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SÉRIE A: TRABALHO DE PESQUISA

STATISTICS OF VISITS TO ZERO ANGLE CORNERS OF BILLIARDS

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Statistics of visits to zero angle corners of billiards

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Abstract

Consider a Sinai billiard table Q (bounded region of the plane, with a finite number of dispersing boundaries ∂Q_i) such that two circular pieces of the boundary are tangent at C. Consider the dynamical system T describing the free motion of a point mass in Q, with elastic reflections on the boundary (angle of incidence with the normal to the curve equal to the angle of reflection).

We prove that the sequence of successive entrance times in a certain small neighbourhood of the corner C converges in law, when suitable normalized, to a Poisson point process.

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1 Introduction

1. Consider three arcs of circumferences of radius one (each one with length $\pi/3$) whose centers are located in the vertices of an equilateral triangle of side 2 (see Figure 1). They determine a "triangular region" Q -the billiard table- bounded by ∂Q, the union of ∂Q_i, $i = 1, 2, 3$, three pieces of circumferences, that are tangent at the vertices C_i of Q. Let be T the dynamical system describing the free motion of a point mass in Q, with elastic reflections on ∂Q (angle of incidence with the normal to the curve equal to the angle of reflection).

All our results are valid if instead of the "triangular billiard" we take any billiard which dispersing boundaries (Sinai billiard) contains tangent circles, or tangent curves with the order of tangency of two circles. In order to simplify the computations and general presentation of the paper we will work with the "triangular billiard" described at the beginning of this Section.

The fact that in our billiard the curves defining each corner have angle zero in the intersection will make us face problems that are analogous to the case of indifferent fixed points for expanding maps (see, for example, [5], [6], [7], [17]). Trajectories that hit on the boundary very close to the vertices, have unstable derivatives very close to one. This dynamical system is mixing [14], but the neutrality of these derivatives has an immediate consequence: one can not extend all

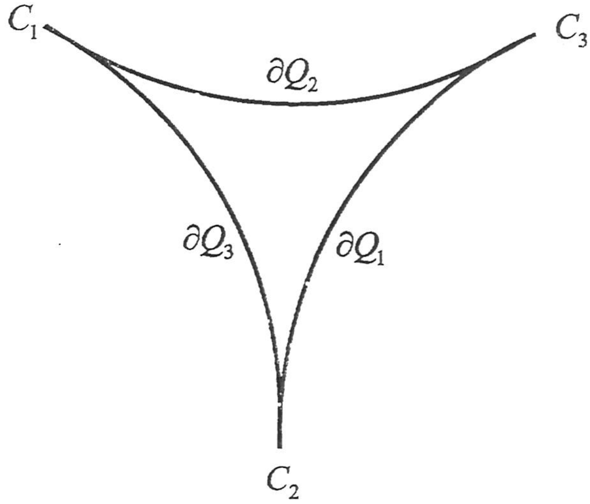

Figure 1:

the results that are true for uniformly hyperbolic planar Sinai billiards (the curves defining each corner have non-zero angle in the intersection).

The main purpose of our work is to study the influence of these trajectories on the decay of correlation of the dynamical system defined by such billiard. We claim that the decay of correlation will be of polynomial order (and not exponential as in the case of the Sinai billiards which have non zero angles) and this will be shown in a forthcoming paper. There are numerical evidences supporting our claim [10], [1]. We point out that the exponential decay of Sinai billiards was finally completely proved by Chernov [4] as a consequence of results by himself, Bunimovich, Sinai and L.S. Young.

In this paper we prove that the number of iterations a trajectory stays close to the vertices (where hyperbolicity is not good) and away from them (where hyperbolicity is good), are almost independent. We study the sequence of entrances into a small neighbourhood J_K of the vertices; K is the number of iterations the trajectory stays in this neighbourhood. Let be $f_K^1(x)$ the number of iterations the billiard map needs to send the point x for the first time to J_K. In fact, we will prove that there exists a normalizing factor β_K (diverging to infinity with K) such that the sequence of normalized stopping times $f_K^1 \beta_K^{-1}$ converges in law to a mean-one exponential random law on $\mathbb{R}^+$ when $K \to \infty$.

The main point of our proof is to estimate the measure of the points that remain more than K times in the bad region R_ε (see Proposition 1). After that (see Section 5) we follow very closely the paper by Collet and Galves [6] changing their proof in some places.

We claim that all our results about statistics of visits are true when two dispersing boundary curves δQ_i touch themselves in a zero angle corner. In fact we prove all our theorems using some estimates that are somewhat worse than the ones we evaluate (for perfect circles) at the end of Section 3. An important question to be adressed in the future is the exact relation between the order of tangency and the asymptotic polynomial power of the decay of correlation.

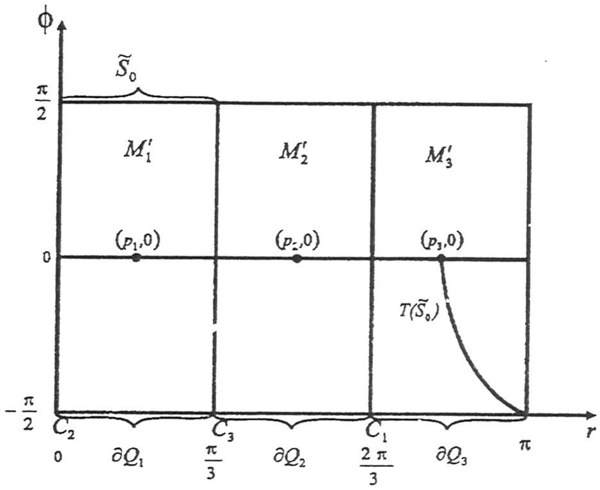

Figure 2:

2. A plane billiard is the dynamical system describing the free motion of a point mass inside an open bounded connected region Q of the plane, with elastic reflections at the boundary. The boundary consists of a finite set of closed C^{k+1}-curves $\partial Q_i, k \geq 2$. Let be $n(q)$ the unit normal to the curve ∂Q at the point q pointing out to the interior of the billiard table.

The phase space (see figure 2) of such a dynamical system is

$$M' = \{(q, v); \quad q \in \partial Q, \quad |v| = 1, \quad < v, n(q) > \quad \geq 0\}.$$

A coordinate system is defined on M' by the arc length parameter r along ∂Q and the angle ϕ between $n(q)$ and v, measured counterclockwise. Clearly $|\phi| \leq \pi/2$ and $< n(q), v > \; = \; \cos\phi$.

Consider the probability $d\mu \; = \; c\cos\phi \; dr d\phi$, where $c = \; (2|\partial Q|)^{-1}$ is just a normalizing factor and $|\partial Q|$ stands for the total length of ∂Q. If m is the Lebesgue Measure on M', then $d\mu = c \; \cos\phi \; dm$.

Now we define the map T in the following way:

$$T(x_0) = T(q_0, v_0) = (q_1, v_1)$$

with q_1 the point of ∂Q where the oriented line through (q_0, v_0) first hits ∂Q and v_1 the velocity vector of the trajectory after reflection on $q_1 \in \partial Q$. Formally, $v_1 = v_0 - 2 < n(q_1), v_0 > n(q_1)$.

We will denote by $x_i = (q_i, v_i) \in M', \; i \in \mathbb{N}$ the successive hits with the boundary ∂Q of a trajectory beginning at $x_0 = (q_0, v_0)$; that is, $T(q_i, v_i) = (q_{i+1}, v_{i+1})$.

The angle between $n(q_i)$ and v_i will be denoted by ϕ_i and finally, t_i denotes the Euclidean distance between the bounces q_i and $q_{i+1}, i \in \mathbb{N}$ (see fig 1). As the velocity is one, t_i is also the time between succesive bounces q_i, q_{i+1}. The backward orbit $x_i = (q_i, v_i)$, for negative $i \in \mathbb{Z}$, is analogously defined. The main relations are $T(x_i) = x_{i+1}, i \in \mathbb{Z}$, $q_{i+1} = q_i + t_i v_i$.

This map T is not well defined if $q_1 \in \partial Q_i \cap \partial Q_j$ and is not continuous in a neighbourhood of (q_0, v_0) if the oriented line through (q_0, v_0) is tangent to some ∂Q_k, $(\phi_1 = \pm\pi/2)$. In this case T is defined in a half open neighbourhood. In the sequel, when we speak about neighbourhoods we will be considering any one of the possible cases described above.

The map T is called the *billiard map*. It preserves the measure μ and is of class C^k. The sets of points $x_0 = (q_0, v_0) \in M'$ whose forward or backward trajectory is tangent to ∂Q for some $x_i, i \in \mathbb{Z}$, or is in $\partial Q_i \cap \partial Q_j$ have μ-measure zero. T satisfies the following **involutive property**. For $x = (r, \phi) \in M$ let be $-x = (r, -\phi)$; then $T^{-1}x = -T(-x)$.

If $\tilde{x}_1 = (\tilde{q}_1, \tilde{v}_1) = T(\tilde{x}_0)$ is defined for $\tilde{x}_0 = (\tilde{q}_0, \tilde{v}_0)$, then for all $x_0 = (q_0, v_0)$ in a small neighbourhood of $\tilde{x}_0$ the derivative matrix is given by (see, for example, [9]):

$$DT(x_0) = \begin{pmatrix} \frac{t_0 K_0 + \cos\phi_0}{-\cos\phi_1} & \frac{t_0}{\cos\phi_1} \\ K_1 \frac{t_0 K_0 + \cos\phi_0}{\cos\phi_1} + K_0 & -\frac{K_1 t_0}{\cos\phi_1} - 1 \end{pmatrix}, \tag{1}$$

where $K_i = K(x_i)$, $i \in \mathbb{N}$, the curvature of ∂Q at q_i. As we consider here the model where all Q_i, $i = 1, 2, 3$, are circunferences, then the K_i are all constants equal to one. Note that when the image of (q_0, v_0) by T is tangent to ∂Q (that is, $\phi_1 = \pm\pi/2$), then the entries of the above matrix become infinite.

2 Analytical expressions

In this section we will collect several analitical and geometrical results about the billiard with three tangent circles of same radio, specially for trajectories close to the corners $\partial Q_i \cap \partial Q_j$.

Let be $C_i \in \partial Q, i = 1, 2, 3$, located in the intersections of the circunferences. To simplify notations, consider $C_2 = \partial Q_3 \cap \partial Q_1$. If p_2 is the midle point of ∂Q_2 (opposite side to the corner C_2), then $T(p_2, 0)$ is not defined, but by continuity we can consider it equal to $(C_2, \pm\pi/2)$, $\pm$ depending in which ∂Q_j we consider C_2 –if $C_2 \in \partial Q_1$ it is $(C_2, -\pi/2)$. Also by continuity, we can define $T(C_2, -\pi/2) = (p_2, 0)$, and therefore, each $(p_i, 0), i = 1, 2, 3$, (and also $(C_i, -\pi/2)$) should be seen as a period two point for T. Note that the initial condition $(p_i, 0)$ is the only one that can reach the corner point C_i.

The phase space is given by three rectangles M_1', M_2' and M_3'. each one is a copy of a rectangle with base $0 \le r \le \pi/3$ and height $-\pi/2 \le \phi \le \pi/2$. (see Figure 2); M_i' corresponds to the arc ∂Q_i. Then $|\partial Q| = \pi$.

Let $\hat{r}_i$ be the r-coordinate of the vertex C_i. If C_2 corresponds to the origin of the arc length in the boundary ($\hat{r}_2 = 0$) we will use the following notation (see [10]): $\alpha_i = \alpha(x_i)$ will be the parameter denoting the successive positions in ∂Q of a trajectory $x_i = T^i x_0$ entering the corner C_2 : α_i will denote the angle with the horizontal line perpendicular to the common tangent to both smooth components of ∂Q at C_2. α_i will be always positive; θ_i will be the velocity parameter defined by $\theta_i = -|\phi_i|$ where $\phi_i, i \in \mathbb{N}$ denotes, as usually, the corresponding angle with the normal. θ_i will be always negative.

We remark that for $x \in M_3'$, $y \in M_1'$, we have

$$\alpha(x) = \pi - r(x), \quad \theta(x) = \phi(x); \qquad \alpha(y) = r(y), \quad \theta(y) = -\phi(y).$$

The billiard map is explicitelly defined by

$$\alpha_{m+1} + \alpha_m = \theta_{m+1} - \theta_m, \tag{2}$$

$$\sin\alpha_{m+1} + \tan(\alpha_m + \theta_m)\cos\alpha_{m+1} = \sin\alpha_m + (2 - \cos\alpha_m)\tan(\alpha_m + \theta_m), \tag{3}$$

$$-\pi/2 < \theta_m \leq -\alpha_m - \alpha_{m+1}.$$

We now make the following almost trivial but important observation: if

$$\rho_m = \sin\alpha_m + (2 - \cos\alpha_m)\tan(\alpha_m + \theta_m), \ \eta_m = \tan(\alpha_m + \theta_m), \tag{4}$$

we have that (3) means that the line $y \sin\omega + x\cos\omega = \rho_m$ at distance $|\rho_m|$ from the origin and whose normal through the origin forms angle $\omega = \alpha_{m+1}$ with the x-axis is satisfied by

$$(\eta_m, 1) \text{ if } \rho_m \geq 0, \quad \text{or by } \ (-\eta_m, -1) \text{ if } \rho_m < 0.$$

In our case $\rho_m < 0$, and we can obtain α_{m+1} immediately calculating the equation of the tangent line $[T_m]$ to the circle of center $(0, 0)$ and radius $|\rho_m|$, through $(-\eta_m, -1)$, with smaller negative slope (see Figure 3).

From (2) we can obtain for $x_{2l} \in M_1'$ (remember that the angles ϕ_j are alternatively negative and positive, and that q_{2l} is on the left side in Figure 1):

$$\alpha_{2k+3} - \alpha_1 = \phi_1 + 2(\phi_2 + \phi_3 + ... + \phi_{2k+2}) + \phi_{2k+3}, \ \ q(x_{2k+1}) \in \partial Q_1$$

and

$$-\theta_0 + \theta_{m+1} = \alpha_0 + 2(\alpha_1 + \alpha_2 + ... + \alpha_m) + \alpha_{m+1}. \tag{5}$$

Let $S_0 = \partial Q \times \{\phi = \pm\pi/2\}$ be the natural boundary of M'.

We consider now the equation of the image $T(\tilde{S}_0) \subset M_3'$ where $\tilde{S}_0 = \{(q, \phi), \ q \in \partial Q_1, \ \phi = \pi/2 = -\theta\}$ is one of the six parts of the set of tangencies S_0. Denote by $(\tilde{r}(r), \tilde{\phi}(r))$ the parametric equation of $T(\tilde{S}_0)$. It is easy to obtain the equations

$$\tilde{r} = \pi + r - \arccos[2\cos(r) - 1], \qquad \tilde{\phi} = \arccos[2\cos(r) - 1] - \pi/2.$$

From this expressions one can obtain the slope of the tangent to this curve in the image of the corner point C_3 (corresponding in our parametrization to $r = \pi/3, \phi = \pi/2$), by

$$\frac{\tilde{\phi}'}{\tilde{r}'}|_{\pi/3} = \frac{-\sqrt{3}}{1+\sqrt{3}},$$

and in C_2, by

$$\frac{\tilde{\phi}'}{\tilde{r}'}|_0 = \frac{-\sqrt{2}}{1+\sqrt{2}}$$

Then, the curve $T\tilde{S}_0 \subset M_3'$ close to $r = \pi(\alpha = 0), \phi = -\pi/2 \ \ \theta$ is well approximated by the line $\phi + \pi/2 = -\sqrt{2}(r - \pi)/(1 + \sqrt{2})$. This means that for a given small α_0,

$$-\pi/2 < \theta_0 \leq \frac{\sqrt{2}}{1+\sqrt{2}}\alpha_0 - \pi/2 \ \text{ implies } \ T^{-1}(\alpha_0, \theta_0) \in M_2', \ \text{ and } \ \theta_1 > \frac{\sqrt{2}}{1+\sqrt{2}}\alpha_1 - \pi/2. \tag{6}$$

The "last" trajectory that is entering in the corner, from the position α is obtained when the trajectory is perpendicular to the other arc of circumference. The equation of the curve C of these last entering trajectories is determined by

$$\alpha_1 + \alpha = -\theta, \qquad \sin\alpha_1 - \sin\alpha = (2 - \cos\alpha_1 - \cos\alpha)\tan\alpha_1.$$

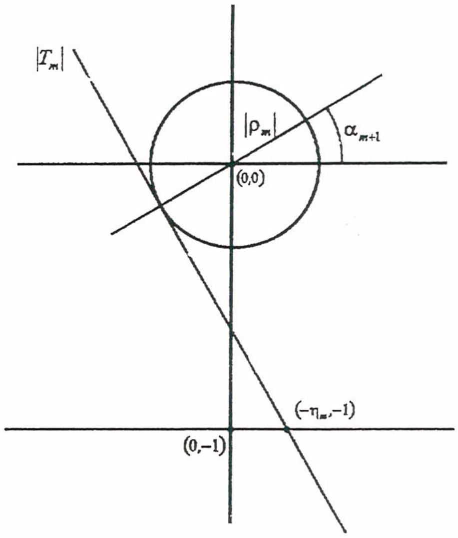

Figure 3:

Its equation in the M_3' component of the phase space is

$$\phi = r - \arctan\frac{\sin r}{2+\cos r} - \pi$$

whose tangent at $(\pi, 0)$ is $\phi = 2(r-\pi)$.

It will also be useful to know how is the image of the line $r = \varepsilon$, entering in the corner. $T(r=\varepsilon)$ is a continuous line that joins the curve $T\tilde{S}_0$ $(r \approx (\sqrt{2}-1)\varepsilon)$ and C $(r \approx \varepsilon)$

Consider a trajectory $(\alpha_i, \phi_i)\, i = 0, 1, 2, ...n$ that is going down to the corner (that is, $\alpha_i > \alpha_{i+1}, i = 0, 1, 2, ..., n-1, \quad -\pi/2 < \theta_m \leq -\alpha_m - \alpha_{m+1}$.) and (α_n, ϕ_n) is the beginning of getting out, that is, $\alpha_n \leq \alpha_{n+1}$.

In the Appendix we prove the fundamental estimates that relate the minimum α-coordinate of the trajectory $x_0, x_1, \ldots, x_n$ entering in a corner, with the initial position α_0, and the initial velocity parameter θ_0; namely, there exists a positive ε such that the $\alpha_{\min}$ of an entering trajectory beginning in $0 < \alpha_0 < \varepsilon$ satisfies

$$\alpha_1 \geq \alpha_{\min} \geq \alpha_1\sqrt{\cos\theta_1}. \tag{7}$$

From this estimate and (5), for $m = n-1$, we obtain

$$2n\alpha_1 > -\theta_1 > 2n\alpha_1\sqrt{\cos\theta_1}.$$

So, the number n of bounces of an entering trajectory with initial position (α_0, θ_0), $0 < \alpha_0 < \varepsilon$, $-\pi/2 < \theta_0 < 0$, satisfy

$$\frac{-\theta_1}{2\alpha_1\sqrt{\cos\theta_1}} \geq n \geq \frac{-\theta_1}{2\alpha_1}. \tag{8}$$

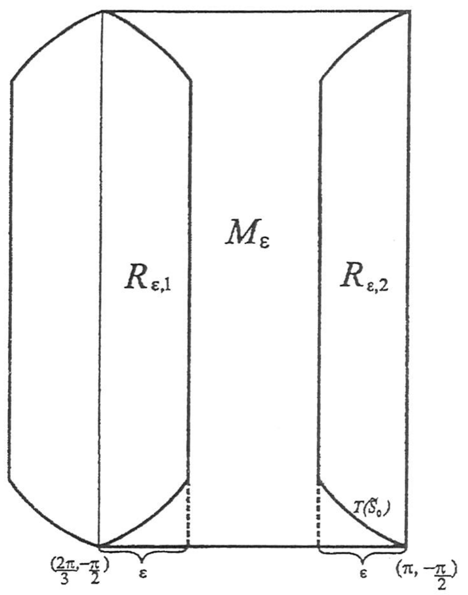

Figure 4:

Consider now the open set $R_{\varepsilon,2}$ bounded by $r = \pi - \varepsilon$, $r = \varepsilon$ and the four parts of $T^{\pm 1} S_0$ that are close to $\tilde{r}_2 = 0$ (see Figure 4). Let be $R_{\varepsilon,i}$, open sets defined in a similar way around the lines $r = \tilde{r}_i$, $i = 1, 2, 3$.

Trajectories that are entering and leaving the corner are essentially symmetric. This is a consequence of the following fact: the trajectory begins to leave the corner in between two extreme cases —"last segment of entering trajectory" is perpendicular to one of the circunferences ($\phi = 0$) or horizontal ($\phi = \alpha$)— that are almost the same one when the trajectory is very close to the vertex: both arcs of circunference are almost "parallel". The previous remark and the involutive property of the billiard map allows to deduce, from (8), that the total number of iterations in the rectangle $R_{\varepsilon,2}$ of a trajectory whose first point is (α_0, θ_0) is approximately $N = N(\alpha_1, \theta_1) \approx 2n$, satisfying

$$\frac{-\theta_1}{\alpha_1 \sqrt{\cos\theta_1}} \geq N \geq \frac{-\theta_1}{\alpha_1}. \tag{9}$$

We are particularly interested in the calculation of the number of iterations that a trajectory stays in the neighbourhood $R_{\alpha_0,2}$ of $\hat{r}_2 = 0 (\alpha_0 < \varepsilon)$ after its first iteration with angle θ_0. The curves $N =$ constant in the phase space that have $N + 1$ iterations in $R_{\alpha_0,2}$ are of the form

$$\alpha_1 = \frac{C}{N\sqrt{cos\theta_1}} \tag{10}$$

and are bounded by $T\tilde{S}_0$ (C bounded away from zero and ∞).

3 The induced first return map F

Let be $R_\epsilon = \cup R_{\epsilon,i}$. The boundaries of R_ε are six vertical lines (the union of them will be called V_ε), six "small" parts of $T^{-1}(S_0)$ and six "small" parts of $T(S_0)$. The choice of ε will be done later.

To sum up the analysis in the previous Section, if (α_0, θ_0) is the first point of a trajectory in R_ε, then it will leave this rectangle in M iterations, and will "penetrate" the corner until $\alpha_{\min}$; these values satisfy inequations (7) and (9).

Now we partition M' into two sets R_ε and $M_\varepsilon = M' \backslash R_\varepsilon$. M_ε is a region with uniform expansion on unstable manifolds, where the technics in [4] and [16] work well. R_ε is a set where the expansion is close to one. All the results valid for uniformly hyperbolic maps with singularities studied in Section 3 of [4] are valid for our induced first return map $F : M_\varepsilon \to M_\varepsilon$.

We will distinguish the iterations of each $x \in M$ by T into two classes: roughly speaking, they will be "bad" if $Tx \in R_\varepsilon$, and "fine", otherwise. For each $x \in M'$ let be

$$b_0(x) = \inf\{i \geq 1 : T^i x \in M_\varepsilon\}.$$

In fact, if $x \in M_\varepsilon$ then $b_0(x)$ is one plus the number of iterations that $T^i x$ is not in M_ε before retruning to it; if $x \in R_\varepsilon$ it is exactly the number of bad iterations before leaving R_ε for the first time.

We define the induced first return transformation $F : M_\varepsilon \to M_\varepsilon$ by $F(x) = T^{b_0(x)}$. Let be $\tilde{J}_k = \{x \in R_\varepsilon : b_0(x) = k,\ T^{-1}x \in M_\varepsilon\} \subset \tilde{H}_k = \{x \in M' : b_0(x) = k,\ T^{-1}x \in M_\varepsilon\}$, $H_N = \cup_{k \geq N} \tilde{H}_k$ and $J_N = \cup_{k \geq N} \tilde{J}_k$ (see Figure 5) All these sets are in between the lines TS_0 and $T^2 S_0$, very close to the phase space points $(\hat{r}_i, \pm\pi/2)$.

The "K-fine" iteration period is defined, for $x \in M_\varepsilon$, by

$$f_K^1(x) = \inf\{n > 0 : T^n x \in J_K\}.$$

This is the number of steps the trajectory starting from $x \in M_\varepsilon$ takes to enter for the first time in a region sufficently close to one corner (after entering it will take more than K iterations to return to M_ε). The sequence of succesive entrance times in J_K is defined by

$$f_K^2(x) = \inf\{n > f_K^1(x) + b_0(T^{f_K^1(x)}x) : T^n x \in J_K\},$$

$$f_K^j(x) = \inf\{n f_K^{j-1}(x) + b_0(T^{f_K^{j-1}(x)}x) : T^n x \in J_K\}.$$

We remark that $T^{f_K^j(x)-1}x$ is the last iteration in M_ε. These (fine) blocks alternate with blocks of (bad) iterations in J_K whose lengths are $b_0(T_1^{f_K^i(x)}x) := b_i(x)$.

As a consequence of (10), there exists a positive integer K_0 such that for $K \geq K_0$, $\tilde{J}_K = \tilde{H}_K$. This means, in particular, that these sets $\tilde{J}_K$ are not bounded by the vertical lines $r = \hat{r}_i \pm \varepsilon$ and that the function $b_0(x)$ is constant on the points of $T^{-1}\tilde{H}_K$ for $K \geq K_0$. So the induced map F for has no singularities of the type $T^{-1}(r = \hat{r}_i \pm \varepsilon)$ on these sets because they enter and go out as a block from R_ε.

Moreover, if $T^j x \in R_\varepsilon$ for $1 \leq j < b_0(x)$ and $x, T^{b_0(x)} \in M_\varepsilon$, then $T^j x$ has "no problems" in R_ε: it will be entering and leaving the corner, but no singularity line wil stop it. So, if

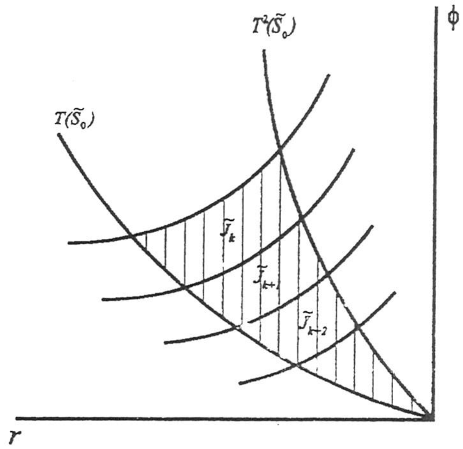

Figure 5:

$W \subset T^{-1}H_K$, $K \geq N_0$ is a smooth non-decreasing curve containing x, and $TW \subset R_\varepsilon$, we conclude that $T_1^{b_0(x)-1} \subset R_\varepsilon$ has no new cutting points. In particular, if W is a smooth component of a LUM, it has spent a lot of time in R_ε, but has not broken, and $T^{b_0(x)-1}W$ will be a bit larger than W, because DT_1 restricted to unstable manifolds in R_ε is (non-uniformly) grater than one. Then we can suppose that W has not entered in R_ε and we must only study which is the smoothness, size, etc. of $FW = T^{b_0(x)}W$. From the same picture, it follows that if W is a non-decreasing curve in $T^{-1}H_K$, $K < N_0$ then FW may only have a finite number of cuts by these singularity lines.

This induced first return map preserves the measure $\mu_{|M_\varepsilon}$. Let us call μ_ϵ the normalized measure. It is essential in this paper to have an estimate of the measure of J_k. We point out that the two measures μ_ε and μ are of the same order on M_ε.

One can decompose the fine set M_ϵ in maximal regions where the induced tranformation F is continuous. We will called I_k, with the indices $k \in \mathbb{N}, k > 0$ the set of points of $z \in M_\varepsilon$ such that $F(z) \in M_\varepsilon$ will take exactly k times to leave for the first time the set R_ε. In other words $T(I_k) = \tilde{J}_k$.

If $z, F(z) \in M_\varepsilon$, then we put $z \in I_0$. In all the cases $z \in I_k$ implies $b_0(z) = k+1$.

As a consequence of the involutive property defined in the Introductgion, **2**, $F(I_k) = -I_k$ and, as our billiard table is symmetric, I_k is symmetric with respect to the points $(p_i, 0)$, see Section 2.

Proposition 1. *For the billiard with three perfect circles there exists a positive constant* C_1 *such that for* $K \geq K_0$

$$\frac{1}{C_1 K^2} \leq \mu(\cup_{k \geq K} I_k) = \mu(J_K) \leq \frac{C_1}{K^2}.$$

Proof. We will do all the calculations on the boundary ∂Q_3, for entering trajectories. We consider the points that are entering in J_K for the first time. More precesily we will consider the points whose second iteration is in R_ε. These points are in between the curves TS_0 andT^2S_0 which equations are well approximated, for r close to π (remember that $\alpha = \pi - r$), by the lines $\phi = -\pi/2 + k_i(\pi - r)$, $0 < k_1 < k_2 < \infty$, respectively (in fact $k_1 = \sqrt{2}/(1+\sqrt{2})$). From (10) we deduce that

$$K \approx \frac{C}{\alpha\sqrt{cos\phi}} \approx \frac{C}{\alpha\sqrt{cos(k\alpha - \pi/2)}} \approx \frac{C}{\alpha^{3/2}}, \quad \text{and}$$

$$\mu(J_K) \approx \int_0^{CK^{-2/3}} d\alpha \int_{-\pi/2+k_1\alpha}^{-\pi/2+k_2\alpha} c\cos\phi d\phi \approx \frac{cC(k_2^2 - k_1^2)}{K^2}.$$

To support our estimations we tested numerically and obtained the following results [1]:
1. Entering and leaving trajectories are symmetric when α_0 is small enough.
2. In (10), $C \approx 1.2$.
3. For each $\alpha < 0.01$ the maximal number of iterations of entering trajectories which first iterations hits on the boundary on a point with coordinate α is

$$N_{\max,\alpha} \approx \frac{2.45}{\alpha^{1.4993}} .$$

We will show in section 5 the statistics of visits for a more general class of billiards because we will assume, instead of the estimations in Proposition 1, that there exist positive constants C_1 and δ such that

$$\frac{1}{C_1 K^{1+\delta}} \le \mu_\varepsilon(J_K) \le \frac{C_1}{K^{1+\delta}}. \tag{11}$$

We claim that the case $0 < \delta < 1$ (respectively $1 < \delta$) corresponds to billiards where the sides are dispersing and the order of tangency in the zero angle is bigger (respectively smaller) than the case for three perfect circles (a litle bit distorted version of figure 1).

4 Exponential decay of F

In this Section, we will follow as much as we can the notation of [4]. We will show that the technics introduced by N. Chernov and Lai-San Young to prove the exponential decay of correlation of Sinai billiards with non zero angle can be applied to our first return map $F : M_\varepsilon \to M_\varepsilon$.

F is uniformly hyperbolic: there exist two families of cones C_x^u, C_x^s in the tangent space T_xM such that $DF(C^u) \subset C_{Fx}^u, DF(C_x^s) \supset C_{Fx}^s$ whenever DF exists, and there exists $\Lambda = \Lambda(\varepsilon) > 1$ such that

$$|DF(v)| \ge \Lambda|v| \ \ \forall v \in C_x^u, \quad |DF^{-1}(u)| \ge \Lambda|u| \ \ \forall u \in C_x^s.$$

$|\cdot|$ is the euclidean metric on TM. See Section 7 of [4] for an interesting discussion about the use of this metric.

Let be $S_{m,n} = \cup_{i=m}^n F^i S_0$; $V_{m,n} = \cup_{i=m}^n F^i V_\varepsilon$ If K_m is the maximal number of smooth curves of $S_{-m,0} \cup V_{-m,0}$ that intersect or terminate at any point of M_ε, we assume (as Chernov

[1]The simulations were done at CVSSP, University of Surrey, England, while A. Pardo was there thanks to an Alfa-Cometas plan, supported by the EU and Universidad de la República, Uruguay.

does in Section 9 of [4]) that K_m does not grow too fast: there is a large enough m such that $\Lambda_1 = \Lambda^{m-m_3} > K_m + 1$ for some fixed m_3.

In fact, all the proofs in [4] are done with F^m and Λ_1 isntead of F and Λ, respectively. But it is known that if T^m has exponential decay of correlations, then so does T: we will maintain F and Λ to avoid some technical difficulties.

We call the DF-image of $\{r\phi \geq 0\}$ the **unstable cone** C^u. Similarly DTF^{-1} maps $\{r\phi \leq 0\}$ stricly into itself, and C^s is defined accordingly. The tangent vectors to the curves in $T^m S_0$ belong in ustable cones for $m \geq 1$ and in stable cones for $m \leq -1$. This property is usually refered to as **alignement**.

One has to partition the neighbourhood of S_0 in M_ε into countably many narrow strips parallel to S_0 in each of which the control of distortions of the derivatives on LUM is possible. For some fixed large $k_0 \geq 1$, (Section 7 of [4]), and for each $k \geq k_0$ define the "homogeinity strips" (introduced in [2])

$$L_k = \{(r,\phi) \in M_\varepsilon : \pi/2 - k^{-2} < \phi < \pi/2 - (k+1)^{-2}\}$$

and

$$L_{-k} = \{(r,\phi) \in M_\varepsilon : -\pi/2 + (k+1)^{-2} < \phi < -\pi/2 - k^{-2}\}.$$

We put

$$L_0 = \{(r,\phi) \in M_\varepsilon : -\pi/2 + k_0^{-2} < \phi < \pi/2 - k_0^{-2}\}.$$

Phase space. In order to apply the results by Chernov we define an open set $M \subset M_\varepsilon$, on which F will satisfy all the required assumptions. We put $M = \cup_{k>k_0} L_k \cup L_0$. The map F restrictred on M has the **singularity set** $\Gamma = S_{-1} \cup T^{-1}(\cup_k \partial L_k) \cup T^{-1} V_\varepsilon$. Since the boundaries of L_k are parallel to S_0, and V_ε is the union of vertical lines, their preimages under T have tangent vectors in stable cones. Then the *alignement* holds *lato sensu* for the curves in Γ. It also holds, in the same sense for all the curves in $\Gamma^{(n)} = \Gamma \cup F^{-1}\Gamma \cup \ldots \cup F^{-n+1}\Gamma$, $n \geq 1$. We will denote also by F the restriction of the induced first return map map F on M.

We fix $\varepsilon < \pi/10$ and k_0 in such a way that $V_\varepsilon \cap \partial L_0 = \emptyset$.

Define

$$M^+ = \{x \in M : F^n x \notin \Gamma, n \geq 0\}, \quad M^- = \bigcap_{n>0} F^n(M \setminus \Gamma^{(n)}), \quad M^0 = M^+ \cap M^-.$$

The sets M^+ and M^- consist, respectively, of points whose future and past iterations by F are defined, and M^0 is the set of points where all the iterations by F are defined.

Let be ρ the riemannian metric on M and ρ_W the metric induced by ρ on any submanifold $W \subset M$. For any $l,\ h > 0$ let

$$M^\pm_{h,l} = \{x \in M^\pm : \rho(F^{\pm n}x, \Gamma \cup \partial M) > le^{-nh} \ \ \forall n \geq 0\},$$

$$M^\pm_h = \bigcup_{l>0} M^\pm_{h,l}, \qquad M^0_h = M^+_h \cap M^-_h.$$

The following result is a standard consequence of the theory of maps with nonzero Lyapunov exponents. See, for example [13]. Sinai billiards have a total measure set of points with nonzero

Lyapunov exponents; see for example [11]. There exist $h, d_l = \epsilon$ such that for every $x \in M^-_{h,l}$ there exists a local unstable manifold (LUM) $W^u(x)$ such that $\rho(x, \partial W^u(x)) \geq \epsilon$. Similarly, the local stable manifold (LSM) $W^s(x)$ is defined for every $x \in M^+_{h,l}$. For the fixed h we will only consider l's such that $\mu(M^0_{h,l}) > 0$.

Now we define **rectangles** as in [4]. A subset $R \subset M^0$ is called a rectangle of there exists $\eta > 0$ such that for any $x, y \in R$ there is a LSM $W^s(x)$ and a LUM $W^u(y)$ both of diameter $\leq \eta$, that meet in exactly one point, which also belongs in R. We assume that δ_0, δ_1 are small enough, so that $A_{\delta_1} = \{x \in M : \text{the unstable disks } W^u_{\delta_1}(x) \text{ exists}\} \neq \emptyset$. The unstable **disk** of radius ϵ, through x, $W^u_\epsilon(x)$, is the LUM which is a ε-ball centered at x in the $\rho_{W^u_\epsilon(x)}$ metric (length along the LUM). W is a δ-LUM if it is a LUM and its length is $\leq \delta$. If W, W' are two δ_0-LUM's, we say that W' **overshadows** W if roughly speaking, the stable cone constructed in any point of W has a common point with W'. In this case we can define

$$\rho^s(W, W') = \sup_{x \in W} \rho^s(x, W')$$

the s-distance from W to W'. $\rho^s(x, W)$ is the supremum of the riemannian distances between x and the points of W, measured along s-disks (disks whose tangent vectors in each point is contained in the stable cone in this point).

δ-Filtration. Let $\delta_0, \delta > 0$, and W be a δ_0-LUM. Two sequences of opens subsets $W = W^1_0 \supset W^1_1 \supset W^1_2 \ldots$ and $W^0_n \subset W^1_n \setminus W^1_{n+1}, n \geq 0$ are said to make a **δ-filtration of** W, denoted by $\{W^1_n, W^0_n\}$,if $\forall n \geq 0$, T^n is defined on W^1_n and W^0_n, each copnnected component of $T^n W^i_n$ has length $\leq \delta_0, i = 0, 1$ and they are constructed inductively in such that a way that the segments W^0_n are taken out from W^1_n if its n-iterate comes too close ($\delta \Lambda^{-n}$ close) to singularity lines at time n, nct earlier. So, points in W whose images come too close to the singularity lines will be set apart and no longer iterated under T. This will create countably many gaps in W in which stable manifolds fail to be long enough. Let be $W^1_\infty = \cap_{n \geq 0} W^i_n$.

We can vary together all the small parameters $\delta_i, i \geq 1$ that appear in the sequel preserving the specified relations between them. For any $z \in A_{\delta_1}$ we define a **canonical rectangle** $R(z)$ as follows: $y \in R(z)$ iff $y = W^s_{\delta_2}(x) \cap W$ for some $x \in W^1_\infty(z)$ and for some LUM W that overshadows $W(z) = W^u_{\delta_1/3}(x)$, and such that $\rho^s(W(z), W) \leq \delta_3$. In Section 4 of [4] it was observed that if $\delta_3/\delta_2 < c'$, where $c' > 0$ is determined by the minimum angle between the stable and unstable cone families, the every W that overshadows $W(z)$ and is δ_3-close to it in the above sense will meet all stable disks $W^s_{\delta_2}(x)$, $W^1_\infty(z)$. In that case $R(z)$ will be a rectangle indeed.

For any connected subdomain $V \subset W(z)$ the set $R_V(z) = \{y \in R(z) : W^s(y) \cap V \neq \emptyset\}$ is an s-subrectangle in $R(z)$ "based on V". For $n \geq 1$, the partition of $W^1_n(z)$ into connected components $\{V\}$, induces a partition of $R(z)$ into s-subrectangles $\{R_V(z)\}$ that are based on those components. If $R_V(z)$ is one of those s-subrectangles, then $F^n R_V(z)$ is a rectangle.

If δ_0 is small enough, then there exists $z_1 \in A_{\delta_1}$ such that $\nu(R(z_1)) > 0$. We fix such a δ_0 and one such z_1. We then denote, for brevity, $R = R(z_1), W = W(z_1), W^1_\infty = W^1_\infty(z_1)$, etc. Let $\mathcal{Z} = \{z_1, z_2, \cdots, z_p\}$ be a finite δ_4-dense subset of A_{δ_1} containing the above point z_1. We call $\mathcal{R} = \cup_i R(z_i)$ the **rectangular estructure**. It is a finite union of rectangles that are likely to overlap and are away from S_0. This means, in particular, that on its points the density $\cos\phi$ is bounded away from zero.

We will partition the set W^1_∞ into a countable collection of subsets $W^1_{\infty,k}, k \geq 0$. For every $k \geq 1$ there is a r_k such that for the s-subrectangle $R_k = \{x \in R = R(z_1) : W^s(x) \cap W^1_\infty \subset W^1_{\infty,k}\}$ based on $W^1_{\infty,k}$, the set $F^{r_k}(R_k)$ will be a u-subrectangle of some $R(z_i)$. This factor is considered a **proper return** and the **return time function** $r(x)$ is defined on W^1_∞ by $r(x) = r_k$ for $x \in W^1_{\infty,k}$, $k \geq 1$ and $r(x) = \infty$ if $x \in W^1_{\infty,0}$ **(leftover set)**.

We say that (F, ν) has **exponential decay of correlations for Hölder continuous functions** if $\forall \eta > 0$, there exists $\gamma(\eta) \in (0,1)$ such that for every η-Hölder functions f, g there exists $C_1(f,g) > 0$ such that

$$\left| \int_M (f \circ F^n) g d\nu - \int_M f d\nu \int_M g d\nu \right| \leq C_1 \gamma^{|n|} \quad \forall n \in \mathbb{Z}.$$

The value of $C_1(f,g)$ will play a subtle role in the proofs of the Propositions and Lemmas of next section. So we remark that in our systems there exists a fixed positive constant C such that

$$C_1(f,g) \leq C \sup_{x \in \mathcal{R}} f(x) \sup_{x \in \mathcal{R}} g(x)$$

where the supremum are taken on the rectangular structure $\mathcal{R}$ (see, [16], Section 4).

We say that (F, ν) satisfies **central limit therem for Hölder continuous functions** if $\forall \eta > 0$, and for every η-Hölder function f, there exists $\sigma = \sigma_f \geq 0$ such that for every interval $A \subset \mathbb{R}$

$$\nu\left(\left\{x \in M : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(F^i(x)) - \int f d\nu \in A\right\}\right) \to \frac{1}{\sqrt{2\pi}\sigma} \int_A e^{\frac{-t^2}{2\sigma^2}} dt \text{ as } \to \infty$$

(Convergence in distribution to $\mathcal{N}(0, \sigma^2)$). Furthermore, $\sigma_f = 0$ iff $f = g \circ F - g$ for some $g \in L^2(\nu)$.

The proofs in [4] can be followed step by step in order to study the measure $m_W\{r(x) > n\}$ (where m_W is the conditional measure on W induced by the Lebesgue measure m). Its exponential decay is enough to deduce the exponential decay of correlations and the central limit theorem, using [16].

Theorem 2. *(F, μ_ε) has exponential decay of correlations and satisfies the central limit theorem for Hölder continuous functions on M_ε*

5 Statistics of visits to the corner

In this section we will present the proofs of our main theorems. We will follow closely the proofs of [6] adapting their results to our context. Their proof applies with small modifications to the present situation. We will point out to the reader in each step of our proof whenever there is an essential difference.

First, we would like to point out that we will consider the two-dimensional bijective dynamical system defined by the billiard on phase-space and not the one-dimensional system defined by the dynamics on unstable leaves. Second, we mention that the measure on M in [6] is σ-finite and here is a probability; the "mixing rate" in our case is much more better.

Denote by $\mathcal{A}$ the collection of the sets I_k defined in Section 3. One can suppose the I_k are closed sets whose interiors are disjoint.

Denote also by $\mathcal{A}_k = \bigvee_{l=0}^{k} F^{-l}\mathcal{A}$ and by $\mathcal{F}_k$ the σ-algebra generated by $\mathcal{A}_k$, $k \in \mathbb{N}$.

The main properties we will use are the hyperbolic character of the induced map F –and consequently the exponential rate of mixing of such system (see Section 4)–, and the estimation of the measure of the sets I_k (Proposition 1).

More precisely we will use the following property whose proof follows the ideas in Proposition 2.2 in [6] :

Proposition 3. - *Let F be the induced map on M_ε, and μ_ε its invariant measure, then there are a constant C_1 and a positive number $\theta < 1$ such that for any positivc integers k and n, for any Hölder continuous function f an any function g constant on atoms of $\mathcal{A}_k$ ($g = \sum c_i I_{A_{k,i}}$; $A_{k,i} \in \mathcal{A}_k$, $A_{k,i} \cap A_{k,j} = \emptyset$ if $i \neq j$)*

$$\left|\int f \circ F^{n+k+1} g d\mu_\varepsilon - \int f d\mu_\varepsilon \int g d\mu_\varepsilon\right| \leq C_1 \theta^n \sup_{x\in\mathcal{R}} f(x) \sup_{x\in\mathcal{R}} g(x)$$

Proof. Let be $d\mu_\varepsilon = hdm$ where $h(x) = c\cos\phi(x)$ is the density of μ_ε with respect to the Lebesgue measure m. Then $\int f \circ F^{n+k+1} gh\, dm = \int \frac{1}{h}\mathcal{L}^{k+1}(gh) f \circ F^n\, hdm$, where $\mathcal{L}$ is the transfer operator for hyperbolic systems (see, for example, Section 4.1 in [15]). Let be $g_1 = \frac{1}{h}\mathcal{L}^{k+1}(gh)$. Then

$$\left|\int f \circ F^{n+k+1} g d\mu_\varepsilon - \int f d\mu_\varepsilon \int g_1 d\mu_\varepsilon\right| \leq C\theta^n \sup_{x\in\mathcal{R}} f(x) \sup_{x\in\mathcal{R}} g_1(x).$$

Let be ψ_i the inverse image of the diffeomorphism F^{k+1} on $A_{k,i}$. As $\det F(x) = \cos\phi(x)/\cos\phi(F(x))$ (see formula (1) and definition $\mathcal{R}$ in section δ-Filtration above), $\cos\phi(x)$ is bounded away from zero, we have

$$g_1(x) = \sum_i \frac{c_i\, h \circ \psi_i(x)}{|\det DF^{k+1} \circ \psi_i(x)|\, h(x)} \leq \tilde{C}_1 \sup_{x\in\mathcal{R}} g(x).$$

Since $\int g_1 d\mu_\varepsilon = \int g d\mu_\varepsilon$, the proof is finished.

We will need a stronger version of the above result. Remember that a positive integer-valued function $\tau : M_\epsilon \to \mathbf{R}$ is called a stopping time with respect to the sequence of sigma-algebras $\mathcal{F}_k$, $k \in \mathbf{N}$, if for each fixed j, the set $\{\tau = j\}$ belongs to $\mathcal{F}_j$ We denote also by $\mathcal{F}_\tau$ the σ-algebra of all measurable sets B such that $B \cap \{\tau = j\}$ belongs to $\mathcal{F}_j$ for any nonnegative integer j.

Following the proof of Corollary 3 in [6] one can easily obtain a similar result:

Corollary 1. *Let F be the induce map and τ a stopping time with respect to the sequence of σ-algebras $\mathcal{F}_k$. Then for any measurable set B and positive integer n*

$$\sup_{A\in\mathcal{A}_k} |\mu_\varepsilon(A \cap F^{-n-\tau-1}(B) - \mu_\epsilon(A)\mu_\varepsilon(B)| \leq C_1\theta^n.$$

Now we state the main theorems. Remember that we consider here a general class of billiards that satisfy (11). The case $\delta = 1$ corresponds to the three perfect circles case shown in figure 1.

Theorem 4. *The value*

$$\beta_K = \min\{n \in I\!N : \mu_\varepsilon\{f_K^1 \geq n\} \leq e^{-1}\}$$

is finite for each K. We have

$$\lim_{K\to\infty} \frac{\beta_K}{K} = \infty.$$

For any positive real number t the following limit holds

$$\lim_{K\to\infty} \mu_\varepsilon\{\beta_K^{-1} f_K^1 > t\} = e^{-t}.$$

Moreover β_K satisfies

$$\lim_{K\to\infty} \beta_K^{-1} \int f_K^1 d\mu_\varepsilon = 1$$

The value β_K is a kind of scale-normalizer to get an exponential process with distribuition e^{-t}. The Theorem says that the time needed to perform the first visit to the interval $[K, \infty)$ is much longer than the typical mixing time, which is the time needed to loose memory from the initial condition in M. Therefore, for large values of K, every unsuccessful trial to overrun level K after the process starts is approximately like a new run from the origin.

Theorem 5. *For any positive integer n and any sequence of positive real numbers $s_1, s_2, \cdots, s_n$ the following holds*

$$\lim_{K\to\infty} \mu_\varepsilon\{f_K^1 > \beta_K s_1, f_K^2 - f_K^1 > \beta_K s_2, \cdots, f_K^n - f_K^{n-1} > \beta_K s_n\} = e^{-\sum_{i=1}^n s_i}.$$

This last theorem allows, for a generic $x \in M_\epsilon$, to consider each renormalized period the orbit stays in $J_K \cup M_\epsilon$ – before the next return to J_K – as a Poisson process.

Using Corollary 1 the proofs of both Theorems are the same as presented in [6] , Theorems 5 and 10.

Nevertheless, in order to prove the theorems some *a priori* lower bounds for f_K^1 are needed. The proofs of this bounds are a little bit different than in [6].

Definition. For $x \in M_\varepsilon$ we will define by $\tau_K^h(x)$ the time of the h-th visit to the interval $b_0^{-1}([K, \infty)$ of the orbit by x under the induced map F:

$$\tau_K^1(x) = \inf\{j \geq 0 : b_0(F^j x) \geq K\},$$

$$\tau_K^h(x) = \inf\{j \geq \tau_K^{h-1}(x) : b_0(F^j x) \geq K\}.$$

It results that up to the h-visit the "fine" iteration period have a total lenght

$$f_K^h = \sum_{j=0}^{\tau_K^h - 1} b_0 \circ F^j + 1, \qquad f_K^1 = \tau_K^1 + 1$$

Definition. We also define the integer valued random variable $N_t(x)$ for $x \in M_\varepsilon$ which counts the number of returns (iterations by F) of the path starting at x to the set M_ε until time t:

$$N_t(x) = \sup\{j \geq 0 : \sum_{i=0}^{j} b_0 \circ F^i(x) \leq t\}.$$

Lemma 6. *There exists an increasing integer valued function* $l : \mathbb{R}^+ \to \mathbb{N}$ *such that for any* $1 > \gamma > 0$

$$\lim_{r\to\infty} \frac{l(r)}{r^{1-2\gamma}} = \infty, \qquad \lim_{r\to\infty} \mu_\varepsilon\{N_{2l(r)} \geq r-1\} = 0$$

Proof. Note that

$$\{N_{2l(r)} \geq r-1\} = \{b_0 + b_0 \circ F + ... + b_0 \circ F^{r-1} \leq 2l(r)\}.$$

In the same way as in Proposition 6 in [6] we begin defining the Laplace transform of the variable

$$W_r = \sum_{i=0}^{r-1} b_0 \circ F^i.$$

For a fixed value of $t \in [0,1[$ to be chosen later, we obtain from Markov inequality

$$\mu_\epsilon(W_r \leq l) \leq t^{-l} \int t^{W_r} d\mu_\epsilon.$$

Let m and s to be chosen later such that $ms \leq r$. Let be $\tilde{W}_r = \sum_{j=0}^{s-1} b_0 \circ F^{jm}$. It satisfies

$$W_r \geq \sum_{j=0}^{m-1} \tilde{W}_r \circ F^j, \qquad \int t^{W_r} d\mu_\varepsilon \leq \int t^{m\tilde{W}_r} d\mu_\varepsilon.$$

The second relation is a consequence of Hölder's inequality applied recursively.

Now we apply Proposition 3 above with $f = g = t^{mb_0}$, $k = jm$. We also observe that $\sup f(x) = t^{\inf mb_0(x)}$ (sup, inf taken on $\mathcal{R}$). But this value $\inf b_0(x)$ can be assumed to be taked on a set of positive μ_ε-measure (b_0 assumes only integer values); then $\sup_{x\in\mathcal{R}} f(x) \leq C_2 \int f d\mu_\varepsilon$.

After a short calculation we get

$$\int t^{W_r} d\mu_\epsilon \leq \left(C_2\theta^m + \int t^{mb_0} d\mu_\varepsilon \right)^{s-1} \leq e^{(s-1)[-g(t^m)+C\theta^{m-1}]}$$

where $g(t) = 1 - \int t^{b_0} d\mu_\varepsilon$.

A standard computation shows that

$$\int t^{b_0} d\mu_\epsilon = 1 + (1-t^{-1})[\sum_{n=1}^{\infty} \mu_\varepsilon\{b_0 \geq n\}\, t^n].$$

and this suggest to consider $u = (1-t)$, $u > 0$ and $\sigma(u) = g(1-u)/u$.

From (11) there exist the finite limit

$$\lim_{t\to 1-} g(t)/(t-1) \in \mathbb{R} \quad \text{and} \quad \sigma(u) = \sum_{n=1}^{\infty} \mu_\varepsilon\{b_0 \geq n\}\,(1-u)^{n-1}$$

is a decreasing function of u.

In the same way as in [6] we have

$$\mu_\varepsilon(W_r \leq l) \leq t^{-l} \int t^{W_r} d\mu_\varepsilon \leq$$

$$\exp\{u\,[l+lu-\sigma(mu-m^2u^2)\,m\,(s-1)\,(1-mu)]+C_2(s-1)\theta^m\}. \qquad (12)$$

Now we set

a) $u(r)=r^{\gamma-1}$ (goes to 0 with $r\to\infty$),

b) $t=(1-u)$ (goes to 1 with r),

c) $s=[\frac{r}{(\log r)^2}]$ (goes to ∞ like r^{1+}),

d) $m=[(\log r)^2]$ (goes to ∞ like r^{0+}),

e) $l(r)=[r^{1-\gamma}\{\sigma(\frac{(\log r)^2}{r})\}^{1/2}]$ (goes to ∞ like $r^{1-\gamma}$ because $\sigma(0)$ is finite),

where $[\cdot]$ denotes the integer part.

From the definitions above we derive that expression (12) goes to zero because

1) $C_2(s-1)\theta^m$ goes to zero,

2) the positive part of the exponent in (12) $lu+lu^2$ is finite,

3) and the negative part of the exponent in (12) $-u[\sigma(mu-m^2u^2)\,m\,(s-1)\,(1-mu)]$ goes to $-\infty$ like $-r^\gamma$ because $\sigma(0)$ is finite and $mu-m^2u^2$ goes to zero.

It is easy to see that $\frac{l(r)}{r^{1-2\gamma}}$ goes to ∞. This is the end of the proof of Lemma 6.

Lemma 7. *Let C be the constant in (11) then:*

i) $C^{-1}\le\mu_\varepsilon\{b_0\ge K\}\int\tau^1_K d\mu_\varepsilon\le C$,

ii) Moreover, for any positive integer k,

$$\mu_\varepsilon\{\tau^1_K\le k\}\le k\mu_\varepsilon\{b_0\ge K\}.$$

The proof of this lemma is equal to Proposition 4 in [6].

Lemma 8. *There exists a decreasing positive valued function L defined in a neighbourhood of $0\in\mathbb{R}$ such that*

$$\lim_{a\to 0}aL(a)=\infty, \quad \text{and} \quad \limsup_{k\to\infty,s\ge 0}\mu_\varepsilon\{s\le f^1_K\le s+L(1/K)\}=0.$$

Proof. We have to set $L(a)=l(r(a))$ where l was defined in Lemma 6 and $r(a)$ will be defined later. First we define the non-decreasing function

$$w(z)=\inf_{y\ge z}\frac{l(y)}{y^{1-2\gamma}}, \quad \gamma>0$$

which diverges when z goes to ∞ according to Lemma 4.

As the non-decreasing funtion $sw(r)^{1/2}$ goes to ∞, one can define $r(a)$ by the equation

$$a=\frac{1}{r(a)}\left(\frac{1}{w(r(a))}\right)^{1/2}.$$

Note that $r(a)$ goes to ∞ and $ar(a)\to 0$ when $a\to 0+$.

Following the same argument as in the proof of Proposition 6 in [6] all we have to prove is that

$$\mu_\varepsilon\{s\le f^1_K<s+l(r(1/K))\}\le\mu_\varepsilon\{b_0\ge K\}r(1/K)+\mu_\varepsilon\{N_{2l(r(1/K))}\ge r(1/K)\} \qquad (13)$$

goes to zero when K goes to ∞.

When $K \to \infty$ the second term of the right-hand-side of (13) goes to zero by Lemma 4 and the first term of the right-hand-side goes to zero because $ar(a) \to 0$ and $\mu_\epsilon\{b_0 \geq K\} < CK^{-(1+\delta)}$ by (11).

Therefore, taking $L(a) = l(r(a))$ the Lemma is proved.

Appendix

In this Appendix it is proved that the fundamental estimates that relate the minimum α-coordinate α_n of the trajectory $x_0, x_1, \ldots, x_n$ entering in a corner, with the initial position α_0, and the initial velocity parameter θ_0 are:

$$\alpha_1 \geq \alpha_{\min} \geq \alpha_1\sqrt{\cos\theta_1}$$

for (α_1, θ_1) determined below.

1. First of all we consider equations (2), (3), with a small modification: instead of $\tan(\alpha_m + \theta_m)$ in (3), we will take only $\tan(\theta_m)$. Then we will have the following equations

$$\hat{\alpha}_{m+1} + \hat{\alpha}_m = \hat{\theta}_{m+1} - \hat{\theta}_m, \tag{14}$$

$$\sin\hat{\alpha}_{m+1} + \tan\hat{\theta}_m \cos\hat{\alpha}_{m+1} = \sin\hat{\alpha}_m + (2 - \cos\hat{\alpha}_m)\tan\hat{\theta}_m, \tag{15}$$

$$-\pi/2 < \hat{\theta}_m \leq -\hat{\alpha}_m - \hat{\alpha}_{m+1}, \quad \hat{\vartheta}_0 = \theta_0, \quad \hat{\alpha}_0 = \alpha_0.$$

We compare the values of the minimum of an "entering" trajectory for both pairs of equations (2), (3) and (14), (15). We will prove that

$$\alpha_{\min} > \hat{\alpha}_{\min}.$$

In fact, if $\alpha_m \geq \hat{\alpha}_m$ and $\theta_m \geq \hat{\theta}_m$, then the same relations (strict inequalities) are valid in the next step $m+1$. $\hat{\alpha}_{m+1}$ can be obtained calculating the tangent line $[\hat{T}_m]$ to the circle of center $(0,0)$ and radius $\hat{\rho}_m$, through $(-\hat{\eta}_m, -1)$. Here $\hat{\rho}_m = \sin\hat{\alpha}_m + (2 - \cos\hat{\alpha}_m)\tan\hat{\theta}_m$, $\hat{\eta}_m = \tan\hat{\theta}_m$.

A short computation gives $|\hat{\eta}_m - \eta_m| < |\hat{\rho}_m - \rho_m|$, and from the comparison of the relative positions of the lines $[T_m]$ and $[\hat{T}_m]$, it results that $\alpha_{m+1} > \hat{\alpha}_{m+1}$. From (2), (14) and the last inequality, it results $\theta_{m+1} > \hat{\theta}_{m+1}$.

2. Let be $h = \hat{\alpha}_1 = \alpha_1$. Now we will reduce equations (14), (15) to differential equations by making the substitutions (see [10])

$$t = mh, \quad y(t) = \frac{\hat{\alpha}_{t/h}}{h}, \quad \theta(t) = \hat{\theta}_{t/h}.$$

We obtain

$$y(t) + y(t+h) = \frac{\theta(t+h) - \theta(t)}{h}$$

$$\sin(hy(t+h)) - \sin(hy(t)) = [2 - \cos(hy(t+h) - \cos(hy(t)]\tan\theta(t).$$

Then we substitute the trigonometric terms in the second equation by its second order Taylor's polynomials:

$$\frac{y(t+h)-y(t)}{h} = \frac{y^2(t+h)+y^2(t)}{2}\tan\theta(t).$$

Finally, if h goes to zero we obtain the following system of differential equations

$$\frac{d\theta(t)}{dt} = 2y(t), \quad \frac{dy(t)}{dt} = y^2(t)\tan\theta(t), \quad y(0)=1, \quad \theta(0)=\theta_1$$

which yields to the solution

$$\overline{y}(t) = \left[\frac{\cos\theta_1}{\cos\theta(t)}\right]^{1/2}, \quad \text{and } y_{\min} \geq \sqrt{\cos\theta_1}.$$

3. As a consequence of convergence results of Euler's method of numerical analysis, we can deduce that the value $\overline{y}(mh)$ of

the exact solution of the differential equation and the value $\hat{\alpha}_m/h$ differ at most in Mh, where M is a constant depending on the maximum of the second derivative of the exact solution in $t \in [0,1]$. For example, in [8] 8.5, it is proved the following result: consider the differential equation $y' = f(t,y), y(0) = y_0, t \in [0,\overline{t}]$, with $||f(t,y)-f(t,z)|| \leq L||y-z||$ (Lipschitz); it has a unique solution $\overline{y}(t)$ that satisfies $\max_{0\leq t\leq \overline{t}} ||y''(t)|| < M_2 < \infty$. Let be

$$y_h(0) = y_0, \quad y_h(t+h) = y_h(t) + hf(t, y_h(t)) + F, \quad ||F|| \leq M_1 h^2.$$

Then, if $t_n = nh$, and $t \in [t_n, t_{n+1}] \subset [0,\overline{t}]$, we have

$$||\overline{y}(t) - y_h(t)|| \leq \frac{h}{2}M_2\left\{\frac{e^{Lt_{n+1}}-1}{L} + \frac{h}{4}\right\}.$$

4. Then $\hat{\alpha}_{\min} \geq \alpha_1\sqrt{\cos\theta_1} + M\alpha_1^2$, if $\sqrt{\cos\theta_1} \gg \alpha_1$. As it was remarked in (6) $\theta_1 > \frac{\sqrt{2}}{1+\sqrt{2}}\alpha_1 - \pi/2$. So, $\cos\theta_1 > \frac{\sqrt{2}}{1+\sqrt{2}}\alpha_1$ and $\sqrt{\cos\theta_1} > c_1\sqrt{\alpha_1} \gg \alpha_1$. Finally, from the results in **1**, **2**, **3** and some numerical computations, we conclude that there exists a positive $\overline{\alpha}$ such that an entering trajectory satisfying $0 < \alpha_1 < \overline{\alpha}$ has its $\alpha_{\min} \geq \alpha_1\sqrt{\cos\theta_1}$ and $\sqrt{\cos\theta_1} > c_1\sqrt{\alpha_1}$ with $c_1 \approx 0.5858$.

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References

[1] R. Artuso, G. Casati and I. Guarneri, *Numerical Experiments on Billiards*, J. Statist. Phys **83** (1996), 145–166.

[2] L.A. Bunimovich, Ya.G. Sinai and N.I. Chernov, *Statistical properties of two-dimensional hyperbolic billiards*, Russ. Math. Surv. **46**:4 (1991), 47–106.

[3] N.I. Chernov, *Statistical properties of piecewise smooth hyperbolic systems in high dimensions*, Discrete and Continuous Dynamical Systems **5** (1999), 425–448 .

[4] N.I. Chernov, *Decay of correlations and dispersing billiards*, J. Statist. Phys. **94** (1999), 513–556.

[5] P. Collet, A. Galves and B. Schmitt, *Unpredictability of the ocurrence time of a long laminar laminar period in a model of temporal intermittency*, Ann. Inst. Henri Poincaré. Phys. Théor. **57** (1992), 319–331.

[6] P. Collet and A. Galves, *Statistics of Close Visits to the Indifferent Fixed Point of an Interval Map*, J. Statist. Phys **72** (1993), 459–478.

[7] A. Fisher and A. Lopes, *Polynomial decay of correlation and the central limit theorem for the equilibrium state of a non-Hölder potential*, Preprint 1998 - UFRGS.

[8] R.T. Gregory and D.M. Young, *A survey of Numerical Mathematics*, Vol I, Adisson Wesley, 1972.

[9] A. Katok and J.-M. Strelcyn, *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lect. Notes Math., **1222**, Springer, New York, 1986.

[10] J. Machta, *Power Law Decay of Correlations in a Billiard Problem*, J. Statist. Phys. **32** (1983), 555–564.

[11] R. Markarian, *Billiards with Pesin region of measure one*, Comm. Math. Phys. **118** (1988), 87–97.

[12] R. Markarian, *New ergodic billiards: exact results*, Nonlinearity **6** (1993), 819–841.

[13] Ya.B. Pesin, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergod. Th. & Dynam. Sys. **12** (1992), 123–151.

[14] Ya.G. Sinai, *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards*, Russ. Math. Surv. **25** :1 (1970), 137–189.

[15] M. Viana, *Stochastic dynamics of deterministic systems*, IMPA, 1997.

[16] L.S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Annals of Mathematics **147** (1998), 585–650.

[17] L.S. Young, *Recurrence times and rates of mixing*, Israel J of Mathematics **110** (1999), 153–188.