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On the Free Terms of Boundary Integral Equations for Thick Plates Undergoing Large Displacements

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Abstract

This work presents a theoretical derivation of the convective terms appearing in integral equations for large displacement analysis of the Mindlin and the Reissner plate models. They are necessary to complete the Somigliana identities of the problem, since the non-linear terms in the Green-Lagrange strain tensor require additional derivative, hypersingular integral equations for the gradient of the displacement field. The attainment of these terms is commonly omitted in the literature, in spite of their presence in the integral equations for most nonlinear elasticity problems. With all the free terms identified, a complete set of integral equations for large displacement analysis of moderately thick plate models is obtained, aiming its BEM implementation. Numerical comparisons are made with available solutions showing good agreement.

Keywords: Boundary element method. Integral equations. Derivative integral equations. Plate bending. Large displacements.

1 Introduction

Numerical solutions for geometrically non-linear bending of moderately thick plates are well reported in the literature. Among the conventional numerical methods used to solve this type of problem, the boundary element method (BEM) has been receiving relative little attention on the subject, in spite of the excellence of the results obtained with the method for linear problems [10, 14, 18, 23]. Many reasons have contributed to prevent the general application of the BEM in non-linear problems. The generality of the finite element method is obviously one of them, but some mathematical aspects inherent to integral equation methods have contributed as well. As one of these aspects one could mention the so called convective (or free) terms that arise in derivative integral equations, as these terms are sometimes misunderstood or even missing from the equations.

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The objective of the present work is to outline the deduction of the convective terms appearing in integral equations for large displacement analysis of Mindlin and Reissner plate models. There are a few works exploring the solution of geometrically nonlinear thick plate bending problems using the BEM [8, 13, 22, 24, 27, 28]. However, most of them do not present the derivation of the free terms and, in addition, no one shows results for maximum trasnverse displacement far beyond the plate thickness magnitude, to the best of author´s knowledge. The present work aims a clear and didactic derivation of such terms, as they are quite common in nonlinear applications using boundary integral equation methods.

The Mindlin and the Reissner plate theories are very well known structural models. In his celebrated work E. Reissner [19] started from a stress field and a mixed variational principle to obtain the equilibrium equations. The Hencky-Bollé-Mindlin (or simply Mindlin, as it is generally known) plate model [2, 16] can be more easily obtained departing from a kinematical point of view, where the Kirchhoff-Love normality (thin plate) condition is relaxed (throughout this work, greek indexes range from 1 to 2 while latin indexes range from 1 to 3):

$$\begin{aligned} U_\alpha(x_1,x_2,x_3) &= \bar{u}_\alpha(x_1,x_2) + x_3 u_\alpha(x_1,x_2) \\ U_3(x_1,x_2,x_3) &= u_3(x_1,x_2) \end{aligned} \tag{1}$$

where $\bar{\mathbf{u}}$ and $\mathbf{u}$ are the membrane and plate displacements, respectively (i.e. $\bar{u}_\alpha$ and u_3 are in and out of plane translations, respectively, while u_α are the plate rotations). All variables are referred to the plates's middle surface. If taken pointwise across the thickness, the displacement field of the Reissner´s model is more complex than postulated in eqs.(1). However, the middle surface field $u_i(x_1,x_2)$ remains valid for the Reissner´s model if it is interpreted as a weighed mean value of the displacement field across the thickness:

$$\begin{aligned} u_\alpha^{\text{Mindlin}} &= \frac{12}{h^3}\int_{-h/2}^{h/2} u_\alpha^{\text{Reissner}}(x_1,x_2,x_3)\, x_3\, dx_3 \\ u_3^{\text{Mindlin}} &= \frac{3}{2h}\int_{-h/2}^{h/2} u_3^{\text{Reissner}}(x_1,x_2,x_3)\left[1-\left(\frac{2x_3}{h}\right)^2\right] dx_3 \end{aligned}$$

where h is the plate thickness.

The in plane displacements are included in eq.(1) because the two-dimensional elasticity behavior will be superimposed to the plate bending equations, aiming the derivation of equilibrium equations for geometrically non-linear bending problems. These are found to be written in terms of resultant stresses as [6]:

$$N_{\alpha\beta,_\beta} + q_\alpha = 0 \tag{2a}$$

$$\left(N_{\alpha\beta} u_{3,\alpha}\right)_{,\beta} + Q_{\alpha,_\alpha} + q_3 = 0 \tag{2b}$$

$$M_{\alpha\beta,_\beta} - Q_\alpha + m_\alpha = 0 \tag{2c}$$

where $N_{\alpha\beta}$ are the inplane (membrane) forces, Q_α are the shear forces and $M_{\alpha\beta}$ are the bending moments. The symbols q_α and q_3 stand for inplane and transverse loadings, respectively, while m_α are the distributed moments. Equations (2) can be recovered in terms of displacements through the stress-displacement relations:

$$N_{\alpha\beta} = C\frac{1-\nu}{2}\left[u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}u_{3,\beta} + \frac{2\nu}{1-\nu}\left(u_{\gamma,\gamma} + \frac{1}{2}u_{3,\gamma}u_{3,\gamma}\right)\delta_{\alpha\beta}\right] \tag{3a}$$

$$M_{\alpha\beta} = D\frac{1-\nu}{2}\left[u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2\nu}{1-\nu}\,u_{\gamma,\gamma}\delta_{\alpha\beta}\right] \tag{3b}$$

$$Q_\alpha = D\lambda^2\frac{1-\nu}{2}\left[u_\alpha + u_{3,\alpha}\right] \tag{3c}$$

where

$$C = \frac{Eh}{(1-\nu^2)} \quad , \qquad D = \frac{Eh^3}{12(1-\nu^2)} \quad , \qquad \lambda^2 = \frac{12\kappa^2}{h^2}$$

and κ^2 is the shear stress correction factor. Regarding the plate theory used, the only visible difference in eqs.(3) is in the expression for the moments, which has an additional term in the Reissner's plate model:

$$M_{\alpha\beta}^{\text{Reissner}} = \text{ R.H.S. of eq.(3b) } + \frac{\nu}{(1-\nu)\lambda^2}\,q_3\,\delta_{\alpha\beta} \tag{4}$$

In order to unify the equilibrium equations under the same computational model, a plate model factor (m_f) is employed [25]:

$$M_{\alpha\beta} = D\frac{1-\nu}{2}\left[\psi_{\alpha,\beta} + \psi_{\beta,\alpha} + \frac{2\nu}{1-\nu}\psi_{\gamma,\gamma}\delta_{\alpha\beta}\right] + m_f\ q_3\,\delta_{\alpha\beta} \tag{5}$$

where

$$m_f = \frac{\nu}{(1-\nu)\lambda^2} \qquad \text{, for the Reissner's model} \tag{6a}$$

$$m_f = 0 \qquad \text{, for the Mindlin's model.} \tag{6b}$$

Eqs.(2) describe moderately thick plate bending problems under large displacements and moderately large rotations regime [6] In view of eqs.(5), they can be used regardless the plate model considered, including the classical Kirchhoff-Love model. The presence of the non-linear terms in eqs.(3) is a consequence of relevant higher order terms kept in the Green-Lagrange strain tensor. Both the linear and nonlinear contributions can be further evidenced by writing:

$$N_{\alpha\beta} = N^l_{\alpha\beta} + N^n_{\alpha\beta} \tag{7a}$$

$$Q_\alpha = Q^l_\alpha + Q^n_\alpha \tag{7b}$$

where

$$N^l_{\alpha\beta} = C\frac{1-\nu}{2}\left[\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha} + \frac{2\nu}{1-\nu}\,\bar{u}_{\gamma,\gamma}\delta_{\alpha\beta}\right] \tag{8a}$$

$$N^n_{\alpha\beta} = C\frac{1-\nu}{2}\left[u_{3,\alpha}u_{3,\beta} + \frac{\nu}{1-\nu}\,u_{3,\gamma}u_{3,\gamma}\delta_{\alpha\beta}\right] \tag{8b}$$

$$Q^l_{\alpha} = D\lambda^2\frac{1-\nu}{2}\,(u_\alpha + u_{3,\alpha}) \tag{8c}$$

$$Q^n_{\alpha} = N_{\alpha\beta}\,u_{3,\beta} \tag{8d}$$

Equations (2) and (3) – with eq.(5) replacing eq.(3b) – are taken herein as a starting point for an incremental integral formulation. Using the weighted residual method [3], the following Somigliana identities for boundary variables are obtained [12, 13, 28]:

$$\begin{aligned}
&{}^mC_{\alpha\beta}(p)\bar{u}_\beta(p) + \int_\Gamma {}^mT_{\alpha\beta}(q,p)\bar{u}_\beta(q)\,d\Gamma_q = \int_\Gamma {}^mU_{\alpha\beta}(q,p)\bar{t}_\beta(q)\,d\Gamma_q + \\
&+ \int_\Omega {}^mV_{\alpha\beta}(Q,p)q_\beta(Q)\,d\Omega_Q - \\
&+ \int_\Omega {}^mU_{\alpha\beta,\gamma}(Q,p)N^n_{\beta\gamma}(Q)\,d\Omega_Q + {}^mv_\alpha(p)
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&{}^fC_{ij}(p)u_j(p) + \int_\Gamma {}^fT_{ij}(q,p)u_j(q)\,d\Gamma_q = \int_\Gamma {}^fU_{ij}(q,p)t_j(q)\,d\Gamma_q + \\
&+ \int_\Omega {}^fV_{ij}(Q,p)q_j(Q)\,d\Omega_Q + \\
&- \int_\Omega {}^fU_{i3,\beta}(Q,p)N_{\alpha\beta}(Q)u_{3,\alpha}(Q)\,d\Omega_Q + {}^fv_i(p)
\end{aligned} \tag{10}$$

where the m and f suffixes refer to the membrane and the bending problem, respectively. The non-integral terms ${}^mv_\beta$ and ${}^fv_i$ were included to account for concentrated loads inside the domain [9]. The symbols p and q denote source (collocation) and field points, respectively (lower case indicates boundary points and upper case indicates domain points). The corresponding displacement (${}^mU_{ij}$ and ${}^fU_{ij}$), traction (${}^mT_{ij}$ and ${}^fT_{ij}$), and the other fundamental solution tensors can be found in the Appendix. Equations (9) and (10) are easily particularized for internal points by making ${}^mC_{\alpha\beta} = \delta_{\alpha\beta}$ and ${}^fC_{ij} = \delta_{ij}$.

From eqs.(9) and (10) it is evident that the evaluation of the derivatives of the transverse displacement (u_3) is required. They are present in the nonlinear membrane forces of the last integral of eq.(9) and also in the last integral of eq.(10). These terms are partially responsible for the membrane-bending coupling. In domain methods like finite elements, it is typical to employ the derivatives of the shape functions, i.e. $u_{i,\alpha} = \phi_{i,\alpha}u_i$, where ϕ_i are the shape functions. Although being very simple, this approach may generate poor results when the global shape function is not able to represent accurately the gradients of the displacement field. Similar approaches can be used for boundary elements, but the use of higher order domain cells becomes mandatory for acceptable results (see, for instance, [24]).

In the case of the boundary element method, there is no need to assume an a priori interpolatory form for the displacements derivatives since eqs.(9) and (10) are already a strong form of the displacement field. Therefore, a more rigorous solution can be obtained by differentiation of these integral equations with respect to the coordinates $x_\alpha(P)$. The procedure leads to the six required additional integral equations for $\bar{u}_{\beta,\alpha}$ and $u_{3,\alpha}$.

Assuming that the displacement derivatives are required only at internal points, the differentiation of eqs.(9) and (10) is straightforward as all their kernels become regular. However, the differentiation of the two last integrals on the right hand side of both equations is not, because the tensors ${}^mV_{\beta\gamma,\alpha}$, ${}^fV_{3i,\alpha}$, ${}^mU_{\alpha\beta,\gamma}$ and ${}^fU_{i3,\beta}$ have weak singularities[1] when $Q \equiv P$. Unfortunately the differentiation of integrals containing singular kernels does not obey the classical calculus rules, and they must be treated by means of the Leibnitz formula [4, 15]. The formal derivation of such derivative integral equations produces the so-called convective terms [3] which must be added to the final expressions for $\bar{u}_{\beta,\alpha}(P)$ and $u_{3,\alpha}(P)$ resulting:

$$
\begin{aligned}
\bar{u}_{\beta,\alpha}(P) - \int_\Gamma {}^mT_{\beta\gamma,\alpha}(q,P)\bar{u}_\gamma(q)\,d\Gamma_q = &- \int_\Gamma {}^mU_{\beta\gamma,\alpha}(q,P)\bar{t}_\gamma(q)\,d\Gamma_q + \\
&- \int_\Omega {}^mV_{\beta\gamma,\alpha}(Q,P)q_\gamma(Q)\,d\Omega_Q + \int_\Omega {}^mU_{\beta\gamma,\delta\alpha}(Q,P)N^n_{\gamma\delta}(Q)\,d\Omega_Q + \\
&+ N^n_{\gamma\delta}(P)\int_{\Gamma_1'} {}^mU_{\beta\gamma,\delta}(Q,P)\,r_{,\alpha}(P)\,d\Gamma_{Q_1} + \\
&- q_\gamma(P)\int_{\Gamma_1'} {}^mV_{\beta\gamma}(Q,P)\,r_{,\alpha}(P)\,d\Gamma_{Q_1} - {}^mv_{\beta,\alpha}(P)
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
u_{3,\alpha}(P) - \int_\Gamma {}^fT_{3i,\alpha}(q,P)u_i(q)\,d\Gamma_q = &- \int_\Gamma {}^fU_{3i,\alpha}(q,P)t_i(q)\,d\Gamma_q + \\
&- \int_\Omega V_{3i,\alpha}(Q,P)q_i(Q)\,d\Omega_Q + \int_\Omega {}^fU_{33,\alpha\gamma}(Q,P)N_{\beta\gamma}(Q)u_{3,\beta}(Q)\,d\Omega_Q + \\
&+ N_{\beta\gamma}(P)u_{3,\beta}(P)\int_{\Gamma_1'} {}^fU_{33,\gamma}(Q,P)\,r_{,\alpha}(P)\,d\Gamma_{Q_1} + \\
&- m_f\, q_i(P)\int_{\Gamma_1'} {}^fV_{3i}(Q,P)\,r_{,\alpha}(P)\,d\Gamma_{Q_1} - {}^fv_{3,\alpha}(P)
\end{aligned} \tag{12}
$$

A negative sign was added to all the integrals as the derivatives are assumed to be taken with respect to $x_\alpha(P)$. The integrals on Γ_1' in eqs.(11) and (12) are the aforementioned convective

[1] Taking into account the dimension of the corresponding integration domains, one can show that the integral containing ${}^f\mathbf{V}$ is singular only in the case of Reissner's plate model, while ${}^m\mathbf{V}$ is always regular [25].

terms ($\mathit{\Gamma}'_1$ stands for a unit circle centered in P), whose derivation is the objective of the present work. The main goal is to resolve the analytical expressions for all four convective terms:

$$ {}^f c^N_{\alpha\beta}(P) = N_{\beta\gamma}(P) u_{3,\beta}(P) \int_{\mathit{\Gamma}'_1} {}^f U_{33,\gamma}(Q,P)\, r_{,\alpha}(P)\, d\mathit{\Gamma}_{Q_1} \tag{13a} $$

$$ {}^f c^q_{\alpha}(P) = m_f\ q_i(P) \int_{\mathit{\Gamma}'_1} {}^f V_{3i}(Q,P)\, r_{,\alpha}(P)\, d\mathit{\Gamma}_{Q_1} \tag{13b} $$

$$ {}^m c^N_{\alpha\beta}(P) = N^n_{\gamma\delta}(P) \int_{\mathit{\Gamma}'_1} {}^m U_{\beta\gamma,\delta}(Q,P)\, r_{,\alpha}(P)\, d\mathit{\Gamma}_{Q_1} \tag{13c} $$

$$ {}^m c^q_{\alpha\beta}(P) = q_\gamma(P) \int_{\mathit{\Gamma}'_1} {}^m V_{\beta\gamma}(Q,P)\, r_{,\alpha}(P)\, d\mathit{\Gamma}_{Q_1} \tag{13d} $$

2 Derivation of the convective terms

This section details the analytical unfolding of eqs.(13) following the steps described in [3]. Once these terms are obtained, the set of derivative integral equations for the translational displacements are eventually completed.

An inspection on eqs.(9) and (10) reveals that the candidate terms to originate convective terms are:

$$ I^N_i = \int_{\mathit{\Omega}} {}^f U_{i3,\beta}(Q,P) N_{\alpha\beta}(Q) u_{3,\alpha}(Q)\, d\mathit{\Omega}_Q \tag{14a} $$

$$ I^q_i = \int_{\mathit{\Omega}} {}^f V_{ij}(Q,P) q_j(Q)\, d\mathit{\Omega}_Q \tag{14b} $$

$$ J^N_\alpha = \int_{\mathit{\Omega}} {}^m U_{\alpha\beta,\delta}(Q,P) N^n_{\beta\delta}(Q)\, d\mathit{\Omega}_Q \tag{14c} $$

$$ J^q_\alpha = \int_{\mathit{\Omega}} {}^m V_{\alpha\beta}(Q,P) q_\beta(Q)\, d\mathit{\Omega}_Q \tag{14d} $$

whose derivation with respect to the coordinate axes lead to a general form for eqs.(13):

$$ \frac{\partial I^N_i}{\partial x_\gamma(P)} = {}^f c^N_{\alpha\beta}(P) \tag{15a} $$

$$ \frac{\partial I^q_i}{\partial x_\gamma(P)} = {}^f c^q_{\alpha}(P) \tag{15b} $$

$$ \frac{\partial J^N_\alpha}{\partial x_\gamma(P)} = {}^m c^N_{\alpha\beta}(P) \tag{15c} $$

$$ \frac{\partial J^q_\alpha}{\partial x_\gamma(P)} = {}^m c^q_{\alpha\beta}(P) \tag{15d} $$

In order to keep the notation concise, the prefixes m and f will be suppressed in the next subsections. Their use become clear by recalling eqs.(15).

2.1 Evaluation of $\dfrac{\partial I_i^N}{\partial x_\gamma\,(P)}$

Equations (14a) is rewritten as:

$$I_i^N = \lim_{\epsilon\to 0}\int_{\Omega-\Omega_\epsilon} U_{i3,\alpha}(Q,P)M_\alpha(Q)\,d\Omega_Q \tag{16}$$

where

$$M_\alpha(Q) = N_{\alpha\beta}(Q)u_{3,\beta}(Q)$$

and Ω_ϵ is a unit circle centered on the source point P. The boundary of Ω_ϵ is denoted $\overline{\Gamma}_\epsilon$. Then one can write:

$$\frac{\partial I_i^N}{\partial x_\gamma\,(P)} = \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial x_\gamma}\int_{\Omega-\Omega_\epsilon} U_{i3,\alpha}(Q,P)M_\alpha(Q)\,d\Omega_Q\right) \tag{17}$$

Using a polar coordinate system $(\bar r,\bar\theta)$ with origin $P\equiv o$ as depicted in Fig. 1, $U_{i3,\alpha}$ is rewritten considering only its strongly singular part (see eq.(54)):

$$U_{i3,\alpha} = \frac{1}{r(\bar r,\bar\theta)}\,\Lambda_{i3,\alpha}(\phi) \tag{18}$$

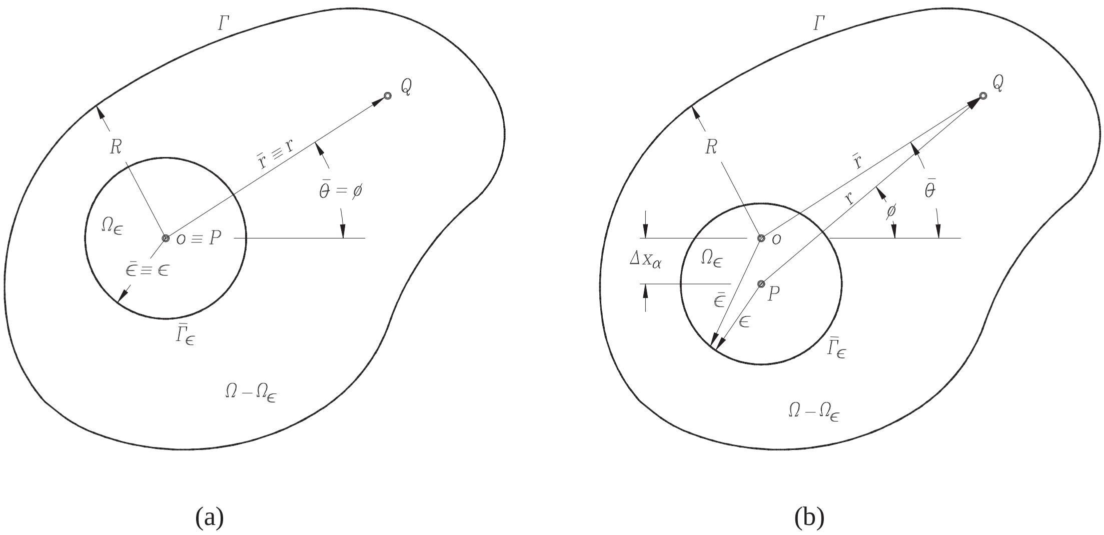

Figure 1: Definition of the boundary $\overline{\Gamma}_\varepsilon$ around the source point. (a) Initial configuration. (b) Effect of an increment Δx_α applied to the source point coordinates.

Figure 1 shows that $r(\bar r,\bar\theta) = \bar r$ and $\phi(\bar r,\bar\theta) = \bar\theta$. However, if the source P is perturbed by a Cartesian increment Δx_α, the parameters r and ϕ differ from $\bar r$ and $\bar\theta$, respectively, and the boundary $\overline{\Gamma}_\epsilon$ changes as well. This means that $\overline{\Gamma}_\epsilon$ is dependent of the load point location. Using the polar coordinate system eq.(17) is written:

$$\frac{\partial I_i^N}{\partial x_\gamma} = \int_0^{2\pi} \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial x_\gamma} \int_{\bar{\epsilon}}^{R(\bar{\theta})} \frac{\Lambda_{i3,\alpha}(\phi)}{r} M_\alpha(Q)\, \bar{r}\, d\bar{r} \right) d\bar{\theta} \tag{19}$$

One should note that in eq.(19) the integration limits varies with the integration variable. When this dependence holds the Leibnitz formula must be used [21]:

$$\frac{d}{d\alpha} \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} f(x,\alpha)\, dx = \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\, dx - f(\phi_1,\alpha)\frac{d\phi_1}{d\alpha} + f(\phi_2,\alpha)\frac{d\phi_2}{d\alpha} \tag{20}$$

Applying eq.(20) directly to eq.(19) results:

$$\begin{aligned} \frac{\partial}{\partial x_\gamma} \int_{\bar{\epsilon}}^{R(\bar{\theta})} \frac{\Lambda_{i3,\alpha}(\phi)}{r} M_\alpha(Q)\, \bar{r}\, d\bar{r} &= \int_{\bar{\epsilon}}^{R(\bar{\theta})} \frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}(\phi)}{r} \right) M_\alpha(Q)\, \bar{r}\, d\bar{r} + \\ & - \frac{\Lambda_{i3,\alpha}(\phi)}{r(\bar{\epsilon},\bar{\theta})} M_\alpha(P)\, \bar{\epsilon} \frac{d\bar{\epsilon}}{dx_\gamma} + \frac{\Lambda_{i3,\alpha}(\phi)}{r(R,\bar{\theta})} M_\alpha(P)\, R \frac{dR}{dx_\gamma} \end{aligned} \tag{21}$$

Because the origin of the coordinate system coincides with the source point P before the imposition of Δx_α, and there it remains after the application of the increment, only $\bar{\epsilon}$ changes with x_α (while R does not). The last term on the right hand side of eq.(21) then vanishes. Taking into account that $r(\bar{\epsilon},\bar{\theta}) = \epsilon = \bar{\epsilon}$ when $P \equiv o$ results:

$$\begin{aligned} \frac{\partial I_i^N}{\partial x_\gamma} &= \int_0^{2\pi} \lim_{\epsilon \to 0} \left[\int_\epsilon^{R(\phi)} \frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}(\phi)}{r} \right) M_\alpha(Q)\, r\, dr \right] d\phi + \\ & - M_\alpha(P) \int_0^{2\pi} \Lambda_{i3,\alpha}(\phi)\, \cos(r, x_\gamma)\, d\phi \end{aligned} \tag{22}$$

Now it is worth to investigate the existence of the first integral on the right hand side of (22). Noting that:

$$\frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}(\phi)}{r} \right) M_\alpha(Q)\, r \ = r^2 \frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}(\phi)}{r} \right) M_\alpha(Q)\, \frac{1}{r} \tag{23}$$

and defining

$$\bar{\Lambda}_{i3,\alpha\gamma}(\phi) = r^2 \frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}(\phi)}{r} \right)$$

the term

$$\int_0^{2\pi} \lim_{\epsilon \to 0} \left[\int_\epsilon^{R} \frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}(\phi)}{r} \right) M_\alpha(P)\, r\, dr \right] d\phi$$

can be added and subtracted from eq.(23), resulting:

$$
\begin{aligned}
&\int_0^{2\pi} \lim_{\epsilon\to 0} \left[\int_\epsilon^R \frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}(\phi)}{r} \right) M_\alpha(Q)\, r\, dr \right] d\phi \; = \\
&= \int_0^{2\pi} \lim_{\epsilon\to 0} \left\{ \bar{\Lambda}_{i3,\alpha\gamma}(\phi) \int_\epsilon^R \left[M_\alpha(Q) - M_\alpha(P) \right] \frac{1}{r}\, dr \right\} d\phi \, + \\
&+ M_\alpha(P) \int_0^{2\pi} \bar{\Lambda}_{i3,\alpha\gamma}(\phi) \ln(R)\, d\phi + \\
&- \lim_{\epsilon\to 0} \left[M_\alpha(P) \ln\epsilon \int_0^{2\pi} \bar{\Lambda}_{i3,\alpha\gamma}(\phi) d\phi \right]
\end{aligned} \tag{24}
$$

All the integrals in eq.(24) are limited, provided that the membrane-bending coupling satisfies the Hölder condition on P:

$$
\| M_\alpha(Q) - M_\alpha(P) \| \le A r^\alpha \quad , \qquad A, \alpha > 0 \tag{25}
$$

Because the tensor $\bar{\Lambda}_{i3,\alpha m}$ satisfies the property $\int_0^{2\pi} \bar{\Lambda}_{i3,\alpha\gamma}(\phi)\, d\phi = 0$ the last two terms in eq.(24) vanish. In addition, the first integral on the right hand side is convergent since:

$$
\lim_{\epsilon\to 0} \left[\bar{\Lambda}_{i3,\alpha m}(\phi) \int_\epsilon^R \frac{A r^\alpha}{r}\, dr \right] = \lim_{\epsilon\to 0} \left[\frac{A\, r^{\alpha+2}}{\alpha - 1} \, \ln(r)\, \frac{\partial}{\partial x_\gamma} \left(\frac{\Lambda_{i3,\alpha}}{r} \right) \right]_\epsilon^R < \infty
$$

which completes the demonstration.

Now $\partial I_i^N / \partial x_\gamma$ can be rewritten again in Cartesian coordinates:

$$
\begin{aligned}
\frac{\partial I_i^N}{\partial x_\gamma} &= - \int_\Omega \frac{\partial U_{i3,\alpha}(Q,P)}{\partial x_\gamma} N_{\alpha\beta}(Q)\, u_{3,\beta}(Q)\, d\Omega_Q + \\
&\quad - N_{\alpha\beta}(P)\, u_{3,\beta}(P) \int_{\Gamma_1'} U_{i3,\alpha}\, r_{,\gamma}\, d\Gamma'
\end{aligned} \tag{26}
$$

where the first integral must be interpreted in its Cauchy principal value (CPV) sense. The second term on the right hand side of (26) is the convective contribution, as it appears from a change in the position of source point. In the present work, the interest remains on the development of the convective term particularized for $i = 3$, as stated in eq.(13a).

Since the exterior normal of Γ_1' points to the center of the circle $r_{,\alpha} = -n_\alpha$, one can write the convective term as:

$$
{}^f\mathbf{c}^N(P) = N_{\alpha\beta}(P) \int_{\Gamma_1'} U^s_{33,\alpha}\, r_{,\gamma}\, d\Gamma' = -N_{\alpha\beta}(P) \int_{\Gamma_1'} U^s_{33,\alpha}\, n_\gamma\, d\Gamma' \tag{27}
$$

where $U^s_{33,\alpha}$ contains only the singular part of $U_{33,\alpha}$. In the present case (see Appendix C):

$$
U^s_{33,\alpha} = \frac{-1}{\pi D (1-\nu) \lambda^2} \frac{r_{,\alpha}}{r}
$$

which validates the representation (18). Using $d\Gamma = r\, d\phi$ eq.(27) is analytically defined as:

$$^f c^N_{\alpha\beta}(P) = \frac{-1}{\pi D(1-\nu)\lambda^2} \left[\int_{2\pi}^{0} n_\gamma n_\alpha \, d\phi \right] N_{\gamma\beta}(P)$$

Recalling from Fig. 1 that $n_1 = -\cos\phi$, $n_2 = -\sin\phi$, and using elementary trigonometric integrals the following result is obtained:

$$^f c^N_{\alpha\beta}(P) = \frac{-N_{\alpha\beta}(P)}{D(1-\nu)\lambda^2} \tag{28}$$

This non-integral term is added to eqs.(12) replacing the first integral on Γ'_1. The correction must be done only in the singular case ($P \equiv Q$). Equation (28) is in agreement with the results obtained by Xiao-Yan et al. [28].

2.2 Evaluation of $\dfrac{\partial I_i^q}{\partial x_\gamma(P)}$

The fundamental solution tensor used to take into account domain bending loadings in both the Mindlin and the Reissner plate models is given by (see Appendix D):

$$^f\mathbf{V} = {}^f\mathbf{U} - m_f\, {}^f\widetilde{\mathbf{U}} = {}^f\mathbf{U} - m_f \begin{bmatrix} 0 & 0 & U_{11,1} + U_{12,2} \\ 0 & 0 & U_{21,1} + U_{22,2} \\ 0 & 0 & U_{31,1} + U_{32,2} \end{bmatrix} \tag{29}$$

Following the procedure outlined in the previous section, eq.(14b) is written:

$$I_i^q = \lim_{\epsilon \to 0} \int_{\Omega - \Omega_\epsilon} V_{ij}(Q,P)\, q_j(Q)\, d\Omega_Q \tag{30}$$

and its derivatives result in:

$$\frac{\partial I_i^N}{\partial x_\gamma(P)} = \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial x_\gamma} \int_{\Omega - \Omega_\epsilon} U_{ij}(Q,P) q_j(Q)\, d\Omega_Q + \right. \tag{31}$$

$$\left. + m_f \frac{\partial}{\partial x_\gamma} \int_{\Omega - \Omega_\epsilon} \widetilde{U}_{ij}(Q,P) q_j(Q)\, d\Omega_Q \right) \tag{32}$$

Therefore, the treatment has to be carried out for the Reissner's model ($m_f = 1$). Otherwise $^f\mathbf{V} = {}^f\mathbf{U}$, and since $\mathbf{U} = O(\ln r)$ the first integral does not manifest strong singularities after the differentiation and will not originate convective terms. The second integral deserves a more careful inspection. Since the interest in upon the derivative of the plate transverse displacement, eq.(31) is particularized from the outset considering only the necessary terms:

$$\frac{\partial I_3^N}{\partial x_\gamma(P)} = \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial x_\gamma} \int_{\Omega - \Omega_\epsilon} U_{3\alpha,\alpha}(Q,P) q_3(Q)\, d\Omega_Q \right) \tag{33}$$

However, as $U_{3\alpha,\alpha}$ is regular on Ω it is not possible to apply the representation

$$U_{3\alpha,\alpha} = \frac{1}{r(\bar{r},\bar{\theta})}\,\Lambda_{3\alpha,\alpha}(\phi) \tag{34}$$

and thus there is no convective contribution, as expected:

$${}^{f}c^{q}_{\alpha\beta}(P) \;= 0 \tag{35}$$

2.3 Evaluation of $\dfrac{\partial J^N_\alpha}{\partial x_\gamma\,(P)}$

In this case, eq.(14c) is rewritten:

$$J^N_\alpha = \lim_{\epsilon\to 0}\int_{\Omega-\Omega_\epsilon} U_{\alpha\beta,\delta}(Q,P)N^n_{\beta\delta}(Q)\,d\Omega_Q \tag{36}$$

Repeating the procedure of the previous sections, one has:

$$\frac{\partial J^N_\alpha}{\partial x_\gamma\,(P)} = \lim_{\epsilon\to 0}\left(\frac{\partial}{\partial x_\gamma}\int_{\Omega-\Omega_\epsilon} U_{\alpha\beta,\delta}(Q,P)N^n_{\beta\delta}(Q)\,d\Omega_Q\right) \tag{37}$$

Introducing

$$U_{\alpha\beta,\delta} = \frac{1}{r(\bar{r},\bar{\theta})}\,\Lambda_{\alpha\beta,\delta}(\phi) \tag{38}$$

result, after the application of the Leibnitz' formula:

$$\frac{\partial J^N_\alpha}{\partial x_\gamma} = -\int_\Omega \frac{U_{\alpha\beta,\delta}}{\partial x_\gamma}N^n_{\beta\delta}(Q)\,d\Omega_Q - N^n_{\beta\delta}(P)\int_{\Gamma'_1} U_{\alpha\beta,\delta}\,r_{,\gamma}\,d\Gamma' \tag{39}$$

where the first integral must be interpreted in the CPV sense, provided the nonlinear membrane forces satisfy the Hölder condition on P:

$$\|N^n_{\beta\delta}(Q) - N^n_{\beta\delta}(P)\| \le Ar^\alpha \quad , \qquad A,\alpha > 0 \tag{40}$$

Therefore, the evaluation of the convective term results in:

$${}^{m}\mathbf{c}^N(P) = N^n_{\beta\delta}(P)\int_{\Gamma'_1} U_{\alpha\beta,\delta}\,r_{,\gamma}\,d\Gamma' = -N^n_{\beta\delta}(P)\int_{\Gamma'_1} U_{\alpha\beta,\delta}\,n_\gamma\,d\Gamma' \tag{41}$$

where $U_{\alpha\beta,\delta}$ is $O(r^{-1})$ (see Appendix C). The eq.(41) is analytically defined as:

$${}^{m}\mathbf{c}^N(P) = \frac{1}{8\pi G\,(1-\nu)}\left\{\int_{2\pi}^{0}\left[(3-4\nu)\,r_{,\gamma}\,\delta_{\alpha\beta} - r_{,\alpha}\,\delta_{\beta\gamma} - r_{,\beta}\,\delta_{\alpha\gamma} + \right.\right.$$
$$\left.\left. + 2r_{,\alpha}r_{,\beta}r_{,\gamma}\right]\,d\phi\right\} N^n_{\beta\delta}(P) \tag{42}$$

Using the relations $n_1 = -\cos\phi$, $n_2 = -\sin\phi$ and elementary integrals of trigonometric powers leads to the following expression:

$$
\begin{aligned}
{}^{m}c^{N}_{\alpha\beta}(P) = \frac{-1}{8G(1-\nu)} \Big[& (3-4\nu)\,\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta} \; + \\
& + \frac{1}{4}\delta_{\alpha\beta}\delta_{\gamma\delta}\,(1+2\delta_{\alpha\gamma}) \Big] \, N^{n}_{\gamma\delta}(P)
\end{aligned}
\tag{43}
$$

2.4 Evaluation of $\dfrac{\partial J^q_\alpha}{\partial x_\gamma\,(P)}$

In this case ${}^{m}\mathbf{V} = {}^{m}\mathbf{U}$, and since $\mathbf{U} = O\,(\ln r)$ the convective term is null:

$$
{}^{m}c^{q}_{\alpha\beta}(P) = 0 \tag{44}
$$

3 Summary of the results

All the relevant expressions obtained in the previous sections can be summarized as:

$$
\begin{aligned}
{}^{m}c^{N}_{\alpha\beta}(P) &= \frac{-1}{8G(1-\nu)} \Big[(3-4\nu)\,\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta} \; + \\
&\qquad - \frac{1}{4}\delta_{\alpha\beta}\delta_{\gamma\delta}\,(1+2\delta_{\alpha\gamma}) \Big] \, N^{n}_{\gamma\delta}(P) \quad , && \text{(45a)} \\
{}^{m}c^{q}_{\alpha\beta}(P) &= 0 \quad , && \text{(45b)} \\
{}^{f}c^{N}_{\alpha\beta}(P) &= -\frac{\delta_{\alpha\gamma}}{D(1-\nu)\lambda^2}\, N_{\gamma\beta}(P) \quad , && \text{(45c)} \\
{}^{f}c^{q}_{\alpha}(P) &= 0 \quad . && \text{(45d)}
\end{aligned}
$$

subjected to

$$
\begin{aligned}
\|M_\alpha(Q) - M_\alpha(P)\| &\le A r^{\alpha} \quad , \qquad A, \alpha > 0 \\
\|N^{n}_{\beta\delta}(Q) - N^{n}_{\beta\delta}(P)\| &\le B r^{\beta} \quad , \qquad B, \beta > 0
\end{aligned}
$$

And finally eqs.(11) and (12) are rewritten in their final form:

$$
\begin{aligned}
\bar{u}_{\beta,\alpha}(P) - \int_{\Gamma} {}^{m}T_{\beta\gamma,\alpha}(q,P)\bar{u}_{\gamma}(q)\, d\Gamma_q = & -\int_{\Gamma} {}^{m}U_{\beta\gamma,\alpha}(q,P)\bar{t}_{\gamma}(q)\, d\Gamma_q \; + \\
& - \int_{\Omega} {}^{m}V_{\beta\gamma,\alpha}(Q,P) q_{\gamma}(Q)\, d\Omega_Q + \int_{\Omega} {}^{m}U_{\beta\gamma,\delta\alpha}(Q,P) N^{n}_{\gamma\delta}(Q)\, d\Omega_Q \; + \\
& + {}^{m}c^{N}_{\alpha\beta}(P) - {}^{m}v_{\beta,\alpha}(P)
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
u_{3,\alpha}(P) - \int_{\Gamma} {}^{f}T_{3i,\alpha}(q,P)u_i(q)\,d\Gamma_q = -\int_{\Gamma} {}^{f}U_{3i,\alpha}(q,P)t_i(q)\,d\Gamma_q + \\
-\int_{\Omega} V_{3i,\alpha}(Q,P)q_i(Q)\,d\Omega_Q + \int_{\Omega} {}^{f}U_{33,\alpha\gamma}(Q,P)N_{\gamma\beta}(Q)u_{3,\beta}(Q)\,d\Omega_Q + \\
+\; c^{N}_{\alpha\beta}(P)u_{3,\beta}(P) - {}^{f}v_{3,\alpha}(P)
\end{aligned}
\tag{47}
$$

Equations (46) and (47) are valid for interior points, only. It is worth to observe the singularities $O\left(1/r^2\right)$ in the integrals on the left hand side, and $O\left(1/r\right)$ and $O\left(1/r^2\right)$ for the first and third integrals on the right hand side. For boundary points, their limit to the boundary must be taken in order to obtain the corresponding geometric factors (C matrix). In that case, the integrals on the left hand side must be interpreted in their Haddamard sense, which demonstrates the hypersingular character of these equations. All remaining integrals are interpreted in the CPV sense.

4 Numerical assessment

Although not being the objective of the present work, the inclusion of a few numerical results is advisable to assert the validity of the expressions obtained. Equations (9-10) and (46-47) were discretized and numerically implemented by the standard direct boundary element method using the convective terms presented herein. An iterative algorithm was used to solve the membrane and the bending equations simultaneously at each load level [12]. Square plates benchmarks were solved for clamped and supported boundary conditions. The Mindlin compatible ($\kappa^2 = \pi^2/12$) plate model was used with $\nu = 0.3$. Loads and displacements were normalized as parameters $R = q_3 a^4/Eh^4$ and $r = w/h$, respectively, where w is the central transverse displacement, and a is the plate´s lateral dimension. Figure 2 compares the equilibrium paths obtained for a thin plate case ($h/a = 0.01$) using regular meshes of linear boundary elements and constant domain cells with other analytical [11, 20] and numerical solutions [17]. Also shown are the results of ref. [27], obtained with 16 constant boundary elements and 25 constant cells. The rapid degeneration of those results for larger r ratios is evident. It is worth to note that BEM results for $r > 1$ are not currently available in the technical literature. The present results show good agreement with analytical solutions, regardless the mesh used.

Table 1 compares the present results with some other numerical solutions for supported plates with $h/a = 0.05$. Worth to note is the results of Xiao-Yan et al. [28], a rare BEM large displacement solution for moderately thick plates. The results of the proposed formulation are presented for both types of thick plate support (hard and soft) for completeness sake.

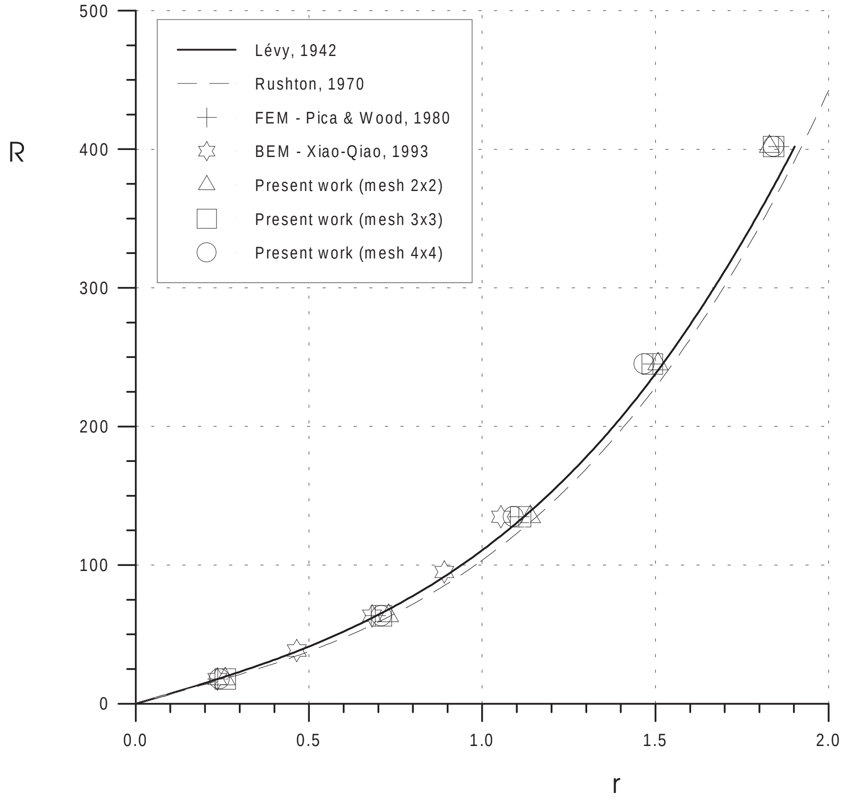

Figure 2: Equilibrium path for a clampled square plate under uniform loading.

5 Conclusions

The analytic derivation of the convective terms for geometrically non-linear BEM analysis of the Mindlin and the Reissner moderately thick plate models is presented. Equations (9-10) and eqs.(46-47) now include these terms, hence completing the Somigliana identities for the displacement derivatives. The existence conditions for the non-null convective terms are clearly stated. Some numerical results are included to test the correctness of these equations, and they show good agreement with available solutions. As a by product, eqs.(46-47) provide a straightforward approach to evaluate the stress and the strain tensors on interior points for plate bending problems in large displacements regime. An integral equation formulation for linear elastic stability problems can be obtained by linearization of these equations.

	R			
	0.9158	4.579	6.868	9.158
Thin plate[1]	0.04053	0.1929	0.2750	0.3467
FSM [1]	0.04205	0.1950	0.2776	0.3494
BEM [28]	0.04090	0.1942	0.2767	0.3489
Present work (hard support)	0.04200	0.1969	0.2775	0.3464
Present work (soft support)	0.04428	0.2056	0.2878	0.3573

[1] Rayleigh-Ritz method [1].

Table 1: Normalized central displacement r for supported square plate under uniform load ($h/a = 0.05$).

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Appendices - Fundamental Solutions

The fundamental solutions for the two dimensional elasticity differential operator are well known from the literature. Fundamental solutions for the Mindlin and the Reissner plate models are described in detail in the works of [23] and [26]. In those works, Hörmander´s method was used [7], and the plate bending displacement fundamental solution ${}^f\mathbf{U}$ tensor was found to be composed by six functions weighted by six parameters. Some of the parameters are obtained by applying distribution theory and regularity conditions at infinity, but two of them do not have any conditions imposed. These parameters are regarded as free, whose values can be judiciously chosen to simplify the final form of the tensors. One of these forms was used in the present work. The bending traction fundamental solution ${}^f\mathbf{T}$ is obtained by traction-displacement relations.

All tensors used in this work are presented below. The following notation was used:

$$\begin{aligned} z &= \lambda r \\ r_\alpha &= x_\alpha(Q) - x_\alpha(P) \\ r &= \|Q-P\| = \sqrt{r_\alpha r_\alpha} \\ r_{,\alpha} &= \frac{\partial r}{\partial x_\alpha(Q)} = \frac{1}{r} r_\alpha \\ A(z) &= K_0(z) + \frac{2}{z}\left(K_1(z) - \frac{1}{z}\right) \\ B(z) &= K_0(z) + \frac{1}{z}\left(K_1(z) - \frac{1}{z}\right) \quad , \end{aligned}$$

where K_0 and K_1 are second kind modified Bessel functions of order 0 and 1, respectively. The terms F_3 and F_6 are the aforementioned free functions [5]. All two dimensional elasticity tensors refer to plane strain state.

A Displacement fundamental solution tensors

2D elasticity:

$$ {}^mU_{\alpha\beta} = \frac{-1}{8\pi G\,(1-\nu)}\left[(3-4\nu)\ln r\,\delta_{\alpha\beta} - r_{,\alpha} r_{,\beta}\right] \tag{48}$$

Plate bending:

$$\begin{aligned} {}^fU_{\alpha\beta} = \frac{1}{8\pi D\,(1-\nu)} \Big\{ &\left[8B - (1-\nu)\,(2\ln z + 1 + 8F_3)\right]\delta_{\alpha\beta} + \\ &- \left[8A + 2\,(1-\nu)\right] r_{,\alpha} r_{,\beta} \Big\} \end{aligned} \tag{49a}$$

$$ {}^fU_{\alpha 3} = \frac{1}{8\pi D}\,(2\ln z + 1 + 8F_3)\, r\, r_{,\alpha} \tag{49b}$$

$$ {}^fU_{3\alpha} = -U_{\alpha 3} \tag{49c}$$

$$\begin{aligned} {}^fU_{33} = \frac{1}{8\pi D\,(1-\nu)\,\lambda^2} \Big\{ &z^2\,(1-\nu)\,(\ln z + 4F_3) - 8\ln z + \\ &- 4\left[(3-\nu)\,(4F_3+1) - (1-\nu)\,F_6\right] \Big\} \end{aligned} \tag{49d}$$

B Traction fundamental solution tensors

2D elasticity:

$$ {}^{m}T_{\alpha\beta} = \frac{-1}{4\pi\,(1-\nu)\,r} \left\{ \left[(1-2\nu)\,\delta_{\alpha\beta} + 2r,_{\alpha}\,r,_{\beta} \right] r,_{n} \; + \right. \tag{50} $$

$$ \left. -\;(1-2\nu)\,(n_{\alpha} r,_{\beta} - n_{\beta} r,_{\alpha}) \right\} \tag{51} $$

Plate bending:

$$ {}^{f}T_{\alpha\beta} = \frac{-1}{4\pi r} \Big[(4A + 2zK_1 + 1 - \nu)(r_{,\beta} n_{\alpha} + r_{,n} \delta_{\alpha\beta}) \; + $$

$$ + \; (4A + 1 + \nu) r_{,\alpha} n_{\beta} - 2(8A + 2zK_1 + 1 - \nu) r_{,\alpha} r_{,\beta} r_{,n} \Big] \tag{52a} $$

$$ {}^{f}T_{\alpha 3} = \frac{\lambda^2}{2\pi} \left(B\, n_{\alpha} - A\, r_{,\alpha} r_{,n} \right) \tag{52b} $$

$$ {}^{f}T_{3\alpha} = \frac{-1}{8\pi} \Big\{ \left[2(1+\nu)\ln z + (1+8F_3) + (3+8F_3)\nu \right] n_{\alpha} \; + $$

$$ + \; 2(1-\nu) r_{,\alpha} r_{,n} \Big\} \tag{52c} $$

$$ {}^{f}T_{33} = \frac{-1}{2\pi r} r_{,n} \tag{52d} $$

C Derivative tensors

All tensors presented below are differentiated with respect to the field point Q.

C.1 First derivatives of U

2D elasticity:

$$ {}^{m}U_{\alpha\beta,\gamma} = \frac{-1}{8\pi G\,(1-\nu)\,r} \Big[(3-4\nu)\, r_{,\gamma}\, \delta_{\alpha\beta} - r_{,\alpha}\, \delta_{\beta\gamma} \; + $$

$$ - \; r_{,\beta}\, \delta_{\alpha\gamma} + 2 r_{,\alpha} r_{,\beta} r_{,\gamma} \Big] \tag{53} $$

Plate bending:

$$ {}^{f}U_{\alpha 3,\beta} = \frac{1}{8\pi D} \left[(2\ln z + 1 + 8F_3)\; \delta_{\alpha\beta} + 2 r_{,\alpha} r_{,\beta} \right] \tag{54a} $$

$$ {}^{f}U_{3\alpha,\beta} = -\, {}^{f}U_{\alpha 3,\beta} \tag{54b} $$

$$ {}^{f}U_{33,\beta} = \frac{1}{8\pi D\,(1-\nu)\,z^2} \left[(2\ln z + 1 + 8F_3)\,(1-\nu)\; z^2 - 8 \right] r\, r_{,\beta} \tag{54c} $$

C.2 First derivatives of $\mathbf{T}$

2D elasticity:

$$
\begin{aligned}
{}^{m}T_{\alpha\beta,\gamma} = \frac{-1}{4\pi\,(1-\nu)\,r^2} \Big\{ \big[\,(1-2\nu)\,r_{,\gamma}\;\delta_{\alpha\beta} - r_{,\alpha}\,\delta_{\beta\gamma} - r_{,\beta}\,\delta_{\alpha\gamma}\;+ \\
+\;4r_{,\alpha}r_{,\beta}r_{,\gamma}\,\big]\,2r_{,n} + (1-2\nu)\,\big[2n_\alpha r_{,\beta}r_{,\gamma}\;+ \\
-\;2n_\beta r_{,\alpha}r_{,\gamma} - n_\alpha\delta_{\beta\gamma} + n_\beta\delta_{\alpha\gamma} - n_\gamma\delta_{\alpha\beta}\big] + \\
-\;2n_\gamma r_{,\alpha}\,r_{,\beta} + \Big\}
\end{aligned}
\tag{55}
$$

Plate bending:

$$
{}^{f}T_{3\alpha,\beta} = \frac{-\,(1-\nu)}{4\pi r}\left[r_{,\alpha}n_\beta + r_{,n}\delta_{\alpha\beta} + \frac{(1+\nu)}{(1-\nu)}r_{,\beta}n_\alpha - 2r_{,\alpha}r_{,\beta}r_{,n}\right] \tag{56a}
$$

$$
{}^{f}T_{33,\beta} = \frac{-1}{2\pi r^2}\,(n_\alpha - 2r_{,\alpha}r_{,n}) \tag{56b}
$$

C.3 Second derivatives of $\mathbf{U}$

Plate bending:

$$
\begin{aligned}
{}^{f}U_{33,\alpha\beta} = \frac{1}{2\pi D}\Bigg\{ \frac{-2\left(\delta_{\alpha\beta} - 2r_{,\alpha}r_{,\beta}\right)}{(1-\nu)\,z^2} + 2F_3\delta_{\alpha\beta}\;+ \\
+\;\frac{1}{4}\big[(2\ln z + 1)\;\delta_{\alpha\beta} + 2r_{,\alpha}r_{,\beta}\,\big]\;\Bigg\}
\end{aligned}
\tag{57}
$$

D Loading fundamental solution tensors

2D elasticity:

$$
{}^{m}V_{\alpha\beta}(Q,P) = {}^{m}U_{\alpha\beta}(Q,P) \tag{58}
$$

Plate bending:

$$
\begin{aligned}
{}^{f}V_{\alpha\beta}(Q,P) &= {}^{f}U_{\alpha\beta}(Q,P) - m_f\,\frac{\partial^f U_{i\alpha}(Q,P)}{\partial x_\alpha(Q)} \\
&= {}^{f}\mathbf{U} - m_f\;{}^{f}\widetilde{\mathbf{U}} \\
&= {}^{f}\mathbf{U} - m_f\begin{bmatrix} 0 & 0 & U_{11,1} + U_{12,2} \\ 0 & 0 & U_{21,1} + U_{22,2} \\ 0 & 0 & U_{31,1} + U_{32,2} \end{bmatrix}
\end{aligned}
\tag{59}
$$