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Algebraic properties of the Bethe ansatz for an spl(2,1)-supersymmetric t–J model

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We investigate the algebraic structure of the supersymmetric $t-J$ model in one dimension. We prove that the Bethe ansatz states are highest-weight vectors of an spl(2,1) superalgebra. By acting with shift operators we construct a complete set of states for this model. In addition we analyse the multiplet structure of the anti-ferromagnetic ground state and some low-lying excitations. It turns out that the ground state is a member of a quartet.

1. Introduction

Since the pioneering work of Bethe [1] and a subsequent work of Faddeev and Takhtajan [2] on the isotropic Heisenberg model, it is known that the Bethe ansatz alone does not provide a complete set of states instead it only determines the highest-weight vectors of multiplets of the underlying SU(2) symmetry group. Recently, Essler et al. [3] proved that for the one-dimensional Hubbard model the Bethe ansatz states are lowest-weight vectors with respect to the SO(4) symmetry. In this paper we show that this feature, which is essential to construct a complete set of states, also appears in the context of a supersymmetric integrable model. However, the algebraic structure is more complicated and exhibits new interesting properties, e.g. the anti-ferromagnetic ground state is not a singlet but a member of a higher multiplet.

We investigate a model of classical statistical physics in two dimensions, an spl(2,1)-supersymmetric 15-vertex model, which is a generalization of the 6-vertex model. Each link in the lattice can assume one of three states where two are

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bosonic and one is fermionic. The results for the spl(2,1)-supersymmetric 15-vertex model are easily translated to the one-dimensional t–J model (for special values of the couplings t and J). Recently this model has attracted much interest in connection with high-T_c superconductivity. It describes a quantum system of electrons on a one-dimensional chain, where at a lattice point there may be an electron with spin up or spin down or a hole. The hamiltonian for a lattice of L sites is given by [4]

$$\mathscr{H} = P\left\{-t\sum_{j,\sigma}\left(c_{j,\sigma}^{\dagger}c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger}c_{j,\sigma}\right)\right\}P + J\sum_{j}\left(\boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1} - \frac{n_j n_{j+1}}{4}\right), \quad (1.1)$$

where the projector $P = \prod_{j=1}^{L}(1 - n_{j\uparrow}n_{j\downarrow})$ restricts the Hilbert space by the constraint of no double occupancy at one lattice point.

We present an explicit construction of the eigenvalues and eigenvectors of the transfer matrix of the spl(2,1)-supersymmetric 15-vertex model using the algebraic nested Bethe ansatz method [5,6]. By this procedure the problem of finding the spectrum is reduced to the problem of solving a system of coupled transcendental equations, the Bethe ansatz equations (BAE). We find three different kinds of BAE, which correspond to three different possible choices of pseudovacua. Two of these forms of BAE were already obtained by Lai [7], Schlottmann [8], Sutherland [9] and Sarkar [10] using similar methods. Moreover, we analyse in detail the algebraic structure of the eigenvectors obtained by this nested construction. From the invariance of the transfer matrix (and consequently of the one-dimensional t–J hamiltonian) with respect to the spl(2,1) superalgebra it follows that the eigenstates are classified in terms of supermultiplets corresponding to irreducible representations of this superalgebra. We analyse the structure of these representations. In addition, we prove that the Bethe ansatz states are highest-weight vectors of the spl(2,1) superalgebra, which was investigated by Scheunert et al. [11]. Therefore, by acting with the spl(2,1) lowering operators on the Bethe states we obtain additional eigenvectors. Finally, the total number of orthogonal eigenvectors generated by this procedure leads to a complete set of states. This result has been already announced in ref. [12].

The paper is organized as follows. In sect. 2 the spl(2,1) vertex model, as well as its transfer matrix, is defined on a two-dimensional lattice. We also give the relation between the transfer matrix and the one-dimensional supersymmetric t–J model. In sect. 3 we diagonalize the transfer matrix using the quantum inverse-scattering method. In sect. 4 the algebraic structure of the Bethe vectors is investigated. Our results for lattices with small and large number of sites are illustrated in sect. 5, where the structure of the ground state is also discussed. In sect. 6 we give details of the proof of the completeness problem of the Bethe states of this model and sect. 7 contains a summary of the main results.

2. The spl(2,1) vertex model and Yang–Baxter algebra

The graded 15-vertex model is a lattice model of classical statistical physics in two dimensions. Its partition function on a $L \times L'$ (L columns and L' rows) periodic square lattice is given as

$$\mathscr{Z} = \sum_{\text{conf.}} \prod_{x \in L \times L'} S(x), \tag{2.1}$$

where the sum extends over all allowed "bond configurations". Each bond can accept one of three states characterised by $\alpha = 1, 2, 3$, which can be bosonic (B) or fermionic (F). In what follows we will adopt the convention $1 = \mathrm{B}$, $2 = \mathrm{B}$, $3 = \mathrm{F}$. We follow the general strategy of the algebraic Bethe ansatz of Faddeev et al. [5]. The vertex weights $S(x)$ are determined by 15 bond configurations at the lattice site x, and take the following values:

$$S(v)_{\alpha\beta}^{\gamma\delta} = \gamma \overset{\delta}{\underset{\beta}{\text{—}\!\!\!|\!\!\!\text{—}}} \alpha = \sigma_{\gamma\delta}\,\delta_\alpha^\gamma \delta_\beta^\delta - \frac{2}{v}\,\delta_\beta^\gamma \delta_\alpha^\delta . \tag{2.2}$$

The parametrization in terms of the spectral parameter "v" has been introduced for later convenience (see eq. (2.11)). The sign factor σ takes care of the statistics,

$$\sigma_{\gamma\delta} = \begin{cases} -1, & \text{if } \gamma = \delta = 3 \text{ (fermionic)} \\ \;\;1, & \text{otherwise.} \end{cases} \tag{2.3}$$

S can be considered as a matrix acting in the tensor product of two three-dimensional auxiliary spaces $\mathbb{C}^3 \times \mathbb{C}^3$ and can be arranged as a 9×9 matrix,

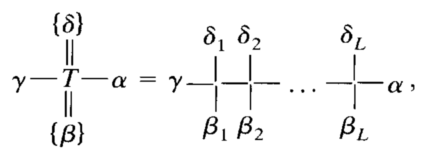
$$S_{\alpha\beta}^{\gamma\delta}(v) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & c & 0 & 0 \\ 0 & c & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & c & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{pmatrix}, \tag{2.4}$$

where

$$a = 1 - \frac{2}{v}, \quad b = 1, \quad c = -\frac{2}{v}, \quad w = -1 - \frac{2}{v}. \tag{2.5}$$

We define the monodromy matrix as the matrix product over the S's in the following way:

$$T^{\gamma\{\delta\}}_{\alpha\{\beta\}}(v) = S^{\gamma\delta_1}_{\alpha_2\beta_1}(v)S^{\alpha_2\delta_2}_{\alpha_3\beta_2}(v)\dots S^{\alpha_L\delta_L}_{\alpha\beta_L}(v),$$


(2.6)

This monodromy matrix acts in the tensor product of an auxiliary space and a "quantum space" $\mathbb{C}^3 \times \mathbb{C}^{3L}$ and can be regarded as a 3×3 matrix of matrices acting in the "quantum space",

$$T^{\gamma}_{\alpha}(v) = \begin{pmatrix} A & B_2 & B_3 \\ C_2 & D_1 & D_2 \\ C_3 & D_3 & D_4 \end{pmatrix}. \tag{2.7}$$

The transfer matrix is defined as a trace of the monodromy matrix in the auxiliary space,

$$\tau^{\{\delta\}}_{\{\beta\}}(v) = \sum_{\alpha} \tilde{T}^{\alpha\{\delta\}}_{\alpha\{\beta\}}(v) = \sum_{\alpha} \sigma_{\alpha\alpha}\sigma_{\alpha\{\delta\}}T^{\alpha\{\delta\}}_{\alpha\{\beta\}}(v), \tag{2.8}$$

where

$$\sigma_{\alpha\{\delta\}} = \prod_i \sigma_{\alpha\delta_i}. \tag{2.9}$$

Here the σ-factors take into account the fact that we are dealing with bosons and fermions.

The thermodynamic properties of the vertex model can be obtained from the solutions of the eigenvalue problem of the transfer matrix,

$$\tau\Psi = \lambda\Psi. \tag{2.10}$$

This eigenvalue problem will be solved in sect. 3 by means of the nested Bethe ansatz.

It can easily be shown that the matrix S given by eq. (2.2) fulfills the Yang–Baxter equation

$$S^{\alpha''\beta''}_{\alpha'\beta'}(v-v')S^{\alpha'\gamma''}_{\alpha\gamma'}(v)S^{\beta'\gamma'}_{\beta\gamma}(v') = S^{\beta''\gamma''}_{\beta'\gamma'}(v')S^{\alpha''\gamma'}_{\alpha'\gamma}(v)S^{\alpha'\beta'}_{\alpha\beta}(v-v'). \tag{2.11}$$

By means of iterations we can also prove the Yang–Baxter relation for the monodromy matrix T,

$$S_{\alpha'\beta'}^{\alpha''\beta''}(v-v')T_{\alpha\ \{\gamma'\}}^{\alpha'\{\gamma''\}}(v)T_{\beta\ \{\gamma\}}^{\beta'\{\gamma'\}}(v') = T_{\beta'\ \{\gamma'\}}^{\beta''\{\gamma''\}}(v')T_{\alpha'\ \{\gamma\}}^{\alpha''\{\gamma'\}}(v)S_{\alpha\ \beta}^{\alpha'\beta'}(v-v'). \quad (2.12)$$

In addition conservation of fermions imply the following property of the T-matrix:

$$\sigma_{\alpha\beta''}\sigma_{\alpha\{\gamma''\}}T_{\beta\ \{\gamma\}}^{\beta''\{\gamma''\}}(v) = \alpha_{\alpha\beta}\sigma_{\alpha\{\gamma\}}T_{\beta\ \{\gamma\}}^{\beta''\{\gamma''\}}(v), \quad (2.13)$$

for all $\alpha = 1, 2$ or 3.

The Yang–Baxter equation for the monodromy matrix (2.12) together with property (2.13) imply the commutativity of the transfer matrix for different spectral parameters,

$$[\tau(v), \tau(v')] = 0. \quad (2.14)$$

This reflects the integrability of the model. In fact, the eigenvalue problem (2.10) can be solved exactly by the Bethe ansatz method.

At the end of this section we will show that the above defined transfer matrix is related to the one-dimensional supersymmetric t–J model, such that if we solve the eigenvalue problem of the transfer matrix τ we will automatically diagonalize the hamiltonian of the one-dimensional supersymmetric t–J model.

The hamiltonian of the t–J model for a one-dimensional lattice of L sites is given as [4]

$$\mathscr{H} = P\left\{-t\sum_{j,\sigma}\left(c_{j,\sigma}^{\dagger}c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger}c_{j,\sigma}\right)\right\}P + J\sum_{j}\left(S_j\cdot S_{j+1} - \frac{n_j n_{j+1}}{4}\right), \quad (2.15)$$

where the $c_{j\pm}^{(\dagger)}$ are spin up or down annihilation (creation) operators, the S_j spin matrices and the n_j occupation numbers of electrons at lattice site j. The projector $P = \prod_{j=1}^{L}(1 - n_{j\uparrow}n_{j\downarrow})$ restricts the Hilbert space by the constraint of no double occupancy at one lattice point. Therefore, at each lattice site we have three possibilities $(1, 2, 3) \equiv (\uparrow, \downarrow, 0)$, i.e. an electron with spin up or down or no electron (hole). This hamiltonian can be rewritten in terms of Hubbard's projection operators [13],

$$X_j^{\alpha\beta} = |\alpha_j\rangle\langle\beta_j| \quad (\alpha, \beta = 1, 2, 3), \quad (2.16)$$

where $|1_j(2_j)\rangle$ denotes an electron with spin up (down) and $|3_j\rangle$ a hole at site j. Using (2.16), up to a chemical potential the hamiltonian reads

$$\mathscr{H} = -t\sum_{\alpha=1}^{2}\sum_{j=1}^{L}\left(X_j^{\alpha 3}X_{j+1}^{3\alpha} + X_{j+1}^{\alpha 3}X_j^{3\alpha}\right) + \tfrac{1}{2}J\sum_{j=1}^{L}\left(\sum_{\alpha,\beta=1}^{2}X_j^{\alpha\beta}X_{j+1}^{\beta\alpha} - X_j^{33}X_{j+1}^{33}\right). \quad (2.17)$$

For convenience we will consider the hole operators as fermions and the spin operators as bosons. In fact, this choice is possible since in one-dimension there exists a transformation exchanging bosons and fermions. Therefore, the spectrum of the t–J model with two fermions and one boson is equivalent to the spectrum of the t–J model with two bosons and one fermion (for even L) [10].

For $J = 2t$ the t–J model is "supersymmetric" and connected to the previously defined vertex model through the relation

$$\mathscr{H} = -2\frac{\partial}{\partial v}\ln\left(v^L \tau(v)\right)\bigg|_{v=0}. \tag{2.18}$$

The proof of this identity is analogous to the one for the isotropic Heisenberg model [14].

3. Construction of Bethe eigenvectors

The main subject of this section will be solving the eigenvalue problem of the transfer matrix

$$\tau\Psi = \lambda\Psi \tag{3.1}$$

through an algebraic construction [5] based on the Yang–Baxter algebra of the monodromy matrices

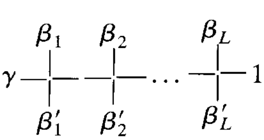
$$S_{\alpha'\,\beta'}^{\alpha''\beta''}(v-v')T_{\alpha\ \{\gamma'\}}^{\alpha'\{\gamma''\}}(v)T_{\beta\ \{\gamma\}}^{\beta'\{\gamma'\}}(v') = T_{\beta'\ \{\gamma'\}}^{\beta''\{\gamma''\}}(v')T_{\alpha'\ \{\gamma\}}^{\alpha''\{\gamma'\}}(v)S_{\alpha\,\beta}^{\alpha'\beta'}(v-v'). \tag{3.2}$$

The monodromy matrix T can be written as a 3×3 matrix,

$$\left(\begin{array}{c|cc} A & B_2 & B_3 \\ \hline C_2 & D_1 & D_2 \\ C_3 & D_3 & D_4 \end{array}\right). \tag{3.3}$$

This suggests solving the problem by means of the nested Bethe ansatz with two levels [6]. The transfer matrix is given by a trace of the monodromy matrix T (see eq. (2.8)). For the first-level Bethe ansatz the operators B_α (C_α) ($\alpha = 2, 3$) play the role of creation (annihilation) operators of "pseudoparticles". The first-level "pseudovacuum" Φ is defined by the equation

$$C_{\gamma\{\beta'\}}^{\{\beta\}}\,\Phi^{\{\beta'\}} = 0,$$



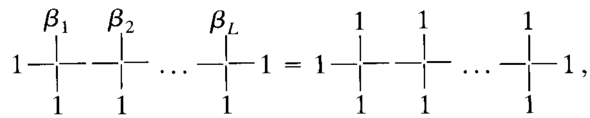
$\Phi^{\{\beta'\}} = 0$ for $\gamma = 2, 3$. (3.4)

Since at a vertex a generalized "ice rule" holds (see eq. (2.2)) the solution of this equation is

$$\Phi^{\{\beta\}} = \prod_{i=1}^{L} \delta_{\beta_i 1} = \begin{array}{c}\beta_1\\ |\\ 1\end{array}\ \begin{array}{c}\beta_2\\ |\\ 1\end{array}\ \cdots\ \begin{array}{c}\beta_L\\ |\\ 1\end{array}\ . \tag{3.5}$$

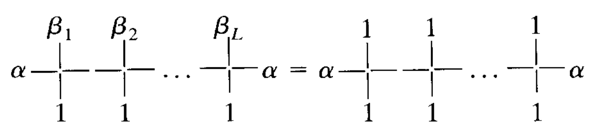
This pseudovacuum is an eigenstate of A,

$$A^{\{\beta\}}_{\{\beta'\}}(v)\,\Phi^{\{\beta'\}} = a^L(v)\,\Phi^{\{\beta\}},$$


(3.6)

and also of D_1 and D_4,

$$D_{1(4)}{}^{\{\beta\}}_{\{\beta'\}}(v)\,\Phi^{\{\beta'\}} = b^L(v)\,\Phi^{\{\beta\}},$$


(3.7)

($\alpha = 2$ and 3, respectively). Because of the special form of the matrix S of eq. (2.2) the summations over the internal lines in eqs. (3.6) and (3.7) are trivial. In eq. (3.6) they can assume only the value 1, and in eq. (3.7) only the fixed value $\alpha = 2$ or 3, respectively. The action of B_α ($\alpha = 2$ or 3) on the "pseudovacuum" yields new states. So, the $\{B_\alpha\}$ can be considered as "creation operators" and the eigenvector of the transfer matrix can be obtained by successive application of the B's according to the first-level Bethe ansatz

$$\Psi^{\{\beta\}} = B_{\alpha_1}{}^{\{\beta\}}_{\{\eta_1\}}(v_1)\,B_{\alpha_2}{}^{\{\eta_1\}}_{\{\eta_2\}}(v_2)\cdots B_{\alpha_N}{}^{\{\eta_{N-1}\}}_{\{\eta_N\}}(v_N)\,\Phi^{\{\eta_N\}}\Psi^{\{\alpha\}}_{(1)}, \tag{3.8}$$

where the summations over the α_i ($i = 1,\dots, N$) are restricted to $\alpha_i = 2$, 3. The coefficients $\Psi^{\{\alpha\}}_{(1)}$ are to be to be determined by the second-level Bethe ansatz. This means the eigenvalue problem of the transfer matrix (3.1) will be solved in a recurrent way (nested Bethe ansatz method). The requirement that Ψ is an eigenvector of τ leads to another eigenvalue problem for a new transfer matrix $\tau_{(1)}$, as will be shown later. Now we start to solve eq. (3.1). Following the general strategy of the algebraic Bethe ansatz [5] we apply the transfer matrix $\tau(v)$ (2.8) to the state Ψ given by eq. (3.8),

$$\tau^{\{\beta''\}}_{\{\beta'\}}(v)\,\Psi^{\{\beta'\}} = \left(A^{\{\beta''\}}_{\{\beta'\}}(v) + \tau_D{}^{\{\beta''\}}_{\{\beta'\}}(v)\right)\Psi^{\{\beta'\}}, \tag{3.9}$$

where

$$\tau_{D\{\beta'\}}^{\{\beta''\}}(v) = \sum_{\alpha=2}^{3} \tilde{T}_{\alpha\,\{\beta'\}}^{\alpha\{\beta''\}}(v) = \sum_{\alpha=2}^{3} \sigma_{\alpha\alpha}\sigma_{\alpha\{\beta''\}} T_{\alpha\,\{\beta'\}}^{\alpha\{\beta''\}}(v). \tag{3.10}$$

In order to commute $A(v)$, $D_1(v)$ and $D_4(v)$ through all $B(v_i)$ towards Φ and then apply (3.6) and (3.7) we use the property (2.13) and the following commutation rules, derived from the Yang–Baxter relation (3.2):

$$A(v)B_\alpha(v') = \frac{a(v'-v)}{b(v'-v)} B_\alpha(v')A(v) - \frac{c(v'-v)}{b(v'-v)} B_\alpha(v)A(v'), \tag{3.11}$$

and

$$T_{\gamma'}^{\gamma}(v)B_\alpha(v') = \frac{1}{b(v-v')}\Big(B_{\alpha'}(v')T_{\gamma''}^{\gamma}(v)S_{\gamma'\,\alpha}^{\gamma''\alpha'}(v-v')$$

$$-c(v-v')B_{\gamma'}(v)T_{\alpha}^{\gamma}(v')\Big) \tag{3.12}$$

$$B_\alpha(v)\ B_\beta(v') = \frac{1}{a(v-v')} B_{\beta'}(v')\ B_{\alpha'}(v) S_{\alpha\,\beta}^{\alpha'\beta'}(v-v'). \tag{3.13}$$

All indices of the auxiliary space in eqs. (3.11), (3.12) and (3.13) assume only the values 2 and 3. Using eq. (3.11) two types of terms arise when A is commuted through B_α. In the first type A and B_α preserve their arguments and in the second type their arguments are exchanged. The first kind of terms are called "wanted terms", since they will give a vector proportional to Ψ and the second type are the "unwanted terms (u.t.)". Then, using eqs. (3.9), (3.8), (2.13), (3.11) and (3.6), we get

$$A_{\{\beta'\}}^{\{\beta''\}}(v)\Psi^{\{\beta'\}} = \lambda_A(v)\Psi^{\{\beta''\}} + \text{u.t.}(A), \tag{3.14}$$

where the coefficient λ_A is given by

$$\lambda_A(v) = a^L(v)\prod_{i=1}^{N}\frac{a(v_i-v)}{b(v_i-v)}. \tag{3.15}$$

Correspondingly we obtain from eq. (3.12) the wanted and unwanted terms in the form

$$\tau_{D\{\beta'\}}^{\{\beta''\}}(v)\Psi^{\{\beta'\}} = b^L(v)\prod_{i=1}^{N}\frac{1}{b(v-v_i)} B_{\alpha'_1\{\eta_1\}}^{\{\beta''\}}(v_1) B_{\alpha'_2\{\eta_2\}}^{\{\eta_1\}}(v_2)\ldots$$

$$\times B_{\alpha'_N\{\eta_N\}}^{\{\eta_{N-1}\}}(v_N)\Phi^{\{\eta_N\}}\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v,\{v_i\})\Psi_{(1)}^{\{\alpha\}} + \text{u.t.}(D), \tag{3.16}$$

where we have introduced a new (the second-level) transfer matrix

$$\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v, \{v_i\}) = \sum_{\beta=2}^{3} \sigma_{\beta\beta}\sigma_{\beta\{\alpha'\}} T_{(1)\beta\{\alpha\}}^{\beta\{\alpha'\}}(v, \{v_i\}) \tag{3.17}$$

as a trace (over only $\beta = 2$ and 3) of the second-level monodromy matrix. This is given by $T_{(1)} = S(v - v_N) \dots S(v - v_1)$ in analogy to eq. (2.6). Now, however, all indices (the external and the internal ones) assume only the values 2 and 3, as in the internal block of the matrix T denoted in eq. (3.3). In order to obtain in eq. (3.16) a "wanted term" proportional to Ψ, the vector $\Psi_{(1)}$ has to fulfill the eigenvalue equation

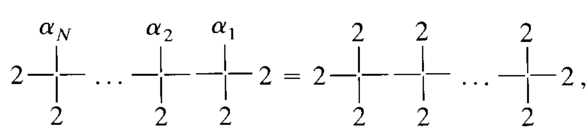
$$\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v, \{v_i\}) \Psi_{(1)}^{\{\alpha\}} = \lambda_{(1)}(v, \{v_i\}) \Psi_{(1)}^{\{\alpha'\}}, \tag{3.18}$$

which is solved by the second-level Bethe ansatz. The monodromy matrix $T_{(1)}$ belongs to an SL(1,1) 6-vertex model slightly modified compared to the SU(2) one due to the presence of fermions. If we identify $T_{(1)2}^{\ \ 2} \equiv A_{(1)}$, $T_{(1)3}^{\ \ 2} \equiv B_{(1)}$, $T_{(1)2}^{\ \ 3} \equiv C_{(1)}$ and $T_{(1)3}^{\ \ 3} \equiv D_{(1)}$ again $B_{(1)}$ ($C_{(1)}$) can be interpreted as a creation (annihilation) operator with respect to the "pseudovacuum" $\Phi_{(1)}$, which is now of the form

$$\Phi_{(1)}^{\{\alpha\}} = \prod_{i=1}^{N} \delta_{\alpha_i, 2} = \begin{array}{c}\alpha_N\\ |\\ 2\end{array} \ \cdots \ \begin{array}{c}\alpha_2\\ |\\ 2\end{array}\ \begin{array}{c}\alpha_1\\ |\\ 2\end{array} \ . \tag{3.19}$$

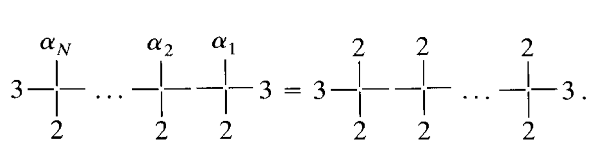
It is an eigenstate of $A_{(1)}$ and $D_{(1)}$, satisfying

$$A_{(1)\{\alpha'\}}^{\{\alpha\}}(v, \{v_i\}) \Phi_{(1)}^{\{\alpha'\}} = \prod_{i=1}^{N} a(v - v_i) \Phi_{(1)}^{\{\alpha\}},$$



(3.20)

$$D_{(1)\{\alpha'\}}^{\{\alpha\}}(v, \{v_i\}) \Phi_{(1)}^{\{\alpha'\}} = \prod_{i=1}^{N} b(v - v_i) \Phi_{(1)}^{\{\alpha\}},$$



(3.21)

The summations over the internal lines in eqs. (3.20) and (3.21) are only over the values 2 and 3, respectively. The eigenvector $\Psi_{(1)}$ of $\tau_{(1)}$ is given by the second-level

Bethe ansatz

$$\Psi_{(1)}^{\{\alpha\}} = B_{(1)\{\eta_1\}}^{\{\alpha\}}(\gamma_1, \{v_i\}) B_{(1)\{\eta_2\}}^{\{\eta_1\}}(\gamma_2, \{v_i\}) \dots B_{(1)\{\eta_M\}}^{\{\eta_{M-1}\}}(\gamma_M, \{v_i\}) \Phi_{(1)}^{\{\eta_M\}}. \quad (3.22)$$

Following a strategy analogous to the one above, we apply $\tau_{(1)}$ to the state $\Psi_{(1)}$ and commute $A_{(1)}(v, \{v_i\})$ and $D_{(1)}(v, \{v_i\})$ through the $B_{(1)}(\gamma_\alpha, \{v_i\})$ towards $\Phi_{(1)}$ and then use eqs. (3.20) and (3.21). Since the Yang–Baxter algebra for the monodromy matrices (3.2) is also valid in the inhomogeneous cases when $T(v)$ is replaced by $T(v, \{v_i\})$ [15], we derive the following commutation relations:

$$A_{(1)}(v, \{v_i\}) B_{(1)}(v', \{v_i\}) = \frac{a(v'-v)}{b(v'-v)} B_{(1)}(v', \{v_i\}) A_{(1)}(v, \{v_i\})$$

$$- \frac{c(v'-v)}{b(v'-v)} B_{(1)}(v, \{v_i\}) A_{(1)}(v', \{v_i\}), \quad (3.23)$$

$$D_{(1)}(v, \{v_i\}) B_{(1)}(v', \{v_i\}) = \frac{w(v-v')}{b(v-v')} B_{(1)}(v', \{v_i\}) D_{(1)}(v, \{v_i\})$$

$$- \frac{c(v-v')}{b(v-v')} B_{(1)}(v, \{v_i\}) D_{(1)}(v', \{v_i\}). \quad (3.24)$$

$$B_{(1)}(v, \{v\}) B_{(1)}(v', \{v\}) = \frac{w(v-v')}{a(v-v')} B_{(1)}(v', \{v\}) B_{(1)}(v, \{v\}). \quad (3.25)$$

Using eqs. (3.17), (3.22), (2.13), (3.23), (3.24), (3.20) and (3.21) as above we obtain again wanted and unwanted terms,

$$\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v, \{v_i\}) \Psi_{(1)}^{\{\alpha\}} = \left(\lambda_{A_{(1)}}(v, \{v_i\}) + \lambda_{D_{(1)}}(v, \{v_i\})\right) \Psi_{(1)}^{\{\alpha'\}} + \text{u.t.}(A_{(1)}) + \text{u.t.}(D_{(1)}), \quad (3.26)$$

where

$$\lambda_{A_{(1)}} = \prod_{i=1}^{N} a(v-v_i) \prod_{\beta=1}^{M} \frac{a(\gamma_\beta - v)}{b(\gamma_\beta - v)}, \quad (3.27)$$

$$\lambda_{D_{(1)}} = -(-1)^M \prod_{i=1}^{N} b(v-v_i) \prod_{\beta=1}^{M} \frac{w(v-\gamma_\beta)}{b(v-\gamma_\beta)}. \quad (3.28)$$

Substituting these equations in (3.16) and taking (3.8) into account we get, in case the unwanted terms u.t.$(A_{(1)})$ and u.t.$(D_{(1)})$ cancel,

$$\tau_{D\{\beta'\}}^{\{\beta''\}}(v) \Psi^{\{\beta'\}} = \left(\lambda_{D_{\rm I}}(v) + \lambda_{D_{\rm II}}(v)\right) \Psi^{\{\beta''\}} + \text{u.t.}(D), \quad (3.29)$$

where $\lambda_{D_{\mathrm{I}}}$ and $\lambda_{D_{\mathrm{II}}}$ are given by

$$\lambda_{D_{\mathrm{I}}} = b^L(v) \prod_{i=1}^{N} \frac{a(v-v_i)}{b(v-v_i)} \prod_{\beta=1}^{M} \frac{a(\gamma_\beta - v)}{b(\gamma_\beta - v)}, \tag{3.30}$$

$$\lambda_{D_{\mathrm{II}}} = -(-1)^M b^L(v) \prod_{\beta=1}^{M} \frac{w(v-\gamma_\beta)}{b(v-\gamma_\beta)}. \tag{3.31}$$

Finally, combining eqs. (3.14) and (3.29) we have, again if the unwanted terms u.t.(A) and u.t.(D) cancel,

$$\tau_{\{\beta'\}}^{\{\beta''\}}(v)\Psi^{\{\beta'\}} = \lambda(v)\Psi^{\{\beta''\}}, \tag{3.32}$$

where

$$\lambda(v) = \lambda_A(v) + \lambda_{D_{\mathrm{I}}}(v) + \lambda_{D_{\mathrm{II}}}(v). \tag{3.33}$$

The cancellation of all unwanted terms ensure that Ψ, as given by eq. (3.8), is an eigenstate of the transfer matrix τ (2.8) with eigenvalue $\lambda(v)$ of eq. (3.33).

In appendix A we show that the unwanted terms indeed vanish if the Bethe ansatz equations hold,

$$\left(\frac{a(v_j)}{b(v_j)}\right)^L \prod_{i=1}^{N} \frac{a(v_i-v_j)}{b(v_i-v_j)} \frac{b(v_j-v_i)}{a(v_j-v_i)} \prod_{\beta=1}^{M} \frac{b(\gamma_\beta - v_j)}{a(\gamma_\beta - v_j)} = -1, \quad j=1,\dots,N, \tag{3.34}$$

$$(-1)^M \prod_{i=1}^{N} \frac{a(\gamma_\alpha - v_i)}{b(\gamma_\alpha - v_i)} \prod_{\beta=1}^{M} \frac{a(\gamma_\beta-\gamma_\alpha)}{b(\gamma_\beta-\gamma_\alpha)} \frac{b(\gamma_\alpha-\gamma_\beta)}{w(\gamma_\alpha-\gamma_\beta)} = 1, \quad \alpha=1,\dots,M, \tag{3.35}$$

where N is the number of holes plus down spins and M is the number of holes. Another way to obtain these equations is to require that the eigenvalue $\lambda(v)$ (3.33) has no poles at $v = v_i$ and $v = \gamma_\beta$. Using (2.5) and making the change of variables $v \to iv + 1$, $\gamma \to i\gamma + 2$ we obtain

$$\left(\frac{v_j+i}{v_j-i}\right)^L = -\prod_{k=1}^{N} \frac{v_j-v_k+2i}{v_j-v_k-2i} \prod_{\beta=1}^{M} \frac{v_j-\gamma_\beta-i}{v_j-\gamma_\beta+i}, \quad j=1,\dots,N, \tag{3.36}$$

$$\prod_{j=1}^{N} \frac{\gamma_\alpha - v_j + i}{\gamma_\alpha - v_j - i} = 1, \quad \alpha = 1,\dots,M. \tag{3.37}$$

This form of the Bethe ansatz equations (BAE) was previously derived by Sutherland [9] and later by Sarkar using a generalized permutation operator [10]. We stress that this procedure could be repeated with two other choices of the pseudovacuum leading to two other forms of the BAE. The pseudovacua of both levels of the Bethe ansatz Φ and $\Phi_{(1)}$ (see eqs. (3.5) and (3.19)), which we used above, consist of states of kind 1 = B and 2 = B, respectively. Basically, the change of pseudovacuum is determined by altering the initial convention (1 = B, 2 = B, 3 = F). Using (1 = F, 2 = B, 3 = B) we get

$$\left(\frac{v_j+i}{v_j-i}\right)^L = \prod_{\beta=1}^{M} \frac{v_j-\gamma_\beta+i}{v_j-\gamma_\beta-i}, \quad j=1,\dots,N, \tag{3.38}$$

$$\prod_{j=1}^{N} \frac{\gamma_\alpha-v_j-i}{\gamma_\alpha-v_j+i} = \prod_{\beta=1}^{M} \frac{\gamma_\alpha-\gamma_\beta-2i}{\gamma_\alpha-\gamma_\beta+2i}, \quad \alpha=1,\dots,M, \tag{3.39}$$

where N is the total number of spins and M is the number of spins down. These equations were already obtained by Lai [7] and Schlottmann [8] using the coordinate Bethe ansatz method.

Finally, the choice (1 = B, 2 = F, 3 = B) leads to a new form of the BAE *,

$$\left(\frac{v_j-i}{v_j+i}\right)^L = \prod_{\beta=1}^{M} \frac{v_j-\gamma_\beta-i}{v_j-\gamma_\beta+i}, \quad j=1,\dots,N, \tag{3.40}$$

$$\prod_{j=1}^{N} \frac{\gamma_\alpha-v_j+i}{\gamma_\alpha-v_j-i} = 1, \quad \alpha=1,\dots,M, \tag{3.41}$$

where N is the number of holes plus spin downs and M is the number of spins down. In the following we will work with the BAE's (3.36) and (3.37), since this is the most convenient form for the present investigation.

We have reduced the eigenvalue problem of the transfer matrix (3.1) to a system of coupled algebraic equations for the parameters $\{v_j\}$ $(j=1,\dots,N)$ and $\{\gamma_\alpha\}$ $(\alpha=1,\dots,M)$. The basic procedure to solve eqs. (3.36) and (3.37) is to adopt the string-conjecture, which means that the v's appear as strings and all roots γ's are real,

$$\begin{aligned} &v^n_{\alpha j}=v^n_\alpha+i(n+1-2j), && j=1,\dots,n, \quad \alpha=1,\dots,N_n, \quad n=1,2,\dots, \\ &\gamma_\beta=\text{real}, && \beta=1,\dots,M, \end{aligned} \tag{3.42}$$

where v^n_α is the position of the center of the string on the real v-axis. The number of n-strings N_n satisfy the relation

$$N=\sum_n nN_n. \tag{3.43}$$

* When this paper was in preparation the authors were informed about a preprint of Essler, Korepin and Schoutens where this new form of the BAE also was obtained.

This hypothesis for the v's can be easily understood by heuristic arguments, analogously to the isotropic Heisenberg model [2,16]. To understand absence of complex roots for the γ's we apply the following argument, which is similar to that one developed by Takahashi for the one-dimensional electron gas with a repulsive delta function [17]. If all v_i are real or appear as complex conjugate pairs, Im $\gamma_\alpha > 0$ implies that the absolute value of the left-hand side of eq. (3.37) is larger than unity. Therefore, Im $\gamma_\alpha > 0$ is not possible. In the same way we can prove that Im $\gamma_\alpha < 0$ is not possible. We can see here the great advantage of using this form of BAE. In the other two forms not only the parameters v but also the roots γ appear as strings. This means that counting the states is much more complicated. Although we are not able to prove the string-conjecture rigorously, we will assume it to be valid. Since Bethe [1], assumptions of this kind have been widely used by many authors (ref. [16] and references therein). Applying this conjecture in (3.36) and (3.37) and taking its logarithm we obtain the coupled equations for the v_α^n and γ_β,

$$L\theta\left(\frac{v_\alpha^n}{n}\right) - \underbrace{\sum_m \sum_{\beta=1}^{N_m}}_{(m,\beta)\neq(n,\alpha)} \Theta_{nm}\left(v_\alpha^n - v_\beta^m\right) + \sum_{\beta=1}^{M} \theta\left(\frac{v_\alpha^n - \gamma_\beta}{n}\right) = 2\pi I_\alpha^n, \quad (3.44)$$

$$\sum_n \sum_{\alpha=1}^{N_n} \theta\left(\frac{v_\alpha^n - \gamma_\beta}{n}\right) = 2\pi J_\beta, \quad (3.45)$$

where $\theta(x) = 2 \arctan x$ and

$$\Theta_{nm}(x) = \begin{cases} \theta\left(\frac{x}{|n-m|}\right) + 2\theta\left(\frac{x}{|n-m|+2}\right) + \ldots + 2\theta\left(\frac{x}{n+m-2}\right) \\ \quad + \theta\left(\frac{x}{n+m}\right) \quad \text{for } n \neq m \\ 2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \ldots + 2\theta\left(\frac{x}{2n-2}\right) + \theta\left(\frac{x}{2n}\right) \quad \text{for } n = m. \end{cases} \quad (3.46)$$

Hence the solutions of eqs. (3.36) and (3.37) are parametrized in terms of the numbers I_α^n and J_β. Here, the I_α^n are integers (half-integers) if $L + M - N_n$ is odd (even) and the J_β are integers (half-integers) if $\Sigma_n N_n$ is even (odd). In addition they are limited to the intervals

$$|I_\alpha^n| \leqslant I_{\max}^n = \tfrac{1}{2}\left(L + M - \sum_m t_{nm} N_m - 1\right), \quad (3.47)$$

$$|J_\beta| \leqslant J_{\max} = \tfrac{1}{2}\left(\sum_n N_n - 2\right), \quad (3.48)$$

where $t_{nm} = 2\min(n, m) - \delta_{nm}$. In fact, all sets $\{I_\alpha^n, J_\beta\}$ where the I's and J's are pairwise different specify all the Bethe vectors $(|\psi_{\text{Bethe}}\rangle_{N,M})$. They are highest-weight vectors of an spl(2,1) superalgebra, as we will show in sect. 4.

In order to avoid misunderstandings we should add some general remarks on the string-conjecture (3.42) and the bounds $I_{\max}^n$ and $J_{\max}$ given by eqs. (3.47) and (3.48). Both statements are to be considered as assumptions, they cannot be proven rigorously. In fact they are not exact. There are finite-size corrections of the string configurations of order $O(e^{-L})$ for fixed string centers v_α^n and of order O(1) near to the boundary $v_{\max}^n$ (given by $I_{\max}^n$), producing "exotic solutions". On the other hand a naive estimate of $I_{\max}^n$ from eq. (3.44) would suggest additional solutions (for $n \geqslant 2$) which are cancelled by assumption (3.47). However, both assumptions together lead to the correct number of states, as is well known for the SU(2) case [2] and will be proven below for the spl(2,1) case. Obviously, the effects of the two phenomena mentioned above compensate for this computation. In addition to the "exotic solutions" mentioned above, there exist also "wide pairs" and "quartets" if the density of real roots is large enough. It is believed that these problems may be avoided and exotic effects may be neglected, if one considers the following thermodynamic limit. Introduce a symmetry breaking magnetic field B and take first the limit $L \to \infty$ and then $B \to 0$. It should be stressed that many features of the Bethe ansatz are not well understood.

In the thermodynamic limit the BAE's are written in terms of densities of roots $(\rho_n(v), \sigma(\lambda))$ and BA-holes * $(\rho_n^{\text{h}}(v), \sigma^{\text{h}}(\lambda))$, such that eqs. (3.44) and (3.45) can be replaced by integral equations for the densities.

At the end of this section we apply the results obtained for the spl(2,1) vertex model to the supersymmetric t–J model. Using the identity (2.18) it is possible to obtain the energy eigenvalues of the t–J model from the eigenvalues of the transfer matrix (3.33). The terms $\lambda_{D_\text{I},D_\text{II}}$ given by eqs. (3.30) and (3.31) do not contribute and from eq. (3.15) we find

$$E = L - \sum_{j=1}^{N} \frac{4}{1+v_j^2}. \tag{3.49}$$

Thermodynamic properties of the model were investigated in ref. [8] using the second form of the BAE (3.38), (3.39). The ground state and the excitation spectrum were discussed in ref. [18] using the first and second form of the BAE.

* Unfortunately, in this paper the meaning of the term hole is ambiguous: A "hole", as denoted above is a physical hole, i.e. a lattice site with no electron. A "BA-hole" corresponds to a non-occupied place in the set of numbers $\{I^n\}$ or $\{J\}$ for a solution of the BAE (see sect. 5 for examples).

4. Algebraic properties of the Bethe states

In this section we analyse the algebraic properties of the Bethe states. By asymptotic expansion ($v \to \infty$) we obtain the generators of spl(2,1) as matrix elements of a matrix $\tilde{M}$ of operators in the "quantum space" defined as follows:

$$T_{\alpha\ \{\gamma\}}^{\alpha''\{\gamma''\}}(v) = \sigma_{\alpha''\{\gamma''\}}\delta_{\alpha}^{\alpha''}\delta_{\{\gamma\}}^{\{\gamma''\}} - \frac{2}{v}\sigma_{\alpha''\alpha}\sigma_{\alpha''\{\gamma''\}}\tilde{M}_{\alpha\ \{\gamma\}}^{\alpha''\{\gamma''\}} + O(v^{-2}). \qquad (4.1)$$

We prove the commutation relations of the entries of $\tilde{M}$ using the Yang–Baxter relation (3.2) for the monodromy matrix and the property (2.13). For $v \to \infty$ we have (in what follows we will omit the quantum space indices and write them only whenever necessary)

$$\tilde{M}_{\alpha}^{\alpha''}\tilde{T}_{\beta}^{\beta''}(v') - \Sigma(\alpha'', \beta'', \alpha, \beta)\tilde{T}_{\beta}^{\beta''}(v')\tilde{M}_{\alpha}^{\alpha''}$$
$$= \tilde{T}_{\alpha}^{\beta''}(v')\delta_{\beta}^{\alpha''} - \Sigma(\alpha'', \beta'', \alpha, \beta)\delta_{\alpha}^{\beta''}\tilde{T}_{\beta}^{\alpha''}(v'). \qquad (4.2)$$

Here the sign function Σ is given by

$$\Sigma(\alpha'', \beta'', \alpha, \beta) = \sigma_{\alpha''\beta''}\sigma_{\alpha''\beta}\sigma_{\alpha\beta''}\sigma_{\alpha\beta}. \qquad (4.3)$$

Furthermore, taking $v' \to \infty$ we get

$$\tilde{M}_{\alpha}^{\alpha''}\tilde{M}_{\beta}^{\beta''} - \Sigma(\alpha'', \beta'', \alpha, \beta)\tilde{M}_{\beta}^{\beta''}\tilde{M}_{\alpha}^{\alpha''}$$
$$= \tilde{M}_{\alpha}^{\beta''}\delta_{\beta}^{\alpha''} - \Sigma(\alpha'', \beta'', \alpha, \beta)\delta_{\alpha}^{\beta''}\tilde{M}_{\beta}^{\alpha''}. \qquad (4.4)$$

This relation represents the commutation and anti-commutation rules of the spl(2,1) superalgebra [11]. The generators $\tilde{M}_3^{\alpha}$, $\tilde{M}_{\alpha}^{3}$ ($\alpha \neq 3$) are fermionic, whereas the $\tilde{M}_3^3$ and $\tilde{M}_{\beta}^{\alpha}$ (α, $\beta \neq 3$) are bosonic. The sign factors Σ take into account the statistics, i.e. $\Sigma = -1$ (1) if we are dealing with odd (even) generators. Eq. (4.4) can be written in the compact form

$$\left[\tilde{M}_{\alpha}^{\alpha''}, \tilde{M}_{\beta}^{\beta''}\right]_{\pm} = \tilde{M}_{\alpha}^{\beta''}\delta_{\beta}^{\alpha''} \pm \delta_{\alpha}^{\beta''}\tilde{M}_{\beta}^{\alpha''}. \qquad (4.5)$$

In addition, from eq. (4.2) it is easy to see that the transfer matrix τ (2.8) is invariant with respect to the spl(2,1) superalgebra, i.e.

$$\left[\tilde{M}_{\alpha}^{\alpha''}, \tau(v')\right] = 0. \qquad (4.6)$$

Notice that the results (4.2), (4.4), (4.5) and (4.6) are also valid if we change the convention (1 = B, 2 = B, 3 = F). The position of the fermion simply determine which are the odd generators.

Let us now consider the matrix $\tilde{M}$,

$$\tilde{M} = \begin{pmatrix} W_1 & \tilde{M}_2^1 & \tilde{M}_3^1 \\ \tilde{M}_1^2 & W_2 & \tilde{M}_3^2 \\ \tilde{M}_1^3 & \tilde{M}_2^3 & W_3 \end{pmatrix}. \tag{4.7}$$

The diagonal elements W_α ($\alpha = 1, 2, 3$) generate the Cartan subalgebra with weights w_α ($\alpha = 1, 2, 3$),

$$W_\alpha \Psi = w_\alpha \Psi. \tag{4.8}$$

In terms of the t–J model the weights are related to the z-component of the SU(2)-spin $S_z = \frac{1}{2}(w_1 - w_2)$ and the number of electrons $Q = w_1 + w_2$. In order to calculate these weights for Bethe ansatz states we substitute (2.5) in eqs. (3.14), (3.15), (3.29), (3.30) and (3.31) and obtain with eq. (4.1) and (4.7) for $v \to \infty$

$$\left(1 - \frac{2}{v} W_1\right)\Psi + \mathrm{O}(v^{-2}) = \left(1 - \frac{2}{v}(L - N)\right)\Psi + \mathrm{O}(v^{-2}),$$

$$\left(1 - \frac{2}{v} W_2\right)\Psi + \mathrm{O}(v^{-2}) = \left(1 - \frac{2}{v}(N - M)\right)\Psi + \mathrm{O}(v^{-2}),$$

$$\left(-1 - \frac{2}{v} W_3\right)\Psi + \mathrm{O}(v^{-2}) = \left(-1 - \frac{2}{v} M\right)\Psi + \mathrm{O}(v^{-2}). \tag{4.9}$$

Therefore, the weights can be expressed in terms of the quantities L (= number of sites), N (= number of first-level roots) and M (= number of second-level roots),

$$w_1 = n_\uparrow = L - N, \qquad w_2 = n_\downarrow = N - M, \qquad w_3 = n_\mathrm{h} = M, \tag{4.10}$$

where $n_\uparrow$, $n_\downarrow$, n_h are the numbers of up-spins, down-spins and holes, respectively. At the end of this section we will derive inequalities between these weights and give a physical interpretation.

Next we show that the Bethe vectors are highest-weight vectors with respect to the spl(2,1) superalgebra, i.e.

$$\tilde{M}_\alpha^\beta \Psi = 0, \quad \beta > \alpha. \tag{4.11}$$

For $\alpha = 1$, $\beta = 2$ or 3 we have, after using eqs. (3.8) and (4.2),

$$\tilde{M}_1^\beta \Psi = \sum_{j=1}^{N} \sigma_{\beta(\alpha_1, \dots, \alpha_{j-1})} B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \dots B_{\alpha_{j-1}}(v_{j-1})$$

$$\times \left[\tilde{M}_1^\beta, B_{\alpha_j}(v_j)\right]_\pm B_{\alpha_{j+1}}(v_{j+1}) \dots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}}, \tag{4.12}$$

where

$$\left[\tilde{M}_1^\beta, B_\alpha(v)\right]_\pm = \tilde{M}_1^\beta B_\alpha(v) - \sigma_{\beta\alpha} B_\alpha(v) \tilde{M}_1^\beta = \delta_\alpha^\beta A(v) - \sigma_{\beta\alpha} \tilde{T}_\alpha^\beta(v). \quad (4.13)$$

In order to commute $A(v_j)$ and $\tilde{T}_\alpha^\beta(v_j)$ through the B_α's toward Φ we use the commutation rules (3.11), (3.12) and the property (2.13). Although many terms appear, it is possible to arrange them as follows:

$$\tilde{M}_1^\beta \Psi = \sum_{j=1}^{N} Y_{j\alpha_j}^\beta(v_j, \{v_i\}) B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \dots B_{\alpha_{j-1}}(v_{j-1})$$

$$\times B_{\alpha_{j+1}}(v_{j+1}) \dots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}}, \quad (4.14)$$

with yet unknown coefficients $Y_{j\alpha_j}^\beta$. The first coefficient, $Y_{1\alpha_1}^\beta$, can be obtained by using the first term in (3.11) and (3.12) when commuting $A(v_1)$ and $\tilde{T}_\alpha^\beta(v_1)$ with $B_{\alpha_2}(v_2) B_{\alpha_3}(v_3) \dots B_{\alpha_N}(v_N)$, since otherwise the argument v_1 re-appears in the B_α. The contribution of the $A(v_1)$ term to $Y_{1\alpha_1}^\beta$ is straightforward, whereas for the $\tilde{T}_\alpha^\beta(v_1)$ term we shall use the relation

$$-\tfrac{1}{2} \operatorname*{Res}_{v'=v} S_{\alpha\beta}^{\gamma\delta}(v - v') = \delta_\beta^\gamma \delta_\alpha^\delta \quad (4.15)$$

to get the eigenvalue problem for the transfer matrix $\tau_{(1)}(v_1, \{v_i\})$ (3.17). Once again, we just take the first term in eqs. (3.23) and (3.24) when commuting $A_{(1)}$ and $D_{(1)}$ with the $B_{(1)}$'s. Then, after some manipulations we have

$$Y_{1\alpha_1}^\beta = \delta_{\alpha_1}^\beta \left(a^L(v_1) \prod_{i=2}^{N} \frac{a(v_i - v_1)}{b(v_i - v_1)} - b^L(v_1) \prod_{i=2}^{N} \frac{a(v_1 - v_i)}{b(v_1 - v_i)} \prod_{\beta=1}^{M} \frac{a(\gamma_\beta - v_1)}{b(\gamma_\beta - v_1)} \right). \quad (4.16)$$

Analogous expressions follow for the other coefficients $Y_{j\alpha_j}^\beta$ ($j \geqslant 2$),

$$Y_{j\alpha_j}^\beta \propto \delta_{\alpha_j}^\beta \left(a^L(v_j) \prod_{i\neq j}^{N} \frac{a(v_i - v_j)}{b(v_i - v_j)} - b^L(v_j) \prod_{i\neq j}^{N} \frac{a(v_j - v_i)}{b(v_j - v_i)} \prod_{\beta=1}^{M} \frac{a(\gamma_\beta - v_j)}{b(\gamma_\beta - v_j)} \right),$$

$$j = 1, \dots, N. \quad (4.17)$$

We observe that the requirement $Y_{j\alpha_j}^\beta = 0$ ($j = 1 \dots N$, $\beta = 2, 3$) is equivalent to the Bethe ansatz equations (3.34), therefore Bethe states fulfill the highest-weight condition $\tilde{M}_1^\beta \Psi = 0$ ($\beta = 2$ or 3).

To calculate $\tilde{M}_2^3\Psi$ we use the relation

$$\tilde{M}_2^3\Psi = \sigma_{3\{\alpha'\}} B_{\alpha'_1}(v_1) B_{\alpha'_2}(v_2) \dots B_{\alpha'_N}(v_N) \Phi \tilde{M}_{(1)2\{\alpha\}}^{3\{\alpha'\}} \Psi_{(1)}^{\{\alpha\}}, \tag{4.18}$$

which follows from (4.2). $\tilde{M}_{(1)}$ is defined by asymptotic expansion of the monodromy $T_{(1)}$, in analogy with $\tilde{M}$ given by eq. (4.1). From (3.22) and commutation relations for $\tilde{M}_{(1)}$ and $\tilde{T}_{(1)}$ analogous to eq. (4.2) we get

$$\tilde{M}_{(1)2}^{\ 3}\Psi_{(1)} = \sum_{\beta=1}^{M} (-1)^{\beta-1} B_{(1)}(\gamma_1, \{v_i\}) \dots B_{(1)}(\gamma_{\beta-1}, \{v_i\})$$

$$\times \left[\tilde{M}_{(1)2}^{\ 3}, B_{(1)}(\gamma_\beta, \{v_i\})\right]_+ B_{(1)}(\gamma_{\beta+1}, \{v_i\}) \dots B_{(1)}(\gamma_M, \{v_i\}) \Phi_{(1)}, \tag{4.19}$$

where

$$\left[\tilde{M}_{(1)2}^{\ 3}, B_{(1)}(\gamma)\right]_+ = A_{(1)}(\gamma) + \tilde{T}_{(1)3}^{\ 3}(\gamma). \tag{4.20}$$

Analogously, by commuting $A_{(1)} + \tilde{T}_{(1)3}^{\ 3}$ through the $B_{(1)}$'s we have

$$\tilde{M}_{(1)2}^{\ 3}\Psi_{(1)} = \sum_{\beta=1}^{M} Y_{(1),\beta}(\gamma_\beta, \{\gamma_\alpha\}, \{v_i\}) B_{(1)}(\gamma_1, \{v_i\}) \dots$$

$$\times B_{(1)}(\gamma_{\beta-1}, \{v_i\}) B_{(1)}(\gamma_{\beta+1}, \{v_i\}) \dots B_{(1)}(\gamma_M, \{v_i\}) \Phi_{(1)}. \tag{4.21}$$

The coefficients $Y_{(1),\beta}$ can be derived in a straightforward way by taking the first terms of the commutation relations (3.23) and (3.24). We get

$$Y_{(1),\beta} = \prod_{i=1}^{N} a(\gamma_\beta - v_i) \prod_{\alpha\neq\beta}^{M} \frac{a(\gamma_\alpha - \gamma_\beta)}{b(\gamma_\alpha - \gamma_\beta)}$$

$$+ (-1)^M \prod_{i=1}^{N} b(\gamma_\beta - v_i) \prod_{\alpha\neq\beta}^{M} \frac{w(\gamma_\beta - \gamma_\alpha)}{b(\gamma_\beta - \gamma_\alpha)},$$

$$\beta = 1, \dots, M. \tag{4.22}$$

The requirement $Y_{(1),\beta} = 0$ ($\beta = 1, \dots, M$) is equivalent to the Bethe ansatz equations (3.35), which implies $\tilde{M}_2^3\Psi = 0$. We stress that the property (4.11) can also be proved for the other two choices of pseudovacuum in a similar way.

At the end of this section we derive some inequalities between the weights w_α ($\alpha = 1, 2, 3$). From eq. (4.5) we have

$$\left[\tilde{M}_\alpha^\beta, \tilde{M}_\beta^\alpha\right]_\pm = W_\alpha^\alpha \pm W_\beta^\beta, \quad \beta > \alpha. \tag{4.23}$$

Using $(\tilde{M}_{\alpha}^{\beta})^{\dagger} = \tilde{M}_{\beta}^{\alpha}$ and the highest-weight property of the Bethe vectors (4.11) we obtain

$$w_1 \geqslant w_2 \geqslant -w_3. \tag{4.24}$$

Combining (4.10) with $w_i \geqslant 0$ ($i = 1, 2, 3$) and (4.24) we find conditions for the numbers N and M of roots in the first- and second-level Bethe ansatz, respectively,

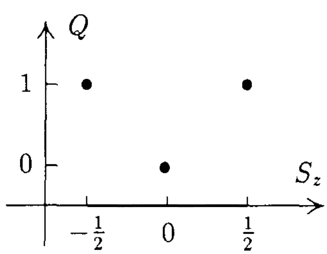
$$M \leqslant N \leqslant \frac{L+M}{2}, \qquad 0 \leqslant M \leqslant L. \tag{4.25}$$

This means in terms of physical quantities that the magnetization $S_z = \frac{1}{2}(n_{\uparrow} - n_{\downarrow}) = \frac{1}{2}(L - 2N + M)$ and the number of electrons $Q = n_{\uparrow} + n_{\downarrow} = L - M$ are restricted to $0 \leqslant S_z \leqslant Q/2 \leqslant L/2$.

5. Results for small and large lattices

In this section we illustrate the algebraic properties of the Bethe states. We begin with a lattice of two sites and then discuss the case of lattices with a large number of sites.

The simple case of one lattice point corresponds to the fundamental representation of spl(2,1) which is given by the following weight diagram in the (S_z, Q) plane, where Q is the number of electrons and S_z the total magnetization of the system:



By diagonalization of the t–J hamiltonian (2.17) (or of the transfer matrix τ) on a lattice with two sites we obtain

$$\Psi_1 = |\uparrow\uparrow\rangle, \qquad E = 2,$$

$$\Psi_2 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \qquad E = 2,$$

$$\Psi_3 = |\downarrow\downarrow\rangle, \qquad E = 2,$$

$$\Psi_4 = \frac{1}{\sqrt{2}}(|0\uparrow\rangle + |\uparrow 0\rangle), \qquad E = 2,$$

$$\Psi_5 = \frac{1}{\sqrt{2}}(|0\downarrow\rangle + |\downarrow 0\rangle), \qquad E = 2, \tag{5.1}$$

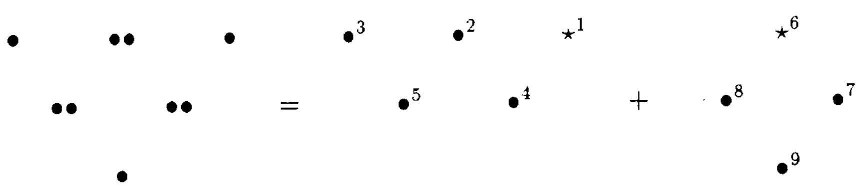
$$\Psi_6 = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), \qquad E = -2,$$

$$\Psi_7 = \frac{1}{\sqrt{2}}(|0\uparrow\rangle - |\uparrow 0\rangle), \qquad E = -2,$$

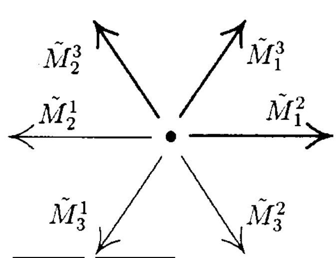
$$\Psi_8 = \frac{1}{\sqrt{2}}(|0\downarrow\rangle - |\downarrow 0\rangle), \qquad E = -2,$$

$$\Psi_9 = |00\rangle, \qquad E = -2, \tag{5.1'}$$

where 0 denotes a hole. This result can be visualized in terms of the following spl(2,1) weight diagrams in the Clebsch–Gordan series $3 \otimes 3 = 5 \oplus 4$:



The numbers in the weight diagrams specify the eigenvectors according to eqs. (5.1) and (5.1′). The symbol ⋆ denotes the highest-weight vectors according to eq. (5.2) below. Notice that the ground state is degenerate and given by a quartet. All states of an irreducible representation can be generated by repeated application of the shift operators $\tilde{M}_\alpha^\beta$ ($\beta \neq \alpha$) to any one of the states. Graphically, the effect of the shift operators on a general state of a representation of spl(2,1) is given by



On the other hand, if we solve the Bethe ansatz equations (3.36) and (3.37) for two sites we obtain only two eigenvectors, $\Psi_1 = \Phi$ and $\Psi_6 = B_{\alpha_1}(v_1 = 0)\Phi\Phi_{(1)}^{\alpha_1}$, with energy eigenvalues 2 and -2, respectively (see eq. (3.49)). In the language of the nested Bethe ansatz Φ and $\Phi_{(1)}$ are the first- and second-level pseudoground states, respectively. We can easily check that these eigenvectors are highest-weight vectors of the spl(2,1) superalgebra, in agreement with our general proof in sect. 4.

$$\tilde{M}_\alpha^\beta \Psi_1 = \tilde{M}_\alpha^\beta \Psi_6 = 0, \quad \beta > \alpha. \tag{5.2}$$

Furthermore, the seven missing eigenvectors can be obtained by successive applications of the shift operators,

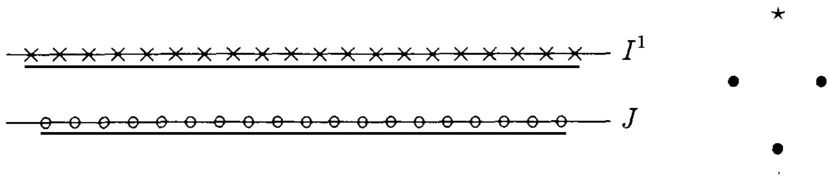
$$\begin{aligned}
\Psi_2 &= \tilde{M}_2^1 \Psi_1, \\
\Psi_3 &= \left(\tilde{M}_2^1\right)^2 \Psi_1, \\
\Psi_4 &= \tilde{M}_3^1 \Psi_1, \\
\Psi_5 &= \tilde{M}_2^1 \tilde{M}_3^1 \Psi_1, \\
\Psi_7 &= \tilde{M}_3^2 \Psi_6, \\
\Psi_8 &= \tilde{M}_3^1 \Psi_6, \\
\Psi_9 &= \tilde{M}_3^1 \tilde{M}_3^2 \Psi_6.
\end{aligned} \tag{5.3}$$

Therefore, the Bethe ansatz together with the supersymmetry of the model provide all 9 eigenvectors for the two-sites model.

We remark that by solving all three different forms of the BAE we get all highest-weight vectors of the SU(2) algebra. Solving eqs. (3.38) and (3.39) we get the eigenvectors Ψ_4 and Ψ_9 and from eqs. (3.40) and (3.41) we obtain the eigenvectors Ψ_1 and Ψ_7.

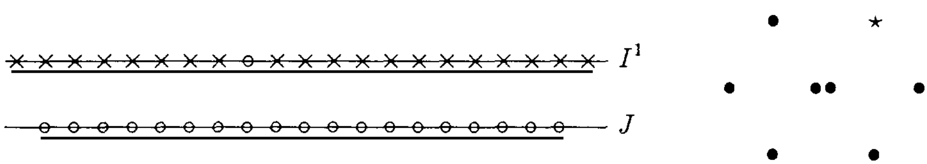
In the case of lattices with a large number of sites the Bethe ansatz method turns out to be crucial, since the effort of an exact diagonalization growths exponentially with the number of sites L. As already pointed out in sects. 3 and 4, by this method, the problem of finding the spectrum of the t–J hamiltonian reduces to the solution of the BAE's (3.36) and (3.37) for the parameters v's and γ's. Adopting the string-conjecture, which has an accuracy of $O(e^{-L})$, the solutions of the BAE's are parametrized in terms of the numbers I_α^n and J_β. Moreover, each set $\{I_\alpha^n, J_\beta\}$ where the I's and J's are pairwise different specify a Bethe vector, which is the leading vector of an spl(2,1) multiplet.

Now we illustrate our results for the ground state and some elementary excitations at "half-filling" $F = Q/L = 1$. The following holds true for any lattice size, especially in the thermodynamic limit $L \to \infty$. The ground state involves only real roots. This can be proved as usual by minimizing the free energy for finite temperature T and taking $T \to 0$ [18]. For example, for a lattice of size $L = 40$ we find $N = 20$ first-level real roots and no BA-holes. There are no second-level real roots ($M = 0$), but 19 BA-holes. Therefore, we have the following distribution of I's and J's:



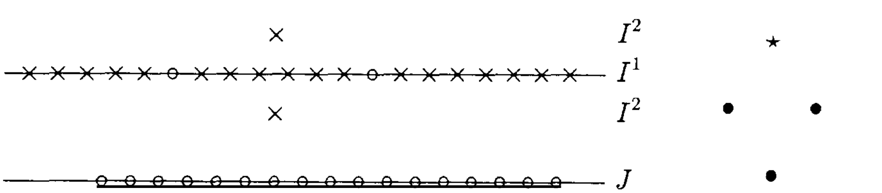
where the numbers corresponding to roots are denoted by × and those corresponding to BA-holes by ○. On the r.h.s. the associated spl(2,1) representation is shown. The quantum numbers of the ground state are $S_z = 0$ and $Q = L = 40$, which means vanishing magnetization and half-filling $F = 1$. The Bethe vector specified by this set of numbers $\{I_\alpha^1\}$ is the highest-weight vector of the irreducible representation of dimension 4, depicted by ⋆. Notice that the ground state is not a singlet but a member of an spl(2,1) quartet. Of course, the state is a singlet with respect to the SU(2) subgroup.

One kind of elementary excitation over the ground state is the "spinon". It is obtained by removing a root from the I^1-axis or introducing a first-level BA-hole, which corresponds to a spin flip. For a lattice of size $L = 41$ * we have $N = 20$ first-level roots and one BA-hole, $M = 0$ second-level roots and 19 BA-holes. The distribution of I's and J's and the corresponding irreducible representation generated by the Bethe vector (⋆) determined by this set of I's and J's are for example illustrated by



The quantum numbers of this state are $S_z = 1/2$ and $Q = L = 41$, which means a spinon is a particle-like excitation with spin $1/2$ and charge 0.

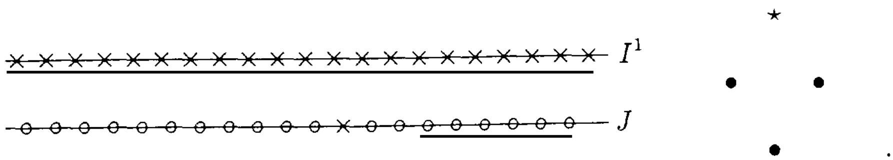
Another excitation is given by the presence of a two-string in the complex v-plane. For $L = 40$ we have $N^1 = 18$, $N^2 = 1$ and $M = 0$,



* Note that a one-spinon state exists only on lattices with an odd number of lattice sites, otherwise spinons appear pairwise.

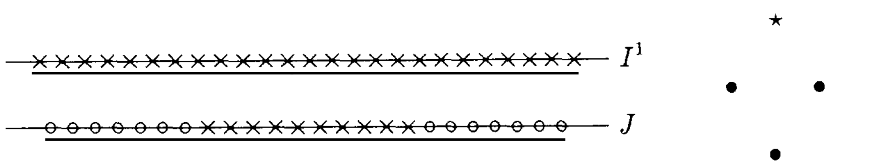
(To support the visual perception we have drawn the I^2-axis twice in order to obtain a picture similar to the corresponding one in the complex v-plane). The quantum numbers are $S_z = 0$ and $Q = L = 40$, which means vanishing magnetization and half-filling $F = 1$. The interpretation of this state is that we have the spin-0 contribution of a two-spinon state.

By filling a vacancy in the J-axis we get another excitation called "holon", i.e. we are removing an electron from the system or introducing a physical hole. For $L = 41$ we have $N^1 = 21$ first-level roots, $M = 1$ second-level root and 19 holes,



The quantum numbers of this state are $S_z = 0$ and $Q = 40 = L - 1$, which means a holon is a particle-like excitation with spin 0 and charge -1.

At arbitrary filling $F < 1$, for the ground state the distribution of the roots in the v-plane also involves only real roots. In contrast to the half-filling case there is now in addition a "sea" of real roots in the γ-plane, such that there appears a nontrivial Fermi level. For example, for a lattice of $L = 40$ sites we find a Bethe ansatz state with $N = 25$ first-level real roots and $M = 10$ second-level real roots and 14 BA-holes,



Also here the ground state is member of a quartet. The quantum numbers are $S_z = 0$ and $Q = 30$, which means spin 0 and filling $F = 1 - 10/40 = 0.75$. Due to the nontrivial Fermi level there exist "holon–antiholon" excitations in this case.

6. Completeness of the Bethe vectors

In this section we show how to construct a complete set of eigenvectors of the t–J hamiltonian for arbitrary chain of length L. This is obtained by combining the Bethe ansatz with the supersymmetry of the model.

From the sect. 3 we know that all collections $\{I_\alpha^n, J_\beta\}$ where the I's and J's are pairwise different specify all the Bethe vectors $(|\psi_{\text{Bethe}}\rangle_{N,M})$. The number of admissible values for the I_α^n and the J_β (for fixed $\{N_n\}$ and M) is $(2I_{\max}^n + 1)$ and $(2J_{\max} + 1)$, respectively. $I_{\max}^n$ and $J_{\max}$ are given by eqs. (3.47) and (3.48). Taking into account that many different string configurations N_n give the same number of roots N (see eq. (3.43)), the number of possible Bethe vectors for fixed N, M is given by

$$Z(N, M) = \sum_{\{N_n\}} \binom{2J_{\max}+1}{M} \prod_n \binom{2I^n_{\max}+1}{N_n}, \tag{6.1}$$

where the sum over $\{N_n\}$ is constrained to $\Sigma_n \; nN_n = N$. It is convenient to introduce the quantity $q = \Sigma_n \; N_n$. Using eqs. (3.47) and (3.48) we write this sum as

$$Z(N, M) + \sum_{q=0}^{N} \binom{q-1}{M} \sum_{\{N_n\}} \prod_n \binom{L - \Sigma_m \, t_{nm} N_m + M}{N_n}, \tag{6.2}$$

where the inner sum is constrained to fixed values of N and q. This expression resembles the one calculated by Bethe in the isotropic Heisenberg model [1,2] and can be simplified to

$$Z(N, M) = \sum_{q=0}^{N} \frac{L+M-2N+1}{L+M-N+1} \binom{q-1}{M} \binom{L+M-N+1}{q} \binom{N-1}{q-1}. \tag{6.3}$$

The total number of Bethe vectors is obtained by summing $Z(N, M)$ over all N, M restricted to (4.25). However, this number is less than 3^L, so that the Bethe ansatz does not yield all the states of the model. In order to construct a complete set we shall invoke the supersymmetry of the transfer matrix. First, from eq. (4.6) it follows that the Bethe vectors are classified by multiplets corresponding to irreducible representations of the superalgebra spl(2,1). Furthermore, from eq. (4.11) follows that the Bethe vectors are highest-weight vectors. Then by acting with the spl(2,1) lowering operators $\tilde{M}_\alpha^\beta (\beta < \alpha)$ on the Bethe states we obtain additional states. Each Bethe state (with fixed N, M in the interval (4.25)) is the highest-weight vector in a multiplet of dimension [11]

$$d(N, M) = \begin{cases} 4S_z + 1 = 2L + 1 & \text{if } N = M = 0 \\ 8(S_z + 1/2) = 4(L - 2N + M + 1) & \text{otherwise.} \end{cases} \tag{6.4}$$

With these considerations, the total number of eigenvectors is

$$Z = \sum_{M=0}^{L} \sum_{N=M}^{(L+M)/2} d(N, M) Z(N, M) = 2L + 1 + Z_1 - 1 + Z_2$$

$$= 2L + 1 + 4 \sum_{N=1}^{L/2} (L - 2N + 1) \frac{L - 2N + 1}{L - N + 1} \sum_{q=1}^{N} \binom{L-N+1}{q} \binom{N-1}{q-1}$$

$$+4\sum_{M=1}^{L}\sum_{N=M}^{(L+M)/2}(L-2N+M+1)\frac{L+M-2N+1}{L+M-N+1}$$

$$\times\sum_{q=1}^{N}\binom{q-1}{M}\binom{L+M-N+1}{q}\binom{N-1}{q-1}. \tag{6.5}$$

The first sum in eq. (6.5) can be performed (see ref. [2]) to give

$$Z_1=4\cdot 2^L-4(L+1). \tag{6.6}$$

The second sum Z_2 deserves special attention. We present the main necessary steps for its evaluation. First, performing the sum over q we get

$$Z_2=4\sum_{M=1}^{L}\sum_{N=M}^{(L+M)/2}(L-2N+M+1)$$

$$\times\frac{L+M-2N+1}{L+M-N+1}\binom{N-1}{M}\binom{L}{N}. \tag{6.7}$$

Employing some combinatorics and making the substitution $N\to x=N-M$ we obtain

$$Z_2=4\sum_{M=1}^{L}\sum_{x=0}^{(L-M)/2}(L-2x-M+1)$$

$$\times\left[\binom{L}{x+M}\binom{x+M-1}{M}-\binom{L}{x-1}\binom{L-x}{M}\right]. \tag{6.8}$$

After some re-arrangements this expression can be rewritten as

$$Z_2=4\sum_{M=1}^{L}\sum_{x=0}^{(L-M)/2}L\left[\left[\binom{L-1}{x+M}\binom{x+M-1}{M}+\binom{L-1}{x-2}\binom{L-x}{M}\right]\right.$$

$$\left.-(M+1)\left[\binom{L}{x+M}\binom{x+M-1}{x-2}+\binom{L}{x-1}\binom{L-x}{M+1}\right]\right]. \tag{6.9}$$

Substituting $x\to L-x-M+1$ in the second and fourth terms of eq. (6.9) we get

$$Z_2=4\sum_{M=1}^{L}\sum_{x=0}^{L-M+1}\left[L\binom{L-1}{x+M}\binom{x+M-1}{M}-(M+1)\left[\binom{L}{x+M}\binom{x+M-1}{M+1}\right]\right]. \tag{6.10}$$

Using the binomial formula we obtain after some re-arrangements

$$Z_2 = 4L! \sum_{M=1}^{L} \frac{1}{M!(L-2-M)!} \int_0^1 p^M \left[(p+1)^{L-M-2} (1-p) \right] \mathrm{d}p. \quad (6.11)$$

Interchanging the sum and the integral and performing the sum gives

$$Z_2 = \frac{4L!}{(L-2)!} \int_0^1 (1-p) \left[(1+2p)^{L-2} - (1+p)^{L-2} \right] \mathrm{d}p. \quad (6.12)$$

This integral can be easily performed, resulting in

$$Z_2 = 3^L - 4 \cdot 2^L + 2L + 3. \quad (6.13)$$

Substituting eqs. (6.6) and (6.13) into (6.5) we get

$$Z = 3^L. \quad (6.14)$$

Thus we have shown that the number of eigenvectors of the t–J hamiltonian is 3^L, which is precisely the number of states in the Hilbert space of a chain of length L, where at each site there may be either a spin-up or a spin-down electron or a hole.

7. Conclusions

In this paper we have shown that the Bethe ansatz states for the one-dimensional supersymmetric t–J model are highest-weight vectors of an spl(2,1) superalgebra. Then by acting with the spl(2,1) lowering operators on the Bethe states we have obtained a complete set of eigenvectors of the t–J hamiltonian.

An interesting extension of this work is an analysis of the $\mathrm{spl}_q(2,1)$ "quantum-group" structure of a "q-deformed" version of this model (see also ref. [19]). This is presently under investigation.

Appendix A

In this appendix we show that the cancellation conditions of the "unwanted terms" u.t. = u.t.(A) + u.t.(D) and $\mathrm{u.t.}_{(1)} = \mathrm{u.t.}(A_{(1)}) + \mathrm{u.t.}(D_{(1)})$ are equivalent to the Bethe ansatz equations (3.34) and (3.35). As already pointed out in sect. 3 all terms whose arguments are exchanged when $A(v)$ and $\tau_D(v)$ is commuted through

$\prod_{j=1}^{N} B_{\alpha_j}(v_j)$ using eqs. (3.11) and (3.12) are called u.t.(A) and u.t.(D), respectively. They can be arranged as follows [15]:

$$\text{u.t.}(A) = \sum_{j=1}^{N} K_j^{(A)}(v_j, \{v_i\}) B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \dots B_{\alpha_{j-1}}(v_{j-1}) B_{\alpha_j}(v)$$

$$\times B_{\alpha_{j+1}}(v_{j+1}) \dots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}}, \tag{A.1}$$

$$\text{u.t.}(D) = \sum_{j=1}^{N} K_j^{(D)}(v_j, \{v_i\}) B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \dots B_{\alpha_{j-1}}(v_{j-1}) B_{\alpha_j}(v)$$

$$\times B_{\alpha_{j+1}}(v_{j+1}) \dots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}}. \tag{A.2}$$

Here $K_j^{(A)}$ and $K_j^{(D)}$ $(j = 1, \dots, N)$ are coefficients to be determined. The first coefficient of eq. (A.1) can be computed using the second term in (3.11) when commuting $A(v)$ with $B_{\alpha_1}(v_1)$ and then using the first term in eq. (3.11) when commuting $A(v_1)$ with the remaining B_α's, since otherwise the argument v_1 reappears in the B_α's. We get

$$K_1^{(A)} = -a^L(v_1) \frac{c(v_1 - v)}{b(v_1 - v)} \prod_{i \neq 1}^{N} \frac{a(v_i - v_1)}{b(v_i - v_1)}. \tag{A.3}$$

In order to calculate $K_j^{(D)}$ we rewrite the second term of eq. (3.12) as

$$-\frac{1}{v - v'} \operatorname*{Res}_{v''=v'} \left(S_{\gamma'\alpha}^{\gamma''\alpha'}(v'' - v') B_{\alpha'}(v) T_{\gamma''}^{\gamma}(v') \right), \tag{A.4}$$

by means of eqs. (2.5) and (4.15). Then, proceeding along the same lines as in the calculation of $K_1^{(A)}$ we get the eigenvalue problem for the transfer matrix $\tau_{(1)}$ (3.17). In addition, just taking the first term in eqs. (3.23) and (3.24) when passing $A_{(1)}$ and $\tilde{T}_3^3$ through the $B_{(1)}$'s we obtain, after some re-arrangements,

$$K_1^{(D)} = -b^L(v_1) \frac{c(v - v_1)}{b(v - v_1)} \prod_{i \neq 1}^{N} \frac{a(v_1 - v_i)}{b(v_1 - v_i)} \prod_{\beta=1}^{M} \frac{\alpha(\gamma_\beta - v_1)}{b(\gamma_{\bar{\beta}} - v_1)} \tag{A.5}$$

To get the other coefficients $K_j^{(A)}$ and $K_j^{(D)}$ $(j = 2, \dots, N)$ we use the commutation rule for the B_α's (3.13) and put $B_{\alpha_j}(v)$ in the first place. Then, repeating the same procedure we obtain analogous expressions with j in the place of 1. Furthermore, the requirement $K_j^{(A)} + K_j^{(D)} = 0$ $(j = 1, \dots, N)$ together with the fact that $c(v)/b(v)$ is an odd function (see eq. (2.5)) leads to the Bethe ansatz equation (3.34).

The "unwanted terms" that appear in the second level of the Bethe ansatz method can be arranged as follows:

$$\text{u.t.}_{(1)} = \sum_{\beta=1}^{M} \left(K_\beta^{(A_{(1)})} + K_\beta^{(D_{(1)})} \right) B_{(1)}(\gamma_1, \{v_i\}) B_{(1)}(\gamma_2, \{v_i\}) \dots$$
$$\times B_{(1)}(\gamma_{\beta-1}, \{v_i\}) B_{(1)}(v, \{v_i\}) B_{(1)}(\gamma_{\beta+1}, \{v_i\}) \dots B_{(1)}(\gamma_M, \{v_i\}) \Phi_{(1)}. \quad \text{(A.6)}$$

By similar arguments as above, the coefficients $K_1^{(A_{(1)})} + K_1^{(D_{(1)})}$ can be computed using the second term in eqs. (3.23) and (3.24) when commuting $A_{(1)}(v, \{v_i\})$ and $\tilde{T}_{(1)3}^{3}(v, \{v_8 \text{ii}\})$ through $B_{(1)}(\gamma_1, \{v_i\})$ and then using the first term in (3.23) and (3.24) when commuting $A_{(1)}(\gamma_1, \{v_i\})$ and $\tilde{T}_{(1)3}^{3}(\gamma_1, \{v_i\})$ with the remaining $B_{(1)}$'s.

$$K_\beta^{(A_{(1)})} + K_\beta^{(D_{(1)})} = -\prod_{i=1}^{N} a(\gamma_\beta - v_i) \frac{c(\gamma_\beta - v)}{b(\gamma_\beta - v)} \prod_{\alpha \neq \beta}^{M} \frac{a(\gamma_\alpha - \gamma_\beta)}{b(\gamma_\alpha - \gamma_\beta)}$$
$$-(-1)^M \frac{c(v - \gamma_\beta)}{b(v - \gamma_\beta)} \prod_{i=1}^{N} b(\gamma_\beta - v_i) \prod_{\alpha \neq \beta}^{M} \frac{w(\gamma_\beta - \gamma_\alpha)}{b(\gamma_\beta - \gamma_\alpha)}. \quad \text{(A.7)}$$

Once again, the other coefficients can be obtained using the commutation rules (3.25). The requirement $K_\beta^{(A_{(1)})} + K_\beta^{(D_{(1)})} = 0$ $(\beta = 1, \dots, M)$ together with the fact that $c(v)/b(v)$ is an odd function yields the Bethe ansatz equation (3.35).

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