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Rotational Compact Stars in 5-Dimensional Brane-World Gravity

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Estrelas Compactas Rotacionais na Gravitação de Mundo-Brana 5-dimensional

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Resumo

Aplicamos o método perturbativo de Hartle para estrelas com rotação lenta na teoria gravitacional de Mundo-Brana 5D, cuja geometria é do tipo Randall-Sundrum 2 (RS2). Obtemos as equações diferenciais modificadas para o arrasto de inercial (frame-dragging), monopolo e quadrupolo, bem como o momento de inercia modificado. A análise para as contribuições do arrasto e do monopolo nas massas maximas das estrelas é feita baseando-se na aproximação linear $\mathcal{P} = \alpha\mathcal{U}$, além das contribuições do Bulk serem tomadas como nulas fora da estrela. Cinco equações de estado são utilizadas para modelar a estrutura nuclear estelar: GDH3, BBB2, BPAL12 e APR. Soluções exteriores são obtidas para as equações de arrasto e monopolo.

Palavras-chave: Hartle, Rotação, Mundo-Brana, Relatividade Geral.

Abstract

We apply Hartle's perturbative method for slow rotational stars in 5D Brane-World Gravity theory with Randall-Sundrum geometry of the second kind (RS2). We obtain the modified frame-dragging, monopole and quadrupole differential equations as well as moment of inertia. The analysis is done for frame-dragging and monopole contributions to maximum star masses based on linear approach $\mathcal{P} = \alpha\mathcal{U}$, and also with Bulk terms outside star surface negligible. We use four EoS to model stars nuclear composition: GDH3, BBB2, BPAL12 and APR. Exterior solutions for frame-dragging and monopole are also obtained.

Keywords: Hartle, Rotation, Brane-World, General Relativity.

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Notation and Conventions

- Latin indices i,j,k, run over spacial coordinates taken to be 1,2 and 3, with exception of chapters 1 and 2 where no dimension is fixed.
- Capital Latin indices A,B,C, etc. goes from 0 to 4, where 4 is meant to be the fifth dimension index.
- Greek indices μ, ν etc. run over 0 to 3, except in chapters 1 and 2.
- When dealing with tensorial quantities and its products it is easier to use Einstein's sum convention for omitting the summation symbol as presented below:

$$\sum_{\mu} A_{\mu} A^{\mu} \equiv A_{\mu} A^{\mu}.$$

- partial derivatives are commonly abbreviated as $\frac{\partial}{\partial x^i} \equiv \partial_i$.
- Throughout the text $g_{\mu\nu}$ will usually denote, unless said otherwise, the 4-dimensional spacetime metric tensor with signature $(- + ++)$.
- The metric tensor $q_{\mu\nu}$ denotes the 5-dimensional spacetime geometry with signature $(- + + + +)$.
- Physical quantities with a tilde above them indicate they belong to the higher dimensional spaces.
- The symbol δ^{ν}_{μ} stands for the usual Kronecker's delta.
- Squared brackets $[,]$ represent the commutator of any given quantities.
- The symbol $M_{\odot}$ stands for solar mass unit whose value in SI is $M_{\odot} = 1.988 \times 10^{30}$ kg.

Introduction

It was 1915 when Albert Einstein first published his theory of General Relativity (GR) which generalizes Special Relativity and Newton's Gravitational Law. At that time, the new theory brought a different, even revolutionary, way of dealing with measurable quantities in physics. In its concepts GR promotes the geometry of spacetime to an agent of dynamics, thus, the choice of geometry becomes a central key in equations of motion. Despite GR success in describing gravity, the increasing amount of astronomical data of distant objects (other galaxies and galaxy clusters) revealed new puzzles that are still unsolved.

Through the years, predictions made by GR have been confirmed in several experimental tests [1], the most famous being the gravitational deflection of light, Mercury's perihelion precession and the recently discovered gravitational waves. Despite GR success, the first challenge to Einstein's theory made its appearance through observation of distant galaxies. Vera Rubin, in the 1970s, made use of galaxy rotation curves to measure the velocity curves of stars in spiral galaxies [2]. She then discovered these galaxies should be 6 times more massive than the visible mass measured. This invisible matter received the iconic name of Dark Matter. The second strike on GR arose from cosmological scale observations, which brought evidence of an universe in accelerated expansion, thus contradicting the expected attractiveness property of observable matter in gravitation [1, 3]. To add in this behavior into a gravitation theory, many new propositions and modifications were made [4] where none of them are completely satisfactory so far. Latter this repulsive energy was named as Dark Energy whose origin and cause remains unknown.

But things started to become pretty bad, besides macroscopic observations, when was acknowledged that GR is incompatible with microscopical physics theories. Contemporary to GR, Quantum Mechanics, and further Quantum Field Theory (QFT) use the quantization description to evaluate microscopical interactions. Nevertheless, when one tries to apply QFT description to GR turns up that the theory is non-renormalizable and demonstrates a lack of predictive power [5]. There is a vast field of research dealing with gravity quantization where proposed theories, like Loop Quantum Gravity and String theory, introduces a bunch of new ideas and interesting analysis [6].

With GR flaws being revealed during the second half of 20th century, the natural response in physics was to develop new, or modified, gravitational theories trying to solve specific, and even general, problems on gravity description [4]. In particular, there is a singular way to create new formulations of gravitational phenomena by changing the structure of spacetime through considerations of extra dimensions. Dimensional extensions of gravity are known since the Kaluza-Klein (KK) theory, but it is rather difficult to give physical meaning to these extra structures [7]. Even so, it is possible to consider them to be compactified, a whole new concept, and then yield an effective theory of gravity. This method was first introduced by Klein. However, compactification is not necessarily required to reduce the effective number of dimensions and actually may be avoided considering the Randall-Sundrum alternative, a technique developed within string theory context.

This scenario allows extra dimensions to be taken as very large, instead of compactified periodically, and the higher-dimensional graviton would be confined to small regions within space, since curved background can support a graviton bound-state [8]. Thus, is possible to work only with the 4-dimensional quantities and projections of higher dimensional ones by attaching them to a sub-manifold, namely "Brane". In particular, one might consider a 5-dimensional spacetime geometry and develop an effective 4-dimensional gravity theory [9, 10]. This effective theory for gravity is the basis of our work.

The differences appearing between this effective theory and GR are given by correction terms in field equations, some of them are local and some are global. The presence of global terms is due to gravity propagation in higher dimensions (also known as "Bulk"), while the Standard Model (SM) interactions are confined to the Brane. Thus, the energy-momentum tensor remains the same as in GR and can be treated in the usual way we are used to. Following this property it is possible to study stars within Brane-World gravity context [11] by considering spherical symmetric geometries. The analogy between GR and Brane World gravity (BW) is surprisingly simple to understand with BW solutions contrasting with those of GR.

Since there is evidence that BW equations are solvable for Schwarzschild like geometries [11], one may attempt to apply Hartle's treatment of slow rotation in Brane Gravity stars. This calculation procedure was not yet performed for BW, leading us to analyze and uncover all possibilities and conditions for the procedure to work properly. The rotational analysis for stars is important since we know these massive objects aren't static [12, 13, 14, 15] and, by including rotation, they can become more massive than static predictions and even be deformed. In order to achieve the analogous rotation equations of GR, we have to modify and analyze the correction terms inserted by the high dimensions approach and then apply the rotational perturbations.

The perturbative method introduced by Hartle is valid for slow enough rotations in order to avoid mass shedding. In this context, rotation will only be allowed when centrifugal forces is sufficiently smaller than gravitational force. When both are equal the frequency is called the Kepler frequency Ω_K. This method received serious criticism in the 70s due to poor agreement with properties obtained from exact numerical methods. However, it took two decades to be unravelled that discrepancies were caused by the wrong usage of classical Newtonian Ω_K as a stability criterion, instead of its GR counterpart

[15]. Hartle's method was put back on the map when a self-consistent relation in $\Omega_{\boldsymbol{K}}$ was worked out for GR [16], and its prediction power become visible in the next years as a tool to study rotational effects in gravity. For this reasons we have chosen Hartle's method as the machinery for the description of rotation with several references to compare (*e.g.* [17, 18, 19]).

In the following we begin with a simple and straightforward review on the mathematical foundations of GR and Brane-World Gravity. Hence, differential geometry, and Riemannian geometry thereafter, are discussed, explained and analyzed. The first two chapters are dedicated to them; therefore, the reader who is familiar with these subjects may want to jump these chapters. In the sequence, GR and Brane-World gravity make their appearance. We discuss GR main principles and ideas finishing with a brief formulation of its equations through variational principle. For Brane-World gravity a detailed discussion is given where we explain the use of Randall-Sundrum alternative in the context. The field equations are analyzed and we show how they behave for static spherical geometries.

Finally, the last part of the text involves Hartle's method, its analysis and usage. First we give a brief introduction to the method and then apply it to the Brane structure where several conditions and approximations are made. Exterior solutions for monopole and frame-dragging equations are also given. After we find all the perturbed equations, we integrate them numerically for five selected equations of estate (EoS) and compare the results with other references. The last chapter consists of future ideas to be calculated and some considerations regarding our approximations.

1 Notions of Differential Geometry

The advances in physics and mathematics in the past century have risen differential geometry to one of the pillars of modern physics. It is not a surprise that it is now a fundamental subject for understanding nature's peculiarities. However, a primary contact to differential geometry may sound confusing or annoying. Even so, when difficulties are surpassed, the beauty of the formalism naturally arises as physical interpretations become more and more intuitive and clear from mathematical results. Its applicability to physical phenomena is very rich and fruitful with plenty of examples [20].

In the special relativity field of research, it was already known that spacetime have hyperbolic structure due to spacetime invariance regarding Lorentz transformations. There also is common notion that any event can be localized by a set of four numbers and this is assumed as globally true, i.e., one could establish a straightforward correspondence between $\mathbb{R}^4$ and spacetime continuum. Therefore, to comprise the prior concepts into a general curved spacetime, which is the dynamical field of the theory, it is necessary to introduce the formal mathematical concepts of differential geometry. GR is the pioneer theory that gives geometry a whole new meaning inside physics. Its mathematical concepts are entirely based on a special case of differential geometry, called Riemannian geometry.

Riemann's geometry is a direct generalization of Euclidean geometry which is more intuitive when working with curved spaces. Nowadays, it can be seen as an old-fashioned differential geometry whose formalism is based on Tensor Calculus and Analysis imbued into a metric space defined by a metric tensor; all of this inside a well-defined coordinate system. Once the metric is given, one can perform its derivatives and construct tensors that describe the space curvature point-to-point. Moreover, important notions such as parallel transport and covariance appear naturally in curved spaces formulation. Another meaningful aspect is the definition of distance between points that is not required to be positive, but can also be zero or negative, a feature explored by relativists to explain causality.

Though Riemannian geometry is a good primary contact with the subject, there is a more elegant and modern way to reach its results by using the manifold formalism. This formalism was first introduced by Riemann and later extensively studied by great names such as H. Poincaré, H. Weyl and V. Arnold. The main advantage of using manifolds relies on the fact that we do not necessarily need to define a coordinate system, thus providing straightforward calculations and easier interpretations. We shall, from now on, use this formalism to derive the important aspects and quantities of geometry. This chapter's purpose comprehends the definition of manifolds, vector spaces, connections as well as their analyses and discussion.

1.1 Manifolds

A manifold is basically a space that looks like Euclidean space at every "piece" of it and that all can be "glued" together. To understand this statement we shall first remark that Euclidean space $\mathbb{R}^n$ is a topological space with the so-called *ball topology*, to be presented soon. A topological space is a pair $(A, \mathcal{T})$ consisting of a set A and a collection $\mathcal{T}$ of *open subsets* satisfying the following primordial properties:

(1) If $U_i \in \mathcal{T}$ is open, then $\cup_i U_i \in \mathcal{T}$ and is open.

(2) The intersection of a finite number of open subsets in $\mathcal{T}$ is in $\mathcal{T}$. If $U_1, \cdots, U_n \in \mathcal{T}$, then $\cap_{i=1}^n U_i \in \mathcal{T}$.

(3) The entire set A and the empty set $\emptyset$ are in $\mathcal{T}$ and are open.

The collection $\mathcal{T}$ is called a *topology* on A. For Euclidean space, let $x = (x^1, \cdots, x^n)$ and $y = (y^1, \cdots, y^n)$ be points $x, y \in \mathbb{R}^n$, we define an *open ball* in $\mathbb{R}^n$ with radius r, centered in y, when the x points are such that $|x - y| < r$, with

$$|x - y| = \left[\sum_{i=1}^{n} (x^i - y^i)^2\right]^{1/2} . \tag{1.1}$$

The union of open balls creates open sets in $\mathbb{R}^n$, hence this space is classified as topological with the ball topology. Manifolds can be defined as topological spaces too and we will consider only manifolds that are *Hausdorff* and *paracompact*. Actually, paracompactness is a property that comes from second-countability in the usual mathematical definition of manifold. However, we can avoid this technicality since manifolds with second-countability are paracompact [21].

A topological space A is Hausdorff when for each pair of points $p, q \in A$, with $p \neq q$, one can find open sets $U_p, U_q \in \mathcal{T}$ such that $p \in U_p$, $q \in U_q$ and $U_p \cap U_q = \emptyset$. The space A is said to be paracompact if every open cover $\{U_\alpha\}$ of A has a locally finite refinement $\left\{V_\beta\right\}$. The open cover $\left\{V_\beta\right\}$ is a refinement of $\{U_\alpha\}$ if for each V_β there exists U_α such that $V_\beta \subset U_\alpha$. Finally, $\left\{V_\beta\right\}$ is said to be locally finite if each $a \in A$ has an open neighborhood W such that only finitely many V_β satisfy $W \cap V_\beta \neq \emptyset$. Good examples of Hausdorff and paracompact manifolds are $\mathbb{R}^n$ itself, the m-sphere $\mathbb{S}^m$ and the k-torus $\mathbb{T}^k$.

All these definitions yield two important results for our manifold. First is that paracompact Hausdorff spaces admit partitions of the unity, i.e., for A paracompact and Hausdorff with an open cover $\{B_\alpha\}$ there exists a collection of functions $\{f_\alpha\} \in A$ such that

(1) The support[1] of f_α is contained within B_α.

(2) $0 \leq f_\alpha \leq 1$ and $\sum_\alpha f_\alpha = 1$.

Partitions of unity allow us to define integration in a piece of space and then extend it to the entire manifold. The second result is that paracompact manifold M admits a Riemannian metric [20].

Another important aspect to be developed is a smooth structure for a manifold. This structure provides ways to relate two different manifolds through smooth maps and also gives a sense to introduce known calculus notions. A function $f : U \to V$ for $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ is said to be *smooth* (C^∞ or infinitely differentiable) if it has partial derivatives of all orders. In addition, if f is bijective and has smooth inverse map, then f is called a *diffeomorphism*. We shall denote the set of smooth functions in a manifold M by $C^\infty(M)$. Finally we can define a n-dimensional, C^∞, real manifold M with a collection of subsets $\mathcal{T}$ by the following three properties:

(1) The collection $\mathcal{T}$ cover the entire set M, that means for each $p \in M$ there is at least one $U_\alpha \in \mathcal{T}$ such that $p \in U_\alpha$.

(2) For each $U_\alpha \in \mathcal{T}$ there is an homeomorphism[2] $\psi_\alpha : U_\alpha \to \hat{U}_\alpha$, where $\hat{U}_\alpha$ is an open subset of $\mathbb{R}^n$.

(3) If $U_\alpha \cap U_\beta \neq \emptyset$, then there is a map $\psi_\beta \circ \psi_\alpha^{-1}$ which takes points in $\psi_\alpha[U_\alpha \cap U_\beta] \subset \hat{U}_\alpha \subset \mathbb{R}^n$ to points in $\psi_\beta[U_\alpha \cap U_\beta] \subset \hat{U}_\beta \subset \mathbb{R}^n$ (see Fig. 1).

The pair (U, ψ) is generally called *chart* or *coordinate system*. It is very convenient to add the requirement that the cover $\mathcal{T}$ and the chart family $\{\psi_\alpha\}$ are maximal, meaning that all compatible coordinate systems satisfying (2) and (3) are included. This prevents us from defining new manifolds when deleting or adding in new coordinate systems. Note that we can define a topology on M by demanding that all maps ψ_α in the maximal collection be homeomorphisms.

Given a chart (U, ψ), the set U is called the *coordinate neighborhood* and, if $\psi(p) = 0$, we say that this chart is centered at p. In addition, the action $\psi(U) \equiv \hat{U} \subseteq \mathbb{R}^n$, grants ψ the title of *local coordinate map*. Under this actions, the component functions $(x^1, \cdots, x^n)$, defined as $\psi(p) = (x^1(p), \cdots, x^n(p))$, are called *local coordinates* on U. Sometimes we want to emphasize the components instead of ψ which is done by rewriting the chart notation as (U, x^i).

[1] Support is the closure of the set where f_α is nonvanishing.

[2] A homeomorphism is a continuous, one-to-one, onto function between topological spaces that has a continuous inverse function. Two topological spaces are said to be homeomorphic when a homeomorphism connects them, they have same topological structure.

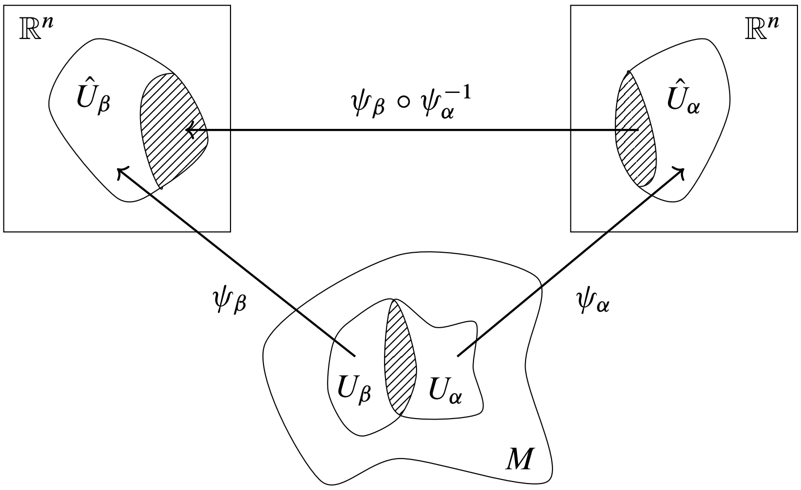


Figure 1 – Diagram illustrating the action of ψ's functions.

1.2 Tangent Vectors

The fact that each neighborhood of a given point in M resembles $\mathbb{R}^n$ is an indication that we can construct a tangent space at that point. The notion of a tangent space is important because we know how to use the advanced calculus machinery in linear spaces. Therefore, we will be able to operate in each neighborhood of $p \in M$ similarly as we do in Euclidean space, but one should keep in mind that we cannot, yet, connect different tangent spaces for two distinct points of M.

Take for example the sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$, it is possible to attach tangent planes in different points of $\mathbb{S}^2$, these planes are formed by a collection of vectors which are all orthogonal with a common vector. Now let us consider a tangent space $T_p\mathbb{S}^2$ for $p \in \mathbb{S}^2$ and another space $T_q\mathbb{S}^2$ for $q \in \mathbb{S}^2$, we are tempted to ask: How a vector $v_p \in T_p$ is related to a vector $v_q \in T_q$? In order to answer this, first we have to connect p and q by a smooth curve, however, there exists infinite ways of doing so. Hence, the relation between v_p and v_q is path-dependent and we must be able to express how the vectors behave in these curves.

First of all we introduce how vectors are treated in $\mathbb{R}^n$. Our intuition says that we can attach to each point $a \in \mathbb{R}^n$ an n-tuple $(v^i, \cdots, v^n)$ having a modulus and direction. We may summarize it by saying that, for a given point a, the space $\mathbb{R}^n_a \equiv \{a\} \times \mathbb{R}^n = \{v_a, v \in \mathbb{R}^n\}$ is the space of the geometric tangent vectors for that point. The set $\mathbb{R}^n_a$ forms a real *vector space* under natural operations

$$(v + w)_a = v_a + w_a, \quad (cv)_a = c(v)_a.$$

There is also a set of linear independent vectors $\{\hat{e}_i\} \in \mathbb{R}^n_a$, for $i = 1, \cdots, n$, that forms a basis for $\mathbb{R}^n_a$. The sets $\mathbb{R}^n$ and $\mathbb{R}^n_a$ are topologically the same, but for two different points the sets $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ are disjoint and we can distinguish them.

Having the geometric tangent vectors we will be able to settle the directional derivative of smooth functions. The vector $v_a \in \mathbb{R}^n_a$ will provide a linear map $D_{v_a} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which takes the derivative in the direction v at a:

$$D_{v_a} f = D_v f(a), \tag{1.2}$$

this operation is linear and satisfies Leibnitz rule:

$$D_{v_a}(fg) = f(a) D_{v_a} g + g(a) D_{v_a} f. \tag{1.3}$$

If we express v_a in terms of the standard basis, i.e., $v_a = v^i \hat{e}_i \big|_a$, then $D_{v_a} f$ can be rewritten as

$$D_{v_a} f = v^i \frac{\partial f}{\partial x^i}\bigg|_a . \tag{1.4}$$

This construction for tangent vectors is very similar to the definition of a *derivation*. Indeed, a derivation at a is defined by a map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ that is linear and satisfies the Leibnitz rule (1.3). Now, let $T_a\mathbb{R}^n$ denote the collection of all derivations of smooth functions at a. Then, $T_a\mathbb{R}^n$ will be a vector space under the operations:

$$(w_1 + w_2) f = w_1 f + w_2 f, \quad (cw) f = c(wf).$$

There is a proposition showing that derivations have a one-to-one correspondence with geometric tangent vectors.

Proposition 1.1. *Let $a \in \mathbb{R}^n$.*

(a) *For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map $D_{v_a} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at a.*

(b) *The map $v_a \mapsto D_{v_a}$ is an isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$.*

Accordingly to item (b), $T_a\mathbb{R}^n$ has the same dimension n of $\mathbb{R}^n$, hence the tangent space can be spanned by n-vectors. From this proposition follows a corollary that establishes a basis for the tangent space $T_a\mathbb{R}^n$ in terms of directional derivatives

Corollary 1.2. *For $a \in \mathbb{R}^n$ the n derivations*

$$\frac{\partial}{\partial x^1}\bigg|_a, \cdots, \frac{\partial}{\partial x^1}\bigg|_a \quad \text{defined by} \quad \frac{\partial}{\partial x^i}\bigg|_a f = \frac{\partial f}{\partial x^i}\bigg|_a$$

forms a basis for $T_a\mathbb{R}^n$.

For proofs of the proposition and its corollary the reader is referred to [21] page 53.

The next step is to define tangent vectors for smooth manifolds inspired by our definition of geometric tangent vectors. Let M be a smooth manifold and let p be a point of M. A linear map $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a derivation at p if it satisfies

$$v(fg) = f(p) vg + g(p) vf, \text{ for all } f, g \in C^\infty(M).$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the tangent space of M at p. An element of T_pM is called a tangent vector at p. Since we have a description for tangent spaces, we can proceed to understand how they are affected by smooth maps. This will help to connect tangent spaces for different manifolds and later will lead us towards a less abstract picture of them.

For now we will introduce a coordinate chart (U, x) and a smooth function f near a point $p \in M$. Thus, in this local coordinate system, $f = f(x^1, \cdots, x^n)$ [3], where x^i are components of x. Using the vector v at p as a derivation we can define its action on f by

$$v(f) := v^i \frac{\partial f}{\partial x^i}(p). \tag{1.5}$$

This seems a coordinate dependent definition, but it is not the case. Consider now another coordinate chart (V, x') containing p, then we also have $f = f(x')$ from our smooth manifold structure. Let v act on f as before for the patch (U, x), from the chain rule we see:

$$\begin{aligned} v(f) = v^i \frac{\partial f}{\partial x^i} &= \frac{\partial f}{\partial x'^j}\left[\frac{\partial x'^j}{\partial x^i} v^i\right] \\ &= \frac{\partial f}{\partial x'^j} v'^j = v'(f). \end{aligned}$$

Therefore we have a general description of vectors on manifolds. Now is very clear that we can associate any tangent vector to a differential operator, evaluated at p, given a coordinate system. The tangent vector takes the form

$$v_p = v^i(p) \frac{\partial}{\partial x^i}\bigg|_p . \tag{1.6}$$

If we take in the last demonstration $f = x^i$, i.e. we are changing coordinates from U to V, we would have a transformation law of the components functions of v_p as

$$v^i = \frac{\partial x^i}{\partial x'^j}(p) v'^j,$$

and this is why tangent vectors are called *contravariant* vectors, because they transform in the "opposite way" of the coordinates partial derivatives.

Each one of the operators $\partial/\partial x^i$ defines a vector at p thus forming a basis to describe tangent vectors. Therefore the tangent space T_pM has a coordinate basis spanned by the n vectors

$$\frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p .$$

[3] Note that this is the same as $f \circ x^{-1}$.

It is usual that some spaces have a collection of real-valued functionals which acts on its functions. This collection yields the *dual space* that will play an important role later. For the tangent space T_pM we can form a dual space from linear functionals defined by a map $\omega : T_pM \to \mathbb{R}$ which will be denoted as T_p^*M. The elements $\omega \in T_p^*M$ are called tangent *covectors* and T_p^*M receive the name of *cotangent space*.

An important feature of the dual space is that its basis is orthogonal to the basis of the original space. Let V be a real-valued vector space with basis $(e_1, \cdots, e_n)$, thus the covectors $\epsilon^1, \cdots, \epsilon^n \in V^*$ form a basis for V^* when

$$\epsilon^i(e_j) = \delta^i{}_j,$$

where $\delta^i{}_j$ is the Kronecker delta. Analogously, a covector $\omega \in T_p^*M$ can be expanded in a basis as $\omega = \omega_i \epsilon^i\big|_p$. In the next section we shall discuss in more detail what is this basis for the cotangent and its meaning. But first we have to understand the differentials.

1.3 Differentials

Consider now M and N two smooth manifolds and let $F : M \to N$ be a smooth map, for each element $p \in M$ we define

$$dF_p : T_pM \to T_{F(p)}N,$$

called the *differential* of F at p as follows. Given $v \in T_pM$ we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

This map is linear and if F is a diffeomorphism, then dF_p is an isomorphism. Therefore, it follows that T_pM and $T_{F(p)}N$ have same dimensions, i.e., $\dim T_pM = \dim T_{F(p)}N$. For instance, if we consider a coordinate system (U, ϕ) with $U \subset M$ and ϕ a diffeomorphism from U onto $\hat{U} \subseteq \mathbb{R}^n$, we can show that $\dim T_pM = \dim T_{d\phi(p)}\mathbb{R}^n$ [21]. The differential map is commonly known as *pushfoward*, since it "pushes" tangent vectors forward from the domain manifold to the codomain.

The differential is a generalization of the notion of Jacobian matrix. Moreover, we can also work out the representation of the differential in a coordinate system. In a general case we have two smooth manifolds and the function $F : M \to N$, now choose a smooth chart (U, ϕ) for M containing p and (V, ψ) for N containing $F(p)$. A coordinate representation would be given by $\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(V)$, see Fig. 2. The representation of p will be denoted as $\hat{p} = \phi(p)$, and we compute

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = dF_p\left(d(\phi^{-1})_{\hat{p}}\left(\frac{\partial}{\partial x^i}\bigg|_{\hat{p}}\right)\right), \tag{1.7}$$

using $F \circ \phi^{-1} = \psi^{-1} \circ \hat{F}$, then

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = d(\psi^{-1})_{\hat{F}(\hat{p})}\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\bigg|_{\hat{F}(\hat{p})}\right)$$
$$= \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial y^j}\bigg|_{F(p)}. \tag{1.8}$$

The result of our derivation for dF_p is just a Jacobian matrix of F represented in coordinate basis. In fact, if we take M and N as Euclidean spaces we would obtain the usual Jacobian of F. Note that in a general manifold this "matrix" is a local one, because tangent spaces are locally defined.

It is worth to say that there is a definition of *rank of F at p* as the rank of dF_p; it is the rank of the Jacobian matrix, or yet, the $\operatorname{Im} dF_p \subseteq T_{F(p)}N$. For that, F is called an *immersion* if its differential is injective for every $p \in M$, i.e., $rankF = \dim M$. One particular kind o immersion is an *embedding* which, in addition, is a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology. Such terminology is used to define submanifolds.

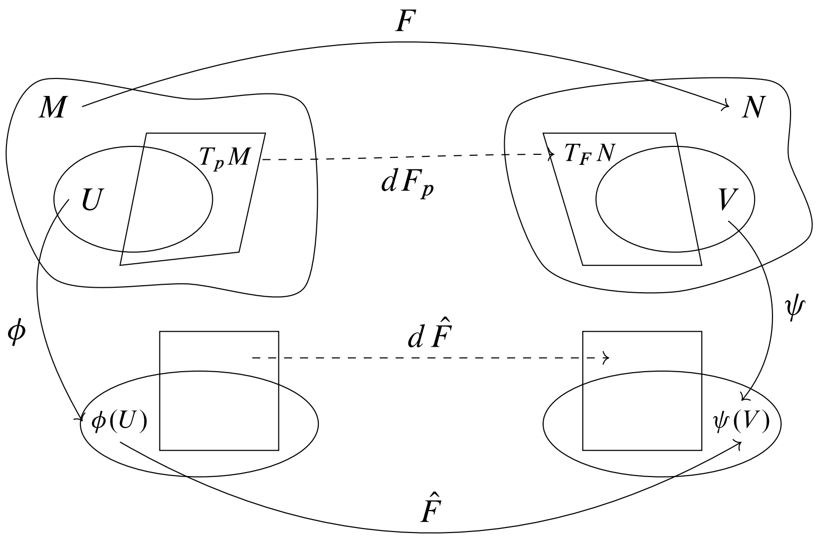


Figure 2 – Diagram of differentials in different coordinate systems.

Interesting results arise when we choose $N = \mathbb{R}$ in the previous reasoning. Here the differential map becomes $dF_p : T_pM \to \mathbb{R}$ and is precisely the definition of a functional on T_pM given earlier. Therefore, dF_p is a covector of T_p^*M. From our definition, if we apply the differential on a smooth function f at p yields

$$df_p(v) = vf(p) \quad \text{for } v \in T_pM,$$

and recall that the differential can be written in coordinates (U, x^i) by $df_p = A_i(p)\epsilon^i\big|_p$ where $A_i : U \to \mathbb{R}$. Thus, from the action of df_p on the vector basis $\partial/\partial x^i$ we find:

$$df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = A_i(p) = \frac{\partial f}{\partial x^i}(p) \tag{1.9}$$

which yields the formula

$$df_p = \left.\frac{\partial f}{\partial x^i}\epsilon^i\right|_p \tag{1.10}$$

To find the explicit formula for covectors basis we just apply (1.10) to coordinate functions. Consequently, we have

$$dx^i_p = \left.\frac{\partial x^i}{\partial x^j}\epsilon^j\right|_p = \left.\delta^i_j\epsilon^j\right|_p = \left.\epsilon^i\right|_p, \tag{1.11}$$

and the basis of cotangent space is none other than the differential dx^i of the coordinate system! In sequence, we want to extend our definitions of vectors and covectors to create fields over the manifold structure and finally arrive to tensors fields.

1.4 Tangent Bundle

A useful construction is that of a tangent bundle. Once given the tangent spaces T_pM for all $p \in M$, the *tangent bundle* is simply the disjoint union of all these spaces:

$$TM = \bigsqcup_{p\in M} T_pM.$$

A point in TM consists of a pair (p, v), where $p \in M$ and $v \in T_pM$. If we choose a local coordinate system $(U, \{x^i\})$, the point p will be described by the n-tuple $(x^1(p), \cdots, x^n(p))$ and v by $v = v^i\, \partial/\partial x^i$. Thus, (p, v) is completely described by the $2n$-tuple

$$x^1(p), \cdots, x^n(p), v^1, \cdots, v^n.$$

Note that these $2n$ functions give TM a natural topology of form $(U \subset \mathbb{R}^n)\times\mathbb{R}^n \subset \mathbb{R}^{2n}$, since the x's functions take their values in a portion U of $\mathbb{R}^n$ and v's fill out an entire $\mathbb{R}^n$. Now if p also lies in another coordinate patch (U', x') there is a new set of $2n$ functions that describes the pair (p, v)

$$x'^1(p), \cdots, x'^n(p), v'^1, \cdots, v'^n$$

and they are related with the old ones by $2n$ equations:

$$x'^i = x'^i(x^1, \cdots, x^n),$$
$$v'^i = v^j\frac{\partial x'^i}{\partial x^j}.$$

This characterizes TM as a $2n$ dimensional smooth manifold. In mechanics, for example, TM is the space of all generalized velocities with points $(q, \dot{q})$, i.e., the configuration space [20].

The tangent bundle comes naturally equipped with the projection map $\pi : TM \to M$ acting on pairs as $\pi(p, v) = p$ or in local coordinates

$$\pi(x^1, \cdots, x^n, v^1, \cdots, v^n) = (x^1, \cdots, x^n),$$

which is clearly differentiable and smooth. On the other hand, the inversion $\pi^{-1}(x)$ represents all tangent vectors to M at x and is often called a *fiber* over x. We can regard TM locally as a product $U \times \mathbb{R}^n$ by considering the map $\pi^{-1}(U)$ of the patch (U, x) and the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ is called a *local trivialization*. If M is entirely covered by this single chart, then TM is diffeomorphic to $M \times \mathbb{R}^n$.

If we choose a local chart (U, ϕ) with coordinates (x^i) and considering a fiber $\pi^{-1}(U)$ we construct a smooth coordinate chart for TM by defining the trivialization $\Psi : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n$ as

$$\Psi\left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = (x^1(p), \cdots, x^n(p), v^1, \cdots, v^n),$$

i.e. given any smooth chart in M we can extend it to TM.

A vector field is a smooth section of the map π. In other words, a vector field X is a continuous smooth map $X : TM \to M$ with the property $\pi \circ X = \mathrm{Id}_M$. For every $p \in M$ we have $X_p \in T_pM$, where X_p is the field evaluated at p. In local coordinates we can drop the p subscript and write the vector field as

$$X = X^i \frac{\partial}{\partial x^i}. \tag{1.12}$$

Note that these fields acts as derivations on smooth functions similarly to tangent vectors. They are required to be linear and smooth, thus the components X^i are differentiable.

1.5 Cotangent Bundle

Similarly to tangent bundle construction we can create the cotangent bundle of M as a disjoint union of cotangent spaces of all $p \in M$, such that

$$T^*M = \bigsqcup_{p \in M} T_p^*M.$$

It also has a projection map $\pi : T^*M \to M$ and, as before, given a local coordinate system $\{x^i\}$ for open $U \subseteq M$ we denote the basis for T_p^*M by the differentials $\left\{ dx^i \big|_p \right\}$.

This defines the maps $dx^1, \cdots, dx^n : U \to T^*M$ which are the coordinate covector fields. Hence, a basis for T^*M is yielded by the pair (x^i, ω_i) where ω_i are the components of a covector $\omega = \omega_i dx^i$ in local frame.

A covector field is a (local or global) section of the projection π given as $\Omega : T^*M \to M$ and they are written in local coordinates as

$$\Omega = \Omega_i dx^i.$$

We can also characterize covector fields by their action on vector fields as $\Omega(X) : M \to \mathbb{R}$ such that

$$\Omega(X) = \Omega_i X^i,$$

Covector fields are also known as differential 1-form, or just 1-form. This terminology is very used in exterior calculus where differential forms provide a generalization for cross product, Jacobian determinant, curl and divergence, besides its role on integration as a coordinate invariant.

Another useful concept is that of *pullback* of covector fields. The pullback is a dualization of the differential map of section 1.3 that take covector fields in N to covectors fields in M, thus in the opposite direction of the pushforward. Explicitly, given $F : M \to N$ a map between two smooth manifolds and its differential $dF_P : T_pM \to T_{F(p)}N$, the dualization yields then

$$dF_P^* : T_{F(p)}^*N \to T_p^*M.$$

Unraveling the definitions, we see that dF_p^* is characterized by

$$dF_p^*(\omega)(v) = \omega(dF_p(v)), \tag{1.13}$$

The pullback of a covector field ω by any function $F : M \to N$ defines another covector field $F^*\omega \in T_p^*M$ by

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)}),$$

and additionally, if u is any smooth real-valued function in N leads to the properties

$$F^*(u\omega) = (u \circ F)F^*\omega,$$
$$F^*du = d(u \circ F).$$

Pullbacks are very important to restrict covector fields to submanifolds. Later when we will be dealing with submanifolds of Riemannian spaces, we shall use pullbacks to create important quantities in these submanifolds.

Following our schematics, we introduce local coordinate systems $(x^i) \in M$ near p and $(y^j) \in N$ near $F(p)$, where $y \equiv F(x)$. Hence, T_pM has basis $\partial/\partial x^i$ and $\partial/\partial y^j$ for $T_{F(p)}N$. Now let us act F^* on a covector field $\omega = \omega_i dy^i$ living in $T^*_{F(p)}N$, then

$$\begin{aligned}
F^*\omega = F^*(\omega)\left(\frac{\partial}{\partial x^i}\right)dx^i &= \omega\left(dF\left(\frac{\partial}{\partial x^i}\right)\right)dx^i && \text{(action (1.13))}\\
&= \omega\left(\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}\right)dx^i && \text{(differential (1.8))}\\
&= \left(\frac{\partial y^j}{\partial x^i}\right)\omega\left(\frac{\partial}{\partial y^j}\right)dx^i\\
&= \omega_j\frac{\partial y^j}{\partial x^i}dx^i.
\end{aligned}$$

The cotagent bundle also has an analogy with classical mechanics. It represents the phase space (q, p) of the coordinates q's and its generalized momenta p's [22].

1.6 Tensors

In general, tensors are real-valued multilinear functions of one or more variables. Suppose there are n vector spaces $V_1, \cdots V_n$, then a map $T : V_1 \times \cdots \times V_n \to \mathbb{R}$ is multilinear if it is linear as a function of each variable separately when the others are held fixed. That is,

$$T(v_1, \cdots, av_i + bv'_i, \cdots, v_n) = aT(v_1, \cdots, v_i, \cdots, v_n) + bT(v_1, \cdots, v'_i, \cdots, v_n).$$

There is an especial function called the tensor product which help us create new tensors from given ones. Consider the vector spaces $V_1, \cdots, V_n, W_1, \cdots, W_k$ and the multilinear maps $T : V_1 \times \cdots \times V_n \to \mathbb{R}$ and $G : W_1 \times \cdots \times W_k \to \mathbb{R}$. A tensor product is defined by the function

$$T \otimes G : V_1 \times \cdots \times V_n \times W_1 \times \cdots \times W_k \to \mathbb{R}$$

as $T \otimes G(v_1, \cdots, v_n, w_1, \cdots, w_k) = T(v_1, \cdots, v_n)G(w_1, \cdots, w_k)$. Therefore, the tensor product also results in a multilinear function. From this we can form a basis for any tensor by taking all possible tensor products of the basis of vectors spaces which forms it.

Consider now just one vector space V, the tensor formed by

$$T : \underbrace{V \times \cdots \times V}_{\text{n copies}} \to \mathbb{R},$$

is called a purely contravariant tensor of rank n. We may abbreviate the notation as

$$T^{(n,0)}(V) = \underbrace{V \times \cdots \times V}_{\text{n copies}}.$$

Analogously, if V^* is the dual space of V, then a purely covariant tensor of rank k over V^* is given by

$$T : T^{(0,k)}(V) \to \mathbb{R},$$

where we abbreviate $T^k(V^*) = T^{(0,k)}(V)$. Finally, a mixed (n, k)-tensor on V is defined by

$$T : T^{(n,k)}(V) \to \mathbb{R}.$$

The transition of these concepts to manifolds is straightforward. Remember that T_pM and T_p^*M behave as vector spaces, thus we create tensors over the tangent and cotangent spaces at p. The most general tensor would be a mixed tensor of type (n, k) given by

$$T : T^{(n,k)}(T_pM) \to M,$$

if $k = 0$, then T is purely contravariant. On the other hand, if $n = 0$ the tensor is purely covariant.

Tensor bundles are also of importance, we define tensor fields from them, just as in vectors case. A bundle of mixed tensor is the disjoint union for tensors at all $p \in M$ defined by

$$T^{(n,k)}TM = \bigsqcup_{p\in M} T^{(n,k)}(T_pM).$$

This will be a bundle of covariant tensors if $n = 0$, and a bundle of contravariant tensors if $k = 0$. Furthermore, the space of smooth sections of the bundles $\Gamma(T^kT^*M)$, $\Gamma(T^nTM)$ and $\Gamma(T^{(n,k)TM})$ defines tensor fields over M. Considering a coordinate system (x^i) we write a tensor field in its basis according to the space it pertains; therefore

$$T = \begin{cases} T_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}, & T \in \Gamma(T^kT^*M); \\ T^{i_1\cdots i_n} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_n}}, & T \in \Gamma(T^nTM); \\ T^{i_1\cdots i_n}_{j_1\cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_n}} dx^{j_1} \otimes \cdots \otimes dx^{j_k}, & T \in \Gamma(T^{(n,k)}TM). \end{cases}$$

The functions $T_{i_1\cdots i_k}$, $T^{i_1\cdots i_n}$ and $T^{i_1\cdots i_n}_{j_1\cdots j_k}$ are the components of T. Throughout the text we shall use only the components of tensors and vectors which is a very useful practice. One should keep in mind that subscript indices in the functions refer to covariance and superscript to contravariance. Although, their basis are written in the opposite way as we can see above. It follows that changing coordinates the tensor components transforms accordingly to its basis laws. For example consider a change of coordinates $(x^j) \to (y^i)$ then the components of a covariant tensor become

$$T'_{i_1,\cdots,i_n} = \frac{\partial x^{j_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{j_n}}{\partial y^{i_n}} T_{j_1,\cdots,j_n}. \tag{1.14}$$

As last topic we discuss how purely covariant tensor fields can be pulled back from the codomain to the domain, just as covectors. Suppose $F : M \to N$. For any $p \in M$ and any k-tensor $\alpha \in T^k(T^*_{F(p)}N)$ we define a tensor $dF^*_p(\alpha) \in T^k(T^*_pM)$, called the pullback of α by F at p, by

$$dF^*_p(\alpha)(v_1, \cdots, v_k) = \alpha(dF_p(v_1), \cdots, dF_p(v_k))$$

for $v_1, \cdots, v_k \in T_pM$.

1.7 Submanifolds

Submanifolds are subsets of M preserving some features of its original manifold, such as local topology and smoothness. We can classify submanifolds in two main categories, the embedded and the immersed. Embedded submanifolds are the exactly image of their containing manifold since they inherit the subspace topology. They are modeled locally on linear subspaces of Euclidean spaces. On the other hand, immersed submanifolds are images of injective immersions and behave locally as embedded submanifolds. However, they do not preserve the global topology behavior of its containing manifold. We shall focus now on defining embedded submanifolds.

Let M be a smooth manifold. Thus, an *embedded submanifold of M* is a subset $S \subseteq M$ that is also a manifold in the subspace topology. The smooth structure of S is given by the inclusion map[4] $\iota : S \hookrightarrow M$ when it is an embedding map. Moreover, the difference $\dim M - \dim S$ is called the *codimension* of S in M and M is called the *ambient manifold*. For example, an *embedded hypersurface* is an embedded submanifold of codimension 1. The easiest manifolds to understand are those with condimension 0, since we just have to restrict the charts of M to S and the subspace topology follows naturally from one to another. These codimension 0 submanifolds are called *open submanifolds*.

We can think of tangent spaces of S at p as subspaces of T_pM. Since we have the smooth inclusion map ι, at each point $p \in S$ we have an injective linear map given by $d\iota_p : T_pS \to T_pM$. For any vector $v \in T_pS$, the image vector $\widetilde{v} = d\iota_p(v) \in T_pM$ acts on smooth functions on M by

$$\widetilde{v}f = d\iota_p(v)f = v(f \circ \iota) = v(f|_S),$$

where $f : M \to \mathbb{R}$ and $f|_S = S \to \mathbb{R}$. Thus, v acts over a restriction of f on S. Now T_pS is identified as its image under this map, that is, as a linear subspace of T_pM. Consequently T_pS is then characterized by

$$T_pS = \{v \in T_pM : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0\}.$$

[4] Inclusion maps are used when we want to formally say that every element of a subset is also an element of the set containing it.

In words, whenever we have a restricted function $f|_S = 0$ the tangent vector v must vanish when acted upon f. Otherwise if $vf = 0$ is identically zero, then exists a vector $w \in T_pS$ such as $v = d\iota_p(w)$ and T_pS is fully characterized. In the next chapter we use the notions of submanifolds to study hypersurfaces over curved spaces geometry. We will need to restrict functions and tensors of a ambient manifold M to a codimension 1 submanifold.

Let now M be an embedded submanifold of $\widetilde{M}$. Using our formal definition of submanifolds we have the inclusion map as $\iota : M \hookrightarrow \widetilde{M}$, and through ι any vector field $V \in TM$ assigns a tangent vector $V_p \in T_pM \subset T_p\widetilde{M}$. But each tangent space T_pM is, by definition, a nondegenerate subspace of $T_p\widetilde{M}$; therefore, we make the decomposition

$$T_p\widetilde{M} = T_pM \oplus N_pM,$$

where $N_pM := (T_pM)^{\perp}$ is the *normal space* with vectors normal to M and dimension equal to the codimension of M, for hypersurfaces $\dim N_pM = 1$. This decomposition permits every vector $v \in T_pM$ to be split in its tangent and normal part as

$$v = \mathrm{tang}(v) + \mathrm{norm}(v)$$

This result is important to understand how curvature and other quantities behave when "projected" into the submanifold. Tensors in $\widetilde{M}$ can also be decomposed, but note that they are formed by direct products of tangent spaces. Hence, this decomposition leads to nontrivial forms of "projected" tensors in M.

2 Geometry of Curved Spaces

Now that we settled the basic properties and mathematical results of manifolds and its inner structures, we are able to define lengths, angles and curvature, since these are the measurable quantities of relevance in physics. This chapter focus on mathematical description and definitions of important quantities which will be given physical meaning throughout the text.

First, we need to define a function that will give us the sense of inner product in each tangent space of M, because this is the primordial way to acquire lengths and angles. This function will be none other than the metric itself and here we will be dealing only with *pseudo-Riemannian metrics*. Specifically, the geometry of special and general relativities are created based on a type of pseudo-Riemannian metric called *Lorenztian metric*.

After metrics are defined, connections arise to introduce how we can find curves that resembles Euclidean straight lines over the manifold. That is, we want to find something similar to minimum length curves. And instead of using the distance property, it will be presented a rather general aspect of curves to do so. Such curves are called *geodesics* which play a key role for interpreting mathematically some principles of GR. Furthermore, we present *linear connections* that provides a bridge between metrics and derivatives and curvature thereafter.

Provided the metric and connection, we then apply some constraint to them in order to achieve their proper physical interpretations. Then we proceed to acquire a definition and mathematical expression for curvature and derivatives of tensor fields. Important quantities, such as Riemann and Ricci tensors, will be defined and analyzed. For more detailed discussions and definitions the reader is referred to [23, 24] which our study is based on.

2.1 Pseudo-Riemannian Metrics

A pseudo-Riemannian metric on a smooth manifold M is a 2-tensor field $g \in \Gamma(T^2T^*M)$ that is symmetric (i.e. $g(X,Y) = g(Y,X)$) and can assume any value in $\mathbb{R}$. A pseudo-Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written $\langle X,Y\rangle := g(X,Y)$. The manifold M equipped with this kind of metric is called a pseudo-Riemannian manifold (M,g).

We define lengths and angles similarly to Euclidean case. The length of a vector field X thus is defined $|X| := [g(X,X)]^{1/2}$. The angle between two nonzero vectors

$X, Y \in T_pM$ is a unique $\theta \in [0, \pi]$ satisfying $\cos\theta = g(X,Y)/(|X||Y|)$. Two vectors are said to be orthogonal when $\theta = \pi/2$ or equivalently $g(X,Y) = 0$. We also require g to be nondegenerate i.e. the only vector orthogonal to everything is the zero vector. Formally speaking, $g(X,Y) = 0$ for $Y \in T_pM$ if and only if $X = 0$.

The Riemannian geometry is the study of properties of Riemannian manifolds that are invariant under an special diffeomorphism: the isometry. A diffeomorphism ϕ between two pseudo-Riemannian manifolds (M, g) and $(\widetilde{M}, \tilde{g})$ is called *isometry* if $\phi^*\tilde{g} = g$. In this case the manifolds are said to be isometric. Compositions of isometries and the inverse of an isometry are again isometries. It can be shown that the set of isometries forms a group with a finite-dimensional Lie algebra [23].

If we have local coordinates (x^i) on M, the basis of cotagent space will be (dx^i) and the pseudo-Riemannian metric takes the form

$$g = g_{ij}dx^i \otimes dx^j,$$

where the coefficients are symmetric $g_{ij} = g_{ji} = g(x^i, x^j)$. It is common to shorten the notation introducing the symmetric product of two 1-forms ω and η

$$\omega\eta = \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega),$$

and with the symmetry of the coefficients g_{ij} we have

$$g = g_{ij}dx^i dx^j.$$

A important aspect of pseudo-Riemannian metrics is that they allows us to convert vectors to covectors and vice-versa. Given g on M we can define a map from TM to T^*M by sending a vector X to the covector ξ defined by

$$\xi(Y) := g(X, Y).$$

In coordinates

$$\xi = g(X^i\partial_i, \cdot) = g_{ij}X^i dx^j,$$

and we can identify $\xi = X_j dx^j$, where $X_j = g_{ij}X^i$. This is just the operation of lowering indices. Analogously, we can *raise indices* by considering the action of $g^{ij} = (g_{ij})^{-1}$ on a vector ω_i sending it to a covector ω^j, given by

$$\omega^j = g^{ij}\omega_i.$$

Using index notation becomes easy to extrapolate the inner product to tensors. If we have smooth coordinates (x^i) over the bundle TM and (dx^i) for its dual T^*M, then any tensor fields $F, G \in \Gamma(T^{(n,k)}TM)$ can be described on these coordinates. The metric for the vector bundle TM is also a metric for tensor bundles acting as an inner product on each fiber of them. Thus, the inner product for two tensor, in indices, is given by

$$g(F,G) = g^{i_1 r_1} \cdots g^{i_k r_k} g_{j_1 s_1} \cdots g_{j_n s_n} F^{j_1 \cdots j_n}_{i_1 \cdots i_k} G^{s_1 \cdots s_n}_{r_1 \cdots r_k}.$$

We can also extract or create new vectors and tensors from other tensors. This would be given by acting g not for each fiber of the tensor bundle, but only in a few of them. As an example (in indices again), considering a tensor $T \in \Gamma(T^3(TM))$ we create a vector as

$$g_{ij} T^{ijk} = T_i{}^{ik} := X^k.$$

On any oriented manifold (M, g) there is a unique n-form dV satisfying the property that $dV(E_1, \cdots, E_n) = 1$ whenever $(E_1, \cdots, E_n)$ is an oriented orthonormal basis for some T_pM. We call this form the volume element. Using coordinates (x^i) we write dV as[1]

$$dV = \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n. \tag{2.1}$$

Now we are able to perform integrals over functions on M. Considering a smooth function f its integral would be $\int_M f dV$.

One example of pseudo-Riemannian manifold is the 4-dimensional Minkowski spacetime $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ where it has a product topology of two Euclidean spaces. A local frame on this space is often written (t, x, y, z) and its metric assume the form

$$g = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.2}$$

which configures a pseudo-Riemannian metric because of the minus sign on dt^2.

2.2 Linear Connections

We want now to study geodesics, which are the generalization of straight lines on curved spaces. However, it is of some difficulty to study them as minimum length curves such as in Euclidean case. Because of that we use another property of Euclidean straight lines in order to extend this concept to a pseudo-Riemannian manifold. A Curve in Euclidean space is a straight line if and only if its acceleration is identically zero. Thus, connections are introduced as tools that will give us an invariant interpretation of the acceleration of a curve.

To understand what is acceleration on M we have to analyze it more deeply. Consider a curve $\gamma : (a, b) \to M$, then the velocity vector $\dot{\gamma}(t)$ has a coordinate-independent

[1] Formally, this construction is given by the wedge product of the Grassman Algebra. But here it suffices to say that for a dual basis (dx^i) is given in this way.

meaning for each $t \in M$. On a coordinate system we would have $\dot{\gamma}(t) = (\dot{\gamma}^1(t), \cdots, \dot{\gamma}^n(t))$. It is tempting trying to create the acceleration vector by taking the second derivative of $\gamma(t)$. However, this process turns out to be not invariant for different coordinate systems [23].

Actually, the most appropriate way would be to take a difference quotient between two vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(t_0)$; but these two vector are in different tangent spaces, namely $T_{\gamma(t_0)}M$ and $T_{\gamma(t)}M$, and does not make sense to subtract them. The differentiation $\dot{\gamma}(t)$ is an example of vector field along a curve and we need a coordinate-invariant way of differentiate these vector fields. To do so, we have to "connect" nearby tangent spaces on the curve and compare its values in different points.

We proceed to define formally a linear connection on M. Let $X, Y \in TM$ be vector fields on M. Hence, A linear connection on M is a connection in TM, i.e., a map

$$\nabla : TM \times TM \to TM$$

written $(X, Y) \to \nabla_X Y$ satisfying the properties:

(a) $\nabla_X Y$ is linear over $C^\infty(M)$ in X:

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y, \quad \text{for} \quad f, g \in C^\infty(M).$$

(b) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2 \quad \text{for} \quad a, b \in \mathbb{R}.$$

(c) ∇ satisfies the following product rule:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y \quad \text{for} \quad f \in C^\infty(M).$$

The symbol $\nabla_X Y$ is called the *covariant derivative of Y in the direction of X*. Linear connections are often called *affine connections*. Although the definition above resembles that of a tensor of rank $(2, 1)$, it it not a tensor field because it is not linear over $C^\infty(M)$ in Y, but instead satisfy the product rule.

Next step is to determine the components of ∇. Let (∂_i) be a local coordinate frame for TM. For a general indices choice we take the action of ∇ over ∂_i yielding

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

This defines n^3 functions Γ^k_{ij} called the *Christoffel symbols* of ∇. Also in this frame we have $X = X^i\partial_i$ and $Y = Y^j\partial_j$ for $X, Y \in TM$, then

$$\nabla_X Y = (XY^k + X^iY^j\Gamma^k_{ij})\partial_k.$$

By definition, the connection can also be used to compute derivatives of tensor fields. Since the derivative $\nabla_X Y$ is linear over $C^\infty(M)$ in X, it can be used to create another tensor field called the total derivative. Consider a tensor field $F \in \Gamma(T^{n,k}TM)$, the 1-forms (ω^k) and vector fields (Y^n), the map $\nabla F : TM \times \cdots \times TM \times T^*M \times \cdots \times T^*M \to C^\infty(M)$, given by

$$\nabla F(\omega^1, \ldots, \omega^k, Y^1, \ldots, Y^n, X) = \nabla_X F(\omega^1, \ldots, \omega^k, Y^1, \ldots, Y^n),$$

defines a $(n+1,k)$-tensor. Considering a local frame, the total covariant derivative of a vector $Y = Y^i \partial_i$ can be written as

$$\nabla Y = \nabla_j Y^i \partial_i \otimes dx^j,$$

with

$$\nabla_j Y^i = \partial_j Y^i + Y^k \Gamma^i_{jk}.$$

For any (n,k)-tensor field F, the components of the total covariant derivative can be expressed in coordinates by its components as

$$\nabla_l F^{j_1 \cdots j_n}_{i_1 \cdots i_k} = \partial_l F^{j_1 \cdots j_n}_{i_1 \cdots i_k} + \sum_{s=1}^{n} F^{j_1 \cdots p \cdots j_n}_{i_1 \cdots i_k} \Gamma^{j_s}_{lp} - \sum_{s=1}^{k} F^{j_1 \cdots j_n}_{i_1 \cdots p \cdots i_k} \Gamma^{p}_{l i_s}.$$

Since we are dealing with pseudo-Riemannian manifolds, we want to understand now the relation between the metric and the connection. First we say that the connection ∇ is *compatible* with g, for any vector fields X, Y and Z, if it satisfies the following product rule:

$$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

This is equivalent to say that $\nabla g \equiv 0$. Another important feature is to require that our connection is symmetric. This involves the torsion tensor of the connection $\tau : TM \times TM \to TM$ defined by

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X + [X,Y],$$

where $[X,Y]$ is the commutator. A symmetric connection thus is that with null torsion, i.e., $\tau(X,Y) = 0$. An important result arise from symmetry that is: For the (pseudo-)Riemannian manifold (M,g) the symmetric and compatible connection is unique and is called the Levi-Civita connection[2]. In coordinates it appears as a symmetry on the lower indices of Christoffel symbols , i.e., $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Yet in this context, the last important feature of the Levi-Civita connection is that it is uniquely determined by the metric tensor. In coordinates, as usual, the statement is represented by the equation

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{2.3}$$

[2] This is the fundamental Lemma of pseudo-Riemannian manifolds [23].

With connections we can create covariant derivatives along curves. Consider an interval $I \subset \mathbb{R}$ and let $\gamma : I \to M$ be a curve. We can always choose a parameter $t \in I$ to invariantly define a velocity $\dot{\gamma}(t)$ of γ as the push-forward $\gamma_*(\mathrm{d}/\mathrm{d}t)$. It act on functions by

$$\dot{\gamma}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t)$$

and, in coordinates, have the components $\dot{\gamma}(t) = \dot{\gamma}^i(t)\partial_i$. Now to attach vector fields along curves we take a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$. Thus, we have a vector $V(t)$ on each tangent space centered in points $\gamma(t)$.

For each curve $\gamma(t)$, the connection ∇ will provide us with a unique derivative operator $D_t : \mathscr{T}(\gamma) \to \mathscr{T}(\gamma)$ which takes the directional derivative of vector fields along curves. Here $\mathscr{T}(\gamma)$ denotes the space of vector fields along curve γ. This operator is a covariant derivative and has the following properties:

(a2) Linearity over $\mathbb{R}$:

$$D_t(aV + bW) = aD_tV + bD_tW, \quad \text{for } a, b \in \mathbb{R}.$$

(b2) Product rule:

$$D_t(fV) = \dot{f}V + fD_tV \quad \text{for } f \in C^\infty(I).$$

Choosing coordinates, the action of D_t on a vector $V(t) = V^j(t)\partial_j$ in a small neighborhood $(t_0 - \epsilon, t_0 + \epsilon)$ is given by the components

$$D_tV(t_0) = \left[\dot{V}^k(t_0) + V^j(t_0)\dot{\gamma}^i(t_0)\Gamma^k_{ij}(\gamma(t_0))\right]\partial_k.$$

2.3 Parallel Transport and Geodesics

We shall define now what is the acceleration of curves and geodesics in the following. Considering our manifold M equipped with the connection ∇ and a curve γ, we say that the *acceleration* of γ is the vector given by $D_t\dot{\gamma}$. A geodesic with respect to ∇ is thus a curve γ with null acceleration $D_t\dot{\gamma} \equiv 0$. If we apply the operator D_t using a coordinate system as above we find the geodesic equation. Now we choose conveniently $\gamma(t) = (x^1(t), \cdots, x^n, (t))$ which lead us to the equation

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(x(t)) = 0. \tag{2.4}$$

This is a set of n second-order differential equations. Sometimes it is useful to separate them into $2n$ first-order equations by identifying $\dot{x}^k = v^k$. Thus, we have

$$\begin{aligned}\dot{x}^k(t) &= v^k(t)\\ \dot{v}^k(t) &= -v^i(t)v^j(t)\Gamma^k_{ij}(x(t)).\end{aligned}$$

Another construction associated with covariant derivatives along curves is the parallel transport. We define parallelism along a curve γ by the action of D_t on a vector field V when $D_t V \equiv 0$. Therefore, a geodesic can be described as a curve whose velocity vector field is parallel along the curve. We say that a vector field is *parallel* when its total covariant derivative ∇V vanishes identically. That is, a vector field is parallel if it is parallel along every curve on M. The equation that characterizes parallel transport is then:

$$\dot{V}^k(t_0) + V^j(t_0)\dot{\gamma}^i(t_0)\Gamma^k_{ij}(\gamma(t_0)) = 0.$$

Note that the second term is the responsible for maintaining the vector direction (see Fig. 3). The connection "connects" V to the perpendicular vector $\dot{\gamma}^i(t_0)$ for each t_0. Thus, curvature in this case is precisely represented by this product.

Geodesics and parallel transport have some interesting mathematical properties such as uniqueness and existence [23]. These features provide a powerful machinery to describe physical systems because they guarantee invariance of the equations for geodesics and parallelism. Hence, once chosen a coordinate system we do not have to worry about the form of the operators since they will be invariant. Our work is briefly resumed on finding the relevant physical quantities (momentum, positions, interaction fields, etc.) and study how they behave when affected by operators.

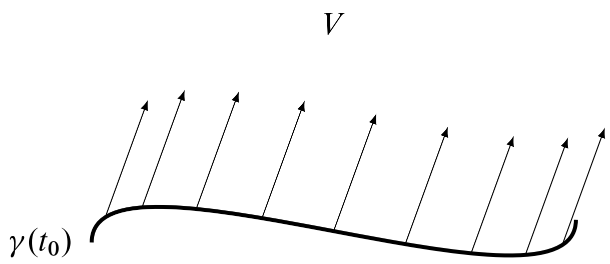


Figure 3 – Parallel transport of vector V along $\gamma(t_0)$.

2.4 Curvature

We know very well that partial derivatives commute on $\mathbb{R}^n$ and for covariant derivatives in $\mathbb{R}^n$ wouldn't be different. In fact, if we apply covariant derivatives on any vector field Z in Euclidean space, given two coordinates systems (∂_1) and (∂_2) we would "discover" that

$$\nabla_{\partial_1}\nabla_{\partial_2}Z - \nabla_{\partial_2}\nabla_{\partial_1}Z = 0.$$

This is nothing too far from our intuition. However, if we push our reasoning a little further and substitute the vector fields (∂_1) and (∂_2) for two more general ones, say X and Y, the condition then becomes

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z. \tag{2.5}$$

Since we treat Euclidean space as a flat one, the condition above would be respected by any general manifold which is isometric to Euclidean space. We call the last equation the *flatness criterion*.

The expression (2.5) gives a glimpse of what curvature means. It may be interpreted as the failure of covariant derivatives to commute. Also, there is the fact that manifolds with curvature are not isometric to Euclidean space, thus meaning that we need to provide some new quantity that locally measures curvature. We achieve this by defining a $(3,1)$-tensor $\mathbf{Riem} : \mathscr{T}(M) \times \mathscr{T}(M) \times \mathscr{T}(M) \to \mathscr{T}(M)$ as

$$\mathbf{Riem}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

called the *Riemann curvature tensor*. In a local frame (x^i) we write the Riemann tensor as $(3,1)$-tensor field. Thus, we have

$$\mathbf{Riem} = R_{ijk}{}^{l} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l, \tag{2.6}$$

and we can also create a 4-tensor acting the metrics on **Riem** to lower the upper index. On general vector fields it would be

$$Rm(X,Y,Z,W) = g(R(X,Y)Z,W),$$

and in coordinates

$$Rm = R_{ijkl}\ dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

with

$$R^k{}_{ijl} = \partial_j \Gamma^k{}_{li} - \partial_l \Gamma^k{}_{ij} + \Gamma^k{}_{jn} \Gamma^n{}_{il} - \Gamma^k{}_{ln} \Gamma^n{}_{ji} \tag{2.7}$$

and $R_{ijkl} = g_{lm} R_{ijk}{}^{m}$. From its definition, the Riemann tensor has very useful symmetries that we write down on components in the following:

(a) $R_{ijkl} = -R_{jikl}$,

(b) $R_{ijkl} = -R_{ijlk}$,

(c) $R_{ijkl} = R_{klij}$,

(d) $R_{ijkl} + R_{jkil} + R_{kijl} = 0$.

The property (d) is the famous *Bianchi identity* or sometimes called the first Bianchi identity. The *second Bianchi identity* is increased with covariant derivatives, that is

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0.$$

There still two more important quantities we can extract from Riemann tensor, namely the Ricci and scalar curvatures. These quantities are created by taking the traces of ${R_{ijk}}^l$. The Ricci curvature is a 2-tensor field often denoted in the literature by Rc and with component R_{ij} given by

$$R_{ij} := {R_{kij}}^k = g^{km} R_{kijm},$$

the components are explicitly written in terms of the Christoffel symbol as

$$R_{ij} = \partial_k {\Gamma^k}_{ij} - \partial_j {\Gamma^k}_{ik} + {\Gamma^k}_{kl} {\Gamma^l}_{ij} - {\Gamma^k}_{jl} {\Gamma^l}_{ki}. \tag{2.8}$$

The scalar curvature is the function R defined as the trace of the Ricci tensor:

$$R := g^{ij} R_{ij} = {R_i}^i$$

Now, it follows from symmetry properties of Riemann tensor and Ricci's definition that

$$R_{ij} = R_{ji} = {{R_{ik}}^k}_j = -{{R_{ki}}^k}_j = -{R_{ikj}}^k.$$

A good exercise is to analyze how the second Bianchi identity behaves when we contract the indices i, l and, then again on j, k after raising one index of each pair. What we obtain is the following equation:

$$\nabla^j R_{ij} = \frac{1}{2} \nabla_i R. \tag{2.9}$$

We call a metric g *Einstenian metric* when its Ricci tensor is proportional to g at every point for some function λ, that is $R_{ij} = \lambda g_{ij}$. Taking the traces on both sides we obtain

$$R = \lambda g^{ij} g_{ij} = n\lambda$$

where $n = \dim M$. Thus, the Einstein condition is rewritten as

$$R_{ij} = \frac{1}{n} R g_{ij}.$$

Now taking the covariant derivative of this equation and further tracing indices j and k we obtain:

$$\nabla_k R_{ij} = \frac{1}{n} g_{ij} \nabla_k R$$
$$\nabla^j R_{ij} = \frac{1}{n} \nabla_i R,$$

Compare this result with the identity (2.9) and we find

$$\frac{1}{n}\nabla_i R = \frac{1}{2}\nabla_i R.$$

This proves that for manifolds with $\dim M \geq 3$ its scalar curvature must obey $\nabla_i R = 0$. In fact, for Einstenian metrics, the Bianchi identity leads us to the equation:

$$R_{ij} - \frac{1}{2}R g_{ij} = 0, \tag{2.10}$$

which is precisely GR vacuum equation.

2.5 Curvature on Hypersurfaces

Consider an embedded submanifold (M, g) and its ambient pseudo-Riemannian manifold $(\widetilde{M}, \tilde{g})$. Recalling our definitions we then have the inclusion embedding $\iota : M \to \widetilde{M}$. Our first task is to relate the metric $\tilde{g}$ to g which can be achieved by pulling back g with respet to ι, or simply

$$g := \iota^* \tilde{g}.$$

In this case ι is an *isometric embedding* and M is a pseudo-Riemannian submanifold. The metric g is called the *induced metric* and can be expressed in coordinates if we define a set of coordinates x^μ on $\widetilde{M}$ and y^i on M. In instance, we create a hypersurface by setting one of the functions x^i as constant, say $z = cte \in x^i$. Hence, ι is the embedding such that

$$\iota : y^i \to x^\mu = (z, y^i)$$

and the induced metric becomes

$$g_{ij} = (\iota^* \tilde{g})_{ij} = \frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}\tilde{g}_{\mu\nu}. \tag{2.11}$$

With z given, it is possible to define normal vectors by taking its covariant derivative $\xi^\mu := \tilde{g}^{\mu\nu}\nabla_\nu z$. If ξ^μ is spacelike, the hypersurface is said to be timelike; if ξ^μ is timelike the hypersurface is spacelike, and if ξ^μ is null the hypersurface is also null. We can define a normalized normal vector as

$$n^\mu = \pm \frac{\xi^\mu}{\left|\xi^\mu \xi_\mu\right|^{1/2}}.$$

.

Then $n^\mu n_\mu = \pm 1 \equiv \sigma$, where $\sigma = +1$ stands for n^μ spacelike and $\sigma = -1$ for timelike. We choose the sign σ usually to make n^μ be future-oriented. Using this definitions in (2.11) we achieve the *first fundamental form* of pseudo-Riemannian hypersurfaces

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \sigma n_\mu n_\nu. \tag{2.12}$$

This form is useful as a projection operator, if we want to project any quantity of $\widetilde{M}$ on M we shall use it. Note that the normal vector n^μ have dimension one since hypersurfaces have codimension one, then we represent it in M as a vector with one label equal one and the rest as zero, e.g. for 4 dimensional M: $n_\mu = (1, 0, 0, 0)$.

There remains other two main quantities to be studied. The obvious one is the connection of the hypersurface which will lead to Riemann tensor. As connections take its values on $T\widetilde{M}$ we can split it in normal and tangent parts. Consider the general vector fields $X, Y \in \mathscr{T}(M)$, then we write

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^\top + (\widetilde{\nabla}_X Y)^\perp,$$

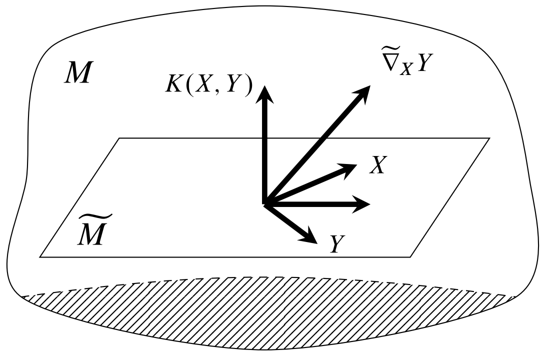


Figure 4 – Decomposition of $K(X, Y)$.

where the symbol $\top$ stands for tangential projection. We define the bilinear form $K(X, Y) : \mathscr{T}(M) \times \mathscr{T}(M) \to \mathscr{N}(M)$ as the normal projection (see Fig. 4)

$$K(X, Y) := (\widetilde{\nabla}_X Y)^\perp.$$

Sometimes $K(X, Y)$ is called the *second fundamental form* and will be our definition of *extrinsic curvature*. The symmetry of $K(X, Y)$ follows directly from the connection symmetry

$$K(X, Y) - K(Y, X) = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^\perp = [X, Y]^\perp$$

and in our standard case, where $X, Y \equiv \partial_i$, we have $[X, Y]^\perp = 0$ leading to a symmetric extrinsic curvature $K(X, Y) = K(Y, X)$. Establishing the hypersurface by the coordinates $x^\mu = (z, y^i)$ the extrinsic curvature can be written in terms of the normal vector n^μ as

$$K_{\mu\nu} = \nabla_\mu n_\nu - \sigma n_\mu a_\nu \tag{2.13}$$

where $a_\mu = n^\nu \nabla_\nu n_\mu$ is the acceleration of n_μ measured by a geodesic. If n^μ is defined to always sit in a geodesic, then $a_\mu = 0$.

The tangential part of the decomposition is determined by the Gauss formula. If we define $(\widetilde{\nabla}_X Y)^\top : \mathscr{T}(M) \times \mathscr{T}(M) \to \mathscr{T}(M)$, then it follows that this tangential connection is also a connection of M and we simply acquire $\widetilde{\nabla}^{\top} = \nabla$. Consequently, our decomposition is rewritten as

$$\widetilde{\nabla}_X Y = \nabla_X Y + K(X, Y).$$

Riemann tensor for hypersurfaces is now obtained applying the non-commutation condition on the decomposition and we obtain Gauss equation, for any $X, Y, W, Z \in T_p M$:

$$\begin{aligned} \mathbf{Riem}(X, Y, Z, W) =& \widetilde{\mathbf{Riem}}(X, Y, Z, W) \\ &+ g(K(X, W), K(Y, Z)) - g(K(X, Z), K(Y, W)) \end{aligned} \tag{2.14}$$

and inserting the usual coordinate system on the ambient manifold, Gauss equation becomes

$$R^{\varepsilon}{}_{\sigma\mu\nu} = g^{\varepsilon}{}_{\alpha} g^{\beta}{}_{\sigma} g^{\gamma}{}_{\mu} g^{\delta}{}_{\nu} \widetilde{R}^{\alpha}{}_{\beta\gamma\delta} + \sigma(K^{\varepsilon}{}_{\mu} K_{\sigma\nu} - K^{\varepsilon}{}_{\nu} K_{\sigma\mu}). \tag{2.15}$$

Ricci tensor and curvature scalar are also transformed, as an example the curvature scalar turns out to be

$$R = \widetilde{R} + \sigma(2\widetilde{R}_{\mu\nu} n^{\mu} n^{\nu} + K^2 - K^{\mu\nu} K_{\mu\nu}), \tag{2.16}$$

where $K = K^{\mu}{}_{\mu}$. We also have Codacci's equation closing the set of projection equations

$$\nabla_{\mu} K_{\nu}{}^{\mu} - \nabla_{\nu} K = \frac{1}{2} g^{\sigma}{}_{\nu} \widetilde{R}_{\varepsilon\sigma} n^{\varepsilon}. \tag{2.17}$$

Finally, we learn that curvature on hypersurfaces is described by two main tensors: its Riemann intrinsic tensor $\mathbf{Riem}(X, Y, W, Z)$ and the extrinsic curvature tensor $K(X, Y)$. The former is a inner measure of the hypersurface curvature such as described in the previous section. The second is an additional structure measuring the curvature of M related to its ambient manifold structure, which is new information and modifies our interpretation of the total curvature of space. This will change our conception of GR spacetime and, of course, its field equations once we will be dealing with extra dimensions.

3 General Relativity

As physics evolved in the late 1800s and early 1900s some interesting ideas and theories emerged to correct and evolve their predecessors. Yet that Maxwell electromagnetism and Newtonian gravity contemplated the experimental data of that time, they had significant issues to be dealt with. Problems as the asymmetry of Maxwell equations for observers in motion [3] and Newton's absolute spacetime were put on trial. Special Relativity and GR are among these new theories whose final results revealed new aspects of nature and had shaken the structures of its contemporary physics. All of this commotion is due to the great effort of names such as A. Einstein, E. Mach, H. Poincaré, among others, to revisit the philosophical foundation and its consequences on the mathematical description of physics. On this scope, several questions were raised: What is inertia and how is it measured? Is spacetime really absolute? Do the dynamical equations need to be the same for any observer? Depending on the answer given to any of these questions what kind of theory would then give the proper physical description of phenomena?

The idea that any physical equation must be equal to all (inertial) observers is something always taken for granted and is worth some discussion. Since it seems rather intuitive and a simple quote, we do not give much attention to it. However, nature is not anthropic and may not respect our common logic and senses. In order to measure and understand it we must assure that observations for distinct spacetime points will be equivalent. This statement is true only if all inertial frames (observers) are said to be equivalent, not allowing preferred observers and thus preserving physics' laws invariance. In fact, Einstein formulated it as a *relativity principle* in response to the ether puzzle raised by the Michelson-Morley null experimental result about same subject. Accepting the equivalence of inertial frames, aside with the knowledge that light has the same speed for any observer, led to the creation of special relativity. All this thinking was mathematically sustained by the Lorentz transformations which make Maxwell equations invariant for observers in motion.

Despite the advances, Newton's gravitational theory still was the best theory for gravity description. This changed when incompatibilities were found between the relativity principle and absolute space. The Newtonian picture defines inertia over an absolute spacetime background which makes one choose preferred observers, whilst the principle of relativity asserts the contrary. Moreover, for physical laws to be invariant we must be able to construct the inertial frame according to a given problem. All of the mystery lies around of why Descartes' absolute conception of spacetime should be the recipe followed in the physics framework. Now add into this the fact that Newtonian gravity allows instantaneous exchange of interaction and a serious a problem is created in what concerns causality.

There was hope, however, that if absolute spacetime was abandoned it could give place to new physics. But what kind of mathematical structure would then do the job properly? In fact we have already studied it and was where Einstein's genius had struck fiercely. By abdicating absolute spacetime and combining Riemannian geometry together with a bunch of principles, he was able to produce what we know as General Relativity.

In the following we shall summarize some of the most important principles that underlie GR. One should always have them in mind and their understanding is the primary requirement for those who want to search new theories in gravitation framework. Explanations on the principles and important discussions are given in [1, 25, 26, 27]. A beautiful and pictorial discussion is also given in [28]. Yet, we give a quick introduction to the mathematical results of GR based on its action since its one of the most modern approaches to the theory.

3.1 Principles

General Relativity was conceived in a turmoil of new ideas and built upon what we understand as three separate principles: general covariance, Mach's principle and the Equivalence principle. These were the philosophical keys that guided Einstein in his search for a new theory of gravity. At the time there were concern about Newton's conception of inertia and absolute space was being abolished through the results of Special Relativity, since absolute spacetime is unable to achieve total equivalence between inertial frames.

Mach's principle

The term "Mach's principle" was coined by Einstein for the whole complex of ideas written by Mach [29]. Mach claimed that the inertia of a given object is due to some interaction (not specified) with all other masses present in the universe and that mechanics is built upon the relative motion of the object to all these masses. It is a simple idea to conceive philosophically, but not so mathematically.

There actually is a great difference when one transfers the standard "measure" of inertia from absolute space to all masses in the universe. If we have for example a spinning spherical elastic body it would bulge at its equator. For Newton, the body would only know that it is spinning because it "felts" the action of absolute space, due to centrifugal forces, and hence it bulges. But for Mach space is not a "thing" in its own right, it is merely a relational quantity between two objects and thus the body knows it is spinning by the action of its surrounding masses in the universe. To absorb this idea mathematically into a theory Einstein not only promoted spacetime to a dynamical field, but also assumed that gravitational mass and inertial mass sould be equivalent since now gravity would be the cause of inertia. Thus, the spacetime field, namely the metric tensor, would have to encompass every gravitational source we can imagine, if usage of the equivalence principle was not made previously.

Despite Einstein effort, doubt still remains if GR fully contemplates Mach's principle. Some believe that GR equation partially covers the problem of defining inertial frames [4]. Such discussion raised alternative theories of GR in the expectation to fully contemplate Mach's principle into a physical theory. Among these ideas we can cite Brans and Dicke creation of a very distinct scalar-tensor theory for gravity [30]. The primary idea is to insert a new scalar field into GR Lagrangian with a coupling parameter ω which encodes Mach's conception of inertia and see what would result from that. One interesting feature is that GR field equation is recovered when $\omega \to \infty$ [31]. However, the dynamics derived from such equations become increasingly difficult to understand.

General Covariance

General Covariance is very similar to the relativity principle but it is mathematically more sophisticated. Its primary definition is that the laws, and equations thereafter, of physics are invariant under any kind of diffeomorphism. Because of that, the principle is also referred as diffeomorphism invariance. This formulation also reveals a deeper feature of the principle, which is nature's independence of coordinated systems. Indeed, nature does not have any intrinsic or preferred choice of coordinated system, it is merely a tool to describe physical interactions. The direct consequence of this reasoning is that we get a new appropriated mathematical tool for physics: tensors!

Remember that when Einstein showed up with this principle the tensor formalism was a fresh newly mathematical development. Thus, taking tensors as ideal physical quantities was a great step towards a more concise mathematical formulation of equations. One great advance and an example of elegance and concision was achieved in electromagnetic theory worked out in tensor language. Moreover, the power of this mathematical tool becomes evident when we prove that equations of general gravitational fields hold in the absence of matter sources, that is, we must recover special relativity in this limit. Such thing is achieved by replacing the metric tensor for Minkowski spacetime flat metric.

Using tensors means that only the quantities which really "pertains to space" should appear in the laws of physics. But the metric is the only structure known to satisfy this requirement. There is not any vector field or basis preferable that pertains to space and, therefore, should not appear in the equations. Furthermore, the notion "pertains to space" is rather vague since it is not well defined. We can violate the principle by having a theory with a preferable vector v^a for which we should have to rewrite our equations to insert it, but for that we would take a specific coordinate system. After acquiring the new set of equations we would find that they are not invariant with respect to general coordinate transformations; therefore, not transforming according to (1.14). Nonetheless, our incapability to explicitly create a covariant equation having v_a means that we might not be fully incorporating the geometry of the problem.

A good way to understand this difficulty is to look at the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$. Since they do not transform as the tensor law (1.14), they could not appear directly in the equations of motion. However, if we look to (2.10) we can see the symbols appearing under derivatives. Indeed, they are attached to the structures ∂_α which pertains to space. Now this makes (2.10) covariant and hence respecting the principle.

Equivalence Principle

The equivalence principle is by far the most known principle, it states that the gravitational mass is equivalent to the inertial mass. It presents itself very simply but this one is very tricky. Here we have to pay attention to what we really want to say by inertial mass and gravitational mass. Which one is the body real mass and the dynamical mass? Let's try to shed light upon it.

We understand inertial mass as an assignment to a body's resistance towards motion. This follows directly from Newton's second law $\vec{F} = m_I\vec{a}$, where m_I is the inertial mass. Therefore, m_I is some kind of "inertia measure" when a given force acts on a body. On the other hand, gravitational mass is understood as the passive or active agent occurring in Newton's gravitational force equation

$$\vec{F} = \frac{Gm_G m'_G}{r^2}\hat{r},$$

where G is the gravitational constant. Experiments to measure the proportionality between m_I and m_G were done by Eötvös in 1922 and by Roll, Krotkov and Dicke (Princeton, 1964) showing that their values differ by a part in 10^9 in the former and by a part in 10^{11} in the later [25]. If m_G is then equivalent to m_I, then gravitation can be regarded as a "measure" of inertia and vice-versa. This kind of reasoning led Einstein to propose his famous mental elevator experiment where freely falling objects constitute inertial frames.

Intuitively and experimentally the equivalence principle might seem not a big deal. However, by assuming it as true, Einstein had determined that all local, freely falling, non-rotating frames are equivalent and thus, if the frames are at free fall or near large masses, they will be equivalent to describe any physical phenomena. Every body feels gravitation in exactly the same way when in presence of gravitational fields. Furthermore, these local frames are the same presented in special relativity which makes it a locally applicable theory.

The direct consequences of this idea are surprising. For the observer in free fall, gravity can be ruled out, where it will only be measured when the observer experiences acceleration. Equivalence Principle also implies that light bends in the presence of gravitational fields, since any local inertial frame is equivalent when experimenting gravity so does light when it free "falls" (propagate near a massive body). Moreover, this suggests that we discovered not a new property of light, but instead, a property of space: curvature. A second result is about the possibility of light suffer blueshift.

3.2 Formal Development

We start the formal development of GR somewhat different of the usual literature (*e.g.* [1, 25, 26]). It is common to reach GR equations analyzing how the gravitational potential behaves and thus identifying its source as the metric tensor $g_{\mu\nu}$. Furthermore, using the mathematical apparatus of the previous chapters, one accomplishes Einstein's field equations as we did in Section 2.4. However, a more elegant approach is construct the gravitational field Lagrangian and then use the variational principle.

The search for a proper Lagrangian of the gravitational field was not an easy task. It is due to Hilbert [32] the analysis and discovery that the appropriate Lagrangian is the Ricci scalar. R is the simplest known scalar and depends on the second order derivative of $g_{\mu\nu}$[1] unravelling that we shall achieve a set of second order differential equations in the variational procedure. All this said we now consider a 4-dimensional pseudo-Riemannian smooth manifold M. Thus, the action of a physical system on M is given by the integral

$$S = \int_V \mathcal{L}\, dV \tag{3.1}$$

which, for gravitation, turns out to be

$$S[g] = \int R\sqrt{-g}\; d^4x \tag{3.2}$$

where dV is the same as in (2.1) with $g = \det\big(g_{\mu\nu}\big)$. Note that this is the gravitational field alone in the action. If we want to add matter fields, then we have to add another Lagrangian function, say $\mathcal{L}_m$, and we have

$$S[g] = \int \left(R\sqrt{-g} + \mathcal{L}_m\right) d^4x. \tag{3.3}$$

There are two ways to vary the action above. The first is to consider $g_{\mu\nu}$ as the only independent variable and then make the variation as $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. On the other hand, one can make use of the Palatini procedure which considers the connection $\Gamma^{\lambda}{}_{\mu\nu}$ an independent variable as well as the metric [33]. In this case, the variations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and $\Gamma^{\varepsilon}_{\mu\nu} \to \Gamma^{\varepsilon}_{\mu\nu} + \delta\Gamma^{\varepsilon}_{\mu\nu}$ are made simultaneously. Anyway, both procedures reach the same result which is nothing else than Einstein's field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3.4}$$

[1] See equation (2.3), and take the trace of (2.8).

where $T_{\mu\nu}$ is the energy-momentum tensor defined as

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}. \tag{3.5}$$

We know from §2 that if we act a covariant derivative on (3.4) the left side will vanish due to Bianchi identity. Hence, for the right side we just get

$$\nabla^{\mu} T_{\mu\nu} = 0,$$

which is an explicit appearance of the energy-momentum conservation in the theory. In the cosmological scenario, one commonly adds the cosmological constant Λ to the action to study the different possibilities of the universe fate. Thus, the "complete" action becomes

$$S[g] = \int \left[(R+\Lambda)\sqrt{-g} + \mathcal{L}_m\right] d^4x. \tag{3.6}$$

yielding the field equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3.7}$$

Equation (3.4) represents a set of 10 second-order non-homogeneous and nonlinear PDEs instead of 16 since all tensors are symmetric on their indices. In order to solve them, one needs to specify a form for the energy-momentum tensor and a geometry to be analyzed. Quite after the theory's release, the first solutions were achieved considering highly symmetric geometries, such as Schwarzschild vacuum solution given by the line element

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{3.8}$$

where m is the mass of the object being studied. Other solutions were constructed along the time, even for cosmological scenarios enlarging this branch of physics. For this case the FLRW (Friedmann-Lemaître-Robertson-Walker) line element gives a solution of (3.4):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right) \tag{3.9}$$

where $a(t)$ is known as scale factor and k a constant representing curvature.

The last important aspect of Einstein equation is its weak field limit. In this case we consider a flat gravitational field plus small fluctuations functions $h^{\mu\nu}(x^{\alpha})$, that is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{3.10}$$

Note that we do not have any strong constraint to characterize $h_{\mu\nu}$ as small since there is no positive definite metric in this space and no natural "norm" for the metric tensor.

Although there is not a formal constraint, we can establish that all functions satisfies $h_{\mu\nu} \ll 1$ individually.

The procedure is summarized as using (3.10) into (3.4) left side and extract only the terms that are linear in $h_{\mu\nu}$. It is sometimes referred as linearized gravity. Consequently, we recover the equation

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = 16\pi T_{\mu\nu}, \tag{3.11}$$

where we defined

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{3.12}$$

with $h \equiv h^\alpha{}_\alpha$. The result obtained is very remarkable, where we can recover Newton's gravity by the existence of a global inertial frame such that $T_{\mu\nu} \approx \varepsilon t_\mu t_\nu$ with t_μ the time directions, whereas it also leads to graviton interpretation in the vacuum $T_{\mu\nu} = 0$. For a discussion of the Newtonian limit see *e.g.* [26].

The graviton interpretation was achieved by Pauli and Fierz [34] when they were analyzing possible equations representing particles with spin greater than 1 on electromagnetic theory. They realized that equations for spin 2 particles resemble (3.11) in the vacuum including a gauge freedom in respect to coordinate transformation. We shall see how (3.11) helps when interpreting gravitons behavior in higher dimensional spaces.

3.3 Tolman-Oppenheimer-Volkoff Equations

The Tolman-Oppenheimer-Volkoff (TOV) is a system of equations describing the dynamics of perfect fluids[2] in gravitational fields with the use of (3.4). The physical problem was first analyzed by R. Tolman and then the equations were developed by J. Oppenheimer and G. Volkoff [35]. The equations describe hydrodynamics in isotropic, spherical symmetric spacetime, thus becoming a convenient tool to study gravitational star structure and phenomenology. The system is acquired when we consider an isotropic perfect fluid whose energy tensor has the form:

$$T_{\mu\nu} = (\varepsilon + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{3.13}$$

with u_μ the unitary 4-velocity, i.e., $u_\mu u^\mu = -1$, $p = p(r)$ the pressure and $\varepsilon = \varepsilon(r)$ the energy density. Spacetimes that are spherical, symmetric and static can be described by a rather general metric tensor given by

$$\mathrm{d}s^2 = -e^{2\Phi(r)}\mathrm{d}t^2 + e^{2\Lambda(r)}\mathrm{d}r^2 + r^2\big[\mathrm{d}\phi^2 + \sin^2\phi\,\mathrm{d}\theta^2\big]. \tag{3.14}$$

[2] A fluid is considered perfect when it does not presente viscosity, sheer stress and heat conduction.

Substituting these tensors in (3.4) we get the so-called TOV system of equations:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon, \tag{3.15}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -(\varepsilon + p)\frac{\mathrm{d}\Phi}{\mathrm{d}r}, \tag{3.16}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m + 4\pi r^3 p}{r(r-2m)}, \tag{3.17}$$

with $m(r)$ the function defined to be the mass of the object being studied. Here $\Phi(r)$ is known as the gravitational potential. The function $\Lambda(r)$ is uniquely determined as

$$e^{2\Lambda(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1}. \tag{3.18}$$

A step-by-step demonstration of this result can be found in [12]. Note that this result resembles the Schwarzschild metric (3.8) which is one important aspect explored in Hartle's perturbation method.

The equations are to be solved once given some initial conditions. One integrates outwards of the spherical structures beginning at $r = 0$ where is considered that $m(0) = 0$ and a initial central energy density, $\varepsilon_c \equiv \varepsilon(0)$, is given. To properly solve the system one must choose an equation of state $\varepsilon \equiv \varepsilon(p)$ which describes the desired inner stellar structure. Finally, the integration is finished when the pressure reaches to zero at some max radius R, that is, $p(R) = 0$. Thus, configuring the star final form with radius R and final mass $M \equiv m(R)$.

4 Brane World Gravity

As a classical theory, General Relativity breaks down at high enough energies suggesting it should be replaced by its quantum version. But to find a quantum theory of gravity is still an open problem once our quantum description of phenomena seems incompatible with GR. Essentially, the advanced quantum field theory is built upon a Minkowski spacetime, the space of Special Relativity, while GR promotes spacetime to be the agent of gravitational field itself. So how do we construct the base and proper Hilbert space for the theory? Geometry might seem the proper answer. Indeed, the two main research programs that have risen to attack this problem are based on geometrical analysis of nature, they are: The canonical and covariant [6] programs. The canonical approach successfully led to Loop Quantum Gravity while the covariant developed M-theory. In the covariant program, M-theory and String theory are known to demand extra dimensions and also try to unify the fundamental interactions.

In the 1980s to the 1990s there were at least five distinct $1 + 9$-dimensional superstring theories, all leading to quantum theories of gravity. They are all related via duality transformations with the $(1 + 10)$-dimensional supergravity, a feature that wasn't discovered until the mid-1990s with the proposal of the AdS/CFT correspondence by Maldacena [36]. This proposition unravel that theories living in Anti-de Sitter spaces (AdS) can be related to Conformal Field Theories (CFT) via specific dualities. Soon after, supergravity as well as superstring theories begun to be treated as different limits of a single theory which is now called M-theory. At low energies M-theory behaves like supergravity.

Brane World gravity have rise in the framework of String Theory and M-theory as a possible effective theory for gravity in $(4 + d)$ dimensions projected 4 dimensions, where d are the extra dimensions. Here the 4-dimensional Planck scale M_P is no longer the fundamental scale, it is actually replaced by M_{4+d}. We can see that from the modified gravitational potential in the weak field limit $g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$ where it becomes:

$$V(r) \propto \frac{8\pi G_{4+d}}{r^{1+d}},$$

with $G_{4+d} = 1/M_{4+d}^{2+d}$ is the new gravitational constant. The change of fundamental scales provides an explanation to the hierarchy problem[1] in the Brane-world scenario. Since gravitational field is the geometry of spacetime itself (recall the metric tensor used in the last chapter), it propagates in all possible dimensions of the theory, namely bulk, whilst the other fundamental interactions are confined in a hypersurface structure called Brane.

[1] This one is about why gravity is by far the weaker of the 4 fundamental forces and how it became so.

Therefore, as gravity spreads across entire space, it becomes weaker than other interactions as can be seen in the potential expression [37].

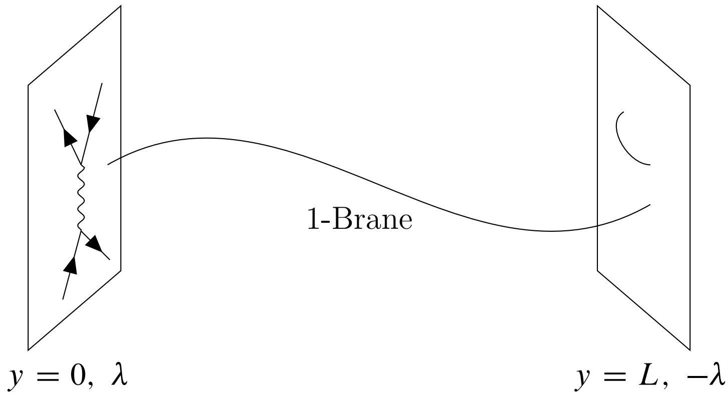


Figure 5 – Illustration of SM interactions in Brane World.

If we take L as a length scale[2] for the extra dimensions and assume $r > L$ thus the potential is not affected by the extra dimensions and behaves like the 4-dimensional potential. On the other hand, if $r < L$ we have $V \sim r^{-(1+d)}$ as above, and for $r \sim L$ the potential is just $V \sim L^d r^{-1}$. Hence, the usual Planck scale will be an effective coupling constant related to the fundamental scale of any given theory by the volume of the extra dimensions:

$$M_P^2 \sim M_{4+d}^{2+d} L^d .$$

As stated before, Branes can confine the usual SM field interactions. In the String theory framework, the p-Branes[3], responsible for encapsulating the Standard Model fields, are connected by open strings (1-Branes) whose end points attached to p-Branes (illustrated in Fig. 5). Gravity propagates through strings and radiative/matter fields through p-Branes. Considering this, Horava and Witten proposed a solution [38] where gauge fields of SM are confined on two $(1+9)$-Branes located at the end points of an S^1/Z_2[4] orbifold, i.e., a circle folded on itself across diameter. They showed that the 10-dimensional $E_8 \times E_8$[5] heterotic string is related to the 11-dimensional orbifold $\mathbb{R}^{10} \times S^1/Z_2$. The 6 extra dimensions are felt by their behavior as $5D$ scalar fields since the dimensions are compactified on a very small scale close to the fundamental one. But the Horava-Witten solution can also be worked out in 5 dimensions [39] which turn up to be an effective theory of 5 dimensions, simplifying its former version.

This completes the lore about Branes that provided the foundation for Randall and Sundrum to propose an alternative to compactification [8]. Using a 2-Brane they modeled

[2] The length scale in units $\hbar = c = 1$ is given by $L = \sqrt{G}$. In the case considered here it is $L = \sqrt{G_{4+d}}$.
[3] P-dimensional Branes.
[4] Z_2 is the symmetry group, the mirror symmetry.
[5] E_8 represents any exceptional simple Lie group or Lie algebra with dimension 248. Therefore, the vectors of its root system are immersed in 8-dimensional Euclidean space.

a 5D gravity where the extra dimension scale is taken to be infinitely large. Although the scale is infinite, it is treated as "warped" or curved in opposition to its predecessor model, the Arkani-Dimopoulos-Dvali (ADD) models where extra dimensions are flat [40].

The Randall-Sundrum (RS) analysis is separeted in two types: the RS1 which considers 2 Branes and the RS2 having only one Brane. The first attempts to solve the hierarchy problem by considering two Branes positioned in the extra dimension y at $y = 0$ and $y = L$, with Z_2-symmetry considered. One of them has positive tension λ while the other is negative, where

$$\lambda = \frac{3M_P^2}{4\pi l^2},$$

and l is the AdS_5 curvature radius. The Brane with negative tension is "visible" meaning that SM is confined to it, while its partner is "hidden" with fundamental scale M_5. The scales are related by

$$M_P^2 = M_5^3\left(e^{-2L/l} - 1\right),$$

and hence we can recover $M_P \sim 10^{16}$ TeV by choosing L/l large enough even if $M_5 \sim l^{-1} \sim$ TeV.

The second RS model can be achieved by sending the negative tension Brane of RS1 to infinity, $L \to \infty$. Thus, additional structure is not required, making it more simple and geometrical appealing. It is also used for studying Ads/CFT correspondence because the space of the model is AdS and the extra dimension is not compactified. All the subsequent results presented in this work are based on this model, so in the next section we provide a discussion about it and after we present its covariant analysis.

4.1 Randall-Sundrum Alternative

We need to go back for a while and analyze the graviton behavior in higher dimensional spacetime. This consists on studying an analog of (3.10) and analyze the possibility of bound states for the graviton. If such modes exists we call them KK modes, named after Kaluza-Klein theory. Let us consider a trivial extension of GR into a 5D spacetime in an inertial frame with coordinates (x^μ, z) whose metric field is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + dz^2.$$

Hence, our action for vacuum gravity in this system is written as

$$S = \int \widetilde{R}\sqrt{-\tilde{g}}\ \mathrm{d}^4x\mathrm{d}z, \tag{4.1}$$

which yields the field equation:

$$\widetilde{R}_{AB} - \frac{1}{2}\widetilde{R}\tilde{g}_{AB} = 0. \tag{4.2}$$

By applying the weak field perturbation, $\tilde{g}_{AB} \to \eta_{AB} + h_{AB}(x^\mu, z)$ we get a linear equation

$$(\partial_\mu \partial^\mu + \partial_4 \partial^4 - V(z))\bar{h}_{AB}(x^\mu, z) = 0 \tag{4.3}$$

where $V(z)$ is a non-trivial "potential" arising from the curvature. We now rewrite the general fluctuations as a superposition of modes $\bar{h}(z) = e^{ip\cdot x}\bar{\psi}(z)$, where the functions $\bar{\psi}$ are eigenmodes satisfying the equation:

$$(\partial_4 \partial^4 - V(z))\bar{\psi} = -m^2 \bar{\psi} \tag{4.4}$$

and $p^2 = m^2$. The last equation is analog to a non-relativistic quantum mechanical problem which means that it may have a zero-mode, which is guaranteed if the background preserves four-dimensional Poincaré invariance. This procedure thus implements KK-reduction of the five-dimensional fluctuations in terms of the 4D KK-modes, with eigenvalues m^2. Additionally, compactified theories usually have the zero-mode followed by a tower of KK-modes separated by a gap. The 4D gravity is reproduced up to scale determined by this gap proportional to graviton mass. But it can be modified by adding Branes to the action used before a take one of them to infinity.

In the following we shall consider a gravity action summed up with two Branes actions. In first approach the Branes are suited to be the boundary of the finite fifth dimension in order to produce the right quantization of the set-up. After that, we remove one of the Branes by taking it to infinity. The action of the system then is

$$\begin{aligned} S &= S_{\text{grav}} + S_{\text{b1}} + S_{\text{b2}}, \\ S_{\text{grav}} &= \int (-\tilde{\Lambda} + 2M_5^3 \tilde{R})\sqrt{-\tilde{g}}\ \mathrm{d}^4x\mathrm{d}z, \\ S_{\text{b}} &= \int (\Lambda_{\text{b}} + \mathcal{L}_b)\sqrt{-g}\ \mathrm{d}^4x. \end{aligned} \tag{4.5}$$

where M_5 is the fundamental scale the five-dimensional space.

Although we already specified the Branes in the boundary, we need to define the limits for z such that this spacetime will not fill the entire space. We parameterize z as an angular variable with periodicity, identifying the boundaries as (x^μ, z) and $(x^\mu, -z)$ which characterizes a S^1/Z_2 space. Moreover, the parameter is taken in the range $z \in [-\pi, \pi]$ whereas the metric is completely specified in the range $z \in [0, \pi]$. Thus, the Branes are positioned at $z = 0$ and $z = \pi$, meaning that bulk's metric, $\tilde{g}$, spacetime and 4D, g, are related by

$$g_{\mu\nu}^{\text{b1}} \equiv \tilde{g}_{\mu\nu}^{\text{b1}}(x^\beta, z = 0) \qquad \text{and} \qquad g_{\mu\nu}^{\text{b2}} \equiv \tilde{g}_{\mu\nu}^{\text{b2}}(x^\beta, z = \pi).$$

All settled we can finally see what equation the action (4.5) will provide, which is:

$$\sqrt{-\tilde{g}}(\tilde{R}_{AB} - \frac{1}{2}\tilde{g}_{AB}\tilde{R}) = \frac{1}{4M_5^3}[\tilde{\Lambda}\tilde{g}_{AB}\sqrt{-\tilde{g}} + \Lambda_{\mathrm{b1}}g^{\mathrm{b1}}_{\mu\nu}\delta^\mu_A\delta^\nu_B\delta(z)\sqrt{-g_{\mathrm{b1}}} + \Lambda_{\mathrm{b2}}g^{\mathrm{b2}}_{\mu\nu}\delta^\mu_A\delta^\nu_B\delta(z-\pi)\sqrt{-g_{\mathrm{b2}}}. \tag{4.6}$$

Assuming there exists a solution and it satisfies Poincaré invariance, it have been showed [37] that a metric respecting this ansatz takes the form

$$\mathrm{d}s^2 = e^{-2\sigma(z)}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + r_c^2\mathrm{d}z^2. \tag{4.7}$$

Here r_c is the "compactification radius" independent of z and acts as a proportionality in the extra dimension circle prior to orbifolding. After orbifolding the interval in z will be πr_c. Now using the ansatz in (4.6), it reduces to:

$$\frac{6\sigma'^2}{r_c^2} = -\frac{\tilde{\Lambda}}{4M_5^3}, \tag{4.8}$$

$$\frac{3\sigma''}{r_c^2} = \frac{\Lambda_{\mathrm{b1}}}{4M_5^3 r_c}\delta(z) + \frac{\Lambda_{\mathrm{b2}}}{4M_5^3 r_c}\delta(z-\pi). \tag{4.9}$$

A solution for (4.8) consistent with Z_2 symmetry is given by

$$\sigma = r_c|z|\sqrt{\frac{-\tilde{\Lambda}}{24M_5^3}}, \tag{4.10}$$

and is assumed from now on that $\tilde{\Lambda} < 0$ in order to the solution make any physical sense. This is not absurd since the space is AdS and has negative curvature constant. For the equation (4.9) we recall the fact that z lies in $[-\pi, \pi]$ and use the previous solution to construct

$$\sigma'' = 2r_c\sqrt{\frac{-\tilde{\Lambda}}{24M_5^3}}[\delta(z) - \delta(z-\pi)]. \tag{4.11}$$

Therefore, we can see that the only viable solution in conformity with (4.9) is given when Λ_{b1} and Λ_{b2} are related in terms of a single scale k,

$$\Lambda_{\mathrm{b1}} = -\Lambda_{\mathrm{b2}} = 24M_5^3 k = -\tilde{\Lambda}k. \tag{4.12}$$

The final solution for the bulk corresponds to the metric:

$$\mathrm{d}s^2 = e^{-2kr_c|z|}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + r_c^2\mathrm{d}z^2, \tag{4.13}$$

and r_c is effectively an arbitrary integration constant. In contrast with compactified product spaces, where the Placnk scale is related to M_5 by $M_P^2 = M_5^3\pi r_c$, we get another relation in the RS2 which gives a clue that we might have a good effective theory in four-dimensions.

To see that, we substitute the Minkowski metric in (4.13) by a 4D non-flat metric, $\bar{g}_{\mu\nu}$, and using it in the action (4.5) we get an effective action

$$S_{eff} \supset \int\int_0^{\pi} 2M_5^3 r_c e^{-2kr_c|z|}\bar{R}\sqrt{-\bar{g}}\ \mathrm{d}z\mathrm{d}^4x. \tag{4.14}$$

As the integrand has simple functions of z and we have a effective 4-dimensional field we can perform the integral in z to get an effective 4D action. From this one derives

$$M_P^2 = 2r_c M_5^3 \int_0^{\pi} e^{-2kr_c|z|}\mathrm{d}z = \frac{M_5^3}{k}\Big[1 - e^{-2kr_c\pi}\Big]. \tag{4.15}$$

Hence, in the limit $r_c \to \infty$ we recover a finite Planck mass, whereas in compactified theories M_p is not preserved in this limit. Taking the limit of r_c to infinity is the same as removing the second Brane from the set-up which configures the RS2 model. We shall assume now that the second Brane has gone to infinity, thus all the subsequent analysis relies in the remaining Brane.

Now we turn our attention to how the graviton is bounded to the Brane. We already know how to acquire the weak field limit, but in this case we do general perturbations in the "warped" metric, instead of the flat one. That is, in 5D the generalized perturbations are of the form: $\tilde{g}_{\mu\nu} \to e^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu}(x^\beta, y)$, with the redefinition $y \equiv zr_c$. Here the gauge chosen is $\partial^\mu h_{\mu\nu} = h^\mu_\mu = 0$. As before, we separate variables $h(x,y) = \psi(y)e^{ip\cdot x}$, where $p^2 = m^2$ and the eigenfunctions are calculated via the linearized equation

$$\Big[-\frac{m^2}{2}e^{2k|y|} - \frac{1}{2}\partial_y^2 - 2k\delta(y) + 2k^2\Big]\psi(y) = 0. \tag{4.16}$$

Only even solutions are considered because of the conditions imposed at the boundary. The mass m is the 4D mass of the KK excitation. By a change of variables $w \equiv sgn(y)\big(e^{k|y|-1}\big)/k$ we give the eigenvalue equation a more familiar face

$$\Big[-\frac{1}{2}\partial_w^2 + V(z)\Big]\hat{\psi}(w) = m^2\hat{\psi}(w), \tag{4.17}$$

where $\hat{\psi}(w) = \psi(y)e^{k|y|/2}$ and

$$V(w) = \frac{15k^2}{8(k|w|+1)^2} - \frac{3k}{2}\delta(w) \tag{4.18}$$

At a first glance to the potential we can see that it supports a single normalizable state due to the presence of the δ-function. The remaining part produces a continuum spectrum of KK modes. The bound state is the massless graviton of the effective 4D theory. Furthermore, note that there is no gap in the theory, when we take the limit $|w| \to \infty$ in the continumm the potential falls off to zero and the modes become plane waves asymptotically. Yet, the continuum KK modes have all possible $m^2 > 0$. The exact solutions of (4.17) for the continuum are given in terms of Bessel functions for small m [6]:

$$\hat{\psi}_m \sim N_m(|w| + 1/k)^{1/2}\left[Y_2(m(|w| + 1/k)) + \frac{4k^2}{\pi m^2} J_2(m(|w| + 1/k))\right], \tag{4.19}$$

where N_m is a normalization constant. For large mw,

$$\begin{aligned} \sqrt{w} J_2(mw) &\sim \sqrt{\frac{2}{\pi m}} \cos\left(mw - \frac{5\pi}{4}\right), \\ \sqrt{w} Y_2(mw) &\sim \sqrt{\frac{2}{\pi m}} \sin\left(mw - \frac{5\pi}{4}\right). \end{aligned} \tag{4.20}$$

Once found the KK modes in the effective theory, one can compute the gravitational non-relativistic potential between two particles with masses m_1 and m_2. This means that we calculate the static potential generated by exchange of the zero-mode and continuum KK-mode propagators. Thus we have

$$V(r) \sim G\frac{m_1 m_2}{r} + \int_0^\infty \frac{G}{k} \frac{m_1 m_2 e^{-mr}}{r} \frac{m}{k} \, \mathrm{d}m \tag{4.21}$$

with G the gravitational constant[6]. The Yukawa exponential occurs to suppress the massive Green's functions for $m > 1/r$, and the extra power m/k arises from the continuum wavefunctions suppression at Brane position $z = 0$. The resulting effective potential is then:

$$V(r) = G\frac{m_1 m_2}{r}\left(1 + \frac{1}{r^2 k^2}\right). \tag{4.22}$$

We finally learned that Brane world gravity in a Randall-Sundrum model can produce effective gravity theory and how it does so. Therefore we are able to go on with the covariant formalism to search a set of equations that will give the effective 4-dimensional dynamics of gravity.

4.2 Covariant Description

The covariant analysis of Brane world gravity consists in a projection of the 5-dimensional equations into the 4-dimensional Brane space in a way which will resemble GR plus additional structures. This analysis was done first by Shiromizu, Maeda and Sasaki [9]. We now shall follow their steps into the subject.

We assume a 5-dimensional space as the bulk, a manifold $(M, \tilde{g}_{AB})$ embedding a 3-Brane $(V, g_{\mu\nu})$. The line element of the bulk is written as

$$\mathrm{d}s^2 = g_{\mu\nu}(x^\alpha, y)\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2. \tag{4.23}$$

[6] Recall that in units $\hbar = c = 1$ we have $M_P^2 = 1/G$.

The bulk and Brane metrics are related by the first fundamental form $g_{AB} = \tilde{g}_{AB} - n_A n_B$, which works as a projection operator. Here we extended the indices of $g_{\mu\nu}$ to 5D, where we have $g_{44} = 1 - 1 = 0$ and $g_{A4} = g_{4A} = 0$. In diagonal form we write $\tilde{g}_{AB} = (g_{\mu\nu}, 1)$ and $g_{AB} = (g_{\mu\nu}, 0)$.

The next step is to use Gauss and Codacci equations, respectively:

$$R^{\rho}{}_{\sigma\mu\nu} = g^{\rho}{}_{A} g^{B}{}_{\sigma} g^{C}{}_{\mu} g^{D}{}_{\nu} \widetilde{R}^{A}{}_{BCD} + K^{\rho}{}_{\mu} K_{\sigma\nu} - K^{\rho}{}_{\nu} K_{\sigma\mu}, \tag{4.24}$$

$$\nabla_{\mu} K_{\nu}{}^{\mu} - \nabla_{\nu} K = \frac{1}{2} g^{A}{}_{\nu} \widetilde{R}_{AB} n^{B}. \tag{4.25}$$

where the extrinsic curvature is given by $K_{\mu\nu} = g_{\mu}{}^{A} \widetilde{\nabla}_{A} n_{\nu}$ and K its trace. Taking the traces of (4.24) we produce the Ricci tensor and scalar as

$$R_{\mu\nu} = g^{A}_{\ \mu} g^{B}_{\ \nu} \widetilde{R}_{AB} - \widetilde{R}^{A}_{\ BCD} n_A n^C + K K_{\mu\nu} - K_{\mu}{}^{\alpha} K_{\nu\alpha}, \tag{4.26}$$

$$R = \widetilde{R} + 2\widetilde{R}_{AB} n^A n^B + K^2 - K^{\mu\nu} K_{\mu\nu}. \tag{4.27}$$

Now we reconstruct Einstein tensor $G_{\mu\nu}$ with these relations which can be seen as its projection. Thus, we have

$$\begin{aligned} G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \left(\widetilde{R}_{AB} - \frac{1}{2}\tilde{g}_{AB}\widetilde{R}\right) g^{A}_{\mu} g^{B}_{\nu} + \widetilde{R}_{AB} n^A n^B \\ + K K_{\mu\nu} - K_{\mu}{}^{\alpha} K_{\nu\alpha} - \frac{1}{2} g_{\mu\nu}(K^2 - K^{\alpha\beta} K_{\alpha\beta}) - \hat{\mathcal{E}}_{\mu\nu}, \end{aligned} \tag{4.28}$$

where we defined

$$\mathcal{E}_{\mu\nu} \equiv \widetilde{R}^{A}_{BCD} n_A n^C g^{B}{}_{\mu} g^{D}{}_{\nu}. \tag{4.29}$$

Let us consider the bulk gravity as given by the equation

$$\widetilde{R}_{AB} - \frac{1}{2}\tilde{g}_{AB}\widetilde{R} = -\widetilde{\Lambda}\tilde{g}_{AB} + \tilde{\kappa}^2 \widetilde{T}_{AB}, \tag{4.30}$$

with 5D energy-momentum tensor $\widetilde{T}_{AB}$ and $\widetilde{\Lambda}$ the bulk's cosmological constant. Together with the 5D Weyl curvature traceless tensor, built upon decomposition of Riemann tensor;

$$\widetilde{C}_{ABCD} = \widetilde{R}_{ABCD} - \frac{2}{3}\left(\tilde{g}_{A[C}\widetilde{R}_{D]B} - \tilde{g}_{B[C}\widetilde{R}_{D]A}\right) + \frac{1}{6}\tilde{g}_{A[C}\tilde{g}_{D]B}, \tag{4.31}$$

we rewrite Einstein projected equation as

$$\begin{aligned} G_{\mu\nu} = -\widetilde{\Lambda} g_{\mu\nu} + \frac{2\tilde{\kappa}^2}{3}\left[\widetilde{T}_{AB} g_{\mu}{}^{A} g_{\nu}{}^{B} + \left(\widetilde{T}_{AB} n^A n^B - \frac{1}{4}\widetilde{T}\right) g_{\mu\nu}\right] \\ + K K_{\mu\nu} - K_{\mu}{}^{\alpha} K_{\nu\alpha} - \frac{1}{2} g_{\mu\nu}(K^2 - K^{\alpha\beta} K_{\alpha\beta}) - \mathcal{E}_{\mu\nu}, \end{aligned} \tag{4.32}$$

with the bulk Weyl tensor projection

$$\mathcal{E}_{\mu\nu} = \widetilde{C}_{ABCD} n^C n^D g_{\mu}{}^{A} g_{\nu}{}^{B}, \tag{4.33}$$

along the normal $\mathcal{E}_{AB}n^A = 0$.

Some quantities are still blur to us at the moment, the extrinsic curvature and 5D energy tensor for example. A few definitions will do the work. We define the total energy-momentum tensor on the Brane

$$T^{'}_{\mu\nu} \equiv T_{\mu\nu} - \lambda g_{\mu\nu}, \tag{4.34}$$

with the parameter λ known as Brane tension. It can be thought as Brane's permeability towards gravitation interactions with the bulk. To include the idea that SM lives only in the Brane we change the 5D energy-momentum tensor to

$$\widetilde{T}_{AB} \rightarrow \widetilde{T}_{AB} + T^{'}_{\mu\nu}\delta(y);$$

with Brane contribution attributed by the delta. Hence, the Bulk dynamics is given by the equation

$$\tilde{g}_{AB} = -\widetilde{\Lambda}\tilde{g}_{AB} + \tilde{\kappa}^2\Big[\widetilde{T}_{AB} + T^{'}_{\mu\nu}\delta(y)\Big], \tag{4.35}$$

Now we produce the Israel-Darmois junction conditions [41] integrating along y from $y = -\epsilon$ to $y = \epsilon$ and taking the limit $\epsilon \rightarrow 0$. The conditions are then

$$g^{+}_{\mu\nu} - g^{-}_{\mu\nu} = 0 \tag{4.36}$$

$$K^{+}_{\mu\nu} - K^{-}_{\mu\nu} = -\tilde{\kappa}^2\left[T^{'}_{\mu\nu} - \frac{1}{3}T^{'}g_{\mu\nu}\right] \tag{4.37}$$

where minus and plus superscript specifies quantities from the left and right of the Brane position $y = 0$, respectively.

Here the use of Z_2-symmetry has a very interesting interpretation. When you approach the Brane from its left side and go through it, you emerge into a bulk that looks the same, but with normal reversed $n^A \rightarrow -n^A$. Thus, use of the symmetry in $K_{\mu\nu}$ definition leads to

$$K^{+}_{\mu\nu} = -K^{-}_{\mu\nu}, \tag{4.38}$$

implicating that we have a non-degenerate unique $K_{\mu\nu}$ tensor calculated via Israel-Darmois junction conditions

$$K_{\mu\nu} = -\frac{\tilde{\kappa}^2}{2}\left[T_{\mu\nu} + \frac{1}{3}(\lambda - T)g_{\mu\nu}\right]. \tag{4.39}$$

Finally we arrive at the induced field equations on the Brane by substituting (4.39) into (4.32):

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \frac{6\kappa^2}{\lambda}S_{\mu\nu} - \mathcal{E}_{\mu\nu} + \frac{4k^2}{\lambda}\mathcal{F}_{\mu\nu}, \tag{4.40}$$

which is a modified Einstein equation correcting GR with additional terms. The 4-dimensional constant κ is inherited from the bulk fundamental coupling constant, and the 4D cosmological constant, Λ, is balanced by 4D coupling constant and 5D cosmological constant

$$\kappa^2 \equiv \frac{1}{6}\lambda\tilde{\kappa}^4, \tag{4.41}$$

$$\Lambda = \frac{1}{2}(\widetilde{\Lambda} + \kappa^2\lambda). \tag{4.42}$$

For Einstein's equation corrections we have $S_{\mu\nu}$ that is quadratic in $T_{\mu\nu}$, the bulk Weyl projection $\mathcal{E}_{\mu\nu}$, and $F_{\mu\nu}$ the generalization of any bulk stress apart from the cosmological contant. They are given by

$$S_{\mu\nu} = \frac{1}{12}T_{\mu\nu}T - \frac{1}{4}T_{\mu\alpha}T^{\alpha}_{\nu} + \frac{1}{24}g_{\mu\nu}[3T_{\alpha\beta}T^{\alpha\beta} - T^2]; \tag{4.43}$$

$$\mathcal{F}_{\mu\nu} = \widetilde{T}_{AB}g_{\mu}{}^{A}g_{\nu}{}^{B} + \left(\widetilde{T}_{AB}n^{A}n^{B} - \frac{1}{4}\widetilde{T}\right)g_{\mu\nu} \tag{4.44}$$

Next step is to see how conservation equations behave in this scenario. This is where we use Codacci's equation (4.25) together with (4.30) and (4.39) yields

$$\nabla^{\nu}T_{\mu\nu} = -2\widetilde{T}_{AB}n^{A}g^{B}{}_{\mu}, \tag{4.45}$$

which means that, in general, the bulk and the Brane may exchange energy. To avoid this we assume from now on that:

$$\widetilde{T}_{AB} = 0 \quad \Rightarrow \quad \mathcal{F}_{\mu\nu} = 0, \tag{4.46}$$

recovering the standard conservation equation $\nabla^{\nu}T_{\mu\nu} = 0$ and simplifying the modified equation to

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \frac{6\kappa^2}{\lambda}S_{\mu\nu} - \mathcal{E}_{\mu\nu}, \tag{4.47}$$

By this assumption we have imposed that bulk and Brane interaction is strictly gravitational. If we apply the covariant derivative in (4.47) and use the conservation equations we obtain

$$\nabla^{\nu}\mathcal{E}_{\mu\nu} = \frac{6\kappa^2}{\lambda}\nabla^{\nu}S_{\mu\nu}, \tag{4.48}$$

evidencing qualitatively that 4D variations in matter-radiation on the Brane source KK modes.

The system of equations we found is not closed yet. We need to find the equations that will determine $\mathcal{E}_{\mu\nu}$ from the 5D Eisntein equation and Bianchi identity. This leads to the system

$$\mathscr{L}_n K_{\mu\nu} = K_{\mu\alpha} K^{\alpha}_{\ \nu} - \mathcal{E}_{\mu\nu} - \frac{1}{6}\widetilde{\Lambda} g_{\mu\nu}, \tag{4.49}$$

$$\begin{aligned}\mathscr{L}_n \mathcal{E}_{\mu\nu} &= \nabla^{\alpha}\mathcal{B}_{\alpha(\mu\nu)} + \frac{1}{6}\widetilde{\Lambda}(K_{\mu\nu} - g_{\mu\nu}K) + K^{\alpha\beta}R_{\mu\alpha\nu\beta} - K\mathcal{E}_{\mu\nu} \\ &\quad + 3K^{\alpha}_{\ (\mu}\mathcal{E}_{\nu)\alpha} + (K_{\mu\alpha}K_{\nu\beta} - K_{\alpha\beta}K_{\mu\nu})K^{\alpha\beta},\end{aligned} \tag{4.50}$$

$$\mathscr{L}_n \mathcal{B}_{\mu\nu\alpha} = -2\nabla_{[\mu}\mathcal{E}_{\nu]\alpha} + K_{\alpha}^{\ \beta}\mathcal{B}_{\mu\nu\beta} - 2\mathcal{B}_{\alpha\beta[\mu}K_{\nu]}^{\ \ \beta}, \tag{4.51}$$

$$\mathscr{L}_n R_{\mu\nu\alpha\beta} = -2R_{\mu\nu\gamma[\alpha}K^{\gamma}_{\beta]} - \nabla_{\mu}\mathcal{B}_{\alpha\beta\nu} + \nabla_{\mu}\mathcal{B}_{\beta\alpha\nu}. \tag{4.52}$$

with the symbol $\mathscr{L}_n$ referring to the Lie derivative along the normal n^{μ}. A brief discussion regarding Lie derivatives can be seen in appendix B. The symbol $\mathcal{B}_{\mu\nu\alpha}$ is called the "magnetic" part of the bulk Weyl tensor, counterpart of the "electric" part $\mathcal{E}_{\mu\nu}$, given by

$$\mathcal{B}_{\mu\nu\alpha} = g_{\mu}^{\ A} g_{\nu}^{\ B} g_{\alpha}^{\ C} C_{ABCD} n^{D}. \tag{4.53}$$

The system is to be solved subject to boundary condtions on the Brane [10].

Finally, we attach an observer to the Brane to study how bulk corrections over Einstein modified equation changes the observer perspective of the world. For general matter field, perfect fluids, scalar fields the 4D energy-momentum tensor acquires the form

$$T_{\mu\nu} = (\varepsilon + p)u_{\mu}u_{\nu} + p g_{\mu\nu}, \tag{4.54}$$

where for the sake of brevity we aren't considering anisotropic pressure terms. Here ε and p are respectively the fluid energy density and pressure. The 4-vector u_{μ} is the normalized observer's 4-velocity. From this we found $S_{\mu\nu}$ with a surprisingly simple form

$$S_{\mu\nu} = \frac{\varepsilon}{12}\big[2(\varepsilon + p)u_{\mu}u_{\nu} + (\varepsilon + 2p)g_{\mu\nu}\big]. \tag{4.55}$$

The last piece is the traceless $\mathcal{E}_{\mu\nu}$ which can be found by solving the system presented before. However, the system is pretty hard to solve and, actually, the observer on the Brane does not have any further information about the bulk. Indeed, it "feels" only gravitational modifications enforced by the bulk into the Brane. Therefore, we have to propose a form to Weyl tensor "electric" part and we choose a form similar to a perfect fluid

$$-\frac{1}{\kappa^2}\mathcal{E}_{\mu\nu} = \mathcal{U}\left[\frac{4}{3}u_{\mu}u_{\nu} + \frac{1}{3}g_{\mu\nu}\right] + q_{\mu}u_{\nu} + q_{\nu}u_{\mu} + \mathcal{P}_{\mu\nu}, \tag{4.56}$$

with $\mathcal{U}$ often called the "dark radiation" that incorporates the spin-0 mode of graviton. As we are interested in static and spherical symmetric objects in this work, we can neglect the momentum density, $q_{\mu} = 0$, giving then

$$-\frac{1}{\kappa^2}\mathcal{E}_{\mu\nu} = \mathcal{U}\left[\frac{4}{3}u_{\mu}u_{\nu} + \frac{1}{3}g_{\mu\nu}\right] + \mathcal{P}_{\mu\nu}, \tag{4.57}$$

remaining only the nonlocal anisotropic stress $\mathcal{P}_{\mu\nu}$, due to graviton excited modes, to be worked out. Again, with static spherical symmetric the anisotropic term obeys the conservation equation

$$\partial_0 \mathcal{P}_{\mu\nu} = 0. \tag{4.58}$$

Thus, a solution satisfying this equation is given by

$$\mathcal{P}_{\mu\nu} = \mathcal{P}\left[r_\mu r_\nu - \frac{1}{3}(u_\mu u_\nu + g_{\mu\nu})\right], \tag{4.59}$$

where r_ν is a radial unit vector and $\mathcal{P}$ a scalar.

We have found the basis to go further and describe stars in 5D Brane World theory. There are other analysis with equation (4.47) and symmetries. There are also great works for cosmological scenarios, collapse and black holes that can be found in [10, 42].

4.3 Stars in the Brane

Computing the solutions of the effective gravity equations is a very cumbersome task. However, due to the effort of focused researches we have interesting analysis to discuss in this subject. One can derive a TOV-like solution for the problem [11], or yet, find a minimal geometrical deformation (MGD) solution [43]. We shall focus on the TOV-like solution from now on.

Let us put all the remaining pieces together and see how they look like now. We must combine equations (4.54), (4.55), (4.57), (4.59) into (4.47). Thus, we get

$$G_{\mu\nu} = \kappa^2 T^{\text{eff}}_{\mu\nu} \equiv \kappa^2\left[(\varepsilon_{\text{eff}} + p_{\text{eff}})u_\mu u_\nu + p_{\text{eff}} g_{\mu\nu} + \frac{6}{\kappa^4\lambda}\mathcal{P}_{\mu\nu}\right] \tag{4.60}$$

where the effective energy density and effective pressure are respectively:

$$\varepsilon_{\text{eff}} \equiv \varepsilon\left(1 + \frac{\varepsilon}{2\lambda}\right) + \frac{6}{\kappa^4\lambda}\mathcal{U}, \tag{4.61}$$

$$p_{\text{eff}} \equiv p + \frac{\varepsilon}{2\lambda}(\varepsilon + 2p) + \frac{2}{\kappa^4\lambda}\mathcal{U}. \tag{4.62}$$

A few comments about this result are worth quoting. First, we regain GR theory in the limit $\lambda \to \infty$. That is, in the case when the Brane is very resistant towards bulk effects, only bounded 4-dimensional graviton modes will be measured. Yet this could be seen through (4.47), but here is more evident. As second point we note that Brane's tension has values constrained by phenomenological results. From Big Bang nucleosynthesis the constraint is $\lambda > 1$ MeV^4, while a much stronger bound is obtained from null results of sub-millimeter tests of Newton's Law which gives $\lambda > 10^8$ GeV^4 [44]. Lastly, local Bulk effects are imprinted by the term $S_{\mu\nu} \sim (T_{\mu\nu})^2/\lambda$ and are significant when at high energies $\varepsilon > \lambda$.

The careful reader may have noticed that in effective Brane gravity we need two more equations to complete this analysis. Note that we have GR usual quantities ε and p, but also new quantities to be taken on consideration, $\mathcal{U}$ and $\mathcal{P}$. Here enters two conservation equations that will do the job: $\nabla^\mu T_{\mu\nu} = 0$ and $\nabla^\mu T^{\text{eff}}_{\mu\nu} = 0$. The first is the usual one and the second appears from Bianchi identity that we commented in the last section. Opening the equation explicitly gives

$$\nabla_\mu p + (\varepsilon + p)a_\mu = 0 \tag{4.63}$$

$$\nabla_\mu \mathcal{U} + 4\mathcal{U}a_\mu + \nabla^\nu \mathcal{P}_{\mu\nu} = -\frac{\kappa^4}{2}(p+\varepsilon)\nabla_\mu \varepsilon \tag{4.64}$$

where 4_μ is the four-acceleration. Now we make use of the static spherical metric, the same as in the TOV case, which is

$$\mathrm{d}s^2 = -e^{2\phi(r)}\mathrm{d}t^2 + e^{2\Lambda(r)}\mathrm{d}r^2 + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right], \tag{4.65}$$

resulting in the system of equations, or the TOV in the Brane world gravity:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \varepsilon_{\text{eff}}, \tag{4.66}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -(\varepsilon + p)\frac{\mathrm{d}\phi}{\mathrm{d}r}, \tag{4.67}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{m + 4\pi r^3\left(p_{\text{eff}} + \frac{4\mathcal{P}}{\kappa^4\lambda}\right)}{r(r-2m)}, \tag{4.68}$$

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}r} + (4\mathcal{U} + 2\mathcal{P})\frac{\mathrm{d}\phi}{\mathrm{d}r} = -32\pi^2(\varepsilon + p)\frac{\mathrm{d}\varepsilon}{\mathrm{d}r} - 2\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}r} - \frac{6\mathcal{P}}{r}, \tag{4.69}$$

where the definition of the mass in the first equation generates a solution for $\Lambda(r)$ function in the metric, that is:

$$e^{2\Lambda(r)} = \left[1 - \frac{2m(r)}{r}\right]^{-1}. \tag{4.70}$$

In general, the region outside the star is characterized by:

$$\varepsilon = 0 = p, \quad \mathcal{U} = \mathcal{U}^+, \quad \mathcal{P} = \mathcal{P}^+, \tag{4.71}$$

so that ε_{eff} and p_{eff} do not vanish in the exterior. The Weyl bulk stresses are radiative since the energy momentum tensor is traceless, i.e., $\varepsilon_{\text{eff}} = 3p_{\text{eff}}$. To close the system of equations we should impose a condition in $\mathcal{U}$ and $\mathcal{P}$ as well as their exterior part. In a purely Schwarzschild case, for example, we would choose $\mathcal{U}^+ = 0 = \mathcal{P}^+$. As in GR case, to solve the system we integrate all equations outwards from star center with the same initial conditions for energy density and pressure. But now, we must also include a condition $\mathcal{U}(0) = 0$ and some relation between bulk pressure and radiative energy.

One proposition is a linear relation between $\mathcal{U}$ and $\mathcal{P}$, *i.e.* $\mathcal{P} = \alpha\mathcal{U}$ [45]. This assumption simplifies equation (4.69) to

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}r} = -\frac{2}{1+2\alpha}\left[16\pi^2(\varepsilon + p)\frac{\mathrm{d}\varepsilon}{\mathrm{d}r} + \frac{3\alpha}{r}\mathcal{U} + (2+\alpha)\mathcal{U}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right]. \tag{4.72}$$

We shall use this argument in our numerical analysis in §6 regarding the results found in [45].

It is possible to find a lower bound for Brane tension parameter λ [11]. Consider a scenario where bulk's terms are not present at the interior of the star, that is, $\mathcal{U}^- = \mathcal{P}^- = 0$. Hence, only the local high-energy terms $\sim (T_{\mu\nu})^2$ modifies GR uniform-density solutions. The inner mass function is

$$m = M\left[1 + \frac{3M}{8\pi\lambda R^3}\right]\left(\frac{r}{R}\right)^3, \tag{4.73}$$

where $M = 4\pi R^3\varepsilon/3$. Therefore, metric function $e^{2\Lambda(r)}$ for $r < R$ becomes

$$e^{2\Lambda(r)} = \frac{1}{\Delta(r)}, \tag{4.74}$$

giving the pressure a form of

$$\frac{p(r)}{\varepsilon} = \frac{[\Delta(r) - \Delta(R)](\lambda + \varepsilon)}{\lambda[3\Delta(R) - \Delta(r)] + \varepsilon[3\Delta(R) - 2\Delta(r)]}, \tag{4.75}$$

with

$$\Delta(r) = \left[1 - \frac{2M}{r}\left(\frac{r}{R}\right)^3\left\{1 + \frac{\varepsilon}{2\lambda}\right\}\right]^{1/2}. \tag{4.76}$$

The function $\Delta(r)$ must be real and this gives an astrophysical lower limit on λ for uniform stars given by

$$\lambda \geq \left(\frac{M}{R-2M}\right)\varepsilon, \tag{4.77}$$

which for a typical neutron star with $\varepsilon \sim 10^{15}$ g/cm^3, $M \sim 3 \times 10^{33}$ g and $R \sim 10$ km gives

$$\lambda > 10^{35}\ \text{dyn/cm}^2. \tag{4.78}$$

Below this limit stable neutron stars cannot exist on the Brane. In this scale, the quadratic deviations from GR are significant, but they decrease as λ grows, *i.e.*, when gravity propagation from bulk to Brane becomes "harder". For example, it was verified that for $\lambda \sim 10^{38}$ dyn/cm^2 the Brane stellar solutions are very close to the General Relativity ones [45].

5 Rotating Stars

So far we have seen that stars can be described by TOV equations representing the simplest possible scenario for these objects. Indeed, star dynamics is far richer than spherical static objects; they rotate, evolve, collapse and also have a vivid nucleosynthesis due to atomic fusion. Therefore, a more realistic treatment is needed if we want to get a better picture of reality, which can be achieved by considering rotational stars for example. Before the 1980s most neutron stars studies were done using the non-rotating formalism, but everything changed when the first milisecond pulsar was discovered in 1982 [46] redirecting the attention of scientific community into the subject.

Though rotation may look like a simple idea to conceive, in this case it changes drastically the structure equations of stars. One of its direct consequences is the star deformation due to the different rotation velocities, orthogonal to the radius r, along the polar coordinate θ. On second place, rotation stabilizes a star against gravitational collapse, meaning they can carry more mass in contrast with its non-rotating case. However, more mass means an alteration in spacetime geometry, modifying the metric tensor which now becomes dependent of star rotation. The last important feature is the dragging of inertial frames occasioned by rotation, which grants a non-diagonal term to the metric. All of this must be taken into account when performing the calculations as we will see in this chapter.

Yet still, rotation is not a freely tractable feature such as evidenced by one of its constraints: Kepler rotational frequency Ω_K. As the rotational frequency increases, the star becomes more and more oblate until the point it starts to lose mass (mass shedding), where gravity is not able to hold rapidly rotating matter anymore. The parameter Ω_K is responsible to encode the upper limit that frequency must reach in order to trigger mass shedding. In that sense our study can be restricted by Kepler frequency as actually follows in Hartle analysis, where slow rotation is considered, *i.e.*, $\Omega \ll \Omega_K$. There are several ways to calculate Ω_K [12], where the most simple is to use the Newtonian equation for balancing centrifugal and gravitational forces. For typical neutron stars one has Kepler frequency lying in the range 9.5 kHz $< \Omega_K <$ 15 kHz [12, 16].

The term "slow" is misleading since it was shown the treatment can also be used when frequencies close to Ω_K are considered [47]. When dealing with this frequency scale one must take into account a self-consistency equation between Ω and Ω_K arising from GR while integration of equations is being performed. Thus Ω_K must be replaced in every step as soon as Ω, for the present configuration, is calculated. Nonetheless, rotational analysis is not restricted to slow rotations and works were developed in the highly rotating regime, *e.g.*, [48, 49].

In the following we give a basic introduction to Hartle's method based on the pioneering works of Hartle [50, 51] and a very detailed discussion given by F. Weber in his book [15]. After that, we adapt the method as we apply it to Brane World star solution. We present the hypothesis taken and further calculations in this process of implementation. The structure equations for rotations are obtained, discussed and analyzed. We also highlight the differences that the Brane gravitational structure imprints in analogy with GR differential equations. Moreover, Highly rotating setups and stellar instabilities are not of our concern in this work, thus, all calculations were done considering slow configurations.

5.1 Hartle's Perturbative Method

We begin by outlining the form of the metric tensor in spherical coordinates for a stationary rotating, axially symmetric equilibrium configuration:

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + e^{2\psi}(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + e^{2\mu}\mathrm{d}\theta^2 + e^{2\beta}\mathrm{d}r^2, \tag{5.1}$$

where ω is the angular velocity of local inertial frames. Each of the functions ν, ψ, μ, and β depends on the radial coordinate, polar angle θ, and implicitly on the star angular velocity Ω. Moreover, ω is also a function of r, θ, and depends implicitly on Ω since the inertial frames velocities are related to the mass that surrounds them, which varies with Ω. Both star and frame velocities are of interest to define the relative velocity as

$$\bar{\omega}(r, \theta, \Omega) \equiv \Omega - \omega(r, \theta, \Omega). \tag{5.2}$$

This is the star velocity relative to the inertial frame of motion. Such quantity has great importance on understanding the rotational flow of the fluid inside the star. In GR the inertial frames are not at rest with respect to distant stars because of the fluid motion. The frames are dragged along star rotation, a phenomenon that does not happen on Newtonian gravitation.

Note that the metric functions do not depend on time and axial coordinate assuring the properties of stationary rotation and axial symmetry, respectively. Hence, the star does not evolve in time and its rotation occurs parallel to the axial plane. In fact, we can input this assumptions in a general way where the metric is to assume the form $g_{\mu\nu} = g_{\mu\nu}(r, \theta)$. Under transformations as $t \to t^{'} + f_t(r, \theta)$ and $\phi \to \phi^{'} + f_t(r, \theta)$, $g_{\mu\nu}$ remains a function of r and θ alone. If we appropriately choose f_t and f_ϕ we could always find a coordinate system which $g_{12} = g_{13} = 0$. Besides, if we require invariance under $t \to -t$ and $\phi \to -\phi$ we also obtain $g_{01} = g_{02}$ [52, 53]. Although all of this is present in (5.1), we see only from the general arguments that this metric is invariant under transformations

$$r = r(r', \theta'), \quad \theta = \theta(r', \theta'), \tag{5.3}$$

which guarantees that the element $g_{11}\,\mathrm{d}r^2 + g_{22}\,\mathrm{d}\theta^2$ can be put in the form $f(r,\theta)[\mathrm{d}r^2 + \mathrm{d}\theta^2]$.

The symmetry of reflection with respect to the axial plane of rotation ($\phi \to -\phi$) is required whenever one assumes the star does not radiate rotational energy in form of gravitational radiation. However, this can also be seen as a consequence of slow rotation instead of a requirement. An important aspect of our configuration is that Ω is taken as constant over time as well as throughout the stellar fluid. It, therefore, determines uniform rotation (rigid body rotation), which means the fluid rotates at same rate in all its extension. The uniform rotation is a very simple approximation for the behaviour of the fluid inside the complex star structure.

From the line element (5.1) the expressions for the four-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ are derived as:

$$\begin{aligned} u^t &= \frac{1}{\sqrt{e^{2\nu} - \bar{\omega}^2 e^{2\psi}}}, \quad u_t = -\frac{e^{2\nu} + \omega\bar{\omega}e^{2\psi}}{\sqrt{e^{2\nu} - \bar{\omega}^2 e^{2\psi}}} \\ u^\phi &= \frac{\Omega}{\sqrt{e^{2\nu} - \bar{\omega}^2 e^{2\psi}}}, \quad u_\phi = \frac{\bar{\omega}e^{2\psi}}{\sqrt{e^{2\nu} - \bar{\omega}^2 e^{2\psi}}} \\ & u^r = u^\theta = 0, \end{aligned} \tag{5.4}$$

where in this deduction we identify $\Omega \equiv \mathrm{d}\phi/\mathrm{d}t$. Note that the u^t and u^ϕ components are related by $u^\phi = \Omega u^t$. The vector u^μ thus satisfies the normalization condition $u^\mu u_\mu = -1$. The explicit form of the metric components, Christoffel symbols, and of the curvature tensor can be found at [15]. So far we have all the basic ingredients that appear on Einstein equation with the energy tensor of a perfect fluid, that is:

$$G_{\mu\nu} = 8\pi[(p+\varepsilon)u_\mu u_\nu + p g_{\mu\nu}], \tag{5.5}$$

with $g_{\mu\nu}$ given in (5.1).

The basic idea in Hartle's treatment is to perform perturbation of order $\mathcal{O}(\Omega^2)$ on the metric tensor, which assumes the Schwarzschild-like solution (3.14). The variation is a multipole expansion in powers of Ω since this is the parameter that characterizes rotational velocity. We should obtain a perturbed metric of the form:

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + e^{2\psi}(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + e^{2\mu}\mathrm{d}\theta^2 + e^{2\beta}\mathrm{d}r^2 + \mathcal{O}(\Omega^3). \tag{5.6}$$

The functions ν, ψ, μ, and β are expanded up to second order in Ω, while the frame dragging is only of first order. This is a direct consequence of metric behaviour under reversal in the direction of rotation ($\phi \to -\phi$) as well as time reversal ($t \to -t$) due to the symmetries adopted. The multipole expansion of the functions in the metric yields:

$$e^{2\nu} = e^{2\Phi}[1 + 2h(r, \theta, \Omega)], \tag{5.7}$$

$$e^{2\psi} = r^2 \sin^2\theta[1 + 2(v_2(r, \Omega) - h_2(r, \Omega))P_2(\cos\theta)], \tag{5.8}$$

$$e^{2\mu} = r^2[1 + 2(v_2(r, \Omega) - h_2(r, \Omega))P_2(\cos\theta)], \tag{5.9}$$

$$e^{2\beta} = e^{2\Lambda}\left[1 + \frac{2m(r, \theta, \Omega)}{r - 2M(r)}\right], \tag{5.10}$$

where we have introduced the second-order terms

$$h(r, \theta, \Omega) = h_0(r, \Omega) + h_2(r, \Omega)P_2(\cos\theta) + \cdots, \tag{5.11}$$

$$v(r, \theta, \Omega) = v_0(r, \Omega) + v_2(r, \Omega)P_2(\cos\theta) + \cdots, \tag{5.12}$$

$$m(r, \theta, \Omega) = m_0(r, \Omega) + m_2(r, \Omega)P_2(\cos\theta) + \cdots, \tag{5.13}$$

and $P_2(x) = (3x^2 - 1)/2$ denotes the second order Legendre Polynomial. $M(r)$ is the mass calculated at r in the non-rotation setup. We can simplify the metric using its invariance under transformations of the form $r \to f(r)$ guaranteeing that $v_0 = 0$.

As said before, we expect the star inner pressure and energy density differ from non-rotational configuration. The centrifugal forces distorts the structure of the star changing the pressure and energy density by amounts of ΔP and $\Delta\varepsilon$, respectively. Performing a multipole expansion up to second-order in the pressure gives:

$$\Delta p = (\varepsilon + p)(p_0 + p_2 P_2(\cos\theta)), \tag{5.14}$$

$$\Delta\varepsilon = \Delta p \frac{\partial\varepsilon}{\partial p}. \tag{5.15}$$

The total energy-momentum tensor, therefore, will be modified as

$$T_{\mu\nu} \equiv (\varepsilon + \Delta\varepsilon + p + \Delta p)u_\mu u_\nu + (p + \Delta p)g_{\mu\nu}. \tag{5.16}$$

Note that this expansion is a consequence of rotation and therefore is used *a posteriori* of the calculations. Not only the pressure changes, but also the radius of star is modified to:

$$r \to r + \xi(r, \theta) + \mathcal{O}(\Omega^4), \tag{5.17}$$

where the new function $\xi(r, \theta)$ encodes the deviations caused by rotation (see Fig. 6), also called deformation function. Its dependency on θ reflects the change on star surface to an oblate form. As customary now, the deformation function is expressed in the multipolar expansion as

$$\xi(r, \theta) = \xi_0(r) + \xi_2(r)P_2(\cos\theta). \tag{5.18}$$

In hydrostatic equilibrium we have that deformations are directly related to deviations in pressure and energy density of the fluid [50]; thus, Δp and ξ are related by:

$$\Delta p(r,\theta) = \xi(r,\theta)\frac{\mathrm{d}p}{\mathrm{d}r}, \tag{5.19}$$

$$(\varepsilon + p)p_0(r) = -\xi_0(r)\frac{\mathrm{d}p}{\mathrm{d}r} \quad l = 0, \tag{5.20}$$

$$(\varepsilon + p)p_2(r) = -\xi_2(r)\frac{\mathrm{d}p}{\mathrm{d}r} \quad l = 2. \tag{5.21}$$

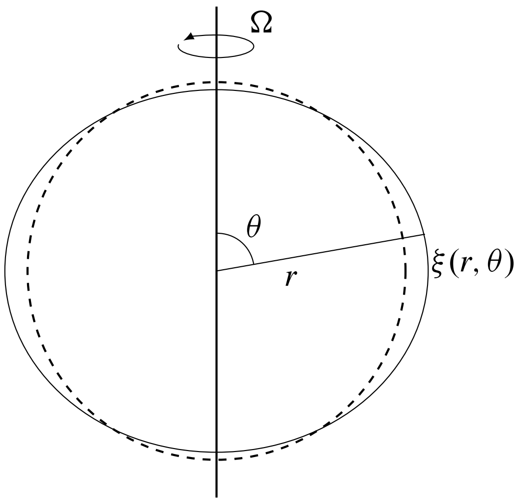


Figure 6 – Star deformation under rotational effects. The dashed surface represents the non-rotational profile.

Any changes in geometry of the problem should reflect on the left side of Einstein equation. It is, therefore, expected that we have deviations $\delta G_{\mu\nu}$ of the original Einstein tensor. These deviations are written in terms of the deformation functions as [50]:

$$\delta G_{\mu\nu}(r,\theta) = \xi\frac{\mathrm{d}}{\mathrm{d}r}[G_{\mu\nu}(r,\theta)]. \tag{5.22}$$

But, by using $G_{\mu\nu} = 8\pi T_{\mu\nu}$ we can rewrite this in a simpler manner, because it is easier to deal with $T_{\mu\nu}$ than with $G_{\mu\nu}$. We then get:

$$\delta G_{\mu\nu}(r,\theta) = 8\pi\xi(r,\theta)\frac{\mathrm{d}}{\mathrm{d}r}[T_{\mu\nu}(r,\theta)]. \tag{5.23}$$

When we have for example $\mu = \nu = t$ this result leads to

$$\delta G_t{}^t(r,\theta) = -8\pi\xi(r,\theta)\frac{\mathrm{d}\varepsilon}{\mathrm{d}r} = -8\pi\xi(r,\theta)\frac{\mathrm{d}\varepsilon}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}r} = -8\pi\Delta p\frac{\mathrm{d}\varepsilon}{\mathrm{d}p} \tag{5.24}$$

where we have used a one-parameter equation of state, *i.e.*, $\varepsilon = \varepsilon(p)$ and relation (5.19). This last line reveals the equivalence between working the corrections on the left-hand side of Einstein equation and its right-hand side. The last reasoning was taken from [50] where

Hartle works through a change of coordinated systems, something that we do not do here because of the equivalence highlighted before, which makes the subsequent calculations simpler. The final Einstein tensor to be calculated in second order is then

$$\Delta G_{\mu}{}^{\nu} = G_{\mu}{}^{\nu} + 8\pi\xi(r,\theta)\frac{\mathrm{d}}{\mathrm{d}r}[T_{\mu}{}^{\nu}(r,\theta)]. \tag{5.25}$$

Our problem is now resumed in computing the expanded functions via Einstein equation (5.5) with left-hand side (5.25). We have eight quantities to be found: p_0, p_2, h_0, h_2, m_0, m_2 and v_2 aside with $\bar{\omega}$. Hence, at least 8 equations must be found, but for our luck some of the functions are related to each other. The methodology to find them is a repetitive process, summarized as follows:

- The equation (5.5) is preferrably put in its mixed form, that is, $G_{\mu}{}^{\nu} = 8\pi T_{\mu}{}^{\nu}$;
- First we introduce the expanded metric (5.6) in Einstein tensor $G_{\mu\nu}$ which gives the first term of (5.25), they are listed at appendix A. The modification $\delta G^{\mu\nu}(r,\theta)$ is easily calculated and summed to $G_{\mu\nu}$, then completing (5.25);
- Next we compute the terms of the energy-momentum tensor by making use of the four velocity components(5.4) and separating its terms according to Ω order. Note here we do not use the expanded form of the metric because this term refers to matter, not geometry. Also, the deviations of pressure and energy are already being inserted via (5.25);
- Final step is to reunite left and right sides and equate the terms proportional to same order to obtain all substantial equations.

5.1.1 Frame Dragging Equation

The frame dragging equation is obtained by analyzing the non-diagonal component $G_{\phi}{}^{t} = R_{\phi}{}^{t} = 8\pi T_{\phi}{}^{t}$ up to first order in Ω. Thus, equation (5.25) does not need to be used here. To illustrate how the calculation procedure is managed we will present here the full, non-expanded, left-hand side of Einstein equation, taken from [15]. We then have:

$$R_{\phi}{}^{t} = -\frac{1}{2}\left[e^{2\mu}\frac{\partial^2\omega}{\partial r^2} + e^{2\beta}\frac{\partial^2\omega}{\partial\theta^2} + e^{2\beta}\frac{\partial\beta}{\partial\theta}\frac{\partial\omega}{\partial\theta} + e^{2\mu}\frac{\partial\mu}{\partial r}\frac{\partial\omega}{\partial r}\right. \tag{5.26}$$

$$+3e^{2\beta}\frac{\partial\psi}{\partial\theta}\frac{\partial\omega}{\partial\theta} - e^{2\mu}\frac{\partial\beta}{\partial r}\frac{\partial\omega}{\partial r} - e^{2\mu}\frac{\partial\nu}{\partial r}\frac{\partial\omega}{\partial r} - e^{2\beta}\frac{\partial\nu}{\partial\theta}\frac{\partial\omega}{\partial\theta} \tag{5.27}$$

$$\left.+3e^{2\mu}\frac{\partial\psi}{\partial r}\frac{\partial\omega}{\partial r} - e^{2\beta}\frac{\partial\mu}{\partial\theta}\frac{\partial\omega}{\partial\theta}\right] e^{2(\psi-\nu-\beta-\mu)}, \tag{5.28}$$

with the functions depending on r and θ. Now we use the first order approximation, which gives $\partial\beta/\partial\theta = \partial\nu/\partial\theta = \partial\mu/\partial\theta = 0$ provided by expanded functions (5.7)-(5.10). After substituting the functions in remaining terms we get

$$G_{\phi}{}^{t} \approx \frac{e^{-(\nu+\beta)}\sin^2\theta}{2r^2}\frac{\partial}{\partial r}\left(r^4 e^{-(\beta+\nu)}\frac{\partial\bar{\omega}}{\partial r}\right) + \frac{e^{-2\nu}}{2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\omega}}{\partial\theta}\right). \tag{5.29}$$

Now we move our attention to the right-hand side of Einstein's equation. The non-diagonal term of energy-momentum tensor is:

$$T_{\phi}{}^{t} = (\varepsilon + p)u_{\phi}u^{t} + pg_{\phi}{}^{t} = (\varepsilon + p)\frac{\bar{\omega}e^{2\psi}}{e^{2\nu} - \bar{\omega}^2 e^{2\psi}}, \tag{5.30}$$

But since terms proportional to Ω^2 are excluded we have just

$$T_{\phi}{}^{t} = (\varepsilon + p)\bar{\omega}e^{2\psi-2\nu} = (\varepsilon + p)\bar{\omega}e^{-2\nu}r^2\sin^2\theta. \tag{5.31}$$

Before gluing the equation together again, we shall define a new quantity from our TOV system (3.17), that is:

$$j(r) \equiv e^{-(\Phi(r)+\Lambda(r))} = e^{-\Phi(r)}\sqrt{1 - 2M(r)/r}, \tag{5.32}$$

and its derivative given by

$$\frac{\mathrm{d}j}{\mathrm{d}r} = -\frac{4\pi r e^{-\Phi}}{\sqrt{1 - 2m/r}}(\varepsilon + p). \tag{5.33}$$

Putting together the left and right side and using $j(r)$ along with its derivative yield the equation:

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\bar{\omega}}{\partial r}\right) + \frac{e^{\beta-\nu}}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\omega}}{\partial\theta}\right) + \frac{4\bar{\omega}}{r}\frac{\mathrm{d}j}{\mathrm{d}r} = 0. \tag{5.34}$$

Since ω transforms like a vector under rotation [50, 52], we can expand it in spherical harmonics

$$\bar{\omega}(r,\theta) = \sum_{n=1}^{\infty}\bar{\omega}_n(r)\left(-\frac{1}{\sin\theta}\frac{\mathrm{d}P_n}{\mathrm{d}\theta}\right), \tag{5.35}$$

and (5.34) can be separeted with $\bar{\omega}_l(r)$ obeying the equation

$$\frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j(r)\frac{\mathrm{d}\bar{\omega}_n}{\mathrm{d}r}\right) + \left(\frac{4}{r}\frac{\mathrm{d}j(r)}{\mathrm{d}r} - e^{\beta-\nu}\frac{n(n+1)-2}{r^2}\right)\bar{\omega}_n = 0. \tag{5.36}$$

From general arguments regarding the asymptotical behaviour of the solutions to this equation, it follows that only $n = 1$ has non-vanishing value in the expansion [50]. We demand that space be regular for small values of r, *i.e.*, $r \to 0$ and for large enough r to be flat. Close to the origin the equation has a behavior of the form

$$\begin{aligned} \bar{\omega}_l(r) &\to c_1 r^{S_+} + c_2 r^{S_-}, \quad r \to 0; \\ S_{\pm} &= -\frac{3}{2} \pm \left[\frac{9}{4} + \frac{n(n+1)-2}{j(0)}\right]^{1/2}, \end{aligned} \tag{5.37}$$

with constants c_1 and c_2. The solution will be regular only if $c_2 = 0$, which now we keep fixed. At large r, $j(r)$ becomes unity and $\bar{\omega}_l$ assumes the form

$$\bar{\omega}_l(r) \rightarrow c_3 r^{-n-2} + c_4 r^{n-1}, \tag{5.38}$$

where the constants c_3 and c_4 have been already determined when we fixed the solution at the origin. Therefore, they only vanish if both do. Flatness consideration is guaranteed if ω decreases faster than $1/r^3$ which is obtained only for non-vanishing $n = 1$ term in the expansion.

The equation for frame dragging is then simplified to

$$\frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right) + \frac{4}{r}\frac{\mathrm{d}j}{\mathrm{d}r}\bar{\omega} = 0. \tag{5.39}$$

This equation is to be integrated outward from star center with initial value $\bar{\omega}(0) = \text{const.}$ Besides, The local rate of rotation, $\bar{\omega}(r)$ is in the direction of Ω and at large r, $\bar{\omega} \rightarrow \Omega$ always having the same sign of Ω. Therefore, $|\bar{\omega}(r)| = |\Omega - \omega(r)|$ always increases whereas $|\omega(r)|$ decreases, meaning that in the star core occurs the largest rate of frame-dragging. At no point of the star $\bar{\omega}$ will exceed or be equal to Ω [50].

The frame-dragging effect is also known as Lense-Thirring effect conceived after Mach's idea of inertia [54]. The initial idea was that nearby masses surrounding the inertial frame gravitationally acts dragging it towards the rotation motion. Thus, inertia, or momentum of inertia, is modified accordingly to inertial frame motion which in fact is written in terms of the star total angular momentum J,

$$I = J(\Omega)/\Omega. \tag{5.40}$$

The quantity J is found from (5.39) in the region outside the star, $r > R$, where $j(r) = 1$, which has the following solution.

$$\bar{\omega}(r) = \Omega - \frac{2J}{r^3}, \quad r > R. \tag{5.41}$$

and

$$J = \frac{R^4}{6}\left.\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right|_{r=R}. \tag{5.42}$$

For the moment of inertia we can derive a general form regarding rotational configuration once given the expression [55]

$$I(\Omega) = \frac{1}{\Omega}\int T_\phi{}^t(r,\theta,\phi,\Omega)\sqrt{-g(r,\theta,\phi,\Omega)}\;\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi, \tag{5.43}$$

where the energy-momentum tensor component is explicited in (5.30) and $\sqrt{-g} = e^{\nu+\psi+\beta+\mu}$. Using the expressions in the moment of inertia definition leads to

$$I(\Omega) = 2\pi \int_0^{\pi} \mathrm{d}\theta \int_0^{R(\theta)} \mathrm{d}r\; \frac{(\varepsilon + p)}{e^{2\nu-2\psi} - \bar{\omega}^2}\,\frac{\bar{\omega}}{\Omega} e^{\nu+\psi+\beta+\mu}. \tag{5.44}$$

5.1.2 Monopole Equations

The monopole equations are obtained by the $l = 0$ terms in the multipolar expansion. From them we determine p_0, m_0 and h_0. But first we can find the primary integral equation from hydrostatic equilibrium analysis of a one-parameter equation of state in a rotating setup [52]. In this case, general angular momentum is built from invariance of GR action and the injection energy imprinted in the fluid is found to be a constant of integration, which is given by:

$$\text{const} \equiv \mu_c = \frac{\varepsilon + p}{u^t} \exp\left(-\int \frac{\mathrm{d}\varepsilon}{\varepsilon + p}\right). \tag{5.45}$$

For an isentropic configuration μ_c is identified with the chemical potential. Both sides are now expanded in powers of Ω^2. The injection energy is expanded in the form

$$\mu_c = \mu\big[1 + h_{0c} + \mathcal{O}(\Omega^4)\big], \tag{5.46}$$

with μ the non-rotating injection energy and h_{0c} a constant. This is combined with the non-rotating term in the right hand side, giving then

$$\mu = (\varepsilon + p)e^{\nu} \exp\left(-\int \frac{\mathrm{d}\varepsilon}{\varepsilon + p}\right). \tag{5.47}$$

For the terms of order Ω^2 we identify the pressure perturbation p^* and function h with h_{0c} given by:

$$h_{0c} = p^*(r,\theta) + h(r,\theta) - \frac{1}{2}e^{-2\nu}\bar{\omega}^2 r^2 \sin^2\theta, \tag{5.48}$$

and expansion in spherical harmonics yields

$$l = 0 \quad \Rightarrow \quad h_{0c} = p_0(r) + h_0(r) - \frac{1}{3}e^{-2\nu} r^2 \bar{\omega}^2, \tag{5.49}$$

$$l = 2 \quad \Rightarrow \quad 0 = p_2(r) + h_2(r) + \frac{1}{3}e^{-2\nu} r^2 \bar{\omega}^2. \tag{5.50}$$

To find h_0 and m_0 we use the Einstein equations ${G_t}^t = 8\pi {T_t}^t$ and ${G_r}^r = 8\pi {T_r}^r$. For the t coordinate in the $l = 0$ terms we will have

$$(\Delta {G_t}^t)_{l=0} = -\frac{2}{r^2}\frac{\mathrm{d}m_0}{\mathrm{d}r} + \frac{j}{6r^2}\left[jr^4\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 + 8r^3\frac{\mathrm{d}j}{\mathrm{d}r}\omega\bar{\omega}\right] - 8\pi(\varepsilon + p)p_0\frac{\mathrm{d}\varepsilon}{\mathrm{d}p} \tag{5.51}$$

$$({T_t}^t)_{l=0} = \frac{2}{3}\frac{\mathrm{d}(j^2)}{\mathrm{d}r}\Omega\bar{\omega}r, \tag{5.52}$$

and equating the terms we get

$$\frac{\mathrm{d}m_0}{\mathrm{d}r} = 4\pi r^2(\varepsilon + p)p_0\frac{\mathrm{d}\varepsilon}{\mathrm{d}p} + \frac{1}{12}j^2 r^4\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 + \frac{8\pi}{3}r^5 j^2\frac{\varepsilon + p}{r - 2M}\bar{\omega}^2. \tag{5.53}$$

The last equation is obtained from the $l = 0$ expansion term in the r coordinate of Einstein equation. The expanded $T_{\mu\nu}$ and $G_{\mu\nu}$ tensors are then:

$$(\Delta G_r{}^r)_{l=0} = -\frac{2m_0}{r-2M}\left(8\pi p + \frac{1}{r^2}\right) + \left(1 - \frac{2M}{r}\right)\frac{2}{r}\frac{\mathrm{d}h_0}{\mathrm{d}r} + \frac{1}{6}r^2 j^2\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 + 8\pi(\varepsilon + p)p_0, \tag{5.54}$$

$$(T_r{}^r)_{l=0} = 0, \tag{5.55}$$

where equation (3.17) has been used to substitute $\mathrm{d}\nu/\mathrm{d}r$ up to first order. Now we differentiate (5.49) to eliminate h_0 of the left side, and after reuniting both sides, we get:

$$\frac{\mathrm{d}p_0}{\mathrm{d}r} = -\frac{1+8\pi r^2 p}{(r-2M)^2}m_0 - 4\pi\frac{(\varepsilon+p)r^2}{r-2M}p_0 + \frac{1}{12}\frac{r^4 j^2}{r-2M}\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 + \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3 j^2 \bar{\omega}^2}{r-2M}\right). \tag{5.56}$$

Finally we have obtained two coupled equations, (5.53) and (5.56), expressing the monopole behaviour and its role in rotation. Furthermore, h_0 is uniquely determined by (5.49) where h_{0c} is determined by continuity condition of $h(r)$ across star surface. These equations must be solved together subjected to the boundary conditions $m_0 \to 0$ and $p_0 \to 0$ for $r \to 0$. The change in gravitational mass due to rotation can be expressed in terms of m_0 by using (5.41) into (5.53) and (5.56):

$$\Delta M(\Omega) = m_0(\Omega) + \frac{1}{r^3}J(\Omega)^2, \quad r > R \tag{5.57}$$

and we also obtain

$$h_0(r) = -\frac{\Delta M}{r-2M} + \frac{J^2}{r^3(r-2M)}, \quad r > R. \tag{5.58}$$

Note that $h_0(r)$ is calculated once $\bar{\omega}$, p_0, J and ΔM are known.

5.1.3 Quadrupole Equations

Quadrupole equations give information about the shape of the star and how it deviates from non-rotating case. Once we saw how to obtain monopole equations, it follows directly how to get the quadrupole ones, since the procedure is equivalent. We have to find equations for $l = 2$ functions m_2, v_2, h_2 and also the pressure perturbation p_2. We already have equation (5.50) in hand relating p_2 and h_2, which means we need three more to go.

The equations chosen are those providing the simplest non-trivial expressions. The first one vanishes identically in the case of no rotation

$$R_\theta{}^\theta - R_\phi{}^\phi = 8\pi(T_\theta{}^\theta - T_\phi{}^\phi), \tag{5.59}$$

whose right-hand side to order Ω^2 is

$$T_\theta{}^\theta - T_\phi{}^\phi = -(\varepsilon + p)u_\phi u^\phi \approx -(\varepsilon + p)e^{-2\nu}\Omega\bar{\omega}r^2\sin^2\theta. \tag{5.60}$$

Now putting together the left-hand side already expanded (equation (A.7)) and right-hand side we get a useful first integral of the field equations, that is:

$$h_2 + \frac{m_2}{r - 2M} = -\frac{r^3}{3}\frac{\mathrm{d}(j^2)}{\mathrm{d}r}\bar{\omega}^2 + \frac{r^4}{6}j^2\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2. \tag{5.61}$$

Further information is provided taking the equation $R_r{}^\theta = 0$ whose terms of order $\mathcal{O}(\Omega^2)$ yield the equation

$$\frac{\mathrm{d}v_2}{\mathrm{d}r} = h_2\left(\frac{1}{r} - \frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + \frac{m_2}{r - 2M}\left(\frac{1}{r} + \frac{\mathrm{d}\phi}{\mathrm{d}r}\right), \tag{5.62}$$

where $\mathrm{d}\phi/\mathrm{d}r$ is replaced by (3.17), but here we chose to shorten the notation. Using (5.61) to substitute m_2 in the last expression gives:

$$\frac{\mathrm{d}v_2}{\mathrm{d}r} = -2h_2\frac{\mathrm{d}\phi}{\mathrm{d}r} + \left[\frac{1}{r} + \frac{\mathrm{d}\phi}{\mathrm{d}r}\right]\left[-\frac{r^3}{3}\frac{\mathrm{d}(j^2)}{\mathrm{d}r}\bar{\omega}^2 + \frac{r^4}{6}j^2\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)\right] \tag{5.63}$$

Last equation is obtained from expression

$$G_r{}^r = 8\pi T_r{}^r = 8\pi p, \tag{5.64}$$

where left side in expanded form is given in (A.4) while right-hand side does not present any second order term, *i.e.*, $(T_r{}^r)_{l=2} = 0$. The deformation factor causes a modification $(\delta G_r{}^r)_{l=2} = 8\pi(\varepsilon + p)p_2$ on Einstein tensor. Equating the resulting sides we have

$$\begin{aligned} &\frac{2(r-2M)}{r^2}\left[\frac{\mathrm{d}v_2}{\mathrm{d}r} + r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\left(\frac{\mathrm{d}v_2}{\mathrm{d}r} - \frac{\mathrm{d}h_2}{\mathrm{d}r}\right)\right] - \frac{2m_2}{r^2}\left(2\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{1}{r}\right) \\ &- \frac{(4v_2 + 2h_2)}{r^2} - \frac{r^2j^2}{6}\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 - 8\pi(\varepsilon + p)p_2 = 0 \end{aligned} \tag{5.65}$$

Now if we use (5.50) and (5.61) to isolate $\mathrm{d}h_2/\mathrm{d}r$ we produce:

$$\begin{aligned} \frac{\mathrm{d}h_2}{\mathrm{d}r} = h_2&\left\{-2\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{2r}{r-2M}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^{-1}\left[2\pi(\varepsilon + p) - \frac{M}{r^3}\right]\right\} \\ &- \frac{2v_2}{r(r-2M)}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^{-1} \\ &+ \frac{1}{6}\left[r\frac{\mathrm{d}\phi}{\mathrm{d}r} - \frac{1}{2(r-2M)}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^{-1}\right]r^3j^2\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 \\ &- \frac{1}{3}\left[r\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{1}{2(r-2M)}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^{-1}\right](r\bar{\omega})^2\frac{\mathrm{d}j^2}{\mathrm{d}r}. \end{aligned} \tag{5.66}$$

The quadrupole part of the expansion is fully described by the four equations (5.50), (5.61), (5.63) and (5.66). The differential equations (5.63) and (5.66) are to be simultaneously integrated radially outward from the star origin. Their boundary conditions are $h_2(0) = 0 = v_2(0)$ and $h_2(r \to \infty) = 0 = v_2(r \to \infty)$ with regular solutions at the origin.

5.2 Rotation in the Brane World

On §4 we have demonstrated that Brane world gravity contains a TOV-like system of equations describing stars. The basic ingredients of this result were the metric (4.65) and equation (4.47). All of this is a first indication that Hartle's method is applicable to branes, since the expanded functions are perturbations of (4.65). As before, we will consider rotations characterized by Kepler rotational frequency $\Omega \ll \Omega_K$ together with inertial frames motion ω described by line element (5.1). We assume that rotation influences and distorts only purely four-dimensional quantities meaning that $\mathcal{U}$ and $\mathcal{P}$ are not perturbed. Since they come from higher dimensional structure, their action is of a gravitational steady fluid-like source permeating the inner star[1]. Moreover, for regions outside the star $r > R$ we will consider $\mathcal{U}(r > R) \equiv \mathcal{U}^+ = 0$ and $\mathcal{P}(r > R) \equiv \mathcal{P}^+ = 0$. All the remaining quantities are perturbed equivalently to the GR case.

Remember that Brane gravity is effectively four-dimensional and any 5th dimension contribution was already discussed and accounted in the field equation. The governing dynamical equation (4.47), when perfect fluids are considered, assumes the form (4.60) which for better visualization we express as

$$G_{\mu\nu} = \kappa^2\left(\varepsilon_{\text{eff}} + p_{\text{eff}} - \frac{2\mathcal{P}}{\lambda\kappa^4}\right)u_\mu u_\nu + \kappa^2\left(p_{\text{eff}} - \frac{2\mathcal{P}}{\lambda\kappa^4}\right)g_{\mu\nu} + \frac{6\mathcal{P}}{\lambda\kappa^2}r_\mu r_\nu \equiv \kappa^2\mathcal{T}_{\mu\nu}, \quad (5.67)$$

with p_{eff} and ε_{eff} given in (4.62) and (4.61), respectively. The vector $r^\mu = (0, e^{-\beta}, 0, 0)$ is the unitary spacelike radial vector. The four-velocity u_μ is given in (5.4)[2]. The radial vector r_μ is one major difference between GR and Brane gravity rotational setup, since it introduces an anisotropic term into energy-momentum tensor. Moreover, the effective pressure and energy density also constitute significant differences from GR. We, thus, explicitly see these quantities will be the source of any modifications on rotation equations that Brane Gravity will have apart from GR.

The calculations follow same path of last section. The Einstein tensor is, of course, the same as in general relativity and its deviations are given by (5.25). Yet, structural deviations in geometry calculated earlier are taken to hold here since for typical star energy

[1] To consider perturbations in the non-local bulk functions we should solve and apply the method to the entire system (4.49) concomitantly with effective field equations which would be a very difficult task.

[2] Remember that four-velocity vector is calculated from line element (5.1) which is the same in both Brane and GR cases, so does is u_μ.

density $\varepsilon \sim 10^{15}$ g/cm^3 and Brane tension $\lambda \sim 10^{38}$ dyn/cm^2 give small local corrections $(T_{\mu\nu})^2 \sim \varepsilon^2/\lambda \sim 10^{-8}$ erg.

5.2.1 Modified Frame-dragging Equation

The frame-dragging equation is obtained from non-diagonal term of effective field equation $G_\phi{}^t = 8\pi \mathcal{T}_\phi{}^t$. The left-hand side was already calculated at (5.29), so only remains the right side to be analyzed. Indeed, using the definitions of u_μ and r_μ in order $\mathcal{O}(\Omega)$ we get for the right-hand side:

$$\kappa^2 \mathcal{T}_\phi{}^t = \kappa^2 \bar{\omega} e^{-2\nu} \left[\varepsilon_{\text{eff}} + p_{\text{eff}} - \frac{2\mathcal{P}}{\lambda\kappa^4} \right] r^2 \sin^2\theta. \tag{5.68}$$

This is slightly different from (5.31) and equating with (5.29) one yields

$$\begin{aligned} \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 e^{-(\beta+\nu)} \frac{\partial \bar{\omega}}{\partial r} \right) + \frac{e^{\beta-\nu}}{r^2 \sin^3\theta} \frac{\partial}{\partial \theta}\left(\sin^3\theta \frac{\partial \bar{\omega}}{\partial \theta} \right) & \\ = 16\pi \bar{\omega} e^{\beta-\nu} \left[\varepsilon_{\text{eff}} + p_{\text{eff}} - \frac{2\mathcal{P}}{\lambda\kappa^4} \right] &. \end{aligned} \tag{5.69}$$

To proceed we use same definition of GR case

$$j(r) \equiv e^{-(\Phi(r)+\Lambda(r))} = e^{-\Phi}\sqrt{1 - 2M(r)/r}. \tag{5.70}$$

Despite being mathematically the same, the quantities involved are very different. In fact, the potential $\Phi(r)$ is obtained by integrating (4.68), the Brane potential. The mass function is defined over integration of (4.66),

$$M(r) \equiv \int_0^r 4\pi r^2 \varepsilon_{\text{eff}}\, \mathrm{d}r, \tag{5.71}$$

and $\Lambda(r)$ is a solution of the Brane-TOV system given in (4.70). The derivative of $j(r)$ is calculated as

$$\frac{\mathrm{d}j}{\mathrm{d}r} = \frac{-4\pi r e^{-\Phi}}{\sqrt{1-2m/r}} \left[\varepsilon_{\text{eff}} + p_{\text{eff}} + \frac{4\mathcal{P}}{\lambda\kappa^4} \right]. \tag{5.72}$$

Using the definitions on (5.69) produces

$$\begin{aligned} \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j \frac{\partial \bar{\omega}}{\partial r} \right) + \frac{e^{\beta-\nu}}{r^2 \sin^3\theta} \frac{\partial}{\partial \theta}\left(\sin^3\theta \frac{\partial \bar{\omega}}{\partial \theta} \right) & \\ = -\frac{4\bar{\omega}}{r}\frac{\mathrm{d}j}{\mathrm{d}r} - \frac{3j\bar{\omega}\mathcal{P}}{2\lambda\pi}\left[\frac{r}{r-2M} \right] &. \end{aligned} \tag{5.73}$$

In this scenario $\bar{\omega}$ still transforms as a vector under rotations and an expansion in spherical harmonics, as in (5.35), can be done. Nonetheless, the component $\bar{\omega}_n(r)$ will obey the equation

$$\frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j(r) \frac{\mathrm{d}\bar{\omega}_n}{\mathrm{d}r} \right) + \left(\frac{4}{r}\frac{\mathrm{d}j(r)}{\mathrm{d}r} + \frac{3jr\mathcal{P}}{2\pi\lambda(r-2M)} - e^{\beta-\nu}\frac{n(n+1)-2}{r^2} \right)\bar{\omega}_n = 0, \tag{5.74}$$

whose solutions in the limits $r \to 0$ and $r \to \infty$ have same behaviour of GR case. This is because of "dark pressure" behaviour in these limits, where $\mathcal{P} \to 0$ in both extremes and $M(r)$ is a constant for r large than star final radius. Moreover, when working on the low energy $\varepsilon/\lambda \ll 1$ scale, the Brane corrections can be neglected in the extremes [11]. In conclusion, this reasoning leads to only one non-trivial equation when $n = 1$ yielding the Brane frame-dragging equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j(r)\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right) + \left(4r^3\frac{\mathrm{d}j(r)}{\mathrm{d}r} + \frac{3jr^5\mathcal{P}}{2\pi\lambda(r-2M)}\right)\bar{\omega} = 0. \tag{5.75}$$

As before, the moment of inertia is given by expression (5.43), but now we use the energy-momentum tensor calculate on the Brane which is

$$\mathcal{T}_\phi{}^t = \left(\varepsilon_{\text{eff}} + p_{\text{eff}} - \frac{2\mathcal{P}}{\lambda k^4}\right)\frac{\bar{\omega}e^{2\psi}}{e^{2\nu} - \bar{\omega}^2 e^{2\psi}}. \tag{5.76}$$

After substitution the modified moment of inertia becomes

$$I(\Omega) = \frac{2\pi}{\Omega}\int_0^\pi \mathrm{d}\theta \int_0^{R(\theta)} \mathrm{d}r \left(\varepsilon_{\text{eff}} + p_{\text{eff}} - \frac{2\mathcal{P}}{\lambda k^4}\right)\frac{\bar{\omega}\ e^{\nu+\psi+\beta+\mu}}{e^{2\nu-2\psi} - \bar{\omega}^2}. \tag{5.77}$$

For the exterior solution we have to consider $\mathcal{P}^+ = 0$ and $j = 1$ which leads to same expression relating total angular momentum (5.42) and $\bar{\omega}$ (5.41). From this we are permitted to calculate the moment of inertia through equation (5.40) which is far simpler to use in numerical integration.

5.2.2 Modified Monopole Equations

In GR scenario we had a first integral equation coming from hydrostatic equilibrium, then providing a relation between pressure perturbation Δp and second order function $h(r,\theta)$. We, therefore, expect that Brane gravity should provide a rather different condition for hydrostatic equilibrium to happen, once the field equations are distinct. However, once again, the approximation argument of low energies prevents difficulties and a first integral of the equations is also given by the constant injection energy:

$$\mu = (\varepsilon + p)e^{\nu}\exp\left(-\int \frac{\mathrm{d}\varepsilon}{\varepsilon + p}\right). \tag{5.78}$$

By same arguments of earlier section we would get the relations (5.49) and (5.50) for the Brane star.

Going further with the analysis of the effective field equations, we will use $G_t{}^t = 8\pi\mathcal{T}_t{}^t$ to find m_0 function and, analogously $G_r{}^r = 8\pi\mathcal{T}_r{}^r$ to find h_0. The left-hand side of both equations were already obtained, up to second order with $l = 0$, they are given by

(5.51) and (5.54). The right-hand sides are easily calculated to be:

$$\begin{aligned}\kappa^2(\mathcal{T}_t{}^t)_{l=0} &= -\frac{16\pi}{3}\left[\varepsilon_{\rm eff}+p_{\rm eff}-\frac{2\mathcal{P}}{\lambda\kappa^4}\right]\Omega\bar{\omega}r^2e^{-2\nu},\\ &= -\frac{2r\Omega\bar{\omega}}{3}\frac{\mathrm{d}(j^2)}{\mathrm{d}r}+\frac{\Omega\bar{\omega}r^2\mathcal{P}}{2\lambda\pi}e^{-2\Phi}\end{aligned} \tag{5.79}$$

$$\kappa^2(\mathcal{T}_r{}^r)_{l=0} = 0. \tag{5.80}$$

The equation for m_0 will change by a term only, where the differential equation turns out to be

$$\begin{aligned}\frac{\mathrm{d}m_0}{\mathrm{d}r} &= 4\pi r^2(\varepsilon+p)p_0\frac{\mathrm{d}\varepsilon}{\mathrm{d}p}+\frac{1}{12}j^2r^4\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2\\ &+\frac{8\pi}{3}r^5j^2\frac{\varepsilon+p}{r-2M}\bar{\omega}^2-\frac{\Omega\bar{\omega}r^2\mathcal{P}}{2\lambda\pi}e^{-2\Phi},\end{aligned} \tag{5.81}$$

whereas for p_0 we get the same relation as in GR case, since no changes arise on second order expansion; thus,

$$\begin{aligned}\frac{\mathrm{d}p_0}{\mathrm{d}r} &= -\frac{1+8\pi r^2p}{(r-2M)^2}m_0-4\pi\frac{(\varepsilon+p)r^2}{r-2M}p_0+\frac{1}{12}\frac{r^4j^2}{r-2M}\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2\\ &+\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3j^2\bar{\omega}^2}{r-2M}\right).\end{aligned} \tag{5.82}$$

This finishes the analysis of monopole functions. The set of equations (5.81), (5.82), (5.49) are the closed system for monopole ($l=0$) perturbations up to order $\mathcal{O}(\Omega^2)$. Equations (5.81) and (5.82) are to be integrated simultaneously outwards from star center. In comparison with GR derived equations, the changes are minimal, by one term indeed, and shall not modify the asymptotic behaviour of the equations. Nonetheless, in regions where $\mathcal{P}$ has significant value (*e.g.* inside the star) we shall expect different solutions.

5.2.3 Modified Quadrupole Equations

The first quadrupole equation to be derived follows from

$$G_\theta{}^\theta - G_\phi{}^\phi = \kappa^2(\mathcal{T}_\theta{}^\theta-\mathcal{T}_\phi{}^\phi), \tag{5.83}$$

whose left-hand side is given in (A.7). In the $l=2$ expansion terms the right-hand side behaves as

$$\kappa^2(\mathcal{T}_\theta{}^\theta-\mathcal{T}_\phi{}^\phi)_{l=2} = -\kappa^2\left(\varepsilon_{\rm eff}+p_{\rm eff}-\frac{2\mathcal{P}}{\lambda\kappa^4}\right)\Omega\bar{\omega}r^2e^{-2\nu}\sin^2\theta, \tag{5.84}$$

and with definitions (5.70) and (5.72) it becomes:

$$\kappa^2(\mathcal{T}_\theta{}^\theta-\mathcal{T}_\phi{}^\phi)_{l=2} = r\Omega\bar{\omega}\frac{\mathrm{d}(j^2)}{\mathrm{d}r}\sin^2\theta+\frac{3\mathcal{P}}{4\pi\lambda}\Omega\bar{\omega}e^{-2\Phi}\sin^2\theta. \tag{5.85}$$

We equate this result with (A.7) to yield the first quadrupole equation

$$h_2 + \frac{m_2}{r-2M} = \frac{j^2 r^4}{6}\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 - \frac{\bar{\omega}^2 r^3}{3}\frac{\mathrm{d}(j^2)}{\mathrm{d}r} - \frac{\Omega\bar{\omega}\mathcal{P}}{4\pi\lambda} r^2 e^{-2\Phi}, \tag{5.86}$$

where the potential function $\Phi(r)$ is related to Brane equation (4.68). Therefore, any factor such as $e^{-2\Phi}$ is calculated once (3.17) is integrated.

The next non-trivial relation is obtained from the Brane effective field equation ${G_r}^{\theta} = \kappa^2 {\mathcal{T}_r}^{\theta}$. The left-hand side in $l = 2$ term is given in (A.6) whereas ${\mathcal{T}_r}^{\theta} = 0$ from definitions (5.67), (5.4) and r_μ. Therefore, the equation provided by this term does not present any modifications from the GR form

$$\frac{\mathrm{d}v_2}{\mathrm{d}r} = h_2\left(\frac{1}{r} - \frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + \frac{m_2}{r-2M}\left(\frac{1}{r} + \frac{\mathrm{d}\phi}{\mathrm{d}r}\right). \tag{5.87}$$

And then the function m_2 is replaced by isolating it with respect to equation (5.86), which yields the modified differential equation

$$\frac{\mathrm{d}v_2}{\mathrm{d}r} = -2h_2\frac{\mathrm{d}\phi}{\mathrm{d}r} + \left[\frac{j^2 r^4}{6}\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 - \frac{\bar{\omega}^2 r^3}{3}\frac{\mathrm{d}(j^2)}{\mathrm{d}r} - \frac{\Omega\bar{\omega}\mathcal{P}}{4\pi\lambda} r^2 e^{-2\Phi}\right]\left(\frac{1}{r} + \frac{\mathrm{d}\phi}{\mathrm{d}r}\right). \tag{5.88}$$

Our final relation is obtained from $l = 2$ component of ${G_r}^{r} = \kappa^2 {\mathcal{T}_r}^{r}$. As always, Einstein tensor in this approximations is given in (A.4). The right-hand side for $l = 2$ is just $({\mathcal{T}_r}^{r})_{l=0} = 0$. After equating both sides we get same mathematical result of GR

$$\begin{aligned} &\frac{2(r-2M)}{r^2}\left[\frac{\mathrm{d}v_2}{\mathrm{d}r} + r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\left(\frac{\mathrm{d}v_2}{\mathrm{d}r} - \frac{\mathrm{d}h_2}{\mathrm{d}r}\right)\right] - \frac{2m_2}{r^2}\left(2\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{1}{r}\right) \\ &- \frac{(4v_2 + 2h_2)}{r^2} - \frac{r^2 j^2}{6}\left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 - 8\pi(\varepsilon + p)p_2 = 0. \end{aligned} \tag{5.89}$$

Now, though, we replace m_2 by making use of (5.86) whilst p_2 will be replaced using (5.50). This procedure then yields the last piece of the rotation "puzzle", which is:

$$\begin{aligned} \frac{\mathrm{d}h_2}{\mathrm{d}r} = h_2 &\left\{-2\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{2r}{r-2M}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^{-1}\left[2\pi(\varepsilon + p) - \frac{M}{r^3}\right]\right\} \\ &- \frac{2v_2}{r(r-2M)}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^{-1} - \frac{\Omega\bar{\omega}\mathcal{P}}{4\pi\lambda}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) r^3 e^{-2\Phi} \\ &+ \frac{1}{6}\left[r\frac{\mathrm{d}\phi}{\mathrm{d}r} - \frac{1}{2(r-2M)}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^{-1}\right] r^3 j^2 \left(\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r}\right)^2 \\ &- \frac{1}{3}\left[r\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{1}{2(r-2M)}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)^{-1}\right](r\bar{\omega})^2\frac{\mathrm{d}(j^2)}{\mathrm{d}r}. \end{aligned} \tag{5.90}$$

Finally the $l = 2$ components of the expansion are fully described by equations (5.50), (5.61), (5.88) and (5.90). The corrections from Brane structure are similar to those introduced on $l = 0$ components, proportional to $\mathcal{P}/\lambda$, giving the already known asymptotic behaviour for this equations [50, 15]. However, we still lack a generalized solution as given in [50], since we must solve the homogeneous and particular differential equations for v_2 and h_2 which are coupled to (4.69) in the Brane case. Therefore, the criteria to exclude possible solutions must be analyzed carefully.

6 Numerical Analysis for Neutron Stars

Neutron stars are astrophysical structures classified as compact stars which are the remnant core of collapsed massive stars ($> 8M_\odot$). They where first proposed by Baade and Zwicky in 1933, one year after Chadwick's neutron discovery [12]. Few years later, Tolman, Oppenheimer and Volkoff studied the possible structure of such astrophysical bodies, consisting of neutral particles, and predicted their maximum masses to be $0.75M_\odot$[1] with radius in the order of $R \sim 10$ km [35, 56]. This combination of masses closer to the Sun's and very small radius (in astrophysical scale) pictures neutron stars as compact objects, but also reveals the difficulty to detect them. In fact, it took the next 34 years for the first neutron star to be spotted by chance in a radio telescope experiment designed to study scintillation phenomena [12]. The researchers behind this project, Hewish, Bell *et al.* detected a source of very small pulsed radio signal outside the solar system, opening the "hunting season" of so-called pulsars in the 70's.

Meanwhile, in the feverish search of pulsars and neutron stars, there were those who were researching explanations on the new objects structures, forms and properties. Hartle, Thorn, as we have already seen, together with Friedmann [48] are examples of researchers working on modifications of GR field equations to explain the macroscopic properties and form of pulsars. On the other hand, another research branch had its attention turned to the use of nuclear and quantum physics, exploring the possibilities on the inner structure and particle constituents of the compact objects. The latter provides an input to be used in the integration of our relativistic equations, namely the equation of state (EoS), relating pressure and energy density by $p = p(\varepsilon)$ depending on the thermal bath the particles are inserted in. We will be working only with cold nuclear matter equations of state.

We must then choose suitable EoS in order to solve the field equations we have found in §5. Hartle and Thorn in their pioneer analysis [51] used the Harrison-Wheeler (HW) and V_γ EoS, which are outdated nowadays and we shall discard them from our study. For comparison purposes we select four EoS that are used in rather recent references regarding Brane Gravity and Hartle method [18, 19, 45, 57]. The EoS chosen are: GDH3 [58], BBB2 [59], BPAL12 [60] and APR [61].

The nuclear interactions within a neutron star core are calculated considering a variety of nuclear particles and are carried out following mainly two different approaches: non-relativistic nuclear many-body theory (NMBT) and relativistic mean field theory (RMFT). APR and BBB2 represents the NMBT formalism, where the first is formed of neutrons, protons, electron and muons in weak equilibrium. The latter is constituted

[1] The symbol $M_\odot$ represents the solar mass. In units $c = G = 1$ it has length dimension with value $M_\odot = 1.4677$ km, and in SI units it is given by $M_\odot = 1.988 \times 10^{30}$ kg.

of protons and neutrons. The remaining EoS are representatives of RMFT. For GDH3 the constituents are neutrons, protons, electrons, muons and hyperons (Σ, Λ and Ξ) in equilibrium with hyperons and mesons added only after the density reaches $\varepsilon \simeq 2\varepsilon_0$. The last EoS, BPAL12, is composed of neutrons, protons, muons, electrons in weak equilibrium.

The study of EoS not only provides insight about the inner structure of stars, but also gives constraints on their maximum masses. General Relativity imposes a limit in maximum mass to be $3.2M_\odot$, after which the star collapses into a black hole. The minimum theoretical mass is calculated to be $0.7M_\odot$ by considering a very soft EoS of Fermi gas [12]. However, the lower bound can be raised to $1.44M_\odot$ by the most accurate measured mass taken from the binary pulsar PSR 1913+16 [14, 62]. When crossing the data with the knowledge of EoS, the maximum masses allowed can be determined up to some uncertainty with respect to the EoS used, see for example [63, 64] where maximum masses vary in the range $2.0M_\odot < M_{\text{max}} < 2.6M_\odot$. Most of what we know about neutron star masses comes from measures of pulsars in binary systems and determining its bounds and constraints still is a subject of intense debate. For a detailed and rich discussion into these difficulties the reader is referred to [13].

We shall now move forward to the numerical analysis and discussion of modified structure equations imprinted by the Brane Gravity in rotational configurations. We summarize the numerical procedure used as well as the parameters that leads to our results. The analysis is done by checking what differences appear in stellar properties, such as mass and radius, when comparing results obtained from GR against those got from Brane Gravity.

6.1 Numerical Procedure

First of all we have to remember that our equations were deduced on the unit system $c = G = \hbar = 1$. Thus, it follows that all relevant quantities in the analysis have unit in powers of distance, which is very useful in numerical procedures. But, if we want to correctly interpret and compare our data with the literature, we have to rescale them properly. To this purpose we use a list of relations between most frequent units in stellar studies given in [12].

To integrate the field equations and the rotational modifications for GR and Brane World Gravity we follow the algorithm discussed in [65]. All integrations were done on a C-language routine which we have developed. All the differential equations are integrated using the four-point Runge-Kutta method with fixed step-size $\delta = 0.01$ km. Before solving any equation we provide the program one of the EoS above in its tabular form, whose columns form a pair (ε_i, p_i). The tables were taken from the free software LORENE <https://lorene.obspm.fr/>. If any of the quantities are localized between two table entries, say (ε_i, p_i) and $(\varepsilon_{i+1}, p_{i+1})$, its pair is calculated via logarithmic interpolation:

$$\frac{\log p - \log p_i}{\log \varepsilon - \log \varepsilon_i} = \frac{\log p_{i+1} - \log p_i}{\log \varepsilon_{i+1} - \log \varepsilon_i}. \tag{6.1}$$

Each RK4 step evaluates the integrated functions for a radius r_i and thus the next radius will be $r_{i+1} = r_i + \delta$, all the obtained quantities are stored in tables. The routine only stops when pressure reaches $p(R) = p_N = 0$, characterizing the star surface. At this point we obtain the maximum number N of steps taken, maximum radius $R = r_N$, final mass $M(R) = m_N$ as well as other relevant quantities ($\bar{\omega}(R)$, $p_0(R)$, $j(R)$, *etc*). The two theories are integrated separately in different routines.

We begin by solving the static equations for both theories. Thus, for GR we integrate equations (3.15)-(3.17) while for the Brane they are (4.66)-(4.68) and (4.72). Note that we are already considering the linear approach $\mathcal{P}(r) = \alpha\mathcal{U}(r)$. The equations are simultaneously integrated with initial conditions $\varepsilon(r_1) = \varepsilon_c$, $p(r_1) = p_c$, $m(r_1) = 0$ and $r_1 = 0.5$ km, or $r_1 = 0.8$ km [2] in specific cases. The Brane requires two additional initial conditions that are: $\mathcal{U}(0) = 0$ and $\mathcal{P}(r_1) = 0$. The central value for the potential is arbitrarily chosen to be $\phi^{\mathrm{arb}}(r_1) = \phi_1^{\mathrm{arb}} = 10^{-5}$ [65]. But it gives a result $\phi_N^{\mathrm{arb}} \neq \phi(R)$ incompatible with the exterior solution (which is the same for GR and Brane cases assuming $\mathcal{P}^+ = 0$):

$$\phi(R) = \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right). \tag{6.2}$$

Thus, ϕ_i^{arb} must be rescaled after the integration process is finished by [65]:

$$\phi_i = \phi_i^{\mathrm{arb}} + \phi(R) - \phi_N^{\mathrm{arb}}. \tag{6.3}$$

Once the static case is solved we work on the rotational equations, beginning with the frame-dragging equations. First we calculate $j(r)$ and its derivative given by equations (5.32), (5.33), (5.72). The frame dragging equations (5.39) and (5.75) are splitted into two differential equations:

$$\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}r} = \eta \tag{6.4}$$

$$\mathrm{GR}: \frac{\mathrm{d}\eta}{\mathrm{d}r} = -\frac{4j\eta + (r\eta + 4\bar{\omega})\frac{\mathrm{d}j}{\mathrm{d}r}}{rj} \tag{6.5}$$

$$\mathrm{Brane}: \frac{\mathrm{d}\eta}{\mathrm{d}r} = -\frac{4j\eta + (r\eta + 4\bar{\omega})\frac{\mathrm{d}j}{\mathrm{d}r}}{rj} - \frac{3r\mathcal{P}\bar{\omega}}{2\pi\lambda(r - 2m)}. \tag{6.6}$$

We integrate the frame-dragging equations for the N points given in the static case. An initial value for $\bar{\omega}(0)$ is arbitrarily chosen, where we take it to be $\bar{\omega}_1^{\mathrm{arb}} = 1.82342$ s^{-1} [65, 17]. This will yield an arbitrary rotation frequency (from (5.41) and (5.42))

$$\Omega^{\mathrm{arb}} = \bar{\omega}_N^{\mathrm{arb}} + \frac{R}{3}\eta_N^{\mathrm{arb}}, \tag{6.7}$$

[2] Small values provided unstable numerical solutions, mainly for Brane equations.

and if we want a specific rotation frequency we have to rescale the solutions by [12, 17, 65]:

$$\bar{\omega}_i = \frac{\Omega}{\Omega^{\mathrm{arb}}}\bar{\omega}_i^{\mathrm{arb}}, \tag{6.8}$$

$$\eta_i = \frac{\Omega}{\Omega^{\mathrm{arb}}}\eta_i^{\mathrm{arb}}. \tag{6.9}$$

Once obtained the N points for $\bar{\omega}$ and its derivative for some given Ω, we can calculate the total angular momentum via (5.42) and the moment of inertia. For this we use the results stored in the Nth point.

For the monopole we must integrate equations (5.53), (5.56), (5.81) and (5.82). The initial conditions we provide are $m_0(r_1) = 0$ and $p_0(r_1) = 0$. After the integration procedure we use (5.57) to calculate the total mass variation due to rotation. In this first analysis we choose only to integrate the frame-dragging equation and monopole differential equations.

The central density lies in the range 2.0×10^{14} g/cm^3 $< \varepsilon_c < \varepsilon_f$, where ε_f is given by the last entry in the table of the EoS used. It is typically of order $\varepsilon_f \sim 10^{16}$ g/cm^3. The range is in accordance with the expected energy density in the core of neutron stars [12]. This variation gives several different stars characteristics which is used to obtain, for example, the mass-radius relations for the EoS as well as the behaviour of the rotation configurations we choose.

6.2 Analysis and Results

Our primary task was to verify our numerical procedure with the references [45, 57, 18, 19]. For this purpose we fix the Brane tension to $\lambda = 10^{38}$ dyn/cm^2 in order to acquire Brane solutions close to GR ones regardless of the linear parameter α chosen (see Fig.7). It is observed that Brane solutions always produce smaller masses than GR, and this effect is minimized when $\lambda = 10^{38}$ dyn/cm^2 [45]. The results in Fig.7, for example, show that Brane Gravity produces maximum masses 1.78% and 2.25% smaller than GR for GDH3 and BBB2, respectively. These differences are the greatest obtained in Fig.7 which corresponds to $\alpha = 0.1$. However, when compared to other values of α, there is no significant difference between these curves confirming the analysis done in [45].

Once α and λ are held fixed we obtain the curves for the static case (Fig.8) that we will use later as a basis of comparison for the rotational configurations. Note that the choice for α is due to numerical purposes only, later on this analysis we shall change its value, but for now we keep it as $\alpha = 2$. Furthermore, we can see that the Brane behaviour is the same for all EoS. The maximum masses taken of Fig.8 are listed in table 1 and are in good shape when compared with [18, 19, 57].

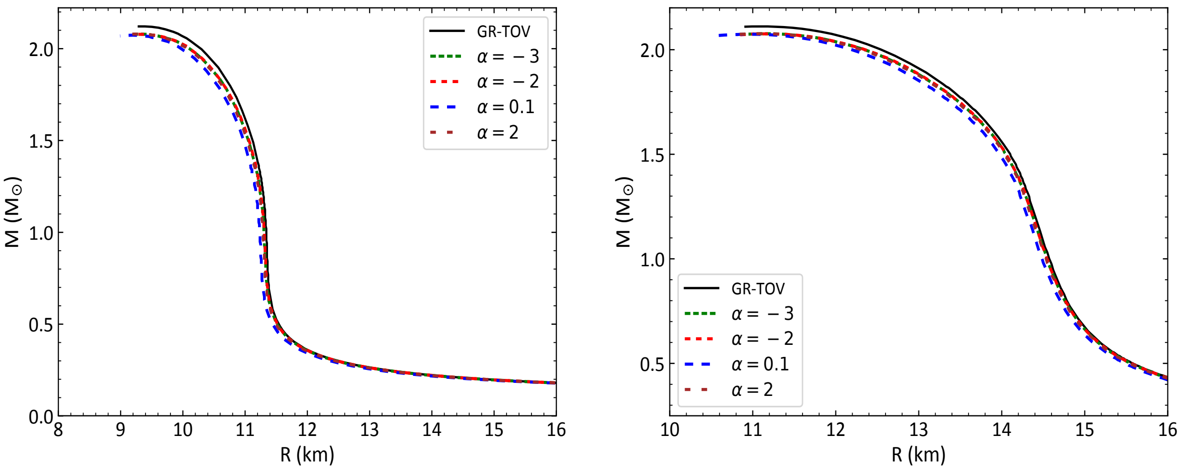


Figure 7 – Two graphics representing the mass-radius relation for BBB2 (left) and GDH3 (right) in the static case. The TOV curve represents usual GR solutions while curves labeled with α represent Brane solutions with respective linear parameter.

	GR			BW ($\lambda = 10^{38}$dyn/cm^2, $\alpha = 2$)		
EoS	$M_{\max}(M_\odot)$	R(km)	ε_c(g/cm^3)	$M_{\max}(M_\odot)$	R(km)	ε_c(g/cm^3)
GDH3	2.111	11.091	2.70×10^{15}	2.076	11.130	2.50×10^{15}
BBB2	2.121	9.351	3.60×10^{15}	2.080	9.270	3.40×10^{15}
BPAL12	1.591	8.781	4.40×10^{15}	1.549	8.620	4.30×10^{15}
APR	2.431	9.791	3.10×10^{15}	2.393	9.740	3.00×10^{15}

Table 1 – Maximum masses, $M_{\max}$, with its respective radius, R, and central density ε_c taken from Fig.8 data.

Now we are able to test what modifications rotation would give for the mass and radius of stars and what are the deviations of Brane Gravity from GR. Measures on radio pulsars rotation speed are the main source of knowledge we have for rotating neutron stars. These measured frequencies range from ~ 0.1 Hz up to $\sim$ kHz, where the fastest currently-known pulsar is PSR J1748-2446ad [66] with rotational frequency of 716Hz. At this frequency order Hartle's method is applicable, but we will also use frequencies above this range to cover and test the versatility of the modified Brane equations.

We calculate rotational effects for four distinct frequencies, two measured and two fictional. In search of a more realistic analysis we use the measured frequencies obtained from observations of a neutron star within the atoll source 4U 1636-53 [18, 67] which are $f_{\text{rot}} = 580$ Hz and $f_{\text{rot}} = 290$ Hz (if we observe doubled radiating structure). For the fictional frequencies we choose $f_{\text{rot}} = 1000$ Hz which would correspond to a milisecond pulsar with rotation period $P = 1$ ms. The last frequency is a limiting case for Hartle's treatment, $f_{\text{rot}} = 2000$ Hz, corresponding to a very rapidly rotating star. Such large fictional frequencies can be allowed in the treatment depending on the nuclear matter being used [15]. We can estimate, for example, the Kepler frequency for the most massive

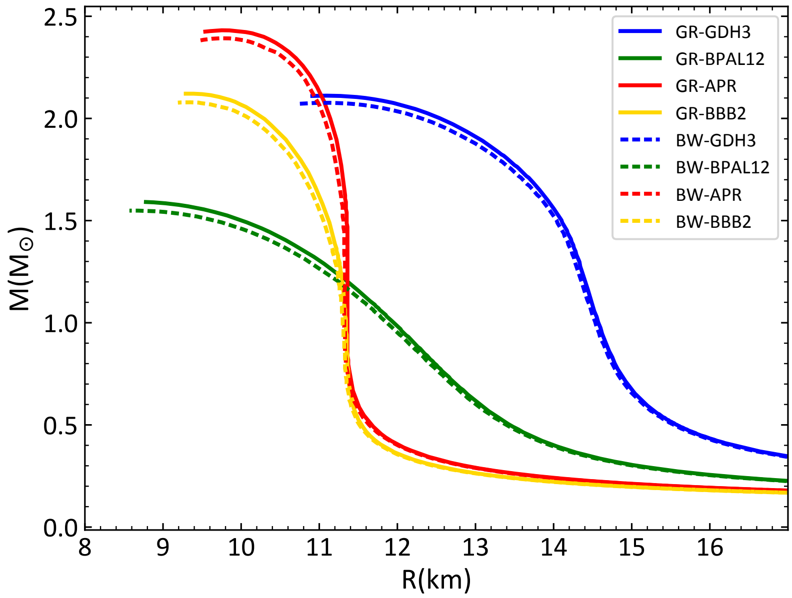


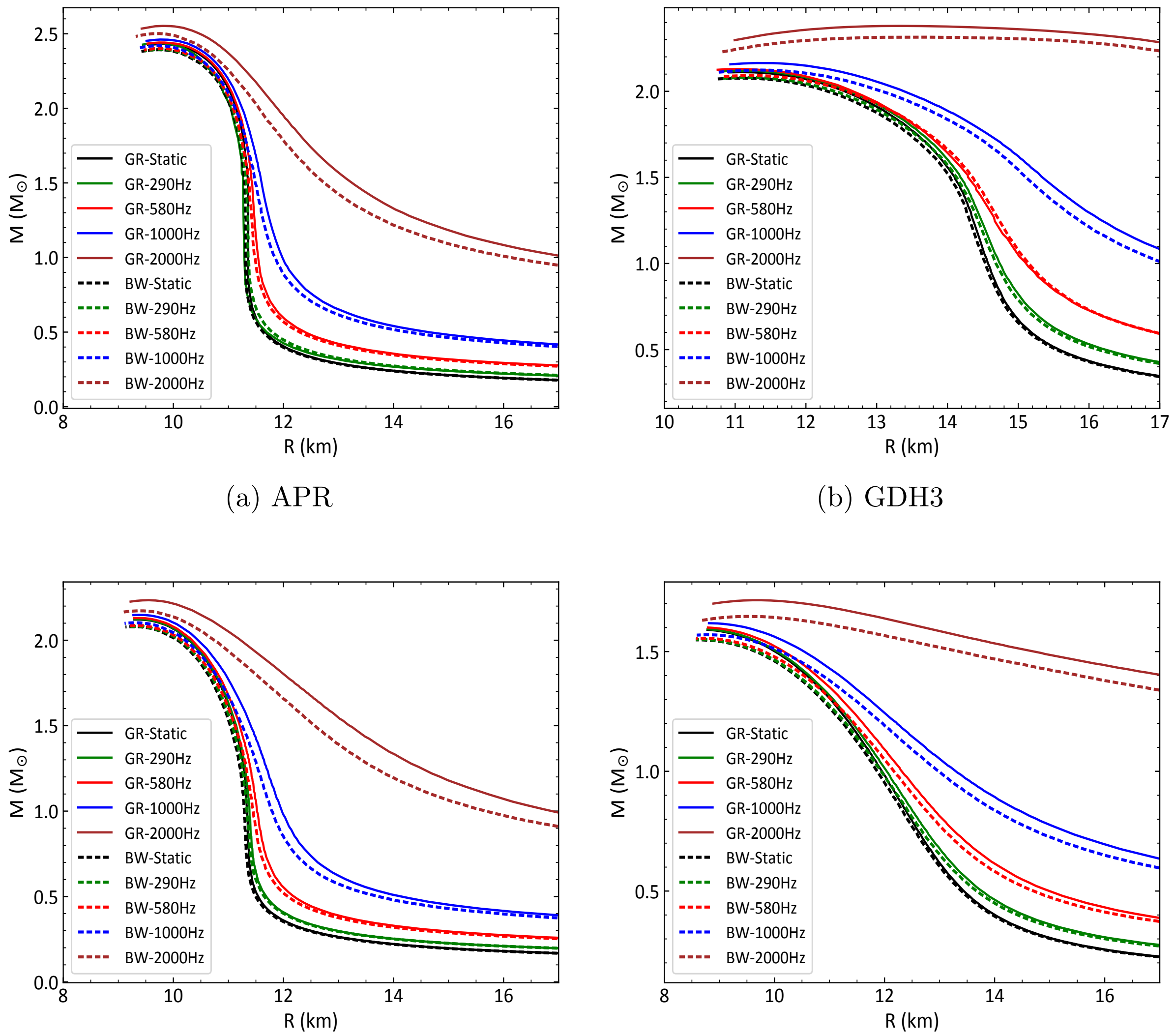
Figure 8 – Mass-radius relation for the four EoS considered. Curves labelled with GR represents the usual TOV solutions. Those labelled with BW are the Brane World Gravity solutions with $\alpha = 2.0$ and $\lambda = 10^{38}$ dyn/cm^2.

star obtained in Fig.8 when its centrifugal force equals gravitational force. This is given by [15]

$$\Omega_K = \sqrt{\left(\frac{c}{R}\right)^2 \frac{MG}{Rc^2}} = 36\sqrt{\frac{M/M_\odot}{(R/\text{km})^3}} \times 10^4 \text{ s}^{-1}, \tag{6.10}$$

where c is the speed of light and G the gravitational constant. Now using $M = 2.431 M_\odot$ and its radius 9.791 km we obtain $f_K = \Omega_K/2\pi \approx 2933.873$ Hz. Therefore, we have our limiting case $f_{\text{rot}} = 2000$ Hz very close of the Kepler frequency where is expected that the method begins to fail (remember the condition $\Omega \ll \Omega_K$). Note that Ω_K varies from star to star, but for those with $1.4\ M_\odot < M < 2.5\ M_\odot$, the Kepler frequency does not change much from f_K calculated above and still greater than 2000 Hz.

Using this considerations we obtain the results in Fig.9 and Fig.9b. All Brane solutions, static and rotational, have masses below GR solutions with the discrepancy increasing as rotation raises. Although the high speed rotation curves, with $f_{\text{rot}} = 2000$ Hz produce the most visible differences between rotational GR and Brane Gravity, they indeed distort the mass-radius curves astonishingly due to their proximity with Kepler frequency. At this point, these data yield unphysical results where central densities in the order $\varepsilon \sim 5 \times 10^{14}$ g/cm^3. For clarity, consider the results in Fig.9 for BPAL12 EoS, the most visible gap in maximum masses for rotational configurations. In the static case the maximum masses are $M_{BW} = 1.549\ M_\odot$ and $M_{GR} = 1.591\ M_\odot$ for Brane Gravity and GR, respectively. Now looking into the rotation profile with $f_{\text{rot}} = 2000$ Hz we obtain



(a) APR

(b) GDH3

(c) BBB2

(d) BPAL12

Figure 9 – Mass-radius relations for static and rotational configurations in GR (solid lines) and Brane Gravity (dashed). For rotation curves 290Hz (green), 580Hz (red), 1000Hz (blue) and 2000Hz (brown) we are plotting the modified mass and radius due to rotation. The Brane parameters are $\alpha = 2$ and $\lambda = 10^{38}$ dyn/cm^2.

the maximum masses $M_{BW,2000} = 1.631 M_\odot$ and $M_{GR,2000} = 1.700 M_\odot$. Therefore, the percentual difference in maximum masses in this case is 5.02% for Brane Gravity and 6.05% for GR, while $1 - M_{BW,2000}/M_{GR,2000} = 4.06\%$. All other similar calculations yield differences below than 5% with slower rotation configuration giving differences lesser than 3%.

The small differences produced highlight an intrinsic difficulty in what could be a Brane star from a GR star. In the observational level we have that, for the most accurate mass measured in the binary PSR 1913+16, the uncertainty are less than the 5% obtained. However, uncertainty in mass measures grows as mass increases [62] with most of them above the percentual difference between Brane and GR maximum masses. Therefore, within the parameters used, the differences of Brane Gravity curves from GR are much smaller than most experimental errors (see [68], for example).

Despite the results obtained for mass-radius profile, we can investigate if other star properties remain the same or are affected by Brane corrections. From equations (5.42), (5.40) we calculate the total angular momentum J and the moment of inertia $I(\Omega)$. For instance, we choose the rotational configuration with $f_{\text{rot}} = 580$Hz to analyze a real case and simplify the data. First we note that for any EoS chosen the behaviour is the same when varying α with fixed $\lambda = 10^{38}$ dyn/cm^2. A demonstration of this behaviour is given in Fig.10 where we follow [57] for comparison. Note that Brane inertia is always bellow GR since they have almost same mass-radius relation from Fig.9 but different $I(\Omega)$ as we can see.

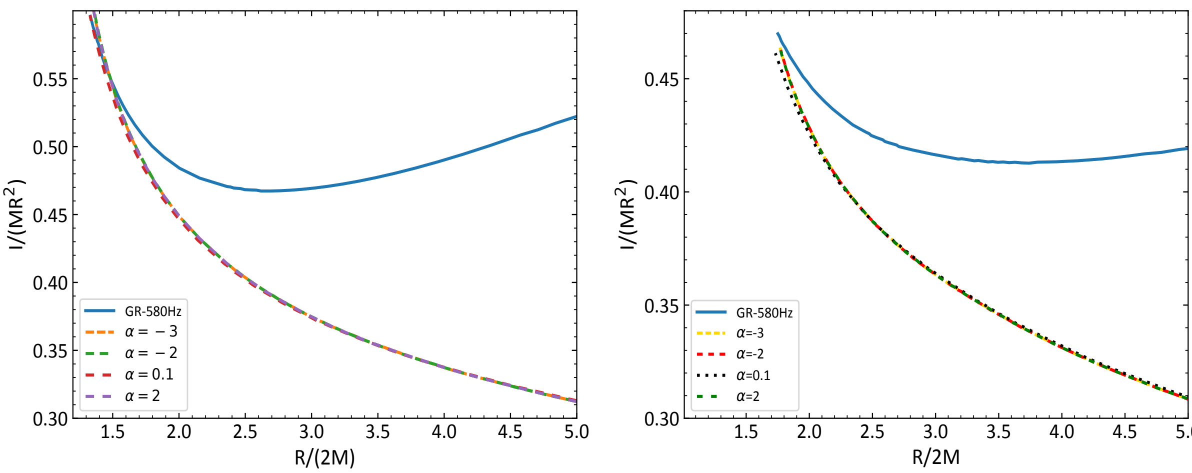


Figure 10 – Dimensionless moment of inertia against compactness for APR (left) and GDH3 (right). Few α are tested with $\lambda = 10^{38}$ dyn/cm^2 fixed.

As variation in α does not change the inertia in the Brane, a similar plot of Fig.10 is done for the remaining EoS in Fig.11 but now only with $\alpha = 2$. Hence, it is evident that Brane structure modifies the moment of inertia regardless of the EoS used. This behaviour can be explained by equation (5.42) which is the same for both theories. However, $J(\Omega)$ is lower for Brane Gravity (see Fig.12) since the anisotropy term in the modified equation (5.75) lowers the value of $\bar{\omega}(r)$ and its derivative. Once J is lower, the moment of inertia $I = J/\Omega$ will also decrease.

If we compare, however, the absolute value for the moment of inertia we have only small fractions of difference among the theories. The most expressive differences are given by the BBB2 EoS, where we have $I_{BW} = 1.905 \times 10^{45}$ g cm^2 and $I_{GR} = 1.993 \times 10^{45}$ g cm^2 with respect to maximum masses in Brane Gravity and GR, respectively. Thus, moment of inertia in GR is 4.62% greater than Brane's inertia in this case which is the greatest difference obtained. The interesting here is that stars with same compactness in GR and

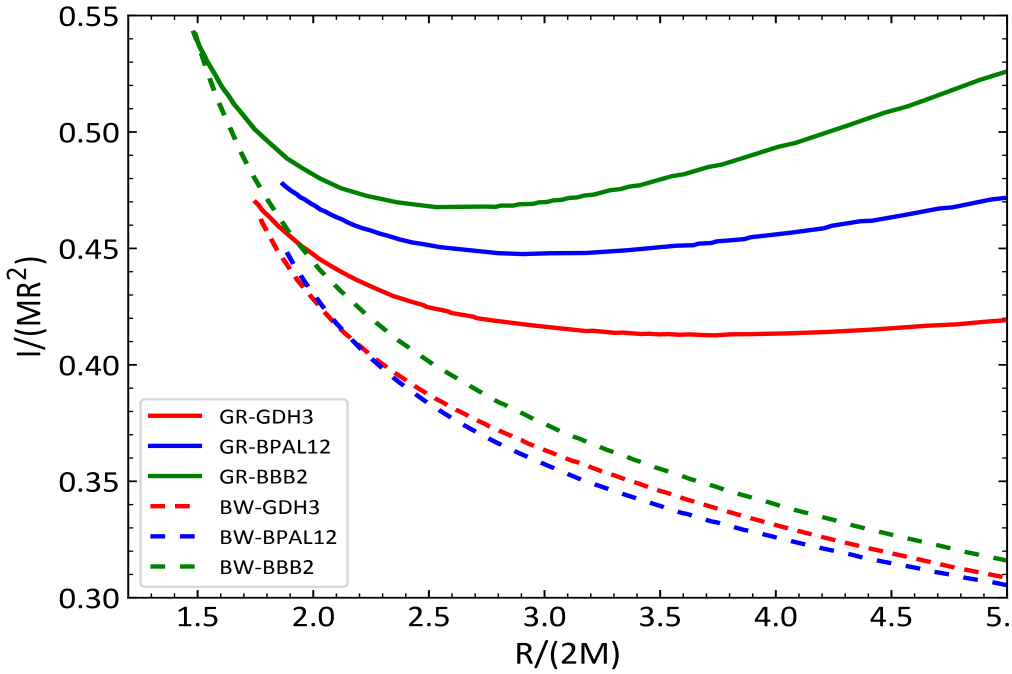


Figure 11 – Dimensionless moment of inertia against compactness for GR (solid) and Brane Gravity (dashed) for rotation frequency $f_{\rm rot} = 580$Hz. Brane parameters are $\alpha = 2$ and $\lambda = 10^{38}$ dyn/cm^2.

Brane Gravity have a very small difference in moment of inertia. The Brane, hence, allows lower moments of inertia for stars with similar mass, radius and equal rotational frequency of GR due to the presence of Bulk's anisotropic term $\mathcal{P}$.

Now we proceed to the final part of the analysis which is focused on relaxing some of the parameters. So far we have seen that modifications imprinted by the Brane for $\lambda = 10^{38}$ dyn/cm^2 are of the order of $\sim 10^{-2} M_\odot$ for the masses. It is indeed known that lowering λ can severely modify the mass-radius relations for the EoS [45]. However, this behaviour was only demonstrated for static profiles, which lead us to verify if this would also be valid for rotations. Let us fix the linear parameter as $\alpha = 2$ and vary the Brane tension in the range 10^{35} dyn/cm^2 $< \lambda <$ 10^{38} dyn/cm^2 [42, 45] for a rotation frequency $f_{\rm rot} = 580$Hz. The results obtained for modified mass $M + \Delta M$ and radius $R + \Delta R$ are given in Fig.13 and Fig.13b to all EoS considered.

By varying λ for a fixed α, it becomes pretty clear that we reproduce same behaviour obtained in [45], but for rotational perturbation solutions instead. Note that curves with $\lambda = 10^{35}$ dyn/cm^2 present dynamics completely dominated by Bulk anisotropic term producing results far from those conceived by observations of neutron stars [62, 13]. This can also be inferred to the curves built with $\lambda = 10^{36}$ dyn/cm^2. On the other hand we obtain good-shaped and well-behaved curves for $\lambda = 10^{37}$ dyn/cm^2 whose values in Table 2 are a bit lower than those of Table 1 for GR. Hence, all of this suggests that Brane

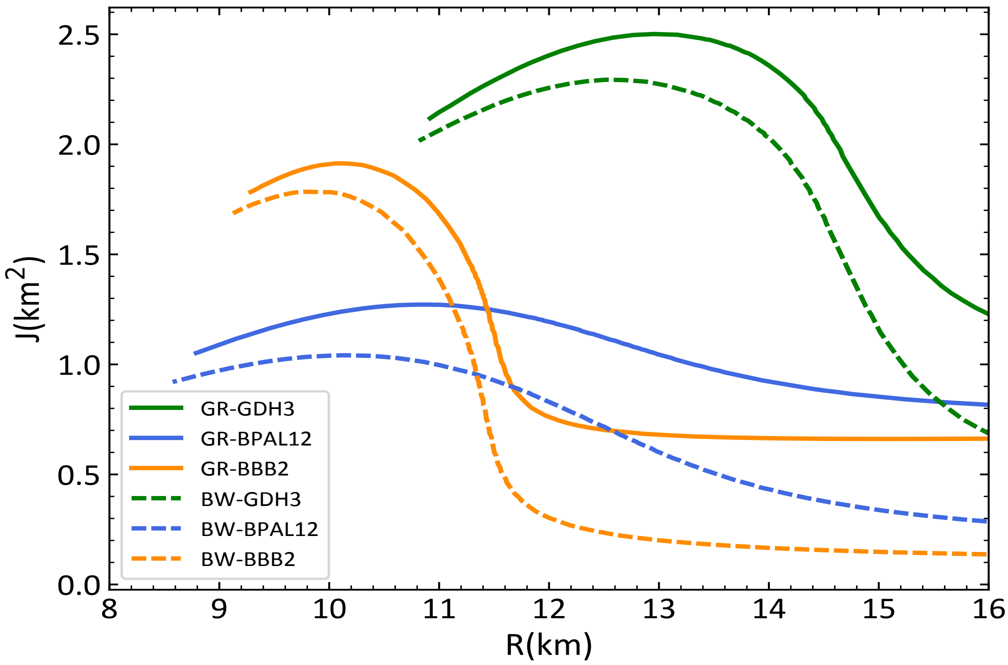


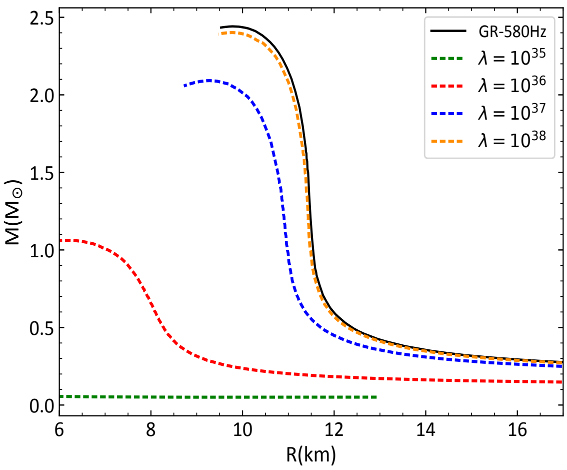
Figure 12 – Total angular momentum against radius for GR (solid) and Brane Gravity (dashed) with respect to parameters used in Fig.11

	$\lambda = 10^{37}$dyn/cm^2			$\lambda = 10^{38}$dyn/cm^2		
EoS	$M_{\max}(M_\odot)$	R(km)	ε_c(g/cm^3)	$M_{\max}(M_\odot)$	R(km)	ε_c(g/cm^3)
GDH3	1.819	10.910	2.10×10^{15}	2.092	11.199	2.50×10^{15}
BBB2	1.766	8.861	2.90×10^{15}	2.087	9.276	3.40×10^{15}
BPAL12	1.252	8.454	3.20×10^{15}	1.556	8.626	4.30×10^{15}
APR	2.092	9.287	2.70×10^{15}	2.402	9.781	3.00×10^{15}

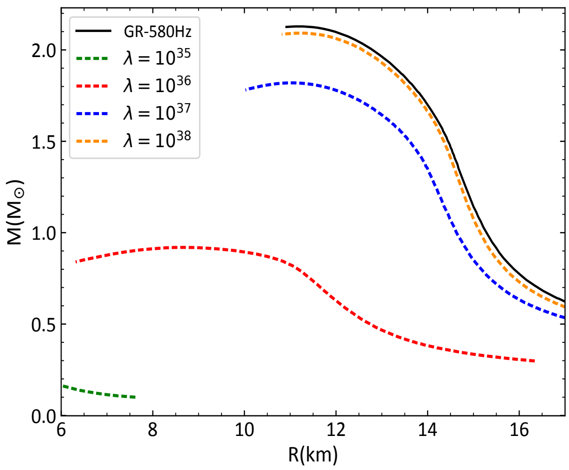
Table 2 – Perturbed maximum masses, $M_{\max}$, with its respective perturbed radius, R, and central density ε_c for $f_{\mathrm{rot}} = 580$Hz. These quantities refer to Fig.13 and Fig.13b

tension could be adjusted to give any already known (or measured) value for the maximum masses, a feature which was already noted in [45].

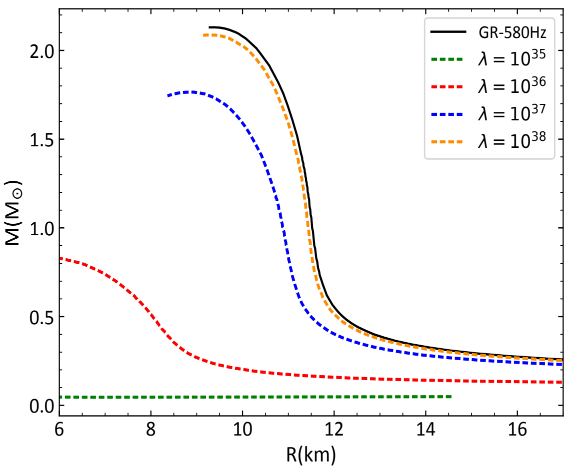
Nonetheless, a far more interesting result comes in hand with these graphics. When Brane tension is softer, the anisotropic term, proportional to $\mathcal{P}$, plays an important role by allowing rotational curves with lower masses than expected of GR solutions. To see the power of this effect in action we plot a similar variation in λ, but now excluding $\lambda = 10^{35}$ dyn/cm^2 and $\lambda = 10^{36}$ dyn/cm^2 given in Fig.14. Despite the curves for rotation keeping their shape, is very evident that masses are lowered (and radius as well) by impressive amounts. Look for example the curve with $\lambda = 5 \times 10^{37}$dyn/cm^2 for APR (Fig. 14); the maximum mass obtained is $1.929 M_\odot$ which is 20.65% lower than GR static result $2.431 M_\odot$, thus configuring a very radical difference from the previous analysis. However, in the radius obtained regarding these maximum masses we obtain



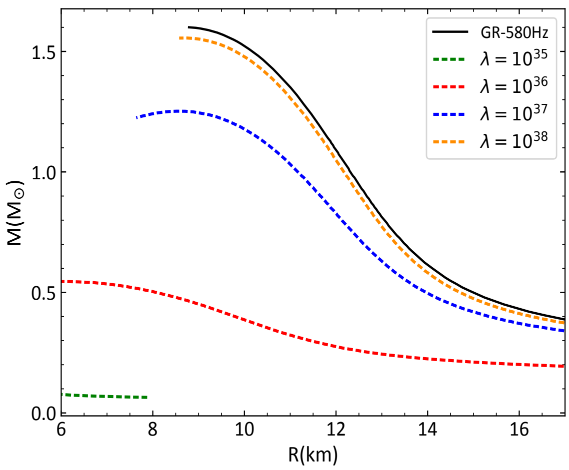
(a) APR



(b) GDH3



(c) BBB2



(d) BPAL12

Figure 13 – Rotational mass-radius relation for different Brane tension parameters. Linear parameter is fixed in $\alpha = 2$. The GR solution is plotted together as a reference curve. Parameter λ is given in units of dyn/cm^2.

$R_{BW}/R_{GR} = 8.776/9.791 = 0.8963$. As Brane radius is 10.36% lower than static GR, the relation in compactness $a = R/2M$ between static GR and rotational BW with $\lambda = 5 \times 10^{36}$ dyn/cm^2 is then $a_{BW}/a_{GR} = 1.1295$.

The interesting result here is that, for rotational configurations with lower λ values, the Brane allows masses below GR static solutions, but with very similar compactness. Note that in this last calculation we have used a rotation near the limit of the requisition $\Omega \ll \Omega_K$. The Kepler frequency Ω_K was calculated using a Newtownian approach that is taken as controversial (in GR framework) and have been calculated by several other means [12], including empirical formulations. But the point is in this approach the anisotropic force inputted by the Brane is not accounted. In this way, the anisotropy could help to

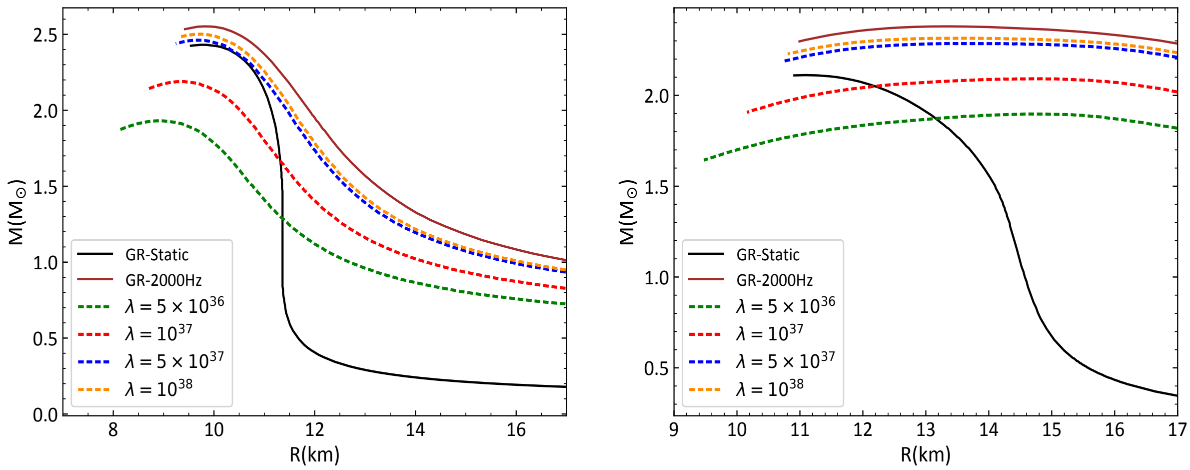


Figure 14 – Modified mass-radius relation for rotation frequency $f_{\mathrm{rot}} = 2000$Hz plotted for several λ and compared with GR static and rotational curves. The EoS used are APR (left) and GDH3 (right).

extend the values Ω_K can assume and, therefore, enhancing the validity of the method. Although the approach is questionable and perhaps inaccurate, it should be noted that its use do not consist in an error for this case. That is because we have calculated the expected modification for second order corrections inserted by Brane Gravity in tensor energy to be $\sim 10^{-8}$erg (see §4).

7 Final Considerations and Discussion

For our final remarks and considerations the reader have realized that we covered several distinct subjects, some of them connected and some apparently not. From Riemannian Geometry to Hartle's formalism and its usage in Brane World Gravity, we have been able to create a perturbative description for slowly rotating stars in Brane's framework. As any other perturbartive method, Hartle's treatment have several requirements to be fulfilled, leading us to a deep analysis and thinking on what adjustments should be performed in order to Brane theory become adequate for the method implementation. This was done based on an intensive step-by-step tracking of the procedure constructed in [50, 51] for General Relativity and compared with Brane Gravity.

The study on §4 demonstrates how the Brane affects spacetime dynamics and inserts new quantities yielding an effective gravity theory far richer in complexity than GR. Hypotheses were made not only on the second order structure terms, as in $\rho^2/\lambda \approx 10^{-8}$erg, but also in the "mysterious" Bulk functions $\mathcal{U}$ and $\mathcal{P}$. In what concerns the latter quantities, there is an open debate to what they could really stand for [11, 44, 45] since we lack solutions for the complete set of equations (4.49). Therefore, we had liberty to constrain the Bulk terms in a suitable manner which would not conflict with Hartle's method. Outside the star surface we were looking for a Schwarzschild-like spacetime which is compatible with $\mathcal{U}^+ = 0$ and $\mathcal{P}^+ = 0$, and inside the star a non-vanishing configuration was assumed. Hydrostatic equilibrium was assumed to be the same of GR [52, 53] in the approximations considered.

After all calculations and methodological procedure we have generated differential equations and relations for the perturbed functions m_0, p_0, h_0, p_2, m_2 and $\bar{\omega}$ which are corrected in comparison with GR perturbed functions. From equations (5.49), (5.75), (5.82), (5.81) we have constructed neutron star profiles (radius and mass for different ε_c) considering four EoS: BPAL12, BBB2, APR and GDH3. An important step taken to precisely solve this equations was to consider the linear relation $\mathcal{P} = \alpha\mathcal{U}$ [45] with constant parameter α. The results obtained reproduce the behaviours highlighted in [45], but for rotation configurations instead, showing the coherence of our work and unravelling yet unexplored Brane aspects. Brane modifications in rotation are closely related to the Brane tension chosen, where we verified that GR and Brane curves are slightly distinguishable for $\lambda = 10^{38}$ dyn/cm^2, with only a maximum difference of $\sim 5\%$.

Moreover, the control of parameter λ over the final results are remarkable (*e.g.* Fig13 and Fig.13b). It can be adjusted to reproduce observational data of pulsars and allow highly rotating stars (Ω near Ω_K) with lower masses and radius than static solutions obtained from GR (See Fig.14). The role of Brane tension can be seen as a filter that allows Brane structure modifications to gravitationally spread easier or harder as the parameter changes. We observe this very behaviour in rotational configurations, where contributions of Bulk and second order terms are enhanced as much as we want.

Despite the well-behaved results obtained and the functionality of the method, there are plenty of hypotheses, considerations and approximations that can be improved in order to give even more reliable results and other interesting analyses. One of the first things we can change or improve is the linear approach $\mathcal{P} = \alpha\mathcal{U}$ to any different expression we may find suitable. This kind of analyses would be the most direct of all, since we do not need to recalculate the perturbed equations.

A more general solution for the Brane would consider $\mathcal{P}^+ \neq 0 \neq \mathcal{U}^+$ which severely alters the exterior form of spacetime [11, 69]. By making this hypothesis we would have to modify Hartle method to be applicable for a Reissner-Nördstrom-like spacetime with

$$(e^{2\Phi})^+ = (e^{-2\Lambda})^+ = 1 - \frac{2M}{r}\left(1 - \frac{\rho}{\lambda}\right) + \frac{q}{r^2}, \tag{7.1}$$

$$\mathcal{U}^+ = -\frac{\mathcal{P}}{2} = \frac{4\pi}{3r^4} q\lambda, \tag{7.2}$$

where $q = -3Mr\rho/\lambda$. Or an even more complicated new solution given in [11]. From the equations above we observe a drastic difference of the metric used (4.65) upon which we perform Hartle perturbation.

Another point of interest is the method used to calculate Ω_K. In fact, there are several methods to do so [12, 15]. However, it was shown [16, 15, 47] that the Newtonian expression used leads to unstable solutions and have to be replaced by its GR analogue:

$$\Omega = e^{2\nu(\Omega) - 2\Psi(\Omega)} V(\Omega) + \omega(\Omega), \tag{7.3}$$

$$V(\Omega) \equiv \frac{\omega'}{2\psi'} e^{\psi - \nu} \sqrt{\frac{\nu'}{\psi'} + \left(\frac{\omega'}{2\psi'} e^{\psi - \nu}\right)^2}, \tag{7.4}$$

where the primes indicate derivatives with respect to radius r. This expression is also a self-consistent condition for rotation frequency. Thus, integration of rotational perturbations must be performed together with (7.3) to yield the appropriate stable solutions. In our case this would only configure a minor procedure in the integration method use. Furthermore, if we take the corrections imprinted by Brane to be large, we would have to formulate a new expression for (7.3) which should also include Bulk contributions and second order terms interactions.

We finish our discussion by presenting the possibility to calculate a hydrostatic equilibrium by performing variations on the action of the theory [52]. This analysis consist on a rather general formulation of quantities preserved via Noether's theorem upon symmetric transformations on the action. Hartle and Sharp then showed that it is possible to construct an invariant quantity encoding hydrostatic equilibrium in GR framework using rather general argument over its action. The quantity is named as the chemical potential in GR, which establishes the relation between pressure, energy density and the baryon number. It is given by

$$\mu = (\varepsilon + p)e^{\nu} \exp\left(-\int \frac{\mathrm{d}\varepsilon}{\varepsilon + p}\right). \tag{7.5}$$

In other words, we recover the same result of $\nabla_\mu T^{\mu\nu} = 0$. Nonetheless, to perform such calculation in the Brane World theory we have to consider the action (4.5) or an effective version of it that leads to (4.47) and work on possible 5D symmetries that would give a chemical potential μ_{BW}. The procedure, in GR, uses the variational principle to extremize the total mass M constrained by rotation[1], J_z, and total baryon number, A. Thus, by using Lagrange multipliers, the quantity we extremize is [52]:

$$\Lambda_c = M - \Omega J_z - \mu_c A. \tag{7.6}$$

We can see from equation above the quantities that would need to be replaced by their BW analogues, if we want to implement this procedure. For quantities J_z and A are solely constructed from the energy-momentum tensor, which is modified in BW gravity.

[1] Here, this is given by the total angular momentum in a chosen direction, commonly coordinate z.

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Appendix

A List of $G_{\mu\nu}$ Terms in Multipolar Expansion

Here we present the list of calculated $G_{\mu\nu}$ terms used in obtaining the structure equations for rotating spherical objects in Hartle's treatment. This results were taken from [50] which we bring here for elucidation purpose on the calculations performed in chapter 5. The quantities are calculated based on the general form of $G_{\mu\nu}$ once metric (5.1) is given, which can be found explicitly in appendix G of [15]. Moreover, up to order $\mathcal{O}(\Omega^2)$, the Einstein tensor is written in terms of the functions j, $\bar{\omega}$, h_0, m_0, m_2 and h_2. Thus, they are:

$$(G_t{}^t)_{l=0} = \frac{j}{6r^2}\left[8r^3\omega\bar{\omega}\frac{\mathrm{d}j}{\mathrm{d}r} + jr^4\left(\frac{\mathrm{d}\omega}{\mathrm{d}r}\right)^2\right] - \frac{2}{r^2}\frac{\mathrm{d}m_0}{\mathrm{d}r} \tag{A.1}$$

$$\begin{aligned}(G_t{}^t)_{l=2} = &-\frac{j}{6r^2}\left[8r^3\omega\bar{\omega}\frac{\mathrm{d}j}{\mathrm{d}r} - jr^4\left(\frac{\mathrm{d}\omega}{\mathrm{d}r}\right)^2\right] - \frac{2}{r^2}\frac{\mathrm{d}m_2}{\mathrm{d}r} - \frac{6m_2}{r^2(r-2M)} \\ &+ \left(1-\frac{2M}{r}\right)\left(2\frac{\mathrm{d}^2k}{\mathrm{d}r^2} + \frac{6}{r}\frac{\mathrm{d}k}{\mathrm{d}r}\right) - 2\frac{\mathrm{d}k}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{M}{r}\right) - \frac{6k_2}{r^2},\end{aligned} \tag{A.2}$$

$$(G_r{}^r)_{l=0} = \frac{j^2}{6r^2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}r}\right)^2 - \frac{2m_0}{r^2}\left(2\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{1}{r}\right) + \frac{2}{r}\left(1-\frac{2M}{r}\right)\frac{\mathrm{d}h_0}{\mathrm{d}r} \tag{A.3}$$

$$\begin{aligned}(G_r{}^r)_{l=2} = &-\frac{j^2}{6r^2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}r}\right)^2 - \frac{2m_2}{r^2}\left(2\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{1}{r}\right) + \frac{2}{r}\left(1-\frac{2M}{r}\right)\frac{\mathrm{d}h_2}{\mathrm{d}r} \\ &- \frac{6h_2}{r^2} + \left(1-\frac{2M}{r}\right)\left(2\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \frac{2}{r}\right)\frac{\mathrm{d}k_2}{\mathrm{d}r} - \frac{4k_2}{r^2}\end{aligned} \tag{A.4}$$

$$(G_r^\theta)_{l=0} = 0 \tag{A.5}$$

$$(G_r{}^\theta)_{l=2} = -\frac{\mathrm{d}h_2}{\mathrm{d}r} + h(r,\theta)\left(\frac{1}{r} - \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) - \frac{\mathrm{d}k_2}{\mathrm{d}r} + \frac{m(r,\theta)}{r-2M}\left(\frac{1}{r} + \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \tag{A.6}$$

$$(G_\theta{}^\theta - G_\phi{}^\phi)_{l=2} = \sin^2\theta\left[-\frac{3}{r^2}\left(h_2 + \frac{m_2}{r-2M}\right) + \frac{j^2r^2}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}r}\right)^2 + r\omega\bar{\omega}\frac{\mathrm{d}j^2}{\mathrm{d}r}\right]. \tag{A.7}$$

where the function $k(r,\theta)$ in our definitions is written as $k(r,\theta) = v(r,\theta) - h(r,\theta)$. The function $M = M(r)$ is the star mass calculated at radius r. For GR this is given by TOV equation

$$M(r) = \int_0^r 4\pi r^2 \varepsilon \;\mathrm{d}r, \tag{A.8}$$

whilst for Brane world gravity it is

$$M(r) = \int_0^r 4\pi r^2 \varepsilon_{\mathrm{eff}} \;\mathrm{d}r, \tag{A.9}$$

where $\varepsilon_{\mathrm{eff}}$ is given in (4.61).

B Brief Explanation on Lie Derivatives

We have been through a lot of explanation about geometry, manifolds, maps, diffeomorphisms, *etc.* But it would obviously take much more writing to fully explain all aspects of the subject. One lacking piece that can be added to the text, for clarity purposes, is the definition of Lie derivatives. We proceed to a brief definition on this theme.

Let M be a manifold and let Φ_t be a one-parameter group of diffeomorphisms. We know that any map $\phi_t : M_p \to M_{\phi(p)}$ can be generated by a vector field v^μ. In fact, the action of ϕ_t on a general tensor $T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}$ is that of carrying the tensor along the vector field, so it is pushed forward from the domain to the codomain. From the multilinearity of tensors, the pushforward action ϕ_t^* in a tensor of type $(0, n)$ is

$$(\phi_t^* T)_{i_1,\cdots,i_n}(v_1)^{i_1}\cdots(v_n)^{i_n} = T_{i_1,\cdots,i_n}(\phi_t^* v_1)^{i_1}\cdots(\phi_t^* v_n)^{i_n}, \tag{B.1}$$

calculated at some point $p \in M$. If we introduce a coordinate system $(x^1, \cdots, x^n)$, such that in this system $v = \left(\partial/\partial x^1, 0, \cdots, 0\right)$, the pushforward action will become just

$$(\phi_t^* T)_{i_1,\cdots,i_n}(x^1, \cdots, x^n) = T_{i_1,\cdots,i_n}(x^1 - t, x^2, \cdots, x^n), \tag{B.2}$$

Now if we take the parameter t as infinitesimal, we can compare the pushforward in the opposite direction ϕ_{-t}^* to the original tensor point at p and yield the definition of a Lie derivative as

$$\mathscr{L}_v\left(T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}\right) = \lim_{t\to 0}\frac{\phi_{-t}^* T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n} - T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}}{t} \tag{B.3}$$

This operation is a derivation in the sense explained on Chapter 1. The action of $\mathscr{L}_v$ on a smooth function $f : M \to \mathrm{R}$ is just

$$\mathscr{L}_v(f) = v(f) \tag{B.4}$$

Also, if $\mathscr{L}_v\left(T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}\right) = 0$ everywhere if and only if for all t, then ϕ_t is a symmetry transformation for the tensor. In the coordinate system previously introduced, the Lie derivative has the action

$$\mathscr{L}_v\left(T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}\right)(x^1, \cdots, x^n) = \frac{\partial T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}}{\partial x^1}, \tag{B.5}$$

which means that ϕ_t will be a symmetry only if all components of $T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}$ are independent of x^1 coordinate.

Applying the Lie derivative on a generic vector w^μ is given by

$$\mathscr{L}_v w^\mu = v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \equiv [v, w]^\mu, \tag{B.6}$$

and by Leibnitz rule we get a the Lie derivative of a general product between two fields, u^μ and w^ν, as

$$\mathscr{L}_v(u_\mu w^\mu) = w^\mu \mathscr{L}_v u_\mu + u_\mu [v, w]^\mu \tag{B.7}$$

Things change a bit when we work on a Riemannian (or pseudo-Riemannian) manifold $(M, g_{\mu\nu})$ equipped with a connection along with its defined covariant derivative. In this case, the commutator becomes

$$[v, w]^\mu = v^\nu \nabla_\nu w^\mu - w^\nu \nabla_\nu v^\mu, \tag{B.8}$$

which gives the Lie derivative of a vector field as

$$\mathscr{L}_v w_\mu = v^\nu \nabla_\nu w^\mu + w^\nu \nabla_\nu v^\mu. \tag{B.9}$$

Moreover, for a general tensor one has

$$\begin{aligned} \mathscr{L}_v\left(T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n}\right) = v^\mu \nabla_\mu T^{j_1,\cdots,j_m}_{i_1,\cdots,i_n} + \sum_{k=1}^{n} T^{j_1,\cdots,j_m}_{i_1,\cdots,\mu,\cdots,i_n} \nabla_{b_k} v^\mu \\ - \sum_{k=1}^{m} T^{j_1,\cdots,\mu,\cdots,j_m}_{i_1,\cdots,i_n} \nabla_\mu v^{j_k} \end{aligned} \tag{B.10}$$

In curved spaces we can also analyze the Lie derivative of the metric tensor along the vector field v^μ. In such spaces, when we act the diffeomorphism ϕ_t on $g_{\mu\nu}$ we generate the space $(M, \phi_t^* g_{\mu\nu})$ which represents the same physical space of $(M, g_{\mu\nu})$ [26]. The Lie derivative of the metric tensor is then

$$\begin{aligned} \mathscr{L}_v g_{\mu\nu} &= v^\alpha \nabla_\alpha g_{\mu\nu} + g_{\alpha\nu} \nabla_\mu v^\alpha + g_{\mu\alpha} \nabla_\nu v^\alpha \\ &= \nabla_\mu v^\nu + \nabla_\nu v^\mu. \end{aligned} \tag{B.11}$$

The vector v^μ gains a special name when $\mathscr{L}_v g_{\mu\nu} = 0$ of a Killing vector, which satisfies Killing equation

$$\nabla_\mu v^\nu + \nabla_\nu v^\mu = 0. \tag{B.12}$$

When this happens, we say that ϕ_t is an isometry, or conversely, when ϕ_t is an isometry, then $\mathscr{L}_v g_{\mu\nu} = 0$.