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Decaimento assintótico de soluções das equações da Magneto-Hidrodinâmica incompressível em $\mathbb{R}^n, 2 \leq n \leq 4$.

por

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Tese de doutorado

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Resumo

O intuito deste trabalho foi estudar as equações da Magneto-Hidrodinâmica (MHD) incompressível em $\mathbb{R}^n$, com $2 \leq n \leq 4$. Começamos investigando o problema assintótico de Leray para essas equações e a partir da prova generalizamos este resultado, analisando o comportamento das soluções em outras normas e estendendo para suas derivadas de ordem m. Além disso, uma desigualdade fundamental recentemente descoberta para as equações de Navier-Stokes foi aqui estendida para o sistema MHD.
Palavras-chave Equações da Magneto-Hidrodinâmica. Problema de Leray. Comportamento assintótico.

Abstract

The purpose of this paper was to study the incompressible Magneto-Hydrodynamic (MHD) equations in $\mathbb{R}^n$, with $2 \leq n \leq 4$. We begin by investigating the Leray's assymptotic problem for these equations and from its proof, we generalize this result by analyzing the behavior of the solutions in others norms and we also extend the analysis to their derivatives of ordem m. Moreover, a fundamental inequality recently discovered for solutions of the Navier-Stokes equations was extended here to the larger systems of the MHD equations.
Keywords Magneto-hydrodynamic equations. Leray's problem. Asymptotic behavior.

Sumário

Capítulo 1

Introdução

Por volta de 1920, começou-se o estudo da área conhecida como Magneto-Hidrodinâmica (MHD). Área que estuda a interação entre campos magnéticos e fluidos em movimento. Estes campos magnéticos influenciam fluidos naturais como por exemplo a formação das nuvens, estrelas e manchas solares, e também, os produzidos pelos homens, como o aquecimento dos metais líquidos. A ideia é que campos magnéticos afetam o movimento dos fluidos e tais fenômenos, por sua vez, modificam o campo magnético novamente.

Nas indústrias, temos um exemplo bem prático de um sistema que as equações MHD modelam. Devido aos sais minerais, acontece a cristalização dos minerais nas tubulações. O campo magnético centraliza as moléculas do fluido ao longo do fluxo, afetando a cinética da cristalização do minerais. Com isso, é possível estudar mecanismos para mantê-los dissolvidos e evitar a incrustação nas superfícies internas das tubulações.

O sistema MHD aplica-se em diversas áreas, como geofísica, física nuclear, engenharia e matemática. Em um primeiro momento este campo foi estudado por engenheiros, um dos que mais destacou-se foi o engenheiro J.Hatmann, que neste período investigou a construção da bomba eletrodinâmica. Porém, devido à falta de investimentos, o desenvolvimento neste campo ficou comprometido por um longo período e só retornou a partir de 1960.

Já na física houve uma consolidação desta área de pesquisa em 1940. Os físicos de plasmas e astrofísicos começaram o estudo sobre os sistemas MHD com interesse em compreender melhor eventos relacionados ao Universo, como o surgimento da Terra. Um grande reconhecimento deste ramo deu-se através do físico A.Hannes, pesquisador que estudou as equações MHD e suas aplicações na física de plasmas. E em virtude deste trabalho ganhou o prêmio Nobel da Física, em 1970.

A MHD está diretamente relacionada com duas outras grandes áreas, o eletromagnetismo e a fluidodinâmica, de onde decorrem os estudos das equações de

Maxwell e Navier-Stokes, respectivamente. Então, em nosso sistema, temos a equação que representa o movimento do fluido, dada por:

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \mu \Delta \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \tag{1.1}$$

descrevendo a evolução do campo de velocidade. Esta equação modela, por exemplo, correntes oceânicas, propagação de fumaça em incêndios e fluxos ao redor de turbinas de aviões.
Já para descrever a evolução do campo magnético, temos:

$$\boldsymbol{b}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{b} = \nu \Delta \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{u}, \tag{1.2}$$

também conhecida como equação de indução, que retrata como um campo magnético interage com um circuito elétrico, para produzir uma força. Temos como exemplos os motores elétricos, transformadores e geradores. Sua derivação segue das equações de Ohm's, Ampère e Faraday.

Neste trabalho os fluidos são incompressíveis, ou seja, fluidos que apresentam resistência à redução de seu volume, como líquidos, gases e plasmas. Esta condição em nosso sistema é descrita pela equação:

$$\nabla \cdot \boldsymbol{u}(\cdot, t) = 0. \tag{1.3}$$

E por último, temos a equação:

$$\nabla \cdot \boldsymbol{b}(\cdot, t) = 0, \tag{1.4}$$

conhecida como Lei de Gauss do magnetismo.

Em resumo, temos dois sistemas acoplados, onde (1.1) e (1.3) resultam em um problema de mecânica dos fluidos, e (1.2) e (1.4), de eletromagnetismo. Em [4, 10, 26, 32], temos um estudo mais detalhado destas equações e suas derivações.

Na matemática também existe um grande interesse no estudo dessas equações. Em 1934, o matemático Leray, em seu famoso artigo [14], provou a existência de soluções fracas de energia finita $\boldsymbol{u}(\cdot, t) \in L^\infty([0,\infty), \mathbf{L}^2_\sigma(\mathbb{R}^3)) \cap C_w([0,\infty), \mathbf{L}^2(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{\mathbf{H}}^1(\mathbb{R}^3))$ para o problema de Navier Stokes em $\mathbb{R}^3$. Apesar da unicidade destas soluções ainda estar em aberto, Leray provou que, para um instante de tempo $t_* \geq 0$ suficientemente grande,

$$\boldsymbol{u} \in C^\infty(\mathbb{R}^3 \times [t_*, \infty))$$

e para $m \geq 0$ qualquer,

$$\boldsymbol{u}(\cdot, t) \in C^0([t_*, \infty), \mathbf{H}^m(\mathbb{R}^3)),$$

onde $\mathbf{H}^m(\mathbb{R}^3) = H^m(\mathbb{R}^3)\times H^m(\mathbb{R}^3)\times H^m(\mathbb{R}^3),\ H^m(\mathbb{R}^3) = \{v \in L^2(\mathbb{R}^3) : D^m v \in L^2(\mathbb{R}^3)\}$.
Analogamente, podemos construir soluções fracas de Leray para o sistema MHD e também obter que existe $t_* \geq 0$, suficientemente grande, tal que

$$(\boldsymbol{u},\boldsymbol{b}) \ \in C^\infty(\mathbb{R}^3 \times [t_*,\infty)) \tag{1.5}$$

e, para $m \geq 0$ qualquer,

$$(\boldsymbol{u},\boldsymbol{b})(\cdot,t) \ \in \ C^0([t_*,\infty),\mathbf{H}^m(\mathbb{R}^3)). \tag{1.6}$$

Neste trabalho, vamos usar os resultados (1.5) e (1.6) e estudamos propriedades de soluções análogas de Leray

$$(\boldsymbol{u},\boldsymbol{b})(\cdot,t) \ \in \ L^\infty((0,\infty),\boldsymbol{L}^2(\mathbb{R}^n)) \cap L^2((0,\infty),\dot{\boldsymbol{H}}^1(\mathbb{R}^n)) \cap C^0_w([0,\infty),\boldsymbol{L}^2(\mathbb{R}^n))$$

para o problema MHD incompressível em dimensão $2 \leq n \leq 4$, dado por:

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \mu\Delta\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}, \\ \boldsymbol{b}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \nu\,\Delta\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \qquad \nabla\cdot\boldsymbol{b}(\cdot,t) = 0, \end{cases} \tag{1.7}$$

onde $\boldsymbol{u} = \boldsymbol{u}(x,t)$ denota a velocidade do fluido, $\boldsymbol{b} = \boldsymbol{b}(x,t)$ o campo magnético e $p = p(x,t)$ a pressão total do fluido. E ainda, $\mu > 0$ conhecida como constante cinemática, $\nu > 0$ constante de difusão magnética e $(\boldsymbol{u}_0,\boldsymbol{b}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n)\times\boldsymbol{L}^2_\sigma(\mathbb{R}^n)$ dados iniciais, onde $\boldsymbol{L}^2_\sigma(\mathbb{R}^n) = \{\mathbf{v} \in \ \boldsymbol{L}^2(\mathbb{R}^n) : \nabla\cdot\mathbf{v} = 0,$ no sentido das distribuições$\}$.
Para o problema acima em dimensão n=2, tem-se a existência e unicidade de solução clássica, a prova foi dada por Sermange e Teman [23]. Já o caso em que a dimensão é n=3, essa questão segue em aberto. Muitos autores investigam existência e unicidade de soluções em MHD, alguns resultados podem ser encontrados em [3, 5, 11, 15, 22, 30, 34].
É interessante observar que sem o campo magnético em nosso sistema, temos as equações de Navier-Stokes, existindo assim uma analogia entre o estudo dessas equações. Para conhecer alguns resultados sobre as equações de Navier-Stokes, pode-se consultar [6, 7, 9, 16, 18, 24, 33].

O objetivo principal deste trabalho foi investigar resultados de decaimento assintótico para as soluções de (1.7), sempre visando melhorar as propriedades encontradas. Obtemos, assim, uma desigualdade fundamental, onde para $\alpha \geq 0$ e $m \geq 0$ qualquer, sempre que $\limsup_{t\to\infty} t^\alpha \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty$, tem-se

$$\lim_{t\to\infty}\sup t^{\alpha+\frac{m}{2}}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha,m)C^{-m/2}\lim_{t\to\infty}\sup t^\alpha\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \tag{1.8}$$

onde $C = \min\{\mu, \nu\}$ e $2 \leq n \leq 4$.
Utilizamos a técnica desenvolvida em [27] para as equações de Navier-Stokes. Com esta prova, foi possível estender alguns resultados anteriores.
Condições adicionais sobre $(\boldsymbol{u_0}, \boldsymbol{b_0}) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n) \times \boldsymbol{L}^2_\sigma(\mathbb{R}^n)$ garantindo que se tenha $\limsup_{t\to\infty} t^\alpha \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty$ são dadas em [9] (no caso de Navier-Stokes, ver [8, 24, 35]). Por exemplo: se $(\boldsymbol{u_0}, \boldsymbol{b_0}) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n) \times \boldsymbol{L}^2_\sigma(\mathbb{R}^n) \cap \boldsymbol{L}^1(\mathbb{R}^n) \times \boldsymbol{L}^1(\mathbb{R}^n)$, então segue que

$$\limsup t^{n/4} \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty.$$

(Ver [8, 9, 22, 24]).
Sob condições muito especiais (a saber $\hat{\boldsymbol{u}}_{\boldsymbol{0}}(\boldsymbol{\xi})$=0, $\hat{\boldsymbol{b}}_{\boldsymbol{0}}(\boldsymbol{\xi})$=0 para todo ξ numa vizinhança de zero), as condições podem decair exponencialmente, tendo-se

$$\limsup_{t\to\infty} e^{\lambda t} \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty,$$

para algum $\lambda > 0$. Neste caso, as técnicas apresentadas no Capítulo 7 para obtenção de (1.8) produzem, de modo análogo, para todo $m \geq 1$ e $C = \min\{\mu, \nu\}$ a desigualdade

$$\limsup_{t\to\infty} e^{\lambda t} t^{-\frac{m}{2}} \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq K(\lambda, m) C^{-m/2} \limsup_{t\to\infty} e^{\lambda t} \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \quad (1.9)$$

onde $K(\lambda, m) = \frac{1}{\sqrt{m!}} 2^{m/2} \lambda^m$ e $2 \leq n \leq 4$.

Todos os resultados aqui discutidos supõem dimensão $n \leq 4$; para $n \geq 5$, os argumentos aqui usados não se aplicam (dado que não valem as desigualdades de Sobolev que seriam então necessárias). Alguns dos resultados obtidos podem ser estendidos para $n \geq 5$ (ver [24]), mas condições adicionais são necessárias.

Essa tese foi dividida do seguinte modo: começamos pelo capítulo 2, onde esclarecemos a notação usada e revisamos vários resultados preliminares que serão utilizados nos capítulos seguintes.

No capítulo 3, provamos que as soluções do problema (1.7), satisfazem a desigualdade de energia:

$$\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} + 2\lambda \int_{t_0}^t \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau \leq \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t_0)\|^2_{L^2(\mathbb{R}^n)}, \quad (1.10)$$

$\forall\ t > t_0$ ($t_0 > 0$ suficientemente grande) e $\lambda = \min(\mu, \nu)$.

No capítulo 4, estabelecemos a desigualdade referente às derivadas de primeira ordem da solução $(\boldsymbol{u}, \boldsymbol{b})$:

$$\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} + C \int_{t_0}^t \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot, t_0)\|^2_{L^2(\mathbb{R}^n)}, \quad (1.11)$$

para $C > 0$ constante, e $\forall t > t_0$ ($t_0 > 0$ suficientemente grande).
Uma importante decorrência deste fato é a propriedade de monotonicidade das derivadas primeiras.
Com a demonstração da desigualdade (1.11) e a consequência acima, obtemos o primeiro resultado assintótico para as derivadas primeiras das soluções, dado por:

$$\lim_{t\to\infty} t^{1/2}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0, \tag{1.12}$$

prova adaptada de [8, 9].

No capítulo 5, utilizando os resultados anteriores, foi possível resolver o chamado **problema de Leray**: dadas soluções do problema MHD incompressível, temos

$$\lim_{t\to\infty} \|(\boldsymbol{u}, \boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0, \tag{1.13}$$

$2 \leq n \leq 4$.

A motivação para o estudo deste problema deu-se através da questão deixada em aberto por Leray em seu trabalho de 1934 [14], p.248. Trata-se da validade ou não de (1.13) acima, para as equações de Navier Stokes. Esta questão foi finalmente resolvida em 1984, por Kato [33], Masuda [16], e outros. É interessante observar que em [18], os autores provaram este teorema utilizando resultados obtidos por Leray e somente técnicas matemáticas conhecidas em 1934.

No capítulo 6, generalizamos as estimativas (1.10) e (1.11), provando que para $m \geq 0$ qualquer, as soluções do sistema (1.7) satisfazem para $2 \leq n \leq 4$:

$$\begin{aligned}(t-t_0)^m\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + C_m\int_{t_0}^{t}(\tau-t_0)^m\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\ \leq m\int_{t_0}^{t}(\tau-t_0)^{m-1}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau,\end{aligned} \tag{1.14}$$

$\forall\, t >\ t_0$ ($t_0 > 0$ suficientemente grande), onde $C_m > 0$.
E através da desigualdade (1.14) provamos que, para $m \geq 0$ qualquer, vale a propriedade mais geral:

$$\lim_{t\to\infty} t^{m/2}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0, \tag{1.15}$$

$2 \leq n \leq 4$, onde $D^m\vec{v}(.,t)$ denota as derivadas espaciais de ordem m de $\vec{v}(.,t)$.
Note que os casos m=1 e m=0, já foram estabelecidos em (1.12) e (1.13), respectivamente.
O resultado (1.15) foi obtido anteriormente para as equações de Navier-Stokes em [12, 20] no caso n=2 e em [21] no caso n=3, [8] nos casos n=2,3.

Decorrente de (1.15) e Teoremas de interpolação que estão expostos no capítulo 2, segue algumas estimativas de decaimento para soluções do sistema (1.7), dadas por:

$$\lim_{t\to\infty} t^{\frac{s}{2}} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} = 0,$$

$$\lim_{t\to\infty} t^{\frac{n}{4}} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = 0,$$

$$\lim_{t\to\infty} t^{\frac{n}{4}-\frac{n}{2q}} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0,$$

para $s \geq 0$, $2 \leq n \leq 4$ e $2 \leq q \leq \infty$.
E ainda para derivadas de ordem m:

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{4}} \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = 0,$$

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{4}-\frac{n}{2q}} \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0,$$

para $s \geq 0$, $2 \leq n \leq 4$ e $2 \leq q \leq \infty$.
Para $q < 2$, é preciso impor condições adicionais sobre $(\boldsymbol{u_0},\boldsymbol{b_0}) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n)\times\boldsymbol{L}^2_\sigma(\mathbb{R}^n)$ como, por exemplo, $(\boldsymbol{u_0},\boldsymbol{b_0}) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n)\times\boldsymbol{L}^2_\sigma(\mathbb{R}^n) \cap \boldsymbol{L}^1(\mathbb{R}^n)\times\boldsymbol{L}^1(\mathbb{R}^n)$ para que se possa ter $(\boldsymbol{u},\boldsymbol{b})(\cdot,t) \in \boldsymbol{L}^q(\mathbb{R}^n)\times\boldsymbol{L}^q(\mathbb{R}^n)$. Mesmo sob estas condições adicionais quase nada é conhecido sobre o comportamento de $\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}$ (ou $\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}$) ao $t\to\infty$.

E por fim, no capítulo 8, temos o resultado principal (1.8), onde conseguimos generalizar as demais estimativas de decaimento, obtidas nos capítulos anteriores, pois com essa prova conseguimos mostrar que, para todo $s \geq 0$:

$$\limsup_{t\to\infty} t^{\alpha+s/2} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} < \infty$$

se tivermos $\limsup_{t\to\infty} t^{\alpha} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty$, e também

$$\limsup_{t\to\infty} t^{\alpha+s/2} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} = 0\ ,$$

se tivermos $\limsup_{t\to\infty} t^{\alpha} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0$.

Neste trabalho, o leitor irá perceber que alguns dos resultados acima foram provados para n=3. O caso n=2 foi omitido, devido seguir de forma análoga ao caso anterior, utilizando nas demonstrações as desigualdades de Sobolev referentes à dimensão n=2. E, por último, provamos o caso n=4.

Capítulo 2

Preliminares

Neste capítulo, apresentaremos algumas notações, definições e teoremas que serão importantes e aparecerão nas demonstrações ao longo do texto. Em [1, 7, 17, 29], encontra-se as provas destes resultados e uma abordagem mais geral sobre estes assuntos.

2.1 Notações

• $\boldsymbol{u}(x,t) = (u_1(x,t), ..., u_n(x,t))$, denota uma grandeza vetorial.

• $\nabla p \equiv \nabla p(\cdot, t)$, denota o gradiente espacial de $p(\cdot, t)$.

• $\nabla \cdot\ \boldsymbol{u} = D_1 u_1 + ... + D_n u_n$, denota o divergente espacial de $\boldsymbol{u}(\cdot, t)$.

• $\| \cdot \|_{L^q(\mathbb{R}^n)},$ com $1 \leq q < \infty$, denota a norma tradicional do espaço de Lebesgue $L^q(\mathbb{R}^n) = \{\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}; \boldsymbol{u}$ é mensurável e $\int |\boldsymbol{u}|^p \ dx < \infty\}$, então podemos definir:

$$\begin{aligned}
\|\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} &= \left\{ \sum_{i=1}^{n} \int_{\mathbb{R}^n} |u_i(x,t)|^q \ dx \right\}^{1/q}, \\
\|D\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} &= \left\{ \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} |D_j u_i(x,t)|^q \ dx \right\}^{1/q}, \\
\|D^2\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} &= \left\{ \sum_{i,j,l=1}^{n} \int_{\mathbb{R}^n} |D_j D_l u_i(x,t)|^q \ dx \right\}^{1/q}.
\end{aligned}$$

Em geral, para $m \geq 1$:

$$\|D^m\boldsymbol{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \;=\; \left\{\sum_{i=1}^{n}\sum_{j_1=1}^{n}...\sum_{j_m=1}^{n}\int_{\mathbb{R}^n}|D_{j_1}...D_{j_m}u_i(x,t)|^q\,dx\right\}^{1/q}.$$

Para $q=\infty$, temos :

$\|\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;=\; \left\{\max\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} : 1\le i\le n\right\},$
que denota o supremo essencial de $\boldsymbol{u}(\cdot,t)$. Analogamente, denota-se para $\|D\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$, $\|D^2\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$, etc.

• $\mathbf{L}^p_\sigma(\mathbb{R}^n)$ denota o espaço das funções $\boldsymbol{u}\in\mathbf{L}^p(\mathbb{R}^n)$, com $\nabla\cdot\boldsymbol{u}=0$ no sentido das distribuições.

• $\|(\boldsymbol{u},\boldsymbol{b})\|^q_{L^q(\mathbb{R}^n)} := \|\boldsymbol{u}\|^q_{L^q(\mathbb{R}^n)} + \|\boldsymbol{b}\|^q_{L^q(\mathbb{R}^n)}$ denota a norma do par $(\boldsymbol{u},\boldsymbol{b})$. Para $m\ge 1$ inteiro, denota-se:

$$\|(D^m\boldsymbol{u},D^m\boldsymbol{b})\|^q_{L^q(\mathbb{R}^n)} := \|D^m\boldsymbol{u}\|^q_{L^q(\mathbb{R}^n)} + \|D^m\boldsymbol{b}\|^q_{L^q(\mathbb{R}^n)}, \tag{2.1}$$

para todo $1\le q<\infty$. Para $q=\infty$,

$$\|(D^m\boldsymbol{u},D^m\boldsymbol{b})\|_{L^q(\mathbb{R}^n)} := \max\left\{\|D^m\boldsymbol{u}\|_{L^q(\mathbb{R}^n)}, \|D^m\boldsymbol{b}\|_{L^q(\mathbb{R}^n)}\right\}.$$

• Para $s\ge 0$,

$$\|(\boldsymbol{u},\boldsymbol{b})\|^2_{\dot{H}^s(\mathbb{R}^n)} := \|\boldsymbol{u}\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|\boldsymbol{b}\|^2_{\dot{H}^s(\mathbb{R}^n)},$$

onde,

$$\|\boldsymbol{u}\|_{\dot{H}^s(\mathbb{R}^n)} = \left\{\sum_{i=1}^{n}\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{u}_i(\xi)|^2\,d\xi\right\}^{1/2},$$

onde $\hat{\text{u}}_i$ denota a Transformada de Fourier de u_i e $\dot{H}^s(\mathbb{R}^n) = \{\, u\in S'(\mathbb{R}^n), \hat{u}\in L^1_{loc}(\mathbb{R}^n) : \|\hat{u}\|_{\dot{H}^s(\mathbb{R}^n)}<\infty\}$ é o Espaço Sobolev Homogêneo de ordem s.

• Denotaremos as constantes por K, C e C'(que representaram diversos valores numéricos mesmo quando o mesmo símbolo foi usado).

2.2 Teoremas de Análise

Teorema 2.2.1. ***(Teorema de Fubini)***
Suponhamos que $F\in L^1(\Omega_1\times\Omega_2)$. Então, para quase todo $x\in\Omega_1$,

$$F(x,y)\in L^1_y(\Omega_2)\ e\ \int_{\Omega_2}F(x,y)\,dy\;\in\;L^1_x(\Omega_1).$$

De maneira análoga, para quase todo $y \in \Omega_2$, *temos:*

$$F(x,y) \in L^1_x(\Omega_1) \text{ e } \int_{\Omega_1} F(x,y)\, dx \in L^2_y(\Omega_2).$$

Além disso,

$$\int_{\Omega_1} dx \int_{\Omega_2} F(x,y)dy = \int_{\Omega_2} dy \int_{\Omega_1} F(x,y)dx.$$

Teorema 2.2.2. *Se* $f \in L^p(\Omega)$, *para* $1 \leq p < \infty$, *então dado* $\varepsilon > 0$, *existe* $\delta > 0$ *tal que, se* $A \subset \Omega$ *e* $\mu(A) < \delta$, *então*

$$\int_A |f|^p d\mu < \varepsilon.$$

Teorema 2.2.3. ***(Teorema da Convergência Dominada de Lebesgue)*** *Seja* (f_n) *uma sequência de funções integráveis, tal que* $f_n \to f$ *q.t.p. Se existe uma função integrável* g *tal que* $|f_n| \leq g$, $\forall n \in \mathbb{N}$, *então* f *é integrável e*

$$\int f dx = \lim_{n\to\infty} \int f_n\, dx.$$

Teorema 2.2.4. ***(Teorema da Convergência Monótona)*** *Sendo* (f_n) *uma sequência monótona crescente de funções em* $M^+(X, \boldsymbol{X})$ *que converge para f, então*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Teorema 2.2.5. ***(Cauchy-Schwarz)*** *Sendo* $f, g \in L^2(\Omega)$, *então* $f, g \in L^1(\Omega)$ *e*

$$\|f.g\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Teorema 2.2.6. ***(Desigualdade de Hölder)*** *Sejam* $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *com* $1 < p < \infty$ *e* $\frac{1}{p} + \frac{1}{q} = 1$. *Então,* $f.g \in L^1(\Omega)$ *e*

$$\|f.g\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)}\, \|g\|_{L^q(\Omega)}.$$

Teorema 2.2.7. ***(Desigualdade de Minkovski)*** *Sejam* $f, g \in L^p(\Omega)$, *com* $1 \leq p \leq \infty$. *Então,* $f + g \in L^p(\Omega)$ *e*

$$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Teorema 2.2.8. ***(Teorema da Divergência)*** *Se* $u, v \in C^2(\Omega)$ *e* Ω *é um conjunto aberto, com fronteira suave, então:*

$$\int_\Omega u\Delta v\, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \eta} dS - \int_\Omega \nabla u \nabla v\, dx.$$

2.3 Teoremas de interpolação

Teorema 2.3.1. *Se $f \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ para algum $1 \leq p < \infty$, então $f \in L^q(\mathbb{R}^n)$, para cada $p \leq q \leq \infty$, e ainda,*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{q}} \|f\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{p}{q}}.$$

Teorema 2.3.2. *Se $f \in L^2(\mathbb{R}^n) \cap \dot{H}^{s_1}(\mathbb{R}^n)$, para $s_1 > 0$ então, tem-se $f \in \dot{H}^s(\mathbb{R}^n)$, para cada $0 < s < s_1$, com*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{s_1}} \|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}^{\frac{s}{s_1}}.$$

2.4 Desigualdades de Sobolev

Teorema 2.4.1. ***(1ª Desigualdade de Sobolev-Nirenberg-Gagliardo)*** *Para qualquer $0 < p < r \leq \infty$, $1 \leq q \leq \infty$, temos*

$$\|\boldsymbol{u}\|_{L^r(\mathbb{R}^n)} \leq K(m,n,p,q,r) \|\boldsymbol{u}\|_{L^p(\mathbb{R}^n)}^{1-\theta} \|D^m \boldsymbol{u}\|_{L^q(\mathbb{R}^n)}^{\theta},$$

tal que:

$$\frac{1}{r} = \theta \left(\frac{1}{q} - \frac{m}{n} \right) + (1-\theta)\frac{1}{p}, \ \theta \in (0,1),$$

$\forall \ \boldsymbol{u} \in \ L^p(\mathbb{R}^n)$, *tal que* $D^m \boldsymbol{u} \ \in L^q(\mathbb{R}^n)$.

Teorema 2.4.2. ***(2ª Desigualdade de Sobolev-Nirenberg-Gagliardo)*** *Para qualquer $0 < p < r \leq \infty$, $1 \leq q \leq \infty$, $0 \leq j < m$, temos*

$$\|D^j \boldsymbol{u}\|_{L^r(\mathbb{R}^n)} \leq K(m,n,p,q,r) \|\boldsymbol{u}\|_{L^p(\mathbb{R}^n)}^{1-\theta} \|D^m \boldsymbol{u}\|_{L^q(\mathbb{R}^n)}^{\theta},$$

tal que:

$$\frac{1}{r} = \frac{j}{n} + \theta \left(\frac{1}{q} - \frac{m}{n} \right) + (1-\theta)\frac{1}{p}, \ \theta \in (0,1)$$

$\forall \ \boldsymbol{u} \in \ L^p(\mathbb{R}^n)$, *tal que* $D^m \boldsymbol{u} \ \in L^q(\mathbb{R}^n)$.

2.5 Equação do Calor

Dado o problema,

$$\begin{aligned} \boldsymbol{u}_t &= \nu \Delta \boldsymbol{u}, \ t > 0, x \in \mathbb{R}^n \\ \boldsymbol{u}(.,0) &= \boldsymbol{u}_0 \in L^{q_0}(\mathbb{R}^n), \end{aligned}$$

onde $1 \leq q_0 < \infty$. Temos a seguinte estimativa:

$$\|D^\alpha e^{\nu\Delta(t-\tau)}\boldsymbol{u}(.,\tau)\|_{L^q(\mathbb{R}^n)} \leq K(m,n)\|\boldsymbol{u}(.,\tau)\|_{L^r(\mathbb{R}^n)}(\nu(t-\tau))^{\frac{-n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{|\alpha|}{2}}.$$

$\forall\, t > \tau \geq 0, q_0 \leq r \leq q, n \geq 1$ arbitrário e $m = |\alpha|$, onde α é o multi-índice.

Capítulo 3

Desigualdade de Energia para as soluções ($\boldsymbol{u}$,$\boldsymbol{b}$)

Relembrando o sistema MHD incompressível em dimensão $2 \leq n \leq 4$:

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \mu \Delta \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \tag{3.1}$$

$$\boldsymbol{b}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{b} = \nu \Delta \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{u}, \tag{3.2}$$

$$\nabla \cdot \boldsymbol{u}(\cdot, t) = 0, \qquad \nabla \cdot \boldsymbol{b}(\cdot, t) = 0, \tag{3.3}$$

com dados iniciais $(\boldsymbol{u}_0, \boldsymbol{b}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n) \times \boldsymbol{L}^2_\sigma(\mathbb{R}^n)$.

Estamos considerando soluções de Leray $(\boldsymbol{u},\boldsymbol{b})(\cdot, t)$ nas hipóteses comentadas na introdução, ou seja, existe um instante de regularidade $t_* \geq 0$, tal que valem (1.5) e (1.6). Nessas condições, iremos mostrar que vale o seguinte resultado:

Teorema 3.0.1. *Dada $(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)$ solução de Leray do problema MHD incompressível, vale a desigualdade de energia:*

$$\| (\boldsymbol{u}, \boldsymbol{b})(\cdot, t) \|^2_{L^2(\mathbb{R}^n)} + 2\lambda \int_{t_0}^{t} \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \| (\boldsymbol{u}, \boldsymbol{b})(\cdot, t_0) \|^2_{L^2(\mathbb{R}^n)},$$

$\forall\, t > t_0$ ($t_0 > 0$ suficientemente grande) e $\lambda = \min(\mu, \nu)$.

Demonstração. Para que seja válida a desigualdade a ser mostrada, introduziremos uma função de corte auxiliar, onde para $R > 0$ dado, $\phi_R \in C^\infty_0(\mathbb{R}^n)$ definida, por:

$$\phi_R(x) := \begin{cases} 1, \ |x| \leq \mathrm{R} \\ \varphi(x), \mathrm{R} < |x| < \mathrm{R}{+}1 \\ 0, \ |x| \geq \mathrm{R} + 1, \end{cases}$$

com $0 \leq \varphi(x) \leq 1 \in C^\infty(\mathbb{R}^n)$.
Primeiro multiplicando a equação (3.1) por $2u_i\phi_R$ e reescrevendo em coordenadas, temos

$$(u_i^2)_t\,\phi_R + \sum_{j=1}^n D_j(u_i^2)u_j\,\phi_R + 2\,(D_i\,p)\,u_i\phi_R \;=\; 2\mu\sum_{j=1}^n (D_jD_ju_i)\,u_i\phi_R + \sum_{j=1}^n b_j(D_jb_i)\,2u_i\phi_R. \qquad (3.4)$$

Integrando a igualdade (3.4) na região $\mathbb{R}^n_{B_{R+1}} \times [t_0, t]$, para $t > t_0 > t_*$

$$\underbrace{\int_{t_0}^t\int_{B_{R+1}} (u_i^2(x,\tau))_\tau\,\phi_R(x)\,dxd\tau}_{(I)} + \underbrace{\int_{t_0}^t\int_{B_{R+1}}\sum_{j=1}^n D_j(u_i^2(x,\tau))u_j(x,\tau)\,\phi_R(x)\,dxd\tau}_{(II)}$$
$$\underbrace{+2\int_{t_0}^t\int_{B_{R+1}} D_i\,p\,u_i(x,\tau)\phi_R(x)\,dxd\tau}_{(III)} \;=\; \underbrace{2\mu\int_{t_0}^t\int_{B_{R+1}}\sum_{j=1}^n D_jD_ju_i(x,\tau)\,u_i(x,\tau)\phi_R(x)\,dxd\tau}_{(IV)}$$
$$\underbrace{+\,2\int_{t_0}^t\int_{B_{R+1}}\sum_{j=1}^n b_j(x,\tau)D_jb_i(x,\tau)\,u_i(x,\tau)\phi_R(x)\,dxd\tau}_{(V)}\,. \qquad (3.5)$$

Agora, vamos estimar cada termo da igualdade (3.5).
Aplicando Teorema de Fubini e o Teorema Fundamental do Cálculo no primeiro termo, temos

$$(I) = \int_{t_0}^t\int_{B_{R+1}} (u_i^2(x,\tau))_\tau\,\phi_R(x)\,dxd\tau = \int_{B_{R+1}} [{u_i}^2(x,t) - {u_i}^2(x,t_0)]\,\phi_R(x)dx.$$

Para os demais termos, note que $\phi_R(x) = 0, \forall x$ tal que $|x| = R+1$, logo ao integrarmos por partes (II),(III),(IV) e (V), não temos termos de fronteira e, com isso, ficamos:

$$
\begin{aligned}
(II) &= \int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} D_j(u_i^2(x,\tau))u_j(x,\tau)\,\phi_R(x)\,dxd\tau \\
&= -\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} u_i^2(x,\tau)D_ju_j(x,\tau)\,\phi_R(x)\,dxd\tau \\
&- \int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{n}(u_i^2(x,\tau)u_j(x,\tau))\,D_j\phi_R(x)\,dxd\tau, \\
(III) &= 2\int_{t_0}^{t}\int_{B_{R+1}} u_i(x,\tau)\,D_i\,p(x,\tau)\,\phi_R(x)\,dxd\tau \\
&= -2\int_{t_0}^{t}\int_{R<|x|<R+1} u_i(x,\tau)\,p(x,\tau)\,D_i\,\phi_R(x)\,dxd\tau \\
&- 2\int_{t_0}^{t}\int_{B_{R+1}} D_i\,u_i(x,\tau)\,p(x,\tau)\,\phi_R(x)\,dxd\tau, \\
(IV) &= 2\mu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} u_i(x,\tau)D_jD_ju_i(x,\tau)\,\phi_R(x)\,dxd\tau \\
&= -2\mu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n}(D_ju_i(x,\tau))^2\,\phi_R(x)\,dxd\tau \\
&- 2\mu\int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{n} u_i(x,\tau)\,D_ju_i(x,\tau)\,D_j\phi_R(x)dxd\tau, \\
(V) &= 2\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} b_j(x,\tau)D_jb_i(x,\tau)\,u_i(x,\tau)\phi_R(x)dxd\tau \\
&= -2\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} b_j(x,\tau)b_i(x,\tau)\,D_ju_i(x,\tau)\phi_R(x)dxd\tau \\
&- 2\int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{n} b_j(x,\tau)b_i(x,\tau)\,u_i(x,\tau)D_j\phi_R(x)dxd\tau.
\end{aligned}
$$

Fazendo $R \to \infty$ nos termos anteriores, note que onde aparece $D_j\phi_R$ irá para zero, pois $D_j\phi_R = 0,\ \forall x$ tal que $|x| \leq R$ ou $|x| \geq R+1$. Logo, utilizando este fato, a definição da ϕ_R e o Teorema da Convergência Monótona, obtemos

$$
\begin{aligned}
(I) &= \int_{\mathbb{R}^n} {u_i}^2(x,t)dx - \int_{\mathbb{R}^n} {u_i}^2(x,t_0)\,dx = \|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} - \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)},\\
(II) &= -\int_{t_0}^{t}\int_{\mathbb{R}^n}\sum_{j=1}^{n} {u_i}^2(x,\tau)D_j u_j(x,\tau)\,dxd\tau,\\
(III) &= -2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_i u_i(x,\tau)p(x,\tau)\,dxd\tau,\\
(IV) &= -2\mu\int_{t_0}^{t}\int_{\mathbb{R}^n}\sum_{j,i=1}^{n}(D_j u_i(x,\tau))^2\,dxd\tau,\\
(V) &= -2\int_{t_0}^{t}\int_{\mathbb{R}^n}\sum_{j=1}^{n} b_j(x,\tau)b_i(x,\tau)\,D_j u_i(x,\tau)dxd\tau.
\end{aligned}
$$

Então, reescrevendo (3.5)

$$
\begin{aligned}
\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &- \int_{t_0}^{t}\int_{\mathbb{R}^n}\sum_{j=1}^{n}{u_i}^2(x,\tau)D_j u_j(x,\tau)\,dxd\tau - 2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_i u_i(x,\tau)p(x,\tau)\,dxd\tau\\
&+2\mu\int_{t_0}^{t}\int_{\mathbb{R}^n}\sum_{j=1}^{n}(D_j u_i(x,\tau))^2dxd\tau + 2\int_{t_0}^{t}\int_{\mathbb{R}^n}\sum_{j=1}^{n} b_j(x,\tau)b_i(x,\tau)D_j u_i(x,\tau)dxd\tau\\
&= \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}
\tag{3.6}
$$

Agora, iremos trabalhar com a equação (3.2) utilizando as mesmas ideias anteriores. Caso o leitor já esteja familiarizado com as contas, poderá seguir direto para a desigualdade (3.10), onde temos as estimativas encontradas para (3.1) e (3.2). Multiplicando a equação (3.2) por $2b_i\phi_R$ e reescrevendo em coordenadas, temos

$$({b_i}^2)_t\,\phi_R \;+\; \sum_{j=1}^{n} D_j({b_i}^2)u_j\phi_R = 2\nu\sum_{j=1}^{n}\; b_i(D_jD_jb_i)\phi_R \;+\; 2\sum_{j=1}^{n}\; b_i(b_jD_ju_i)\phi_R. \tag{3.7}$$

Integrando (3.7) na região $\mathbb{R}^n_{B_{R+1}} \times [t_0, t]$, para $t_0 > t_*$

$$\underbrace{\int_{t_0}^{t}\int_{B_{R+1}} (b_i(x,\tau))^2_\tau \ \phi_R(x)\ dxd\tau}_{(I)} + \underbrace{\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} D_j(b_i^2(x,\tau))u_j(x,\tau)\phi_R(x)\ dxd\tau}_{(II)}$$
$$= \underbrace{2\nu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} b_i(x,\tau)D_jD_jb_i(x,\tau)\phi_R(x)\ dxd\tau}_{(III)}$$
$$\underbrace{+2\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} b_i(x,\tau)\ b_j(x,\tau)D_ju_i(x,\tau)\phi_R(x)\ dxd\tau}_{(IV)}. \tag{3.8}$$

Vamos estimar os termos da igualdade acima, exceto o termo (IV). No termo (I), vamos aplicar os Teoremas de Fubini e Fundamental do Cálculo, respectivamente. E nos termos (II) e (III), integração por partes.

$$\begin{aligned}
(I) &= \int_{t_0}^{t}\int_{B_{R+1}} (b_i(x,\tau))^2_\tau \ \phi_R(x)dxd\tau = \int_{B_{R+1}} [{b_i}^2(x,t) - \ {b_i}^2(x,t_0)]\ \phi_R(x)dx \\
(II) &= \int_{t_0}^{t}\int_{B_{R+1}} \sum_{j,=1}^{n} D_j(b_i^2(x,\tau))u_j(x,\tau)\phi_R(x)\ dxd\tau \\
&= -\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n} D_ju_j(x,\tau)b_i^2(x,\tau)\phi_R(x)\ dxd\tau \\
&- \int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{n}\ u_j(x,\tau)b_i^2(x,\tau)D_j\phi_R(x)\ dxd\tau, \\
(III) &= 2\nu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n}\ b_i(x,\tau)D_jD_jb_i(x,\tau)\phi_R(x)\ dxd\tau \\
&= -2\nu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n}(D_jb_i(x,\tau))^2\phi_R(x)\ dxd\tau \\
&- \ 2\nu\int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{n}\ b_i(x,\tau)D_jb_i(x,\tau)D_j\phi_R(x)\ dxd\tau.
\end{aligned}$$

Pelos mesmos argumentos usados na equação (3.1), quando $R \to \infty$ as derivadas de ϕ_R irão para zero. E ainda o Teorema da Convergência Monótona em (I), a definição da ϕ_R em (II), (III) e substituindo estas estimativas em (3.8), ficamos com a igualdade:

$$\|b_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} - \int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j=1}^{n} D_j u_j(x,\tau) b_i^2(x,\tau)\, dxd\tau + 2\nu \int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j=1}^{n}(D_j b_i(x,\tau))^2\, dxd\tau$$
$$-2\int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j=1}^{n} b_i(x,\tau) b_j(x,\tau) D_j u_i(x,\tau)\, dxd\tau = \|b_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}. \tag{3.9}$$

Pelas igualdades (3.6) e (3.9),

$$\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \|b_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} - \sum_{j=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} {u_i}^2(x,\tau) D_j u_j(x,\tau)\, dxd\tau$$
$$-2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_i u_i(x,\tau) p(x,\tau)\, dxd\tau + 2\mu \int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{n}(D_j u_i(x,\tau))^2 dxd\tau$$
$$-2\int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j=1}^{n} b_j(x,\tau) b_i(x,\tau) D_j u_i(x,\tau) dxd\tau - \int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j=1}^{n} D_j u_j(x,\tau) b_i^2(x,\tau)\, dxd\tau$$
$$+2\nu \int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j=1}^{n}(D_j b_i(x,\tau))^2\, dxd\tau + 2\int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j=1}^{n} b_i(x,\tau) b_j(x,\tau) D_j u_i(x,\tau)\, dxd\tau$$
$$= \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} + \|b_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}. \tag{3.10}$$

Somando em i a igualdade (3.10), note que alguns termos se anulam, pois temos por hipótese que $\nabla \cdot \boldsymbol{u}(\cdot,t) = 0$. Logo, pela definição (2.1) e tomando mínimo entre ν e μ, temos a desigualdade desejada:

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\lambda \int_{t_0}^{t} \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)},$$

$\forall\, t > t_0$ ($t_0 > 0$ suficientemente grande) e $\lambda = \min(\mu,\nu)$. □

Observação 3.0.2. *Note que como consequência da prova, temos a partir de um instante de tempo o controle da norma L^2 das soluções de Leray pela norma L^2 das condições iniciais, e, em particular, que $\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}$ é integrável em $[t_0,\infty)$, ou seja,*

$$\int_{t_0}^{\infty} \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\, dt \leq \frac{1}{2\lambda}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

Observação 3.0.3. *Neste teorema foi utilizada a função de corte ϕ_R. De forma análoga, rigorosamente deveríamos utiliza-lá nos próximos resultados. Porém, foi observado que os mesmos passos feitos antes valem e tornam-se menos massantes quando não a utilizamos.*

Capítulo 4

Desigualdade de energia para as derivadas de primeira ordem das soluções($\boldsymbol{u}$,$\boldsymbol{b}$)

Neste capítulo, fizemos contas análogas ao Teorema 3.0.1, com o objetivo de derivar a Desigualdade de Energia para derivadas de primeira ordem. Abaixo seguem os Lemas que utilizamos na demonstração do teorema desta seção. Estes resultados são decorrentes das desigualdades (2.4.1) e (2.4.2) de Sobolev e suas provas encontram-se no apêndice.

4.1 Desigualdades Relevantes

Referentes as dimensões n=2,3,4, temos as seguintes desigualdades:

Lema 4.1.1. *Para n=2,*

$$\begin{aligned}
\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Em geral, para $m \geq 2$, $0 \leq l \leq m-2$:

$$\|(D^l\boldsymbol{u},D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},$$

para $C > 0$.

Lema 4.1.2. *Para n=3,*

$$\begin{aligned}
\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/2}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/2}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/2}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/2}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/2}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/2}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{3/4}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/4}\|(D^4\boldsymbol{u},D^4\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{3/4}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{1/4}\|(D^5\boldsymbol{u},D^5\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Em geral, para $m \geq 3$, $0 \leq l \leq m-3$*:*

$$\begin{aligned}
&\|(D^l\boldsymbol{u},D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\
&\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{\frac{l+3/2}{l+2}}\|(D^{l+2}\boldsymbol{u},D^{l+2}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}^{\frac{1/2}{l+2}}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

para $C > 0$.

Lema 4.1.3. *Para n=4,* $m \geq 1,\ 0 \leq l \leq m-1$:

$$\|(D^l\boldsymbol{u},D^l\boldsymbol{b})\|_{L^4(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})\|_{L^4(\mathbb{R}^n)} \leq C\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},$$

para $C > 0$.

4.2 Desigualdade de Energia

Para as derivadas de primeira ordem da solução de Leray, vale o seguinte resultado:

Teorema 4.2.1. *Dada* $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$ *solução de Leray do problema MHD incompressível, vale a desigualdade de energia:*

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^2 + C\int_{t_0}^{t}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}^2 d\tau \leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}^2,$$

$\forall\ t > t_0$ ($t_0 > 0$ *suficientemente grande),* $C > 0$ *constante.*

Demonstração. • **Caso n=3**

Derivando a equação (3.1), em relação x_l, com $1 \leq l \leq 3$ e reescrevendo em coordenadas, obtemos:

$$D_l(u_i)_t + \sum_{j,l=1}^{n} D_l(u_jD_ju_i) + D_l(D_ip) = \mu\sum_{j,l=1}^{n} D_jD_jD_lu_i + \sum_{j,l=1}^{n} D_l(b_jD_jb_i)\,. \quad (4.1)$$

Multiplicando (4.1) por $2D_l u_i$, encontramos:

$$\begin{aligned} &D_l(u_i^2)_t \,+\, 2\sum_{j,l=1}^{n} D_l u_i D_l(u_j D_j u_i) \,+\, 2D_l u_i D_l(D_i\, p) \\ &= 2\mu \sum_{j,l=1}^{n} D_l u_i D_j D_j D_l u_i + 2\sum_{j,l=1}^{n} D_l u_i D_l(b_j D_j b_i). \end{aligned} \tag{4.2}$$

Integrando (4.2) na região $\mathbb{R}^n \times [t_0, t]$ para $t > t_0 > t_*$, chegamos a:

$$\begin{aligned} &\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n} (D_l u_i(x,\tau))_\tau^2 \, dxd\tau}_{(I)} \,+\, \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(u_j(x,\tau)D_j u_i(x,\tau))\, dxd\tau}_{(II)} \\ &+\, \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(D_i\, p(x,\tau))\, dxd\tau}_{(III)} \,=\, \underbrace{2\mu \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l(u_i(x,\tau)) D_j D_j D_l u_i(x,\tau)\, dxd\tau}_{(IV)} \\ &\qquad + \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(b_j(x,\tau) D_j b_i(x,\tau))\, dxd\tau}_{(V)}\,. \end{aligned} \tag{4.3}$$

Vamos analisar cada termo da igualdade (4.3).
Aplicando os Teoremas Fundamental do Cálculo e Fubini no termo (I), temos

$$\int_{t_0}^{t}\int_{\mathbb{R}^n} (D_l u_i(x,\tau))_\tau^2 \, dxd\tau \;=\; \|D_l u_i(\cdot,t)\|_{L^2(\mathbb{R}^n)}^2 - \|D_l u_i(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}^2.$$

Em (II) aplicaremos integração por partes, desigualdade de Cauchy-Schwarz, Hölder e $\|\boldsymbol{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} := \max_i \|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$. Logo

$$\begin{aligned}
(II) &= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(u_j(x,\tau)D_j u_i(x,\tau))\, dxd\tau \\
&= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l u_j(x,\tau) D_j u_i(x,\tau)\, dxd\tau \\
&+ \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} u_j(x,\tau) D_j(D_l u_i(x,\tau))^2\, dxd\tau \\
&= -2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} u_i(x,\tau) D_l u_j(x,\tau) D_j D_l u_i(x,\tau)\, dxd\tau \\
&- \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_j u_j(x,\tau)(D_l u_i(x,\tau))^2\, dxd\tau \\
&\leq 2\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\sum_{j,l=1}^{n}|D_l u_j(x,\tau)||D_j D_l u_i(x,\tau)|\, dxd\tau \\
&+ \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_j u_j(x,\tau)(D_l u_i(x,\tau))^2\, dxd\tau \\
&\leq K\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D_l\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D_l D_l\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, dxd\tau \\
&+ \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_j u_j(x,\tau)(D_l u_i(x,\tau))^2\, dxd\tau,
\end{aligned}$$

somando em i e usando $\nabla\cdot\boldsymbol{u}(\cdot,t)=0$, segue

$$(II) \leq K\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, dxd\tau.$$

Com contas análogas para o termo (V), chegamos:

$$\begin{aligned}
(V) &= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(b_j(x,\tau)D_j b_i(x,\tau))\, dxd\tau \\
&\leq K\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, dxd\tau \\
&+ K\int_{t_0}^{t}\|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, dxd\tau.
\end{aligned}$$

Nos demais termos aplicaremos apenas integração por partes.

$$\begin{aligned}
(III) &= 2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l D_i\, p(x,\tau)\, dxd\tau \\
&= -2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l D_i u_i(x,\tau) D_l p(x,\tau)\, dxd\tau. \\
(IV) &= 2\mu \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_j D_j D_l u_i(x,\tau)\, dxd\tau \\
&= -2\mu \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} (D_j D_l u_i(x,\tau))^2\, dxd\tau.
\end{aligned}$$

Reescrevendo (4.3), temos

$$\begin{aligned}
&\|D_l u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\mu \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} (D_j D_l u_i(x,\tau))^2\, dxd\tau \\
&-2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l D_i u_i(x,\tau) D_l\, p(x,\tau)\, dxd\tau \leq \|D_l u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} \\
&+K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&+K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau. \\
&+K\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, dxd\tau.
\end{aligned}$$

Somando em i,l os termos acima, obtemos

$$\begin{aligned}
&\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\mu \int_{t_0}^{t} \|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, dxd\tau \leq \|D\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} \\
&+K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&+K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau. \\
&K\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, dxd\tau. \qquad (4.4)
\end{aligned}$$

Abaixo segue argumentos análogos para a equação (3.2).
Começamos derivando (3.2) em relação a x_l, com $1 \leq l \leq 3$, multiplicando por $2D_l b_i$ e reescrevendo em coordenadas, chegando em

$$(D_l b_i)^2_t + 2\sum_{j,l=1}^{n} D_l(u_j D_j b_i) D_l b_i = 2\nu \sum_{j,l=1}^{n} D_j D_j D_l b_i D_l b_i + 2\sum_{j,l=1}^{n} D_l(b_j D_j u_i) D_l b_i \quad (4.5)$$

Integrando (4.5) na região $\mathbb{R}^n \times [t_0, t]$, para $t > t_0 > t_*$

$$\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n} (D_l b_i(x,\tau))_\tau^2 \, dx d\tau}_{(I)} \;+\; \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j,l=1}^{n} D_l(u_j(x,\tau) D_j b_i(x,\tau)) D_l b_i(x,\tau)\, dx d\tau}_{(II)}$$
$$= \underbrace{2\nu \int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j,l=1}^{n} D_j D_j D_l b_i(x,\tau) D_l b_i(x,\tau)\, dx d\tau}_{(III)}$$
$$+ \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^n} \sum_{j,l=1}^{n} D_l(b_j(x,\tau) D_j u_i(x,\tau)) D_l b_i(x,\tau)\, dx d\tau}_{(IV)}\,. \tag{4.6}$$

Utilizando em (I) os Teorema Fundamental do Cálculo e o de Fubini, e, nos demais termos integração por partes, (4.6), torna-se somando em i:

$$\begin{aligned}\|D\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\nu \int_{t_0}^{t} \|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau &\le \|D\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} \\ +K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)} \|D\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\ +K\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbf{R}^n)} \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau. \end{aligned} \tag{4.7}$$

Pelos termos (4.4) e (4.7), segue a desigualdade:

$$\begin{aligned}\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \|D\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\mu\int_{t_0}^{t} \|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau + 2\nu \int_{t_0}^{t} \|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\ \le \|D\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} + \|D\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} \\ +K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\ +K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\ +K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)} \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\ +K\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau. \\ +K\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau. \end{aligned} \tag{4.8}$$

Pela definição (2.1) e tomando o mínimo entre μ e ν, temos

$$\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau \leq \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}$$
$$+K'\int_{t_0}^{t} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\, \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau. \tag{4.9}$$

Agora vamos estimar a desigualdade acima, usando o Lema 4.1.2, que trata da norma de Sobolev referente a dimensão n=3, exposta no início do capítulo.
Então,

$$\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau \leq \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}$$
$$+C\int_{t_0}^{t} \Big(\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\Big)^{1/2} \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau. \tag{4.10}$$

Em particular, decorrente do Teorema 3.0.1, obtemos

$$\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau \leq \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}$$
$$+C\int_{t_0}^{t} \Big(\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}\, \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\Big)^{1/2} \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau, \tag{4.11}$$

$\forall\, t > t_0$.
Também pelo Teorema 3.0.1, existe $t_0 > t_*$ tal que

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C}.$$

Afirmamos:

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C},\ \forall\, \tau \geq t_0. \tag{4.12}$$

Demonstração. Suponhamos que a afirmação seja falsa. Então, existe $t_1 > t_0$ tal que

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C},\ \forall \tau \in [t_0, t_1),$$

com

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_1)\|^{1/2}_{L^2(\mathbb{R}^n)} = \frac{\min(\mu,\nu)}{C}.$$

Tomando $t = t_1$ na desigualdade (4.11), temos:

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_1)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t_1}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau$$
$$\leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t_1}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau.$$

Portanto,

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_1)\|_{L^2(\mathbb{R}^n)} \leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}. \tag{4.13}$$

Com isso,

$$\begin{aligned}\frac{\min(\mu,\nu)}{C} &= \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_1)\|^{1/2}_{L^2(\mathbb{R}^n)}\\ &\leq \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C}.\end{aligned}$$

Absurdo. □

Logo a afirmação é verdadeira e utilizando em (4.11), segue que para n=3

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + C\int_{t_0}^{t}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau$$
$$\leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)},$$

para constante $C = 2\min(\mu,\nu) - \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)} > 0$ e $\forall\, t > t_0$ ($t_0 > 0$ suficientemente grande).

- **Caso n=4**

Note que podemos utilizar os mesmos argumentos dados em n=3 para as equações (3.1) e (3.2), ou seja, derivando as equações em relação a x_l, com $1 \leq l \leq 4$, multiplicando por $2D_l u_i$ e $2D_l b_i$, respectivamente. E ainda integrando na região $\mathbb{R}^n \times [t_0, t]$, temos:

$$\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n}(D_l u_i(x,\tau))^2_\tau \, dxd\tau}_{(I)} + \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(u_j(x,\tau)D_j u_i(x,\tau))\, dxd\tau}_{(II)}$$

$$+ \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l D_i p(x,\tau)\, dxd\tau}_{(III)} = \underbrace{2\mu\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_j D_j D_l u_i(x,\tau)\, dxd\tau}_{(IV)}$$

$$+ \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(b_j(x,\tau)D_j b_i(x,\tau))\, dxd\tau}_{(V)}$$

e

$$\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n}(D_l b_i(x,\tau))^2_\tau \, dxd\tau}_{(A)} + \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l(u_j(x,\tau)D_j b_i(x,\tau)) D_l b_i(x,\tau)\, dxd\tau}_{(B)}$$

$$= \underbrace{2\nu\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_j D_j D_l b_i(x,\tau) D_l b_i(x,\tau)\, dxd\tau}_{(C)}$$

$$+ \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l(b_j(x,\tau)D_j u_i(x,\tau)) D_l b_i(x,\tau)\, dxd\tau}_{(D)}.$$

Nos termos (I) e (A), utilizaremos os Teoremas de Fubini e Fundamental do Cálculo e em (III), (IV) e (C) integração por partes. Então as desigualdades acima quando somadas em i, tornam-se:

$$\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\mu\int_{t_0}^{t}\|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau \le \|D\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}$$
$$+ \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(u_j(x,\tau)D_j u_i(x,\tau))\, dxd\tau}_{(II)}$$
$$+ \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(b_j(x,\tau)D_j b_i(x,\tau))\, dxd\tau}_{(V)} \qquad (4.14)$$

e

$$
\begin{aligned}
\|D\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\nu \int_{t_0}^{t} \|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau \leq \|D\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} \\
\underbrace{+2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_l(u_j(x,\tau)D_j b_i(x,\tau))\, dxd\tau}_{(B)} \\
\underbrace{+\,2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_l(b_j(x,\tau)D_j u_i(x,\tau))\, dxd\tau}_{(D)}\,. \qquad (4.15)
\end{aligned}
$$

Agora aplicaremos integração por partes nos termos restantes.

$$
\begin{aligned}
(II) &= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(u_j(x,\tau)D_j u_i(x,\tau))\, dxd\tau \\
&= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l u_j(x,\tau) D_j u_i(x,\tau)\, dxd\tau \\
&+ \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} u_j(x,\tau) D_j(D_l u_i(x,\tau)) dxd\tau \\
&= -2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} u_i(x,\tau) D_l u_j(x,\tau) D_l D_j u_i(x,\tau)\, dxd\tau \\
&- \sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_j u_j(x,\tau) D_l u_i(x,\tau) dxd\tau,
\end{aligned}
$$

$$
\begin{aligned}
(V) &= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l(b_j(x,\tau) D_j b_i(x,\tau))\; dxd\tau \\
&= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l b_j(x,\tau) D_j b_i(x,\tau)\; dxd\tau \\
&+ 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_l b_j(x,\tau) D_j D_l b_i(x,\tau)\; dxd\tau \\
&= -2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} u_i(x,\tau) D_l b_j(x,\tau) D_j D_l b_i(x,\tau)\; dxd\tau \\
&- 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l u_i(x,\tau) D_j b_j(x,\tau) D_l b_i(x,\tau)\; dxd\tau,
\end{aligned}
$$

$$
\begin{aligned}
(B) &= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_l(u_j(x,\tau) D_j b_i(x,\tau))\; dxd\tau \\
&= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_l u_j(x,\tau) D_j b_i(x,\tau)\; dxd\tau \\
&+ 2\sum_{j,i,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) u_j(x,\tau) D_j D_l b_i(x,\tau)\; dxd\tau \\
&= -2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} b_i(x,\tau) D_l u_j(x,\tau) D_j D_l b_i(x,\tau)\; dxd\tau \\
&- 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_j u_j(x,\tau) D_l b_i(x,\tau)\; dxd\tau,
\end{aligned}
$$

$$
\begin{aligned}
(D) &= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_l(b_j(x,\tau)D_j u_i(x,\tau))\ dxd\tau \\
&= 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_l b_j(x,\tau)D_j u_i(x,\tau)\ dxd\tau \\
&+ 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} b_i(x,\tau) D_l b_j(x,\tau)D_j D_l u_i(x,\tau)\ dxd\tau \\
&= -2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} b_i(x,\tau) D_l b_j(x,\tau)D_j D_l u_i(x,\tau)\ dxd\tau \\
&- 2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n} D_l b_i(x,\tau) D_j b_j(x,\tau)D_l u_i(x,\tau)\ dxd\tau.
\end{aligned}
$$

Somando em i,l, aplicando Cauchy-Schwartz, Hölder e a definição (2.1), temos:

$$
\begin{aligned}
(II) &\leq K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau, \\
(V) &\leq K\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau, \\
(B) &\leq K\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau, \\
(D) &\leq K\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau.
\end{aligned}
$$

Substituindo as estimativas em (4.14) e (4.15), ficamos com

$$
\begin{aligned}
&\|D\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^2 + 2\mu\int_{t_0}^{t}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}^2\ d\tau \\
&\leq \|D\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}^2 + K'\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau \\
&\quad + K'\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau \\
&\leq \|D\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}^2 + K\int_{t_0}^{t}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau
\end{aligned}
$$

e

$$
\begin{aligned}
&\|D\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\nu\int_{t_0}^t \|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
&\leq \|D\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} + K'\int_{t_0}^t \|\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
&\quad + K'\int_{t_0}^t \|\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
&\leq \|D\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} + K\int_{t_0}^t \|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}
$$

Pelas desigualdades acima, tomando o mínimo μ e ν, e utilizando novamente a definição (2.1), temos:

$$
\begin{aligned}
&\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^t \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} \\
&\quad + K\int_{t_0}^t \|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau.
\end{aligned}
$$

Decorre do Lema 4.1.3:

$$
\begin{aligned}
&\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^t \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
&\leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)} + C\int_{t_0}^t \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau,
\end{aligned}
$$

onde $C > 0$.
Pelo Teorema 3.0.1, exite $t_0 > t_*$ tal que

$$
\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C}.
$$

Por um argumento análogo ao feito na prova da Afirmação (4.12), temos

$$
\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C}, \forall \tau \geq t_0.
$$

Logo para n=4,

$$
\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + C\int_{t_0}^t \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}.
$$

$\forall\, t > t_0$ ($t_0 > 0$ suficientemente grande) e $C > 0$.

Ficando provado o teorema para $2 \leq n \leq 4$. □

Observação 4.2.2. *Como consequência da prova do Teorema 4.2.1, temos a monotonicidade das derivadas em $L^2(\mathbb{R}^n)$, ou seja,*

$$
\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}, \forall\, t \geq t_0.
$$

Este fato é muito importante e usaremos em alguns resultados posteriormente.

Capítulo 5

Problema de Leray em $L^2(\mathbb{R}^n), 2 \leq n \leq 4$

No presente capítulo, primeiramente apresentamos os Lemas (5.1.1) e (5.1.2), que foram de extrema importância para a prova do teorema principal deste capítulo, cujas demonstrações encontram-se em [2]. Após, foi provado uma propriedade que trata do decaimento assintótico da norma L^2 das derivadas primeiras de $(\boldsymbol{u}, \boldsymbol{b})$, onde obtemos uma taxa de decrescimento $t^{1/2}$. E por último, utilizando a mesma técnica feita [18], provamos que o problema de Leray é um resultado válido para o sistema MHD incompressível, ou seja, temos o decaimento assintótico de $\|(\boldsymbol{u}, \boldsymbol{b})\|_{L^2(\mathbb{R}^n)}$.

5.1 Lemas

Lema 5.1.1. *Para* $t > 1$,

$$\int_{t_0}^{t} (t-\tau)^{-3/4}\, \tau^{-1/2}\, d\tau < K.$$

Lema 5.1.2. *Para* $\tau > 0$, *tem-se*

$$\begin{aligned}
\|e^{\mu\Delta(t-\tau)}\, \boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} &\leq K\sqrt{n}\mu^{-n/4}(t-\tau)^{-n/4}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)},\\
\|e^{\mu\Delta(t-\tau)}\, \boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} &\leq K\sqrt{n}\mu^{-n/4}(t-\tau)^{-n/4}\|\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)},\\
\|e^{\mu\Delta(t-\tau)}\, \boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} &\leq K\sqrt{n}\mu^{-n/4}(t-\tau)^{-n/4}\|\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)},\\
\|e^{\mu\Delta(t-\tau)}\, \boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} &\leq K\sqrt{n}\mu^{-n/4}(t-\tau)^{-n/4}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Lema 5.1.3. *Para* $t > 0$, *considere*

$$Q_1(\cdot,t) = -\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p + \boldsymbol{b}\cdot\nabla\boldsymbol{b}, \tag{5.1}$$

então

$$\nabla \cdot Q_1(\cdot, t) = 0.$$

Demonstração. Note que podemos reescrever (3.1) como:

$$\boldsymbol{u}_t = \mu \Delta \boldsymbol{u} + Q_1,$$

para Q_1 da forma (5.1).
Aplicando o divergente em ambos os lados da igualdade acima, temos

$$(\nabla \cdot \boldsymbol{u})_t = \mu \Delta (\nabla \cdot \boldsymbol{u}) + \nabla \cdot Q_1,$$

usando a hipótese que $\nabla \cdot \boldsymbol{u}(\cdot, t) = 0$, segue

$$\nabla \cdot Q_1 = 0.$$

□

Observação 5.1.4. *Uma consequência importante do Lema 5.1.3 e que iremos utilizar bastante á partir de agora, é o fato de podermos através dos resultados (8.2.1) e (8.2.2) do apêndice, escrever:*

$$Q_1 = \mathbb{P}_H[-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}],$$

onde $\mathbb{P}_H$ é o projetor de Helmholtz ou conhecido também, como projetor de Leray. No apêndice encontra-se uma melhor explicação e alguns resultados.
Note que esse projetor tem papel fundamental, pois ao o utilizarmos conseguimos eliminar a pressão e com isso, conseguimos estimar (3.1). Em nosso caso, queremos realmente eliminar a pressão, porém é possível obter estimativas para este termo.
A ideia seria aplicar divergente na equação (3.1) e usar que $\nabla \cdot \boldsymbol{u}(\cdot, t) = 0$. Ao isolarmos o termo que envolve a pressão, obtemos:

$$\Delta p = -\sum_{i,j} D_i u_j D_j u_i + D_i b_j D_j b_i,$$

ou seja, que a pressão satisfaz a equação de Poisson. Caso o leitor esteja interessado, é possível encontrar em [31] estimativas para a pressão utilizando a teoria de Calderón-Zygmund.

5.2 Comportamento assintótico da Norma L^2

Abaixo seguem os principais resultados deste capítulo.

Teorema 5.2.1. *Seja* $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$ *solução de Leray do problema MHD incompressível, então*

$$\lim_{t\to\infty} t^{1/2}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}=0.$$

Demonstração. Suponhamos que o teorema seja falso. Então, existiria uma sequência crescente $t_j\to\infty$ (com $t_j\geq t^*$ e $t_j\geq 2t_{j-1}$ para todo j) e uma constante $\eta>0$, tal que,

$$t_j\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_j)\|^2_{L^2(\mathbb{R}^n)}\geq\eta,\ \forall\, j.$$

Em particular, teríamos

$$\begin{aligned}\int_{t_{j-1}}^{t_j}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}dt &\geq (t_j-t_{j-1})\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_j)\|^2_{L^2(\mathbb{R}^n)}\\ &\geq \frac{1}{2}t_j\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_j)\|^2_{L^2(\mathbb{R}^n)}\\ &\geq \frac{1}{2}\eta,\end{aligned}$$

o que contradiz os Teoremas 3.0.1 e a desigualdade 3.11. □

Teorema 5.2.2. *Considere* $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$ *solução de Leray do problema MHD incompressível, então*

$$\lim_{t\to\infty}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}=0,$$

$2\leq n\leq 4$.

Demonstração. Começaremos com caso n=3.
Primeiramente reescrevemos as equações (3.1) e (3.2), como

$$\boldsymbol{u}_t=\mu\Delta\boldsymbol{u}+Q_1$$

e

$$\boldsymbol{b}_t=\nu\Delta\boldsymbol{b}+Q_2,$$

onde

$$Q_1=-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nabla p+\boldsymbol{b}\cdot\nabla\boldsymbol{b}$$

e

$$Q_2=-\boldsymbol{u}\cdot\nabla\boldsymbol{b}+\boldsymbol{b}\cdot\nabla\boldsymbol{u}.$$

Como para $t_*\geq 0$, suficientemente grande as soluções $(\boldsymbol{u},\boldsymbol{b})$ são suaves e suas derivadas estão em L^2, podemos utilizar o principio de Duhamel e escrever

$$\boldsymbol{u}(x,t)\ = e^{\mu\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)+\int_{t_0}^{t}e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\,d\tau \tag{5.2}$$

e

$$\boldsymbol{b}(x,t) \ = e^{\nu\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0) + \int_{t_0}^{t} e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\ d\tau, \tag{5.3}$$

$\forall\ t > t_0$.

Aplicando a norma L^2 em (5.2) e (5.3) e, em seguida a desigualdade de Minkowski, temos

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \ \leq \underbrace{\|e^{\mu\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}}_{(I)} + \underbrace{\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau}_{(II)} \tag{5.4}$$

e

$$\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \ \leq \underbrace{\|e^{\nu\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}}_{(III)} + \underbrace{\int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau}_{(IV)}. \tag{5.5}$$

Agora, vamos estimar os termos $(I), (II), (III)$ e (IV). Começaremos por (I) e (III), onde temos a norma L^2 das soluções da equação do calor, com condição inicial $\boldsymbol{u}(\cdot,t_0)$ e $\boldsymbol{b}(\cdot,t_0)$, respectivamente.

Note que, utilizando a propriedade de Leray para a equação do Calor, resultado provado em [2], temos que, dado $\varepsilon > 0$, existe $t_0 > t_*$ suficientemente grande tal que

$$\|e^{\mu\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{4} \tag{5.6}$$

e

$$\|e^{\nu\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{4}. \tag{5.7}$$

Para o termo (II), iremos utilizar o Lema (5.1.3), pois como havíamos comentado anteriormente, teremos

$$Q_1 = \mathbb{P}_H[-\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}],$$

onde $\mathbb{P}_H$ é o projetor de Helmholtz. Logo,

$$\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\ d\tau = \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbb{P}_H[-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{b}\cdot\nabla\boldsymbol{b}\ ]\ \|_{L^2(\mathbb{R}^n)}\ d\tau. \tag{5.8}$$

Pelos resultados do apêndice, sabemos que o projetor de Helmholtz comuta com o Heat Kernel e é ortogonal na norma L^2. Com esses fatos e além disso, utilizando a desigualdade de Minkowski, o Lema (5.1.2) e a definição (2.1), chegaremos em

$$
\begin{aligned}
\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau &= \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbb{P}_H(-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{b}\cdot\nabla\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} d\tau \\
&= \int_{t_0}^{t} \|\mathbb{P}_H\left[e^{\mu\Delta(t-\tau)}(-\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\boldsymbol{b}\cdot\nabla\boldsymbol{b})\right]\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&\leq \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&+ \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau))\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&\leq C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&+ C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&\leq 2C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau.
\end{aligned}
\tag{5.9}
$$

Dado $\varepsilon > 0$ existe $t_0 > t_*$ tal que

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \varepsilon t^{-1/2}, \tag{5.10}$$

$\forall\, t > t_0$ pelo resultado (5.2.1). Então note que podemos reescrever

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon t^{-1/2}}{8KC\sqrt{3}\mu^{-3/4}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}}, \tag{5.11}$$

onde K é tal que, pelo Lema (5.1.1),

$$\int_{t_0}^{t}(t-\tau)^{-3/4}\tau^{-1/2}d\tau < K.$$

Usando (5.11) e (3.0.1), temos que

$$
\begin{aligned}
(II) &= \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&\leq 2C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&\leq 2C\sqrt{3}\mu^{-3/4}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}\int_{t_0}^{t}(t-\tau)^{-3/4}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&\leq \frac{\varepsilon}{4},
\end{aligned}
\tag{5.12}
$$

para todo $t > t_0$.
Por (5.6) e (5.12), obtemos

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{2},\ \forall t > t_0. \tag{5.13}$$

Agora, iremos trabalhar com o termo

$$(IV) = \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau.$$

Note que a estimativa para o termo $Q_2(.,\tau)$ será mais simples, pois não teremos que eliminar a pressão, ou seja, não iremos utilizar o projetor $\mathbb{P}_H$. Logo, aplicando a desigualdade de Minkowski, o Lema (5.1.2) e a definição (2.1), tem-se

$$\begin{aligned}
\int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau &\leq \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}[\boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau)]\|_{L^2(\mathbb{R}^n)} d\tau \\
&+ \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}[\boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)]\|_{L^2(\mathbb{R}^n)}\, d\tau \\
&\leq C\sqrt{3}\nu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&+ C\sqrt{3}\nu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&\leq 2C\sqrt{3}\nu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} d\tau.
\end{aligned}$$

Por (3.0.1), (5.7) e (5.11),

$$\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{2},\ \forall t > t_0. \tag{5.14}$$

Usando a definição (2.1), para os termos (5.13) e (5.14), segue que dado $\varepsilon > 0$, existe $t_0 > t_*$ tal que

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \varepsilon,$$

para todo $t > t_0$. O que prova o teorema para n=3.

Agora, mostraremos que o teorema é válido para n=4.
Pela teoria de Leray, existe $t_* \geq 0$, suficientemente grande, tal que $(\boldsymbol{u},\boldsymbol{b})$ são suaves e suas derivadas parciais estão em L^2. Logo podemos aplicar o princípio de Duhamel, obtendo

$$\boldsymbol{u}(x,t) = e^{\mu\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t} e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\, d\tau \tag{5.15}$$

e

$$\boldsymbol{b}(x,t) = e^{\nu\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0) + \int_{t_0}^{t} e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\,d\tau, \tag{5.16}$$

para todo $t > t_0$, onde

$$Q_1 = -\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p + \boldsymbol{b}\cdot\nabla\boldsymbol{b}$$

e

$$Q_2 = -\boldsymbol{u}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}.$$

Aplicando a norma L^2 em (5.15) e (5.16) e, em seguida a desigualdade de Minkowski, temos

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \underbrace{\|e^{\mu\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}}_{(I)} + \underbrace{\int_{t_0}^{t}\|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau}_{(II)} \tag{5.17}$$

e

$$\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \underbrace{\|e^{\nu\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)}}_{(III)} + \underbrace{\int_{t_0}^{t}\|e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau}_{(IV)}. \tag{5.18}$$

Nos termos (I) e (III), temos que dado $\varepsilon > 0$, existe $t_0 > t_*$ suficientemente grande, tal que

$$\|e^{\mu\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^4)} \le \frac{\varepsilon}{4} \tag{5.19}$$

e

$$\|e^{\nu\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^4)} \le \frac{\varepsilon}{4}, \tag{5.20}$$

por [2].

A parte que difere do caso n=3, são as estimativas para os termos (II) e (IV). Utilizaremos uma desigualdade já conhecida da literatura, oriunda do seguinte fato: dada a equação do calor $v_t = \Delta v,\ v(\cdot,0) = v_0 \in L^2(\mathbb{R}^n),\ t > 0$, vale:

$$\|v(\cdot,t)\|_{L^2(\mathbb{R}^n)} = \|e^{\Delta t}v_0\|_{L^2(\mathbb{R}^n)} \le \|v_0\|_{L^2(\mathbb{R}^n)}.$$

Obtemos, assim:

$$(II) = \int_{t_0}^{t}\|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \le \int_{t_0}^{t}\|Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \tag{5.21}$$

e

$$(IV) = \int_{t_0}^{t}\|e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \le \int_{t_0}^{t}\|Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau. \tag{5.22}$$

No primeiro termo, utilizaremos o fato que

$$Q_1 = \mathbb{P}_H[-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}].$$

Pela sua ortogonalidade em L^2 e a Desigualdade de Minkowski, tem-se

$$\begin{aligned} \|Q_1\|_{L^2(\mathbb{R}^n)} &= \|\mathbb{P}_H[-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}]\|_{L^2(\mathbb{R}^n)} \\ &\leq \|-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}\|_{L^2(\mathbb{R}^n)} \\ &\leq \|\boldsymbol{u} \cdot \nabla \boldsymbol{u}\|_{L^2(\mathbb{R}^n)} + \|\boldsymbol{b} \cdot \nabla \boldsymbol{b}\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Aplicando as desigualdades de Cauchy-Schwarz e Hölder nos últimos termos da desigualdade anterior,

$$\begin{aligned} \| & \boldsymbol{u}(\cdot,\tau) \cdot \nabla \boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} + \|\boldsymbol{b}(\cdot,\tau) \cdot \nabla \boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \\ = & \left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \left(\sum_{j=1}^4 u_j D_j u_i\right)^2 dx\right)^{1/2} + \left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \left(\sum_{j=1}^4 b_j D_j b_i\right)^2 dx\right)^{1/2} \\ \leq & \sqrt{2}\left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \left(\sum_{j=1}^4 u_j^4\right)^{1/2} \left(\sum_{j=1}^4 (D_j u_i)^4\right)^{1/2} dx\right)^{1/2} \\ + & \sqrt{2}\left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \left(\sum_{j} b_j^4\right)^{1/2} \left(\sum_{j=1}^4 (D_j b_i)^4\right)^{1/2} dx\right)^{1/2} \\ \leq & \sqrt{2}\left(\int_{\mathbb{R}^4} \left(\sum_{i=1}^4 \sum_{j=1}^4 u_j^4\right)^{1/2} \left(\sum_{i=1}^4 \sum_{j=1}^4 (D_j u_i)^4\right)^{1/2} dx\right)^{1/2} \\ + & \sqrt{2}\left(\int_{\mathbb{R}^4} \left(\sum_{i=1}^4 \sum_{j=1}^4 b_j^4\right)^{1/2} \left(\sum_{i=1}^4 \sum_{j=1}^4 (D_j b_i)^4\right)^{1/2} dx\right)^{1/2} \\ \leq & \sqrt{2}\left(\left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \sum_{j=1}^4 u_j^4\right)^{1/2} \left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \sum_{j=1}^4 (D_j u_i)^4\right)^{1/2} dx\right)^{1/2} \\ + & \sqrt{2}\left(\left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \sum_{j=1}^4 b_j^4\right)^{1/2} \left(\int_{\mathbb{R}^4} \sum_{i=1}^4 \sum_{j=1}^4 (D_j b_i)^4\right)^{1/2} dx\right)^{1/2} \\ \leq & \sqrt{2}\|\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)} + \sqrt{2}\|\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}. \end{aligned}$$

Então, usando a desigualdade acima, a definição (2.1) e a desigualdade de Sobolev

(4.1.3), temos:

$$\begin{aligned}\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau &\leq \sqrt{2}\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}d\tau\\ &+ \sqrt{2}\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}d\tau\\ &\leq 2\sqrt{2}\int_{t_0}^{t} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}d\tau\\ &\leq 2\sqrt{2}C\int_{t_0}^{t} \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^{\,2}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau.\end{aligned} \tag{5.23}$$

Pela observação (4.2.2) e Teorema 5.2.1, vale:

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{8\sqrt{2}C\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}}, \tag{5.24}$$

para todo $t > t_0$. Usando este fato e o Teorema 4.2.1 em (5.23), tem-se

$$\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \leq \frac{\varepsilon}{4\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}}\int_{t_0}^{t} \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau \leq \frac{\varepsilon}{4},$$

$\forall t > t_0$.
Logo,

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\mu\Delta(t-\tau)}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} + \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_1(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\, d\tau \leq \frac{\varepsilon}{4}+\frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}, \forall t > t_0. \tag{5.25}$$

Agora iremos estimar a norma L^2 de $Q_2 = -\boldsymbol{u}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}$. Usando a Desigualdades de Minkowski, Cauchy- Shwarz e Hölder:

$$\begin{aligned}\|Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} &\leq \|\boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} + \|\boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\\ &\leq \sqrt{2}\|\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)} + \sqrt{2}\|\boldsymbol{b}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}.\end{aligned}$$

Aplicando a Desigualdade de Sobolev (4.1.3):

$$\begin{aligned}\|Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} &\leq \sqrt{2}C\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\\ &+ \sqrt{2}C\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Com isso e a definição (2.1), resulta em

$$\begin{aligned}
\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau &\leq \int_{t_0}^{t} \|Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau \\
&\leq \sqrt{2}C\int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau \\
&+ \sqrt{2}C\int_{t_0}^{t} \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau \\
&\leq 2\sqrt{2}C\int_{t_0}^{t} \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}$$

Por fim, pelos Teoremas 3.0.1, 5.2.1, 4.2.1 e pela desigualdade (5.24), vale

$$\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \leq \frac{\varepsilon}{4\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}}\int_{t_0}^{t} \|(D^2\boldsymbol{u},D^2\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau \leq \frac{\varepsilon}{4},$$

para todo $t > t_0$.
Logo, vale

$$\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \|e^{\nu\Delta(t-\tau)}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^n)} + \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}Q_2(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \leq \frac{\varepsilon}{4}+\frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}, \tag{5.26}$$

para todo $t > t_0$.
Aplicando a definição (2.1) em (5.25) e (5.26), obtemos:

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \varepsilon,\ \forall t > t_0,$$

o que prova o teorema para n=4.
Segue que o resultado é válido para $2 \leq n \leq 4$. □

Capítulo 6

Problema de Leray em $L^q(\mathbb{R}^n)$, com $2 \leq n \leq 4$ e $2 \leq q \leq \infty$

Nesta etapa do trabalho estendemos as desigualdades de energia (3.0.1) e (4.2.1) até as derivadas de ordem m. Além disso, foi possível obter uma generalização do problema de Leray dada no item 3 do Teorema 6.1.1, para as derivadas de ordem m, obtendo uma taxa de decrescimento de $t^{m/2}$. Com essa prova tivemos a possibilidade de estender esses resultados para norma L^q, $2 \leq q \leq \infty$, corolários que serão provados na seção 6.2 que segue.

6.1 Desigualdade de Energia para derivadas de ordem m das soluções $(\boldsymbol{u},\boldsymbol{b})$

Teorema 6.1.1. *Seja $(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)$ solução de Leray do problema MHD incompressível, então para $m \geq 1$ qualquer e $2 \leq n \leq 4$ vale:*

1.

$$(t - t_0)^m \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} + C_m \int_{t_0}^{t} (\tau - t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau$$
$$\leq m \int_{t_0}^{t} (\tau - t_0)^{m-1} \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau,$$

2.

$$\int_{t_0}^{\infty} (\tau - t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^n)} d\tau < \infty,$$

3.

$$\lim_{t\to\infty} t^{m/2} \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot, \tau)\|_{L^2(\mathbb{R}^n)} = 0,$$

$\forall\, t > t_0$ ($t_0 > 0$ *suficientemente grande) e* $C_m > 0$.

Demonstração. • **Caso n=3**

Começaremos provando o teorema para $m = 1$.
Relembrando a equação (3.1) :

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p \;=\; \mu\Delta\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}.$$

Diferenciando a equação acima em relação a x_l, com $1 \leq l \leq 3$, reescrevendo em coordenadas, temos

$$D_l u_{it} + \sum_{j,l=1}^{n} D_l(u_j D_j u_i) + D_i(D_l p) \;=\; \mu\sum_{j,l=1}^{n} D_j D_j(D_l u_i) + \sum_{j,l=1}^{n} D_l(b_j D_j b_i). \tag{6.1}$$

Multiplicando (6.1) por $2(t-t_0)D_l u_i$ e integrando na região $\mathbb{R}^n \times [t_0, t]$,

$$\begin{aligned}
&\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau - t_0)(D_l u_i(x,\tau))_\tau^2\, dxd\tau}_{(I)} + \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau - t_0)D_l u_i(x,\tau)D_l(u_j(x,\tau)D_j u_i(x,\tau))\, dxd\tau}_{(II)} \\
&\qquad + \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau - t_0)D_l u_i(x,\tau)D_i(D_l p(x,\tau))\, dxd\tau}_{(III)} \\
&= \underbrace{2\mu\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}}(\tau - t_0)\, D_l u_i(x,\tau)D_j D_j(D_l u_i(x,\tau))\, dxd\tau}_{(IV)} \\
&\qquad + \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau - t_0)D_l u_i(x,\tau)D_l(b_j(x,\tau)D_j b_i(x,\tau))\, dxd\tau}_{(V)}\,.
\end{aligned} \tag{6.2}$$

No termo (I), aplicaremos Fubini e integração por partes. Nos termos (III) e (IV), apenas integração por partes. Note que neste caso o termo (III) que possui a pressão, será zero quando somarmos em i, pois $\nabla\cdot\boldsymbol{u}(\cdot,t) = 0$. Então, a igualdade (6.2) torna-se:

$$
\begin{aligned}
(t-t_0) \quad & \|D_l u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)D_l u_i(x,\tau)D_l(u_j(x,\tau)D_j u_i(x,\tau))\,dxd\tau}_{(II)} \\
= \quad & \int_{t_0}^{t}\|D_l u_i(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau - 2\mu\sum_{j,l=1}^{n}\int_{t_0}^{t}(\tau-t_0)\,\|D_j D_l u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & \underbrace{2\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)D_l u_i(x,\tau)D_l(b_j(x,\tau)D_j b_i(x,\tau))\,dxd\tau}_{(V)}\,. \qquad (6.3)
\end{aligned}
$$

Nos termos (II) e (V), aplicaremos integração por partes a desigualdade de Cauchy Schwarz e somaremos em i,j,l, para obter:

$$
\begin{aligned}
(II) \quad \leq \quad & 2\left|\int_{t_0}^{t}\int_{\mathbb{R}^n}\sum_{j,l=1}^{n}(\tau-t_0)u_i(x,\tau)D_l u_j(x,\tau)D_j D_l u_i(x,\tau)\,dxd\tau\right| \\
\leq 2 \quad & \int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\sum_{j,l=1}^{n}|D_l u_j(x,\tau)||D_j D_l u_i(x,\tau)|\,dxd\tau \\
\leq \quad & 2\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau, \qquad (6.4)
\end{aligned}
$$

$$
\begin{aligned}
(V) \quad \leq \quad & 2\left|\sum_{j,l=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)u_i(x,\tau)D_l b_j(x,\tau)D_j D_l b_i(x,\tau)\,dxd\tau\right| \\
\leq \quad & 2\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\sum_{j,l=1}^{n}|D_l b_j(x,\tau)||D_j D_l b_i(x,\tau)|\,dxd\tau \\
\leq \quad & 2\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau. \qquad (6.5)
\end{aligned}
$$

A penúltima desigualdade nos termos (II) e (V) são válidas, pois $\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} :=$

$\max_i \|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$. Logo, somando em i podemos reescrever (6.3) como

$$\begin{aligned}
(t-t_0) \quad & \|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\mu\int_{t_0}^{t}(\tau-t_0)\,\|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
\leq \quad & \int_{t_0}^{t}\|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau.
\end{aligned} \tag{6.6}$$

Agora, considere a equação

$$\boldsymbol{b}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \nu\,\Delta\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}.$$

Por um processo similar feito para primeira equação, chegamos a

$$\begin{aligned}
(t-t_0) \quad & \|D\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\nu\int_{t_0}^{t}(\tau-t_0)\,\|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
\leq \quad & \int_{t_0}^{t}\|D\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau.
\end{aligned} \tag{6.7}$$

Em (6.6) e (6.7), tomando o mínimo entre μ e ν e usando a definição (2.1), temos:

$$\begin{aligned}
(t-t_0) \quad & \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t}(\tau - t_0)\,\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
\leq \quad & \int_{t_0}^{t}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)\|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau \\
\leq \quad & \int_{t_0}^{t}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & 4K\int_{t_0}^{t}(\tau-t_0)\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\,d\tau.
\end{aligned} \tag{6.8}$$

Utilizando o Lema 4.1.2, no último termo da desigualdade anterior:

$$\begin{aligned}
(t-t_0) \quad & \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t}(\tau - t_0)\,\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
\leq \quad & \int_{t_0}^{t}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,d\tau \\
+ \quad & C\int_{t_0}^{t}(\tau-t_0)\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned} \tag{6.9}$$

Pelo mesmo argumento dado na Afirmação 4.12 e pelo Teorema 3.0.1, existe $t_0 > t_*$ suficientemente grande, tal que

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C},\ \forall \tau \geq t_0.$$

Com isso, podemos reescrever (6.9) da seguinte forma:

$$\begin{aligned}
(t-t_0) \quad & \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + C_1\int_{t_0}^{t}(\tau - t_0)\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
\leq \quad & \int_{t_0}^{t}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau,
\end{aligned} \tag{6.10}$$

para $C_1 > 0$. O que prova o item 1 para n=3 e m=1.
O item 2 considerando novamente n=3 e m=1 é uma consequência do Teorema 3.0.1, onde temos

$$\int_{t_0}^{\infty} \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau < \infty.$$

Aplicando este resultado em (6.10), segue que

$$\int_{t_0}^{\infty} (\tau - t_0)\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau < \infty. \tag{6.11}$$

Agora vamos provar o item 3. Pelo Teorema 5.2.2, dado $\epsilon > 0$, existe $t_0 > 0$, tal que

$$\|(\boldsymbol{u}, \boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon,$$

para todo $t \geq t_0$. Aplicando este resultado no Teorema 3.0.1, obtemos

$$\int_{t_0}^{t} \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \frac{1}{2\lambda}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^n)} < \epsilon,$$

utilizando esta desigualdade em (6.10), temos

$$\int_{t_0}^{t} (\tau - t_0)\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau < \epsilon. \tag{6.12}$$

E também,

$$t\Big(\frac{t-t_0}{t}\Big)\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = (t-t_0)\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \epsilon,$$

então

$$t^{1/2}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \Big(\frac{t}{t-t_0}\Big)^{1/2}\epsilon^{1/2}. \tag{6.13}$$

Como $\epsilon > 0$ é arbitrário,

$$t^{1/2}\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0,\ t \to \infty.$$

Demonstração para m=2.

Considerando a equação

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \mu\Delta\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}.$$

Começaremos derivando em relação a $x_{l_1}x_{l_2}$, com $1 \leq l_1, l_2 \leq 3$, após, multiplicaremos por $2(t-t_0)^2 D_{l_1}D_{l_2}u_i$ e, por último, integraremos na região $\mathbb{R}^n \times [t_0, t]$, o que resulta em

$$
\begin{aligned}
&\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau-t_0)^2 (D_{l_1}D_{l_2}u_i(x,\tau))^2_\tau \, dxd\tau}_{(I)} \\
&+ \underbrace{\sum_{j,l_1,l_2=1}^{n} 2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau) D_{l_1}D_{l_2}(u_j(x,\tau)D_j u_i(x,\tau)) dxd\tau}_{(II)} \\
&+ \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau) D_{l_1}D_{l_2}D_i p(x,\tau) \, dxd\tau}_{(III)} \\
&= \underbrace{2\mu \sum_{j,l_1,l_2=1}^{n} \int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau) D_j D_j (D_{l_1}D_{l_2}u_i(x,\tau)) \, dxd\tau}_{(IV)} \\
&+ \underbrace{\sum_{j,l_1,l_2=1}^{n} 2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau) D_{l_1}D_{l_2}(b_j(x,\tau)D_j \, b_i(x,\tau)) \, dxd\tau}_{(V)}. \qquad (6.14)
\end{aligned}
$$

No termo (I), aplicaremos teorema de Fubini e após integração por partes. Também será aplicado integração por partes nos termos (III) e (IV). Observe que (III) anula-se usando a hipótese que $\nabla \cdot \boldsymbol{u}(\cdot, t) = 0$. Então, podemos reescrever (6.14) como

$$
\begin{aligned}
(t-t_0)^2 \quad & \|D_{l_1}D_{l_2}u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
+ \quad & \underbrace{2\sum_{j,l_1,l_2=1}^{n} \int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau) D_{l_1}D_{l_2}(u_j(x,\tau)D_j u_i(x,\tau)) dxd\tau}_{(II)} \\
= \quad & 2\int_{t_0}^{t} (\tau-t_0)\|D_{l_1}D_{l_2}u_i(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\
- \quad & 2\mu \sum_{j,l_1,l_2=1}^{n} \int_{t_0}^{t} (\tau-t_0)^2 \|D_j D_{l_1}D_{l_2}u_i(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\
- \quad & \underbrace{2\sum_{j,l_1,l_2=1}^{n} \int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau) D_{l_1}D_{l_2}(b_j(x,\tau)D_j b_i(x,\tau)) \, dxd\tau}_{(V)}.
\end{aligned}
\qquad (6.15)
$$

Agora, estimaremos os termos (II) e (V) com integração por partes e a desigualdade de Cauchy-Schwarz,

$$
\begin{aligned}
(II) &= 2\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau)D_{l_1}D_{l_2}(u_j(x,\tau)D_ju_i(x,\tau))\,dxd\tau\\
&= 2\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau)D_{l_1}D_{l_2}u_j(x,\tau)D_ju_i(x,\tau)\,dxd\tau\\
&+ 2\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau)D_{l_2}u_j(x,\tau)D_{l_1}D_ju_i(x,\tau)\,dxd\tau\\
&+ 2\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau)D_{l_1}u_j(x,\tau)D_{l_2}D_ju_i(x,\tau)\,dxd\tau\\
&+ 2\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau)u_j(x,\tau)D_{l_1}D_{l_2}D_ju_i(x,\tau)\,dxd\tau\\
&\leq -K\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_jD_{l_1}D_{l_2}u_i(x,\tau)D_{l_1}D_{l_2}u_j(x,\tau)u_i(x,\tau)\,dxd\tau\\
&- K\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_jD_{l_1}D_{l_2}u_i(x,\tau)D_{l_2}u_j(x,\tau)D_{l_1}u_i(x,\tau)\,dxd\tau\\
&- K\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_jD_{l_1}D_{l_2}u_i(x,\tau)D_{l_1}u_j(x,\tau)D_{l_2}u_i(x,\tau)\,dxd\tau\\
&- \underbrace{K\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau)D_ju_j(x,\tau)D_{l_1}D_{l_2}u_i(x,\tau)\,dxd\tau}_{=0,\ \sum_j D_ju_j=0}\\
&\leq K\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\,d\tau\\
&+ K'\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\,d\tau.
\end{aligned}
\tag{6.16}
$$

Usando os mesmos argumentos do termo anterior, temos

$$
\begin{aligned}
(V) &= 2\sum_{j,l_1,l_2=1}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^2 D_{l_1}D_{l_2}u_i(x,\tau)D_{l_1}D_{l_2}(b_j(x,\tau)D_jb_i(x,\tau))\,dxd\tau\\
&\le K'\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\,d\tau\\
&+ K\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\;d\tau.
\end{aligned}
\tag{6.17}
$$

Reescrevendo (6.15), com as estimativas para (II) e (V). Após, somando em i, l_1, l_2, usando a definição (2.1) e o Lema 4.1.2, temos

$$
\begin{aligned}
(t-t_0)^2 \quad & \|D^2\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\mu\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\\
\le \quad & 2\int_{t_0}^{t}(\tau-t_0)\|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\\
+ \quad & C''\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\Big[\ \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\\
+ \quad & \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\Big]\ d\tau\\
+ \quad & C'\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\Big[\ \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\\
+ \quad & \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\Big]\ d\tau\\
\le \quad & 2\int_{t_0}^{t}(\tau-t_0)\|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\Big[\ \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\\
+ \quad & \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\Big]\ d\tau\\
\le \quad & 2\int_{t_0}^{t}(\tau-t_0)\|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\ \|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\ \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}
\tag{6.18}
$$

O mesmo processo poderá ser feito com

$$
\boldsymbol{b}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \nu\,\Delta\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}
$$

e chegaremos em

$$
\begin{aligned}
(t-t_0)^2 \quad & \|D^2\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\nu\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
\leq \quad & 2\int_{t_0}^{t}(\tau-t_0)\|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\Big[\,\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)} \\
+ \quad & \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\Big]\,d\tau \\
\leq \quad & 2\int_{t_0}^{t}(\tau-t_0)\|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
+ \quad & K\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\,\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}
\tag{6.19}
$$

Logo, por (6.18) e (6.19)

$$
\begin{aligned}
(t-t_0)^2 \quad & \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
\leq \quad & 2\int_{t_0}^{t}(\tau-t_0)\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
+ \quad & C\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\,\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\,\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}
$$

Pelo Teorema 3.0.1, existe $t_0 > t_*$ suficientemente grande, tal que

$$
\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)} < \frac{\min(\mu,\nu)}{C}, \forall\tau\geq t_0, \tag{6.20}
$$

como visto na Afirmação (4.12).
Segue,

$$
\begin{aligned}
(t-t_0)^2 \quad & \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + C_2\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau, \\
\leq \quad & 2\int_{t_0}^{t}(\tau-t_0)\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau, \forall\, t\geq t_0
\end{aligned}
\tag{6.21}
$$

onde $C_2 > 0$. Isto prova o item 1.
O item 2, segue aplicando (6.11) na desigualdade (6.21), pois com isso vale

$$
\int_{t_0}^{\infty}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau < \infty. \tag{6.22}
$$

Para a prova do item 3, vamos usar (6.12) em (6.21) e obter

$$\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau<\epsilon. \tag{6.23}$$

e também, por (6.21)

$$t^2\Big(\frac{t-t_0}{t}\Big)^2\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}=(t-t_0)^2\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\le\epsilon.$$

Consequentemente, temos:

$$t\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\le\Big(\frac{t}{t-t_0}\Big)\epsilon^{1/2}.$$

Logo, como $\epsilon>0$ é arbitrário,

$$t\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\to 0,\ t\to\infty.$$

Mais geralmente para $m\ge 3$, também reescrevendo em coordenadas a equação para $\boldsymbol{u}$ e $\boldsymbol{b}$, derivando-as m vezes, em relação a $x_{l_1},...,x_{l_m}$, multiplicando-as por $2(t-t_0)D_{l_1}...D_{l_m}u_i$, $2(t-t_0)D_{l_1}...D_{l_m}b_i$ respectivamente e integrando-as em $\mathbb{R}^n\times[t_0,t]$, obtemos

$$\begin{aligned}&\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m(D_{l_1}...D_{l_m}u_i(x,\tau))^2_\tau\,dxd\tau\\
&+\sum_{j,l_1,...,l_m=1}^{n}2\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m D_{l_1}...D_{l_m}u_i(x,\tau)D_{l_1}...D_{l_m}(u_j(x,\tau)D_ju_i(x,\tau))dxd\tau\\
&+2\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m D_{l_1}...D_{l_m}u_i(x,\tau)D_{l_1}...D_{l_m}D_ip(x,\tau)\,dxd\tau\\
&=2\mu\sum_{j,l_1,...,l_m}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m D_{l_1}...D_{l_m}u_i(x,\tau)D_jD_j(D_{l_1}...D_{l_m}u_i(x,\tau))\,dxd\tau\\
&+\sum_{j,l_1,...,l_m=1}^{n}2\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m D_{l_1}...D_{l_m}u_i(x,\tau)D_{l_1}...D_{l_m}(b_j(x,\tau)D_j\ b_i(x,\tau))\,dxd\tau,\end{aligned}$$

e

$$\begin{aligned}&\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m(D_{l_1}...D_{l_m}b_i(x,\tau))^2_\tau\,dxd\tau\\
&+\sum_{j,l_1,...,l_m=1}^{n}2\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m D_{l_1}...D_{l_m}b_i(x,\tau)D_{l_1}...D_{l_m}(u_j(x,\tau)D_jb_i(x,\tau))dxd\tau\\
&=2\nu\sum_{j,l_1,...,l_m}^{n}\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m D_{l_1}...D_{l_m}b_i(x,\tau)D_jD_j(D_{l_1}...D_{l_m}b_i(x,\tau))\,dxd\tau\\
&+\sum_{j,l_1,...,l_m=1}^{n}2\int_{t_0}^{t}\int_{\mathbb{R}^n}(\tau-t_0)^m D_{l_1}...D_{l_m}b_i(x,\tau)D_{l_1}...D_{l_m}(b_j(x,\tau)D_ju_i(x,\tau))\,dxd\tau.\end{aligned}$$

Fazendo estimativas análogas aos casos m=1, m=2 e somando $i, l_1, ..., l_m$:

$$(t-t_0)^m \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau$$
$$\leq m\int_{t_0}^{t} (\tau-t_0)^{m-1} \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau$$
$$+C\int_{t_0}^{t} (\tau-t_0)^m \sum_{l=0}^{[m]/2} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{\frac{l+3/2}{l+2}}_{L^2(\mathbb{R}^n)} \|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})(\cdot,\tau)\|^{\frac{l/2}{l+2}}_{L^2(\mathbb{R}^n)} \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau. \quad (6.$$

Pelo Teorema 3.0.1, existe $t_0 > t_*$ suficientemente grande, tal que

$$\sum_{l=0}^{[m]/2} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{\frac{l+3/2}{l+2}}_{L^2(\mathbb{R}^n)} \|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})(\cdot,\tau)\|^{\frac{l/2}{l+2}}_{L^2(\mathbb{R}^n)} < \frac{2\min(\mu,\nu)}{C},\ \forall \tau \geq t_0.$$

Logo, podemos reescrever (6.24):

$$(t-t_0)^m \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + C_m\int_{t_0}^{t} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau$$
$$\leq m\int_{t_0}^{t} (\tau-t_0)^{m-1} \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau,$$

onde $C_m > 0$. Isto prova item 1.
Utilizando os raciocínios anteriores, os itens 2 e 3, seguem do passo m-1, então vale

$$\int_{t_0}^{\infty} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau < \infty.$$

Isto prova o item 2.
Como $\epsilon > 0$ é arbitrário, segue

$$t^{m/2} \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \to 0,\ t\to\infty.$$

Isto garante a veracidade do item 3.

- **Caso n=4**

Para provar este caso, utilizamos o Lema 4.1.3, onde temos a desigualdade de Sobolev referente a dimensão n=4, que vale para $m \geq 1$.

Reescrevendo em coordenadas a equação para $\boldsymbol{u}$ e $\boldsymbol{b}$, derivando-as m vezes, em relação $x_{l_1}, ..., x_{l_m}$, após multiplicando-as por $2(t-t_0)D_{l_1}...D_{l_m}u_i$, $2(t-t_0)D_{l_1}...D_{l_m}b_i$

respectivamente e integrando-as em $\mathbb{R}^n \times [t_0, t]$, obtemos

$$\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m (D_{l_1}...D_{l_m} u_i(x,\tau))_\tau^2 \, dxd\tau}_{(I)}$$

$$+ \underbrace{\sum_{j,l_1,...,l_m=1}^{n} 2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m D_{l_1}...D_{l_m} u_i(x,\tau) D_{l_1}...D_{l_m}(u_j(x,\tau) D_j u_i(x,\tau)) dxd\tau}_{(II)}$$

$$+ \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m D_{l_1}...D_{l_m} u_i(x,\tau) D_{l_1}...D_{l_m} D_i p(x,\tau) \, dxd\tau}_{(III)}$$

$$= \underbrace{2\mu \sum_{j,l_1,...,l_m=1}^{n} \int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m D_{l_1}...D_{l_m} u_i(x,\tau) D_j D_j (D_{l_1}...D_{l_m} u_i(x,\tau)) \, dxd\tau}_{(IV)}$$

$$+ \underbrace{\sum_{j,l_1,...,l_m=1}^{n} 2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m D_{l_1}...D_{l_m} u_i(x,\tau) D_{l_1}...D_{l_m}(b_j(x,\tau) D_j b_i(x,\tau)) \, dxd\tau}_{(V)} .$$

e

$$\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m (D_{l_1}...D_{l_m} b_i(x,\tau))_\tau^2 \, dxd\tau}_{(A)}$$

$$+ \underbrace{\sum_{j,l_1,...,l_m=1}^{n} 2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m D_{l_1}...D_{l_m} b_i(x,\tau) D_{l_1}...D_{l_m}(u_j(x,\tau) D_j b_i(x,\tau)) dxd\tau}_{(B)}$$

$$= \underbrace{2\nu \sum_{j,l_1,...,l_m}^{n} \int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m D_{l_1}...D_{l_m} b_i(x,\tau) D_j D_j (D_{l_1}...D_{l_m} b_i(x,\tau)) \, dxd\tau}_{(C)}$$

$$+ \underbrace{\sum_{j,l_1,...,l_m=1}^{n} 2\int_{t_0}^{t}\int_{\mathbb{R}^n} (\tau - t_0)^m D_{l_1}...D_{l_m} b_i(x,\tau) D_{l_1}...D_{l_m}(b_j(x,\tau) D_j u_i(x,\tau)) \, dxd\tau}_{(D)} .$$

Fazemos estimativas análogas ao caso n=3, nos termos (I) e (A), aplicando Fubini e integração por partes. Nos termos (III), (IV) e (C) também integramos por partes, sendo que (C) será zero, pois $\nabla \cdot \boldsymbol{u}(\cdot, t) = 0$. Em (II),(V),(B) e (D) estimaremos não apenas com integração por partes, mas também com a desigualdade de Cauchy-

Schwarz e após, somando em $i, l_1, ..., l_m$, e aplicando o Lema 4.1.3, chegamos em

$$(t-t_0)^m \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\min(\mu,\nu)\int_{t_0}^{t}(\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau$$
$$\leq m\int_{t_0}^{t}(\tau-t_0)^{m-1}\|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau$$
$$+C\int_{t_0}^{t}(\tau-t_0)^m \|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau.$$

Pelo Teorema 3.0.1, existe $t > t_0$ tal que

$$\|(D\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} < \frac{2\min\{\mu,\nu\}}{C},\ \forall \tau \geq t_0.$$

Segue

$$(t-t_0)^m \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + C_m \int_{t_0}^{t}(\tau-t_0)^m\ \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau$$
$$\leq\ m\int_{t_0}^{t}(\tau-t_0)^{m-1}\|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau,$$

para $C_m > 0$ e $\forall t > t_0 (t_0 > 0$ suficientemente grande).
Isto prova o item 1.
Consequentemente, pelo caso m-1

$$\int_{t_0}^{\infty}(\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau < \infty,$$

vale o item 2.
E para o item 3 temos

$$\int_{t_0}^{t}(\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau < \epsilon,$$

o que implica em

$$t^m\Big(\frac{t-t_0}{t}\Big)^m \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} = (t-t_0)^m \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} \leq \epsilon,$$

para $\epsilon > 0$ arbitrário.
Fazendo $t \to \infty$, segue

$$t^{m/2}\|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \to 0.$$

Isto prova item 3.
Com isso, finalizamos a prova do teorema. □

6.2 Comportamento assintótico das normas $\dot{H}^s$ e L^q

Abaixo, temos os corolários decorrentes da prova do Teorema 6.1.1.

Corolário 6.2.1. *Seja* $(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)$ *solução de Leray do problema MHD incompressível, então*

$$\lim_{t\to\infty} t^{s/2}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} = 0,$$

para todo $s \geq 0$ *e* $2 \leq n \leq 4$.

Demonstração. Vamos utilizar na prova o Teorema de interpolação 2.3.2, o qual garante a desigualdade abaixo:

$$\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)}^{\frac{s}{m}}.$$

Note que, podemos reescrever a desigualdade acima como

$$\begin{aligned} t^{\gamma}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} &\leq t^{\gamma}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)}^{\frac{s}{m}} \\ &\leq \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}}\left[t^{\gamma\frac{m}{s}}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)}\right]^{\frac{s}{m}}, \end{aligned}$$

e precisamos da condição $\gamma\frac{m}{s} = \frac{m}{2}$, pois assim $\gamma = \frac{s}{2}$, para $s \geq 0$ qualquer.
Então, dado $\epsilon > 0$, existe $t_0 > 0$ tal que

$$t^{s/2}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}}\left[t^{\frac{m}{2}}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)}\right]^{\frac{s}{m}} < \epsilon^{(1-\frac{s}{m})}\epsilon^{\frac{s}{m}} = \epsilon,$$

pelos Teoremas 5.2.2 e 6.1.1 e Parseval. Logo,

$$t^{s/2}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0,\ t \to \infty.$$

□

Corolário 6.2.2. *Dada solução de Leray* $(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)$ *para o problema MHD incompressível, temos*

$$\lim_{t\to\infty} t^{\frac{n}{4}}\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = 0,$$

para $2 \leq n \leq 4$.

Demonstração. Começaremos com o caso n=2.
Pela desigualdade de Sobolev (2.4.1), temos

$$\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1/2}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1/2}.$$

Então, dado $\epsilon > 0$, existe $t_0 > 0$ tal que

$$t^{1/2}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{1/2}\left(t\ \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{1/2} < C\epsilon^{1/2}\epsilon^{1/2} = C\epsilon,$$

pelo Teorema 5.2.2 e Corolário 6.2.1.
Logo,

$$t^{1/2}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq C\epsilon,\ \forall t \geq t_0.$$

Isto nos diz que

$$\lim_{t\to\infty} t^{\frac{2}{4}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} = 0.$$

Os casos n=3 e n=4 seguem a mesma ideia. Então provaremos estes resultados simultaneamente.
Utilizando a desigualdade de Sobolev (2.4.1),

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq C\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{1/4}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{3/4},$$

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^4)} \leq C\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^4)}.$$

Logo, podemos reescrever as desigualdades anteriores como:

$$t^{3/4}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq C\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{1/4}\left(t\ \|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right)^{3/4},$$

$$t\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^4)} \leq C\ t\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^4)}.$$

Utilizando o Teorema 5.2.2 e o Corolário 6.2.1, dado $\epsilon > 0$, existe $t_0 > 0$ tal que

$$t^{3/4}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq C\epsilon^{1/4}\epsilon^{3/4} = C\epsilon,$$

$$t\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^4)} \leq C\epsilon.$$

Com isso, temos

$$\lim_{t\to\infty} t^{\frac{3}{4}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = 0,$$
$$\lim_{t\to\infty} t^{\frac{4}{4}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^4)} = 0.$$

Ficando provado o teorema para $2 \leq n \leq 4$. □

Corolário 6.2.3. *Dada solução de Leray $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$ para o problema MHD incompressível, temos*

$$\lim_{t\to\infty} t^{\frac{n}{4}-\frac{n}{2q}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0,$$

para $2 \leq n \leq 4$.

Demonstração. Pelo Teorema de interpolação 2.3.1 para p=2, temos

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{\frac{2}{q}} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{2}{q}},$$

então,

$$t^\gamma \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{\frac{2}{q}} \left(t^{\frac{\gamma}{1-\frac{2}{q}}} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \right)^{1-\frac{2}{q}}.$$

Note que, fazendo

$$\frac{\gamma}{1-\frac{2}{q}} = \frac{n}{4},$$

temos

$$\gamma = \frac{n}{4} - \frac{n}{2q}.$$

Pelo Teorema 5.2.2 e Corolário 6.2.2, temos que dado $\epsilon > 0$, existe $t_0 > 0$ tal que

$$t^{\frac{n}{4}-\frac{n}{2q}} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq \epsilon^{\frac{2}{q}} \epsilon^{(1-\frac{2}{q})} = \epsilon, \ \forall t \geq t_0.$$

Isto nos diz que

$$\lim_{t\to\infty} t^{\frac{n}{4}-\frac{n}{2q}} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0.$$

□

Corolário 6.2.4. *Dada solução de Leray $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$ para o problema MHD incompressível, temos*

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{4}} \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = 0,$$

para $2 \leq n \leq 4$.

Demonstração. Vamos utilizar o Teorema de interpolação 2.4.1, o qual diz que:

$$\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{1/2} \|(D^{m+n}\boldsymbol{u}, D^{m+n}\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{1/2}.$$

Reescrevendo a desigualdade acima, obtemos

$$t^\gamma \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \left(t^{2\gamma_1} \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{1/2} \left(t^{2\gamma_2} \|(D^{m+n}\boldsymbol{u}, D^{m+n}\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{1/2},$$

com $\gamma = \gamma_1 + \gamma_2$.
Note que precisamos que

$$2\gamma_1 = \frac{m}{2} \ \text{ e } \ 2\gamma_2 = \frac{n+m}{2},$$

pois assim

$$\gamma = \frac{m}{4} + \frac{n+m}{4} = \frac{m}{2} + \frac{n}{4}.$$

Então, dado $\epsilon > 0$, existe $t_0 > 0$ tal que

$$\begin{aligned} t^{\frac{m}{2}+\frac{n}{4}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\leq \left(t^{\frac{m}{2}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{1/2} \\ &\times \left(t^{\frac{m+n}{2}}\|(D^{m+n}\boldsymbol{u}, D^{m+n}\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{1/2} \leq \epsilon^{1/2}\epsilon^{1/2} = \epsilon \ \forall t \geq t_0, \end{aligned}$$

utilizando o Teorema 6.1.1.
Com isso,

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{4}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

□

Corolário 6.2.5. *Dada solução de Leray $(\boldsymbol{u},\boldsymbol{b})(\cdot,t)$ para o problema MHD incompressível, temos*

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{4}-\frac{n}{2q}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0,$$

para $2 \leq n \leq 4$.

Demonstração. Nesta prova utilizaremos Teorema de interpolação 2.4.1, então

$$\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq \|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{\frac{2}{q}}\|(D^{m+n}\boldsymbol{u}, D^{m+n}\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{2}{q}},$$

que pode ser reescrito, como:

$$\begin{aligned} t^\gamma\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} &\leq \left(t^{\frac{\gamma_1}{\frac{2}{q}}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{\frac{2}{q}} \\ &\times \left(t^{\frac{\gamma_2}{1-\frac{2}{q}}}\|(D^{m+n}\boldsymbol{u}, D^{m+n}\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\right)^{1-\frac{2}{q}}, \end{aligned}$$

com $\gamma = \gamma_1 + \gamma_2$.
Tomando

$$\frac{\gamma_1}{\frac{2}{q}} = \frac{m}{2}$$

e

$$\frac{\gamma_2}{1-\frac{2}{q}} = \frac{m}{2} + \frac{n}{4},$$

teremos

$$\gamma_1 = \frac{m}{q}$$

e

$$\gamma_2 = \frac{m}{2} - \frac{m}{q} + \frac{n}{4} - \frac{n}{2q}.$$

Desta forma
$$\gamma = \gamma_1 + \gamma_2 = \frac{m}{2} + \frac{n}{4} - \frac{n}{2q}.$$
Utilizando o Teorema 6.1.1 e o Corolário anterior, temos que dado $\epsilon > 0$, existe $t_0 > 0$ tal que

$$\begin{aligned} t^{\frac{m}{2}+\frac{n}{4}-\frac{n}{2q}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} &\leq \left(t^{\frac{m}{2}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{\frac{2}{q}} \\ &\times \left(t^{\frac{m}{2}+\frac{n}{4}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\right)^{1-\frac{2}{q}} \leq \epsilon^{\frac{2}{q}}\epsilon^{1-\frac{2}{q}} \leq \epsilon, \forall t \geq t_0. \end{aligned}$$

Logo,

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{4}-\frac{n}{2q}}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0.$$

□

Capítulo 7

Desigualdade Fundamental

A próxima desigualdade é o resultado mais importante do trabalho, pois com esta demonstração é possível obter taxas de decrescimento melhores e estender o problema de Leray. Uma observação importante é que podemos resgatar alguns resultados provados nos capítulos anteriores dessa tese. Lembrando que a desigualdade utiliza técnicas desenvolvidas em [27].

Na demonstração do Teorema 7.0.1, as desigualdades (7.3), (7.10), (7.22), (7.26), (7.31) e (7.37) são derivadas de forma análogas ao item 1 do Teorema 6.1.1. Sendo assim, omitiremos estas provas e usaremos estes resultados diretamente. Após a prova do Teorema 7.0.1, temos alguns corolários.

Teorema 7.0.1. *Dadas* $(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)$ *soluções de Leray para o problema MHD incompressível, com* $2 \leq n \leq 4$, *tal que,* $\limsup_{t\to\infty} t^{\alpha} \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty$ *para todo* $\alpha \geq 0$, *temos:*

$$\lim_{t\to\infty} \sup t^{\alpha+\frac{m}{2}} \|(D^m \boldsymbol{u}, D^m \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha, m) C^{-m/2} \lim_{t\to\infty} \sup t^{\alpha} \|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)}$$

para todo $m \geq 1$, *onde*

$$K(\alpha, m) = \min_{\delta>0} \left[\delta^{-1/2} \prod_{j=0}^{m} (\alpha + j/2 + \delta)^{1/2} \right].$$

Demonstração. Já sabemos oriundo da teoria de regularidade de Leray, que existe $t_* \geq 0$, tal que:

$$(\boldsymbol{u}, \boldsymbol{b}) \in C^{\infty}(\mathbb{R}^n \times [t_*, \infty))$$

e

$$(\boldsymbol{u}, \boldsymbol{b})(\cdot, t) \in C^0([t_*, \infty), H^m(\mathbb{R}^n)),$$

$\forall\, m \geq 0$. Vamos usar estas hipóteses e também que

$$\lim_{t\to\infty} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0 \tag{7.1}$$

e

$$\lim_{t\to\infty} \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0, \tag{7.2}$$

resultados provados em capítulos anteriores.

- **Caso n=3**

Multiplicando (1.1) e (1.2), por $2(t-t_0)^{2\alpha+\delta}u_i(x,t)$ e $2(t-t_0)^{2\alpha+\delta}\, b_i(x,t)$ para $\delta > 0$ e $m = 0$, respectivamente. Após integramos por parte em $\mathbb{R}^n \times [t_0,t]$ e utilizando a hipótese que $\nabla\cdot\boldsymbol{u}(\cdot,t)=0$, obtemos

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ \leq (2\alpha+\delta)\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta-1}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau,\end{aligned} \tag{7.3}$$

$\forall\, t \geq t_0 \geq t_*$ e $C = \min\{\mu,\nu\} > 0$.
Pela hipótese

$$\lim_{t\to\infty}\sup t^\alpha \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} := \lambda_0(\alpha) < \infty,$$

temos

$$t^{2\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq (\lambda_0(\alpha)+\epsilon)^2, \text{ para t grande.} \tag{7.4}$$

Utilizando (7.4) em (7.3), segue

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ \leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta-1}\tau^{-2\alpha}d\tau.\end{aligned} \tag{7.5}$$

Note que

$$(\tau-t_0)^{2\alpha+\delta-1}\tau^{-2\alpha} \leq (\tau-t_0)^{\delta-1}. \tag{7.6}$$

Então aplicando a desigualdade acima em (7.5)

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ \leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^t (\tau-t_0)^{\delta-1}d\tau.\end{aligned} \tag{7.7}$$

Resolvendo o lado direito da desigualdade anterior, temos

$$(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta}, \quad (7.8)$$

com isso

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \leq \frac{1}{2C}(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta}, \quad (7.9)$$

$\forall t \geq t_0$ ($t_0 \geq t_*$ suficientemente grande).

Derivando (1.1) e (1.2) em relação a D_l, multiplicando por $2(t-t_0)^{2\alpha+1+\delta}D_l u_i(x,t)$ e $2(t-t_0)^{2\alpha+1+\delta}D_l b_i(x,t)$. Integrando por partes em $[t_0,t]\times\mathbb{R}^n$ e utilizando a hipótese que $\nabla\cdot\boldsymbol{u}(\cdot,t)=0$, obtemos para $t\geq t_0>t_*$ e $m=1$:

$$(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq (2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+K_1\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau. \quad (7.10)$$

Pela Desigualdade de Sobolev dada em (4.1.2), vale

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq C\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}$$
$$\times\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}. \quad (7.11)$$

Aplicando em (7.10)

$$(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq (2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+K_1\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau. \quad (7.12)$$

Pelos resultados (7.1) e (7.2), temos que dado $\epsilon>0$, existe $t_0>0$, tal que $\forall\, t\geq t_0$

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^n)} \leq \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}\frac{\epsilon\, C}{K_1\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^n)}}. \quad (7.13)$$

Então (7.12), torna-se

$$(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}+2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq(2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+\epsilon\, C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau. \tag{7.14}$$

Agrupando os termos anteriores que envolvem a derivada segunda e utilizando (7.9), temos

$$(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}+(2-\epsilon)C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq\frac{1}{2C}(2\alpha+1+\delta)\frac{(2\alpha+\delta)}{\delta}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}. \tag{7.15}$$

Segue da desigualdade (7.15) que

$$\begin{aligned}(t-t_0)^{2\alpha+1}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq \frac{1}{2C}(2\alpha+1+\delta)\frac{(2\alpha+\delta)}{\delta}(\lambda_0(\alpha)+\epsilon)^2\\ &\leq \frac{1}{(2-\epsilon)C}(2\alpha+1+\delta)\frac{(2\alpha+\delta)}{\delta}(\lambda_0(\alpha)+\epsilon)^2\end{aligned} \tag{7.16}$$

e

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\leq\frac{(2\alpha+1+\delta)(2\alpha+\delta)}{\delta[2C^2(2-\epsilon)]}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}$$
$$\leq\frac{(2\alpha+1+\delta)(2\alpha+\delta)}{\delta[(2-\epsilon)C]^2}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}, \tag{7.17}$$

$\forall t\geq t_0$.
Em um processo análogo feito em (7.10) para m=1, encontramos para m=2 a desigualdade:

$$(t-t_0)^{2\alpha+2+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}+2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+2+\delta}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq(2\alpha+2+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+K_2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+2+\delta}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau. \tag{7.18}$$

Pelo (4.1.2), temos que

$$(t-t_0)^{2\alpha+2+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}+2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+2+\delta}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq(2\alpha+2+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+K_2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+2+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau. \tag{7.19}$$

Utilizando (7.1) e (7.2) na desigualdade anterior, conseguimos a estimativa

$$(t-t_0)^{2\alpha+2+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}+C(2-\epsilon)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+2+\delta}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq(2\alpha+2+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau. \tag{7.20}$$

Segue de (7.17)

$$(t-t_0)^{2\alpha+2+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}+C(2-\epsilon)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+2+\delta}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq(2\alpha+2+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq\frac{(2\alpha+2+\delta)(2\alpha+1+\delta)(2\alpha+\delta)}{\delta[(2-\epsilon)C]^2}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}. \tag{7.21}$$

Finalmente por (7.21)

$$(t-t_0)^{2\alpha+2}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\leq\frac{(2\alpha+2+\delta)(2\alpha+1+\delta)(2\alpha+\delta)}{\delta[(2-\epsilon)C]^2}(\lambda_0(\alpha)+\epsilon)^2$$

e

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+2+\delta}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\leq\frac{(2\alpha+2+\delta)(2\alpha+1+\delta)(2\alpha+\delta)}{\delta[(2-\epsilon)C]^3}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}$$

Prosseguindo nesse caminho, fazendo indutivamente para m=3,4..., obtemos no m-ésimo passo:

$$
\begin{aligned}
(t-t_0)^{2\alpha+m+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &+ 2C\int_{t_0}^t(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
&\leq (2\alpha+m+\delta)\int_{t_0}^t(\tau-t_0)^{2\alpha+(m-1)+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
&\quad +K_m\int_{t_0}^t(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)} \\
&\quad \times\sum_{l=0}^{[m/2]}\|(D^l\boldsymbol{u},D^l\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau,
\end{aligned}
\tag{7.22}
$$

para $t\geq t_0,\ K_m>0$ e $[m/2]$ denotando a parte inteira de m/2.
Para $m\geq 3,\ 0\leq l\leq m-3$, vale a desigualdade de Sobolev:

$$
\begin{aligned}
&\|(D^l\boldsymbol{u},D^l\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \\
&\leq C\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^{\frac{l+3/2}{l+2}}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u},D^{l+2}\boldsymbol{b})(\cdot,t)\|^{\frac{1/2}{l+2}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

onde $C>0$.
Aplicando a desigualdade acima em (7.22), encontramos

$$
\begin{aligned}
(t-t_0)^{2\alpha+m+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &+ 2C\int_{t_0}^t(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
&\leq (2\alpha+m+\delta)\int_{t_0}^t(\tau-t_0)^{2\alpha+(m-1)+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
&\quad +K_m\left(1+\left[\frac{m}{2}\right]\right)\int_{t_0}^t(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \times\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^{\frac{l+3/2}{l+2}}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u},D^{l+2}\boldsymbol{b})(\cdot,\tau)\|^{\frac{1/2}{l+2}}_{L^2(\mathbb{R}^n)}d\tau.
\end{aligned}
\tag{7.23}
$$

Nessa etapa, sabemos que vale

$$
(t-t_0)^{2\alpha+m}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\leq\frac{1}{\delta[(2-\epsilon)C]^m}\left[\prod_{j=0}^m(2\alpha+j+\delta)\right](\lambda_0(\alpha)+\epsilon)^2
\tag{7.24}
$$

e

$$
\begin{aligned}
\int_{t_0}^t(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau &\leq\frac{\delta^{-1}}{[(2-\epsilon)C]^{m+1}}\left[\prod_{j=0}^m(2\alpha+j+\delta)\right] \\
&\quad \times(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^\delta,
\end{aligned}
\tag{7.25}
$$

para $t \geq t_0$ e $m \geq 1$.
Reescrevendo a desigualdade (7.24), temos

$$(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{\delta^{1/2}[(2-\epsilon)C]^{m/2}}\left[\prod_{j=0}^{m}(\alpha+j/2+\delta)^{1/2}\right](\lambda_0(\alpha)+\epsilon).$$

E ainda para $\delta > 0,\ 0 < \epsilon < 2$, vale

$$(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha,m)C^{-m/2}\lambda_0(\alpha),$$

com

$$K(\alpha,m) = \min_{\delta>0}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+j/2+\delta)^{1/2}\right].$$

Note que podemos escrever

$$\begin{aligned} t^{\alpha+m/2}\left(\frac{t-t_0}{t}\right)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} &= (t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \\ &\leq K(\alpha,m)C^{-m/2}\lambda_0(\alpha). \end{aligned}$$

Fazendo $t \to \infty$ e como ϵ é arbitrário, temos

$$\limsup_{t\to\infty} t^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha,m)C^{-m/2}\lambda_0(\alpha),$$

ou seja,

$$\limsup_{t\to\infty} t^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha,m)C^{-m/2}\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)},$$

o que prova o resultado para n=3.

- **Caso n=4**

Começamos multiplicando (1.1) e (1.2), por $2(t-t_0)^{2\alpha+\delta}u_i(x,t)$ e $2(t-t_0)^{2\alpha+\delta}b_i(x,t)$, respectivamente, depois integramos por partes em $[t_0,t]\times\mathbb{R}^n$ e utilizamos a hipótese $\nabla\cdot\boldsymbol{u}(\cdot,t) = 0$, com objetivo de obter para $t \geq t_0 > t_*$ e m=0 a desigualdade:

$$\begin{aligned} (t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} &+ 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ &\leq (2\alpha+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta-1}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau, \quad (7.26) \end{aligned}$$

$\forall\, t \geq t_0 \geq t_*$ e $C = \min\{\mu, \nu\} > 0$.
Pela hipótese (7.4) , obtemos

$$(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta-1}\tau^{-2\alpha}d\tau. \quad (7.27)$$

Como

$$(\tau-t_0)^{2\alpha+\delta-1}\tau^{-2\alpha} \leq (\tau-t_0)^{\delta-1},$$

temos que (7.27), torna-se

$$(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^{t}(\tau-t_0)^{\delta-1}d\tau. \quad (7.28)$$

Resolvendo a integral no lado direito, temos

$$(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta}. \quad (7.29)$$

Donde segue

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}d\tau \leq \frac{1}{2C}(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta}, \quad (7.30)$$

$\forall t \geq t_0$ ($t_0 \geq t_*$ suficientemente grande).

Derivando (1.1) e (1.2) em relação a D_l, multiplicando o resultado obtido por $2(t-t_0)^{2\alpha+1+\delta}D_l u_i(x,t)$ e $2(t-t_0)^{2\alpha+1+\delta}D_l b_i(x,t)$, respectivamente e então integrando por partes em $[t_0,t]\times\mathbb{R}^n$ e utilizando a hipótese $\nabla\cdot\boldsymbol{u}(\cdot,t)=0$, obtemos para $t \geq t_0 > t_*$ e $m=1$, o seguinte:

$$(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq (2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+K_1\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}d\tau. \quad (7.31)$$

Utilizando em (7.31) a Desigualdade de Sobolev 4.1.3, para $m \geq 1$ e $0 \leq l \leq m-1$:

$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^4(\mathbb{R}^4)}\|(D\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^4(\mathbb{R}^4)} \leq C\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^4)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^4)}$,

onde $C > 0$, temos

$$\begin{aligned}(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ \leq (2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ +K_1\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau.\end{aligned} \tag{7.32}$$

Pelo resultado (7.2), vale

$$\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon\, C}{K_1}, \forall t > t_0.$$

Aplicando em (7.32) temos

$$\begin{aligned}(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ \leq (2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ +\epsilon\, C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau.\end{aligned} \tag{7.33}$$

Agrupando as integrais da desigualdade anterior que envolvem as derivadas de segunda ordem, obtemos

$$\begin{aligned}(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + (2-\epsilon)C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ \leq (2\alpha+1+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau.\end{aligned}$$

Pela desigualdade (7.30), chegamos a

$$\begin{aligned}(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + (2-\epsilon)C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}d\tau \\ \leq \frac{1}{2C}(2\alpha+1+\delta)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta}.\end{aligned} \tag{7.34}$$

Segue da desigualdade anterior que

$$(t-t_0)^{2\alpha+1+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \frac{1}{2C}(2\alpha+1+\delta)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^\delta}{\delta},$$

então

$$\begin{aligned}(t-t_0)^{2\alpha+1}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq \frac{1}{2C}(2\alpha+1+\delta)\frac{(2\alpha+\delta)}{\delta}(\lambda_0(\alpha)+\epsilon)^2,\\ &\leq \frac{1}{(2-\epsilon)C}(2\alpha+1+\delta)\frac{(2\alpha+\delta)}{\delta}(\lambda_0(\alpha)+\epsilon)^2. \quad (7.35)\end{aligned}$$

E também

$$\begin{aligned}\int_{t_0}^t(\tau-t_0)^{2\alpha+1+\delta}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau &\leq \frac{(2\alpha+1+\delta)(2\alpha+\delta)}{\delta[2C^2(2-\epsilon)]}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^\delta\\ &\leq \frac{(2\alpha+1+\delta)(2\alpha+\delta)}{\delta[(2-\epsilon)C]^2}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^\delta,\end{aligned} \tag{7.36}$$

$\forall t \geq t_0$.

Prosseguindo nesse caminho, fazendo indutivamente para m=2,3..., obtemos no m-ésimo passo:

$$\begin{aligned}(t-t_0)^{2\alpha+m+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2C\int_{t_0}^t(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\\ \leq (2\alpha+m+\delta)\int_{t_0}^t(\tau-t_0)^{2\alpha+(m-1)+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau\\ +K_m\int_{t_0}^t(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\\ \times\sum_{l=0}^{[m/2]}\|(D^l\boldsymbol{u},D^l\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})(\cdot,\tau)\|_{L^4(\mathbb{R}^n)}d\tau,\end{aligned} \tag{7.37}$$

para $t \geq t_0,\ K_m > 0$ e $[m/2]$ denotando a parte inteira de m/2.
Temos que vale pela Desigualdade Sobolev 4.1.3, para $m \geq 1$ e $0 \leq l \leq m-1$:

$$\begin{aligned}&\|(D^l\boldsymbol{u},D^l\boldsymbol{b})(\cdot,t)\|_{L^4(\mathbb{R}^4)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})(\cdot,t)\|_{L^4(\mathbb{R}^4)}\\ &\leq \|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

onde $C > 0$.

Aplicando em (7.37) tal desigualdade, obtemos

$$(t-t_0)^{2\alpha+m+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}+2C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$\leq(2\alpha+m+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+(m-1)+\delta}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+K_m\,(1+[m/2])\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\|(D\boldsymbol{u},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}.$$

Note que

$$(t-t_0)^{2\alpha+m}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\leq\frac{1}{\delta[(2-\epsilon)C]^m}\prod_{j=0}^{m}(2\alpha+j+\delta)(\lambda_0(\alpha)+\epsilon)^2 \tag{7.38}$$

e

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}d\tau\leq\frac{\delta^{-1}}{[(2-\epsilon)C]^{m+1}}\left[\prod_{j=0}^{k}(2\alpha+j+\delta)\right]\times(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}, \tag{7.39}$$

$\forall\ t\geq t_0$ e $m\geq 1$.

Logo para t suficientemente grande, dados $\delta>0$ e $0<\epsilon<2$ arbitrários, podemos reescrever (7.38) como:

$$(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\leq\frac{1}{\delta^{1/2}C^{m/2}}\left[\prod_{j=0}^{m}(\alpha+j/2+\delta)^{1/2}\right](\lambda_0(\alpha)+\epsilon), \tag{7.40}$$

onde

$$K(\alpha,m)=\min_{\delta>0}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+j/2+\delta)^{1/2}\right].$$

Podemos reescrever (7.40), como

$$(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\leq K(\alpha,m)C^{-m/2}\lambda_0(\alpha),$$

e ainda,

$$t^{\alpha+m/2}\left(\frac{t-t_0}{t}\right)^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\leq K(\alpha,m)C^{-m/2}\lambda_0(\alpha).$$

Fazendo $t\to\infty$, chegamos que

$$\limsup_{t\to\infty}t^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\leq K(\alpha,m)C^{-m/2}\limsup_{t\to\infty}t^{\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)},$$

vale para n=4, fechando a prova do resultado. □

Corolário 7.0.2. *Dados $\alpha > 0$ e $s \geq 0$, então*

$$(i)\limsup_{t\to\infty} t^{\alpha+s/2}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} < \infty,$$

se tivermos $\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty$.

$$(ii)\limsup_{t\to\infty} t^{\alpha+s/2}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} = 0,$$

se tivermos $\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0$.

Demonstração. Pelo Teorema de interpolação 2.4.2, temos

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)}^{\frac{s}{m}},$$

então, reescrevendo esta desigualdade, chegamos a

$$t^{\gamma}\ \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \left[t^{\frac{\gamma_1}{1-\frac{s}{m}}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right]^{1-\frac{s}{m}}\left[t^{\frac{\gamma_2}{\frac{s}{m}}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)}\right]^{\frac{s}{m}} \quad (7.41)$$

onde $\gamma = \gamma_1 + \gamma_2$.

Note que devemos ter

$$\frac{\gamma_1}{1-\frac{s}{m}} = \alpha \ \Rightarrow \gamma_1 = \alpha(1-\frac{s}{m})$$

e

$$\frac{\gamma_2}{\frac{s}{m}} = \alpha + \frac{m}{2} \Rightarrow \gamma_2 = (\alpha + \frac{m}{2})\frac{s}{m}.$$

Com isso, obtemos

$$\gamma = \gamma_1 + \gamma_2 = \alpha + \frac{s}{2}.$$

Então (7.41) torna-se

$$t^{\alpha+\frac{s}{2}}\ \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \left[t^{\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right]^{1-\frac{s}{m}}\left[t^{\alpha+\frac{m}{2}}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)}\right]^{\frac{s}{m}}.$$

Fazendo $t \to \infty$, aplicando a hipótese

$$\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty$$

e o Teorema 7.0.1 inferimos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\ \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} < \infty.$$

E usando novamente a hipótese

$$\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0,$$

e o Teorema 7.0.1, temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\ \|(\boldsymbol{u},\boldsymbol{b})(.,t)\|_{\dot{H}^s(\mathbb{R}^n)} = 0,$$

para todo $s \geq 0$. □

Capítulo 8

Apêndice

8.1 Desigualdades Relevantes

Abaixo, segue a prova das propriedades utilizadas como Lemas nos capítulos anteriores. Lembrando que essas propriedades são demonstradas utilizando as desigualdades (2.4.1) e (2.4.2) de Sobolev e que as constantes representam diferentes valores numéricos.

Lema 8.1.1. *Para n=2,*

$$\begin{aligned}
&(1) \quad \|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \le C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
&(2) \quad \|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \le C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\\
&(3) \quad \|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \le C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Em geral, para $m \ge 2$, $0 \le l \le m-2$*:*

$$\|(D^l\boldsymbol{u},D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \le C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u},D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.$$

para $C > 0$.

Demonstração. Para a prova do item (1), considere as desigualdades de Sobolev (2.4.1) e (2.4.2), referentes a dimensão n=2 e a definição (2.1):

$$\begin{aligned}
(i) &= \|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)} \le C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)},\\
(ii) &= \|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \le C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Por (i) e (ii), temos:

$$\begin{aligned}
\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\le C \quad \|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\\
&\times C \quad \|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\\
&\le C \quad \|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

o que prova o item (1).

Para a prova de (2), iremos utilizar a desigualdade (i) e também

$$(iii) = \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/3}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|^{2/3}_{L^2(\mathbb{R}^n)}.$$

Logo

$$\begin{aligned}
\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} &\leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\
&= C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{3/2}_{L^2(\mathbb{R}^n)}\\
&\leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\
&\leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

E por último, provaremos (3), utilizando o item (1) para (D$\boldsymbol{u}$, D$\boldsymbol{b}$), (ii) e (iii).

$$\begin{aligned}
\| \; & (D\boldsymbol{u}, D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\\
= \; & C\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{-1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{-1/2}_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{-1/6}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|^{-2/6}_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{-1/6}_{L^2(\mathbb{R}^n)}\\
\times \; & \|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|^{-2/6}_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/6}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\\
\times \; & \|(\boldsymbol{u},\boldsymbol{b})\|^{-1/6}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|^{-2/6}_{L^2(\mathbb{R}^n)}\\
= \; & C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Continuando o processo, pelas desigualdades (i) e (2.4.2), temos:

$$\begin{aligned}
\| \; & (D^l\boldsymbol{u}, D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{\frac{l+1}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{m-l}{m+1}}_{L^2(\mathbb{R}^n)}\\
\leq \; & C(\|(\boldsymbol{u},\boldsymbol{b})\|^{\frac{m-l+1}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{l}{m+1}}_{L^2(\mathbb{R}^n)})^{1/2}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{\frac{l+1}{m+1}}_{L^2(\mathbb{R}^n)}\\
\times \; & \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{m-l}{m+1}}_{L^2(\mathbb{R}^n)}\\
\leq \; & C\|(\boldsymbol{u},\boldsymbol{b})\|^{\frac{m+l+3}{2(m+1)}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{2m-l}{2(m+1)}}_{L^2(\mathbb{R}^n)}(\|(\boldsymbol{u},\boldsymbol{b})\|^{\frac{m-l-1}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{l+2}{m+1}}_{L^2(\mathbb{R}^n)})^{1/2}\\
= \; & C\|(\boldsymbol{u},\boldsymbol{b})\|^{\frac{m+l+3}{2(m+1)}+\frac{m-l-1}{2(m+1)}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{2m-l}{2(m+1)}+\frac{l+2}{2(m+1)}}_{L^2(\mathbb{R}^n)}\\
= \; & C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

ou seja, que vale:

$$\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},$$

para $m \geq 2$, $0 \leq l \leq m-2$ e $C > 0$. □

Lema 8.1.2. *Para n=3,*

(1) $\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}$,

(2) $\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}$,

(3) $\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}$,

(4) $\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^4\boldsymbol{u}, D^4\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}$,

(5) $\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^5\boldsymbol{u}, D^5\boldsymbol{b})\|_{L^2(\mathbb{R}^n}$

Em geral, para $m \geq 3$, $0 \leq l \leq m-3$*:*

$$\begin{aligned}&\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\ \leq\ & C\|(\boldsymbol{u},\boldsymbol{b})\|^{\frac{l+3/2}{l+2}}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{\frac{1/2}{l+2}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

para $C > 0$.

Demonstração. Para n=3, temos pelas desigualdades (2.4.1), (2.4.2) de Sobolev e a definição (2.1):

$$\begin{aligned}(i) &= \|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)},\\ (ii) &= \|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Então segue de (i) e (ii),

$$\begin{aligned}\| &\ (\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\ \leq &\ C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\\ = &\ C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\\ \leq &\ C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\\ = &\ C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u}, D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

o que prova o item (1).
Para a prova da segunda desigualdade, considere

$$(iii) = \|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \leq C\|(D^2\boldsymbol{u}, D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u}, D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}.$$

Então por (1), (i), (ii) e (iii), segue

$$
\begin{aligned}
\| & (\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^\infty(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^\infty(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{-1/2}_{L^2(\mathbb{R}^n)} \\
\times & \|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\times & \|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{-1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)} \\
\times & \|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{-1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Agora iremos provar os próximos itens, utilizando (1) para (D$\boldsymbol{u}$,D$\boldsymbol{b}$), (ii) e (iii),

$$
\begin{aligned}
\| & (D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \\
\le & C\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)} \\
= & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

$$
\begin{aligned}
\| & (D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \\
= & C\|(D\boldsymbol{u},D\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(\mathrm{D}\boldsymbol{u},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)} \\
= & C\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^3\boldsymbol{u},D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \\
\le & C\|(D\boldsymbol{u},D\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^4\boldsymbol{u},D^4\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^4\boldsymbol{u},D^4\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u},\boldsymbol{b})\|^{1/2}_{L^2(\mathbb{R}^n)}\|(D^4\boldsymbol{u},D^4\boldsymbol{u})\|^{1/2}_{L^2(\mathbb{R}^n)} \\
\le & C\|(\boldsymbol{u},\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D^2\boldsymbol{u},D^2\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^4\boldsymbol{u},D^4\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Continuando o processo, usando as desigualdades (i) e (2.4.2), temos:

$$\begin{aligned}
\| & (D^l\boldsymbol{u}, D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \\
\leq & C\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \\
\leq & C\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|^{1/4}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{l+1}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{m-l}{m+1}}_{L^2(\mathbb{R}^n)} \\
\leq & C(\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{m+1-j}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{j}{m+1}}_{L^2(\mathbb{R}^n)})^{1/4}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{3/4}_{L^2(\mathbb{R}^n)}\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{l+1}{m+1}}_{L^2(\mathbb{R}^n)} \\
\times & \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{m-l}{m+1}}_{L^2(\mathbb{R}^n)} \\
= & C\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{1}{4}\frac{m+3l+5}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{\frac{3}{4}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{m-\frac{3}{4}j}{m+1}}_{L^2(\mathbb{R}^n)} \\
= & C\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{1}{4}\frac{m+3l+5}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{\frac{3}{4}(1-\beta)}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{\frac{3}{4}\beta}_{L^2(\mathbb{R}^n)} \\
\times & \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{m-\frac{3}{4}j}{m+1}}_{L^2(\mathbb{R}^n)},\ \beta \in (0,1). \\
= & C\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{1}{4}\frac{m+3l+5}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{\frac{3}{4}(1-\beta)}_{L^2(\mathbb{R}^n)}(\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{m-l-1}{m+1}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{j+2}{m+1}}_{L^2(\mathbb{R}^n)})^{\frac{3}{4}\beta} \\
\times & \|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|^{\frac{m-\frac{3j}{4}}{m+1}}_{L^2(\mathbb{R}^n)}).
\end{aligned}$$

Note que queremos:

$$\frac{3}{4}\beta\frac{j+2}{m+1} + \frac{m-\frac{3j}{4}}{m+1} = 1,\ \text{isto é },\ \beta = \frac{\frac{3j}{4}+1}{\frac{3j}{4}+\frac{3}{4}}\ \epsilon\ (0,1).$$

Com isso, temos:

$$\begin{aligned}
&\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)} \\
\leq C&\|(\boldsymbol{u}, \boldsymbol{b})\|^{\frac{l+3/2}{l+2}}_{L^2(\mathbb{R}^n)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\boldsymbol{b})\|^{\frac{1/2}{l+2}}_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

para $m \geq 3$, $0 \leq l \leq m-3$ e $C > 0$. □

Lema 8.1.3. *Para n=4, $m \geq 1$, $0 \leq l \leq m-1$:*

$$\|(D^l\boldsymbol{u}, D^l\boldsymbol{b})\|_{L^4(\mathbb{R}^n)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\boldsymbol{b})\|_{L^4(\mathbb{R}^n)} \leq C\|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\boldsymbol{b})\|_{L^2(\mathbb{R}^n)},$$

para $C > 0$.

Demonstração. A prova deste Lema segue diretamente da Desigualdade de Sobolev para n=4, dada por

$$\|(\boldsymbol{u}, \boldsymbol{b})\|_{L^4(\mathbb{R}^n)} \leq \|(D\boldsymbol{u}, D\boldsymbol{b})\|_{L^2(\mathbb{R}^n)}.$$

□

8.2 Projetor de Helmholtz

Nesta seção iremos comentar sobre alguns resultados e propriedades do projetor de Helmholtz.

Teorema 8.2.1. *Seja* $f = (f_1, ..., f_n) \in L^q(\mathbb{R}^n)$, *com* $n \geq 2$, $1 < q < \infty$. *Então, existe* $p \in L^q_{loc}(\mathbb{R}^n)$ *com* $\nabla p \ \in L^q(\mathbb{R}^n)$ *e* $f_0 \in L^q_\sigma(\mathbb{R}^n)$, *unicamente determinados, tais que*

$$f = f_0 + \nabla p,$$

$$\|f_0\|_{L^q(\mathbb{R}^n)} + \|\nabla p\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}.$$

Definição 8.2.2. *Seja* $f \in L^q(\mathbb{R}^n)$, *considere a decomposição anterior. Com isso, definimos o projetor de Helmholtz como:*

$$\begin{array}{rcl} \mathbb{P}_H : L^q(\mathbb{R}^n) & \longmapsto & L^q_\sigma(\mathbb{R}^n) \\ f & \longmapsto & f_0 = \mathbb{P}_H[f], \end{array}$$

para $1 < q < \infty$.

Um dos resultados principais deste projetor e que foi utilizado no texto é a propriedade de comutar com Heat Kernel, ou seja,

$$\mathbb{P}_H[e^{\nu\Delta t}\boldsymbol{u}] = e^{\nu\Delta t}[\mathbb{P}_H\boldsymbol{u}], \ \forall \ \boldsymbol{u} \ \in L^q(\mathbb{R}^n).$$

Abaixo seguem outras propriedades e teoremas:

• $\mathbb{P}_H$ é linear,
• $\mathbb{P}_H$ é limitada($\mathbb{P}_H \ \in \ B(L^q(\mathbb{R}^n))$),
• $\mathbb{P}_H^2 = \mathbb{P}_H$,
• $\nabla.\mathbb{P}(\boldsymbol{u}) = 0$, no sentido das distribuições.
• $\mathbb{P}_H$ é contínuo.

E ainda, vale a ortogonalidade do projetor em L^2, de modo que $\mathbb{P}_H : L^2(\mathbb{R}^n) \to L^2_\sigma(\mathbb{R}^n)$ satisfaz

$$\|\mathbb{P}_H(\boldsymbol{u})\|_{L^2(\mathbb{R}^n)} \leq \|\boldsymbol{u}\|_{L^2(\mathbb{R}^n)}.$$

Demonstração. Seja $\boldsymbol{u} = \mathbb{P}_H(\boldsymbol{u}) + \boldsymbol{w}$, *com* $\mathbb{P}_H(\boldsymbol{u}) \in L^2_\sigma(\mathbb{R}^n)$, $\boldsymbol{w} \in G_2 = [\ \nabla p : p \in L^2_{loc}(\mathbb{R}^n), \nabla p \in L^2(\mathbb{R}^n)\ ] \subseteq L^2(\mathbb{R}^n)$ e $L^2_\sigma(\mathbb{R}^n) \perp G_2$. Então,

$$\|\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)} = \|\mathbb{P}_H(\boldsymbol{u})\|^2_{L^2(\mathbb{R}^n)} + \|\boldsymbol{w}\|^2_{L^2(\mathbb{R}^n)},$$

logo

$$\|\mathbb{P}_H(\boldsymbol{u})\|^2_{L^2(\mathbb{R}^n)} \leq \|\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)}.$$

□

A prova que o projetor comuta com Heat Kernel, segue como consequência do seguinte resultado:

Teorema 8.2.3. *Seja* $1 < q < \infty$, $\boldsymbol{u} \in L^q(\mathbb{R}^n)$ *com* $D_j\boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$ *com* $1 < q_1 < \infty$. *Então*

$$D_j\mathbb{P}_H(\boldsymbol{u}) = \mathbb{P}_H(D_j\boldsymbol{u}) \in L^{q_1}_\sigma(\mathbb{R}^n).$$

Teorema 8.2.4. *(Desigualdade do Laplaciano)*
Seja $1 < q < \infty$. *Sendo* $\boldsymbol{u} \in L^1_{loc}(\mathbb{R}^n)$ *com* $\Delta\boldsymbol{u} \in L^q(\mathbb{R}^n)$ *então:*
se $\boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$ *para algum* $1 \leq q_0 \leq \infty$, *então* $D^2\boldsymbol{u} \in L^q(\mathbb{R}^n)$ *e*

$$\|D^2\boldsymbol{u}\|_{L^q(\mathbb{R}^n)} \leq C(n,q)\|\Delta\boldsymbol{u}\|_{L^q(\mathbb{R}^n)}.$$

Corolário 8.2.5. *(Comutatividade com o Laplaciano)*
Sejam $1 < q < \infty$, $1 < q_2 < \infty$ *e* $\boldsymbol{u} \in L^q(\mathbb{R}^n)$ *tal que* $\Delta\boldsymbol{u} \in L^{q_2}(\mathbb{R}^n)$. *Então*

$$\Delta\mathbb{P}_H(\boldsymbol{u}) = \mathbb{P}_H(\Delta\boldsymbol{u}).$$

Demonstração. Pelo Teorema 8.2.4, temos $D^2\boldsymbol{u} \in L^{q_2}(\mathbb{R}^n)$. Daí, sendo $q_1 \in (1,\infty)$ dado por

$$\frac{1}{q_1} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q_2}\right),$$

temos, pela desigualdade de Sobolev (2.4.2),

$$\|D\boldsymbol{u}\|_{L^{q_1}(\mathbb{R}^n)} \leq K\|\boldsymbol{u}\|^{1/2}_{L^q(\mathbb{R}^n)}\|D^2\boldsymbol{u}\|^{1/2}_{L^{q_2}(\mathbb{R}^n)},$$

Segue então, pelo Teorema 8.2.3 (aplicado duas vezes)

$$\Delta\mathbb{P}_H = \sum_j D_j(D_j\mathbb{P}_H(\boldsymbol{u})) = \sum_j D_j(\mathbb{P}_H(D_j\boldsymbol{u})) = \sum_j \mathbb{P}_H(D_jD_j\boldsymbol{u}) = \mathbb{P}_H(\Delta\boldsymbol{u}).$$

□

Teorema 8.2.6. *(Comutatividade com o Heat Kernel)*
Sendo $1 < q < \infty$, $\nu > 0$, *então*

$$\mathbb{P}_H[e^{\nu\Delta t}\boldsymbol{u}] = e^{\nu\Delta t}[\mathbb{P}_H\boldsymbol{u}],\ \forall\ \boldsymbol{u} \in L^q(\mathbb{R}^n), \forall t > 0.$$

Demonstração. Sejam $\boldsymbol{v}(\cdot,t) = e^{\nu\Delta t}\boldsymbol{u}$ e $\boldsymbol{w}(\cdot,t) = e^{\nu\Delta t}\mathbb{P}_H(\boldsymbol{u})$.
Então $\boldsymbol{v},\boldsymbol{w} \in C^0([0,\infty), L^q(\mathbb{R}^n))$ e

$$\boldsymbol{v}_t = \nu\Delta\boldsymbol{v},\ \boldsymbol{v}(\cdot,0) = \boldsymbol{u}$$

$$\boldsymbol{w}_t = \nu\Delta\boldsymbol{w},\ \boldsymbol{w}(\cdot,0) = \mathbb{P}_H(\boldsymbol{u}).$$

Temos que provas que $\boldsymbol{w}(\cdot,t) = \mathbb{P}_H(\boldsymbol{v}(\cdot,t))\ \forall t > 0$.
Seja $\boldsymbol{z}(\cdot,t) := \mathbb{P}_H(\boldsymbol{v}(\cdot,t))$. Temos que mostrar:

$$\boldsymbol{z}(\cdot,t) = \boldsymbol{w}(\cdot,t).$$

Como $\boldsymbol{v}(\cdot,t)) \in C^0([0,\infty), L^q(\mathbb{R}^n))$ e $\mathbb{P}_H : L^q(\mathbb{R}^n) \to L^q_\sigma(\mathbb{R}^n)$ é contínuo, então

$$\boldsymbol{z}(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n)).$$

Mostraremos a seguir que $\boldsymbol{z}(\cdot,t) = \nu\Delta\boldsymbol{z}(\cdot,t)\ \forall t > 0$.
Como

$$\boldsymbol{z}(\cdot,0) = \mathbb{P}_H(\boldsymbol{v}(\cdot,0)) = \mathbb{P}_H(\boldsymbol{u}) = \boldsymbol{w}(\cdot,0),$$

resulta daí (pela unicidade das soluções da equação do calor em $C^0([0,\infty), L^q(\mathbb{R}^n)))$ que

$$\boldsymbol{w}(\cdot,t) = \boldsymbol{z}(.,t),$$

demonstrando o resultado. Para mostrar que $\boldsymbol{z}_t = \nu\Delta\boldsymbol{z}$, podemos proceder como segue.
Dado $t > 0$ (fixo), tem-se

$$\begin{aligned} &\|\frac{1}{h}[\boldsymbol{z}(.,t+h) - \boldsymbol{z}(.,t)] - \mathbb{P}_H(\boldsymbol{v}_t(.,t))\|_{L^q(\mathbb{R}^n)} \\ &= \|\mathbb{P}_H\left[\frac{\boldsymbol{v}(.,t+h) - \boldsymbol{v}(.,t)}{h} - \boldsymbol{v}_t(.,t)]\right]\|_{L^q(\mathbb{R}^n)} \to 0 \end{aligned} \tag{8.1}$$

ao $h \to 0$, visto que

$$\frac{\boldsymbol{v}(.,t+h) - \boldsymbol{v}(.,t)}{h} \to \boldsymbol{v}_t(.,t) \text{ em } L^q(\mathbb{R}^n)$$

(pois $\boldsymbol{v}(.,t) = e^{\nu\Delta t}\boldsymbol{u}$) e $\mathbb{P}_H : L^q(\mathbb{R}^n) \to L^q_\sigma(\mathbb{R}^n)$ é contínuo. Portanto, $\boldsymbol{z}_t(\cdot,t)$ existe e é dado por $\boldsymbol{z}_t(\cdot,t) = \mathbb{P}_H(\boldsymbol{v}_t(\cdot,t))$.
Então,

$$\boldsymbol{z}_t(\cdot,t) = \mathbb{P}_H(\boldsymbol{v}_t(\cdot,t)) = \mathbb{P}_H(\nu\Delta\boldsymbol{v}) = \nu\mathbb{P}_H(\Delta\boldsymbol{v}(.,t)) = \nu\Delta(\mathbb{P}_H(\boldsymbol{v}(\cdot,t))) = \nu\Delta\boldsymbol{z},$$

pelo Corolário 8.2.5. □

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