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Contribuições à teoria matemática de sistemas micropolares

por

Robert Guterres

Tese submetida como requisito parcial
para a obtenção do grau de

Doutor em Matemática Aplicada

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Sumário

RESUMO

O objetivo inicial do presente trabalho foi provar o problema de Leray para o sistema micropolar seguindo uma solução simples recentemente obtida em [30] para as equações de Navier-Stokes. Em [20], Leray deixou um problema de decaimento assintótico em aberto que foi resolvido posteriormente por Kato [16]. Tal problema diz que a norma L^2 da solução da equação de Navier-Stokes incompressível decai assintoticamente a zero, para tempo grande. Ao provar o problema de Leray, observamos uma taxa de decaimento mais rápida para a velocidade microrrotacional $\mathbf{w}$ em relação ao campo $\boldsymbol{u}$ da velocidade. A partir disso, mostramos algumas generalizações naturais dessa propriedade. Mais especificamente, obtemos informações mais precisas a respeito do decaimento de outras normas como, por exemplo, a norma L^∞, o decaimento L^2 das derivadas de ordem mais alta e, por conseguinte, o decaimento em espaços de Sobolev. Por fim, vamos generalizar os resultados obtidos em [14], mostrando uma sequência de desigualdades fundamentais sobre a norma $\dot{H}^s$ das soluções. Além disso, são apresentados alguns resultados básicos de análise, desigualdades de Sobolev e várias propriedades sobre a equação do Calor, dado que tais propriedades se fazem necessárias em nossa análise.

Palavras-chave: *sistema micropolar, problema de Leray, comportamento assintótico, decaimento em espaços de Sobolev.*

ABSTRACT

The first goal in this work was to prove the Leray problem for the micropolar system following a simple solution recently obtained in [30] to Navier-Stokes equations. In [20], Leray left a open problem about asymptotic decay that was solved later by Kato. Such problem says that the L^2 norm for solutions of incompressible Navier-Stokes equations decay to zero asymptotically at large time. In solving the problem, we observe that the micro-rotational velocity decay faster than the velocity field $\boldsymbol{u}$. Hence, we show some natural extensions of this property. More specifically, we get more detailed information about the L^∞ norm, high order derivatives L^2 norm decay and Sobolev spaces decay. Finally, we will generalize the results obtained in [14], showing a sequence of fundamental inequalities about the $\dot{H}^s$ norm of the solutions.Furthermore, we present some analysis results, Sobolev inequalities e some Heat Kernel property that will be necessary in our analysis.

Keywords: *micropolar fluid, Leray Problem, asymptotic behavior.*

1 INTRODUÇÃO

Sejam $(\boldsymbol{u}(\cdot,t), \mathbf{w}(\cdot,t)) \in X$ soluções globais fracas do sistema

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (\mu+\chi)\,\Delta\boldsymbol{u} + \chi\,\nabla\times\mathbf{w}, \tag{1.1a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \nabla(\nabla\cdot\,\mathbf{w}) + \chi\,\nabla\times\,\boldsymbol{u} - 2\,\chi\,\mathbf{w}, \tag{1.1b}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{1.1c}$$

com dados iniciais $(\boldsymbol{u}_0, \mathbf{w}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^3)\times\boldsymbol{L}^2(\mathbb{R}^3)$, sendo X dado por

$$X = L^\infty((0,\infty), \boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2((0,\infty), \dot{\boldsymbol{H}}^1(\mathbb{R}^3)) \cap C^0_w([0,\infty), \boldsymbol{L}^2(\mathbb{R}^3)). \tag{1.2}$$

Onde $\boldsymbol{u}(x,t) \in \mathbb{R}^n$ denota o campo de velocidade, o campo $\mathbf{w}(x,t) \in \mathbb{R}^n$ descreve a velocidade microrrotacional e $p(x,t) \in \mathbb{R}$ a pressão total do fluido no ponto (x,t). As constantes $\mu > 0$, $\gamma > 0$ e $\chi \geq 0$ são, respectivamente, a viscosidade cinemática, a viscosidade de giro e a viscosidade de vórtice. Eventualmente, consideraremos o caso em que $\chi > 0$.

O modelo bidimensional é um caso especial do sistema 3D (1.1) descrito acima, onde $\boldsymbol{u}(x,t) = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0)$ e, como $\mathbf{w}$ é a velocidade de micro-rotação, escreve-se $\mathbf{w}(x,t) = (0,0,w(x_1,x_2,t))$, como feito em [6], por exemplo. Desta maneira, substituindo $\boldsymbol{u}$ e $\mathbf{w}$ escrito na forma acima no sistema (1.1), obtemos as seguintes equações governantes em 2D.

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (\mu+\chi)\,\Delta\boldsymbol{u} + \chi\,\nabla\times\mathbf{w}, \tag{1.3a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \chi\,\nabla\times\,\boldsymbol{u} - 2\,\chi\,\mathbf{w}, \tag{1.3b}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = \nabla\cdot\mathbf{w} = 0, \tag{1.3c}$$

com condições iniciais $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\mathbb{R}^2)$ e $\mathbf{w}_0 = (0,0,w_0)$, onde $w_0 \in L^2(\mathbb{R}^2)$.

Em 1966, Eringen propôs, em seu trabalho intitulado *Theory of micropolar fluids* (veja [7]), um estudo de (1.1). Na literatura, tais fluidos são chamados de micropolares. Fisicamente, tais fluidos representam escoamentos que contém partículas rígidas como uma certa estrutura suspensas em um meio viscoso, onde a deformação destas partículas é ignorada, podendo ser utilizados para descrever vários fenômenos vindos de fluidos mais complexos, tais como circulação sanguínea de animais, cristais líquidos, lubrificantes, etc. Em virtude disso, estes fluidos encontram-se em muitas aplicações envolvendo problemas em engenharia (e.g., em [32] o autor examinou a performance de rolamentos circulares analisando os parâmetros das equações para diferentes valores) e fisiologia (e.g., em [24] os autores estudaram o fluxo sanguíneo através de uma artéria com estenose). Para maiores informações a respeito destes tipo de fluidos, bem como a derivação das equações, veja [22].

Há muitos resultados de existência e unicidade de soluções para problemas relacionados ao problema (1.1) (veja, por exemplo, [3, 4, 5, 7, 9, 22, 25, 35]). Mais precisamente, em 1977, G.P. Galdi e S. Rionero [9] mostraram a existência e unicidade de soluções fracas para o problema de valor inicial do sistema Micropolar.[1] Em 1997, M. A. Rojas-Medar [27] mostrou a existência e unicidade local de soluções clássicas. J.L. Boldrini and M.A. Rojas-Medar, em 1998, estudaram a existência de soluções fracas em [4] para duas e três dimensões espaciais. Em particular, eles mostraram (em 2D) a unicidade de tais soluções. E.E. Ortega-Torres e M.A. Rojas-Medar em 1999, assumindo dados iniciais pequenos, provaram a existência global de uma solução clássica em $\Omega \subset \mathbb{R}^3$ um aberto limitado (veja [25]). Os resultados desses últimos três trabalhos foram obtidos através de um método espectral de Galerkin. Em 2010, J.L. Boldrini, M. Durán and M.A. Rojas-Medar [3] provaram a existência e unicidade de soluções clássicas em $L^q(\Omega)$, para $q > 3$.

[1]Em [9], os autores mostraram tal propriedade para o sistema micropolar em abertos conexos Ω com a solução se anulando em $\partial\Omega \times [0, T]$.

Em seu célebre artigo [20], de 1934, Leray construiu soluções (fracas) globais de energia finita

$$\boldsymbol{u}(\cdot,t) \in L^{\infty}([0,\infty), L^2_{\sigma}(\mathbb{R}^3)) \cap C_w([0,\infty), L^2(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{H}^1(\mathbb{R}^3))$$

para as equações de Navier-Stokes em $\mathbb{R}^3$. Tais soluções (fracas) de Leray podem ser construídas de maneira análoga para o sistema Micropolar (1.1), obtendo assim, as mesmas importantes propriedades existentes para a equação de Navier-Stokes. Como por exemplo, a existência de $t_* \geq 0$ (em $\mathbb{R}^2$, temos $t_* = 0$) suficientemente grande, tal que

$$(\boldsymbol{u}, \mathbf{w})(\cdot,t) \in C^{\infty}(\mathbb{R}^n \times (t_*,\infty)), \quad n = 2, 3.$$

Um problema básico importante deixado aberto por Leray em 1934 foi (denotando $W(t) := \|\, \boldsymbol{u}(\cdot,t)\,\|^2_{L^2(\mathbb{R}^3)}$, como em [20]):

J'ignore si W(t) tend nécessairement vers 0 *quand* t *augmente indéfiniment,*

J. Leray ([20], p. 248)

ou seja, se vale (ou não) que

$$\lim_{t\to\infty} \|\, \boldsymbol{u}(\cdot,t)\,\|_{L^2(\mathbb{R}^3)} = 0. \tag{1.4}$$

Esta questão somente foi resolvida (positivamente) 50 anos mais tarde por Kato [16] e subsequentemente também por outros autores [15, 23, 34]. Vários desenvolvimentos e extensões importantes de (1.4) vem sendo estabelecidos (ver e.g. [2, 18, 28, 30]). Mais especificamente, os autores [30] provaram o mesmo resultado usando uma técnica diferente e, a partir daí, conseguiram responder outras questões (mais gerais) de decaimento assintótico como, por exemplo,

$$\lim_{t\to\infty} t^{3/4} \|\boldsymbol{u}(\cdot,t)\|_{L^{\infty}(\mathbb{R}^3)} = 0.$$

Nosso objetivo é estender tais propriedades para o sistema (1.1), além de obter resultados assintóticos mais profundos, como os descritos no capítulo 5 onde obtemos uma desigualdade assintótica para as derivadas das soluções de Leray que são inéditas para o problema micropolar.

Em todos os resultados deste trabalho, o caso 2D é em geral, mais simples que o caso 3D, uma vez que a principal diferença está nas desigualdades de Sobolev utilizadas (que são mais simples em 2D), e no fato de que em 2D temos $\nabla \cdot \mathbf{w} = 0$. De fato, as demonstrações nos dois casos são muito semelhantes. Devido a isso, vamos focar no caso 3D e ao fim de cada resultado vamos indicar as diferenças na demonstração do caso 2D.

No Capítulo 1, apresentaremos as primeiras desigualdades de energia obtidas, além de obtermos que $t^{\frac{1}{2}} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$ ao $t \to \infty$. Esses resultados serão essenciais para a discussão dos próximos capítulos, não só pelo resultados obtidos, mas também pelas ideias empregadas nas demonstrações, que serão reutilizadas, principalmente no capítulo 3.

No Capítulo 2, vamos obter importantes resultados sobre o comportamento das normas L^q de $(\boldsymbol{u}, \mathbf{w})$. Obtemos que
$\|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$ e $t^{\frac{3}{4}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \to 0$ ao $t \to \infty$. Conseguimos então, por interpolação, obter que $t^{\frac{n}{4} - \frac{n}{2p}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^q(\mathbb{R}^n)} \to 0$ ao $t \to \infty$ $(2 \leq q \leq \infty)$. Além disso, provamos que no caso em que $\chi > 0$, temos uma taxa de decaimento maior para a velocidade microrrotacional $\mathbf{w}$, a saber, $t^{\frac{1}{2}} \|\mathbf{w}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$ ao $t \to \infty$.

No Capítulo 3, vamos obter versões mais gerais dos resultados obtidos no Capítulo 1, obtendo uma sequência de desigualdades para as normas L^2 das derivadas. Como consequência, segue que $t^{s/2} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0$ ao $t \to \infty$.

No Capítulo 4, serão obtidos resultados melhores do que os obtidos no Capítulo 3 para $\mathbf{w}$, no caso em que $\chi > 0$. Obtemos em particular que

$t^{\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0$ e $t^{\frac{n+2}{4}-\frac{n}{2q}}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0$ ao $t \to \infty$ para todo $s > 0$ e $2 \leq q \leq \infty$.

No Capítulo 5, apresentaremos a derivação de uma desigualdade assintótica para as derivadas das soluções de Leray do sistema (1.1). Tal desigualdade foi recentemente obtida para a equação de Navier-Stokes em [14]. Sua versão para o sistemas aqui considerados é a seguinte: Dado $\alpha \geq 0$, se

$$\lim_{t\to\infty} \sup t^{\alpha}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty. \tag{1.5}$$

Então para todo $s \geq 0$, temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{\dot{H}(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)},$$

onde

$$K(\alpha,m) = \frac{1}{\min(\mu,\gamma)^{m/2}} \min_{\delta>0}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+\delta+\frac{j}{2})^{1/2}\right].$$

No caso em que $\chi > 0$ e $\lambda_0(\alpha) < \infty$, obtemos os seguintes resultados mais fortes:

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|\boldsymbol{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s}\left(\frac{K(\alpha,m+1)}{2}\right)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}.$$

Notação. Os espaços $\boldsymbol{L}^2_\sigma(\mathbb{R}^n)$ denotam o espaço dos campos em $\mathbb{R}^n$ em L^2 com divergente nulo, i.e.,

$$\boldsymbol{L}^2_\sigma(\mathbb{R}^n) = \{\boldsymbol{u} = (u_1,u_2,...,u_n); u_i \in L^2(\mathbb{R}^n),\ \text{para } 1 \leq i \leq n \text{ com } \nabla\cdot\boldsymbol{u} = 0\}.$$

Como já visto acima, usaremos (em geral) letras em negrito para grandezas vetoriais, e.g. $\boldsymbol{u}(x,t) = (u_1(x,t), ..., u_n(x,t))$, denotando por $|\cdot|_2$ (ou simplesmente $|\cdot|$) a norma Euclideana em $\mathbb{R}^n$. Como é usual, $\nabla p \equiv \nabla p(\cdot,t)$ denota o gradiente (espacial) de $p(\cdot,t)$, $D_j = \partial/\partial x_j$, e $\nabla\cdot \boldsymbol{u} = D_1 u_1 + ... + D_n u_n$ é o divergente (espacial) de $\boldsymbol{u}(\cdot,t)$; analogamente, $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = u_1 D_1 \boldsymbol{u} + ... + u_n D_n \boldsymbol{u}$. $\|\cdot\|_{L^q(\mathbb{R}^n)}$, $1 \leq q \leq \infty$, denota a norma tradicional do espaço de Lebesgue $\boldsymbol{L}^q(\mathbb{R}^n)$, definimos,

$$\| \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i=1}^{n} \int_{\mathbb{R}^n} | u_i(x,t) |^q \, dx \Big\}^{1/q} \tag{1.6a}$$

$$\| D\boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i,\,j=1}^{n} \int_{\mathbb{R}^n} | D_j \, u_i(x,t) |^q \, dx \Big\}^{1/q} \tag{1.6b}$$

$$\| D^2\boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i,\,j,\,\ell=1}^{n} \int_{\mathbb{R}^n} | D_j D_\ell \, u_i(x,t) |^q \, dx \Big\}^{1/q} \tag{1.6c}$$

e, mais geralmente, para $m \geq 1$ inteiro:

$$\| D^m\boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} = \Bigg(\sum_{i=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \int_{\mathbb{R}^n} | D_{j_1} \cdots D_{j_m} u_i(x,t) |^q \, dx \Bigg)^{1/q}. \tag{1.6d}$$

Definiremos também, por conveniência:

$$\|(\boldsymbol{u},\mathbf{w})\|^q_{L^q(\mathbb{R}^n)} = \|\boldsymbol{u}\|^q_{L^q(\mathbb{R}^n)} + \|\mathbf{w}\|^q_{L^q(\mathbb{R}^n)}, \tag{1.6e}$$

para $1 \leq q < \infty$ e, quando $q = \infty$,

$$\|(\boldsymbol{u},\mathbf{w})\|_{L^\infty(\mathbb{R}^n)} = \max\{\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^n)}, \|\mathbf{w}\|_{L^\infty(\mathbb{R}^n)}\} \tag{1.6f}$$

denotamos por $\| \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} = \max \big\{ \| u_i(\cdot,t) \|_{L^\infty(\mathbb{R}^n)} \colon 1 \leq i \leq n \big\}$ onde $\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R})}$ é o supremo (essencial) de $u_i(\cdot,t)$, similarmente para $\| D\,\boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}$, $\| D^2\boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}$ e para as derivadas de ordem mais alta. É conveniente, também, definir as normas $\dot{H}^s(\mathbb{R}^n)$:

$$\|\boldsymbol{u}\|^2_{\dot{H}^s(\mathbb{R}^n)} = \sum_{i=1}^{3} \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}_i(\xi)|^2 d\xi,$$

$$\|(\boldsymbol{u},\mathbf{w})\|^2_{\dot{H}^s(\mathbb{R}^n)} = \|\boldsymbol{u}\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|\mathbf{w}\|^2_{\dot{H}^s(\mathbb{R}^n)},$$

onde $\hat{u}_i$ denota a transformada de Fourier de u_i. Para maiores detalhes veja o apêndice sobre espaços de Sobolev.

Com estas definições, temos $\| \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} \to \| \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}$ ao $q \to \infty$, assim como, mais geralmente, $\| D^m \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} \to \| D^m \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}$, para todo m inteiro não negativo.[2] Ocasionalmente, resulta também conveniente usar a seguinte definição alternativa para a norma do sup de $\boldsymbol{u}(\cdot,t)$,

$$\| \boldsymbol{u}(\cdot,t) \|_\infty = \text{ ess sup } \big\{ \, | \boldsymbol{u}(x,t) |_2 \colon \; x \in \mathbb{R}^n \, \big\}.$$

Podemos também utilizar $\| \boldsymbol{u}(\cdot,t) \|_{L^q}$ no lugar de $\| \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)}$, por simplicidade. Constantes serão usualmente representadas pelas letras C, c, K; escrevemos $C(\lambda)$ para indicar que o valor da constante referida depende de um dado parâmetro λ. Por conveniência e economia, usamos tipicamente o mesmo símbolo para denotar constantes com diferentes valores numéricos (por exemplo, escrevemos C^2 ou $10\,C$ novamente como C, e assim por diante), como usualmente feito na literatura. O leitor ainda notará que no transcorrer do texto as demostrações passarão a ser mais diretas, dada a familiaridade que o mesmo terá com os argumentos outrora apresentados.

[2] Convém observar que, com as definições (1.16), (1.17), se uma desigualdade de tipo Nirenberg-Gagliardo $\| \mathsf{u} \|_{L^q} \leq K \| \mathsf{u} \|_{L^{r_1}}^{1-\theta} \| \nabla \mathsf{u} \|_{L^{r_2}}^{\theta}$, $0 \leq \theta \leq 1$, valer para funções *escalares* u ($K > 0$ constante), então ela será automaticamente válida para funções *vetoriais* $\boldsymbol{u}$ com a *mesma* constante K do caso escalar. Ademais, temos $\| D^m \boldsymbol{u}(\cdot,t) \|_{L^q} \leq \| D^m \boldsymbol{u}(\cdot,t) \|_{L^{q_1}}^{1-\theta} \| D^m \boldsymbol{u}(\cdot,t) \|_{L^{q_2}}^{\theta}$ se $1/q = (1-\theta)/q_1 + \theta/q_2$, $0 \leq \theta \leq 1$, e assim por diante.

2 PRIMEIRAS ESTIMATIVAS DE ENERGIA

Neste capítulo serão desenvolvidas as primeiras desigualdades de energia para o sistema Micropolar, que vão nos ajudar a obter resultados mais avançados nos próximos capítulos.

Dado o sistema Micropolar

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (\mu+\chi)\,\Delta\boldsymbol{u} + \chi\,\nabla\times\mathbf{w}, \tag{2.1a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \nabla(\nabla\cdot\,\mathbf{w}) + \chi\,\nabla\times\,\boldsymbol{u} - 2\,\chi\,\mathbf{w}, \tag{2.1b}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \tag{2.1c}$$

com dados iniciais $(\boldsymbol{u}_0, \mathbf{w}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n)\times\boldsymbol{L}^2(\mathbb{R}^n)$, sendo X dado por

$$X = L^\infty((0,\infty), \boldsymbol{L}^2(\mathbb{R}^n)) \cap L^2((0,\infty), \dot{\boldsymbol{H}}^1(\mathbb{R}^n)) \cap C^0_w([0,\infty), \boldsymbol{L}^2(\mathbb{R}^n)), \tag{2.2}$$

$n = 2, 3$.

Pela teoria desenvolvida por Leray [20] para Navier-Stokes, existe $t_* > 0$ (dependendo dos dados $\mu, \nu, \gamma, \chi, (\boldsymbol{u}_0, \mathbf{w}_0)$ fornecidos) tal que

$$(\boldsymbol{u}, \mathbf{w}) \in C^\infty(\mathbb{R}^n \times [t_*, \infty)) \tag{2.3a}$$

$$(\boldsymbol{u}, \mathbf{w})(\cdot, t) \in L^\infty_{\text{loc}}([t_*, \infty), H^m(\mathbb{R}^n)) \tag{2.3b}$$

para cada $m \geq 0$ inteiro.

Teorema 2.1. *Sejam $\boldsymbol{u}$ e $\mathbf{w}$ soluções de Leray do sistema Micropolar, então*

$$\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\mu\int_{t_0}^t \|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau + 2\gamma\int_{t_0}^t \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau$$
$$+2\int_{t_0}^t \|\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau + 2\chi\int_{t_0}^t \|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \leq \|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)},$$

$\forall\; t > t_0 > t_*$.

Demonstração. **Caso 3D**

Como foi dito na introdução, aqui e em todos os resultados obtidos neste trabalho, vamos focar no caso 3D e fazer alguns comentários sobre as diferenças do caso 2D no final da demonstração.

Vamos primeiramente definir uma função de corte.

Seja $\phi \in C^{\infty}(\mathbb{R})$ não-crescente tal que $\phi(r) = 1 \; \forall \; r \leq 0$ e $\phi(r) = 0 \; \forall \; r \geq 1$.

Seja $\phi_R : \mathbb{R}^3 \to \mathbb{R}$, definida por

$$\phi_R(x) = \phi(|x| - R).$$

De modo que $\phi_R(x) = 1 \; \forall \; x \in \mathbb{R}^3$ tal que $|x| \leq R$ e $\phi_R(x) = 0 \; \forall \; x \in \mathbb{R}^3$ tal que $|x| \geq R + 1$.

Escrevendo a equação (2.1a) em coordenadas, temos que

$$u_{i,t} + \sum_{j=1}^{3} u_j D_j u_i + D_i p = (\mu + \chi) \sum_{j=1}^{3} D_j D_j u_i + \chi \sum_{j,k=1}^{3} \epsilon_{ijk} D_j w_k, \tag{2.4}$$

onde $\epsilon_{i,j,k}$ é o símbolo de Levi-Civita, i.e.,

$$\epsilon_{i,j,k} = \begin{cases} 1, \text{ se } \{i,j,k\} \text{ é permutação par;} \\ -1, \text{ se } \{i,j,k\} \text{ é permutação ímpar;} \\ 0, \text{ se } i = j, \text{ ou } j = k, \text{ ou } i = k. \end{cases}$$

Seja $t_0 > t_*$, multiplicando a equação (2.4) por $2\phi_R u_i$, integrando em $[t_0, t] \times \mathbb{R}^3$ temos que

$$\int_{t_0}^{t}\int_{\mathbb{R}^3} 2\phi_R u_i u_{i,\tau} dx d\tau + \sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} 2\phi_R u_i u_j D_j u_i dx d\tau + \int_{t_0}^{t}\int_{\mathbb{R}^3} 2\phi_R u_i D_i p dx d\tau =$$
$$= (\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} 2\phi_R u_i D_j D_j u_i dx d\tau + \chi \sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} 2\phi_R \epsilon_{ijk} u_i D_j w_k dx d\tau. \tag{2.5}$$

Usando Fubini no primeiro termo, e o fato de que ϕ_R tem suporte compacto, podemos reescrever (2.5) como

$$\int_{B_{R+1}} \phi_R \int_{t_0}^{t} (u_i^2)_\tau d\tau dx + \sum_{j=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} \phi_R u_j D_j(u_i^2) dx d\tau + 2\int_{t_0}^{t}\int_{B_{R+1}} \phi_R u_i D_i p dx d\tau =$$
$$= 2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} \phi_R u_i D_j D_j u_i dx d\tau + 2\chi \sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} \phi_R \epsilon_{ijk} u_i D_j w_k dx d\tau.$$

Usando o Teorema Fundamental do Cálculo no primeiro termo, e integrando por parte as outras integrais, temos que

$$\int_{B_{R+1}} \phi_R(u_i^2(\cdot,t) - u_i^2(\cdot,t_0))dx - \sum_{j=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} D_j\phi_R u_j (u_i^2) dx d\tau$$
$$-\sum_{j=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} \phi_R D_j u_j (u_i^2) dx d\tau - 2\int_{t_0}^{t}\int_{B_{R+1}} D_i\phi_R u_i p dx d\tau$$
$$-2\int_{t_0}^{t}\int_{B_{R+1}} \phi_R D_i u_i p dx d\tau =$$
$$-2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} D_j\phi_R u_i D_j u_i dx d\tau - 2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} \phi_R D_j u_i D_j u_i dx d\tau$$
$$-2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} D_j\phi_R \epsilon_{ijk} u_i w_k - 2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{B_{R+1}} \phi_R \epsilon_{ijk} D_j u_i w_k dx d\tau.$$

Observe que não há termos de fronteira, pois $\phi_R(x) = 0\ \forall\ x$ tal que $|x| = R+1$.

Fazendo $R \nearrow \infty$, todos os termos que envolvem as derivadas de ϕ_R vão a zero, pois $D_j\phi_R = 0\ \forall\ x$ tal que $|x| < R$ ou $|x| > R+1$. Logo, ao $R \nearrow \infty$, temos que

$$\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} - \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} - \sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_ju_j(u_i^2)dxd\tau - 2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_iu_ipdxd\tau =$$

$$-2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_ju_iD_ju_idxd\tau - 2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{ijk}D_ju_iw_kdxd\tau.$$

Reorganizando os termos,

$$\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_ju_i)^2dxd\tau = \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}$$

$$+\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3} D_ju_j(u_i^2)dxd\tau + 2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_iu_ipdxd\tau + 2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{kji}w_kD_ju_idxd\tau.$$

Como $\sum_{j=1}^{3} D_ju_j = 0$, a equação se torna

$$\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_ju_i)^2dxd\tau = \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}$$

$$+2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_iu_ipdxd\tau + 2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{kji}w_kD_ju_idxd\tau.$$

Somando em i, temos

$$\sum_{i=1}^{3}\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\int_{t_0}^{t}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}(D_ju_i)^2dxd\tau = \sum_{i=1}^{3}\|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}$$

$$+2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\sum_{i=1}^{3} D_iu_i)pdxd\tau + 2\chi\sum_{i,j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{kji}w_kD_ju_idxd\tau.$$

Usando as definições de norma e o fato de que $\nabla\cdot\boldsymbol{u} = 0$, temos

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\int_{t_0}^{t}\|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau = \|\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\ +2\chi\sum_{i,j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{ijk}w_iD_ju_kdxd\tau.\end{aligned} \tag{2.6}$$

Repetindo as contas de forma análoga para a equação (2.1b) em coordenadas

$$w_{i,t} + \sum_{j=1}^{3} u_jD_jw_i = \gamma\sum_{j=1}^{3} D_jD_jw_i + D_i\sum_{j=1}^{3} D_jw_j + \chi\sum_{j,k=1}^{3}\epsilon_{ijk}D_ju_k - 2\chi w_i,$$

obtemos que

$$\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\gamma \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\int_{t_0}^{t} \|\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$
$$+4\chi \int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau = \|\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + 2\chi \sum_{i,j,k=1}^{3} \int_{t_0}^{t}\int_{\mathbb{R}^3} \epsilon_{ijk} w_i D_j u_k \, dx d\tau. \tag{2.7}$$

Somando as equações (2.6) e (2.7) obtemos

$$\begin{aligned}\|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\ +2\gamma \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\int_{t_0}^{t} \|\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\ +4\chi \int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau = \|\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\ +4\chi \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} \epsilon_{ijk} w_i D_j u_k \, dx d\tau.\end{aligned} \tag{2.8}$$

Agora vamos estimar o termo

$$\int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} \epsilon_{ijk} w_i D_j u_k.$$

Fazemos isso usando a Desigualdade de Young e a identidade

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

Com efeito,

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\sum_{i,j,k=1}^{3}\epsilon_{ijk}w_iD_ju_kdx\Big| = \Big|\sum_{i=1}^{3}\int_{\mathbb{R}^3}w_i\sum_{j,k=1}^{3}\epsilon_{ijk}D_ju_kdx\Big|\\
&\leq\sum_{i=1}^{3}\int_{\mathbb{R}^3}|w_i|\Big|\sum_{j,k=1}^{3}\epsilon_{ijk}D_ju_k\Big|dx\\
&\leq\frac{1}{2}\sum_{i=1}^{3}\int_{\mathbb{R}^3}w_i^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\sum_{i=1}^{3}\Big(\sum_{j,k=1}^{3}\epsilon_{ijk}D_ju_k\Big)^2dx\\
&=\frac{1}{2}\|\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\int_{\mathbb{R}^3}\sum_{i=1}^{3}\Big(\sum_{j,k=1}^{3}\epsilon_{ijk}D_ju_k\sum_{l,m=1}^{3}\epsilon_{ilm}D_lu_m\Big)\\
&=\frac{1}{2}\|\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\int_{\mathbb{R}^3}\sum_{j,k,l,m=1}^{3}\Big(\sum_{i=1}^{3}\epsilon_{ijk}\epsilon_{ilm}\Big)D_ju_kD_lu_mdx\\
&=\frac{1}{2}\|\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\int_{\mathbb{R}^3}\sum_{j,k,l,m=1}^{3}(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})D_ju_kD_lu_mdx\\
&=\frac{1}{2}\|\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}(D_ju_k)^2dx-\frac{1}{2}\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}D_ju_kD_ku_jdx\\
&=\frac{1}{2}\|\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\|D\boldsymbol{u}\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Aplicando essa estimativa à equação (2.8), temos que

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + 2(\mu+\chi)\int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau$$

$$+2\gamma\int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau + 2\int_{t_0}^{t} \|\nabla\cdot\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau$$

$$+4\chi\int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau = \|\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2 + \|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2$$

$$+4\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{i,j,k=1}^{3} \epsilon_{ijk} w_i D_j u_k dx d\tau \leq \|\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2 + \|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2$$

$$+4\chi\left|\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{i,j,k=1}^{3} \epsilon_{ijk} w_i D_j u_k dx d\tau\right| \leq \|\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2 + \|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2$$

$$+4\chi\int_{t_0}^{t}\left(\frac{1}{2}\|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2\right) d\tau$$

$$= \|\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2 + \|\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2$$

$$+2\chi\int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau + 2\chi\int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau.$$

Juntando alguns termos, obtemos que

$$\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + 2\mu\int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau + 2\gamma\int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau$$

$$+2\int_{t_0}^{t} \|\nabla\cdot\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau + 2\chi\int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau \leq \|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}^2.$$

Caso 2D

O caso 2D é completamente análogo, com o fato adicional de que em 2D, temos $\nabla\cdot\mathbf{w} = 0$, o que simplifica um pouco a desigualdade de energia, que se torna

$$\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}^2 + 2\mu\int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau + 2\gamma\int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau$$

$$+2\chi\int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq \|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^2)}^2$$

$\forall\ t > t_0 > t_*$.

□

Observação: O Teorema 2.1 fornece importantes informações a repeito de $\boldsymbol{u}$ e $\mathbf{w}$, tais como o fato de $\|(\boldsymbol{u}, \mathbf{w})\|_{L^2(\mathbb{R}^n)}$ ser não crescente no tempo a partir de t_* e que $\|(D\boldsymbol{u}, D\mathbf{w})\|^2_{L^2(\mathbb{R}^n)}$ é integrável em $[t_0, \infty)$.

Na demonstração acima utilizamos funções de corte para justificar formalmente integrações por partes. Nos próximos resultados, não utilizaremos mais essas funções de corte, pois a omissão dessas funções simplificam consideravelmente as demonstrações e os mesmos passos feitos, usando tais funções, podem ser repetidos analogamente.

Nosso objetivo agora será obter uma desigualdade semelhante a apresentada no Teorema 2.1, para as derivadas de $\boldsymbol{u}$ e $\mathbf{w}$.

Teorema 2.2. *Sejam $\boldsymbol{u}$ e $\mathbf{w}$ soluções de Leray do sistema Navier-Stokes Micropolar, então existem $t_{**} > t_*$ e $\eta > 0$ tais que*

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} + \eta \int_{t_0}^{t} \|D^2\boldsymbol{u}, D^2\mathbf{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$
$$+2 \int_{t_0}^{t} \|D\nabla \cdot \mathbf{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\chi \int_{t_0}^{t} \|D\mathbf{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau < \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)}, \tag{2.9}$$

$\forall\ t > t_0 > t_{**} > t_*$.

Demonstração. Escrevendo a equação (2.1a) em coordenadas, derivando em relação a x_l e multiplicando por $2D_l u_i$, obtemos

$$2D_l u_i D_l u_{i,t} + 2\sum_{j=1}^{3} D_l u_i D_l(u_j D_j u_i) + 2D_l u_i D_l D_i p = 2(\mu + \chi) \sum_{j=1}^{3} D_l u_i D_l D_j D_j u_i$$
$$+2\chi \sum_{j,k=1}^{3} \epsilon_{ijk} D_l u_i D_l D_j w_k.$$

Integrando em $[t_0, t] \times \mathbb{R}^3$, temos

$$\int_{t_0}^{t} \int_{\mathbb{R}^3} [(D_l u_i)^2]_t dx d\tau + 2 \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j=1}^{3} D_l u_i D_l u_j D_j u_i dx d\tau +$$

$$\int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j=1}^{3} u_j D_j (D_l u_i)^2 dx d\tau dx d\tau + 2 \int_{t_0}^{t} \int_{\mathbb{R}^3} D_l u_i D_l D_i p$$

$$= 2(\mu + \chi) \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j=1}^{3} D_l u_i D_l D_j D_j u_i dx d\tau + 2\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j,k=1}^{3} \epsilon_{ijk} D_l u_i D_l D_j w_k dx d\tau.$$

Usando o Teorema Fundamental do Cálculo e integrando por partes, obtemos

$$\|D_l u_i(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} - \|D_l u_i(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)} - 2 \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j=1}^{3} u_i D_l D_j u_i D_l u_j dx d\tau$$

$$-2 \int_{t_0}^{t} \int_{\mathbb{R}^3} u_i D_l u_i D_l \sum_{j=1}^{3} D_j u_j dx d\tau - \int_{t_0}^{t} \int_{\mathbb{R}^3} (D_l u_i)^2 \sum_{j=1}^{3} D_j u_j dx d\tau$$

$$-2 \int_{t_0}^{t} \int_{\mathbb{R}^3} D_l D_i u_i D_l p dx d\tau = -2(\mu + \chi) \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j=1}^{3} D_j D_l u_i D_j D_l u_i dx d\tau$$

$$-2\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j,k=1}^{3} \epsilon_{ijk} D_l w_k D_l D_j u_i dx d\tau.$$

Somando em i e l, e usando o fato de que $\nabla \cdot \boldsymbol{u} = 0$, temos que

$$\|D\boldsymbol{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} - \|D\boldsymbol{u}(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)} - 2 \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{i,j,l=1}^{3} u_i D_l D_j u_i D_l u_j dx d\tau$$

$$-2 \sum_{l=1}^{3} \int_{t_0}^{t} \int_{\mathbb{R}^3} D_l p D_l \sum_{i=1}^{3} D_i u_i dx d\tau = -2(\mu + \chi) \sum_{i,j,l=1}^{3} \int_{t_0}^{t} \int_{\mathbb{R}^3} (D_j D_l u_i)^2 dx d\tau$$

$$+2\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{i,j,l,k=1}^{3} \epsilon_{kji} D_l w_k D_l D_j u_i dx d\tau.$$

A relação acima pode ser escrita como

$$\|D\boldsymbol{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu + \chi) \int_{t_0}^{t} \|D^2 \boldsymbol{u}(\cdot, \tau)\| d\tau = \|D\boldsymbol{u}(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)}$$
$$+2 \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{i,jl=1}^{3} u_i D_l D_j u_i D_l u_j dx d\tau + 2\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{i,j,l,k=1}^{3} \epsilon_{ijk} D_l w_i D_l D_j u_k dx d\tau. \tag{2.10}$$

Agora vamos trabalhar com a equação (2.1b). Escrevendo em coordenadas, derivando em relação a x_l e multiplicando por $2D_l w_i$, obtemos

$$2D_l w_i D_l w_{i,t} + 2\sum_{j=1}^{3} D_l w_i D_l(u_j D_j w_i) = 2\gamma \sum_{j=1}^{3} D_l w_i D_l D_j D_j w_i$$

$$+2\sum_{j=1}^{3} D_l w_i D_l D_i D_j w_j + 2\chi \sum_{j,k=1}^{3} \epsilon_{ijk} D_l w_i D_l D_j u_k - 4\chi D_l w_i D_l w_i.$$

Integrando, obtemos

$$\int_{t_0}^{t}\int_{\mathbb{R}^3} [(D_l w_i)^2]_t dx d\tau + 2\int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} D_l w_i D_l u_j D_j w_i dx d\tau$$

$$+ \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} u_j D_j [(D_l w_i)^2] dx d\tau$$

$$= 2\gamma \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} D_l w_i D_l D_j D_j w_i dx d\tau + 2\int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} D_l w_i D_l D_i D_j w_j dx d\tau$$

$$+2\chi \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j,k=1}^{3} \epsilon_{ijk} D_l w_i D_l D_j u_k dx d\tau - 4\chi \int_{t_0}^{t}\int_{\mathbb{R}^3} (D_l w_i)^2 dx d\tau.$$

Usando o Teorema Fundamental do Cálculo e integrando por partes, temos

$$\|D_l w_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} - \|D_l w_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} - 2\int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} w_i D_l D_j w_i D_l u_j dx d\tau$$

$$-2\int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} w_i D_l w_i D_l \sum_{j=1}^{3} D_j u_j dx d\tau - \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} D_j u_j (D_l w_i)^2 dx d\tau$$

$$= -2\gamma \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} (D_j D_l w_i)^2 dx d\tau - 2\int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} D_l D_i w_i D_l D_j w_j dx d\tau$$

$$+2\chi \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j,k=1}^{3} \epsilon_{ijk} D_l w_i D_l D_j u_k dx d\tau - 4\chi \int_{t_0}^{t}\int_{\mathbb{R}^3} (D_l w_i)^2 dx d\tau.$$

Somando em i e l, e usando o fato de que $\nabla \cdot \boldsymbol{u} = 0$, temos

$$\|D\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} - \|D\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} - 2\int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,j,l=1}^3 w_i D_l u_j D_l D_j w_i dx d\tau$$

$$= -2\gamma \int_{t_0}^t \|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau - 2\int_{t_0}^t \int_{\mathbb{R}^3} \sum_{l=1}^3 D_l\Big(\sum_{i=1}^3 D_i w_i\Big) D_l\Big(\sum_{j=1}^3 D_j w_j\Big) dx d\tau$$

$$+2\chi \int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^3 \epsilon_{ijk} D_l w_i D_l D_j u_k dx d\tau - 4\chi \int_{t_0}^t \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau.$$

A relação acima pode ser escrita como

$$\|D\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\gamma \int_{t_0}^t \|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\int_{t_0}^t \|D\nabla \cdot \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$

$$+4\chi \int_{t_0}^t \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau = \|D\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,j,l=1}^3 w_i D_l u_j D_l D_j w_i dx d\tau$$

$$+2\chi \int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^3 \epsilon_{ijk} D_l w_i D_l D_j u_k dx d\tau. \tag{2.11}$$

Somando as equações (2.10) e (2.11), obtemos

$$\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|D\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi) \int_{t_0}^t \|D^2\boldsymbol{u}(\cdot,\tau)\| d\tau$$

$$+2\gamma \int_{t_0}^t \|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\int_{t_0}^t \|D\nabla \cdot \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$

$$+4\chi \int_{t_0}^t \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$

$$= \|D\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|D\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,j,l=1}^3 u_i D_l u_j D_l D_j u_i dx d\tau$$

$$+2\int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,j,l=1}^3 w_i D_l u_j D_l D_j w_i dx d\tau + 4\chi \int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^3 \epsilon_{ijk} D_l w_i D_l D_j u_k dx d\tau. \tag{2.12}$$

Observe que, por Cauchy?Schwarz, temos que

$$
\begin{aligned}
\left|\int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,jl=1}^3 u_i D_l u_j D_l D_j u_i dx d\tau\right| &\le C \int_{t_0}^t \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty} \int_{\mathbb{R}^n} \sum_{i,jl=1}^3 |D_l u_j||D_l D_j u_i| dx d\tau \\
&\le C \int_{t_0}^t \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\le C \int_{t_0}^t \|(\boldsymbol{u},\mathbf{w})(\cdot,\tau)\|_{L^\infty} \|(D\boldsymbol{u},D\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau.
\end{aligned} \tag{2.13}
$$

Da mesma forma

$$
\begin{aligned}
&\left|\int_{t_0}^t \int_{\mathbb{R}^3} \sum_{i,jl=1}^3 w_i D_l u_j D_l D_j w_i dx d\tau\right| \\
&\le C \int_{t_0}^t \|(\boldsymbol{u},\mathbf{w})(\cdot,\tau)\|_{L^\infty} \|(D\boldsymbol{u},D\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau.
\end{aligned} \tag{2.14}
$$

Além disso, temos

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^3 \epsilon_{ijk} D_l w_i D_l D_j u_k dx\right| = \left|\sum_{i,l=1}^3 \int_{\mathbb{R}^3} D_l w_i \sum_{j,k=1}^3 \epsilon_{ijk} D_l D_j u_k dx\right| \\
&\le \sum_{i,l=1}^3 \int_{\mathbb{R}^3} |D_l w_i| \left|\sum_{j,k=1}^3 \epsilon_{ijk} D_l D_j u_k\right| dx \le \frac{1}{2} \sum_{i,l=1}^3 \int_{\mathbb{R}^3} (D_l w_i)^2 + \left(\sum_{j,k=1}^3 \epsilon_{ijk} D_l D_j u_k\right)^2 dx \\
&= \frac{1}{2} \sum_{i,l=1}^3 \int_{\mathbb{R}^3} (D_l w_i)^2 dx + \frac{1}{2} \sum_{i,l=1}^3 \int_{\mathbb{R}^3} \left(\sum_{j,k=1}^3 \epsilon_{ijk} D_l D_j u_k \sum_{p,q=1}^3 \epsilon_{ipq} D_l D_p u_q\right) dx \\
&= \frac{1}{2}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_{\mathbb{R}^3} \sum_{l,j,k,p,q=1}^3 \left(\sum_{i=1}^3 \epsilon_{ijk}\epsilon_{ipq}\right) D_l D_j u_k D_l D_p u_q dx \\
&= \frac{1}{2}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_{\mathbb{R}^3} \sum_{l,j,k,p,q=1}^3 (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) D_l D_j u_k D_l D_p u_q dx \\
&= \frac{1}{2}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \sum_{l,j,k=1}^3 \int_{\mathbb{R}^3} (D_l D_j u_k)^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \sum_{l,j,k=1}^3 D_l D_j u_k D_l D_k u_j dx \\
&= \frac{1}{2}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|D^2\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned} \tag{2.15}
$$

Utilizando as estimativas (2.13), (2.14) e (2.15) na equação (2.12), obtemos

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu\int_{t_0}^{t}\|D^2\boldsymbol{u}(\cdot,\tau)\|^2 d\tau + 2\gamma\int_{t_0}^{t}\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau$$
$$+2\int_{t_0}^{t}\|D\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \leq \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}$$
$$+C\int_{t_0}^{t}\|(\boldsymbol{u},\mathbf{w})(\cdot,\tau)\|_{L^\infty}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau. \tag{2.16}$$

Agora vamos estimar o termo

$$\|(\boldsymbol{u},\mathbf{w})\|_{L^\infty}\|(D\boldsymbol{u}, D\mathbf{w})\|_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})\|_{L^2(\mathbb{R}^3)}. \tag{2.17}$$

Fazendo $\mathbf{z} = (\boldsymbol{u}, \mathbf{w})$ e utilizando as desigualdades de Sobolev (A.1) e (A.2), temos que para todo $t > t_0 > t_*$,

$$\begin{aligned}
&\|\mathbf{z}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\\
&\leq K\|\mathbf{z}(\cdot,t)\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,t)\|^{3/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\\
&= K\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}(\|\mathbf{z}(\cdot,t)\|^{1/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,t)\|^{-1/4}_{L^2(\mathbb{R}^3)})\\
&\leq K\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}(\|\mathbf{z}(\cdot,t)\|^{1/2}\|D^{\mathbf{z}}(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)})\\
&\leq K\|\mathbf{z}(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Afirmação: Existe $t_{**} > t_*$ tal que $\forall\ t > t_0 > t_{**}$ vale

$$\|\boldsymbol{u}(\cdot,t_0), \mathbf{w}(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,t), D\mathbf{w}(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)} < \frac{1}{CK}\min(\mu,\gamma).$$

Demonstração da afirmação: Pelo teorema 2.1 temos que

$$\liminf_{t\to\infty}\|D\boldsymbol{u}(\cdot,t), D\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0,$$

logo existe t_1 tal que

$$\|\boldsymbol{u}(\cdot,t_0), \mathbf{w}(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,t_1), D\mathbf{w}(\cdot,t_1)\|^{1/2}_{L^2(\mathbb{R}^3)} < \frac{1}{CK}\min(\mu,\gamma).$$

Suponha que a afirmação seja falsa. Então existe $t_2 > t_1$ tal que $\forall$ $t \in [t_1, t_2)$

$$\|\boldsymbol{u}(\cdot,t_0),\mathbf{w}(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,t),D\mathbf{w}(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)} < \frac{1}{CK}\min(\mu,\gamma)$$

e

$$\|\boldsymbol{u}(\cdot,t_0),\mathbf{w}(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,t_2),D\mathbf{w}(\cdot,t_2)\|^{1/2}_{L^2(\mathbb{R}^3)} = \frac{1}{CK}\min(\mu,\gamma).$$

Então, por (2.16),

$$\begin{aligned}
&\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_2)\|^2_{L^2(\mathbb{R}^3)} + 2\min(\mu,\gamma)\int_{t_0}^{t_2}\|D^2\boldsymbol{u},D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq \|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
&+C\int_{t_0}^{t_2}\|(\boldsymbol{u},\mathbf{w})(\cdot,\tau)\|_{L^\infty}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\
&\leq \|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
&+CK\int_{t_0}^{t_2}\|\boldsymbol{u},\mathbf{w}(\cdot,t_0)\|^{1/2}\|D\boldsymbol{u},D\mathbf{w}(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\boldsymbol{u},D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq \|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + 2\min(\mu,\gamma)\int_{t_0}^{t_2}\|D^2\boldsymbol{u},D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}$$

E portanto,

$$\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_2)\|^2_{L^2(\mathbb{R}^3)} \leq \|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)},$$

mas então,

$$\begin{aligned}
\left(\frac{1}{CK}\right)^2\min(\mu,\gamma)^2 &= \|\boldsymbol{u}(\cdot,t_0),\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_2)\|_{L^2(\mathbb{R}^3)} \\
&\leq \|\boldsymbol{u}(\cdot,t_0),\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} < \left(\frac{1}{CK}\right)^2\min(\mu,\gamma)^2.
\end{aligned}$$

Absurdo, logo existe t_{**}.

Assim, substituindo em (2.16), obtemos para todo $t > t_0 > t_{**} > t_*$,

$$\begin{aligned}
&\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \eta\int_{t_0}^{t}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+2\int_{t_0}^{t}\|D\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau < \|(D\boldsymbol{u},D\mathbf{w})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

onde $\eta > 0$.

$\square$

Vamos mostrar agora a primeira das estimativas assintóticas que iremos obter ao longo do texto.

Teorema 2.3. *Afirmamos que o limite abaixo é verdadeiro.*

$$\lim_{t\to\infty} t^{\frac{1}{2}} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Demonstração. **Caso 3D**

Pelo Teorema 2.1 sabemos que

$$\int_{t_0}^{t} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau < \infty,$$

para todo $t > t_0 > t_*$.

De modo que

$$\lim_{t\to\infty} \int_{\frac{t}{2}}^{t} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau = 0.$$

Além disso, pelo Teorema 2.2, temos que $\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ é decrescente para todo $t > t_{**}$.

Portanto

$$\begin{aligned}\lim_{t\to\infty} t\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &= \lim_{t\to\infty} 2\int_{\frac{t}{2}}^{t} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} d\tau \\ &\le \lim_{t\to\infty} 2\int_{\frac{t}{2}}^{t} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau = 0.\end{aligned}$$

Logo,

$$\lim_{t\to\infty} t^{\frac{1}{2}} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0.$$

Caso 2D

Derivando o sistema (2.1) e fazendo o produto interno de $(\boldsymbol{u}, \mathbf{w})$ com (2.1a) e (2.1b), respectivamente, integrando por partes em $\mathbb{R}^2$ e somando tudo, temos

$$\frac{d}{dt}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\chi\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq -2\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}$$
$$+\|(\boldsymbol{u}, \mathbf{w})(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)},$$

para todo $t > t_*$.

Usando a desigualdade de Sobolev (A.17) e o Teorema 2.1, obtemos

$$\begin{aligned}
\frac{d}{dt}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\chi\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} &\leq -2\min(\mu,\gamma)\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \\
&+\|(\boldsymbol{u}, \mathbf{w})(\cdot,t)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)} \\
&\leq -2\min(\mu,\gamma)\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \\
&+\|(\boldsymbol{u}, \mathbf{w})(\cdot,t_0)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)} \\
&\leq -2\min(\mu,\gamma)\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \\
&+C\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)},
\end{aligned}$$

para algum $C > 0$.

Usando os expoentes $\alpha = 4$ e $\beta = 4/3$ (satisfazendo $\alpha^{-1} + \beta^{-1} = 1$), temos pela desigualdade de Young, que

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}$$
$$\leq C_\epsilon\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^4_{L^2(\mathbb{R}^2)} + \epsilon\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}$$

e portanto

$$\frac{d}{dt}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq C\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^4_{L^2(\mathbb{R}^2)}.$$

Como, pelo Teorema 2.1, $\int_0^\infty \|(D\boldsymbol{u}, D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds < \infty$, temos (por Gronwall)

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq C\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}, \text{ para todo } 0 \leq s \leq t,$$

ou, fazendo $\eta = 1/C > 0$,

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \geq \eta\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}, \text{ para todo } 0 \leq s \leq t.$$

Agora, suponha que o Teorema 2.3 seja falso. Então, existe $\delta > 0$ e uma sequência $t_n \to \infty$ tais que

$$t_n \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t_n)\|^2_{L^2(\mathbb{R}^2)} > \delta > 0 \quad \text{para todo } n \in \mathbb{N},$$

e podemos assumir que $t_{n+1} \geq 2t_n$. Assim, temos pelo Teorema 2.2,

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, s)\|^2_{L^2(\mathbb{R}^2)} ds &\geq \eta \int_{t_n}^{t_{n+1}} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t_{n+1})\|^2_{L^2(\mathbb{R}^2)} ds \\
&\geq \eta (t_{n+1} - t_n) \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t_{n+1})\|^2_{L^2(\mathbb{R}^2)} \geq \eta\delta(t_{n+1} - t_n)/t_{n+1} \\
&\geq \eta\delta(1 - t_n/t_{n+1}) \geq \eta\delta/2.
\end{aligned}$$

Isto contradiz o fato de $\int_0^\infty \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, s)\|^2_{L^2(\mathbb{R}^2)} ds < \infty$, e portanto, o Teorema 2.3 está provado no caso 2D. □

Este resultado será muito utilizado nos próximos capítulos para obtermos estimativas a respeito das normas de $\boldsymbol{u}$ e $\mathbf{w}$.

Os próximos dois Teoremas deste capítulo tratam dos dois primeiros casos de uma sequência de desigualdades que serão apresentadas no capítulo 4. Esses teoremas serão importantes no estudo das normas $L^2(\mathbb{R}^n)$ e $L^\infty(\mathbb{R}^n)$ de $\boldsymbol{u}$ e $\mathbf{w}$.

Teorema 2.4. *Sejam $\boldsymbol{u}$ e $\mathbf{w}$ soluções de Leray do Sistema Navier-Stokes Micropolar, então existe t_0 suficientemente grande tal que*

1.

$$\begin{aligned}(t-t_0)\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\\ +\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds\\ +2(n-2)\int_{t_0}^{t}(s-t_0)\|D(\nabla\cdot w)(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds\\ +2\chi\int_{t_0}^{t}(s-t_0)\|Dw(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds\le 2\int_{t_0}^{t}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds,\end{aligned} \tag{2.18}$$

para todo $t > t_0 > t_*$.

2.

$$\int_{t_0}^{\infty}(s-t_0)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds<\infty. \tag{2.19}$$

Demonstração. **Caso 3D.**

Pelos Teoremas 2.3 e 2.1, existe $t_0 > t_*$ suficientemente grande tal que para todo $t \ge t_0$

$$C\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}<\min(\mu,\gamma), \tag{2.20}$$

onde $C > 0$ é a constante que aparece em (2.28) adiante.

Escrevendo a equação (2.1a) em coordenadas, temos que

$$u_{i,t}+\sum_{j=1}^{3}u_jD_ju_i+D_ip=(\mu+\chi)\sum_{j=1}^{3}D_jD_ju_i+\chi\sum_{j,k=1}^{3}\epsilon_{ijk}D_jw_k. \tag{2.21}$$

Derivando (2.21) em relação a x_l, multiplicando por $2(s-t_0)D_lu_i$ e integrando em $\mathbb{R}^3\times(t_0,t)$, obtemos

$$\int_{\mathbb{R}^3}\int_{t_0}^{t}(s-t_0)[(D_lu_i)^2]_t dsdx+2\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{j=1}^{3}D_lu_iD_lu_jD_ju_idxds$$
$$+\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{j=1}^{3}u_jD_j[(D_lu_i)^2]dxds+2\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)D_lu_iD_lD_ipdxds$$
$$=2(\mu+\chi)\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{j=1}^{3}D_lu_iD_lD_jD_ju_idxds$$
$$+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{j,k=1}^{3}\epsilon_{ijk}D_lu_iD_lD_jw_kdxds.$$

Integrando por partes e somando em i e l, temos que

$$(t-t_0)\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2(\mu+\chi)\int_{t_0}^{t}(s-t_0)\|D^2\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$\leq\int_{t_0}^{t}\|D\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds+2\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{i,j,l=1}^{3}u_iD_lu_jD_lD_ju_idxds \tag{2.22}$$
$$+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{i,j,l,k=1}^{3}\epsilon_{kji}D_lu_iD_lD_jw_kdxds.$$

Como

$$\int_{\mathbb{R}^3}\sum_{i,j,l=1}^{3}u_iD_lu_jD_lD_ju_idx=\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}u_i\sum_{l=1}^{3}D_lu_jD_lD_ju_idx$$
$$\leq\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}u_i\Big(\sum_{l=1}^{3}(D_lu_j)^2\Big)^{1/2}\Big(\sum_{l=1}^{3}(D_lD_ju_i)^2\Big)^{1/2}dx$$
$$\leq\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)}\sum_{i,j=1}^{3}\Big(\int_{\mathbb{R}^3}\sum_{l=1}^{3}(D_lu_j)^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3}\sum_{l=1}^{3}(D_lD_ju_i)^2dx\Big)^{1/2} \tag{2.23}$$
$$\leq\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)}\sum_{i}^{3}\Big(\int_{\mathbb{R}^3}\sum_{j,l=1}^{3}(D_lu_j)^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^3}\sum_{j,l=1}^{3}(D_lD_ju_i)^2dx\Big)^{1/2}$$
$$\leq C\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)}\|D\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}\|D^2\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}$$
$$\leq C\|\boldsymbol{u},\mathbf{w}\|_{L^\infty(\mathbb{R}^3)}\|D\boldsymbol{u},D\mathbf{w}\|_{L^2(\mathbb{R}^3)}\|D^2\boldsymbol{u},D^2\mathbf{w}\|_{L^2(\mathbb{R}^3)}.$$

A equação (2.22) se torna

$$(t-t_0)\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\int_{t_0}^{t}(s-t_0)\|D^2\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$\leq \int_{t_0}^{t}\|D\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + 2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{i,j,l,k=1}^{3}\epsilon_{kji}D_l u_i D_l D_j w_k dxds$$
$$+C\int_{t_0}^{t}(s-t_0)\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds. \tag{2.24}$$

Analogamente, fazendo as mesmas contas para a equação (2.1b), obtemos que

$$(t-t_0)\|D\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} + 2\gamma\int_{t_0}^{t}(s-t_0)\|D^2\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$+2\int_{t_0}^{t}(s-t_0)\|D(\nabla\cdot\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + 4\chi\int_{t_0}^{t}(s-t_0)\|D\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$\leq \int_{t_0}^{t}\|D\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + 2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{i,j,l,k=1}^{3}\epsilon_{kji}D_l u_i D_l D_j w_k dxds$$
$$+C\int_{t_0}^{t}(s-t_0)\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds. \tag{2.25}$$

Somando as equações (2.24) e (2.25), temos que

$$(t-t_0)\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\int_{t_0}^{t}(s-t_0)\|D^2\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$+2\gamma\int_{t_0}^{t}(s-t_0)\|D^2\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + 2\int_{t_0}^{t}(s-t_0)\|D(\nabla\cdot\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$+4\chi\int_{t_0}^{t}(s-t_0)\|D\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \leq \int_{t_0}^{t}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$+4\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)\sum_{i,j,l,k=1}^{3}\epsilon_{kji}D_l u_i D_l D_j w_k dxds$$
$$+C\int_{t_0}^{t}(s-t_0)\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds. \tag{2.26}$$

Usando a desigualdade (A.7) e o fato de que o termo

$$4\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{i,j,l,k=1}^{3} \epsilon_{kji} D_l u_i D_l D_j w_k dx ds$$

pode ser estimado por (uma vez que $\nabla \cdot \boldsymbol{u} = 0$)

$$\begin{aligned}
\Big| \int_{\mathbb{R}^3} \sum_{i,j,k,l=1}^{3} \epsilon_{ijk} D_l w_i D_l D_j u_k dx \Big| &= \Big| \sum_{i,l=1}^{3} \int_{\mathbb{R}^3} D_l w_i \sum_{j,k=1}^{3} \epsilon_{ijk} D_l D_j u_k dx \Big| \\
\leq \sum_{i,l=1}^{3} \int_{\mathbb{R}^3} |D_l w_i| \Big| \sum_{j,k=1}^{3} \epsilon_{ijk} D_l D_j u_k \Big| dx &\leq \frac{1}{2} \sum_{i,l=1}^{3} \int_{\mathbb{R}^3} (D_l w_i)^2 + \Big(\sum_{j,k=1}^{3} \epsilon_{ijk} D_l D_j u_k \Big)^2 dx \\
&= \frac{1}{2} \sum_{i,l=1}^{3} \int_{\mathbb{R}^3} (D_l w_i)^2 dx + \frac{1}{2} \sum_{i,l=1}^{3} \int_{\mathbb{R}^3} \Big(\sum_{j,k=1}^{3} \epsilon_{ijk} D_l D_j u_k \sum_{p,q=1}^{3} \epsilon_{ipq} D_l D_p u_q \Big) dx \\
&= \frac{1}{2} \|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_{\mathbb{R}^3} \sum_{l,j,k,p,q=1}^{3} \Big(\sum_{i=1}^{3} \epsilon_{ijk} \epsilon_{ipq} \Big) D_l D_j u_k D_l D_p u_q dx \\
&= \frac{1}{2} \|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_{\mathbb{R}^3} \sum_{l,j,k,p,q=1}^{3} (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) D_l D_j u_k D_l D_p u_q dx \\
&= \frac{1}{2} \|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \sum_{l,j,k=1}^{3} \int_{\mathbb{R}^3} (D_l D_j u_k)^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \sum_{l,j,k=1}^{3} D_l D_j u_k D_l D_k u_j dx \\
&= \frac{1}{2} \|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \|D^2 \boldsymbol{u}\|_{L^2(\mathbb{R}^3)}^2,
\end{aligned} \tag{2.27}$$

obtemos que

$$\begin{aligned}
(t - t_0) \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + 2\mu \int_{t_0}^{t} (s - t_0) \|D^2 \boldsymbol{u}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds \\
+2\gamma \int_{t_0}^{t} (s - t_0) \|D^2 \mathbf{w}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds + 2 \int_{t_0}^{t} (s - t_0) \|D(\nabla \cdot \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds \\
+2\chi \int_{t_0}^{t} (s - t_0) \|D\mathbf{w}(\cdot, s)\| ds \leq \int_{t_0}^{t} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 ds \\
+C \int_{t_0}^{t} (s - t_0) \|(\boldsymbol{u}, \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^{1/2} \|(D\boldsymbol{u}, D\mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^{1/2} \|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds.
\end{aligned} \tag{2.28}$$

Aplicando (2.20) obtemos 1.:

$$(t-t_0)\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$+2\int_{t_0}^{t}(s-t_0)\|D(\nabla\cdot\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + 2\chi\int_{t_0}^{t}(s-t_0)\|D\mathbf{w}(\cdot,s)\|ds$$
$$\leq \int_{t_0}^{t}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}ds.$$

O item 2 é uma consequência direta do item 1, pois pelo Teorema 2.1 sabemos que

$$\int_{t_0}^{\infty}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}ds < \infty.$$

Caso 2D

O caso 2D é análogo, com o fato adicional de que em 2D, temos $\nabla\cdot\mathbf{w} = 0$, o que simplifica um pouco a desigualdade de energia

□

Teorema 2.5. *Sejam $\boldsymbol{u}$ e $\mathbf{w}$ soluções de Leray do sistema Micropolar, então existe t_0 suficientemente grande tal que*

1.

$$\begin{aligned}(t-t_0)^2\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\\ +\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^2\|(D^3\boldsymbol{u},D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds\\ +2(n-2)\int_{t_0}^{t}(s-t_0)^2\|D^2(\nabla\cdot w)(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds\\ +2\chi\int_{t_0}^{t}(s-t_0)^2\|D^2w(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds\\ \leq 2\int_{t_0}^{t}(s-t_0)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds,\end{aligned} \tag{2.29}$$

para todo $t > t_0 > t_*$.

2.

$$\int_{t_0}^{\infty} (s-t_0)^2 \|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)} ds < \infty. \tag{2.30}$$

3.

$$\lim_{t\to\infty} t\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0. \tag{2.31}$$

Demonstração. **Caso 3D.**

A prova é similar a do Teorema (2.4).

Dado $\epsilon > 0$ existe, pelo Teorema 2.4, $t_0 > t_*$ suficientemente grande tal que

$$\int_{t_0}^{\infty} (s-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds < \epsilon^2.$$

Além disso, pelos Teoremas 2.3 e 2.1, podemos supor que t_0 é suficientemente grande tal que para todo $t \geq t_0$

$$C\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)} < \min(\mu,\gamma) \tag{2.32}$$

onde $C > 0$ é a constante que aparece em (2.37) adiante.

Escrevendo a equação (2.1a) em coordenadas, temos que

$$u_{i,t} + \sum_{j=1}^{3} u_j D_j u_i + D_i p = (\mu+\chi)\sum_{j=1}^{3} D_j D_j u_i + \chi \sum_{j,k=1}^{3} \epsilon_{ijk} D_j w_k.$$

Derivando duas vezes, em relação a x_{l_1} e x_{l_2}, multiplicando por $2(s-t_0)^2 D_{l_1}D_{l_2}u_i$ e integrando em $\mathbb{R}^3 \times (t_0,t)$, obtemos que

$$\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2[(D_{l_1}D_{l_2}u_i)^2]_s dxds + 2\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2\sum_{j=1}^{3}D_{l_1}D_{l_2}u_i(D_{l_1}D_{l_2}u_jD_ju_i$$
$$+D_{l_2}u_jD_{l_1}D_ju_i + D_{l_1}u_jD_{l_2}D_ju_i + u_jD_{l_1}D_{l_2}D_ju_i)dxds$$
$$+2\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2D_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}D_ipdxds$$
$$= 2(\mu+\chi)\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2\sum_{j=1}^{3}D_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}D_jD_ju_idxds$$
$$+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2\sum_{j,k=1}^{3}\epsilon_{ijk}D_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}D_jw_kdxds.$$

De maneira análoga a (2.23), podemos obter as seguintes desigualdades:

$$\int_{\mathbb{R}^3}\sum_{i,j,l_1,l_2=1}^{3}u_iD_{l_1}D_{l_2}u_jD_{l_1}D_{l_2}D_ju_idx$$
$$\leq C\|\boldsymbol{u},\mathbf{w}\|_{L^\infty(\mathbb{R}^3)}\|D^2\boldsymbol{u},D^2\mathbf{w}\|_{L^2(\mathbb{R}^3)}\|D^3\boldsymbol{u},D^3\mathbf{w}\|_{L^2(\mathbb{R}^3)} \tag{2.33}$$

e

$$\int_{\mathbb{R}^3}\sum_{i,j,l_1,l_2=1}^{3}D_{l_1}u_iD_{l_2}u_jD_{l_1}D_{l_2}D_ju_idx \leq$$
$$\leq C\|D\boldsymbol{u},D\mathbf{w}\|_{L^\infty(\mathbb{R}^3)}\|D\boldsymbol{u},D\mathbf{w}\|_{L^2(\mathbb{R}^3)}\|D^3\boldsymbol{u},D^3\mathbf{w}\|_{L^2(\mathbb{R}^3)}. \tag{2.34}$$

Integrando por partes, utilizando que $\nabla \cdot \boldsymbol{u} = 0$ e as desigualdades (2.33) e (2.34), e somando em i, l_1 e l_2, temos que

$$\begin{aligned}
&(t-t_0)^2\|D^2\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\int_{t_0}^{t}(s-t_0)^2\|D^3\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&\leq 2\int_{t_0}^{t}(s-t_0)\|D^2\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + C\int_{t_0}^{t}(s-t_0)^2\|(D^3\boldsymbol{u},D^3\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\times \\
&\times\big[\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
&+\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\big]ds \\
&+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2\sum_{i,j,k,l_1,l_2=1}^{3}\epsilon_{ijk}D_{l_1}D_{l_2}w_iD_{l_1}D_{l_2}D_ju_kdxds.
\end{aligned} \tag{2.35}$$

De maneira análoga podemos trabalhar com a equação (2.1b) para obter que

$$\begin{aligned}
&(t-t_0)^2\|D^2\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\gamma\int_{t_0}^{t}(s-t_0)^2\|D^3\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&+2\int_{t_0}^{t}(s-t_0)^2\|D^2\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + 4\chi\int_{t_0}^{t}(s-t_0)^2\|D^2\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} \\
&\leq 2\int_{t_0}^{t}(s-t_0)\|D^2\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + C\int_{t_0}^{t}(s-t_0)^2\|(D^3\boldsymbol{u},D^3\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\times \\
&\times\big[\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
&+\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\big]ds \\
&+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2\sum_{i,j,k,l_1,l_2=1}^{3}\epsilon_{ijk}D_{l_1}D_{l_2}w_iD_{l_1}D_{l_2}D_ju_kdxds.
\end{aligned} \tag{2.36}$$

Somando as equações (2.35) e (2.36), temos que

$$
\begin{aligned}
(t-t_0)^2\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi)\int_{t_0}^{t}(s-t_0)^2\|D^3\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
+2\gamma\int_{t_0}^{t}(s-t_0)^2\|D^3\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
+2\int_{t_0}^{t}(s-t_0)^2\|D^2\nabla\cdot\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)} + 4\chi\int_{t_0}^{t}(s-t_0)^2\|D^2\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds \\
\leq 2\int_{t_0}^{t}(s-t_0)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + C\int_{t_0}^{t}(s-t_0)^2\|(D^3\boldsymbol{u},D^3\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\times \\
\times\big[\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
+\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\big]ds \\
+4\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(s-t_0)^2\sum_{i,j,k,l_1,l_2=1}^{3}\epsilon_{ijk}D_{l_1}D_{l_2}w_iD_{l_1}D_{l_2}D_ju_kdxds.
\end{aligned}
$$

O termo

$$\int_{\mathbb{R}^3}\sum_{i,j,k,l_1,l_2=1}^{3}\epsilon_{ijk}D_{l_1}D_{l_2}w_iD_{l_1}D_{l_2}D_ju_kdx$$

pode ser estimado como na desigualdade (2.27) da prova do Teorema 2.4.

$$\left|\int_{\mathbb{R}^3}\sum_{i,j,k,l_1,l_2=1}^{3}\epsilon_{ijk}D_{l_1}D_{l_2}w_iD_{l_1}D_{l_2}D_ju_kdx\right| \leq \frac{1}{2}\|D^2\mathbf{w}\|_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|D^3\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}.$$

Já o termo

$$\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}+\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}$$

pode ser estimado pelas desigualdades de Sobolev (A.8) e (A.9).

Obtemos então que

$$
\begin{aligned}
\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
+\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \qquad (2.37) \\
\leq C\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D^3\boldsymbol{u},D^3\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Aplicando (2.32), temos que

$$(t-t_0)^2\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\min(\mu,\gamma)\int_{t_0}^t(s-t_0)^2\|(D^3\boldsymbol{u},D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$+2\int_{t_0}^t(s-t_0)^2\|D^2(\nabla\cdot\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds+2\chi\int_{t_0}^t(s-t_0)^2\|D^2\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds$$
$$\leq 2\int_{t_0}^t(s-t_0)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds.$$

Isto prova o item 1. O item 2 é uma consequência direta do item 1, Pois pelo Teorema 2.4 sabemos que

$$\int_{t_0}^\infty(t-s)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}ds<\infty.$$

Para mostrar o item 3, note que

$$t^2\left(\frac{t-t_0}{t}\right)^2\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}=(t-t_0)^2\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}$$
$$\leq\int_{t_0}^\infty(s-t_0)\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds<\epsilon^2.$$

Portanto,

$$t\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}<\frac{t}{t-t_0}\epsilon.$$

Observe que $\frac{t}{t-t_0}\leq 2$ para todo $t\geq 2t_0$. Portanto, como $\epsilon>0$ é arbitrário,

$$\lim_{t\to\infty}t\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}=0. \tag{2.38}$$

Caso 2D

O caso 2D é análogo, com o fato adicional de que em 2D, temos $\nabla\cdot\mathbf{w}=0$, o que simplifica um pouco a desigualdade de energia

□

Isto conclui os resultados preliminares que serão úteis nos próximos capítulos.

3 COMPORTAMENTO DAS NORMAS L^P

Agora vamos obter nossos primeiros resultados sobre o comportamento assintótico das normas de $\boldsymbol{u}$ e $\mathbf{w}$.

Teorema 3.1. *Seja* $\mathbf{w}$ *solução de Leray do sistema (2.1).*

Se $\chi = 0$*, então*

$$\lim_{t\to\infty} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Demonstração. **Caso 3D**

Vamos reescrever a equação (2.1b) como

$$\mathbf{w}_t = \gamma\Delta\mathbf{w} + Q,$$

onde

$$Q = -\boldsymbol{u}\cdot\nabla\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}).$$

Dado $\epsilon > 0$ arbitrário, temos, pelo Teorema 2.3, que existe t_0 suficientemente grande tal que $\forall\ t > t_0$ vale

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)} < \epsilon t^{-\frac{1}{2}}.$$

Pelo princípio de Duhamel, temos que

$$\mathbf{w}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-s)}Q(\cdot,s)ds,$$

que pode ser reescrito como

$$\mathbf{w}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) - \int_{t_0}^{t} e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}ds + \int_{t_0}^{t} e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})ds.$$

Aplicando a norma L^2 e usando a Desigualdade de Minkowski, temos que

$$\begin{aligned}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \\ + \int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds.\end{aligned} \tag{3.1}$$

Vamos analisar cada termo da desigualdade (3.1) separadamente.

O termo

$$e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)$$

representa a solução da equação do calor, com condição inicial $\mathbf{w}(\cdot,t_0)$ e

$$\lim_{t\to\infty} \|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0, \tag{3.2}$$

cuja a prova pode ser encontrada no Corolário C.4.

Agora vamos analisar o termo

$$\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds.$$

Para isso vamos utilizar o Lema C.3.

$$\begin{aligned}\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds &\leq K\gamma^{-3/4}\int_{t_0}^{t}(t-s)^{-3/4}\|\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^1(\mathbb{R})}ds \\ &\leq K\gamma^{-3/4}\int_{t_0}^{t}(t-s)^{-3/4}\|\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \\ &\leq K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\int_{t_0}^{t}(t-s)^{-3/4}s^{-1/2}ds\end{aligned}$$

e

$$\begin{aligned}\int_{t_0}^{t}(t-s)^{-3/4}s^{-1/2}ds &= \int_{t_0}^{\frac{t}{2}}(t-s)^{-3/4}s^{-1/2}ds + \int_{\frac{t}{2}}^{t}(t-s)^{-3/4}s^{-1/2}ds \\ &\leq K\left[t^{-3/4}\int_{t_0}^{\frac{t}{2}}s^{-1/2}ds + t^{-1/2}\int_{\frac{t}{2}}^{t}(t-s)^{-3/4}ds\right] \leq Kt^{-1/4}.\end{aligned}$$

De modo que

$$\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \leq K\epsilon\gamma^{-3/4}t^{-1/4}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}.$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty}\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{3.3}$$

Agora vamos analisar o termo

$$\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds.$$

Utilizando a desigualdade (C.2), temos que

$$\begin{aligned}\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds &\leq K\gamma^{-1/2}\int_{t_0}^{t}(t-s)^{-1/2}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds\\ &\leq K\epsilon\gamma^{-1/2}\int_{t_0}^{t}(t-s)^{-1/2}s^{-1/2}ds\end{aligned}$$

e

$$\begin{aligned}\int_{t_0}^{t}(t-s)^{-1/2}s^{-1/2}ds &= \int_{t_0}^{\frac{t}{2}}(t-s)^{-1/2}s^{-1/2}ds + \int_{\frac{t}{2}}^{t}(t-s)^{-1/2}s^{-1/2}ds\\ &\leq K\left[t^{-1/2}\int_{t_0}^{\frac{t}{2}} s^{-1/2}ds + t^{-1/2}\int_{\frac{t}{2}}^{t}(t-s)^{-1/2}ds\right] \leq K.\end{aligned}$$

De modo que

$$\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds \leq K\epsilon\gamma^{-1/2}.$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty}\int_{t_0}^{t} \|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{3.4}$$

Juntando as equações (3.2), (3.3) e (3.4) com (3.1), temos que

$$\lim_{t\to\infty}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0.$$

Caso 2D

As únicas diferenças entre os casos 2D e 3D são as estimativas do apêndice C que são utilizadas (uma vez que elas dependem da dimensão). No entanto essas mudanças não afetam a demonstração, que segue análoga ao caso 3D.

□

Teorema 3.2. *Seja* $\mathbf{w}$ *solução de Leray do sistema (2.1).*

Se $\chi > 0$*, então*

$$\lim_{t\to\infty} t^{\frac{1}{2}} \|\mathbf{w}(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Demonstração. **Caso 3D**

Vamos reescrever a equação (2.1b) como

$$\mathbf{w}_t = \gamma\Delta\mathbf{w} + Q - 2\chi\mathbf{w}, \tag{3.5}$$

onde

$$Q = -\boldsymbol{u} \cdot \nabla\mathbf{w} + \nabla(\nabla \cdot \mathbf{w}) + \chi\nabla \times \boldsymbol{u}.$$

Fazendo a mudança de variável

$$\mathbf{W} = e^{2\chi t}\mathbf{w}$$

e multiplicando a equação (3.5) por $e^{2\chi t}$, obtemos

$$\mathbf{W}_t = \gamma\Delta\mathbf{W} + e^{2\chi t}Q.$$

Dado $\epsilon > 0$ arbitrário, temos, pelo Teorema 2.3, que existe t_0 suficientemente grande tal que $\forall\ t > t_0$ vale que

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^3)} < \epsilon t^{-\frac{1}{2}}.$$

Pelo princípio de Duhamel, temos que

$$\mathbf{W}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{W}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-s)}e^{2\chi s}Q(\cdot,s)ds.$$

Que pode ser reescrito como

$$\begin{aligned}\mathbf{w}(\cdot,t) = e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) - \int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}ds \\ + \int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})ds + \chi\int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}(\nabla\times\boldsymbol{u})ds.\end{aligned}$$

Multiplicando a equação acima por $t^{\frac{1}{2}}$, aplicando a norma L^2 e usando a Desigualdade de Minkowski, temos que

$$\begin{aligned} t^{\frac{1}{2}}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq t^{\frac{1}{2}}e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} \\ &+ t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \\ &+ t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds \\ &+ \chi t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds. \end{aligned} \tag{3.6}$$

Vamos analisar cada termo separadamente.

O termo

$$e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)$$

representa a solução da equação do calor, com condição inicial $\mathbf{w}(\cdot,t_0)$ e

$$\lim_{t\to\infty}\|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0, \tag{3.7}$$

cuja a prova pode ser encontrada no Corolário C.4.

De modo que

$$\lim_{t\to\infty} t^{\frac{1}{2}}e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{3.8}$$

Vamos agora analisar o termo

$$t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds.$$

Para isso vamos utilizar o Lema C.3.

$$\begin{aligned}
t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \\
\leq K\gamma^{-3/4}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}\|\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^1(\mathbb{R}^3)}ds \\
\leq K\gamma^{-3/4}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}\|\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \\
\leq K\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \\
\leq K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}s^{-1/2}ds
\end{aligned}$$

e

$$\begin{aligned}
t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}s^{-1/2}ds = \\
= t^{1/2}\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)}(t-s)^{-3/4}s^{-1/2}ds + t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}s^{-1/2}ds \\
\leq t^{1/2}e^{-\chi t}\left(\frac{t}{2}\right)^{-3/4}\int_{t_0}^{\frac{t}{2}} s^{-1/2}ds + t^{1/2}t^{-1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(t-s)^{-3/4}ds \\
\leq Ce^{-\chi t}t^{1/4} + (2\chi)^{-1/4}\int_0^{\infty} e^{-\tau}\tau^{-3/4}d\tau \\
\leq Ce^{-\chi t}t^{1/4} + (2\chi)^{-1/4}\Gamma\left(\frac{1}{4}\right).
\end{aligned}$$

Assim,

$$\begin{aligned}
t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds \leq \\
\leq K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\left[e^{-\chi t}t^{1/4} + (2\chi)^{-1/4}\Gamma\left(\frac{1}{4}\right)\right].
\end{aligned}$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{3.9}$$

Agora vamos analisar o termo

$$t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds.$$

Para isso vamos utilizar novamente a desigualdade (C.2).

$$\begin{aligned}
&t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds\\
&\leq K\gamma^{-1/2}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-1/2}\|D\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds\\
&\leq K\epsilon\gamma^{-1/2}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-1/2}s^{-1/2}ds.
\end{aligned}$$

Por outro lado,

$$\begin{aligned}
&t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}(t-s)^{-1/2}s^{-1/2}ds \leq\\
&\leq t^{1/2}\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)}(t-s)^{-1/2}s^{-1/2}ds + t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(t-s)^{-1/2}s^{-1/2}ds\\
&\leq t^{1/2}\left(\frac{t}{2}\right)^{-1/2} e^{-\chi t}\int_{t_0}^{\frac{t}{2}} s^{-1/2}ds + t^{1/2}t^{-1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}(t-s)^{-1/2}ds\\
&\leq e^{-\chi t}t^{1/2} + (2\chi)^{-1/2}\int_0^{\infty} e^{-\tau}\tau^{-1/2}d\tau\\
&\leq Ce^{-\chi t}t^{1/2} + (2\chi)^{-1/2}\Gamma\left(\frac{1}{2}\right) = Ce^{-\chi t}t^{1/2} + (2\chi)^{-1/2}\sqrt{\pi}.
\end{aligned}$$

Assim,

$$\begin{aligned}
&t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds \leq\\
&\leq K\epsilon\gamma^{-1/2}\left[e^{-\chi t}t^{1/2} + (2\chi)^{-1/2}\sqrt{\pi}\right].
\end{aligned}$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{3.10}$$

Vamos agora analisar o último termo,

$$\chi t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds.$$

Para isso vamos utilizar a desigualdade (C.2).

$$\chi t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds \leq$$

$$\leq \chi t^{1/2}\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds$$

$$+\chi t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds$$

$$\leq K\chi\gamma^{-1/2}t^{1/2}e^{-\chi t}\int_{t_0}^{\frac{t}{2}}(t-s)^{-1/2}\|\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds + \chi\epsilon t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}s^{-1/2}ds$$

$$\leq K\left[\chi\gamma^{-1/2}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}e^{-\chi t}\int_{t_0}^{\frac{t}{2}}(t-s)^{-1/2}ds + \chi\epsilon t^{1/2}t^{-1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}ds\right]$$

$$\leq K\left[\chi\gamma^{-1/2}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}e^{-\chi t} + \chi\epsilon\frac{1-e^{-\chi t}}{2\chi}\right].$$

De modo que

$$\chi t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds$$
$$\leq K\left[\chi\gamma^{-1/2}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}e^{-\chi t} + \chi\epsilon\frac{1-e^{-\chi t}}{2\chi}\right].$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty}\chi t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}\nabla\times\mathbf{u}\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{3.11}$$

Juntando as equações (3.8), (3.9), (3.10) e (3.11) com (3.6), temos que

$$\lim_{t\to\infty} t^{\frac{1}{2}}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0.$$

Caso 2D

As únicas diferenças entre os casos 2D e 3D são as estimativas do apêndice C que são utilizadas (uma vez que elas dependem da dimensão). No entanto essas mudanças não afetam a demonstração, que segue análoga ao caso 3D.

□

Teorema 3.3. *Seja $\boldsymbol{u}$ solução de Leray do sistema (2.1).*

Então,

$$\lim_{t\to\infty} \|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Demonstração. **Caso 3D**

Reescrevemos a equação (2.1a) como

$$\boldsymbol{u}_t = (\mu+\chi)\Delta\boldsymbol{u} + Q_1,$$

onde

$$Q_1 = \chi\nabla\times\mathbf{w} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla p.$$

Usando a identidade (D.1), podemos escrever Q_1 como

$$Q_1 = \mathbb{P}_h[\chi\nabla\times\mathbf{w} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}].$$

Dado $\epsilon > 0$ arbitrário, temos, pelo Teorema 2.3, que existe t_0 suficientemente grande tal que $\forall\ t > t_0$ vale

$$\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)} < \epsilon t^{-\frac{1}{2}}.$$

Pelo princípio de Duhamel, temos que

$$\boldsymbol{u}(\cdot,t) = e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t} e^{(\mu+\chi)\Delta(t-s)}Q_1(\cdot,s)ds. \tag{3.12}$$

Usando a definição de Q_1 e a identidade (D.5), a equação (3.12) pode ser reescrita como

$$\boldsymbol{u}(\cdot,t) = e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t} \mathbb{P}_h[e^{(\mu+\chi)\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{u}]ds$$
$$+ \chi\int_{t_0}^{t} \mathbb{P}_h[e^{(\mu+\chi)\Delta(t-s)}\nabla\times\mathbf{w}]ds.$$

Aplicando a norma L^2 e usando a Desigualdade de Minkowski, temos que

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \|e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds + \chi\int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\nabla\times\mathbf{w}\|_{L^2(\mathbb{R}^3)}ds. \tag{3.13}$$

Vamos analisar cada termo da desigualdade (3.13) separadamente.

O termo

$$e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)$$

representa a solução da equação do calor, com condição inicial $\boldsymbol{u}(\cdot,t_0)$ e

$$\lim_{t\to\infty} \|e^{(\mu+\chi)\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0, \tag{3.14}$$

cuja a prova pode ser encontrada no Corolário C.4.

Agora vamos analisar o termo

$$\int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds.$$

Para isso vamos utilizar o Lema C.3 e a desigualdade de Cauchy-Schwarz.

$$\begin{aligned}
\int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds &\leq K(\mu+\chi)^{-\frac{3}{4}}\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}\|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^1(\mathbb{R}^3)}ds \\
&\leq K(\mu+\chi)^{-\frac{3}{4}}\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}\|\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds \\
&\leq K(\mu+\chi)^{-\frac{3}{4}}\|(\boldsymbol{u},\mathbf{w})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}\|D\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds \\
&< K\epsilon\int_{t_0}^{t}(t-s)^{-\frac{3}{4}}s^{-\frac{1}{2}}ds \leq K\epsilon t^{-\frac{1}{4}}.
\end{aligned}$$

Portanto

$$\lim_{t\to\infty}\int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{3.15}$$

O termo

$$\chi \int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\nabla \times \mathbf{w}\|_{L^2(\mathbb{R}^3)} ds$$

só precisa ser considerado, no caso em que $\chi > 0$, neste caso podemos utilizar o Teorema 3.2 e assumir que t_0 também é grande o suficiente para que dado $\epsilon > 0$ arbitrário, valha, $\forall\ t > t_0$, que

$$\|\mathbf{w}(\cdot, t)\|_{L^2(\mathbb{R}^3)} < \epsilon t^{-\frac{1}{2}}.$$

De modo que, pela desigualdade (C.2),

$$\begin{aligned}
\chi \int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\nabla \times \mathbf{w}\|_{L^2(\mathbb{R}^3)} ds &\leq K\chi(\mu+\chi)^{-1/2} \int_{t_0}^{t} (t-s)^{-1/2} \|\mathbf{w}\|_{L^2(\mathbb{R}^3)} ds \\
&\leq K\chi\epsilon(\mu+\chi)^{-1/2} \int_{t_0}^{t} (t-s)^{-1/2} s^{-1/2} ds \\
&\leq K\chi\epsilon(\mu+\chi)^{-1/2}.
\end{aligned}$$

Portanto

$$\lim_{t\to\infty} \chi \int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-s)}\nabla \times \mathbf{w}\|_{L^2(\mathbb{R}^3)} ds = 0. \tag{3.16}$$

Juntando (3.14), (3.15) e (3.16) com (3.13), temos que

$$\lim_{t\to\infty} \|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^3)} = 0.$$

Caso 2D

As únicas diferenças entre os casos 2D e 3D são as estimativas do apêndice C que são utilizadas (uma vez que elas dependem da dimensão). No entanto essas mudanças não afetam a demonstração, que segue análoga ao caso 3D.

□

Um corolário obtido diretamente das definições de normas é

Corolário 3.4.

$$\lim_{t\to\infty} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0. \tag{3.17}$$

Corolário 3.5. *Seja* $(\boldsymbol{u}, \mathbf{w})(\cdot, t)$ *solução de Leray do problema (2.1). Então,*

$$\lim_{t\to\infty} t^{\frac{n}{4}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

Demonstração. Dado $\epsilon > 0$, pelo Teorema 2.5 e Corolário 3.4 exite t_0 suficientemente grande tal que, para todo $t > t_0$,

$$\|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \epsilon,$$

$$t\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \epsilon.$$

Usando a desigualdade (A.1) (no caso 3D) e (A.17) (no caso 2D), temos que, para todo $t > t_0$,

$$t^{\frac{n}{4}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{\frac{4-n}{4}} \left(t\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \right)^{\frac{n}{4}} < \epsilon^{\frac{4-n}{4}} \epsilon^{\frac{n}{4}} = \epsilon.$$

Como $\epsilon > 0$ é arbitrário,

$$\lim_{t\to\infty} t^{\frac{n}{4}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

□

Corolário 3.6. *Seja* $(\boldsymbol{u}, \mathbf{w})(\cdot, t)$ *solução de Leray do problema (2.1). Então,*

$$\lim_{t\to\infty} t^{\frac{n}{4} - \frac{n}{2p}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^p(\mathbb{R}^n)} = 0,$$

para todo $2 \leq n < \infty$.

Demonstração. Aplicando uma simples interpolação (veja o Lema B.3 no apêndice), temos

$$t^{\frac{n}{4} - \frac{n}{2p}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq K \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{2/p} \left(t^{\left(\frac{n}{4} - \frac{n}{2p}\right)\frac{1}{1-2/p}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \right)^{1-2/p}$$

$$= K \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{2/p} \left(t^{\frac{n}{4}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \right)^{1-2/p}.$$

Pelo Corolário 3.4, encontramos

$$\lim_{t\to\infty} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Temos também, pelo Corolário 3.5, que

$$\lim_{t\to\infty} t^{\frac{n}{4}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = 0.$$

Portanto,

$$\lim_{t\to\infty} t^{\frac{n}{4}-\frac{n}{2p}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^p(\mathbb{R}^n)} = 0.$$

□

Como mostrado neste capítulo no Teorema 3.2, quando $\chi > 0$, temos uma taxa de decaimento maior para as normas $L^q(\mathbb{R}^n)$ de $\mathbf{w}$. Essa propriedade na verdade se estende também para as derivadas de $\mathbf{w}$, como veremos nos próximos capítulos.

4 COMPORTAMENTO DAS NORMAS $\dot{H}^S$

Neste capítulo iremos obter uma sequência de desigualdades de energia e estimaremos as normas $\dot{H}^s(\mathbb{R}^n)$ das soluções de Leray do Sistema Naviere-Stokes Micropolar.

Teorema 4.1. *Dado $m \in \mathbb{N}$, sejam $\boldsymbol{u}$ e $\mathbf{w}$ soluções de Leray do sistema Navier-Stokes Micropolar, então existe t_0 suficientemente grande tal que*

1.

$$\begin{aligned}
(t-t_0)^m \|(D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
+\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds \\
+2\int_{t_0}^{t}(s-t_0)^m \|D^m \nabla\cdot \mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds \\
+2\chi\int_{t_0}^{t}(s-t_0)^m \|D^m \mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds \\
\leq m\int_{t_0}^{t}(s-t_0)^{m-1} \|(D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds,
\end{aligned} \tag{4.1}$$

para todo $t > t_0$, com t_0 suficientemente grande.

2.

$$\int_{t_0}^{\infty}(s-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^n)}ds < \infty. \tag{4.2}$$

3.

$$\lim_{t\to\infty} t^{m/2}\|(D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0. \tag{4.3}$$

Demonstração. **Caso 3D**

A prova é feita por indução. Os casos $m = 1$ e $m = 2$ já foram provados nos Teoremas 2.4 e 2.5. A demonstração será feita de maneira mais sucinta, visto que sua análise envolve as mesmas técnicas desenvolvidas para os casos $m = 1$ e $m = 2$. Vamos começar, então, pelo caso $m = 3$.

Derivando três vezes o sistema (2.1) com respeito a x_{l_1}, x_{l_2} e x_{l_3}, multiplicando por $2(s-t_0)^3 D_{l_1} D_{l_2} D_{l_3} u_i$ a equação (2.1a) e pelo mesmo fator a equação (2.1b) só que com w_i, integrando na região $[t_0, t] \times \mathbb{R}^3$ e usando as mesmas técnicas usadas em (2.4) e (2.5), temos

$$
\begin{aligned}
(t-t_0)^3 \|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu \int_{t_0}^{t} (s-t_0)^3 \|D^4\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
+2\gamma \int_{t_0}^{t} (s-t_0)^3 \|D^4\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
+2\int_{t_0}^{t} (s-t_0)^3 \|D^3\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} + 2\chi \int_{t_0}^{t} (s-t_0)^3 \|D^3\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} \\
\leq 3 \int_{t_0}^{t} (s-t_0)^2 \|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
+K \int_{t_0}^{t} (s-t_0)^3 \|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \times \\
\times \Big\{ \|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} \|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
+\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} \|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \Big\} ds,
\end{aligned}
\tag{4.4}
$$

usando as desigualdades de Sobolev (A.10) e (A.12), temos

$$
\begin{aligned}
(t-t_0)^3 \|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu \int_{t_0}^{t} (s-t_0)^3 \|D^4\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
+2\gamma \int_{t_0}^{t} (s-t_0)^3 \|D^4\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
+2\int_{t_0}^{t} (s-t_0)^3 \|D^3\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} + 2\chi \int_{t_0}^{t} (s-t_0)^3 \|D^3\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} \\
\leq 3 \int_{t_0}^{t} (s-t_0)^2 \|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
+K \int_{t_0}^{t} (s-t_0)^3 \|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \times \\
\times \Big\{ 2\| (\boldsymbol{u},\mathbf{w}) \|^{3/4}_{L^2(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|^{1/4}_{L^2(\mathbb{R}^3)} \| (D^4\boldsymbol{u}, D^4\mathbf{w}) \|_{L^2(\mathbb{R}^3)} \Big\} ds.
\end{aligned}
$$

Logo, pelo Teorema 2.1 e por (2.31), existe um t_0 suficientemente grande, tal que

$$
\| (\boldsymbol{u},\mathbf{w})(\cdot,t) \|^{3/4}_{L^2(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t) \|^{1/4}_{L^2(\mathbb{R}^3)} \leq \frac{\min\{\mu,\gamma\}}{2K},
$$

para todo $t > t_0$. Logo,

$$
\begin{aligned}
&(t-t_0)^3\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+\min\{\mu,\gamma\}\int_{t_0}^t (s-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
+2\int_{t_0}^t (s-t_0)^3\|D^3\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} &+ 2\chi\int_{t_0}^t (s-t_0)^3\|D^3\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\\
&\leq 3\int_{t_0}^t (s-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds,
\end{aligned}
\tag{4.5}
$$

o que prova (4.1) para o caso $m = 3$. Agora, por (2.30) e por (4.5), temos que

$$
\int_{t_0}^t (s-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds < \infty, \tag{4.6}
$$

concluindo a prova de (4.2) para $m = 3$. Note que, dado $\epsilon > 0$ por (2.18) e 3.4, existe t_0 suficientemente grande, tal que

$$
\int_{t_0}^t (s-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \leq \epsilon.
$$

Então, por (4.1),

$$
\int_{t_0}^t (s-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \leq \frac{2\epsilon}{\min(\mu,\gamma)}. \tag{4.7}
$$

Logo, por (4.5),

$$
(t-t_0)^3\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq C\epsilon,
$$

para alguma constante $C > 0$. O que implica, repetindo o mesmo argumento em (2.38),

$$
t^{3/2}\|(D^3\boldsymbol{u}, D^3\mathbf{w})\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t\to\infty. \tag{4.8}
$$

O que concluí a prova de 4.1 para $m = 3$. Vamos para o caso $m = 4$. Analogamente ao caso anterior, temos

$$
\begin{aligned}
(t-t_0)^4\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu\int_{t_0}^t (s-t_0)^4\|D^5\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds& \\
+2\gamma\int_{t_0}^t (s-t_0)^4\|D^5\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds& \\
+2\int_{t_0}^t (s-t_0)^4\|D^4\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} + 2\chi\int_{t_0}^t (s-t_0)^4\|D^4\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}& \\
\leq 4\int_{t_0}^t (s-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds& \\
+K\int_{t_0}^t (s-t_0)^4\|(D^5\boldsymbol{u}, D^5\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\times& \\
\times\Big\{\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}& \\
+\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}& \\
+\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}\Big\}ds.&
\end{aligned} \tag{4.9}
$$

Usando as desigualdades de Sobolev (A.11) e (A.12) provadas no apêndice, obtemos

$$
\begin{aligned}
(t-t_0)^4\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu\int_{t_0}^t (s-t_0)^4\|D^5\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds& \\
+2\gamma\int_{t_0}^t (s-t_0)^4\|D^5\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds& \\
+2\int_{t_0}^t (s-t_0)^4\|D^4\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} + 2\chi\int_{t_0}^t (s-t_0)^4\|D^4\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}& \\
\leq 4\int_{t_0}^t (s-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds& \\
+K\int_{t_0}^t (s-t_0)^4\|(D^5\boldsymbol{u}, D^5\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\times& \\
\times\Big\{\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|^{3/4}_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^{1/4}_{L^2(\mathbb{R}^3)}& \\
+\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|^{5/6}_{L^2(\mathbb{R}^3)}\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^{1/6}_{L^2(\mathbb{R}^3)}& \\
+\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|^{3/4}_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^{1/4}_{L^2(\mathbb{R}^3)}\Big\}ds.&
\end{aligned}
$$

Pelo mesmo argumento anterior, dado $\epsilon > 0$ existe um $t_0 > t_*$ suficientemente grande tal que por 3.4, (2.31) e (4.8), temos que

$$\begin{aligned} &(t-t_0)^4\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\ &+\min\{\mu,\gamma\}\int_{t_0}^t (s-t_0)^4\|(D^5\boldsymbol{u}, D^5\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\ +2\int_{t_0}^t (s-t_0)^4\|D^4\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} &+ 2\chi\int_{t_0}^t (s-t_0)^4\|D^4\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\ &\leq 4\int_{t_0}^t (s-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds, \end{aligned} \tag{4.10}$$

provando (4.1) para o caso $m = 4$. Por (4.6) e (4.10), temos que

$$\int_{t_0}^t (s-t_0)^4\|(D^5\boldsymbol{u}, D^5\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds < \infty, \tag{4.11}$$

provando (4.2) para o caso $m = 4$. Agora, dado $\epsilon > 0$, por (4.7) e (4.5),

$$\int_{t_0}^t (s-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds < \epsilon.$$

Portanto, por (4.10),

$$(t-t_0)^4\|(D^4\boldsymbol{u}, D^4\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} < 4\epsilon.$$

para todo $t > t_0$. O que implica, pelo mesmo raciocínio usado anteriormente,

$$t^2\|(D^4\boldsymbol{u}, D^4\mathbf{w})\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty. \tag{4.12}$$

Supondo que o resultado é verdadeiro para todo $n \in \mathbb{N}$ tal que $n < m$, vamos provar que ele é verdadeiro para m, onde $m \geq 5$.

Dado $\epsilon > 0$, existe, pelo caso $m-1$, t_0 suficientemente grande tal que

$$\int_{t_0}^\infty (s-t_0)^{m-1}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds < \frac{\epsilon^2}{m}.$$

Além disso, podemos supor que t_0 é suficientemente grande tal que, pela hipótese de indução,

$$\sum_{l=0}^{[m]/2} \|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|^{\frac{l+3/2}{l+2}}_{L^2(\mathbb{R}^3)}\|(D^{l+2}\boldsymbol{u}, D^{l+2}\mathbf{w})(\cdot,t)\|^{\frac{1/2}{l+2}}_{L^2(\mathbb{R}^3)} < \frac{\min(\mu,\gamma)}{C}, \tag{4.13}$$

onde $C > 0$ é a constante que aparece na desigualdade (4.17) adiante.

Escrevendo a equação (2.1a) em coordenadas, temos que

$$u_{i,t} + \sum_{j=1}^{3} u_j D_j u_i + D_i p = (\mu + \chi) \sum_{j=1}^{3} D_j D_j u_i + \chi \sum_{j,k=1}^{3} \epsilon_{ijk} D_j w_k.$$

Derivando m vezes, em relação a x_{l_1}, x_{l_2} , ... x_{l_m}, multiplicando por $2(s-t_0)^m D_{l_1} D_{l_2} ... D_{l_m} u_i$ e integrando em $\mathbb{R}^3 \times (t_0, t)$, obtemos que

$$\begin{aligned}
&\int_{t_0}^{t} \int_{\mathbb{R}^3} (s-t_0)^m [(D_{l_1} D_{l_2} ... D_{l_m} u_i)^2]_s ds dx \\
&+2 \int_{t_0}^{t} \int_{\mathbb{R}^3} (s-t_0)^m D_{l_1} D_{l_2} ... D_{l_m} u_i D_{l_1} D_{l_2} ... D_{l_m} \sum_{j=1}^{3} u_j D_j u_i dx ds \\
&+2 \int_{t_0}^{t} \int_{\mathbb{R}^3} (s-t_0)^m D_{l_1} D_{l_2} ... D_{l_m} u_i D_{l_1} D_{l_2} ... D_{l_m} D_i p dx ds \\
&= 2(\mu + \chi) \int_{t_0}^{t} \int_{\mathbb{R}^3} (s-t_0)^m \sum_{j=1}^{3} D_{l_1} D_{l_2} ... D_{l_m} u_i D_{l_1} D_{l_2} ... D_{l_m} D_j D_j u_i dx ds \\
&+2\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} (s-t_0)^m \sum_{j,k=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} ... D_{l_m} u_i D_{l_1} D_{l_2} ... D_{l_m} D_j w_k dx ds.
\end{aligned}$$

Integrando por partes e somando em i, l_1, l_2,...,l_m, temos que

$$\begin{aligned}
&(t-t_0)^m \|D^m \boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi) \int_{t_0}^{t} (s-t_0)^m \|D^{m+1} \boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&\leq m \int_{t_0}^{t} (s-t_0)^{m-1} \|D^m \boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&+C \int_{t_0}^{t} (s-t_0)^m \|(D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
&\times \sum_{l=0}^{[m]/2} \|(D^l \boldsymbol{u}, D^l \mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} \|(D^{m-l} \boldsymbol{u}, D^{m-l} \mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&+2\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} (s-t_0)^m \sum_{j,k,l_1,l_2,...,l_m=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} ... D_{l_m} w_i D_{l_1} D_{l_2} ... D_{l_m} D_j u_k dx ds.
\end{aligned} \tag{4.14}$$

De forma análoga, trabalhando com a equação (2.1b), temos que

$$
\begin{aligned}
&(t-t_0)^m \|D^m \mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\gamma \int_{t_0}^t (s-t_0)^m \|D^{m+1}\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&+2\int_{t_0}^t (s-t_0)^m \|D^m \nabla \cdot \mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds + 4\chi \int_{t_0}^t (s-t_0)^m \|D^m \mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&\leq m \int_{t_0}^t (s-t_0)^{m-1} \|D^m \mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&+C \int_{t_0}^t (s-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
&\times \sum_{l=0}^{[m]/2} \|(D^l \boldsymbol{u}, D^l \mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} \|(D^{m-l}\boldsymbol{u}, D^{m-l}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&+2\chi \int_{t_0}^t \int_{\mathbb{R}^3} (s-t_0)^m \sum_{j,k,l_1,l_2,\ldots,l_m=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} \ldots D_{l_m} w_i D_{l_1} D_{l_2} \ldots D_{l_m} D_j u_k \, dx ds.
\end{aligned}
\tag{4.15}
$$

Somando as equações (4.14) e (4.15), temos que

$$
\begin{aligned}
&(t-t_0)^m \|(D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2(\mu+\chi) \int_{t_0}^t (s-t_0)^m \|D^{m+1}\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&+2\gamma \int_{t_0}^t (s-t_0)^m \|D^{m+1}\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds + 2\int_{t_0}^t (s-t_0)^m \|D^m \nabla \cdot \mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&+4\chi \int_{t_0}^t (s-t_0)^m \|D^m \mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \leq m \int_{t_0}^t (s-t_0)^{m-1} \|(D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)} ds \\
&+C \int_{t_0}^t (s-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} \\
&\times \sum_{l=0}^{[m]/2} \|(D^l \boldsymbol{u}, D^l \mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^3)} \|(D^{m-l}\boldsymbol{u}, D^{m-l}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&+4\chi \int_{t_0}^t \int_{\mathbb{R}^3} (s-t_0)^m \sum_{j,k,l_1,l_2,\ldots,l_m=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} \ldots D_{l_m} w_i D_{l_1} D_{l_2} \ldots D_{l_m} D_j u_k \, dx ds.
\end{aligned}
\tag{4.16}
$$

O termo

$$\int_{\mathbb{R}^3} \sum_{j,k,l_1,l_2,\ldots,l_m=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} \ldots D_{l_m} w_i D_{l_1} D_{l_2} \ldots D_{l_m} D_j u_k \, dx$$

pode ser estimado de forma análoga a da desigualdade (2.27) na prova do Teorema 2.4,obtendo que

$$\left| \int_{\mathbb{R}^3} \sum_{j,k,l_1,l_2,\ldots,l_m=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} \ldots D_{l_m} w_i D_{l_1} D_{l_2} \ldots D_{l_m} D_j u_k dx \right|$$
$$\leq \frac{1}{2} \|D^m \mathbf{w}\|_{L^2(\mathbb{R}^3)} + \frac{1}{2} \|D^{m+1} \boldsymbol{u}\|_{L^2(\mathbb{R}^3)}.$$

E o termo

$$\sum_{l=0}^{[m]/2} \|(D^l \boldsymbol{u}, D^l \mathbf{w})(\cdot, s)\|_{L^\infty(\mathbb{R}^3)} \|(D^{m-l} \boldsymbol{u}, D^{m-l} \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}$$

pode ser estimado pela desigualdade de Sobolev (A.12):

$$\|(D^l \boldsymbol{u}, D^l \mathbf{w})(\cdot, s)\|_{L^\infty(\mathbb{R}^3)} \|(D^{m-l} \boldsymbol{u}, D^{m-l} \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}$$
$$\leq C \|(\boldsymbol{u}, \mathbf{w})\|_{L^2(\mathbb{R}^3)}^{\frac{l+3/2}{l+2}} \|(D^{l+2} \boldsymbol{u}, D^{l+2} \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^{\frac{1/2}{l+2}} \|(D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)},$$

onde $0 \leq l \leq m - 3$, obtendo então que

$$(t - t_0)^m \|(D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2$$
$$+2 \min\{\mu, \gamma\} \int_{t_0}^{t} (s - t_0)^m \|(D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds$$
$$+2 \int_{t_0}^{t} (s - t_0)^m \|D^m \nabla \cdot \mathbf{w}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds + 2\chi \int_{t_0}^{t} (s - t_0)^m \|D^m \mathbf{w}(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds$$
$$\leq m \int_{t_0}^{t} (s - t_0)^{m-1} \|(D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2 ds$$
$$+C \int_{t_0}^{t} (s - t_0)^m \|(D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^2$$
$$\times \sum_{l=0}^{[m]/2} \|(\boldsymbol{u}, \mathbf{w})\|_{L^2(\mathbb{R}^3)}^{\frac{l+3/2}{l+2}} \|(D^{l+2} \boldsymbol{u}, D^{l+2} . \mathbf{w})(\cdot, s)\|_{L^2(\mathbb{R}^3)}^{\frac{1/2}{l+2}} ds$$

Agora basta aplicar (4.13), para obtermos o item 1. :

$$
\begin{aligned}
(t-t_0)^m\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
+\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^m\|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
+2\int_{t_0}^{t}(s-t_0)^m\|D^m\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
+2\chi\int_{t_0}^{t}(s-t_0)^m\|D^m\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
\leq m\int_{t_0}^{t}(s-t_0)^{m-1}.\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds
\end{aligned}
\tag{4.17}
$$

O item 2. é uma consequência direta do item 1., pois, pela hipótese de indução aplicada ao caso $m-1$, temos que

$$\int_{t_0}^{t}(s-t_0)^{m-1}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds<\infty,$$

para todo $t>t_0$ suficientemente grande.

Para mostrar o item 3. note que

$$
\begin{aligned}
t^m\left(\frac{t-t_0}{t}\right)^m\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &= (t-t_0)^m\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
&\leq m\int_{t_0}^{\infty}(s-t_0)^{m-1}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds<\epsilon^2.
\end{aligned}
$$

Portanto,

$$t^{m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}<\left(\frac{t}{t-t_0}\right)^{m/2}\epsilon.$$

Observe que $\frac{t}{t-t_0}\leq 2$ para todo $t\geq 2t_0$. Portanto, como $\epsilon>0$ é arbitrário,

$$\lim_{t\to\infty}t^{m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}=0.$$

Caso 2D

No caso 2D, estimamos (4.4), (4.9) e (4.16) usando a desigualdade (A.16), além disso, a estimativa (4.13) é substituída por

$$C\left(\frac{[m]}{2}+1\right)\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)} < \min(\mu,\gamma), \tag{4.18}$$

onde $C > 0$ é a constante que aparece na desigualdade (4.22) adiante.

Escrevendo a equação (2.1a) em coordenadas, temos que

$$u_{i,t} + \sum_{j=1}^{3} u_j D_j u_i + D_i p = (\mu+\chi)\sum_{j=1}^{3} D_j D_j u_i + \chi \sum_{j,k=1}^{3} \epsilon_{ijk} D_j w_k.$$

Derivando m vezes, em relação a x_{l_1}, x_{l_2} , ... x_{l_m}, multiplicando por $2(s-t_0)^m D_{l_1}D_{l_2}...D_{l_m}u_i$ e integrando em $\mathbb{R}^3 \times (t_0,t)$, obtemos

$$\begin{aligned}
&\int_{t_0}^{t}\int_{\mathbb{R}^2}(s-t_0)^m[(D_{l_1}D_{l_2}u_i)^2]_s dsdx \\
&+2\int_{t_0}^{t}\int_{\mathbb{R}^2}(s-t_0)^m D_{l_1}D_{l_2}...D_{l_m}u_i D_{l_1}D_{l_2}...D_{l_m}\sum_{j=1}^{2}u_j D_j u_i dxds \\
&+2\int_{t_0}^{t}\int_{\mathbb{R}^2}(s-t_0)^m D_{l_1}D_{l_2}...D_{l_m}u_i D_{l_1}D_{l_2}...D_{l_m}D_i p dxds \\
&= 2(\mu+\chi)\int_{t_0}^{t}\int_{\mathbb{R}^2}(s-t_0)^m \sum_{j=1}^{2} D_{l_1}D_{l_2}...D_{l_m}u_i D_{l_1}D_{l_2}...D_{l_m}D_j D_j u_i dxds \\
&+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^2}(s-t_0)^m \sum_{j,k=1}^{2}\epsilon_{ijk} D_{l_1}D_{l_2}...D_{l_m}u_i D_{l_1}D_{l_2}...D_{l_m}D_j w_k dxds.
\end{aligned}$$

Integrando por partes e somando em i, l_1, l_2,...,l_m, temos que

$$\begin{aligned}
(t-t_0)^m\|D^m\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} &+ 2(\mu+\chi)\int_{t_0}^{t}(s-t_0)^m\|D^{m+1}\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
&\leq m\int_{t_0}^{t}(s-t_0)^{m-1}\|D^m\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
&+C\int_{t_0}^{t}(s-t_0)^m\|(D^{m+1}\boldsymbol{u},D^{m+1}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)} \\
&\times\sum_{l=0}^{[m]/2}\|(D^l\boldsymbol{u},D^l\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)}ds \\
&+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^2}(s-t_0)^m\sum_{j,k,l_1,l_2,...,l_m=1}^{2}\epsilon_{ijk}D_{l_1}D_{l_2}...D_{l_m}w_i D_{l_1}D_{l_2}...D_{l_m}D_j u_k dxds.
\end{aligned} \tag{4.19}$$

De forma análoga, trabalhando com a equação (2.1b), temos que

$$
\begin{aligned}
(t-t_0)^m\|D^m\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\gamma\int_{t_0}^t (s-t_0)^m\|D^{m+1}\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
+4\chi\int_{t_0}^t (s-t_0)^m\|D^m\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
\leq m\int_{t_0}^t (s-t_0)^{m-1}\|D^m\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
+C\int_{t_0}^t (s-t_0)^m\|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)} \\
\times\sum_{l=0}^{[m]/2}\|(D^l\boldsymbol{u}, D^l\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)}ds \\
+2\chi\int_{t_0}^t\int_{\mathbb{R}^2}(s-t_0)^m\sum_{j,k,l_1,l_2,\ldots,l_m=1}^{2}\epsilon_{ijk}D_{l_1}D_{l_2}\ldots D_{l_m}w_iD_{l_1}D_{l_2}\ldots D_{l_m}D_ju_kdxds.
\end{aligned}
\tag{4.20}
$$

Somando as equações (4.19) e (4.20), temos que

$$
\begin{aligned}
(t-t_0)^m\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2(\mu+\chi)\int_{t_0}^t (s-t_0)^m\|D^{m+1}\boldsymbol{u}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
+2\gamma\int_{t_0}^t (s-t_0)^m\|D^{m+1}\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
+4\chi\int_{t_0}^t (s-t_0)^m\|D^m\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
\leq m\int_{t_0}^t (s-t_0)^{m-1}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds \\
+C\int_{t_0}^t (s-t_0)^m\|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)} \\
\times\sum_{l=0}^{[m]/2}\|(D^l\boldsymbol{u}, D^l\mathbf{w})(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\|(D^{m-l}\boldsymbol{u}, D^{m-l}\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)}ds \\
+4\chi\int_{t_0}^t\int_{\mathbb{R}^2}(s-t_0)^m\sum_{j,k,l_1,l_2,\ldots,l_m=1}^{2}\epsilon_{ijk}D_{l_1}D_{l_2}\ldots D_{l_m}w_iD_{l_1}D_{l_2}\ldots D_{l_m}D_ju_kdxds.
\end{aligned}
\tag{4.21}
$$

O termo

$$
\int_{\mathbb{R}^3}\sum_{j,k,l_1,l_2,\ldots,l_m=1}^{3}\epsilon_{ijk}D_{l_1}D_{l_2}\ldots D_{l_m}w_iD_{l_1}D_{l_2}\ldots D_{l_m}D_ju_kdx
$$

pode ser estimado de forma análoga a da desigualdade (2.27) na prova do Teorema 2.4,obtendo que

$$\left| \int_{\mathbb{R}^3} \sum_{j,k,l_1,l_2,...,l_m=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} ... D_{l_m} w_i D_{l_1} D_{l_2} ... D_{l_m} D_j u_k dx \right| \\ \leq \frac{1}{2} \| D^m \mathbf{w} \|_{L^2(\mathbb{R}^2)} + \frac{1}{2} \| D^{m+1} . \boldsymbol{u} \|_{L^2(\mathbb{R}^2)}.$$

E o termo

$$\sum_{l=0}^{[m]/2} \| (D^l \boldsymbol{u}, D^l \mathbf{w})(\cdot, s) \|_{L^\infty(\mathbb{R}^2)} \| (D^{m-l} \boldsymbol{u}, D^{m-l} \mathbf{w})(\cdot, s) \|_{L^2(\mathbb{R}^2)}$$

pode ser estimado pela desigualdade de Sobolev (A.16):

$$\| (D^l \boldsymbol{u}, D^l \mathbf{w})(\cdot, s) \|_{L^\infty(\mathbb{R}^2)} \| (D^{m-l} \boldsymbol{u}, D^{m-l} \mathbf{w})(\cdot, s) \|_{L^2(\mathbb{R}^2)} \\ \leq C \| (\boldsymbol{u}, \mathbf{w})(\cdot, s) \|_{L^2(\mathbb{R}^2)} \| (D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot, s) \|_{L^2(\mathbb{R}^2)},$$

onde $0 \leq l \leq m - 3$. Obtemos então que

$$(t - t_0)^m \| (D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot, t) \|^2_{L^2(\mathbb{R}^2)} \\ + 2 \min\{\mu, \gamma\} \int_{t_0}^{t} (s - t_0)^m \| (D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot, s) \|^2_{L^2(\mathbb{R}^2)} ds \\ + 2\chi \int_{t_0}^{t} (s - t_0)^m \| D^m \mathbf{w}(\cdot, s) \|^2_{L^2(\mathbb{R}^2)} ds \\ \leq m \int_{t_0}^{t} (s - t_0)^{m-1} \| (D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot, s) \|^2_{L^2(\mathbb{R}^2)} ds \\ + C \int_{t_0}^{t} (s - t_0)^m \| (D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot, s) \|^2_{L^2(\mathbb{R}^2)} \left(\frac{[m]}{2} + 1 \right) \| (\boldsymbol{u}, \mathbf{w})(\cdot, t) \|_{L^2(\mathbb{R}^2)} ds.$$

Agora basta aplicar (4.18), para obtermos 1. :

$$(t - t_0)^m \| (D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot, t) \|^2_{L^2(\mathbb{R}^2)} \\ + \min(\mu, \gamma) \int_{t_0}^{t} (s - t_0)^m \| (D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w})(\cdot, s) \|^2_{L^2(\mathbb{R}^2)} ds \\ + 2\chi \int_{t_0}^{t} (s - t_0)^m \| D^m \mathbf{w}(\cdot, s) \|^2_{L^2(\mathbb{R}^2)} ds \\ \leq m \int_{t_0}^{t} (s - t_0)^{m-1} \| (D^m \boldsymbol{u}, D^m \mathbf{w})(\cdot, s) \|^2_{L^2(\mathbb{R}^2)} ds. \tag{4.22}$$

O item 2. é uma consequência direta do item 1., pois pela hipótese de indução aplicada ao caso $m-1$, temos que

$$\int_{t_0}^{\infty}(s-t_0)^{m-1}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)}^2ds<\infty.$$

Para mostrar o item 3. note que para $t>t_0$ suficientemente grande, obtemos

$$\begin{aligned} t^m\left(\frac{t-t_0}{t}\right)^m\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}^2 &= (t-t_0)^m\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}^2 \\ &\leq \int_{t_0}^{\infty}(s-t_0)^{m-1}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^2)}^2ds<\epsilon^2. \end{aligned}$$

Portanto,

$$t^{m/2}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}<\left(\frac{t}{t-t_0}\right)^{m/2}\epsilon$$

Observe que $\frac{t}{t-t_0}\leq 2$ para todo $t\geq 2t_0$. Portanto, como $\epsilon>0$ é arbitrário,

$$\lim_{t\to\infty}t^{m/2}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}=0.$$

□

Como consequência do Teorema 4.1, por interpolação, temos o seguinte resultado:

Corolário 4.2. *Seja $(\boldsymbol{u},\mathbf{w})(\cdot,t)$ solução de Leray do problema (2.1). Então,*

$$t^{s/2}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)}\to 0 \quad \textit{ao} \quad t\to\infty, \tag{4.23}$$

para todo $s\geq 0$, $s\in\mathbb{R}$.

Demonstração. Dado $\epsilon>0$, tome $s>0$ arbitrário. Então, para qualquer $m>s$, existe t_0^1 tal que, pelo Corolário 3.4,

$$\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)}\leq\epsilon,$$

para todo $t > t_0^1$. Além disso, por (4.3), existe t_0^2, tal que, para todo $t > t_0^2$, temos

$$t^{\frac{m}{2}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)} \leq \epsilon,$$

logo, usando o Lema B.5, para $t > \max\{t_0^1, t_0^2\}$ e $\gamma > 0$ (a ser escolhido), temos,

$$\begin{aligned} t^{\gamma} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} &\leq t^{\gamma} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)}^{\frac{s}{m}} \\ &\leq \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}} \left(t^{\gamma \frac{m}{s}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)} \right)^{\frac{s}{m}} \leq \epsilon, \end{aligned}$$

se $\gamma \frac{m}{s} = \frac{m}{2}$, portanto, $\gamma = \frac{s}{2}$, para qualquer $s > 0$. O caso $s = 0$ é uma consequência direta do Corolário 3.4. $\square$

5 COMPORTAMENTO DAS NORMAS $\dot{H}^S$ DE w NO CASO $\chi > 0$

Agora vamos obter estimativas melhores para as normas $\dot{H}^s(\mathbb{R}^n)$ de $\mathbf{w}$, no caso em que $\chi > 0$.

Teorema 5.1. *Seja m inteiro positivo. Se $\chi > 0$ então*

$$\lim_{t\to\infty} t^{\frac{m+1}{2}} \|D^m \mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Demonstração. **Caso 3D**

Reescrevendo a equação (2.1b) como

$$w_t = \gamma\Delta\mathbf{w} + Q - 2\chi\mathbf{w}, \tag{5.1}$$

onde

$$Q = -\boldsymbol{u}\cdot\nabla\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}) + \chi\nabla\times\boldsymbol{u}.$$

Fazendo a mudança de variável

$$\mathbf{W} = e^{2\chi t}\mathbf{w}$$

e multiplicando a equação (5.1) por $e^{2\chi t}$, obtemos que

$$\mathbf{W}_t = \gamma\Delta\mathbf{W} + e^{2\chi t}Q.$$

Pelo o que foi estudado nos capítulos anteriores, em particular pelo Teorema 4.1, dado $\epsilon > 0$, exite $t_0 > t_*$ suficientemente grande tal que para todo $l \in \mathbb{N}$, $1 \leq l \leq m+2$ vale que

$$\|(D^l\boldsymbol{u}, D^l\mathbf{w})(\cdot,t)\| < \epsilon t^{-\frac{l}{2}} \text{ para todo } t > t_0.$$

Pelo princípio de Duhamel, temos que

$$\mathbf{W}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{W}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-s)}e^{2\chi s}Q(\cdot,s)ds.$$

Observe que a equação acima pode ser reescrita como

$$\mathbf{w}(\cdot,t) = e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) - \int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}ds$$
$$+\int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})ds + \chi\int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}(\nabla\times\boldsymbol{u})ds.$$

Multiplicando a expressão acima por $t^{\frac{m+1}{2}}$, derivando m vezes em relação a x, aplicando a norma L^2 e usando a Desigualdade de Minkowski, temos que

$$\begin{aligned} t^{\frac{m+1}{2}}\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq K\Bigg[& \underbrace{t^{\frac{m+1}{2}}e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I} \\ & + \underbrace{t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{II} \\ & + \underbrace{t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{III} \\ & + \underbrace{t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{IV}\Bigg]. \end{aligned} \tag{5.2}$$

Vamos analisar cada termo de (5.2) separadamente.

O termo I representa a solução da equação do calor, com condição inicial $\mathbf{w}(\cdot,t_0)$ e, pelo Teorema C.1,

$$\lim_{t\to\infty}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{5.3}$$

De modo que

$$\lim_{t\to\infty} t^{\frac{m+1}{2}}e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{5.4}$$

Para analisar os demais termos, vamos utilizar a desigualdade (C.2).

Usando (A.19), podemos analisar o termo II, obtendo

$$t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds$$

$$\leq Kt^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\sum_{l=0}^{m}\|D^l\boldsymbol{u}(\cdot,s)\|_{L^4(\mathbb{R}^3)}\|D^{m-l+1}\mathbf{w}(\cdot,s)\|_{L^4(\mathbb{R}^3)}ds$$

$$\leq Kt^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\sum_{l=0}^{m}\|D^l\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^{l+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{3/4}$$

$$\times\|D^{m-l+1}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^{m-l+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{3/4}ds$$

$$\leq Kt^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\sum_{l=0}^{m}\epsilon^{\frac{1}{4}}s^{-\frac{l}{8}}\epsilon^{\frac{3}{4}}s^{-\frac{-3(l+1)}{8}}\epsilon^{\frac{1}{4}}s^{-\frac{m-l+1}{8}}\epsilon^{\frac{3}{4}}s^{\frac{-3(m-l+2)}{8}}ds$$

$$\leq Kt^{\frac{m+1}{2}}\epsilon^2\int_{t_0}^{t} e^{-2\chi(t-s)}s^{-\frac{2m+5}{4}}ds$$

$$= Kt^{\frac{m+1}{2}}\epsilon^2\left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)}s^{-\frac{2m+5}{4}}ds+\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}s^{-\frac{2m+5}{4}}ds\right]$$

$$\leq K\epsilon^2\left(t^{\frac{m+1}{2}}e^{-\chi t}\frac{t_0^{-\frac{2m+1}{4}}}{2m+1}+t^{\frac{m+1}{2}}t^{-\frac{2m+5}{4}}\chi^{-1}\right)\leq K\epsilon^2\left(t^{\frac{m+1}{2}}e^{-\chi t}+t^{-\frac{3}{4}}\chi^{-1}\right).$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{5.5}$$

Podemos usar o mesmo raciocínio para o termo III,

$$t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds$$

$$\leq Kt^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds$$

$$\leq Kt^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\epsilon s^{-\frac{m+2}{2}}ds$$

$$= Kt^{\frac{m+1}{2}}\epsilon\left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)}s^{-\frac{m+2}{2}}ds+\int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)}s^{-\frac{m+2}{2}}ds\right]$$

$$\leq K\epsilon\left(t^{\frac{m+1}{2}}e^{-\chi t}\frac{t_0^{-\frac{m}{2}}}{m}+t^{\frac{m+1}{2}}t^{-\frac{m+2}{2}}\chi^{-1}\right)\leq K\epsilon\left(t^{\frac{m+1}{2}}e^{-\chi t}+t^{-\frac{1}{2}}\chi^{-1}\right).$$

Novamente, como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds = 0. \tag{5.6}$$

Usando mais uma vez o mesmo raciocínio para o termo IV,

$$\begin{aligned}
t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
\leq K t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
\leq K t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \epsilon s^{-\frac{m+1}{2}} ds \\
= K t^{\frac{m+1}{2}} \epsilon \left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)} s^{-\frac{m+1}{2}} ds + \int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)} s^{-\frac{m+1}{2}} ds \right] \\
\leq K\epsilon \left(t^{\frac{m+1}{2}} e^{-\chi t} \frac{t_0^{-\frac{m-1}{2}}}{m-1} + t^{\frac{m+1}{2}} t^{-\frac{m+1}{2}} \chi^{-1} \right) \leq K\epsilon \left(t^{\frac{m+1}{2}} e^{-\chi t} + \chi^{-1} \right).
\end{aligned}$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds = 0. \tag{5.7}$$

Pelas equações (5.2), (5.4), (5.5), (5.6) e (5.7), temos que

$$\lim_{t\to\infty} t^{\frac{m+1}{2}} \|D^m \mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0.$$

Caso 2D

As únicas diferenças entre os casos 2D e 3D são as estimativas do apêndice C que são utilizadas (uma vez que elas dependem da dimensão). No entanto essas mudanças não afetam a demonstração, que segue análoga ao caso 3D.

□

A partir daí, podemos concluir dois interessantes resultados por interpolação.

Corolário 5.2. *Seja* $(\boldsymbol{u}, \mathbf{w})(\cdot,t)$ *solução de Leray de (2.1) e* $\chi > 0$. *Então,*

$$t^{\frac{s+1}{2}} \|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0 \quad ao \quad t \to \infty,$$

para todo $s \geq 0$ *real.*

Demonstração. Dado $\epsilon > 0$, tome $s > 0$ arbitrário. Então, para qualquer $m > s$, existe t_0^1 suficientemente grande tal que, pelo Teorema 3.2,

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon,$$

para todo $t > t_0^1$. Pelo Teorema 5.1, existe t_0^2 tal que

$$t^{\frac{m+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)} \leq \epsilon, \ \forall t > t_0^2.$$

Logo, usando o Lema B.5 do apêndice, para todo $t > \max\{t_0^1, t_0^2\}$ e $\gamma = \gamma_1 + \gamma_2 > 0$ (a ser escolhido), temos

$$t^{\gamma}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \left(t^{\frac{\gamma_1}{1-\frac{s}{m}}}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)}\right)^{1-\frac{s}{m}} \left(t^{\frac{\gamma_2}{s/m}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)}\right)^{s/m} \leq \epsilon,$$

se $\frac{\gamma_1}{1-\frac{s}{m}} = 1/2$ e $\frac{\gamma_2}{s/m} = \frac{m+1}{2}$. Logo, $\gamma = \gamma_1 + \gamma_2 = \frac{s+1}{2}$. Ou seja,

$$t^{\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0 \quad ao \quad t \to \infty,$$

para cada $s \geq 0$ real. □

Corolário 5.3. *Seja $(\boldsymbol{u}, \mathbf{w})(\cdot,t)$ solução de Leray de (2.1) e $\chi > 0$. Então,*

$$t^{\frac{5}{4}-\frac{3}{2q}}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^3)} \to 0 \quad ao \quad t \to \infty,$$

para cada $2 \leq q \leq \infty$.

Demonstração. Pelo Teorema 3.2, dado $\epsilon > 0$, existe t_0^1

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon,$$

para todo $t > t_0^1$. Em particular, pelo Teorema 5.1, existe $t_0^2 > 0$ tal que

$$t^{3/2}\|D^2\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon,$$

para todo $t > t_0^2$. Por (A.4), para $\gamma = \gamma_1 + \gamma_2 > 0$ (a ser escolhido), temos

$$t^{\gamma}\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq \left(t^{4\gamma_1}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right)^{1/4} \left(t^{\frac{4\gamma_2}{3}}\|D^2\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right)^{3/4} \leq \epsilon^{1/4}\epsilon^{3/4} = \epsilon,$$

se $4\gamma_1 = 1/2$ e $\frac{4\gamma_2}{3} = 3/2$, i.e., $\gamma_1 = 1/8$ e $\gamma_2 = 9/8$, para todo $t > \max\{t_0^1, t_0^2\}$. Em outras palavras,

$$t^{5/4}\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty. \tag{5.8}$$

Analogamente, seja $\beta = \beta_1 + \beta_2$. Basta aplicar uma simples interpolação (veja o Lema B.3 no apêndice) para obter que

$$t^\beta \|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq \left(t^{\frac{q}{2}\beta_1} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \right)^{2/q} \left(t^{\frac{\beta_2}{1-2/q}} \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \right)^{1-2/q} \leq \epsilon,$$

se $\frac{q}{2}\beta_1 = \frac{1}{2}$ e $\frac{\beta_2}{1-2/q} = 5/4$, i.e., $\beta = 5/4 - 3/2q$. Portanto,

$$t^{\frac{5}{4}-\frac{3}{2q}} \|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty,$$

para cada $2 \leq q \leq \infty$. □

Corolário 5.4. *Seja $(\boldsymbol{u}, \mathbf{w})(\cdot,t)$ solução de Leray de (2.1) e $\chi > 0$. Então,*

$$t^{1-\frac{1}{q}} \|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^2)} \to 0 \quad \textit{ao} \quad t \to \infty,$$

para cada $2 \leq q \leq \infty$.

Demonstração. Pelo Teorema 3.2, dado $\epsilon > 0$, existe t_0^1

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq \epsilon,$$

para todo $t > t_0^1$. Em particular, pelo Teorema 5.1, existe $t_0^2 > 0$ tal que

$$t^{3/2}\|D^2\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq \epsilon,$$

para todo $t > t_0^2$. Por (A.17), para $\gamma = \gamma_1 + \gamma_2 > 0$ (a ser escolhido), temos

$$t^\gamma \|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \leq \left(t^{2\gamma_1} \|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \right)^{1/2} \left(t^{2\gamma_2} \|D^2\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \right)^{1/2} \leq \epsilon^{1/2}\epsilon^{1/2} = \epsilon,$$

se $2\gamma_1 = 1/2$ e $2\gamma_2 = 3/2$, i.e., $\gamma_1 = 1/4$ e $\gamma_2 = 3/4$, para todo $t > \max\{t_0^1, t_0^2\}$. Em outras palavras,

$$t\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \to 0 \quad \text{ao} \quad t \to \infty. \tag{5.9}$$

Analogamente, seja $\beta = \beta_1 + \beta_2$. Aplicando uma simples interpolação (veja o Lema B.3 no apêndice), obtemos

$$t^{\beta}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^2)} \leq \left(t^{\frac{q}{2}\beta_1}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\right)^{2/q}\left(t^{\frac{\beta_2}{1-2/q}}\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\right)^{1-2/q} \leq \epsilon,$$

se $\frac{q}{2}\beta_1 = \frac{1}{2}$ e $\frac{\beta_2}{1-2/q} = 1$, i.e., $\beta = 1 - 1/q$. Portanto,

$$t^{1-\frac{1}{q}}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^2)} \to 0 \quad \text{ao} \quad t \to \infty,$$

para cada $2 \leq q \leq \infty$. □

6 UMA DESIGUALDADE FUNDAMENTAL PARA AS DERIVADAS EM $\mathbb{R}^N$

Neste capítulo, vamos generalizar os resultados obtidos em [14], mostrando uma sequência de desigualdades fundamentais sobre a norma $\dot{H}^s$ das soluções. Além disso obtemos resultados mais fortes quando $\chi > 0$ (veja os Teoremas 6.3 e 6.4 adiante).

Teorema 6.1. *Dado $\alpha > 0$, se*

$$\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty, \tag{6.1}$$

então para todo $m \geq 0$ inteiro, existe $K(\alpha,m) > 0$ constante tal que

$$\limsup_{t\to\infty} t^{\alpha+\frac{m}{2}}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha,m)\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^n)},$$

onde

$$K(\alpha,m) = \frac{1}{\min(\mu,\gamma)^{m/2}}\min_{\delta>0}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+\delta+\frac{j}{2})^{1/2}\right].$$

Demonstração. **Caso 3D**

Seja $t \geq t_0 > t_*$. Conforme seja necessário, vamos supor que t_0 seja cada vez maior. Isso não é um problema pois estamos interessados no limite quando $t \to \infty$.

Dado $\delta > 0$, multiplicamos (2.1a) e (2.1b) por $2(t-t_0)^{2\alpha+\delta}\boldsymbol{u}(x,t)$ e $2(t-t_0)^{2\alpha+\delta}\mathbf{w}(x,t)$ respectivamente. Integrando por partes em $[t_0,t]\times\mathbb{R}^3$ e usando o fato de que $\nabla\cdot\boldsymbol{u} = 0$, obtemos para $t \geq t_0 > t_*$, a desigualdade

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\ +2\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta}\|\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds + 2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\ \leq (2\alpha+\delta)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta-1}\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds.\end{aligned}$$

Dado $0 < \epsilon < 2$ e escolhendo $t_0 > t_*$ suficientemente grande tal que (usando (6.1)), $t^\alpha \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^3)} < \lambda_0(\alpha) + \epsilon$, temos que

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\min(\mu,\gamma)\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds \\ +2\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|\nabla\cdot\mathbf{w}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds &+ 2\chi\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds \\ &\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^t (s-t_0)^{2\alpha+\delta-1}s^{-2\alpha}ds.\end{aligned} \tag{6.2}$$

Note que

$$(s-t_0)^{2\alpha+\delta-1}s^{-2\alpha} \leq (s-t_0)^{\delta-1}.$$

Aplicando a desigualdade acima em (6.2), temos

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\min(\mu,\gamma)\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds \\ +2\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|\nabla\cdot\mathbf{w}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds &+ 2\chi\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds \\ &\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^t (s-t_0)^{\delta-1}ds.\end{aligned}$$

Resolvendo o lado direito da desigualdade anterior, obtemos

$$\begin{aligned}(t-t_0)^{2\alpha+\delta}\|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\min(\mu,\gamma)\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds \\ +2\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|\nabla\cdot\mathbf{w}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds &+ 2\chi\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds \\ &\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^\delta}{\delta}.\end{aligned}$$

Em particular,

$$(t-t_0)^{2\alpha}\|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} \leq \frac{2\alpha+\delta}{\delta}(\lambda_0(\alpha)+\epsilon)^2$$

e

$$\int_{t_0}^t (s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot, s)\|^2_{L^2(\mathbb{R}^3)}ds \leq \frac{1}{2\min(\mu,\gamma)}(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^\delta}{\delta}, \tag{6.3}$$

para todo $t \geq t_0$.

O próximo passo é derivar (2.1a) e (2.1b) em relação a x_l, multiplicar por $2(t-t_0)^{2\alpha+\delta+1}D_l\boldsymbol{u}(x,t)$ e $2(t-t_0)^{2\alpha+\delta+1}D_l\mathbf{w}(x,t)$ respectivamente, integrar por partes em $[t_0,t]\times\mathbb{R}^3$ e utilizar que $\nabla\cdot\boldsymbol{u}=0$ para obter

$$\begin{aligned}
&(t-t_0)^{2\alpha+\delta+1}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+2\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+2\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|D\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds+2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&\leq(2\alpha+\delta+1)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+C\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\\
&\times\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}ds.
\end{aligned} \tag{6.4}$$

Usando a desigualdade (A.7), temos

$$\begin{aligned}
&(t-t_0)^{2\alpha+\delta+1}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+2\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+2\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|D\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds+2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&\leq(2\alpha+\delta+1)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+C\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\times\|(D^2\boldsymbol{u},D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}ds.
\end{aligned} \tag{6.5}$$

Tomando t_0 suficientemente grande, obtemos, pelos Teoremas 2.1 e 2.3, que

$$\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\leq\epsilon\min(\mu,\gamma),\ \forall t>t_0.$$

Aplicando a desigualdade acima em (6.5), temos que

$$
\begin{aligned}
&(t-t_0)^{2\alpha+\delta+1}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+(2-\epsilon)\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+2\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|D\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds+2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&\le(2\alpha+\delta+1)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds.
\end{aligned}
$$

Segue da última desigualdade e de (6.3), que

$$
(t-t_0)^{2\alpha+1}\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\le\frac{1}{(2-\epsilon)\delta\min(\mu,\gamma)}(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2 \tag{6.6}
$$

e

$$
\begin{aligned}
&\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&\le\frac{1}{\delta((2-\epsilon)\min(\mu,\gamma))^2}(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}.
\end{aligned} \tag{6.7}
$$

Em um processo análogo ao feito anteriormente, obtemos

$$
\begin{aligned}
&(t-t_0)^{2\alpha+\delta+2}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+2\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+2}\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+2\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+2}\|D^2\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&\le(2\alpha+\delta+2)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds\\
&+C\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}\|(\boldsymbol{u},\mathbf{w})(\cdot,s)\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\times\|(D\boldsymbol{u}, D\mathbf{w})(\cdot,s)\|^{1/2}_{L^2(\mathbb{R}^3)}ds.
\end{aligned} \tag{6.8}
$$

Pelos Teoremas 2.1 e 2.3, obtemos

$$\begin{aligned}
&(t-t_0)^{2\alpha+\delta+2}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
&+(2-\epsilon)\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+2}\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&+2\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+2}\|D^2\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&+2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&\leq (2\alpha+\delta+2)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+1}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds.
\end{aligned} \tag{6.9}$$

Utilizando (6.7) e (6.9), temos

$$\begin{aligned}
&(t-t_0)^{2\alpha+2}\|(D^2\boldsymbol{u}, D^2\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
&\leq \frac{1}{\delta((2-\epsilon)\min(\mu,\gamma))^2}(2\alpha+\delta+2)(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2
\end{aligned} \tag{6.10}$$

e

$$\begin{aligned}
&\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+2}\|(D^3\boldsymbol{u}, D^3\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&\leq \frac{1}{\delta((2-\epsilon)\min(\mu,\gamma))^3}(2\alpha+\delta+2)(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}.
\end{aligned} \tag{6.11}$$

Prosseguindo neste caminho, derivamos m vezes as equações (2.1a) e (2.1b) em relação a x_{l_1}, x_{l_2} , ... x_{l_m}, multiplicando por $2(s-t_0)^{2\alpha+\delta+m}D_{l_1}D_{l_2}...D_{l_m}u_i$ e $2(s-t_0)^{2\alpha+\delta+m}D_{l_1}D_{l_2}...D_{l_m}w_i$ respectivamente. Integrando por partes em $\mathbb{R}^3\times[t_0,t]$ e utilizando (A.12), temos que

$$\begin{aligned}
&(t-t_0)^{2\alpha+\delta+m}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
&+(2-\epsilon)\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+m}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&+2\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+m}\|D^m\nabla\cdot\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&+2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds \\
&\leq (2\alpha+\delta+m)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+m-1}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^3)}ds.
\end{aligned}$$

Fazendo indução em m, temos que

$$\begin{aligned}&(t-t_0)^{2\alpha+m}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\ \leq\ & \frac{1}{\delta((2-\epsilon)\min(\mu,\gamma))^m}\left[\prod_{j=0}^{m}(2\alpha+\delta+j)\right](\lambda_0(\alpha)+\epsilon)^2,\end{aligned}\tag{6.12}$$

para $t\geq t_0$ suficientemente grande. Como a desigualdade (6.12) é válida para todo $0<\epsilon<2$, temos que

$$(t-t_0)^{2\alpha+m}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\leq\frac{1}{\delta(2\min(\mu,\gamma))^m}\left[\prod_{j=0}^{m}(2\alpha+\delta+j)\right]\lambda_0(\alpha)^2.$$

Reescrevendo a desigualdade anterior,

$$\begin{aligned}&(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\\ \leq\ & \frac{1}{(2\min(\mu,\gamma))^{m/2}}\left[\delta^{-1/2}\prod_{j=0}^{m}(2\alpha+\delta+j)^{1/2}\right]\lambda_0(\alpha)\\ =\ & \frac{2^{\frac{m+1}{2}}}{(2\min(\mu,\gamma))^{m/2}}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+\delta/2+j/2)^{1/2}\right]\lambda_0(\alpha)\\ =\ & \frac{1}{\min(\mu,\gamma)^{m/2}}\left[\left(\frac{\delta}{2}\right)^{-1/2}\prod_{j=0}^{m}(\alpha+\delta/2+j/2)^{1/2}\right]\lambda_0(\alpha).\end{aligned}\tag{6.13}$$

Como a desigualdade (6.13) é válida para todo $\delta>0$, temos que

$$(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\leq K(\alpha,m)\lambda_0(\alpha),$$

onde

$$K(\alpha,m)=\frac{1}{\min(\mu,\gamma)^{m/2}}\min_{\delta>0}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+\delta+j/2)^{1/2}\right].$$

Note que podemos escrever

$$\begin{aligned}&t^{\alpha+m/2}\left(\frac{t-t_0}{t}\right)^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\\ =\ & (t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\leq K(\alpha,m)\lambda_0(\alpha).\end{aligned}$$

Fazendo $t\to\infty$, temos

$$\limsup_{t\to\infty} t^{\alpha+m/2}\|(D^m\boldsymbol{u},D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\leq K(\alpha,m)\lambda_0(\alpha),$$

ou seja,

$$\limsup_{t\to\infty} t^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq K(\alpha,m)\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^3)},$$

para todo $m \geq 0$ inteiro.

Caso 2D O caso 2D é muito semelhante ao caso 3D, sendo assim, faremos apenas o passo de indução para ilustrarmos a demonstração no caso 2D. Derivando m vezes as equações (2.1a) e (2.1b) em relação a x_{l_1}, x_{l_2} , ... x_{l_m}, multiplicando por $2(s-t_0)^{2\alpha+\delta+m}D_{l_1}D_{l_2}...D_{l_m}u_i$ e $2(s-t_0)^{2\alpha+\delta+m}D_{l_1}D_{l_2}...D_{l_m}w_i$ respectivamente. Integrando por partes em $\mathbb{R}^2\times[t_0,t]$ e utilizando (A.16), temos que

$$\begin{aligned}
&(t-t_0)^{2\alpha+\delta+m}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\\
&+(2-\epsilon)\min(\mu,\gamma)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+m}\|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds\\
&+2\chi\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{w}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds\\
&\leq (2\alpha+\delta+m)\int_{t_0}^{t}(s-t_0)^{2\alpha+\delta+m-1}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,s)\|^2_{L^2(\mathbb{R}^2)}ds.
\end{aligned}$$

Fazendo indução em m, temos que

$$\begin{aligned}
&(t-t_0)^{2\alpha+m}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\\
&\leq \frac{1}{\delta((2-\epsilon)\min(\mu,\gamma))^m}\left[\prod_{j=0}^{m}(2\alpha+\delta+j)\right](\lambda_0(\alpha)+\epsilon)^2,
\end{aligned}\tag{6.14}$$

para $t \geq t_0$ (suficientemente grande). Como a desigualdade (6.14) é válida para todo $0<\epsilon<2$, temos que

$$(t-t_0)^{2\alpha+m}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq \frac{1}{\delta(2\min(\mu,\gamma))^m}\left[\prod_{j=0}^{m}(2\alpha+\delta+j)\right]\lambda_0(\alpha)^2.$$

Reescrevendo a desigualdade anterior, obtemos

$$
\begin{aligned}
&(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}\\
&\leq \frac{1}{(2\min(\mu,\gamma))^{m/2}}\left[\delta^{-1/2}\prod_{j=0}^{m}(2\alpha+\delta+j)^{1/2}\right]\lambda_0(\alpha)\\
&= \frac{2^{\frac{m+1}{2}}}{(2\min(\mu,\gamma))^{m/2}}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+\delta/2+j/2)^{1/2}\right]\lambda_0(\alpha)\\
&= \frac{1}{\min(\mu,\gamma)^{m/2}}\left[\left(\frac{\delta}{2}\right)^{-1/2}\prod_{j=0}^{m}(\alpha+\delta/2+j/2)^{1/2}\right]\lambda_0(\alpha).
\end{aligned}
\tag{6.15}
$$

Como a desigualdade (6.15) é válida para todo $\delta > 0$, temos que

$$(t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq K(\alpha,m)\lambda_0(\alpha),$$

onde

$$K(\alpha,m) = \frac{1}{\min(\mu,\gamma)^{m/2}}\min_{\delta>0}\left[\delta^{-1/2}\prod_{j=0}^{m}(\alpha+\delta+j/2)^{1/2}\right].$$

Note que podemos escrever

$$
\begin{aligned}
&t^{\alpha+m/2}\left(\frac{t-t_0}{t}\right)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)}\\
&= (t-t_0)^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq K(\alpha,m)\lambda_0(\alpha).
\end{aligned}
$$

Fazendo $t\to\infty$, temos

$$\limsup_{t\to\infty} t^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq K(\alpha,m)\lambda_0(\alpha),$$

ou seja,

$$\limsup_{t\to\infty} t^{\alpha+m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq K(\alpha,m)\limsup_{t\to\infty} t^{\alpha}\|(\boldsymbol{u},\mathbf{w})(\cdot,t)\|_{L^2(\mathbb{R}^2)},$$

para todo $m \geq 0$ inteiro.

$\square$

Corolário 6.2. *Dado $\alpha > 0$, se*

$$\lim_{t\to\infty} \sup t^{\alpha} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty, \tag{6.16}$$

então para todo $s \geq 0$ real, vale que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha, m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}.$$

Demonstração. Seja $m > s$ inteiro. Pelo Lema B.5, temos que

$$\|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)}^{\frac{s}{m}}.$$

Escrevendo $\delta = \delta_1 + \delta_2$, temos que

$$t^{\delta} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \left[t^{\frac{\delta_1}{1-\frac{s}{m}}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \right]^{1-\frac{s}{m}} \left[t^{\frac{\delta_2}{\frac{s}{m}}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)} \right]^{\frac{s}{m}}. \tag{6.17}$$

Escolhendo $\delta_1 = \alpha\big(1 - \frac{s}{m}\big)$ e $\delta_2 = \big(\alpha + \frac{m}{2}\big)\frac{s}{m}$, a desigualdade (6.17) se torna

$$t^{\alpha+\frac{s}{2}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \left[t^{\alpha} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \right]^{1-\frac{s}{m}} \left[t^{\alpha+\frac{m}{2}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^m(\mathbb{R}^n)} \right]^{\frac{s}{m}}.$$

Pelo Teorema 6.1

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha, m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}$$

□

Nos Teoremas 6.3 e 6.4 a seguir, vamos assumir que $\chi > 0$. Como visto ao longo do texto, taxas de decaimento melhores podem ser obtidas para o campo $\mathbf{w}$. Estes são os resultados principais obtidos neste trabalho.

Teorema 6.3. *Suponha que $\chi > 0$. Dado $\alpha > 0$, se*

$$\lim_{t\to\infty} \sup t^{\alpha} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty, \tag{6.18}$$

então para todo $s \geq 0$ real, temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}} \|\mathbf{w}(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} \left(\frac{K(\alpha, m+1)}{2} \right)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha} \|(\boldsymbol{u}, \mathbf{w})(\cdot, t)\|_{L^2(\mathbb{R}^n)}.$$

Demonstração. **Caso 3D**

A demonstração deste teorema é semelhante a do Teorema 5.1.

Reescrevendo a equação (2.1b) como

$$\mathbf{w}_t = \gamma\Delta\mathbf{w} + Q - 2\chi\mathbf{w}, \tag{6.19}$$

onde

$$Q = -\boldsymbol{u}\cdot\nabla\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}) + \chi\nabla\times\boldsymbol{u}.$$

Fazendo a mudança de variável

$$\mathbf{W} = e^{2\chi t}\mathbf{w}$$

e multiplicando a equação (6.19) por $e^{2\chi t}$, obtemos que

$$\mathbf{W}_t = \gamma\Delta\mathbf{W} + e^{2\chi t}Q.$$

Pelo o que foi estudado nos capítulos anteriores, em particular pelo Teorema 4.1, dado $\epsilon > 0$, exite $t_0 > t_*$ suficientemente grande tal que para todo $l \in \mathbb{N}$, $1 \leq l \leq m+2$ vale que

$$\|(D^l\boldsymbol{u}, D^l\mathbf{w})(\cdot,t)\| < \epsilon t^{-\frac{l}{2}},\ \forall t > t_0.$$

Pelo princípio de Duhamel, temos que

$$\mathbf{W}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{W}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-s)}e^{2\chi s}Q(\cdot,s)ds.$$

A equação acima pode ser reescrita como

$$\mathbf{w}(\cdot,t) = e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) - \int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}\boldsymbol{u}\cdot\nabla\mathbf{w}ds$$
$$+\int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}\nabla(\nabla\cdot\mathbf{w})ds + \chi\int_{t_0}^{t} e^{-2\chi(t-s)}e^{\gamma\Delta(t-s)}(\nabla\times\boldsymbol{u})ds.$$

Multiplicando a equação acima por $t^{\alpha+\frac{m+1}{2}}$, derivando m vezes em relação a x, aplicando a norma L^2 e usando a Desigualdade de Minkowski, temos que

$$\begin{aligned} t^{\alpha+\frac{m+1}{2}}\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq K\Bigg[& \underbrace{t^{\alpha+\frac{m+1}{2}}e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I} \\ & + \underbrace{t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{II} \\ & + \underbrace{t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{III} \\ & + \underbrace{\chi t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-s)}D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}ds}_{IV}\Bigg]. \end{aligned} \tag{6.20}$$

Vamos analisar cada termo separadamente.

O termo I representa a solução da equação do calor, com condição inicial $\mathbf{w}(\cdot,t_0)$ e, pelo Teorema C.1,

$$\lim_{t\to\infty}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0,$$

de modo que

$$\lim_{t\to\infty} t^{\alpha+\frac{m+1}{2}}e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{6.21}$$

Para analisar os demais termos, vamos utilizar a desigualdade (C.2) e o Teorema 6.1.

Usando (A.19), podemos analisar o termo II, obtendo

$$
\begin{aligned}
& t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^m(\boldsymbol{u}\cdot\nabla \mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&\leq K t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \sum_{l=0}^{m} \|D^l \boldsymbol{u}(\cdot,s)\|_{L^4(\mathbb{R}^3)} \|D^{m-l+1}\mathbf{w}(\cdot,s)\|_{L^4(\mathbb{R}^3)} ds \\
&\leq K t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \sum_{l=0}^{m} \|D^l \boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{1/4} \|D^{l+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{3/4} \\
&\qquad \times \|D^{m-l+1}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{1/4} \|D^{m-l+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)}^{3/4} ds \\
&\leq K(\lambda_0(\alpha)+\epsilon) t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \sum_{l=0}^{m} s^{-\frac{\alpha}{4}-\frac{l}{8}} s^{-\frac{3\alpha}{4}-\frac{3(l+1)}{8}} \epsilon^{\frac{1}{4}} s^{-\frac{m-l+1}{8}} \epsilon^{\frac{3}{4}} s^{\frac{-3(m-l+2)}{8}} ds \\
&\leq K(\lambda_0(\alpha)+\epsilon) t^{\alpha+\frac{m+1}{2}} \epsilon \int_{t_0}^{t} e^{-2\chi(t-s)} s^{-\alpha} s^{-\frac{2m+5}{4}} ds \\
&= K(\lambda_0(\alpha)+\epsilon) t^{\alpha+\frac{m+1}{2}} \epsilon \left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)} s^{-\alpha} s^{-\frac{2m+5}{4}} ds + \int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)} s^{-\alpha} s^{-\frac{2m+5}{4}} ds \right] \\
&\leq K(\lambda_0(\alpha)+\epsilon)\epsilon \left(t^{\alpha+\frac{m+1}{2}} e^{-\chi t} + t^{-\frac{3}{4}} \chi^{-1} \right).
\end{aligned}
$$

Como $\epsilon > 0$ é arbitrário, temos que

$$
\lim_{t\to\infty} t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^m(\boldsymbol{u}\cdot\nabla \mathbf{w})(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds = 0. \tag{6.22}
$$

Podemos usar o mesmo raciocínio para o termo III,

$$
\begin{aligned}
& t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&\leq K t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&\leq K(\lambda_0(\alpha)+\epsilon) t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} s^{-\alpha} s^{-\frac{m+2}{2}} ds \\
&= K(\lambda_0(\alpha)+\epsilon) t^{\alpha+\frac{m+1}{2}} \left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-s)} s^{-\alpha} s^{-\frac{m+2}{2}} ds + \int_{\frac{t}{2}}^{t} e^{-2\chi(t-s)} s^{-\alpha} s^{-\frac{m+2}{2}} ds \right] \\
&\leq K(\lambda_0(\alpha)+\epsilon) \left(t^{\alpha+\frac{m+1}{2}} e^{-\chi t} + t^{-\frac{1}{2}} \chi^{-1} \right).
\end{aligned}
$$

Novamente, temos que

$$
\lim_{t\to\infty} t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^{m+2}\mathbf{w}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds = 0. \tag{6.23}
$$

Por último, analisaremos o termo IV. Para isso, vamos fixar $0 < \lambda < 1$. Assim

$$\begin{aligned}
&\chi t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&\leq \chi t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \\
&\leq K(\alpha, m+1)\chi t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} (\lambda_0(\alpha)+\epsilon) s^{-\alpha-\frac{m+1}{2}} ds \\
&= K(\alpha, m+1)\chi(\lambda_0(\alpha)+\epsilon) t^{\alpha+\frac{m+1}{2}} \\
&\times \left[\int_{t_0}^{\lambda t} e^{-2\chi(t-s)} s^{-\alpha-\frac{m+1}{2}} ds + \int_{\lambda t}^{t} e^{-2\chi(t-s)} s^{-\alpha-\frac{m+1}{2}} ds \right] \\
&\leq K(\alpha, m+1)\chi(\lambda_0(\alpha)+\epsilon) t^{\alpha+\frac{m+1}{2}} \\
&\times \left[e^{-2\chi t(1-\lambda)} \int_{t_0}^{\lambda t} s^{-\alpha-\frac{m+1}{2}} ds + (\lambda t)^{-\alpha-\frac{m+1}{2}} \int_{\lambda t}^{t} e^{-2\chi(t-s)} ds \right] \\
&\leq K(\alpha, m+1)(\lambda_0(\alpha)+\epsilon)\chi \left[t^{\alpha+\frac{m+1}{2}} e^{-2\chi t(1-\lambda)} \left(2\frac{t_0^{-\alpha-\frac{m-1}{2}} - (\lambda t)^{-\alpha-\frac{m-1}{2}}}{2\alpha+m-1} \right) \right. \\
&\left. + \lambda^{-\alpha-\frac{m+1}{2}} \left(\frac{1-e^{-2\chi t(1-\lambda)}}{2\chi} \right) \right].
\end{aligned}$$

Fazendo $\epsilon \to 0$, $t \to \infty$ e $\lambda \to 1$ (nessa ordem), obtemos

$$\limsup_{t\to\infty} \chi t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-s)} \|e^{\gamma\Delta(t-s)} D^{m+1}\boldsymbol{u}(\cdot,s)\|_{L^2(\mathbb{R}^3)} ds \leq \frac{1}{2} K(\alpha, m+1)\lambda_0(\alpha). \tag{6.24}$$

Pelas desigualdades (6.20), (6.21), (6.22), (6.23) e (6.24), temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{m+1}{2}} \|D^m \mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{2} K(\alpha, m+1)\lambda_0(\alpha).$$

para todo $m > 0$ inteiro.

Fazendo uma interpolação semelhante a do Corolário 6.2, obtemos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}} \|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} \left(\frac{K(\alpha, m+1)}{2} \right)^{\frac{s}{m}} \lambda_0(\alpha),$$

para todo $s > 0$ real.

Caso 2D

As únicas diferenças entre os casos 2D e 3D são as estimativas do apêndice C que são utilizadas (uma vez que elas dependem da dimensão). No entanto essas mudanças não afetam a demonstração, que segue análoga ao caso 3D. □

Em posse do Teorema 6.3, podemos enfraquecer a hipótese de que $\lambda_0(\alpha) < \infty$ e obter uma versão mais forte do Teorema 6.1 e do próprio Teorema 6.3. Mais precisamente, obtemos o importante resultado final abaixo

Teorema 6.4. *Suponha que $\chi > 0$. Dado $\alpha > 0$, se*

$$\lim_{t\to\infty} \sup t^{\alpha} \|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty, \tag{6.25}$$

então para todo $s \geq 0$ real, temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}} \|\boldsymbol{u}(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha, m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha} \|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)}$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}} \|\mathbf{w}(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} \left(\frac{K(\alpha, m+1)}{2}\right)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha} \|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)}.$$

Demonstração. Se $\alpha \leq 1/2$, pelo Teorema 3.2, temos que $t^{\alpha}\|\mathbf{w}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$. Portanto, por (6.25), $\lambda_0(\alpha) < \infty$ e assim, pelo Corolário 6.2 e pelo Teorema 6.3, temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}} \|\boldsymbol{u}(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha, m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha} \|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)}$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}} \|\mathbf{w}(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} \left(\frac{K(\alpha, m+1)}{2}\right)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha} \|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)}.$$

Agora considere que $\alpha > 1/2$, então por (6.25), temos que $t^{1/2}\|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$. Assim, pelos Teoremas 3.2 e 6.3 obtemos

$$\limsup_{t\to\infty} t^{\frac{1}{2}+\frac{s+1}{2}} \|\mathbf{w}(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} \left(\frac{K(1/2, m+1)}{2}\right)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\frac{1}{2}} \|\boldsymbol{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0. \tag{6.26}$$

Assim se $\alpha \leq 1$, então por (6.26), temos que $t^{\alpha}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0$. Portanto, $\lambda_0(\alpha) < \infty$, pelo Corolário 6.2 e pelo Teorema 6.3,

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|\boldsymbol{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} \left(\frac{K(\alpha,m+1)}{2}\right)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}.$$

Vamos fazer explicitamente mais um caso.

Se $\alpha > 1$, temos que $t\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0$. Por (6.26) e pelo Teorema 6.3, temos que

$$\limsup_{t\to\infty} t^{1+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq K(1,s) \limsup_{t\to\infty} t\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0. \tag{6.27}$$

Assim, se $\alpha \leq 3/2$, por (6.27), temos que $t^{\alpha}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0$. Portanto, $\lambda_0(\alpha) < \infty$ e pelo Corolário 6.2 e pelo Teorema 6.3,

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|\boldsymbol{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq \min_{m>s} \left(\frac{K(\alpha,m+1)}{2}\right)^{\frac{s}{m}} \limsup_{t\to\infty} t^{\alpha}\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}.$$

Se $\alpha > 3/2$, repetimos este mesmo raciocínio. Em no máximo $2[\alpha]+1$ casos, obtemos o resultado desejado. □

Apêndice A O ESPAÇO DE SOBOLEV HOMOGÊNEO $\dot{H}^S$

Nesta seção, daremos uma rápida noção dos espaços que aqui foram considerados, a começar pelo espaço de Lebesgue L^p que é o espaço "trivial" de Sobolev que, como sabemos, é o espaço das funções f tal que, $\int |f|^p dx < \infty$, para $p \in [1, \infty)$. No caso $p = \infty$, a norma usada é a norma do sup, ou seja, $\|u\|_{L^\infty} = \inf\{C \in \mathbb{R}^+ \ : \ |f| < C\}$ e a medida do conjunto $\{x \ : \ |f(x)| > C\}$ é nula. Para um estudo mais construtivo e detalhado desses espaços e uma boa abordagem a respeito da sua estrutura veja [1].

A partir daí, podemos introduzir os espaços de Sobolev $W^{m,p}(\Omega)$ (que também denotaremos ao longo de nosso texto por $H^m(\Omega)$ quando $p = 2$), onde Ω é um aberto em $\mathbb{R}^n$, significando que a função que pertence a esse espaço e todas as suas derivadas espaciais (fracas) de até ordem m pertencem a L^p, onde $m \geq 1$ inteiro e $1 \leq p \leq \infty$. Tendo isso, introduziremos o espaço de Sobolev homogêneo $\dot{H}^m$.

Definição A.1. *Os espaços homogêneos de Sobolev $\dot{W}^{m,p}$ (similarmente $\dot{H}^m$) é o espaço das funções m vezes fracamente diferenciáveis f tal que $D^\alpha f \in L^p(\Omega)$, em outras palavras,*

$$\|f\|_{\dot{W}^{m,p}(\Omega)} = \left(\sum_{i_1, i_2, \ldots, i_n = 1}^{m} \int_\Omega |D_{i_1} D_{i_2} \ldots D_{i_n} f(x)|^p dx \right)^{1/p} < \infty.$$

Podemos ainda generalizar esse conceito via transformada de Fourier.

Definição A.2. *Dado $s > 0$, denotaremos por $H^s(\mathbb{R}^n)$, o espaço das funções f, tais que,*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty,$$

onde $\hat{f}(\xi)$ é a Transformada de Fourier abaixo,

$$\hat{\mathbf{u}}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix.\xi} \mathbf{u}(x) dx.$$

Esses espaços homogêneos preservam boas propriedades de escala, o que motiva seu estudo. Apresentaremos a seguir uma série de resultados que nos auxiliarão para maior compreensão do texto.

Primeiramente, precisaremos das seguintes desigualdades elementares de So-bolev-Nirenberg-Gagliardo (SNG) para funções $\mathrm{u} \in H^2(\mathbb{R}^3)$ quaisquer:

$$\| \mathrm{u} \|_{L^\infty(\mathbb{R}^3)} \leq \| \mathrm{u} \|^{1/4}_{L^2(\mathbb{R}^3)} \| D^2 \mathrm{u} \|^{3/4}_{L^2(\mathbb{R}^3)}, \tag{A.1}$$

veja e.g. { [33], Proposition 2.4, p. 5 }, ou { [31], Teorema 4.5.1, p. 52 }; e

$$\| D\mathrm{u} \|_{L^2(\mathbb{R}^3)} \leq \| \mathrm{u} \|^{1/2}_{L^2(\mathbb{R}^3)} \| D^2 \mathrm{u} \|^{1/2}_{L^2(\mathbb{R}^3)}, \tag{A.2}$$

facilmente obtida usando a transformada de Fourier. De (A.1), (A.2), obtemos

$$\|\mathrm{u}\|_{L^\infty(\mathbb{R}^3)} \| D\mathrm{u} \|_{L^2(\mathbb{R}^3)} \leq \| \mathrm{u} \|^{1/2}_{L^2(\mathbb{R}^3)} \| D\mathrm{u} \|^{1/2}_{L^2(\mathbb{R}^3)} \| D^2 \mathrm{u} \|_{L^2(\mathbb{R}^3)}, \tag{A.3}$$

Para demonstrar (A.3), basta notar que

$$\begin{aligned} \| \boldsymbol{u}(\cdot,s) \|_{L^\infty(\mathbb{R}^3)} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} &\leq \| \boldsymbol{u}(\cdot,s) \|^{1/4}_{L^2(\mathbb{R}^3)} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D^2\boldsymbol{u}(\cdot,s) \|^{3/4}_{L^2(\mathbb{R}^3)} \\ &= \| D^2\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \Big[\| \boldsymbol{u}(\cdot,s) \|^{1/4}_{L^2(\mathbb{R}^3)} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D^2\boldsymbol{u}(\cdot,s) \|^{-1/4}_{L^2(\mathbb{R}^3)} \Big] \\ &\leq \| D^2\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \Big[\| \boldsymbol{u}(\cdot,s) \|^{1/2}_{L^2(\mathbb{R}^3)} \| D\boldsymbol{u}(\cdot,s) \|^{1/2}_{L^2(\mathbb{R}^3)} \Big]. \end{aligned}$$

No entanto, nosso objetivo é derivar desigualdades do tipo (A.3) para $(\boldsymbol{u}, \mathbf{w})$. Observe que, por definição (veja 1.6, na introdução), temos

$$\|\boldsymbol{u}\|_{L^q(\mathbb{R}^3)} \leq \|(\boldsymbol{u}, \mathbf{w})\|_{L^q(\mathbb{R}^3)}, \text{ para } 1 \leq q \leq \infty.$$

Com isso, por (A.1), obtemos

$$\begin{aligned}\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)} &\leq \|(\boldsymbol{u}, \mathbf{w})\|_{L^2(\mathbb{R}^3)}^{1/4} \|(D^2\boldsymbol{u}, D^2\mathbf{w})\|_{L^2(\mathbb{R}^3)}^{3/4} \\ \|\mathbf{w}\|_{L^\infty(\mathbb{R}^3)} &\leq \|(\boldsymbol{u}, \mathbf{w})\|_{L^2(\mathbb{R}^3)}^{1/4} \|(D^2\boldsymbol{u}, D^2\mathbf{w})\|_{L^2(\mathbb{R}^3)}^{3/4}\end{aligned}$$

o que resulta

$$\begin{aligned}\|(\boldsymbol{u}, \mathbf{w})\|_{L^\infty(\mathbb{R}^3)} &= \max\{\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)}, \mathbf{w}\|_{L^\infty(\mathbb{R}^3)}\} \\ &\leq \|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)} + \|\mathbf{w}\|_{L^\infty(\mathbb{R}^3)} \\ &\leq 3\|(\boldsymbol{u}, \mathbf{w})\|_{L^2(\mathbb{R}^3)}^{1/4} \|(D^2\boldsymbol{u}, D^2\mathbf{w})\|_{L^2(\mathbb{R}^3)}^{3/4},\end{aligned}$$

ou seja,

$$\|(\boldsymbol{u}, \mathbf{w})\|_{L^\infty(\mathbb{R}^3)} \leq 3\|(\boldsymbol{u}, \mathbf{w})\|_{L^2(\mathbb{R}^3)}^{1/4} \|(D^2\boldsymbol{u}, D^2\mathbf{w})\|_{L^2(\mathbb{R}^3)}^{3/4}. \tag{A.4}$$

Analogamente, por (A.2), temos

$$\begin{aligned}\| D\boldsymbol{u} \|_{L^2(\mathbb{R}^3)} &\leq \| (\boldsymbol{u}, \mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2} \\ \| D\mathbf{w} \|_{L^2(\mathbb{R}^3)} &\leq \| (\boldsymbol{u}, \mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2}\end{aligned}$$

Donde,

$$\|(D\boldsymbol{u}, D\mathbf{w})\|_{L^2(\mathbb{R}^3)} = \left(\| D\boldsymbol{u} \|_{L^2(\mathbb{R}^3)}^2 + \| D\mathbf{w} \|_{L^2(\mathbb{R}^3)}^2\right)^{1/2}$$

$$\leq \| D\boldsymbol{u} \|_{L^2(\mathbb{R}^3)} + \| D\mathbf{w} \|_{L^2(\mathbb{R}^3)} \leq 3\| (\boldsymbol{u}, \mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2},$$

ou seja,

$$\|(D\boldsymbol{u}, D\mathbf{w})\|_{L^2(\mathbb{R}^3)} \leq 3\| (\boldsymbol{u}, \mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2}. \tag{A.5}$$

Similarente, no caso geral

$$\begin{aligned}&\|(D^m\boldsymbol{u}, D^m\mathbf{w})\|_{L^2(\mathbb{R}^3)} \\ &\leq 3\| (D^{m-1}\boldsymbol{u}, D^{m-1}\mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}) \|_{L^2(\mathbb{R}^3)}^{1/2}\end{aligned} \tag{A.6a}$$

e

$$
\begin{aligned}
&\|(D^l \boldsymbol{u}, D^l \mathbf{w})\|_{L^2(\mathbb{R}^3)} \\
&\leq 3\| (\boldsymbol{u}, \mathbf{w}) \|^{1-\theta}_{L^2(\mathbb{R}^3)} \| (D^m \boldsymbol{u}, D^m \mathbf{w}) \|^{\theta}_{L^2(\mathbb{R}^3)}, \text{ onde } \theta = l/m,
\end{aligned} \tag{A.6b}
$$

para $0 \leq l \leq m$. O que consequentemente nos fornece o lema a seguir.

Lema A.3.

$$
\begin{aligned}
&\| (\boldsymbol{u}, \mathbf{w}) \|_{L^\infty(\mathbb{R}^3)} \| (D\boldsymbol{u}, D\mathbf{w}) \|_{L^2(\mathbb{R}^3)} \\
&\leq \ C\| (\boldsymbol{u}, \mathbf{w}) \|^{1/2}_{L^2(\mathbb{R}^3)} \| (D\boldsymbol{u}, D\mathbf{w}) \|^{1/2}_{L^2(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
&\| (\boldsymbol{u}, \mathbf{w}) \|_{L^\infty(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)} \\
&\leq \ C\| (\boldsymbol{u}, \mathbf{w}) \|^{1/2}_{L^2(\mathbb{R}^3)} \| (D\boldsymbol{u}, D\mathbf{w}) \|^{1/2}_{L^2(\mathbb{R}^3)} \| (D^3\boldsymbol{u}, D^3\mathbf{w}) \|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
&\| (D\boldsymbol{u}, D\mathbf{w}) \|_{L^\infty(\mathbb{R}^3)} \| (D\boldsymbol{u}, D\mathbf{w}) \|_{L^2(\mathbb{R}^3)} \\
&\leq \ C\| (\boldsymbol{u}, \mathbf{w}) \|^{1/2}_{L^2(\mathbb{R}^3)} \| (D\boldsymbol{u}, D\mathbf{w}) \|^{1/2}_{L^2(\mathbb{R}^3)} \| (D^3\boldsymbol{u}, D^3\mathbf{w}) \|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
&\| (D\boldsymbol{u}, D\mathbf{w}) \|_{L^\infty(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)} \\
&\leq \ C\| (\boldsymbol{u}, \mathbf{w}) \|^{3/4}_{L^2(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|^{1/4}_{L^2(\mathbb{R}^3)} \| (D^4\boldsymbol{u}, D^4\mathbf{w}) \|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
&\| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^\infty(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|_{L^2(\mathbb{R}^3)} \\
&\leq \ C\| (\boldsymbol{u}, \mathbf{w}) \|^{3/4}_{L^2(\mathbb{R}^3)} \| (D^2\boldsymbol{u}, D^2\mathbf{w}) \|^{1/4}_{L^2(\mathbb{R}^3)} \| (D^5\boldsymbol{u}, D^5\mathbf{w}) \|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{A.11}
$$

e, para $m \geq 3$, $0 \leq \ell \leq m-3$*:*

$$
\begin{aligned}
&\| (D^\ell\boldsymbol{u}, D^\ell\mathbf{w}) \|_{L^\infty(\mathbb{R}^3)} \| (D^{m-\ell}\boldsymbol{u}, D^{m-\ell}\mathbf{w}) \|_{L^2(\mathbb{R}^3)} \\
&\leq \ C\| (\boldsymbol{u}, \mathbf{w}) \|^{\frac{\ell+3/2}{\ell+2}}_{L^2(\mathbb{R}^3)} \| (D^{\ell+2}\boldsymbol{u}, D^{\ell+2}\mathbf{w}) \|^{\frac{1/2}{\ell+2}}_{L^2(\mathbb{R}^3)} \| (D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}) \|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{A.12}
$$

Para algum $C > 0$.

Demonstração. Defina $\mathbf{z} := (\boldsymbol{u}, \mathbf{w})$, $D\mathbf{z} = (D\boldsymbol{u}, D\mathbf{w})$ e assim por diante. Para mostrar (A.7), observe que, por (A.4) e por (A.5), temos

$$\begin{aligned}
\|\,\mathbf{z}\,\|_{L^\infty(\mathbb{R}^3)}\|\,D\mathbf{z}\,\|_{L^2(\mathbb{R}^3)} &\leq 3\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|\,D\mathbf{z}\,\|_{L^2(\mathbb{R}^3)} \\
&= 3\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|\,D\mathbf{z}\,\|_{L^2(\mathbb{R}^3)}^{1/2}\|\,D\mathbf{z}\,\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&\leq 3\sqrt{3}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|\,D\mathbf{z}\,\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&= 3\sqrt{3}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Vamos demonstrar agora (A.8). Por (A.4), (A.6) e (A.7),

$$\begin{aligned}
\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&= 3\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}^{1/2}\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}^{1/2}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&\leq 3\sqrt{3\sqrt{3}}\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}^{1/2}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&\leq 3\sqrt{3}\sqrt{3\sqrt{3}}\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}^{1/2}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&= 3\sqrt{3}\sqrt{3\sqrt{3}}\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}^{1/2}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4} \\
&\leq 3\sqrt{3}\sqrt{3\sqrt{3}}\sqrt{3\sqrt{3}}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4} \\
&= 3\sqrt{3}\sqrt{3\sqrt{3}}\sqrt{3\sqrt{3}}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4} \\
&\leq 3\sqrt{3}\sqrt{3}\sqrt{3\sqrt{3}}\sqrt{3\sqrt{3}}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4} \\
&= 27\sqrt{3}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Vamos demonstrar agora (A.9). Analogamente, usando (A.5), (A.4) para $D\mathbf{z}$ e (A.6), obtemos

$$\begin{aligned}
\|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&\leq 3^2\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2} \\
&\leq 3^2\sqrt{3}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4} \\
&= 3^2\sqrt{3}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/2}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Usando (A.1) para $D\mathbf{z}$, (A.5) e (A.6), temos que

$$\begin{aligned}
\|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)} &= \|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3\sqrt{3}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\\
&= 3\sqrt{3}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\sqrt{3}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&= 3^2\sqrt{3}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\sqrt{3}\sqrt{3}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3^4\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|_{L^2(\mathbb{R}^3)}\\
&= 3^4\|\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|_{L^2(\mathbb{R}^3)},
\end{aligned}$$

o que concluí a prova de (A.10). Para verificar (A.11), procedemos da seguinte maneira: usamos (A.4) para $D^2\mathbf{z}$ e (A.6), o que nos fornece,

$$\begin{aligned}
\|D^2\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&= 3^2\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{5/4}_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\left(3\|\mathbf{z}\|^{1/5}_{L^2(\mathbb{R}^3)}\|D^5\mathbf{z}\|^{4/5}_{L^2(\mathbb{R}^3)}\right)^{5/4}\\
&= 3^{13/4}\|\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^5\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Similarmente, no caso geral, dado $m \geq 3$ e $j = \{0, 1, \ldots, m-3\}$, obtemos

$$\begin{aligned}
\|D^\ell \mathbf{z}\|_{L^\infty(\mathbb{R}^3)} \|D^{m-\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|D^\ell \mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)} \|D^{\ell+2}\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \|D^{m-\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)} \\
&\leq 3^2 \|D^\ell \mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)} \|D^{\ell+2}\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \|\mathbf{z}\|^{\frac{\ell+1}{m+1}}_{L^2(\mathbb{R}^3)} \|D^{m+1}\mathbf{z}\|^{\frac{m-\ell}{m+1}}_{L^2(\mathbb{R}^3)} \\
\leq 3^{9/4}\left(\|\mathbf{z}\|^{\frac{m+1-\ell}{m+1}}_{L^2(\mathbb{R}^3)} \|D^{m+1}\mathbf{z}\|^{\frac{\ell}{m+1}}_{L^2(\mathbb{R}^3)} \right)^{1/4} &\|D^{\ell+2}\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \|\mathbf{z}\|^{\frac{\ell+1}{m+1}}_{L^2(\mathbb{R}^3)} \|D^{m+1}\mathbf{z}\|^{\frac{m-\ell}{m+1}}_{L^2(\mathbb{R}^3)} \\
&= 3^{9/4} \|\mathbf{z}\|^{\frac{1}{4}\frac{m+3\ell+5}{m+1}}_{L^2(\mathbb{R}^3)} \|D^{\ell+2}\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \|D^{m+1}\mathbf{z}\|^{\frac{m-\frac{3}{4}\ell}{m+1}}_{L^2(\mathbb{R}^3)} \\
&= 3^{9/4} \|\mathbf{z}\|^{\frac{1}{4}\frac{m+3\ell+5}{m+1}}_{L^2(\mathbb{R}^3)} \|D^{\ell+2}\mathbf{z}\|^{\frac{3}{4}(1-\beta)}_{L^2(\mathbb{R}^3)} \|D^{\ell+2}\mathbf{z}\|^{\frac{3}{4}\beta}_{L^2(\mathbb{R}^3)} \\
&\quad \times \|D^{m+1}\mathbf{z}\|^{\frac{m-\frac{3}{4}\ell}{m+1}}_{L^2(\mathbb{R}^3)} \ (\ \beta \in [0,1] \text{ a ser escolhido }) \\
\leq 3^{9/4} \|\mathbf{z}\|^{\frac{m/4+\frac{3\ell}{4}+5/4}{m+1}}_{L^2(\mathbb{R}^3)} \|D^{\ell+2}\mathbf{z}\|^{\frac{3}{4}(1-\beta)}_{L^2(\mathbb{R}^3)} &\left(3\|\mathbf{z}\|^{\frac{m-\ell-1}{m+1}}_{L^2(\mathbb{R}^3)} \|D^{m+1}\mathbf{z}\|^{\frac{\ell+2}{m+1}}_{L^2(\mathbb{R}^3)} \right)^{\frac{3\beta}{4}} \|D^{m+1}\mathbf{z}\|^{\frac{m-\frac{3}{4}\ell}{m+1}}_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Logo, devemos escolher β tal que

$$\frac{3\beta}{4}\frac{\ell+2}{m+1} + \frac{m-\frac{3}{4}\ell}{m+1} = 1,$$

i.e.,

$$\beta = \frac{\frac{3\ell}{4}+1}{\frac{3\ell}{4}+3/2} \in (0,1).$$

O que implica,

$$\|D^\ell \mathbf{z}\|_{L^\infty(\mathbb{R}^3)} \|D^{m-\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)} \leq K(\ell) \|\mathbf{z}\|^{\frac{\ell+3/2}{\ell+2}}_{L^2(\mathbb{R}^3)} \|D^{\ell+2}\mathbf{z}\|^{\frac{1/2}{\ell+2}}_{L^2(\mathbb{R}^3)} \|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)},$$

onde

$$K(\ell) = 3^{9/4} 3^{\frac{3}{4}\frac{\frac{3\ell}{4}+1}{\frac{3\ell}{4}+3/2}} = 3^{\frac{3}{4}\frac{\frac{3\ell}{4}+1}{\frac{3\ell}{4}+3/2}+9/4}.$$

O que concluí a demonstração. □

Em dimensão $n = 2$, as desigualdades apresentadas no lema acima ficam mais simples.

Lema A.4.

$$\|\,\mathbf{z}\,\|_{L^\infty(\mathbb{R}^2)} \|\, D\mathbf{z}\,\|_{L^2(\mathbb{R}^2)} \leq C\|\,\mathbf{z}\,\|_{L^2(\mathbb{R}^2)} \|\, D^2\mathbf{z}\,\|_{L^2(\mathbb{R}^2)}, \qquad \text{(A.13)}$$

$$\| \mathbf{z} \|_{L^\infty(\mathbb{R}^2)} \| D^2\mathbf{z} \|_{L^2(\mathbb{R}^2)} \leq C\| \mathbf{z} \|_{L^2(\mathbb{R}^2)} \| D^3\mathbf{z} \|_{L^2(\mathbb{R}^2)}, \tag{A.14}$$

$$\| D\mathbf{z} \|_{L^\infty(\mathbb{R}^2)} \| D\mathbf{z} \|_{L^2(\mathbb{R}^2)} \leq C\| \mathbf{z} \|_{L^2(\mathbb{R}^2)} \| D^3\mathbf{z} \|_{L^2(\mathbb{R}^2)}, \tag{A.15}$$

e, para $m \geq 2$, $0 \leq \ell \leq m-2$*:*

$$\| D^\ell\mathbf{z} \|_{L^\infty(\mathbb{R}^2)} \| D^{m-\ell}\mathbf{z} \|_{L^2(\mathbb{R}^2)} \leq C\| \mathbf{z} \|_{L^2(\mathbb{R}^2)} \| D^{m+1}\mathbf{z} \|_{L^2(\mathbb{R}^2)}, \tag{A.16}$$

para algum $C > 0$.

Demonstração. As desigualdades acima podem ser obtidas a partir de

$$\|\mathbf{z}\|_{L^\infty(\mathbb{R}^2)} \leq \|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^2)} \|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^2)} \tag{A.17}$$

e

$$\| D\mathbf{z} \|_{L^2(\mathbb{R}^2)} \leq \| \mathbf{z} \|^{1/2}_{L^2(\mathbb{R}^2)} \| D^2\mathbf{z} \|^{1/2}_{L^2(\mathbb{R}^2)}, \tag{A.18}$$

□

Outra desigualdade (conhecida) de Gagliardo-Nirenberg que será necessária é:

$$\|\mathbf{z}(\cdot,t)\|_{L^4(\mathbb{R}^3)} \leq K\|\mathbf{z}(\cdot,t)\|^{1/4}_{L^2(\mathbb{R}^3)} \|D\mathbf{z}(\cdot,t)\|^{3/4}_{L^2(\mathbb{R}^3)}, \tag{A.19}$$

para algum $K > 0$.

Apêndice B FERRAMENTAS DE ANÁLISE

Faremos aqui, uma pequena reunião de resultados em análise que são úteis para que o leitor tenha um melhor entendimento acerca do que está sendo desenvolvido no texto.

Lema B.1. *Se $f \in L^p(E)$, para $1 \leq p \leq \infty$, então dado $\epsilon > 0$, existe um conjunto $K \subset E$, com $|K| < \infty$, tal que*

$$\|f\|_{L^p(E)} < \|f\|_{L^p(K)} + \epsilon$$

Lema B.2. *Se $f \in L^p(E)$, para $1 \leq p \leq \infty$, então dado $\epsilon > 0$, existe $\delta > 0$ tal que se $H \subset E$ e*

$$\mu(H) < \delta \Rightarrow \int_H |f|^p d\mu < \epsilon$$

Demonstração. Suponha, por absurdo, que existem conjuntos $E_n \subset E$ e $\epsilon > 0$ tal que

$$\mu(E_n) < \frac{1}{2^n} \text{ e } \int_{E_n} |f|^p d\mu \geq \epsilon,$$

para todo $n \in \mathbb{N}$. Seja $F_n = \bigcup_{k=n}^{\infty} E_k$. Logo, $F_{n+1} \subset F_n$, para todo $n \in \mathbb{N}$ e $\mu(F_n) < \frac{1}{2^{n-1}}$, com $\int_{F_n} |f|^p d\mu \geq \epsilon$. Portanto, temos que,

$$\mu(\cap_n F_n) = \lim_n \mu(F_n) = 0$$

e, como a integral é uma medida,

$$\epsilon \leq \lim_n \int_{F_n} |f|^p d\mu = \int_{\cap_n F_n} |f|^p d\mu = 0$$

Absurdo. □

Lema B.3. *Se $f \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ para algum $1 \leq p < \infty$, então $f \in L^q(\mathbb{R}^n)$, para cada $p \leq q \leq \infty$, e, ainda,*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{q}} \|f\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{p}{q}}$$

Lema B.4. *Se* $f \in L^1(\mathbb{R}^n)$ *e* $g \in L^p(\mathbb{R}^n)$ *para algum* $1 \leq p \leq \infty$*, então* $f * g \in L^p(\mathbb{R}^n)$ *e, além disso,*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}$$

Lema B.5. *Sendo* $f \in L^2(\mathbb{R}^n) \cap \dot{H}^{s_1}(\mathbb{R}^n)$*, para* $s_1 > 0$*, então, temos que* $f \in \dot{H}^s(\mathbb{R}^n)$*, para cada* $0 < s < s_1$*, com*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{s_1}}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}^{\frac{s}{s_1}}$$

Para maiores detalhes sobre essas normas, veja Apêndice A.

Demonstração. De fato, basta notar que, pela desigualdade de Hölder, para $1 \leq p, q \leq \infty$

$$\begin{aligned}\|f\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\xi|^{2s}|\hat{f}(\xi)|^{\frac{2}{p}}|\hat{f}(\xi)|^{\frac{2}{q}} d\xi \\ &\leq \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |\xi|^{2sp}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{p}},\end{aligned}$$

logo, devemos ter,

$$2sp = 2s_1$$

O que implica que,

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{s_1}}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}^{\frac{s}{s_1}}$$

□

As demonstrações dos lemas aqui omitidos podem ser encontradas em [11], com exceção do lema B.2. Vale, ainda o caso mais geral:

Lema B.6. *Sendo* $f \in \dot{H}^{s_1}(\mathbb{R}^n) \cap \dot{H}^{s_2}(\mathbb{R}^n)$*, para* $s_1, s_2 > 0$*, então, temos* $f \in \dot{H}^s(\mathbb{R}^n)$*, para cada* $s_1 < s < s_2$*, com*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|^{\theta\frac{s}{s_1}}_{\dot{H}^{s_1}(\mathbb{R}^n)} \|f\|^{(1-\theta)\frac{s}{s_2}}_{\dot{H}^{s_2}(\mathbb{R}^n)}$$

onde $0 \leq \theta \leq 1$, *e* $\frac{1}{s} = \theta\frac{1}{s_1} + (1-\theta)\frac{1}{s_2}$

Demonstração. Observe que, pela desigualdade de Hölder, para $1 \leq p, q \leq \infty$, temos que

$$\begin{aligned} \|f\|^2_{\dot{H}^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\xi|^{\theta 2s} |\xi|^{(1-\theta)2s} |\hat{f}(\xi)|^{\frac{2}{p}} |\hat{f}(\xi)|^{\frac{2}{q}} d\xi \\ &\leq \left(\int_{\mathbb{R}^n} |\xi|^{\theta 2sp} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |\xi|^{(1-\theta)2sq} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{q}}, \end{aligned}$$

logo, deve-se ter $\theta ps = s_1$ e $(1-\theta)qs = s_2$, o que implica que,

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|^{\theta\frac{s}{s_1}}_{\dot{H}^{s_1}(\mathbb{R}^n)} \|f\|^{(1-\theta)\frac{s}{s_2}}_{\dot{H}^{s_2}(\mathbb{R}^n)}$$

com $0 \leq \theta \leq 1$, e $\frac{1}{s} = \theta\frac{1}{s_1} + (1-\theta)\frac{1}{s_2}$ □

Lema B.7. *Dada* $f \in L^2(\mathbb{R}^n)$, *com* $D^n f \in L^2(\mathbb{R}^n)$, *então* $f \in L^\infty(\mathbb{R}^n)$, *com*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq K(n) \|f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)} \|D^n f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)}$$

A demonstração do lema acima encontra-se em [8]. Na verdade, este resultado é um caso particular da seguinte desigualdade geral:

Seja $s > n/2$. *Se* $f \in H^s(\mathbb{R}^n)$, *então*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq K(n,s)\ \|f\|^{1-\frac{n}{2s}}_{L^2(\mathbb{R}^n)}\ \|f\|^{\frac{n}{2s}}_{\dot{H}^s(\mathbb{R}^n)},$$

onde o valor optimal de $K(n,s)$ é dado por

$$K(s,n) = \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2}\Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin(\frac{\pi n}{2s})}{\frac{\pi n}{2s}} \right\}^{-\frac{1}{2}} \left\{ \frac{n}{2s-n} \right\}^{-\frac{n}{4s}} \left\{ \frac{2s}{2s-n} \right\}^{\frac{1}{2}}.$$

Para a demostração da desigualdade acima, veja [29].

Lema B.8. *Sejam* $f \in C^0([t_0, T])$ *e* $w \in L^1([t_0, T])$, *com* $f \geq 0$ *e* $w \geq 0$ *tal que*

$$f(t) \leq A + \int_{t_0}^{t} w(s) f(s) ds,$$

para todo $t \in [t_0, T]$. *Então,*

$$f(t) \leq A e^{\int_{t_0}^{t} w(s) ds},$$

onde A *é uma constante positiva.*

Demonstração. Note que,

$$\frac{f(t)}{A + \int_{t_0}^{t} w(s) f(s) ds} \leq 1$$

definindo $U(t) = A + \int_{t_0}^{t} w(s) f(s) ds$, temos que

$$\frac{U'}{U} \leq w(s)$$

donde, integrando de t_0 a t, segue o resultado. □

Apêndice C EQUAÇÃO DO CALOR: PROPRIEDADES DO *HEAT KERNEL*

Nesta seção, o objetivo é provar algumas propriedades conhecidas da solução da equação do calor usando as técnicas desenvolvidas em [17] e [18] para nos motivarmos a atacar sistemas mais complicados.

Considere o problema,

$$(*) \begin{cases} \mathbf{u}_t = \Delta \mathbf{u}, \ t > 0, \ x \in \mathbb{R}^n \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 \in L^2(\mathbb{R}^n) \end{cases}$$

Então segue o seguinte resultado:

Teorema C.1 (Propriedade de Leray para a equação do Calor)**.** *Dada* $\mathbf{u}(x,t)$ *solução do problema* $(*)$*, então*

$$\lim_{t \to \infty} \|\mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} = 0$$

Provaremos o resultado acima de duas maneiras diferentes, a primeira usando Transformada de Fourier e a segunda usando as técnicas desenvolvidas no presente texto. Posteriormente, compararemos os dois argumentos, explicitando quais as vantagens e desvantagens de cada um.

Demonstração. **Primeiro argumento:**

A maneira mais conhecida de provar esse resultado é usando Transformada de Fourier. Portanto, seja $\hat{\mathbf{u}}$ a transformada de Fourier, definida abaixo,

$$\hat{\mathbf{u}}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix.\xi}\mathbf{u}(x)dx,$$

onde $i = \sqrt{-1}$.

Aplicando a transformada de Fourier no problema $(*)$, obtemos,

$$\hat{\mathbf{u}}_t(\xi, t) = -|\xi|^2\hat{\mathbf{u}}(\xi, t)$$
$$\hat{\mathbf{u}}(\xi, 0) = \hat{\mathbf{u}}_0(\xi)$$

Resolvendo o problema acima, obtemos que,

$$\hat{\mathbf{u}}(\xi, t) = e^{-|\xi|^2 t}\hat{\mathbf{u}}_0(\xi)$$

Sabe-se, pelo Teorema de Plancherel, que $\|\mathbf{u}(x,t)\|^2_{L^2(\mathbb{R}^n)} = \|\hat{\mathbf{u}}(\xi,t)\|^2_{L^2(\mathbb{R}^n)}$

Logo

$$\|\mathbf{u}(x,t)\|^2_{L^2(\mathbb{R}^n)} = \|\hat{\mathbf{u}}(\xi,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi,t)|^2 d\xi = \int_{\mathbb{R}^n} e^{-2|\xi|^2 t}|\hat{\mathbf{u}}_0(\xi)|^2 d\xi \quad \text{(C.1)}$$

Temos que, pelo o lema B.2 dado $\epsilon > 0$, existe $\delta > 0$ tal que

$$\int_{|\xi|<\delta} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi < \frac{\epsilon}{2}$$

Note que, $e^{-2|\xi|^2 t} \leq 1$, para todo $t \geq 0$. Observe, também, que dado $\epsilon > 0$, existe $t_0(\epsilon) > 0$, a saber,

$$t_0 = \ln\left(\frac{2\|\mathbf{u_0}\|^2_{L^2(\mathbb{R}^n)}}{\epsilon}\right)^{\frac{1}{2\delta^2}},$$

tal que para todo $t \geq t_0(\epsilon)$, temos

$$e^{-2\delta^2 t}\|\mathbf{u_0}\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2}$$

Portanto, dado $\epsilon > 0$, existe $\delta > 0$ e $t_0 \geq 0$, como definido acima, tal que, para todo $t \geq t_0$, temos

$$\int_{\mathbb{R}^n} e^{-2|\xi|^2 t}|\hat{\mathbf{u}}_0(\xi)|^2 d\xi = \int_{|\xi|<\delta} e^{-2|\xi|^2 t}|\hat{\mathbf{u}}_0(\xi)|^2 d\xi + \int_{|\xi|\geq\delta} e^{-2|\xi|^2 t}|\hat{\mathbf{u}}_0(\xi)|^2 d\xi \leq$$
$$\int_{|\xi|<\delta} 1.|\hat{\mathbf{u}}_0(\xi)|^2 d\xi + e^{-2\delta^2 t}\int_{|\xi|\geq\delta} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \leq \frac{\epsilon}{2} + e^{-2\delta^2 t}\|\mathbf{u_0}\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

O que implica, por C.1, que

$$\lim_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Assim, obtemos o resultado desejado. $\square$

Uma consequência interessante deste argumento é a seguinte afirmação:

Afirmação C.2. *Dado $T > 0$ suficientemente grande e dado $0 < \lambda < 1$, existe $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$, com $\|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} = 1$ tal que*

$$\|u(x,T)\|_{L^2(\mathbb{R}^n)} \geq \lambda$$

Nota. *Observe que, a afirmação acima diz, em outras palavras que, a norma L^2 da solução equação decresce arbitrariamente lenta ao logo do tempo t.*

Demonstração. Para construir tal função, faremos via Transformada de Fourier. Definimos, então, a seguinte função,

$$\hat{\mathbf{u}}_0(\xi) = \begin{cases} c(r,n), & |\xi| \leq r \\ 0, & |\xi| > r \end{cases}$$

Onde,

$$c(r,n) = \sqrt{\frac{n}{r^n \omega_n}}, \text{ com } \omega_n = \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

Γ é a famosa função Gamma de Euler. Então, seguindo os moldes do argumento anterior, por Plancherel novamente, temos,

$$\|\mathbf{u}(x,T)\|^2_{L^2(\mathbb{R}^n)} = \|\hat{\mathbf{u}}(\xi,T)\|^2_{L^2(\mathbb{R}^n)}$$

$$= \int_{|\xi|\leq r} e^{-2|\xi|^2 T} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \geq e^{-2r^2 T} \int_{|\xi|\leq r} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \geq \lambda^2,$$

pois dado $0 < \lambda < 1$ fixo, existe $r > 0$ tal que $e^{-r^2 T} \geq \lambda$, consequentemente, isso mostra a existência da $\mathbf{u}_0$. $\square$

Portanto uma consequência imediata desse argumento, em outras palavras, é que não podemos dizer o quão rápido $\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}$ vai a zero.

Segundo argumento:

Faremos, agora, outra técnica para demonstrar o teorema C.1 e depois discutiremos as consequências imediatas que tal argumento implicará. Para isso, é necessário o uso do lema a seguir.

Lema C.3. *Dada* $\mathbf{u}(\cdot,\mathbf{t})$ *solução do problema* $(*)$*, com* $\mathbf{u}_0 \in L^1(\mathbb{R}^n)$ *e* $1 \leq p \leq \infty$ *então, temos:*

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^n)} \leq K_n(p)\|\mathbf{u}_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{2}(1-\frac{1}{p})}$$

Demonstração. O resultado é trivial para $p = 1$, basta usar a desigualdade de Young para a convolução provada no lema B.4. Faremos, agora, para $p = \infty$. Note que,

$$|\mathbf{u}(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} |\mathbf{u}_0(y)| dy \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \|\mathbf{u}_0\|_{L^1(\mathbb{R}^n)}$$

Logo,

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{4\pi^{\frac{n}{2}}} \|\mathbf{u}_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{2}}$$

Usando uma simples interpolação, provada no lema B.3, obtemos o resultado desejado. □

Demonstração. Agora, defina,

$$\mathbf{v}_0(x) = \begin{cases} \mathbf{u}_0, & |x| > R_\epsilon \\ 0, & |x| \leq R_\epsilon \end{cases}$$

e, similarmente,

$$\mathbf{w}_0(x) = \begin{cases} 0, & |x| > R_\epsilon \\ \mathbf{u}_0, & |x| \leq R_\epsilon \end{cases}$$

para um certo $R_\epsilon > 0$

Note que,

$$\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{w}_0$$

e, por $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$, temos $\mathbf{w}_0 \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, pois[1]

$$\|\mathbf{w}_0\|_{L^1(\mathbb{R}^n)} = \int_{|\mathbf{w}|>1} |\mathbf{w}_0| dx + \int_{|\mathbf{w}_0|\leq 1} |\mathbf{w}_0| dx \leq \int |\mathbf{w}_0|^2 dx + \frac{\omega_n}{n} R_\epsilon$$

Observe que, pelo lema B.2, temos que dado $\epsilon > 0$, existe $R_\epsilon > 0$ suficientemente grande tal que,

$$\|\mathbf{v}_0\|^2_{L^2(\mathbb{R}^n)} = \int_{|x|>R_\epsilon} |\mathbf{u}_0|^2 dx \leq \left(\frac{\epsilon}{2}\right)^2$$

Sabemos que $\mathbf{v}_0$ e $\mathbf{w}_0$, estão em L^2, mas $\mathbf{w}_0 \in L^1(\mathbb{R}^n)$, então pelo lema C.3, temos,

$$\|e^{\Delta t}\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq K_n(2)\|\mathbf{w}_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{4}}$$

Logo, dado $\epsilon > 0$, existe $t_0(\epsilon) > 0$, tal que $\|e^{\Delta t}\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2}$. O que implica que, para $t > t_0(\epsilon)$

$$\|e^{\Delta t}\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} \leq \|e^{\Delta t}\mathbf{v}_0\|_{L^2(\mathbb{R}^n)} + \|e^{\Delta t}\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq \|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)} + \|\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

O que conclui a demonstração do teorema C.1 usando o outro argumento. □

Note que, usamos a linearidade da equação do calor para fechar a prova do teorema C.1 neste último argumento, mas isso não nos impede de atacar equações não lineares, como será mostrado posteriormente. Além disso, usamos que se $\mathbf{u}$ satisfaz $(*)$, então $\|\mathbf{u}\|_{L^p} \leq \|\mathbf{u}_0\|_{L^p}$, para cada[2] $1 \leq p \leq \infty$, basta usar o lema B.4 para ver isso.

Observe que no primeiro argumento, usamos a identidade de Plancherel, que funciona somente para $p = 2$, agora temos o seguinte resultado como consequência imediata de nosso segundo argumento:

[1] Na verdade, isso vale, não somente para a função $\mathbf{w}_0$ definida acima, mas também, em geral, para qualquer função $\mathbf{w}_0$ com suporte compacto.

[2] Supondo, é claro, que $\mathbf{u}_0 \in L^p$

Corolário C.4. *Dada* $\mathbf{u}(x,t)$ *solução do problema* $(*)$, *com* $\mathbf{u}_0 \in L^p(\mathbb{R}^n)$, *então*

$$\lim_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^n)} = 0,$$

para $1 < p < \infty$.

Demonstração. Basta usar o segundo argumento usado para demonstrar o teorema C.1 para $1 < p < \infty$. O argumento segue analogamente. □

Provaremos agora a propriedade de Leray do problema $(*)$, para a norma

$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R}^n)}$. Antes, precisaremos demonstrar o seguinte lema,

Lema C.5. *Se* $\mathbf{u}(x,t)$ *satisfaz o problema* $(*)$, *com* $\|\mathbf{u}_0\| \in L^p(\mathbb{R}^n)$, *então,*

$$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R}^n)} \leq \tilde{K}_n(p)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)} t^{-\frac{n}{2p}}$$

Demonstração. Basta, notar que

$$|\mathbf{u}(\cdot,t)| \leq (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{|x-y|^2}{4t}} |\mathbf{u}_0(y)|dy \underbrace{\leq}_{\text{Desig. de Hölder}} \frac{1}{(4\pi t)^{\frac{n}{2p}}} \|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)}$$

Logo, em particular,

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi t)^{\frac{n}{2p}}} \|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)} = \tilde{K}_n(p)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)} t^{-\frac{n}{2p}}$$

□

Na verdade, será útil lembrarmos a seguinte estimativa mais geral (bem conhecida):

$$\| D^\alpha [e^{\nu\Delta\tau} \mathrm{u}] \|_{L^2(\mathbb{R}^n)} \;\leq\; K(n,m) \, \| \mathrm{u} \|_{L^r(\mathbb{R}^n)} (\nu\tau)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{|\alpha|}{2}} \tag{C.2}$$

para todo $\tau > 0$, e quaisquer α (multi-índice), $1 \leq r \leq 2$, $\mathrm{u} \in L^r(\mathbb{R}^n)$ considerados,

$n \geq 1$ arbitrário, e onde $m = |\alpha|$. (Para uma derivação de (C.2), ver e.g. [19, 21, 11].)

Com isso, podemos facilmente provar o seguinte resultado,

Teorema C.6. *Dada* $\mathbf{u}$ *solução do problema* $(*)$*, então, temos que*

$$\lim_{t\to\infty} t^{\frac{n}{4}} \|e^{\Delta t}\mathbf{u}_0\|_{L^\infty(\mathbb{R}^n)} = 0$$

Demonstração. Pelo o lema C.5, temos que

$$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R}^n)} \leq \tilde{K}(n,2)\|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} t^{-\frac{n}{4}}$$

Logo,

$$\|e^{\Delta\frac{t}{2}}[e^{\Delta\frac{t}{2}}\mathbf{u}_0]\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2^{\frac{n}{4}}}\tilde{K}(n,2)\|e^{\Delta\frac{t}{2}}\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} t^{-\frac{n}{4}}$$

Portanto,

$$t^{\frac{n}{4}}\|e^{\Delta\frac{t}{2}}[e^{\Delta\frac{t}{2}}\mathbf{u}_0]\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2^{\frac{n}{4}}}\tilde{K}(n,2)\|e^{\Delta\frac{t}{2}}\mathbf{u}_0\|_{L^2(\mathbb{R}^n)}$$

O que implica, ao $t \to \infty$, pelo teorema C.1, que

$$\lim_{t\to\infty} t^{\frac{n}{4}} \|e^{\Delta t}\mathbf{u}_0\|_{L^\infty(\mathbb{R}^n)} = \lim_{t\to\infty} t^{\frac{n}{4}}\|e^{\Delta\frac{t}{2}}[e^{\Delta\frac{t}{2}}\mathbf{u}_0]\|_{L^\infty(\mathbb{R}^n)} = 0$$

□

Corolário C.7. *Dada* $\mathbf{u}(\cdot,t)$ *solução do problema* $(*)$*, temos que*

$$\lim_{t\to\infty} t^{\frac{n}{4}-\frac{n}{2q}}\|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0,$$

para todo $2 \leq q \leq \infty$.

Demonstração. Basta notar que, pelo lema B.3,

$$t^{\gamma}\|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq \|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{\frac{2}{q}} \left(t^{\frac{\gamma}{1-\frac{2}{q}}}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\right)^{1-\frac{2}{q}}$$

Pelo Teorema C.6 e o Teorema C.1, devemos ter,

$$\frac{\gamma}{1-\frac{2}{q}} = \frac{n}{4}$$

Logo,

$$\gamma = \frac{n}{4} - \frac{n}{2q}$$

Observe que, aqui há uma melhora do Corolário C.4 . □

Agora proveremos o que chamamos de Problema completo de Leray, para a equação do calor, i.e.,

$$t^{\frac{s}{2}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0,$$

para qualquer $s \geq 0$. Antes, teremos que provar alguns resultados preliminares.

Proposição C.8. *Dada* $\mathbf{u}(\cdot,t)$ *solução de* $(*)$*, vale a igualdade de energia:*

$$\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t} \|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} d\tau = \|\mathbf{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)},$$

para $0 \leq t_0 < t$.

Demonstração. Observe que o problema $(*)$ está definido em $\mathbb{R}^n$, então, para obtermos a igualdade acima, ao integrarmos por partes, a função deve "decair rápido no infinito". Para formalizarmos este raciocínio, introduziremos uma função auxiliar de corte,

$$\zeta_R(x) := \begin{cases} e^{-\epsilon\sqrt{1+|x|^2}} - e^{-\epsilon\sqrt{1+R^2}}, & |x| < R \\ 0, & |x| \geq R \end{cases},$$

para $R, \epsilon > 0$

Multiplicando a equação do calor em $(*)$ por $2\mathbf{u}\zeta_R$, temos que

$$2\mathbf{u}\mathbf{u}_t\zeta_R = 2\mathbf{u}\Delta\mathbf{u}\zeta_R,$$

como $2\mathbf{u}\mathbf{u}_t = \frac{\partial}{\partial t}(\mathbf{u}^2)$, ao integrarmos de t_0 a t e em $\mathbb{R}^n$, obtemos, pelo Teorema Fundamental do Cálculo e pelo Teorema de Fubini[3], que

$$\int_{|x|<R} \mathbf{u}^2(\cdot,t)\zeta_R dx = \int_{t_0}^{t}\int_{|x|<R} 2\mathbf{u}\Delta\mathbf{u}\zeta_R dx d\tau + \int_{|x|<R} \mathbf{u}^2(\cdot,t_0)\zeta_R dx$$

Fazendo integração por partes no primeiro termo do lado direito da equação acima, como ζ_R se anula em $|x| = R$,

$$\int_{|x|<R} \mathbf{u}^2(\cdot,t)\zeta_R dx + 2\int_{t_0}^{t}\int_{|x|<R} |D\mathbf{u}(\cdot,\tau)|^2\zeta_R dx d\tau =$$
$$-\int_{t_0}^{t}\int_{|x|<R} \langle D\zeta_R, D\mathbf{u}^2(\cdot,\tau)\rangle dx d\tau + \int_{|x|<R} \mathbf{u}^2(\cdot,t_0)\zeta_R dx$$

Integrando por partes, novamente, temos que

$$\int_{|x|<R} \mathbf{u}^2(\cdot,t)\zeta_R dx + 2\int_{t_0}^{t}\int_{|x|<R} |D\mathbf{u}(\cdot,\tau)|^2\zeta_R dx d\tau =$$
$$\int_{t_0}^{t}\int_{|x|<R} \Delta\zeta_R\mathbf{u}^2(\cdot,\tau)dx d\tau - \int_{t_0}^{t}\int_{|x|=R} \langle D\zeta_R, \vec{n}\rangle\mathbf{u}^2(\cdot,\tau)d\sigma d\tau + \int_{|x|<R} \mathbf{u}^2(\cdot,t_0)\zeta_R dx \tag{C.3}$$

ou seja,

$$\int_{|x|<R} \mathbf{u}^2(\cdot,t)\zeta_R dx + 2\int_{t_0}^{t}\int_{|x|<R} |D\mathbf{u}(\cdot,\tau)|^2\zeta_R dx d\tau =$$
$$\int_{t_0}^{t}\int_{|x|<R} \Delta\zeta_R\mathbf{u}^2(\cdot,\tau)dx d\tau + \epsilon\int_{t_0}^{t}\int_{|x|=R} e^{-\epsilon\sqrt{1+R^2}}\sqrt{\frac{R^2}{1+R^2}}\mathbf{u}^2(\cdot,\tau)d\sigma d\tau$$
$$+\int_{|x|<R} \mathbf{u}^2(\cdot,t_0)\zeta_R dx,$$

pois $\vec{n} = \frac{x}{R}$ Nossa intenção é fazer $R \to \infty$, no entanto, temos que analisar se o termo de fronteira não irá divergir. Observe que, $\mathbf{u}_0 \in L^2(\mathbb{R}^n) \Rightarrow \mathbf{u}(\cdot,t) \in L^2(\mathbb{R}^n)$, basta usar o lema B.4.

[3]Para maiores detalhes dos teoremas clássicos de teoria da integração, como os teoremas de Fubini, convergência dominada de Lebesgue, convergência monótona e outros, veja [1]

Logo,

$$\int_{t_0}^{t}\int_{\mathbb{R}^n} \mathbf{u}^2 dx d\tau < \infty$$

Portanto, escrevendo em coordenadas polares e usando o Teorema de Fubini,

$$\int_{t_0}^{t}\int_{\mathbb{R}^n} \mathbf{u}^2 dx d\tau = \int_{t_0}^{t}\int_{0}^{\infty}\int_{|x|=r} \mathbf{u}^2 d\sigma(x) dr d\tau =$$
$$= \int_{0}^{\infty}\int_{t_0}^{t}\int_{|x|=r} \mathbf{u}^2 d\sigma(x) d\tau dr < \infty$$

Logo,

$$\liminf_{r\to\infty}\left[\int_{t_0}^{t}\int_{|x|=r} \mathbf{u}^2 d\sigma(x) d\tau\right] = 0$$

Então, existe uma subsequência R_k tal que

$$\lim_{R_k\to\infty}\left[\int_{t_0}^{t}\int_{|x|=R_k} \mathbf{u}^2 d\sigma(x) d\tau\right] = 0 \tag{C.4}$$

Como,

$$\epsilon\int_{t_0}^{t}\int_{|x|=R} e^{-\epsilon\sqrt{1+R^2}}\sqrt{\frac{R^2}{1+R^2}}\mathbf{u}^2(\cdot,\tau) d\sigma d\tau \leq \epsilon\int_{t_0}^{t}\int_{|x|=R} \mathbf{u}^2(\cdot,\tau) d\sigma d\tau \tag{C.5}$$

Portanto, fazendo $R_k \to \infty$ em C.3, usando o Teorema da Convergência Dominada no primeiro e último termo de C.3, usando C.4 e C.5 no termo de fronteira de C.3 e, finalmente, usando o Teorema da Convergência Monótona nos dois termos restantes de C.3, temos

$$\int_{\mathbb{R}^n} e^{-\epsilon\sqrt{1+|x|^2}}\mathbf{u}^2(\cdot,t)dx + 2\int_{t_0}^{t}\int_{\mathbb{R}^n} e^{-\epsilon\sqrt{1+|x|^2}}|D\mathbf{u}(\cdot,\tau)|^2 dx d\tau =$$
$$\int_{t_0}^{t}\int_{\mathbb{R}^n} \Delta(e^{-\epsilon\sqrt{1+|x|^2}})\mathbf{u}^2(\cdot,\tau) dx d\tau + \int_{\mathbb{R}^n} e^{-\epsilon\sqrt{1+|x|^2}}\mathbf{u}^2(\cdot,t_0)dx,$$

Como,

$$|\mathbf{u}^2\Delta(e^{-\epsilon\sqrt{1+|x|^2}})| \leq \epsilon\mathbf{u}^2,$$

supondo $0 < \epsilon \leq 1$, sem perda de generalidade. Então, tomando $\epsilon \to 0$, usando o Teorema da Convergência Dominada de Lebesgue no termo que contém o laplaciano da igualdade acima e usando o Teorema da Convergência Monótona no termos restantes, obtemos:

$$\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t} \|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} d\tau = \|\mathbf{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}$$

□

Corolário C.9. *Vale, também, as igualdades,*

$$(t-t_0)\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t}(\tau-t_0)\|D^2\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau = \int_{t_0}^{t}\|D\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau$$

$$\begin{aligned}(t-t_0)^2\|D^2\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &+ 2\int_{t_0}^{t}(\tau-t_0)^2\|D^3\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\ &= \int_{t_0}^{t}(\tau-t_0)\|D^2\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau\end{aligned}$$

E assim, sucessivamente,

$$\begin{aligned}(t-t_0)^m\|D^m\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &+ 2\int_{t_0}^{t}(\tau-t_0)^m\|D^{m+1}\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\ &= \int_{t_0}^{t}(\tau-t_0)^{m-1}\|D^m\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau\end{aligned}$$

para $m \geq 1$ inteiro.

Demonstração. Vamos introduzir, novamente, uma função de corte. Seja $\zeta \in C^2(\mathbb{R}^n)$ tal que,

$$\zeta(x) := \begin{cases} 1, & |x| < 1 \\ \Phi(x), & 1 \leq |x| < 2 \\ 0, & |x| \geq 2 \end{cases},$$

onde $\Phi : \mathbb{R}^n \to \mathbb{R}$ é uma função qualquer, obviamente, de classe C^2. Agora, defina:

$$\zeta_R = \zeta\left(\frac{|x|}{R}\right),$$

para $R > 0$.

Derivando a equação do problema $(*)$ com relação a x_i, i.e., aplicando o operador D_i e multiplicando por $2(t-t_0)\zeta_R D_i\mathbf{u}$, temos

$$2(t-t_0)\zeta_R D_i\mathbf{u}(D_i\mathbf{u})_t = 2(t-t_0)\zeta_R D_i\mathbf{u}\Delta(D_i\mathbf{u})$$

Integrando em $[t_0, t]\times\mathbb{R}^n$, temos

$$\int_{|x|<2R}\int_{t_0}^{t}(\tau-t_0)\zeta_R {(D_i\mathbf{u})^2}_t d\tau dx = \int_{|x|<2R}\int_{t_0}^{t} 2(\tau-t_0)\zeta_R D_i\mathbf{u}\Delta(D_i\mathbf{u})d\tau dx$$

Integrando por partes no lado esquerdo, a expressão acima fica,

$$\begin{aligned}(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx &= \int_{|x|<2R}\int_{t_0}^{t} 2(\tau-t_0)\zeta_R D_i\mathbf{u}\Delta(D_i\mathbf{u})d\tau dx\\ &\quad + \int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2 d\tau dx\end{aligned}$$

Agora, integrando por partes em $\mathbb{R}^n$, no primeiro termo do lado direito, temos

$$\begin{aligned}(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx &= -\int_{|x|<2R}\langle\nabla\zeta_R, \nabla(D_i\mathbf{u})^2\rangle dx\\ -\int_{|x|<2R}2\zeta_R|\nabla(D_i\mathbf{u})|^2dx &+ \underbrace{\int_{|x|=2R}2\zeta_R D_i\mathbf{u}\nabla(D_i\mathbf{u})d\sigma(x)}_{=0}\\ &+ \int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2 d\tau dx\end{aligned}$$

Integrando por partes, novamente, em $\mathbb{R}^n$ no primeiro termo do lado direito da equação acima, temos

$$\begin{aligned}(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx &+ \int_{|x|<2R}2\zeta_R|\nabla(D_i\mathbf{u})|^2dx\\ = \int_{|x|<2R}\Delta\zeta_R(D_i\mathbf{u})^2dx &- \int_{|x|=2R}\langle\nabla\zeta_R, \vec{n}\rangle(D_i\mathbf{u})^2d\sigma(x)\\ &+ \int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2 d\tau dx\end{aligned} \tag{C.6}$$

Observe que, $\nabla\zeta_R = 0$, para todo $|x| > 2R$. Sabe-se que, em particular, $\zeta_R \in C^1$, logo,

$$\nabla\zeta_R\Big|_{|x|=2R} = \lim_{|x|\to 2R^+} \nabla\zeta_R = 0$$

Portanto, a equação C.6, fica,

$$\begin{aligned}(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx + \int_{|x|<2R} 2\zeta_R|\nabla(D_i\mathbf{u})|^2dx \\ = \int_{|x|<2R}\Delta\zeta_R(D_i\mathbf{u})^2dx + \int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2d\tau dx\end{aligned} \tag{C.7}$$

Note que, como $\zeta \in C^2$, o laplaciano de ζ, atinge o máximo em $|x| \leq 2R$. Portanto,

$$|\Delta\zeta_R(D_i\mathbf{u})^2| < \frac{M}{R^2}|D_i\mathbf{u}|^2,$$

em $|x| < 2R$. Onde $M > 0$ é o máximo de $\Delta\zeta$. Além disso, pela proposição C.8,

$$\int_{\mathbb{R}^n}|D_i\mathbf{u}|^2dx < \infty$$

Logo, ao $R \to \infty$, podemos usar o teorema da convergência dominada no primeiro termo do lado direito da equação C.7, no primeiro termo do lado esquerdo e no último termo do lado direito(usando a proposição C.8) e convergência monótona no termo restante, a equação se torna, somando em i,

$$(t-t_0)\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t}(\tau-t_0)\|D^2\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau = \int_{t_0}^{t}\|D\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau,$$

Para demonstrar as igualdades seguintes, basta aplicar o operador

$$2\zeta_R(t-t_0)^2D_iD_j\mathbf{u}D_iD_j$$

em ambos o lados na equação do calor definida no problema $(*)$ e usar o mesmo tipo de argumento feito anteriormente. Fazendo de forma indutiva, i.e., aplicado o operador

$$2\zeta_R(t-t_0)^mD_{i1}D_{i2}\ldots D_{im}\mathbf{u}D_{i1}D_{i2}\ldots D_{im}$$

em ambos os lados da equação do calor definida no problema $(*)$ e usando os mesmos argumentos feitos acima, concluímos a demonstração, para $m \geq 1$ inteiro. $\square$

Tendo isso, é fácil obter o seguinte resultado:

Teorema C.10. *Dada* $\mathbf{u}(\cdot,t)$ *solução do problema* $(*)$*, então, vale que*

$$\lim_{t\to\infty} t^{\frac{m}{2}} \|D^m \mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0,$$

Em outras palavras,

$$\lim_{t\to\infty} t^{\frac{m}{2}} \|\mathbf{u}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)} = 0,$$

para $m \geq 0$ *inteiro.*

Demonstração. Pelo teorema C.1, dado $\epsilon > 0$, existe $t_0 > 0$, tal que,

$$\|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \epsilon,$$

para todo $t \geq t_0$. Portanto, pela proposição C.8, temos

$$\int_{t_0}^{t} \|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \frac{\epsilon^2}{2} \tag{C.8}$$

Pelo corolário C.9, obtemos o seguinte

$$(t-t_0)\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = t\left(\frac{t-t_0}{t}\right)\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon^2}{2}$$

Logo,

$$t\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \left(\frac{t}{t-t_0}\right)\frac{\epsilon^2}{2}$$

Observe que $(\frac{t}{t-t_0}) \leq 2$, para $t \geq 2t_0$.

$$t^{\frac{1}{2}}\|D\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon$$

O que prova o Teorema, para $m = 1$

Por C.8 e pelo corolário C.9, temos

$$\int_{t_0}^{t} (\tau - t_0)\|D^2\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \frac{\epsilon^2}{2}$$

Usando a segunda equação do corolário C.9, temos

$$(t-t_0)^2\|D^2\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = t^2\left(\frac{t-t_0}{t}\right)^2\|D^2\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon^2}{2}$$

Portanto, como $(\frac{t}{t-t_0})^2 \leq 2$, para $t \geq \sqrt{2}t_0$,

$$t\|D^2\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon$$

O que prova o Teorema, para $m = 2$. Repetindo o argumento sucessivamente de forma análoga, obtemos o resultado desejado, para $m \geq 1$ inteiro. O resultado para $m = 0$ foi demonstrado no teorema C.1. □

Usando o lema de interpolação B.5 temos o seguinte resultado:

Corolário C.11. *Para qualquer $s > 0$ temos que*

$$t^{\frac{s}{2}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0, \text{ ao } t \to \infty$$

Demonstração. Dado $\epsilon > 0$, tome $s > 0$ arbitrário. Então, para qualquer $m > s$, existe t_* tal que, pelo Teorema C.1,

$$\|\mathbf{u}(\cdot,t)\|^{1-\frac{s}{m}}_{L^2(\mathbb{R}^n)} \leq \sqrt{\epsilon},$$

para todo $t > t_*$. Além disso, pelo Teorema C.10, existe t_{**}, tal que, para todo $t > t_{**}$

$$t^{\frac{m}{2}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)} \leq \left(\sqrt{\epsilon}\right)^{\frac{m}{s}},$$

logo, usando o lema B.5, para $t > \max\{t_*, t_{**}\}$ e $\gamma > 0$(a ser escolhido), temos,

$$t^{\gamma}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq t^{\gamma}\|\mathbf{u}(\cdot,t)\|^{1-\frac{s}{m}}_{L^2(\mathbb{R}^n)}\|\mathbf{u}(\cdot,t)\|^{\frac{s}{m}}_{\dot{H}^m(\mathbb{R}^n)}$$

$$\leq \|\mathbf{u}(\cdot,t)\|^{1-\frac{s}{m}}_{L^2(\mathbb{R}^n)}\left(t^{\gamma\frac{m}{s}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^m}\right)^{\frac{s}{m}} \leq \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon,$$

se $\gamma\frac{m}{s} = \frac{m}{2}$, portanto, $\gamma = \frac{s}{2}$, para qualquer $s > 0$. □

Observe que usando a desigualdade de Sobolev do lema B.7, temos o seguinte resultado:

Corolário C.12. *Dada* $\mathbf{u}$ *solução de* $(*)$*, temos*

$$t^{\frac{m}{2}+\frac{n}{4}}\|D^m\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to 0 \text{ ao } t \to \infty,$$

para qualquer $m \geq 0$ *inteiro.*

Demonstração. Basta notar que pelo lema B.7,

$$t^{2\gamma}\|D^m\mathbf{u}(\cdot,t)\|^2_{L^\infty(\mathbb{R}^n)} \leq (K(n))^2 t^{2\gamma}\|D^m\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}\|D^{m+n}\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

Pelo teorema C.10, $2\gamma = \frac{m}{2} + \frac{m+n}{2}$, o que implica,

$$\gamma = \frac{m}{2} + \frac{n}{4}$$

□

O que encerra a discussão desse tema para a equação do calor.

Apêndice D O PROJETOR DE HELMHOLTZ

Lema D.1. *[10, III, p. 104] Seja* $f = (f_1, f_2, ..., f_n) \in L^q(\mathbb{R}^n)$ *com* $n \geq 2$, $1 < q < \infty$*. Então existem* $p \in L^q_{loc}(\mathbb{R}^n)$ *com* $\nabla p \in L^q(\mathbb{R}^n)$ *e* $f_0 \in L^q_\sigma(\mathbb{R}^n)$*, unicamente determinados, tais que*

$$f = f_0 + \nabla p$$

e

$$\|f_0\|_{L^q(\mathbb{R}^n)} + \|\nabla p\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}$$

Definição D.2. *Seja* $f \in L^q(\mathbb{R}^n)$*, considere a decomposição* $f = f_0 + \nabla p$ *como no lema (D.1), definimos o Projetor de Helmholtz* $\mathbb{P}_h : L^q(\mathbb{R}^n) \to L^q_\sigma(\mathbb{R}^n)$ *como* $\mathbb{P}_h[f] := f_0$*.* $\mathbb{P}_h$ *é linear, limitado e idempotente (Rudin 1973, §5.15(d)).*

Teorema D.3. *Seja* $1 < q_0 < \infty$ *e* $1 < q_1 < \infty$*. Sendo* $\boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$*, com* $D_j\boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$*. Então* $D_j\mathbb{P}_h[\boldsymbol{u}] \in L^{q_1}_\sigma(\mathbb{R}^n)$ *e*

$$D_j\mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[D_j\boldsymbol{u}]$$

Demonstração. Vamos dividir a demonstração deste Teorema em três partes. Primeiro vamos supor que $\boldsymbol{u} \in C_0^\infty(\mathbb{R}^n)$, provado este caso, vamos supor que $\boldsymbol{u}$ tem suporte compacto, e finalmente vamos provar para o caso geral $u \in L^{q_0}(\mathbb{R}^n)$.

Caso I: $\boldsymbol{u} \in C_0^\infty(\mathbb{R}^n)$.

Seja $1 \leq j \leq n$, usando as propriedades do Projetor de Helmholtz, existem p e q tais que

$$\boldsymbol{u} = \mathbb{P}_h[\boldsymbol{u}] + \nabla p \tag{D.1}$$

e

$$D_j \boldsymbol{u} = \mathbb{P}_h[D_j \boldsymbol{u}] + \nabla q. \tag{D.2}$$

Derivando (D.1), temos que

$$D_j \boldsymbol{u} = D_j \mathbb{P}_h[\boldsymbol{u}] + \nabla D_j p. \tag{D.3}$$

Aplicando o divergente nas equações (D.2) e (D.3), e subtraindo (D.2) de (D.3), obtemos que

$$\Delta D_j p - \Delta q = \Delta[(D_j p) - q] = 0. \tag{D.4}$$

Como $(D_j p) - q$ é harmônica e limitada, temos pelo Teorema de Liouville que

$(D_j p) - q = 0$, isto é, $D_j p = q$. Subtraindo (D.2) de (D.3), temos que

$$\mathbb{P}_h[D_j \boldsymbol{u}] = D_j \mathbb{P}_h[\boldsymbol{u}].$$

Caso II: *supp* $\boldsymbol{u} \subset B_R$.

Seja $f \in C_0^\infty$, $f \geq 0$ tal que

$$\int_{\mathbb{R}^n} f(x)dx = 1$$

Dado $\epsilon > 0$, defina

$$\phi_\epsilon := f_\epsilon * \boldsymbol{u}, \text{ onde } \quad f_\epsilon(x) = \epsilon^{-n} f(\frac{x}{\epsilon})$$

Temos que $\phi_\epsilon \to \boldsymbol{u}$ ao $\epsilon \to 0$ em $L^{q_0}(\mathbb{R}^n)$, portanto $\phi_\epsilon \to \boldsymbol{u}$ em $\mathcal{D}'(\mathbb{R}^n)$.

Além disso, $D_j\phi_\epsilon = f_\epsilon * D_j\boldsymbol{u} \to D_j\boldsymbol{u}$ ao $\epsilon \to 0$ em $L^{q_1}(\mathbb{R}^n)$ de modo que $D_j\phi_\epsilon \to D_j\boldsymbol{u}$ em $\mathcal{D}'(\mathbb{R}^n)$.

Como o Projetor de Helmholtz é contínuo, vale que

$$\mathbb{P}_h[\phi_\epsilon] \to \mathbb{P}_h[\boldsymbol{u}] \quad \text{em} \quad L^{q_0}(\mathbb{R}^n)$$

e portanto

$$D_j\mathbb{P}_h[\phi_\epsilon] \to D_j\mathbb{P}_h[\boldsymbol{u}] \quad \text{em} \quad \mathcal{D}'(\mathbb{R}^n).$$

Temos também que

$$\mathbb{P}_h[D_j\phi_\epsilon] \to \mathbb{P}_h[D_j\boldsymbol{u}] \quad \text{em} \quad \mathcal{D}'(\mathbb{R}^n).$$

Como $\mathbb{P}_h[D_j\phi_\epsilon] = D_j\mathbb{P}_h[\phi_\epsilon]$ (já que $\phi_\epsilon \in C_0^\infty(\mathbb{R}^n)$), pela unicidade do limite que

$$D_j\mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[D_j\boldsymbol{u}].$$

Caso III: $u \in L^{q_0}(\mathbb{R}^n)$.

Seja $f \in C_0^\infty(\mathbb{R}^n)$ tal que f é não crescente, $f(0) = 1$ e $f(1) = 0$. Seja f_R definida por

$$f_R(x) = \begin{cases} 1, & |x| < R \\ f(|x| - R), & R < |x| < R + 1 \\ 0, & |x| \geq R + 1 \end{cases}$$

Observe que $\boldsymbol{u} f_R \to \boldsymbol{u}$ em $L^{q_0}(\mathbb{R}^n)$ ao $R \to \infty$. Como $\mathbb{P}_h$ é contínuo, vale que $D_j \mathbb{P}_h[\boldsymbol{u} f_R] \to D_j \mathbb{P}_h[\boldsymbol{u}]$ em $\mathcal{D}'(\mathbb{R}^n)$. Temos também, pelo caso II, que

$$\mathbb{P}_h[D_j(\boldsymbol{u} f_R)] \to D_j \mathbb{P}_h[\boldsymbol{u}] \quad \text{em} \quad \mathcal{D}'(\mathbb{R}^n). \tag{D.5}$$

Por outro lado

$$D_j(\boldsymbol{u} f_R) = (D_j f_R)\boldsymbol{u} + f_R D_j \boldsymbol{u},$$

$$\lim_{R \to \infty} (D_j f_R)\boldsymbol{u} = 0 \quad \text{em} \quad L^{q_0}(\mathbb{R}^n)$$

e

$$\lim_{R \to \infty} f_R D_j \boldsymbol{u} = D_j \boldsymbol{u} \quad \text{em} \quad L^{q_1}(\mathbb{R}^n).$$

Note que $D_j(\boldsymbol{u} f_R) = (D_j f_R)\boldsymbol{u} + f_R D_j \boldsymbol{u}$, $(D_j f_R)\boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$ e $f_R D_j \boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$. Temos por Hölder, que se $q_0 < q_1$, então $f_R D_j \boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$ e se $q_0 > q_1$, então $(D_j f_R)\boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$, de modo que exite r ($r = q_0$ ou r= q_1) tal que $D_j(\boldsymbol{u} f_R) \in L^r(\mathbb{R}^n)$.

Como $\mathbb{P}_h$ é contínuo, temos que

$$\lim_{R\to\infty} \mathbb{P}_h[D_j(\boldsymbol{u} f_R)] = \lim_{R\to\infty} \mathbb{P}_h[(D_j f_R)\boldsymbol{u}] + \lim_{R\to\infty} \mathbb{P}_h[f_R D_j \boldsymbol{u}] = 0 + \mathbb{P}_h[D_j \boldsymbol{u}] \quad \text{em} \quad L^r(\mathbb{R}^n) \tag{D.6}$$

Pelas equações (D.5) e (D.6), e pela unicidade do limite,concluímos que

$$D_j \mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[D_j \boldsymbol{u}].$$

□

Corolário D.4. *Seja* $1 < q_0 < \infty$ *e* $1 < q_1 < \infty$. *Sendo* $\boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$, *com* $D_j \boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$. *Então*

$$\Delta \mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[\Delta \boldsymbol{u}]$$

Demonstração. Basta aplicar o teorema (D.3) duas vezes.

$$\Delta \mathbb{P}_h[\boldsymbol{u}] = \sum_{j=1}^{3} D_j D_j \mathbb{P}_h[\boldsymbol{u}] = \sum_{j=1}^{3} D_j \mathbb{P}_h[D_j \boldsymbol{u}] = \sum_{j=1}^{3} \mathbb{P}_h[D_j D_j \boldsymbol{u}] = \mathbb{P}_h[\sum_{j=1}^{3} D_j D_j \boldsymbol{u}]$$
$$= \mathbb{P}_h[\Delta \boldsymbol{u}].$$

□

Teorema D.5. *Seja* $u \in L^q(\mathbb{R}^n)$, $1 < q < \infty$. *Então*

$$\mathbb{P}_h[e^{\nu\Delta t} u] = e^{\nu\Delta t}[\mathbb{P}_h u]$$

Demonstração. Defina v e w como

$$v(\cdot,t) := e^{\nu\Delta t}u$$
$$w(\cdot,t) := e^{\nu\Delta t}\mathbb{P}_h u.$$

v e w satisfazem, respectivamente, as seguintes equações do calor

$$\begin{aligned} &v_t = \nu\Delta v,\ v(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n)) \\ &v(\cdot,0) = u, \end{aligned}$$

$$\begin{aligned} &w_t = \nu\Delta w,\ w(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n)) \\ &w(\cdot,0) = \mathbb{P}_h u. \end{aligned} \tag{D.7}$$

Vamos mostrar que $w(\cdot,t) = \mathbb{P}_h[v(\cdot,t)]$. Nossa estratégia será mostrar que $\mathbb{P}_h[v(\cdot,t)]$ é solução de (D.7) e que (D.7) tem solução única.

Definindo $z(\cdot,t) := \mathbb{P}_h[v(\cdot,t)]$, como $v(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n))$ e $\mathbb{P}_h : L^q(\mathbb{R}^n) \to L^q_\sigma(\mathbb{R}^n)$ é contínuo, segue que $z(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n))$ e, em particular, $z(\cdot,t) \to \mathbb{P}_h u$ em $L^q(\mathbb{R}^n)$ ao $t \to 0$.

Como v é solução de uma equação do calor com condição inicial em $L^q(\mathbb{R}^n)$, vale que $Dv \in L^q(\mathbb{R}^n)$. Pelo Corolário (D.4), temos que $\Delta\mathbb{P}_h[v] = \mathbb{P}_h[\Delta v]$.

Por outro lado

$$\begin{aligned} \|z_t(\cdot,t) - \mathbb{P}_h[v_t(\cdot,t)]\|_{L^q(\mathbb{R}^n)} =& \left\| \lim_{h\to 0} \frac{z(\cdot,t+h) - z(\cdot,t)}{h} - \mathbb{P}_h[v_t(\cdot,t)] \right\|_{L^q(\mathbb{R}^n)} \\ =& \left\| \lim_{h\to 0} \mathbb{P}_h \left[\frac{v(\cdot,t+h) - v(\cdot,t)}{h} - v_t(\cdot,t) \right] \right\|_{L^q(\mathbb{R}^n)} \\ =& \left\| \mathbb{P}_h \left[\lim_{h\to 0} \frac{v(\cdot,t+h) - v(\cdot,t)}{h} - v_t(\cdot,t) \right] \right\|_{L^q(\mathbb{R}^n)} \\ =& \|\mathbb{P}_h[0]\|_{L^q(\mathbb{R}^n)} = 0. \end{aligned}$$

De modo que $z_t = \mathbb{P}_h[v_t]$. Portanto

$$\Delta z = \Delta \mathbb{P}_h[v] = \mathbb{P}_h[\Delta v] = \mathbb{P}_h[v_t] = z_t.$$

z então satisfaz a equação (D.7).

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