U) \subseteq R[x; \sigma]l_R(U)$ . We clearly have that  $R[x; \sigma]l_R(U) \subseteq l_{R[x; \sigma]}(U)$ . Therefore, we have  $l_{R[x; \sigma]}(U) = R[x; \sigma]l_R(U)$ .

(ii) We only prove for the case  $S = R[x; \sigma]$  because the other cases are similar. Let  $f(x) = \sum_{i=0}^n a_i x^i \in R[x; \sigma]$  such that  $Uf(x) = 0$ . Then  $Ua_i = 0$  for all  $0 \leq i \leq n$  and it follows that  $a_i \in r_R(U)$  for all  $0 \leq i \leq n$ . Hence  $f(x) = \sum_{i=0}^n a_i x^i \in r_R(U)R[x; \sigma]$ . So  $r_{R[x; \sigma]}(U) \subseteq r_R(U)R[x; \sigma]$ . We clearly have that  $r_R(U)R[x; \sigma] \subseteq r_{R[x; \sigma]}(U)$ . Therefore, we have  $r_{R[x; \sigma]}(U) = r_R(U)R[x; \sigma]$ .  $\square$

With the above lemma, we have maps  $\phi : r\text{Ann}_R(2^R) \rightarrow r\text{Ann}_S(2^S)$  defined by  $\phi(I) = IS$  for every  $I \in r\text{Ann}_R(2^R)$  and

$$\Psi : l\text{Ann}_R(2^R) \longrightarrow l\text{Ann}_S(2^S) \quad (2.1)$$

defined by  $\Psi(I) = SI$  for every  $I \in l\text{Ann}_R(2^R)$ . Moreover, we have maps  $\Phi : r\text{Ann}_S(2^S) \rightarrow r\text{Ann}_R(2^R)$  defined by  $\Phi(J) = J \cap R$  for every  $J \in r\text{Ann}_S(2^S)$  and  $\Gamma : l\text{Ann}_S(2^S) \rightarrow l\text{Ann}_R(2^R)$  defined by  $\Gamma(J) = J \cap R$  for every  $J \in l\text{Ann}_S(2^S)$ . Obviously,  $\phi$  is injective and  $\Phi$  is surjective. Clearly,  $\phi$  is surjective if and only if  $\Phi$  is injective, and in this case  $\phi$  and  $\Phi$  are the inverses of each other. Note that  $\Psi$  and  $\Gamma$  satisfy the same relations as above. The first item of the definition below appears in [14, Definition].

**Definition 2.2.** (i) Suppose that  $\sigma$  is an endomorphism of  $R$ . A ring  $R$  satisfies SA1' if for  $f(x) = \sum_{i=0}^n a_i x^i$  and  $g(x) = \sum_{j=0}^m b_j x^j$  in  $R[x; \sigma]$ ,  $f(x)g(x) = 0$  implies that  $a_i \sigma^i(b_j) = 0$  for all  $0 \leq i \leq n$  and  $0 \leq j \leq m$ .

- (ii) Suppose that  $\sigma$  is an endomorphism of  $R$ . A ring  $R$  satisfies SA2' if for  $f(x) = \sum_{i=0}^{\infty} a_i x^i$  and  $g(x) = \sum_{j=0}^{\infty} b_j x^j$  in  $R[[x; \sigma]]$ ,  $f(x)g(x) = 0$  implies that  $a_i \sigma^i(b_j) = 0$  for all  $i \geq 0$ ,  $j \geq 0$ .
- (iii) Suppose that  $\sigma$  is an automorphism of  $R$ . A ring  $R$  satisfies SA3' if for  $f(x) = \sum_{i=s}^q a_i x^i$  and  $g(x) = \sum_{j=t}^n b_j x^j \in R[x, x^{-1}; \sigma]$ ,  $f(x)g(x) = 0$  implies that  $a_i \sigma^i(b_j) = 0$  for all  $s \leq i \leq q$  and  $t \leq j \leq n$ .
- (iv) Suppose that  $\sigma$  is an automorphism of  $R$ . A ring  $R$  satisfies SA4' if for  $f(x) = \sum_{i=s}^{\infty} a_i x^i$  and  $g(x) = \sum_{j=t}^{\infty} b_j x^j \in R[[x, x^{-1}; \sigma]]$ ,  $f(x)g(x) = 0$  implies that  $a_i \sigma^i(b_j) = 0$  for all  $i \geq s$  and  $j \geq t$ .

Note that if  $R$  satisfies one of the conditions above, then all subrings  $S$  of  $R$  such that  $\sigma(S) \subseteq S$  satisfies the same property. The following implications are easy to verify: SA4'  $\Rightarrow$  SA3' and SA2'  $\Rightarrow$  SA1'. Following [15, Example 2.1] when  $\sigma = id_R$ , the last implication is not reversible.

**Lemma 2.3.** *Let  $\sigma$  be an automorphism of  $R$ . Then*

- (i)  *$R$  satisfies SA1' if and only if  $R$  satisfies SA3';*
- (ii)  *$R$  satisfies SA2' if and only if  $R$  satisfies SA4'.*

*Proof.* Let  $f(x), g(x) \in R[x, x^{-1}; \sigma]$  such that  $f(x)g(x) = 0$ , where  $f(x) = \sum_{i=-p}^q a_i x^i$  and  $g(x) = \sum_{j=-t}^s b_j x^j$ . We clearly have  $x^p f(x) \in R[x; \sigma]$  and  $g(x)x^t \in R[x; \sigma]$ , then  $x^p f(x)g(x)x^t = 0$ . By assumption,  $\sigma^p(a_i)\sigma^{i+p}(b_j) = 0$  for all  $-p \leq i \leq q$  and  $-t \leq j \leq s$ . Hence  $a_i \sigma^i(b_j) = 0$  for all  $-p \leq i \leq q$  and  $-t \leq j \leq s$ . Since  $R[x; \sigma] \subseteq R[x, x^{-1}; \sigma]$ , the converse follows.

The proof of the other statement is similar.  $\square$

The following definition appears in [16, Definition 2.1].

**Definition 2.4.** Let  $R$  be a ring and  $\sigma$  an endomorphism of  $R$ . Then  $R$  is said  $\sigma$ -compatible like right  $R$ -module, if  $ar = 0$  if and only if  $a\sigma(r) = 0$  for any  $a \in R$  and  $r \in R$ .

Let  $R$  be a ring and  $\alpha$  an endomorphism of  $R$ . Following [17], the endomorphism  $\alpha$  is said  $\alpha$ -rigid if  $ra(r) = 0$ , then  $r = 0$ . A ring  $R$  is said a rigid ring if it exists a rigid endomorphism  $\alpha$  of  $R$ .

**Proposition 2.5.** *Let  $\sigma$  be an endomorphism of  $R$ . If  $R$  is a reduced ring and  $\sigma$ -compatible like right  $R$ -module, then  $R$  is a  $\sigma$ -rigid ring and hence satisfies SA1' and SA2'.*

*Proof.* We only prove the case of SA2' because the other are similar. We claim that  $R[[x; \sigma]]$  is a reduced ring. In fact, let  $f(x) = \sum_{i=0}^{\infty} a_i x^i$  such that  $(f(x))^2 = 0$ . We have that  $a_0^2 = 0$ . Since  $R$  is reduced, then  $a_0 = 0$ . Next, we have  $a_1 \sigma(a_1) = 0$ , since  $R$  is  $\sigma$ -compatible and reduced, then  $a_1 = 0$ . By induction, we get  $f(x) = 0$ . Hence  $R[[x; \sigma]]$  is reduced. Using the same ideas of [14, Proposition 3], we have that  $R$  is  $\sigma$ -rigid and using similar ideas of [14, Corollary 4], we obtain that  $R$  satisfies SA2'.  $\square$

Without the assumption that  $R$  is  $\sigma$ -compatible, Proposition 2.5 is not true. In fact, let  $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$  and  $\sigma : R \rightarrow R$ , defined by  $\sigma((a, b)) = (b, a)$ . By [14, Example 2],  $R$  does not satisfy SA2' because  $R$  does not satisfy SA1'. Observe that  $(1, 0)(0, 1) = (0, 0)$  but  $(1, 0)\sigma(0, 1) \neq (0, 0)$  and so  $R$  is not  $\sigma$ -compatible. We have the following natural questions.

## Questions

- (i) Let  $\sigma$  be an endomorphism of  $R$ . Suppose that  $R$  satisfies SA2'. Is  $R$   $\sigma$ -compatible like right  $R$ -module?
- (ii) Let  $\sigma$  be an endomorphism of  $R$ . Suppose that  $R$  is  $\sigma$ -compatible like right  $R$ -module. Does  $R$  satisfy SA2'?

The question (i) is false. Let  $R_0$  be any domain and  $R = R_0[x]$ . Let  $\sigma : R \rightarrow R$  be defined by  $\sigma(t) = 0$  and  $\sigma|_{R_0} = id_{R_0}$ . By [16, Example 4.1],  $R$  is not  $\sigma$ -compatible and using the similar ideas of the proof of [14, Proposition 10], we have that  $R$  satisfies SA2' and consequently  $R$  satisfies SA1'.

The question (ii) is false. Let  $R = K[x, y]/(x^2, y^2)$ , where  $K$  is a field of characteristic 2, and consider  $T = M_2(R)$ . In this case, take  $\sigma = id_T$ . By [18, Example 3.6],  $S$  does not satisfy SA2' because  $T$  does not satisfy SA1'. Moreover,  $T$  is  $\sigma$ -compatible like right  $T$ -module.

In [19] the authors introduced the following version of skew Armendariz rings.

- (i) Suppose that  $\sigma$  is an endomorphism of  $R$ . Let  $f(x) = \sum_{i=0}^n a_i x^i$ ,  $g(x) = \sum_{j=0}^m b_j x^j \in R[x; \sigma]$  such that  $f(x)g(x) = 0$  implies  $a_i b_j = 0$  for all  $0 \leq i \leq n$  and  $0 \leq j \leq m$ .
- (ii) Suppose that  $\sigma$  is an endomorphism of  $R$ . Let  $f(x) = \sum_{i \geq 0} a_i x^i$ ,  $g(x) = \sum_{j \geq 0} b_j x^j \in R[[x; \sigma]]$  such that  $f(x)g(x) = 0$  implies  $a_i b_j = 0$  for all  $i \geq 0$  and  $j \geq 0$ .

Note that the item (i) above in [20, Definition 1.1] the authors called it by  $\sigma$ -Armendariz, the item (ii) above is similar with [20, Definition 1.1] and we call it here by  $\sigma$ -power Armendariz.

In the next proposition, we give a relationship between the definition above and the skew versions of Armendariz rings used in this paper. Using [21, Lemma 2.1] and [20, Theorem 1.8], the proof of next proposition is easy to verify.

**Proposition 2.6.** *Let  $\sigma$  be an endomorphism of  $R$  and suppose that  $R$  is  $\sigma$ -compatible like right  $R$ -module. Then*

- (i)  $R$  satisfies SA1' if and only if  $R$  is  $\sigma$ -Armendariz;
- (ii)  $R$  satisfies SA2' if and only if  $R$  is  $\sigma$ -power Armendariz.

The proposition above without the compatibility assumption is not true according to [20, Example 1.9] and the authors in [22, Theorem 2.2] obtained an approach of the result above without the compatibility assumption.

The following proposition is a generalization of [18, Proposition 3.4] and partially generalizes [15, Proposition 2.6].

**Lemma 2.7.** *Let  $S$  be any of the rings  $R[x; \sigma]$  and  $R[[x; \sigma]]$ . The following conditions are equivalent:*

- (i)  $R$  satisfies SA2' (SA1');
- (ii)  $\phi : r\text{Ann}_R(2^R) \rightarrow r\text{Ann}_S(2^S)$  defined by  $\phi(J) = JS$  is bijective;
- (iii)  $\Psi : l\text{Ann}_R(2^R) \rightarrow l\text{Ann}_S(2^S)$  defined by  $\Psi(J) = SJ$  is bijective.

*Proof.* We only prove the proposition in the case of SA2' because the equivalence of (i) and (ii) when  $R$  satisfies SA1' was proved in [23, Proposition 3.2]. The equivalence between (i) and (iii) when  $R$  satisfies SA1' has similar proof.

(i)  $\rightarrow$  (ii). It is only necessary to show that  $\phi$  is surjective. For an element  $f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x; \sigma]]$ , define  $C_{f(x)} = \{\sigma^{-i}(a_i), i \geq 0\}$ , and for a subset  $T$  of  $R[[x; \sigma]]$ , we denote the set

$\cup_{f(x) \in T} C_{f(x)}$  by  $C_T$ . We show that  $r_{R[[x;\sigma]]}(f(x)) = r_{R[[x;\sigma]]}(C_{f(x)})$ . In fact, given  $g(x) = \sum_{j=0}^{\infty} b_j x^j$  in  $r_{R[[x;\sigma]]}(f(x))$ , we have  $f(x)g(x) = 0$ . Since  $R$  satisfies SA2', then  $a_i \sigma^i(b_j) = 0$  for all  $i \geq 0$  and  $j \geq 0$ . In particular,  $\sigma^{-i}(a_i)b_j = 0$  for all  $i \geq 0$  and  $j \geq 0$ . Hence  $g(x) \in r_{R[[x;\sigma]]}(C_{f(x)})$ .

On the other hand, let  $h(x) = \sum_{k=0}^{\infty} c_k x^k$  be an element in  $R[[x;\sigma]]$  such that  $C_{f(x)}h(x) = 0$ . It is clear that  $a_i \sigma^i(c_k) = 0$  for all  $i \geq 0$  and  $k \geq 0$ . So  $f(x)h(x) = (0)$ . Since  $R$  satisfies SA2' then  $r_{R[[x;\sigma]]}(T) = r_{R[[x;\sigma]]}(\cup_{f(x) \in T} C_{f(x)})$ . Thus

$$\begin{aligned} r_{R[[x;\sigma]]}(T) &= \bigcap_{f(x) \in T} r_{R[[x;\sigma]]}(f(x)) = \bigcap_{f(x) \in T} r_{R[[x;\sigma]]}(C_{f(x)}) \\ &= \left( \bigcap_{f(x) \in T} r_R(C_{f(x)}) \right) R[[x;\sigma]] = r_R(C_T) R[[x;\sigma]]. \end{aligned} \quad (2.2)$$

Therefore,  $\phi$  is surjective.

(ii)  $\rightarrow$  (i). Let  $f(x) = \sum_{i=0}^{\infty} a_i x^i$  and  $g(x) = \sum_{j=0}^{\infty} b_j x^j$  be elements in  $R[[x;\sigma]]$  such that  $f(x)g(x) = 0$ . By assumption,  $r_{R[[x;\sigma]]}(f(x)) = BR[[x;\sigma]]$ , for some right ideal  $B$  of  $R$ . Hence  $g(x) \in BR[[x;\sigma]]$  and we have that  $b_j \in B \subset r_{R[[x;\sigma]]}(f(x))$  for all  $j \geq 0$ . So  $a_i \sigma^i(b_j) = 0$  for all  $i \geq 0$  and  $j \geq 0$ .

(iii)  $\rightarrow$  (i). Let  $f(x) = \sum_{i \geq 0} a_i x^i$  and  $g(x) = \sum_{j \geq 0} b_j x^j$  be elements in  $R[[x;\sigma]]$  such that  $f(x)g(x) = 0$ . By assumption,  $l_{R[[x;\sigma]]}(g(x)) = R[[x;\sigma]]B$  for some left ideal  $B$  of  $R$ . We can write  $f(x) = \sum_{i \geq 0} x^i \sigma^{-i}(a_i) \in R[[x;\sigma]]B$ . By the equality of the polynomials with the coefficients on the right side, we have that  $\sigma^{-i}(a_i) \in B \subseteq l_{R[[x;\sigma]]}(g(x))$  for all  $i \geq 0$ . So  $a_i \sigma^i(b_j) = 0$  for all  $i \geq 0$  and  $j \geq 0$ .

(i)  $\rightarrow$  (iii). It is only necessary to show that  $\Psi$  is surjective. Let  $f(x) = \sum_{i \geq 0} a_i x^i \in R[[x;\sigma]]$ . Define  $C_{f(x)} = \{a_i, i \geq 0\}$ , and for a subset  $T$  of  $R[[x;\sigma]]$ , we denote the set  $\cup_{f(x) \in T} C_{f(x)}$  by  $C_T$ . We show that

$$l_{R[[x;\sigma]]}(f(x)) = l_{R[[x;\sigma]]}(C_{f(x)}). \quad (2.3)$$

In fact, given  $g(x) = \sum_{j \geq 0} b_j x^j \in l_{R[[x;\sigma]]}(f(x))$ , we have  $g(x)f(x) = 0$ . Since  $R$  satisfies SA2', then  $b_j \sigma^j(a_i) = 0$  for all  $i \geq 0$  and  $j \geq 0$ . Hence  $g(x) = \sum_{j \geq 0} x^j \sigma^{-j}(b_j) \in l_{R[[x;\sigma]]}(C_{f(x)})$ .

On the other hand, let  $g(x) \in R[[x;\sigma]]$  such that  $g(x)C_{f(x)} = 0$ . Thus  $g(x)a_i = 0$  for all  $i \geq 0$ . So  $g(x) \sum_{i \geq 0} a_i x^i = g(x)f(x) = 0$ , and we have that  $g(x) \in l_{R[[x;\sigma]]}(f(x))$ .

We easily have that for each subset  $T$  of  $R[[x;\sigma]]$ ,

$$l_{R[[x;\sigma]]}(T) = l_{R[[x;\sigma]]} \left( \bigcup_{f(x) \in T} C_{f(x)} \right). \quad (2.4)$$

We claim that  $l_{R[[x;\sigma]]}(C_{f(x)}) = R[[x;\sigma]]l_R(C_{f(x)})$ . In fact, let  $g(x) = \sum_{j \geq 0} b_j x^j$  such that  $g(x)C_{f(x)} = 0$ . Then we have that  $0 = g(x)a_i = \sum_{j \geq 0} b_j x^j a_i = \sum_{j \geq 0} x^j \sigma^{-j}(b_j) a_i$ . Thus  $\sigma^{-j}(b_j) \in l_R(C_{f(x)})$ , and it follows that

$$\sum_{j \geq 0} x^j \sigma^{-j}(b_j) \in R[[x;\sigma]]l_R(C_{f(x)}). \quad (2.5)$$

The other inclusion is trivial. So

$$\begin{aligned} l_{R[[x;\sigma]]}(T) &= \bigcap_{f(x) \in T} l_{R[[x;\sigma]]}(C_{f(x)}) = \bigcap_{f(x) \in T} l_{R[[x;\sigma]]}(C_{f(x)}) \\ &= R[[x;\sigma]] \left( \bigcap_{f(x) \in T} l_R(C_{f(x)}) \right) = R[[x;\sigma]]l_R(C_T). \end{aligned} \quad (2.6)$$

Therefore,  $\Psi$  is surjective.  $\square$

Now we are able to prove the main results of this paper.

**Theorem 2.8.** *Let  $\sigma$  be an automorphism of  $R$ .*

(i) *Suppose that  $R$  satisfies SA1'. The following conditions are equivalent:*

- (a)  *$R$  is a right (left) zip ring;*
- (b)  *$R[x; \sigma]$  is a right (left) zip ring;*
- (c)  *$R[x, x^{-1}, \sigma]$  is a right (left) zip ring.*

(ii) *Suppose that  $R$  satisfies SA2'. The following conditions are equivalent:*

- (a)  *$R$  is right (left) zip ring;*
- (b)  *$R[[x; \sigma]]$  is right (left) zip ring;*
- (c)  *$R[[x, x^{-1}; \sigma]]$  is right (left) zip ring.*

*Proof.* (i) We will show the right case because the left case is similar.

Suppose that  $R[x; \sigma]$  is right zip. Let  $X$  be a subset of  $R$  such that  $r_R(X) = 0$ , and  $f(x) = \sum_{i=0}^n a_i x^i \in R[x; \sigma]$  such that  $Xf(x) = 0$ . Thus  $a_i \in r_R(X) = 0$  and it follows that  $f(x) = 0$ . By assumption, there exists  $X_1 = \{x_0, \dots, x_n\}$  such that  $r_{R[x; \sigma]}(X_1) = 0$ . Hence  $r_R(X_1) = r_{R[x; \sigma]}(X_1) \cap R = (0)$ .

Conversely, let  $Y \subseteq R[x; \sigma]$  such that  $r_{R[x; \sigma]}(Y) = 0$ . By Lemma 2.7,  $r_{R[x; \sigma]}(Y) = r_R(T)R[x; \sigma]$ , where  $T = C_Y = \cup_{f(x) \in Y} C_{f(x)}$  such that  $C_{f(x)} = \{\sigma^{-i}(a_i) : 0 \leq i \leq n\}$  with  $f(x) = \sum_{i=0}^n a_i x^i \in Y$ . We have that  $r_R(T) = 0$  and, by assumption, there exists  $T_1 = \{\sigma^{-i_1}(a_{i_1}), \dots, \sigma^{-i_n}(a_{i_n})\}$  such that  $r_R(T_1) = 0$ . For each  $\sigma^{-i_j}(a_{i_j}) \in T_1$ , there exists  $g_{a_{i_j}}(x) \in Y$  such that some of the coefficients of  $g_{a_{i_j}}(x)$  are  $a_{i_j}$  for each  $1 \leq j \leq n$ . Let  $Y_0$  be a minimal subset of  $Y$  such that  $g_{a_{i_j}}(x) \in Y_0$  for each  $1 \leq j \leq n$ . Then  $Y_0$  is nonempty finite subset of  $Y$ . Set  $T_0 = \cup_{f(x) \in Y_0} (C_{f(x)})$  and we have that  $T_1 \subseteq T_0$ . Hence  $r_R(T_0) \subseteq r_R(T_1) = 0$ . By Lemma 2.7,  $r_{R[x; \sigma]}(Y_0) = r_R(T_0)R[x; \sigma]$  and it follows that  $r_{R[x; \sigma]}(Y_0) = 0$ .

The proofs of (a) $\Leftrightarrow$ (c) and of item (ii) follow similarly.  $\square$

Let  $\sigma$  be an endomorphism of  $R$  and  $\delta : R \rightarrow R$  an additive map of  $R$ . The application  $\delta$  is said to be a  $\sigma$ -derivation if  $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ . The Ore extension  $R[x; \sigma, \delta]$  is the set of polynomials  $\sum_{i=0}^n a_i x^i$  with the usual sum, and the multiplication rule is  $xa = \sigma(a)x + \delta(a)$ .

Following [16],  $R$  is said to be  $(\sigma, \delta)$ -compatible, where  $\sigma$  is an endomorphism of  $R$  and  $\delta$  is a  $\sigma$ -derivation of  $R$  if  $ab = 0 \Leftrightarrow a\sigma(b) = 0$  and  $ab = 0$  implies that  $a\delta(b) = 0$ .

In the next result we obtain a necessary and sufficient condition for  $R[x; \sigma, \delta]$  to be left zip, when  $\sigma$  is an endomorphism of  $R$  using the skew version of Armendariz rings of [19].

**Theorem 2.9.** *Let  $\sigma$  be an endomorphism of  $R$  and  $\delta$  a  $\sigma$ -derivation of  $R$ . Suppose that if  $f(x)g(x) = 0$  for  $f(x) = \sum_{i=0}^n a_i x^i$  and  $g(x) = \sum_{j=0}^m b_j x^j \in R[x; \sigma, \delta]$ , then  $a_i b_j = 0$  for all  $0 \leq i \leq n$  and  $0 \leq j \leq m$ . Then  $R$  is left zip if and only if  $R[x; \sigma, \delta]$  is left zip.*

*Proof.* Let  $X$  be any subset of  $R[x; \sigma, \delta]$  and  $C_X = \cup_{f(x) \in X} C_{f(x)}$ , where  $C_{f(x)} = \{a_i, 0 \leq i \leq n\}$  with  $f(x) = \sum_{i=0}^n a_i x^i$ . Suppose that  $l_{R[x; \sigma, \delta]}(X) = 0$ . We clearly have  $l_R(C_X) = 0$ . By assumption, there exists  $\{b_0, \dots, b_t\} \subseteq C_X$  such that  $l_R(Y) = 0$ . Let  $f_{b_i}(x) \in X$  be an element of  $X$  with some of its coefficients are equal to  $b_i$  for all  $1 \leq i \leq t$ . Take  $X_0$  be a minimal subset of  $X$  with this property. We clearly have that  $X_0$  is a finite set. We claim that  $l_{R[x; \sigma, \delta]}(X_0) = 0$ . In fact, we

easily have  $l_R(C_{X_0}) = 0$ , where  $C_{X_0} = \cup_{f(x) \in X_0} C_{f(x)}$  with  $C_{f(x)}$  being defined as before. Next, let  $g(x) = \sum_{j=0}^m b_j x^j$  such that  $g(x)X_0 = 0$ . Hence for any  $f(x) = \sum_{i=0}^n a_i x^i \in X_0$ ,  $g(x)f(x) = 0$ , and we have, by assumption,  $b_j a_i = 0$  for all  $0 \leq j \leq m$  and  $0 \leq i \leq n$ . Thus  $b_j C_{X_0} = 0$  for all  $0 \leq j \leq m$  and it follows that  $g(x) = 0$ . So  $l_{R[x;\sigma,\delta]}(X_0) = 0$ .

Using the methods of Theorem 2.8, the converse follows.  $\square$

*Remark 2.10.* Let  $R$  be a ring and  $\sigma$  an endomorphism of  $R$ . Suppose that  $R$  is  $\sigma$ -power Armendariz and left zip. Using similar methods of [20, Theorem 1.8],  $R$  satisfies SA2' and with similar ideas of Theorem 2.9, we have that  $R$  is a left zip ring if and only if  $R[[x;\sigma]]$  is a left zip ring.

### 3. Examples

In this section, we present some examples of rings that satisfy SA1' and SA2', and they are zip rings. Moreover, an example of a  $\sigma$ -rigid ring that is a zip ring is given.

*Example 3.1.* Let  $F$  be any field and  $\sigma : F \rightarrow F$  any automorphism of  $F$ . Following [14, page 113], we consider the ring  $T(F, F)$  with automorphism  $\bar{\sigma}(a, b) = (\sigma(a), \sigma(b))$  and we denote it by  $\sigma$ . Note that

$$T(F, F) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in F \right\}. \quad (3.1)$$

By [14, Proposition 15],  $T(F, F)$  satisfies SA1', and using similar methods, we can prove that  $T(F, F)$  satisfies SA2'. We claim that  $T(F, F)$  is a zip ring. In fact, the unique one-sided ideals of  $T(F, F)$  are  $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ ,

$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F \right\}, \quad (3.2)$$

and  $T(F, F)$ . Note that  $r_{T(F, F)}(I) \neq \{0\}$  and  $l_{T(F, F)}(I) \neq 0$ . So we easily have that  $T(F, F)$  is a zip ring.

*Example 3.2.* Let  $F$  be any field and  $\sigma$  a monomorphism of  $F$ , and let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c \in F \right\} \quad (3.3)$$

with usual addition and multiplication of matrix. Note that the monomorphism  $\sigma$  is naturally extended to  $R$ , and  $R$  has the following one-sided ideals:

$$I_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} : a \in F \right\}, \quad I_2 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c \in F \right\}, \quad (3.4)$$

$R$  and the zero ideal. We easily have  $r_R(I_2) \neq 0$ ,  $l_R(I_2) \neq 0$ ,  $r_R(I_1) \neq 0$ , and  $l_R(I_1) \neq 0$ . Now we clearly have that  $R$  is a zip ring and by [14, Proposition 17],  $R$  satisfies SA1', and with similar methods of [14, Proposition 17], we can prove that  $R$  satisfies SA2'.

*Example 3.3.* Let  $D$  be any domain with identity,  $R = D[x]$ ,  $\sigma$  an endomorphism of  $R$  defined by  $\sigma(f(x)) = f(0)$ . Since  $R$  is a domain, then  $R$  is right and left zip. Moreover, using similar methods of [14, Example 5], we have that  $R$  satisfies SA1' and SA2'.

*Example 3.4.* Let  $D$  and  $D_1$  be any domains,  $\sigma$  an monomorphism of  $D$ , and  $\tau$  an monomorphism of  $D_1$ . Set  $R = D \times D_1$  with usual addition and multiplication, and we define an endomorphism  $\gamma$  of  $R$  by  $\gamma(a, b) = (\sigma(a), \tau(b))$ . We easily have that  $\gamma$  is a monomorphism of  $R$ . Since  $D$  is  $\sigma$ -rigid and  $D_1$  is  $\tau$ -rigid, we easily obtain that  $R$  is  $\gamma$ -rigid. We claim that  $R$  is left and right zip. In fact, let  $I$  be any left ideal of  $R$ . It is well known that  $I = A \times B$ , where  $A$  is a left ideal of  $D$  and  $B$  is a left ideal of  $D_1$ . Suppose that  $r_R(I) = 0$ . Then  $A \neq 0$  and  $B \neq 0$ . It is not difficult to show that  $r_D(A) = 0$  and  $r_{D_1}(B) = 0$ . Since  $D$  and  $D_1$  are left zip, then there exists a left finitely generated ideal  $L$  of  $D$  contained in  $A$  such that  $r_D(L) = 0$  and a left finitely generated ideal  $L_1$  of  $D_1$  contained in  $B$  such that  $r_{D_1}(L_1) = 0$ . Thus  $r_R(L \times L_1) = 0$  and  $L \times L_1$  is a left finitely generated ideal of  $R$  contained in  $A \times B$ . Hence  $R$  is left zip. Using similar methods, we have that  $R$  is right zip.

*Example 3.5.* Let  $F$  be a field,  $\sigma$  an automorphism of  $F$ ,

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c \in F \right\}, \quad (3.5)$$

and  $D$  a domain with automorphism  $\tau$ . Set  $T = R \times D$  and we define an endomorphism  $\gamma$  of  $T$  by  $\gamma(a, b) = (\sigma(a), \tau(b))$ . It is clear that  $\gamma$  is an automorphism of  $T$  and it is not difficult to show that  $T$  satisfies SA1' and SA2' because  $R$  and  $D$  satisfy SA1' by [14, Proposition 17] and [14, Proposition 10], respectively, and using similar methods of [14, Proposition 17] and [14, Proposition 10],  $R$  and  $D$  satisfy SA2', respectively.

Using similar methods of Example 3.4, we have that  $T$  is right and left zip and note that  $T$  is not  $\gamma$ -rigid, since  $T$  is not a reduced ring.

## Acknowledgment

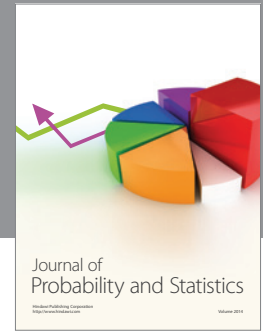
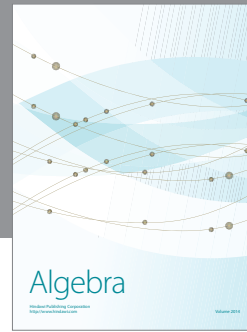
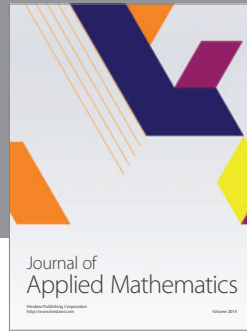
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