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Contribuições à Teoria Matemática de Escoamentos Magneto-micropolares

por

Cilon F. Perusato

Tese submetida como requisito parcial
para a obtenção do grau de

Doutor em Matemática Aplicada

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Sumário

RESUMO

O objetivo inicial do presente trabalho foi provar o problema de Leray para o sistema magneto-micropolar seguindo uma solução simples recentemente obtida em [41] para as equações de Navier-Stokes. Em [26], Leray deixou um problema de decaimento assintótico em aberto que foi resolvido posteriormente por Kato [22]. Tal problema diz que a norma L^2 da solução da equação de Navier-Stokes incompressível decai assintoticamente a zero, para tempo grande. Ao provar o problema de Leray, observamos uma taxa de decaimento mais rápida para a velocidade microrrotacional $\mathbf{w}$ em relação aos outros campos $(\boldsymbol{u}, \boldsymbol{b})$. A partir disso, mostramos algumas generalizações naturais dessa propriedade. Mais especificamente, obtém-se informações mais precisas a respeito do decaimento de outras normas como, por exemplo, a norma L^∞, o decaimento L^2 das derivadas de ordem mais alta e, por conseguinte, o decaimento em espaços de Sobolev, em duas e três dimensões. Por fim, apresentamos a derivação de uma nova desigualdade para o sistema magneto-micropolar (seguindo o recente trabalho obtido para Navier-Stokes [20]) e algumas interessantes consequências. Além disso, são apresentados alguns resultados básicos de análise, desigualdades de Sobolev e várias propriedades sobre a equação do Calor, dado que tais propriedades se fazem necessárias em nossa análise.

ABSTRACT

The first goal in this work was to prove the Leray problem for magneto micropolar system following a simple solution recently obtained in [41] to Navier-Stokes equations. In [26], Leray left a open problem about asymptotic decay that was solved later by Kato [22]. Such problem says that the L^2 norm for solutions of incompressible Navier-Stokes equations decay to zero asymptotically at large time. We observe that the micro-rotational velocity decay faster than the others fields $(\boldsymbol{u}, \boldsymbol{b})$. Hence, we show some natural extensions of this property. More specifically, we get more detailed information about the L^∞ norm, the decay of high order derivatives in L^2 and the Sobolev spaces decay (in two and three dimensions). Finally, we obtain a derivation of a new fundamental inequalty (in the line of [20]) and some interesting consequences. Furthermore, we present some analysis results, Sobolev inequalities e some Heat Kernel property that will be necessary in our analysis.

1 INTRODUÇÃO

O modelo 3D para fluidos micropolares, primeiramente introduzido por Eringen em [11], é uma generalização substancial das clássicas equações de Navier-Stokes, no qual a microestrutura das partículas do fluido é considerada. Além disso, ao levarmos em conta o efeito do campo magnético induzido no movimento, obtemos o sistema mais completo a seguir. Para $t > 0$, considere

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\mathsf{p} = (\mu+\chi)\,\Delta\boldsymbol{u} + \boldsymbol{b}\cdot\nabla\boldsymbol{b} + \chi\,\nabla\wedge\mathbf{w}, \tag{1.1a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}) + \chi\,\nabla\wedge\boldsymbol{u} - 2\,\chi\,\mathbf{w}, \tag{1.1b}$$

$$\boldsymbol{b}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \nu\,\Delta\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \tag{1.1c}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = 0, \qquad \nabla\cdot\boldsymbol{b}(\cdot,t) = 0, \tag{1.1d}$$

com dados iniciais $(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^3)\times\boldsymbol{L}^2(\mathbb{R}^3)\times\boldsymbol{L}^2_\sigma(\mathbb{R}^3)$, e

$$\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t) - (\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{b}_0)\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{quando} \quad t\to\infty.$$

Onde $\boldsymbol{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ denota o campo de velocidade, o campo $\mathbf{w}(x,t) = (w_1(x,t), w_2(x,t), w_3(x,t)) \in \mathbb{R}^3$ descreve a velocidade microrrotacional, $\boldsymbol{b}(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t)) \in \mathbb{R}^3$ o campo magnético e $\mathsf{p}(x,t) \in \mathbb{R}$ a pressão total do fluido no ponto (x,t). As letras μ, γ, ν e χ são constantes positivas que estão associadas a propriedades físicas do material. Mais especificamente, μ e γ são as viscosidades cinemática e de giro (*kinematic and spin viscosities*), respectivamente. Complementando, ν^{-1} é o número magnético de Reynolds e χ é a viscosidade de vórtice (*vortex viscosity*). Eventualmente, consideraremos o caso em que não se tem viscosidade de vortex, isto é, $\chi = 0$, veja, e.g., a seção (2.1).

O modelo bidimensional é um caso especial do sistema 3D (1.1) descrito acima, onde $\boldsymbol{u}(x,t) = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), 0)$, $\boldsymbol{b}(x,t) = (b_1(x_1,x_2,t), b_2(x_1,x_2,t), 0)$ e, como $\mathbf{w}$ é a velocidade de micro-rotação, escreve-se $\mathbf{w}(x,t) = (0, 0, w(x_1,x_2,t))$, como feito em [9], por exemplo. Desta maneira, substituindo $\boldsymbol{u}$, $\mathbf{w}$ e $\boldsymbol{b}$ escrito na forma acima no sistema (1.1), obtemos as seguintes equações governantes em 2D.

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \mathsf{p} = (\mu + \chi) \Delta \boldsymbol{u} + \chi \nabla \wedge \mathbf{w} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \tag{1.2a}$$

$$\mathbf{w}_t + \boldsymbol{u} \cdot \nabla \mathbf{w} = \gamma \Delta \mathbf{w} + \chi \nabla \wedge \boldsymbol{u} - 2 \chi \mathbf{w}, \tag{1.2b}$$

$$\boldsymbol{b}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{b} = \nu \Delta \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{u}, \tag{1.2c}$$

$$\nabla \cdot \boldsymbol{u}(\cdot, t) = \nabla \cdot \boldsymbol{b}(\cdot, t) = 0, \tag{1.2d}$$

com condições iniciais $(\boldsymbol{u}_0, \boldsymbol{w}_0, \boldsymbol{b}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^2) \times \boldsymbol{L}^2(\mathbb{R}^2) \times \boldsymbol{L}^2_\sigma(\mathbb{R}^2)$ e

$$\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t) - (\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|_{L^2(\mathbb{R}^2)} \to 0,$$

ao $t \to 0$. Nos próximos parágrafos apresentaremos a motivação física e algumas aplicações envolvendo os sistemas (1.1 - 1.2). Além disso, faremos um rápido apanhado sobre a teoria de existência que está estabelecida na literatura. Os resultados aqui obtidos estão descritos a partir da página 4. Consequentemente, caso o leitor esteja familiarizado com as equações e com a teoria de existência, aconselha-se ir diretamente para tal página.

Como dito anteriormente, em 1966, Eringen propôs, em seu trabalho intitulado *Theory of micropolar fluids* (veja [11]), um estudo do sistema (1.1) para o caso em que o campo magnético é nulo, ou seja, $\boldsymbol{b} = 0$. Fisicamente, fluidos micropolares representam escoamentos que contém partículas rígidas como uma certa estrutura suspensas em um meio viscoso, onde a deformação das partículas é ignorado. Quando considerado o efeito do campo magnético induzido, obtém-se os sistemas (1.1) e (1.2). Na literatura, tais fluidos não newtonianos são chamados de micropolares ($\boldsymbol{b} = 0$) e magneto-micropolares. Eles descrevem vários fenômenos vindos de fluidos complexos, tais como circulação sanguínea humana e animal, cristais líquidos, lubrificantes[1], fluidos em meios borbulhantes (*bubbly liquids*), ferrofluidos[2],

[1]Em [35] uma interessante aplicação de fluidos micropolares é feita na teoria de lubrificação de rolamentos. Mais recentemente, em [32] também pode ser encontrada uma aplicação envolvendo lubrificação de rolamentos.

[2]Ferrofluido é um líquido que apresenta grande magnetização na presença de um campo magnético. Os ferrofluidos são compostos por partículas ferromagnéticas (geralmente hematita ou magnetita) em escala nanoscópica suspensas em um fluido, geralmente um solvente orgânico ou água.

suspensões poliméricas, e assim por diante. Em virtude disso, encontram-se em muitas aplicações envolvendo problemas em engenharia (e.g., em [43] o autor examinou a performance de rolamentos circulares analisando os parâmetros das equações para diferentes valores) e fisiologia (e.g., em [31] os autores estudaram o fluxo sanguíneo através de uma artéria com estenose) . Para o leitor interessado nos sistemas aqui expostos (ou sistemas análogos), uma boa referência é o livro *Micropolar Fluids. Theory and Applications* [29]. Nesta referência, maiores informações a respeito destes tipo de fluidos poderão ser encontradas, bem como a derivação das equações. Dita a parte física, agora iremos mencionar alguns resultados sobre a teoria matemática que está estabelecida envolvendo os sistemas (1.1) e (1.2).

Há muitos resultados de existência e unicidade de soluções para problemas relacionados a (1.1) e (1.2) (veja, por exemplo, [4, 5, 8, 11, 14, 29, 33, 38, 49]). Mais precisamente, em 1977, G.P. Galdi e S. Rionero [14] mostraram a existência e unicidade de soluções fracas para o problema de valor inicial do sistema micropolar.[3] Em 1997, M. A. Rojas-Medar [36] mostrou a existência e unicidade local de soluções clássicas. J.L. Boldrini and M.A. Rojas-Medar, em 1998, estudaram a existência de soluções fracas em [5] para duas e três dimensões espaciais. Em particular, eles mostraram (em 2D) a unicidade de tais soluções. E.E. Ortega-Torres e M.A. Rojas-Medar em 1999, assumindo dados iniciais pequenos, provaram a existência global de uma solução clássica (veja [33]). Os resultados desses últimos três trabalhos foram obtidos através de um método espectral de Galerkin. Em 2010, J.L. Boldrini, M. Durán and M.A. Rojas-Medar [4] provaram existência e unicidade de soluções clássicas em $L^p(\Omega)$, para $p > 3$. Em [7], os autores mostraram um interessante resultado de decaimento exponencial em domínios limitados e em [6] resultados de existência para sistema micropolar com densidade variável . Para dados iniciais em $L^1 \cap L^2$ e com algumas hipóteses adicionais, há alguns trabalhos em que se obtém resultados de decaimento via as (clássicas) técnicas de *Fourier Splitting* (veja e.g. [44]). O problema bidimensional (1.2) tem

[3]Em [14], os autores mostraram tal propriedade para o sistema micropolar em abertos conexos Ω com a solução se anulando em $\partial\Omega \times [0, T]$.

sido extensivamente estudado e há muitos resultados envolvendo existência e comportamento assintótico da solução considerando viscosidades parciais ou nulas, veja e.g. [9, 10, 28, 45].

Em seu célebre artigo [26] de 1934, Leray construiu soluções (fracas) globais de energia finita $\boldsymbol{u}(\cdot,t) \in L^{\infty}([0,\infty), L^2_{\sigma}(\mathbb{R}^3)) \cap C_w([0,\infty), L^2(\mathbb{R}^3)) \cap L^2([0,\infty), \dot{H}^1(\mathbb{R}^3))$ para as equações de Navier-Stokes em $\mathbb{R}^3$. Tais soluções (fracas) de Leray podem ser construídas de maneira análoga para o seguinte sistema MHD Micropolar (1.1), obtendo assim, as mesmas importantes propriedades existentes para a equação de Navier-Stokes. Por exemplo, a existência de $t_* > 0$ suficientemente grande (dependendo da condição inicial), tal que

$$(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t) \in C^{\infty}(\mathbb{R}^3 \times (t_*,\infty)). \tag{1.3}$$

Um problema básico (e importante) deixado aberto por Leray em 1934 foi (denotando $W(t) := \| \boldsymbol{u}(\cdot,t) \|^2_{L^2(\mathbb{R}^3)}$, como em [26]):

J'ignore si W(t) tend nécessairement vers 0 *quand* t *augmente indéfiniment,*

J. Leray ([26], p. 248)

ou seja, se vale (ou não) que

$$\lim_{t \to \infty} \| \boldsymbol{u}(\cdot,t) \|_{L^2(\mathbb{R}^3)} = 0. \tag{1.4}$$

Esta questão somente foi resolvida (positivamente) 50 anos mais tarde por Kato [22] e subsequentemente também por outros autores [21, 30, 47]. Vários desenvolvimentos e extensões importantes de (1.4) vem sendo estabelecidos (ver e.g. [3, 24, 39, 41]) e até mesmo para o sistema MHD, veja [1] e [38]. Mais especificamente, os autores [41] provaram o mesmo resultado usando uma técnica diferente e, a partir daí, conseguiram responder outras questões (mais gerais) de decaimento assintótico para Navier-Stokes como, por exemplo,

$$\lim_{t\to\infty} t^{3/4} \|\boldsymbol{u}(\cdot,t)\|_{L^{\infty}(\mathbb{R}^3)} = 0.$$

Nosso objetivo foi primeiramente estender tais propriedades para o sistema (1.1) e (1.2). Um ponto chave para demonstrar esses resultados assintóticos é analisar o comportamento das derivadas de primeira ordem. Adaptando o argumento usado em [24], foi possível obter (teorema 2.1, no capítulo 1) que

$$t^{1/2}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{ao} \quad t\to\infty, \tag{1.5}$$

para $n = 2, 3$. A partir daí, no capítulo 2, obtém-se,

$$\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{ao} \quad t\to\infty$$

e, no caso em que $\chi > 0$, observa-se uma taxa de decaimento mais rápida[4]

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \to 0 \text{ ao } t\to\infty, \tag{1.6}$$

para $n = 2, 3$ (veja [17]).

No capítulo 3, estudamos o comportamento assintótico das derivadas de ordem mais alta, donde foi possível demonstrar que[5],

$$\lim_{t\to\infty} t^{m/2}\|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0$$

e no caso em que $\chi > 0$,

$$\lim_{t\to\infty} t^{\frac{m+1}{2}}\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0,$$

para cada $m \geq 0$ inteiro e $n = 2, 3$. O que era de se esperar, tendo em vista os resultados (1.5) e (1.6). Usando desigualdades de Sobolev e interpolação, tem-se

$$t^{s/2}\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0 \quad \textit{ao} \quad t\to\infty,$$

para cada $s \geq 0$ real. Tal propriedade foi demonstrada no final do capítulo 3. Como consequência disto, por interpolação, obtém-se,

$$t^{\frac{n}{4}-\frac{n}{2q}}\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0 \quad \textit{ao} \ \ t\to\infty,$$

[4]Observe que, apagando todos os termos na segunda equação (1.1b) e (1.2b), tem-se um decaimento exponencial o que talvez sugere que o campo $\mathbf{w}$ decaia mais rápido em relação aos campos $(\boldsymbol{u}, \boldsymbol{b})$. Porém o efeito dos outros termos presentes "freiam"tal decaimento.

[5]Em [18] mostramos esta propriedade para o sistema MHD, i.e., quando o campo $\mathbf{w}(\cdot,t)$ é nulo. Isto foi feito em dimensão $n = 2$, $n = 3$ e $n = 4$.

para cada $2 \leq q \leq \infty$. Além do mais, quando $\chi > 0$, tem-se

$$t^{\frac{s+1}{2}} \|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0 \quad ao \quad t \to \infty,$$

o que acontece somente com o campo $\mathbf{w}$, ou seja,

$$t^{s/2} \|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0 \quad ao \quad t \to \infty.$$

Além disso, novamente por interpolação,

$$t^{\frac{n+2}{4}-\frac{n}{2q}} \|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0 \quad ao \quad t \to \infty.$$

A partir daí, obtemos os seguintes resultados gerais de decaimento:
para $n = 2,3$ e $(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t)$ solução de Leray de (1.1) e (1.2), tem-se

$$t^{\gamma(n,m,q)} \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{as } t \to \infty.$$

Além disso, se $\chi > 0$, então

$$t^{\gamma(n,m,q)+1/2} \|D^m\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0 \ \text{ as } \ t \to \infty,$$

onde $\gamma(n,m,q) = \frac{n}{4} + \frac{m}{2} - \frac{n}{2q}$, para cada $m \geq 0$ (inteiro) e $2 \leq q \leq \infty$. A partir dessas propriedades pode-se estimar a pressão total p dos sistemas (1.1) e (1.2) via Calderón–Zygmund, veja [19] e [34].

No capítulo 4, apresentaremos a derivação de uma desigualdade assintótica para as derivadas das soluções de Leray dos sistemas (1.1) e (1.2). Tal desigualdade foi recentemente obtida para a equação de Navier-Stokes em [20]. Sua versão para o sistemas aqui considerados é a seguinte. Para cada $\alpha \geq 0$, seja

$$\lambda_0(\alpha) := \limsup_{t\to\infty} t^{\alpha} \|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)},$$

onde $\mathbf{z}(\cdot,t) := (\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t)$, $D\mathbf{z}(\cdot,t) := (D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})$ e assim por diante.

Se $\lambda_0(\alpha) < \infty$, então

$$\limsup_{t\to\infty} t^{\alpha+m/2} \|D^m\mathbf{z}(\cdot)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha,m)\ \lambda^{-m/2}\ \lambda_0(\alpha) \qquad \forall m \geq 1,$$

para cada $n = 2, 3$, onde

$$K(\alpha, m) := \min_{\delta>0} \left\{ \delta^{-1/2} \prod_{j=0}^{m} (\alpha + \frac{j}{2} + \delta)^{\frac{1}{2}} \right\}$$

e $\lambda := \min\{\mu, \gamma, \nu\}$.

No capítulo 5, consideramos o caso $\chi > 0$. Lá, apresentamos uma melhor versão da desigualdade acima. Defina, por conveniência, $\tilde{\mathbf{z}}(\cdot, t) := (\boldsymbol{u}, \boldsymbol{b})$. Dado $\alpha \geq 0$, seja

$$\tilde{\lambda}_0(\alpha) := \limsup_{t\to\infty} t^{\alpha} \|\tilde{\mathbf{z}}(\cdot, t)\|_{L^2(\mathbb{R}^n)}.$$

Se $\tilde{\lambda}_0(\alpha) < \infty$, então

$$\limsup_{t\to\infty} t^{\alpha+\frac{m}{2}} \|D^m \mathbf{z}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha, m) \lambda^{-\frac{m}{2}} \tilde{\lambda}_0(\alpha).$$

Além disso, para o campo $\mathbf{w}$, obtemos

$$\limsup_{t\to\infty} t^{\alpha+\frac{m}{2}+\frac{1}{2}} \|D^m \mathbf{w}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{2} K(\alpha, m+1) \;\; \lambda^{-\frac{m+1}{2}} \tilde{\lambda}_0(\alpha).$$

Como anteriormente, são apresentados (via interpolação) resultados análogos aos expostos acima.

O estudo do caso 2D é (em geral) facilitado pelo fato de que a propriedade de regularidade (1.3) vale para $t_* = 0$ e o sistema (1.2) é um caso particular ligeiramente mais simples de (1.1). Além disso, as desigualdades de Sobolev usadas são mais simples de se obter (veja, e.g. (A.13)-A.16). Devido a isso, o leitor notará que na última seção de cada capítulo, faremos pequenas notas abordando o caso bidimensional ($n = 2$).

Notação. Os espaços $\boldsymbol{L}^2_\sigma(\mathbb{R}^3)$ denotam o espaço dos campos em $\mathbb{R}^3$ em L^2 com divergente nulo, i.e.,

$$\boldsymbol{L}^2_\sigma(\mathbb{R}^3) = \{\boldsymbol{u} = (u_1, u_2, u_3); u_i \in L^2(\mathbb{R}^3),\ \text{para } 1 \leq i \leq 3 \text{ com } \nabla \cdot \boldsymbol{u} = 0\}.$$

Como já visto acima, usaremos (em geral) letras em negrito para grandezas vetoriais, e.g. $\boldsymbol{u}(x, t) = (u_1(x, t), ..., u_n(x, t))$, denotando por $|\cdot|_2$ (ou simplesmente $|\cdot|$)

a norma Euclideana em $\mathbb{R}^n$. Como é usual, $\nabla \mathsf{p} \equiv \nabla \mathsf{p}(\cdot, t)$ denota o gradiente (espacial) de $\mathsf{p}(\cdot, t)$, $D_j = \partial / \partial x_j$, e $\nabla \cdot \boldsymbol{u} = D_1 u_1 + ... + D_n u_n)$ é o divergente (espacial) de $\boldsymbol{u}(\cdot, t)$; analogamente, $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = u_1 D_1 \boldsymbol{u} + ... + u_n D_n \boldsymbol{u}$. $\| \cdot \|_{L^q(\mathbb{R}^n)}$, $1 \leq q \leq \infty$, denota a norma tradicional do espaço de Lebesgue $\boldsymbol{L}^q(\mathbb{R}^n)$, pondo-se,

$$\| \boldsymbol{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i=1}^{n} \int_{\mathbb{R}^n} | u_i(x, t) |^q \, dx \Big\}^{1/q} \tag{1.7a}$$

$$\| D\boldsymbol{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} | D_j u_i(x, t) |^q \, dx \Big\}^{1/q} \tag{1.7b}$$

$$\| D^2\boldsymbol{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \Big\{ \sum_{i,j,\ell=1}^{n} \int_{\mathbb{R}^n} | D_j D_\ell u_i(x, t) |^q \, dx \Big\}^{1/q} \tag{1.7c}$$

e, mais geralmente, para $m \geq 1$ inteiro:

$$\| D^m\boldsymbol{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left(\sum_{i=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \int_{\mathbb{R}^n} | D_{j_1} \cdots D_{j_m} u_i(x, t) |^q \, dx \right)^{1/q}. \tag{1.7d}$$

Definiremos também, por conveniência:

$$\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})\|^q_{L^q(\mathbb{R}^n)} = \|\boldsymbol{u}\|^q_{L^q(\mathbb{R}^n)} + \|\mathrm{w}\|^q_{L^q(\mathbb{R}^n)} + \|\boldsymbol{b}\|^q_{L^q(\mathbb{R}^n)}, \tag{1.7e}$$

para $1 \leq q < \infty$ e, quando $q = \infty$,

$$\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})\|_{L^\infty(\mathbb{R}^n)} = \max\{\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^n)}, \|\mathrm{w}\|_{L^\infty(\mathbb{R}^n)}, \|\boldsymbol{b}\|_{L^\infty(\mathbb{R}^n)}\} \tag{1.7f}$$

denotando-se por $\| \boldsymbol{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \big\{ \| u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} \colon 1 \leq i \leq n \big\}$ o supremo (essencial) de $\boldsymbol{u}(\cdot, t)$, similarmente para $\| D\boldsymbol{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)}$, $\| D^2\boldsymbol{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)}$ e para as derivadas de ordem mais alta. É conveniente, também, definir as normas $\dot{H}^s(\mathbb{R}^n)$:

$$\|\boldsymbol{u}\|^2_{\dot{H}^s(\mathbb{R}^n)} = \sum_{i=1}^{n} \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u_i}(\xi)|^2 d\xi,$$

$$\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})\|^2_{\dot{H}^s(\mathbb{R}^n)} = \|\boldsymbol{u}\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|\mathrm{w}\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|\boldsymbol{b}\|^2_{\dot{H}^s(\mathbb{R}^n)},$$

onde $\hat{u_i}$ denota a transformada de Fourier de u_i. Para maiores detalhes veja o apêndice sobre espaços de Sobolev.

Com estas definições, tem-se $\| \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} \to \| \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}$ ao $q \to \infty$, assim como, mais geralmente, $\| D^m \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)} \to \| D^m \boldsymbol{u}(\cdot,t) \|_{L^\infty(\mathbb{R}^n)}$, para todo m inteiro não negativo.[6] Ocasionalmente, resulta também conveniente usar a seguinte definição alternativa para a norma do sup de $\boldsymbol{u}(\cdot,t)$,

$$\| \boldsymbol{u}(\cdot,t) \|_\infty = \text{ ess sup } \big\{ \, | \, \boldsymbol{u}(x,t) \, |_2 \colon \; x \in \mathbb{R}^n \, \big\}.$$

Podemos também utilizar $\| \boldsymbol{u}(\cdot,t) \|_{L^q}$ no lugar de $\| \boldsymbol{u}(\cdot,t) \|_{L^q(\mathbb{R}^n)}$, por simplicidade e $e^{\eta\Delta\tau} f$ denota o *Heat kernel*, no sentido que

$$e^{\eta\Delta\tau} f = \frac{1}{(4\pi\eta t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4\eta t}} f(y) dy.$$

Constantes serão usualmente representadas pelas letras C e K; escrevemos $K(\lambda)$ para indicar que o valor da constante referida depende de um dado parâmetro λ. Por conveniência e economia, usamos tipicamente o mesmo símbolo para denotar constantes com diferentes valores numéricos[7] (por exemplo, escrevemos C^2 ou $10\, C$ novamente como C, e assim por diante), como usualmente feito na literatura. O leitor ainda notará que no transcorrer do texto as demostrações passarão a ser mais diretas, dada a familiaridade que o mesmo terá com os argumentos outrora apresentados.

[6] Convém observar que, com as definições (1.16), (1.17), se uma desigualdade de tipo Nirenberg-Gagliardo $\| \mathsf{u} \|_{L^q} \le K \| \mathsf{u} \|_{L^{r_1}}^{1-\theta} \| \nabla \mathsf{u} \|_{L^{r_2}}^{\theta}$, $0 \le \theta \le 1$, valer para funções *escalares* u ($K > 0$ constante), então ela será automaticamente válida para funções *vetoriais* $\boldsymbol{u}$ com a *mesma* constante K do caso escalar. Ademais, tem-se $\| D^m \boldsymbol{u}(\cdot,t) \|_{L^q} \le \| D^m \boldsymbol{u}(\cdot,t) \|_{L^{q_1}}^{1-\theta} \| D^m \boldsymbol{u}(\cdot,t) \|_{L^{q_2}}^{\theta}$ se $1/q = (1-\theta)/q_1 + \theta/q_2$, $0 \le \theta \le 1$, e assim por diante.

[7] Exceto em alguns trechos (nos primeiros capítulos e apêndice A, por exemplo), onde exibimos o valor de algumas constantes para facilitar a clareza do texto.

2 PRELIMINARES

Neste capítulo provaremos algumas propriedades que serão fundamentais na análise dos resultados desenvolvidos nos capítulos posteriores. O principal resultado a ser demonstrado aqui é o que segue.

Teorema 2.1. *Dada $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$ solução de Leray de (1.1) e (1.2), tem-se*

$$t^{1/2} \| (D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t) \|_{L^2(\mathbb{R}^n)} \to 0 \quad \textit{ao} \quad t \to \infty,$$

para $n = 2, 3$.

Além disso, apresentaremos a demostração da desigualdade de energia básica abaixo.

$$\begin{aligned} \| \, (\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t) \, \|^2_{L^2(\mathbb{R}^n)} + 2\mu \int_{t_0}^{t} \| D\boldsymbol{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^n)} d\tau + 2\gamma \int_{t_0}^{t} \| D\mathbf{w}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^n)} d\tau \\ + 2\nu \int_{t_0}^{t} \| D\boldsymbol{b}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^n)} d\tau + 2\chi \int_{t_0}^{t} \| \mathbf{w}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^n)} d\tau + 2(n-2) \int_{t_0}^{t} \| \nabla \cdot \mathbf{w}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^n)} d\tau \\ \leq \| \, (\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t_0) \, \|^2_{L^2(\mathbb{R}^n)}, \end{aligned}$$

para todo[1] $t > t_0 \geq t_*$ e $n = 2, 3$.

Aqui, alguns resultados sobre a regularidade das soluções de Leray do sistema (1.1) também foram incluídos.

[1] Veja (1.3), lembrando que $t_* = 0$ em dimensão $n = 2$.

2.1 Caso $n = 3$

Dado o sistema magneto micropolar

$$\begin{cases} \boldsymbol{u}_t \,+\, \boldsymbol{u}\cdot\nabla\boldsymbol{u} \,+\, \nabla \mathsf{p} \;=\; (\mu+\chi)\,\Delta\boldsymbol{u} \,+\, \boldsymbol{b}\cdot\nabla\boldsymbol{b} \,+\, \chi\,\nabla\wedge\mathbf{w}, \\ \mathbf{w}_t \,+\, \boldsymbol{u}\cdot\nabla\mathbf{w} \;=\; \gamma\,\Delta\mathbf{w} \,+\, \nabla(\nabla\cdot\mathbf{w}) \,+\, \chi\,\nabla\wedge\boldsymbol{u} \,-\, 2\,\chi\,\mathbf{w}, \\ \boldsymbol{b}_t \,+\, \boldsymbol{u}\cdot\nabla\boldsymbol{b} \;=\; \nu\,\Delta\boldsymbol{b} \,+\, \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u}(\cdot,t) \;=\; 0, \qquad \nabla\cdot\boldsymbol{b}(\cdot,t) \;=\; 0, \end{cases} \tag{2.1}$$

com dados iniciais $(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0) \in \boldsymbol{L}^2_\sigma(\mathbb{R}^3) \times \boldsymbol{L}^2(\mathbb{R}^3) \times \boldsymbol{L}^2_\sigma(\mathbb{R}^3)$, considere soluções de Leray $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)$ $\in X$, onde

$$X \;=\; L^\infty((0,\infty), \boldsymbol{L}^2(\mathbb{R}^3)) \cap L^2((0,\infty), \dot{\boldsymbol{H}}^1(\mathbb{R}^3)) \cap C^0_w([0,\infty), \boldsymbol{L}^2(\mathbb{R}^3))$$

(sendo $\dot{\boldsymbol{H}}^1(\mathbb{R}^3) = \dot{H}^1(\mathbb{R}^3)^{3\times 3}$ espaço de Sobolev homogêneo de ordem 1).

Como foi dito anteriormente, pela teoria desenvolvida por Leray [26] para Navier-Stokes, existe $t_* > 0$ (dependendo dos dados $\mu, \nu, \gamma, \chi, (\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)$ fornecidos) tal que

$$(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b}) \in C^\infty(\mathbb{R}^3 \times [t_*, \infty)) \tag{2.2a}$$

e

$$(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t) \in L^\infty_{\text{loc}}([t_*, \infty), H^m(\mathbb{R}^3)) \tag{2.2b}$$

para cada $m \geq 0$ inteiro. As propriedades de regularidade (2.2) também podem ser verificadas apenas explorando o seguinte fato. A solução (fraca) globalmente definida satisfaz a desigualdade (forte) abaixo

$$
\begin{aligned}
\| \,(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)\, \|^2_{L^2(\mathbb{R}^3)} + 2\mu \int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\gamma \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
+ 2\nu \int_{t_0}^{t} \|D\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\chi \int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2 \int_{t_0}^{t} \|\nabla \cdot \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
\leq \| \,(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t_0)\, \|^2_{L^2(\mathbb{R}^3)},
\end{aligned}
\tag{2.3}
$$

para quase todo $t_0 \geq 0$, incluindo $t_0 = 0$ (veja e.g. [26, 12]). Em [48], por exemplo, o autor mostra a existência e unicidade de uma solução (global) clássica, assumindo dados iniciais em $\dot{H}^{1/2}(\mathbb{R}^3)$, i.e., $(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0) \in \dot{H}^{1/2}(\mathbb{R}^3)$. Logo, dado $\epsilon > 0$, pelo lema B.5 (descrito no apêndice), usando a identidade de Parseval e a desigualdade (2.3), existe $\hat{t}$ suficientemente grande, tal que

$$
\begin{aligned}
\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,\hat{t})\|_{\dot{H}^{1/2}(\mathbb{R}^3)} &\leq \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,\hat{t})\|^{1/2}_{\dot{H}^1(\mathbb{R}^3)} \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,\hat{t})\|^{1/2}_{L^2(\mathbb{R}^3)} \\
&= \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\hat{t})\|^{1/2}_{L^2(\mathbb{R}^3)} \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,\hat{t})\|^{1/2}_{L^2(\mathbb{R}^3)} \leq C\epsilon,
\end{aligned}
$$

o que garante a existência de $t_* \gg 1$ apresentado em (2.2).

Como as propriedades aqui estudadas são para tempo $t > 0$ grande, daremos uma demonstração desta desigualdade de energia para $t > t_*$, usando as propriedades de regularidade (2.2). A demostração fica mais simples, pois a solução fica clássica para $t > 0$ grande.

Proposição 2.2. *Sejam $t_0 \geq t_*$ e $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)$ solução de Leray do problema MHD micropolar (2.1), tem-se*

$$
\begin{aligned}
\| \,(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)\, \|^2_{L^2(\mathbb{R}^3)} + 2\lambda \int_{t_0}^{t} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
+2\chi \int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2 \int_{t_0}^{t} \|\nabla \cdot \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \leq \| \,(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t_0)\, \|^2_{L^2(\mathbb{R}^3)},
\end{aligned}
\tag{2.4}
$$

para todo $t > t_0$. Onde $\lambda = \min\{\mu, \gamma, \nu\}$

Demonstração. Daremos a seguir uma demonstração rigorosa deste fato básico usando funções de corte. Dado $R > 0$, seja $\Phi_R \in C_0^\infty(\mathbb{R}^3)$ tal que,

$$\Phi_R = \begin{cases} 1, \ |x| \leq R \\ \varphi(x), \ R \leq |x| \leq R+1 \\ 0, \ |x| \geq R+1 \end{cases}$$

onde $\varphi(x) \in C^\infty$, pois $\Phi_R \in C_0^\infty$. Escrevendo o sistema (2.1) coordenada a coordenada, tem-se

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j D_j u_i + D_i \mathsf{p} = (\mu + \chi) \sum_{j=1}^3 D_j D_j u_i + \sum_{j=1}^3 b_j D_j b_i + \chi \sum_{j,k=1}^3 \epsilon_{i,j,k} D_j w_k \\ \frac{\partial w_i}{\partial t} + \sum_{j=1}^3 u_j D_j w_i = \gamma \sum_{j=1}^3 D_j D_j w_i + \sum_{j=1}^3 D_i(D_j w_j) + \chi \sum_{j,k=1}^3 \epsilon_{i,j,k} D_j u_k - 2\chi w_i \\ \frac{\partial b_i}{\partial t} + \sum_{j=1}^3 u_j D_j b_i = \nu \sum_{j=1}^3 D_j D_j b_i + \sum_{j=1}^3 b_j D_j u_i, \end{cases} \tag{2.5}$$

onde $\epsilon_{i,j,k}$ é o símbolo de Levi-Civita, i.e.,

$$\epsilon_{i,j,k} = \begin{cases} 1, \ \text{se } \{i,j,k\} \text{ é permutação par} \\ -1, \ \text{se } \{i,j,k\} \text{ é permutação ímpar} \\ 0, \ \text{se } i = j, \ \text{ou } j = k, \ \text{ou } i = k \end{cases}$$

Multiplicando a primeira equação do sistema (2.5) por $2u_i\Phi_R$ e integrando na região $\mathbb{R}^3 \times [t_0, t]$ tem-se,

$$
\underbrace{\int_{t_0}^{t}\int_{B_{R+1}} \frac{\partial}{\partial\tau}(u_i^2)\,\Phi_R\,dxd\tau}_{(I)} + \underbrace{\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} D_j(u_i^2u_j)\,\Phi_R\,dxd\tau}_{(II)} + \underbrace{2\int_{t_0}^{t}\int_{B_{R+1}} (D_i\,\mathsf{p})\,u_i\Phi_R\,dxd\tau}_{(III)}
$$

$$
= \underbrace{2(\mu+\chi)\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} D_jD_ju_i\,u_i\Phi_R\,dxd\tau}_{(IV)} + \underbrace{2\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} b_jD_jb_i\,u_i\Phi_R\,dxd\tau}_{(V)}
$$

$$
+ \underbrace{2\chi\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j,k=1}^{3} \epsilon_{i,j,k}D_jw_k\,u_i\Phi_R\,dxd\tau.}_{(VI)} \tag{2.6}
$$

Agora, vamos estimar cada termo da igualdade (2.6). Aplicando o Teorema de Fubini [2] e o Teorema Fundamental do Cálculo no primeiro termo, tem-se

$$
(I) = \int_{t_0}^{t}\int_{B_{R+1}} (u_i(\cdot,\tau)^2)_\tau\,\Phi_R(x)\,dxd\tau = \int_{B_{R+1}} u_i(\cdot,t)^2\Phi_R(x)dx - \int_{B_{R+1}} u_i(\cdot,t_0)^2\,\Phi_R(x)dx.
$$

Note que, $\Phi_R(x) = 0, \forall x$ tal que $|x| = R+1$. Logo, como na integração por partes não aparece termo de fronteira, tem-se

$$
\begin{aligned}
(II) &= \int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} D_j(u_i^2u_j)\,\Phi_R(x)\,dxd\tau = -\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} u_i^2D_ju_j\,\Phi_R(x)\,dxd\tau \\
&\quad - \int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{3}(u_i^2u_j)\,D_j\varphi_R(x)\,dxd\tau, \\
(III) &= 2\int_{t_0}^{t}\int_{B_{R+1}} u_i\,D_i\,\mathsf{p}\,\Phi_R(x)\,dxd\tau = -2\int_{t_0}^{t}\int_{R<|x|<R+1} u_i\,\mathsf{p}\,D_i\,\varphi_R(x)\,dxd\tau \\
&\quad -2\int_{t_0}^{t}\int_{B_{R+1}} D_i\,u_i\,\mathsf{p}\,\Phi_R(x)\,dxd\tau, \\
(IV) &= 2(\mu+\chi)\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} u_iD_jD_ju_i\,\Phi_R(x)\,dxd\tau
\end{aligned}
$$

$$= -2(\mu+\chi)\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j=1}^{3}(D_j u_i)^2\,\Phi_R(x)\,dxd\tau$$

$$-2(\mu+\chi)\int_{t_0}^{t}\int_{R<|x|<R+1}\sum_{j=1}^{3}u_i\,D_j u_i\,D_j\varphi_R(x)dxd\tau,$$

$$(V) = 2\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j=1}^{3}b_j D_j b_i\,u_i\Phi_R(x)dxd\tau = -2\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j=1}^{3}b_j b_i\,D_j u_i\Phi_R(x)dxd\tau$$

$$-2\int_{t_0}^{t}\int_{R<|x|<R+1}\sum_{j=1}^{3}b_j b_i\,u_i D_j\varphi_R(x)dxd\tau,$$

$$(VI) = 2\chi\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j,k=1}^{3}\epsilon_{i,j,k}D_j w_k\,u_i\Phi_R(x)\,dxd\tau$$

$$= -2\chi\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j,k=1}^{3}\epsilon_{i,j,k}w_i\,D_j u_k\Phi_R(x)\,dxd\tau$$

$$-2\chi\int_{t_0}^{t}\int_{R<|x|<R+1}\sum_{j,k=1}^{3}\epsilon_{i,j,k}w_k\,u_i D_j\varphi_R(x)\,dxd\tau.$$

Logo, fazendo $R \to +\infty$, usando a definição da Φ_R, o Teorema da Convergência Monótona [2] e o lema B.1 do apêndice B nos termos anteriores, tem-se

$$(I) = \int_{\mathbb{R}^3} u_i(\cdot,t)^2 dx - \int_{\mathbb{R}^3} u_i(\cdot,t_0)^2\,dx = \|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} - \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)},$$

$$(II) = -\int_{t_0}^{t}\int_{\mathbb{R}^3} u_i^2\sum_{j=1}^{3}D_j u_j\,dxd\tau = 0, \text{ pois } \nabla\cdot\boldsymbol{u} = 0,$$

$$(III) = -2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_i\,u_i\,\mathsf{p}\,dxd\tau,$$

$$(IV) = -2(\mu+\chi)\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3}(D_j u_i)^2\,dxd\tau,$$

$$(V) = -2\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3}b_j b_i\,D_j u_i dxd\tau,$$

$$(VI) = -2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}\epsilon_{i,j,k}w_i\,D_j u_k\,dxd\tau.$$

Então reescrevendo (2.6), tem-se

$$
\begin{aligned}
\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_i\, u_i\, \mathsf{p}\, dxd\tau \\
+2(\mu+\chi)\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3}(D_j u_i)^2 dxd\tau + 2\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3} b_j b_i\, D_j u_i dxd\tau \\
+2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}\epsilon_{i,j,k} w_i\, D_j u_k\, dxd\tau = \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned} \tag{2.7}
$$

Agora iremos trabalhar com a segunda e com a terceira equação do sistema (2.5) utilizando as mesmas ideias anteriores. Caso o leitor já esteja familiarizado com as contas feitas anteriormente, pode seguir direto para desigualdade (2.12).

Multiplicando a segunda equação do sistema (2.5) por $2w_i\Phi_R$ e integrando-a na região $\mathbb{R}^3\times[t_0,t]$, obtém-se

$$
\begin{aligned}
&\underbrace{\int_{t_0}^{t}\int_{B_{R+1}}(w_i^2)_t\;\Phi_R(x)\;dxd\tau}_{(I)} + \underbrace{\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j=1}^{3}D_j(w_i^2 u_i)\;\Phi_R(x)\;dxd\tau}_{(II)} \\
= \;& \underbrace{2\gamma\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j=1}^{3}D_jD_j w_i\; w_i\;\Phi_R(x)\;dxd\tau}_{(III)} + \underbrace{2\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j=1}^{3}D_i(D_j\; w_j)\; w_i\;\Phi_R(x)\;dxd\tau}_{(IV)} \\
&+ \underbrace{2\chi\int_{t_0}^{t}\int_{B_{R+1}}\sum_{j,k=1}^{3}\epsilon_{i,j,k}\,D_j u_k\; w_i\Phi_R(x)\;dxd\tau}_{(V)} - \underbrace{4\,\chi\int_{t_0}^{t}\int_{B_{R+1}} w_i^2\Phi_R(x)\;dxd\tau}_{(VI)}.
\end{aligned} \tag{2.8}
$$

Vamos trabalhar com cada termo da igualdade (2.8). Como não temos termo de fronteira, aplicando o Teorema de Fubini, o Teorema Fundamental do Cálculo em (I) e integração por partes em (II),(III),(IV), temos

$$(I) = \int_{t_0}^{t} \int_{B_{R+1}} (w_i^2)_\tau \ \Phi_R(x) \ dxd\tau = \int_{B_{R+1}} [w_i^2(\cdot, t) - w_i^2(\cdot, t_0)] \ \Phi_R(x) \ dx,$$

$$(II) = \int_{t_0}^{t} \int_{B_{R+1}} \sum_{j=1}^{3} D_j(w_i^2 u_j) \ \Phi_R(x) \ dxd\tau$$

$$= -\int_{t_0}^{t} \int_{B_{R+1}} \underbrace{\sum_{j=1}^{3} w_i^2 D_j u_j}_{=0, \text{ pois } \nabla\cdot\boldsymbol{u}=0} \ \Phi_R(x) \ dxd\tau - \int_{t_0}^{t} \int_{R<|x|<R+1} \sum_{j=1}^{3} w_i^2 u_j \ D_j\varphi_R(x) \ dxd\tau,$$

$$(III) = 2\gamma \int_{t_0}^{t} \int_{B_{R+1}} \sum_{j=1}^{3} D_j D_j w_i \ w_i \ \Phi_R(x) \ dxd\tau$$

$$= -\gamma \int_{t_0}^{t} \int_{R<|x|<R+1} \sum_{j=1}^{3} (w_i)^2 \ D_j\varphi_R(x) \ dxd\tau$$

$$-2\gamma \int_{t_0}^{t} \int_{B_{R+1}} \sum_{j=1}^{3} D_j w_i \ D_j w_i \ \Phi_R(x) \ dxd\tau,$$

$$(IV) = 2 \int_{t_0}^{t} \int_{B_{R+1}} \sum_{j=1}^{3} D_i(D_j \ w_j) \ w_i \ \Phi_R(x) \ dxd\tau$$

$$= -2 \int_{t_0}^{t} \int_{B_{R+1}} \sum_{j=1}^{3} D_i w_i(D_j \ w_j) \ \Phi_R(x) \ dxd\tau$$

$$-2 \int_{t_0}^{t} \int_{R<|x|<R+1} \sum_{j=1}^{3} w_i \ (D_j w_j) \ D_i\varphi_R(x) \ dxd\tau.$$

Fazendo $R \to +\infty$, utilizando nos termos anteriores o Teorema da Convergência Monótona, o lema B.1 do apêndice B e definição da Φ_R, tem-se

$$\|w_i(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + 2\gamma \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j=1}^{3} (D_j w_i)^2 \ dxd\tau$$

$$+2 \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j=1}^{3} D_i w_i(D_j \ w_j) \ dxd\tau + 2\chi \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{j,k=1}^{3} \epsilon_{i,j,k} \, u_i \ D_j w_k \ dxd\tau$$

$$+ 4\,\chi \int_{t_0}^{t} \|w_i(\cdot, \tau)\|^2 \ d\tau = \|w_i(\cdot, t_0)\|_{L^2(\mathbb{R}^3)}^2. \tag{2.9}$$

Por último, analogamente, multiplicamos a terceira equação do sistema (2.5) por $2\Phi(x)b_i$ e integramos na região $\mathbb{R}^3 \times [t_0, t]$

$$\underbrace{\int_{t_0}^{t}\int_{B_{R+1}} (b_i^2)_t\ \Phi_R(x)dxd\tau}_{(I)} + \underbrace{\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} u_j(D_j b_i^2)\Phi_R(x)\ dxd\tau}_{(II)}$$

$$= \underbrace{2\nu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} b_i D_j D_j b_i \Phi_R(x)\ dxd\tau}_{(III)} + \underbrace{2\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} b_i b_j D_j u_i \Phi_R(x)\ dxd\tau}_{(IV)}. \quad (2.10)$$

Vamos estimar cada termo na igualdade (2.10. No termo (I), vamos aplicar os Teoremas de Fubini e Fundamental do Cálculo e nos termos (II),(III), integração por partes.

$$(I) = \int_{t_0}^{t}\int_{B_{R+1}} (b_i^2)_\tau\ \Phi_R(x)dxd\tau = \int_{B_{R+1}} [b_i^2(\cdot,t) - b_i^2(\cdot,t_0)]\ \Phi_R(x)dx,$$

$$(II) = \int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} u_j(D_j b_i^2)\Phi_R(x)\ dxd\tau = -\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} D_j u_j b_i^2 \Phi_R(x)\ dxd\tau$$
$$-\int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{3} u_j b_i^2 D_j \varphi_R(x)\ dxd\tau,$$

$$(III) = 2\nu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} b_i D_j D_j b_i \Phi_R(x)\ dxd\tau = -2\nu\int_{t_0}^{t}\int_{B_{R+1}} \sum_{j=1}^{3} (D_j b_i)^2 \Phi_R(x)\ dxd\tau$$
$$-2\nu\int_{t_0}^{t}\int_{R<|x|<R+1} \sum_{j=1}^{3} b_i D_j b_i D_j \varphi_R(x)\ dxd\tau.$$

Fazendo $R \to \infty$. Usando o Teorema da Convergência Monótona e a definição da Φ nas expressões anteriores ficamos com a igualdade

$$\begin{aligned}\|b_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} D_j u_j b_i^2\ dxd\tau + 2\nu\int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} (D_j b_i)^2\ dxd\tau \\ -2\int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j=1}^{3} b_i b_j D_j u_i\ dxd\tau = \|b_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \quad (2.11)$$

Juntando as igualdades (2.7), (2.9) e (2.11).

$$\|u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|w_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|b_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_i\, u_i\, \mathsf{p}\, dxd\tau$$

$$+2(\mu+\chi)\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3}(D_j u_i)^2 dxd\tau - 2\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3} b_i b_j D_j u_i\, dxd\tau$$

$$+2\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3} b_j b_i\, D_j u_i dxd\tau + \underbrace{2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}\epsilon_{i,j,k} w_i\, D_j u_k\, dxd\tau}_{(*)}$$

$$+2\gamma\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3}(D_j w_i)^2\, dxd\tau + 2\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3} D_i w_i (D_j\, w_j)\, dxd\tau$$

$$\underbrace{+\,2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}\epsilon_{i,j,k}\, w_i\, D_j u_k\ \ dxd\tau}_{(**)} + 4\,\chi\int_{t_0}^{t}\|w_i(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\, d\tau$$

$$+2\nu\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{j=1}^{3}(D_j b_i)^2\, dxd\tau = \|u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|w_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|b_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}.$$

Somando em $i = 1,2,3$ e usando as definições das normas descritas na introdução, tem-se,

$$\|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{i=1}^{3} D_i\, u_i\, \mathsf{p}\, dxd\tau}_{=0,\text{ pois } \nabla\cdot\boldsymbol{u}=0}$$

$$+2(\mu+\chi)\int_{t_0}^{t}\|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + \underbrace{2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{i,j,k=1}^{3}\epsilon_{i,j,k} w_i\, D_j u_k\, dxd\tau}_{(*)}$$

$$+2\gamma\int_{t_0}^{t}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\, d\tau + 2\int_{t_0}^{t}\|\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau$$

$$\underbrace{+\,2\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}\sum_{i,j,k=1}^{3}\epsilon_{i,j,k}\, w_i\, D_j u_k\ \ dxd\tau}_{(**)} + 4\,\chi\int_{t_0}^{t}\|\mathbf{w}(\cdot,\tau)\|^2\, d\tau + 2\nu\int_{t_0}^{t}\|D\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau$$

$$= \|\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}. \tag{2.12}$$

Observe que,

$$\sum_{i=1}^{3} \epsilon_{i,j,k}\epsilon_{i,l,m} = \delta_{j,l}\delta_{k,m} - \delta_{j,m}\delta_{k,l}, \tag{2.13}$$

onde $\delta_{i,j}$ é o tensor delta de Kronecker. Note que podemos somar os termos (*) e (**). Estimando este termo usando a Desigualdade de Young e a identidade (2.13),

$$\begin{aligned}
&|4\chi \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} \epsilon_{i,j,k} w_i\, D_j u_k\, dx d\tau| \\
&\leq 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} |w_i \left(\sum_{j,k} \epsilon_{i,j,k}\, D_j u_k\right)| dx d\tau \\
&\leqslant 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} \frac{w_i^2}{2} + \frac{1}{2}\left(\sum_{j,k} \epsilon_{i,j,k}\, D_j u_k\right)^2 dx d\tau \\
&= 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} \frac{w_i^2}{2} dx d\tau + \frac{4\chi}{2} \sum_i \int_{t_0}^{t}\int_{\mathbb{R}^3} \left(\sum_{j,k} \epsilon_{i,j,k}\, D_j u_k\right)\left(\sum_{l,m} \epsilon_{i,l,m}\, D_l u_m\right) dx d\tau \\
&= 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} \frac{w_i^2}{2} dx d\tau + \frac{4\chi}{2} \sum_i \int_{t_0}^{t}\int_{\mathbb{R}^3} \sum_{j,k,l,m} \epsilon_{i,j,k}\epsilon_{i,l,m}\, D_j u_k\, D_l u_m\, dx d\tau \\
&= 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} \frac{w_i^2}{2} dx d\tau + \frac{4\chi}{2} \sum_{j,k,l,m} \int_{t_0}^{t}\int_{\mathbb{R}^3} (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\, D_j u_k\, D_l u_m\, dx d\tau \\
&= 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} \frac{w_i^2}{2} dx d\tau + \frac{4\chi}{2} \sum_{j,k,l,m} \int_{t_0}^{t}\int_{\mathbb{R}^3} \delta_{jl}\delta_{km}\, D_j u_k\, D_l u_m\, dx d\tau \\
&+ \frac{4\chi}{2} \sum_{j,k,l,m} \int_{t_0}^{t}\int_{\mathbb{R}^3} \delta_{jm}\delta_{kl} D_j u_k\, D_l u_m dx d\tau = 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} \frac{w_i^2}{2} dx d\tau \\
&+ \frac{4\chi}{2} \sum_{j,k} \int_{t_0}^{t}\int_{\mathbb{R}^3} (D_j u_k)^2 dx d\tau \underbrace{- \frac{4\chi}{2} \sum_{j,k} \int_{t_0}^{t}\int_{\mathbb{R}^3} D_j u_k\, D_k u_j dx d\tau}_{=0, \text{ por integração por partes e por } \nabla\cdot\boldsymbol{u}=0} \\
&= 4\chi \int_{t_0}^{t} \sum_i \int_{\mathbb{R}^3} \frac{w_i^2}{2} dx d\tau + \frac{4\chi}{2} \sum_{j,k} \int_{t_0}^{t}\int_{\mathbb{R}^3} (D_j u_k)^2 dx d\tau \\
&= 2\chi \int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\chi \int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau
\end{aligned} \tag{2.14}$$

Por (2.14), a igualdade (2.12) se torna,

$$\begin{aligned}&\|\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\|\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\mu\int_{t_0}^{t}\|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\ &+2\gamma\int_{t_0}^{t}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\nu\int_{t_0}^{t}\|D\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\chi\int_{t_0}^{t}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\ &+2\int_{t_0}^{t}\|\nabla\cdot\ \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\le\|\boldsymbol{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}+\|\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}+\|\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Tomando o mínimo entre os coeficientes (μ,γ,ν) e usando as definições (1.7) descritas na introdução, temos (2.4). O que conclui a demonstração. □

O objetivo agora será prover uma desigualdade semelhante a da proposição anterior para as derivadas de primeira ordem. Então vamos derivar parcialmente com relação a x_l ($\frac{\partial}{x_l}=D_l$) e multiplicar por $2D_lu_i$, $2D_lw_i$ e $2D_lb_i$ a primeira, segunda e terceira equação do sistema (2.5), respectivamente. Seguindo de maneira análoga a do roteiro anterior, integrando em $[t_0,t]\times\mathbb{R}^3$, tem-se o seguinte resultado.[2]

Teorema 2.3. *: Sejam $(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t)$ soluções de Leray do problema MHD micropolar, então existe $t_{**}>t_*$ tal que $\forall t>t_0>t_{**}$, tem-se*

$$\begin{aligned}&\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\eta\int_{t_0}^{t}\|(D^2\boldsymbol{u},D^2\mathbf{w},D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\,d\tau\\ &+2\chi\int_{t_0}^{t}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\int_{t_0}^{t}\|D(\nabla.\ \mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\,d\tau\le\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}\tag{2.15}$$

Onde η é uma constante positiva.

[2] Por simplicidade, não faremos o uso de funções de corte. O mesmo processo pode ser repetido de maneira análoga a feita no teorema anterior.

Demonstração. Derivando a primeira equação do sistema (2.5), em relação D_l, multiplicando por $2D_l u_i$ e integrando na região $[t_0, t] \times \mathbb{R}^3$:

$$\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^3} (D_l u_i)^2_\tau \, dxd\tau}_{(I)} \;+\; \underbrace{2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_i D_l(u_j D_j u_i)\, dxd\tau}_{(II)} \;+\; \underbrace{2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_i D_l(D_i\, \mathsf{p})\, dxd\tau}_{(III)}$$
$$= \underbrace{2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_i D_j D_j D_l u_i \, dxd\tau}_{(IV)} \underbrace{+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_i D_l(b_j D_j b_i)\, dxd\tau}_{(V)}$$
$$\underbrace{+\, 2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \epsilon_{i,j,k} D_l u_i D_l(D_j w_k)\, dxd\tau}_{(VI)}\,. \tag{2.16}$$

Vamos analisar cada termo da igualdade (2.16). Aplicando, no termo (I), o Teorema de Fubini e o Teorema Fundamental do Cálculo,

$$\int_{t_0}^{t}\int_{\mathbb{R}^3} (D_l u_i)^2_\tau \, dxd\tau \;=\; \|D_l u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} - \|D_l u_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}.$$

Integrando por partes no termo (II) e em seguida tomando o módulo,

$$\begin{aligned}
(II) \;&=\; 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_i D_l(u_j D_j u_i)\, dxd\tau \\
&=\; 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_i D_l u_j D_j u_i \, dxd\tau + \sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} u_j D_j (D_l u_i)^2 \, dxd\tau \\
&=\; -2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} u_i D_l u_j D_j D_l u_i \, dxd\tau - \sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_j u_j (D_l u_i)^2 \, dxd\tau \\
&\leq\; 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\, |D_l u_j(\cdot,\tau)||D_j D_l u_i(\cdot,\tau)| dxd\tau
\end{aligned}$$

Usando a mesma ideia anterior no termo (V),

$$(V) \;\leq\; 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} |D_l b_j(\cdot,\tau)||D_j D_l b_i(\cdot,\tau)| dxd\tau.$$

E ainda, integrando por partes os termos

$$
\begin{aligned}
(III) &= 2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_i D_l(D_i\,\mathsf{p})\,dxd\tau = -2\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l(D_i u_i) D_l\mathsf{p}\,dxd\tau,\\
(IV) &= 2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l(u_i)D_jD_jD_lu_i\,dxd\tau,\\
&= -2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_l(u_i))^2\,dxd\tau,\\
(VI) &= 2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{i,j,k}D_lu_iD_l(D_jw_k)\,dxd\tau\\
&= 2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{i,j,k}D_lD_j(u_k)D_lw_i\,dxd\tau.
\end{aligned}
$$

Então, reescrevendo (2.16), tem-se

$$
\begin{aligned}
&\|D_lu_i(\cdot,t)\|^2_{L^2_{(\mathbb{R}^3)}} + 2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_l(u_i))^2\,dxd\tau\\
-2\chi&\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{i,j,k}D_lD_j(u_k)D_lw_i\,dxd\tau \le \|D_lu_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}\\
&+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lu_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau\\
&+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lb_j(\cdot,\tau)||D_jD_lb_i(\cdot,\tau)|dxd\tau.
\end{aligned}
\tag{2.17}
$$

Procederemos de maneira análoga para a segunda e terceira equação do sistema (2.5). Derivando a segunda equação do sistema (2.5), em relação D_l, multiplicando por $2D_lw_i$ e integrando na região $[t_0, t] \times \mathbb{R}^3$, tem-se

$$\underbrace{\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_l w_i)^2_\tau \, dxd\tau}_{(I)} + \underbrace{2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l(u_j D_j w_i) D_l w_i \, dxd\tau}_{(II)}$$

$$= \underbrace{2\gamma\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_j D_j D_l w_i D_l w_i \, dxd\tau}_{(III)} + \underbrace{2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l(D_l w_i D_i(D_j w_j)) D_l w_i \, dxd\tau}_{(IV)}$$

$$+ \underbrace{2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \epsilon_{i,j,k} D_l D_j u_k D_l w_i \, dxd\tau}_{(V)} - \underbrace{4\chi\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_l w_i)^2 \, dxd\tau}_{(VI)}.$$

Aplicando o Teorema de Fubini e Teorema Fundamental do Cálculo,

$$(I) = \int_{t_0}^{t}\int_{\mathbb{R}^3}(D_l w_i)^2_\tau \, dxd\tau = \|D_l w_i(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} - \|D_l w_i(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)}.$$

Integrando por partes,

$$\begin{aligned}
(II) &= 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l(u_j D_j w_i) D_l w_i \, dxd\tau \\
&= \underbrace{2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l u_j D_j w_i D_l w_i \, dxd\tau}_{(A)} + \underbrace{\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} u_j D_j (D_l w_i)^2 \, dxd\tau}_{(B)},
\end{aligned}$$

integrando por partes os termos (A) e (B)

$$\begin{aligned}
(A) &= -2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} w_i D_l u_j D_j D_l w_i \, dxd\tau \\
&\leq 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|w_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} |D_l u_j(\cdot,\tau)||D_j D_l w_i(\cdot,\tau)| dxd\tau,
\end{aligned}$$

$$(B) = \sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} u_j D_j (D_l w_i)^2 \, dxd\tau = -\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_j u_j (D_l w_i)^2 \, dxd\tau = 0.$$

Então,

$$
\begin{aligned}
(II) &= 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l(u_jD_jw_i)D_lw_i\ dxd\tau \\
&\le 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|w_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lu_j(\cdot,\tau)||D_jD_lw_i(\cdot,\tau)|dxd\tau, \\
(III) &= 2\gamma\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_jD_jD_lw_iD_lw_i\ dxd\tau = -2\gamma\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_lw_i)^2dxd\tau, \\
(IV) &= 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_l(D_i(D_jw_j))D_lw_i\ dxd\tau = -2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_lD_jw_jD_lD_iw_i\ dxd\tau.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\|D_lw_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\gamma\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_lw_i(\cdot,\tau))^2\ dxd\tau + 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} D_lD_jw_jD_lD_iw_i\ dxd\tau \\
&+4\chi\int_{t_0}^{t}\|D_lw_i(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\ d\tau \le \|D_lw_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
&+2\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{i,j,k}D_lD_ju_k(\cdot,\tau)D_lw_i(\cdot,\tau)\ dxd\tau \\
&+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|w_i(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}|D_lu_j(\cdot,\tau)||D_jD_lw_i(\cdot,\tau)|dxd\tau.
\end{aligned}
\tag{2.18}
$$

Por último, vamos trabalhar com a terceira equação do sistema (2.5). Derivando-a em relação a D_l, multiplicando por $2D_lb_i$ e integrando na região $[t_0,t]\times\mathbb{R}^3$. Repetindo o processo de maneira análoga, obtém-se,

$$
\begin{aligned}
\|D_lb_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\nu\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_lb_i(\cdot,\tau))^2\ dxd\tau \le \|D_lb_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
&+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lu_j(\cdot,\tau)||D_jD_lb_i(\cdot,\tau)|dxd\tau \\
&+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lb_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau.
\end{aligned}
\tag{2.19}
$$

Somando (2.17), (2.18) e (2.19), obtém-se

$$\|D_l u_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|D_l w_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|D_l b_i(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}$$

$$+2(\mu+\chi)\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_lu_i(\cdot,\tau))^2\,dxd\tau + 2\gamma\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_lw_i(\cdot,\tau))^2\;dxd\tau$$

$$+2\nu\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}(D_jD_lb_i(\cdot,\tau))^2\;dxd\tau + 2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}D_lD_jw_j(\cdot,\tau)D_lD_i\;w_i(\cdot,\tau)\;dxd\tau$$

$$+4\chi\int_{t_0}^{t}\|D_lw_i(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau$$

$$\leq \|D_lu_i(\cdot,t_0)\|^2_{L^2_{(\mathbb{R}^3)}} + \|D_lw_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|D_lb_i(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}$$

$$+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lu_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau$$

$$+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lb_j(\cdot,\tau)||D_jD_lb_i(\cdot,\tau)|\;dxd\tau$$

$$+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|w_i(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}|D_lu_j(\cdot,\tau)||D_jD_lw_i(\cdot,\tau)|dxd\tau$$

$$+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lb_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau$$

$$+2\sum_{j=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_lu_j(\cdot,\tau)||D_jD_lb_i(\cdot,\tau)|dxd\tau$$

$$+4\chi\sum_{j,k=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\epsilon_{i,j,k}D_lD_ju_k(\cdot,\tau)D_lw_i(\cdot,\tau)\;dxd\tau.$$

Somando em i e l, tem-se

$$
\begin{aligned}
&\|D\boldsymbol{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|D\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + \|D\boldsymbol{b}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
&+2(\mu+\chi)\int_{t_0}^{t} \|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\, d\tau + 2\gamma\int_{t_0}^{t} \|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\, d\tau \\
&+2\nu\int_{t_0}^{t} \|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\, d\tau + 2\int_{t_0}^{t} \|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)} d\tau \\
&+4\chi\int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \leq \|D\boldsymbol{u}(\cdot,t_0)\|^2_{L^2_{(\mathbb{R}^3)}} + \|D\mathbf{w}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + \|D\boldsymbol{b}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
&+2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} |D_l u_j(\cdot,\tau)||D_j D_l u_i(\cdot,\tau)| dx d\tau \\
&+2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} |D_l b_j(\cdot,\tau)||D_j D_l b_i(\cdot,\tau)|\, dx d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|w_i(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} |D_l u_j(\cdot,\tau)||D_j D_l w_i(\cdot,\tau)| dx d\tau \\
&+2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} |D_l b_j(\cdot,\tau)||D_j D_l u_i(\cdot,\tau)| dx d\tau \\
&+2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} |D_l u_j(\cdot,\tau)||D_j D_l b_i(\cdot,\tau)| dx d\tau \\
&\underbrace{+\,4\chi\sum_{i,j,k,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3} \epsilon_{i,j,k} D_l D_j u_k(\cdot,\tau) D_l w_i(\cdot,\tau)\, dx d\tau}_{(A)}.
\end{aligned}
$$

Aplicando a Desigualdade de Young no termo (A) da desigualdade anterior resulta que,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n}\sum_{i,j,k,l=1}^{3}\epsilon_{ijk}D_lw_iD_lD_ju_kdx\right| = \left|\sum_{i,l=1}^{3}\int_{\mathbb{R}^n}D_lw_i\sum_{j,k=1}^{3}\epsilon_{ijk}D_lD_ju_kdx\right|\\
&\leq \sum_{i,l=1}^{3}\int_{\mathbb{R}^n}|D_lw_i|\left|\sum_{j,k=1}^{3}\epsilon_{ijk}D_lD_ju_k\right|dx \leq \frac{1}{2}\sum_{i,l=1}^{3}\int_{\mathbb{R}^n}(D_lw_i)^2+\left(\sum_{j,k=1}^{3}\epsilon_{ijk}D_lD_ju_k\right)^2dx\\
&= \frac{1}{2}\sum_{i,l=1}^{3}\int_{\mathbb{R}^n}(D_lw_i)^2dx+\frac{1}{2}\sum_{i,l=1}^{3}\int_{\mathbb{R}^n}\left(\sum_{j,k=1}^{3}\epsilon_{ijk}D_lD_ju_k\sum_{p,q=1}^{3}\epsilon_{ipq}D_lD_pu_q\right)dx\\
&= \frac{1}{2}\|D\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\int_{\mathbb{R}^n}\sum_{l,j,k,p,q=1}^{3}\left(\sum_{i=1}^{3}\epsilon_{ijk}\epsilon_{ipq}\right)D_lD_ju_kD_lD_pu_qdx \qquad (2.20)\\
&= \frac{1}{2}\|D\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\int_{\mathbb{R}^n}\sum_{l,j,k,p,q=1}^{3}(\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp})D_lD_ju_kD_lD_pu_qdx\\
&= \frac{1}{2}\|D\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\sum_{l,j,k=1}^{3}\int_{\mathbb{R}^n}(D_lD_ju_k)^2dx-\frac{1}{2}\int_{\mathbb{R}^n}\sum_{l,j,k=1}^{3}D_lD_ju_kD_lD_ku_jdx\\
&= \frac{1}{2}\|D\mathbf{w}\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\|D^2\boldsymbol{u}\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

Portanto, usando as definições das normas descritas na introdução,

$$
\begin{aligned}
&\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\mu\int_{t_0}^{t}\|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\,d\tau+2\gamma\int_{t_0}^{t}\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}d\tau\\
&+2\nu\int_{t_0}^{t}\|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\,d\tau+2\chi\int_{t_0}^{t}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\int_{t_0}^{t}\|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\,d\tau\\
&\leq \|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}+J_1(\cdot,t)+J_2(\cdot,t)+J_3(\cdot,t)+J_4(\cdot,t)+J_5(\cdot,t).
\end{aligned}
$$

Onde,

$$J_1(\cdot,t) := 2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_l u_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau,$$

$$J_2(\cdot,t) := 2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_l b_j(\cdot,\tau)||D_jD_lb_i(\cdot,\tau)|\ dxd\tau,$$

$$J_3(\cdot,t) := 2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|w_i(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}|D_l u_j(\cdot,\tau)||D_jD_lw_i(\cdot,\tau)|dxd\tau,$$

$$J_4(\cdot,t) := 2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_l b_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau,$$

$$J_5(\cdot,t) := 2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|b_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_l u_j(\cdot,\tau)||D_jD_lb_i(\cdot,\tau)|dxd\tau.$$

Vamos estimar o termo $J_1(\cdot,t)$ definido anteriormente. Pelas desigualdades de Cauchy-Schwarz para soma e de Hölder, tem-se

$$\begin{aligned}
&2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\int_{\mathbb{R}^3}\|u_i(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}|D_l u_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau\\
&\leq 2\sum_{i,j,l=1}^{3}\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}|D_l u_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau\\
&= 2\sum_{i,l=1}^{3}\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\sum_{j=1}^{3}|D_l u_j(\cdot,\tau)||D_jD_lu_i(\cdot,\tau)|dxd\tau\\
&\leq 2\sum_{i=1}^{3}\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\sum_{l=1}^{3}\left(\sum_{j=1}^{3}|D_l u_j(\cdot,\tau)|^2\right)^{1/2}\left(\sum_{j=1}^{3}|D_jD_lu_i(\cdot,\tau)|^2\right)^{1/2}dxd\tau\\
&\leq 2\sum_{i=1}^{3}\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\left(\sum_{j,l=1}^{3}|D_l u_j(\cdot,\tau)|^2\right)^{1/2}\left(\sum_{j,l=1}^{3}|D_jD_lu_i(\cdot,\tau)|^2\right)^{1/2}dxd\tau\\
&= 2\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\sum_{i=1}^{3}\left(\sum_{j,l=1}^{3}|D_l u_j(\cdot,\tau)|^2\right)^{1/2}\left(\sum_{j,l=1}^{3}|D_jD_lu_i(\cdot,\tau)|^2\right)^{1/2}dxd\tau\\
&\leq 2\int_{t_0}^{t}\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}\left(\sum_{i,j,l=1}^{3}|D_l u_j(\cdot,\tau)|^2\right)^{1/2}\left(\sum_{i,j,l=1}^{3}|D_jD_lu_i(\cdot,\tau)|^2\right)^{1/2}dxd\tau
\end{aligned}$$

$$
\begin{aligned}
= 2\sqrt{3}\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \left(\sum_{j,l=1}^{3} |D_l u_j(\cdot,\tau)|^2\right)^{1/2} \left(\sum_{i,j,l=1}^{3} |D_j D_l u_i(\cdot,\tau)|^2\right)^{1/2} dx d\tau \\
\leq 2\sqrt{3}\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau.
\end{aligned}
\tag{2.21}
$$

O mesmo raciocínio pode ser feito para os termos J_2, J_3, J_4 e J_5. Assim, usando as definições das normas (1.7), obtém-se

$$
\begin{aligned}
\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu\int_{t_0}^{t} \|D^2\boldsymbol{u}(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\, d\tau + 2\gamma\int_{t_0}^{t} \|D^2\mathbf{w}(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)} d\tau \\
+2\nu\int_{t_0}^{t} \|D^2\boldsymbol{b}(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\, d\tau + 2\chi\int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
+2\int_{t_0}^{t} \|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\, d\tau \leq \|D(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
+2\sqrt{3}\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
+2\sqrt{3}\int_{t_0}^{t} \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
+2\sqrt{3}\int_{t_0}^{t} \|\mathbf{w}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|D^2\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
+2\sqrt{3}\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
+2\sqrt{3}\int_{t_0}^{t} \|\boldsymbol{b}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|D^2\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} + 10\sqrt{3}\int_{t_0}^{t} \|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \times \\
\times \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau.
\end{aligned}
$$

Logo, usando a desigualdade de Sobolev (A.7) descrita no apêndice e (2.4), tem-se

$$
\begin{aligned}
\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\lambda \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\, d\tau \\
+2\chi \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau &+ 2\int_{t_0}^{t} \|D(\nabla\cdot \mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\, d\tau \\
\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} &+ 10\sqrt{3}\int_{t_0}^{t} \|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\times \\
\times\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} &\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
+10\sqrt{3}K\int_{t_0}^{t} \|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)} &\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\times \\
&\times\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
&\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
+10\sqrt{3}K\|(\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)} &\int_{t_0}^{t} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\times \\
&\times\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau,
\end{aligned}
$$

onde $\lambda := \min\{\mu,\gamma,\nu\}$ como anteriormente. Ou seja,

$$
\begin{aligned}
\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\lambda \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\, d\tau \\
+2\chi \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau &+ 2\int_{t_0}^{t} \|D(\nabla\cdot \mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\, d\tau \\
&\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
+10\sqrt{3}K\|(\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)} &\int_{t_0}^{t} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\times \\
&\times\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau.
\end{aligned}
\tag{2.22}
$$

Note que, por (2.4), segue que[3]

$$
\int_0^\infty \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \leq \frac{\|(\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{b}_0)\|^2_{L^2(\mathbb{R}^3)}}{2\lambda} < \infty.
$$

[3]Como dito anteriormente, é possível mostrar, usando as mesmas técnicas usadas por Leray, que a desigualdade (2.4) vale para quase todo $t_0 > 0$, inclusive em $t_0 = 0$. Veja [26].

O que implica,

$$\liminf_{t \to +\infty} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} = 0.$$

Portanto, existe $t_{**} > t_*$ suficientemente grande tal que,

$$\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t_{**})\|_{L^2(\mathbb{R}^3)} < C,$$

onde

$$C := \left(\frac{\lambda}{5\sqrt{3}K\|(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)}} \right)^2.$$

Na verdade, vale mais:

$$\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, s)\|_{L^2(\mathbb{R}^3)} < C, \text{ para todo } s \geq t_{**}. \tag{2.23}$$

Para demonstrar que (2.23) vale, suponhamos por absurdo, que exista um instante $t_1 > t_{**}$, tal que $\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, s)\|_{L^2(\mathbb{R}^3)} < C$ para todo $s \in [t_{**}, t_1)$ com

$$\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t_1)\|_{L^2(\mathbb{R}^3)} = C.$$

Logo, fazendo $t = t_1$ e $t_0 \geq t_{**}$ em (2.22), obtém-se

$$C = \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t_1)\|_{L^2(\mathbb{R}^3)} \leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t_{**})\|_{L^2(\mathbb{R}^3)} < C.$$

Absurdo. Portanto, usando (2.23) em (2.22), tem-se

$$\begin{aligned}
\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot, \tau)\|^2_{L_2(\mathbb{R}^3)}\, d\tau& \\
+2\chi \int_{t_0}^{t} \|D\mathbf{w}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\int_{t_0}^{t} \|D(\nabla \cdot \mathbf{w}(\cdot, \tau))\|^2_{L^2(\mathbb{R}^3)}\, d\tau& \\
\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)}& \\
+10\sqrt{3}K\|(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\sqrt{C} \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau.& \\
\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot, t_0)\|^2_{L^2(\mathbb{R}^3)}& \\
+2\lambda \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot, \tau)\|^2_{L^2(\mathbb{R}^3)} d\tau.&
\end{aligned}$$

O que implica

$$\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}, \tag{2.24}$$

para todo $t \geq t_0 \geq t_{**}$.

Destarte, usando (2.24) e (2.23) na desigualdade (2.22), tem-se

$$\begin{aligned}
\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\, d\tau \\
+2\chi \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\int_{t_0}^{t} \|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\, d\tau \\
\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
+10\sqrt{3}K\|(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)} \int_{t_0}^{t} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)} \times \\
\times \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)} \\
+10\sqrt{3}K\|(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)} \times \\
\times \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau,
\end{aligned}$$

resultando em

$$\begin{aligned}
\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
+\left(2\lambda - 10\sqrt{3}K\|(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\right) \times \\
\times \int_{t_0}^{t} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\, d\tau \\
+2\chi \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\int_{t_0}^{t} \|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\, d\tau \\
\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

para todo $t \geq t_0 \geq t_{**}$.

Defina

$$\eta := \lambda - 5\sqrt{3}K\|(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)} \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)}.$$

Assim,

$$\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\eta\int_{t_0}^{t}\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\,d\tau$$
$$+2\chi\int_{t_0}^{t}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\int_{t_0}^{t}\|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\,d\tau$$
$$\leq \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}.$$

Note que η é uma constante positiva, pois da definição de $C > 0$ e por (2.23), tem-se

$$5\sqrt{3}K\|(\boldsymbol{u}_0, \mathbf{w}_0, \boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t_0)\|^{1/2}_{L^2(\mathbb{R}^3)} < \lambda.$$

O que concluí a prova de (2.15). □

Em outras palavras, o teorema anterior nos diz que: dada uma solução de Leray para o problema (2.1), existe um instante de tempo $t_{**} > 0$ suficientemente grande tal que $\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ é suave e decrescente em $t \in [t_{**}, \infty)$. Uma importante consequência disto é o teorema seguinte.

Teorema 2.4. *Dada $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)$ uma solução de Leray do problema (2.1), tem-se*

$$t^{1/2}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0,$$

ao $t \to \infty$.

Demonstração. Basta notar que, pelo lema (B.1) do apêndice e pelo fato de $\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ ser decrescente para t grande, vale

$$t\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} = 2\int_{t/2}^{t}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}d\tau$$
$$\leq 2\int_{t/2}^{t}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \to 0,$$

ao $t \to \infty$. □

2.2 Caso $n = 2$

Vamos demostrar aqui o teorema 2.1, para o caso bidimensional, isto é

$$t^{1/2}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \to 0 \ \text{ ao } \ t \to \infty. \tag{2.25}$$

Demonstração. Seguindo o mesmo roteiro feito na seção anterior, podemos facilmente obter a desigualdade de energia (2.4) em 2D. Analogamente, tem-se

$$\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\lambda\int_{t_0}^{t}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2}d\tau + 2\chi\int_{t_0}^{t}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau \leq \|\mathbf{z}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^2)}, \tag{2.26}$$

para todo $t > t_0 \geq 0$. Em particular, pela desigualdade (de energia) anterior, obtém-se

$$\int_0^{\infty}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau < \infty. \tag{2.27}$$

Como anteriormente, iremos repetir o mesmo tipo de raciocínio para obter (2.15) em dimensão $n = 2$. Assim, derivando o sistema (1.2) e fazendo o produto interno de $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})$ com (1.2a), (1.2b) e (1.2c), respectivamente, integrando por partes e somando tudo, tem-se

$$\begin{aligned}\frac{d}{dt}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\chi\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} &\leq -2\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\\ &+\|\mathbf{z}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|D^2\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Usando a desigualdade de Sobolev (A.17) descrita no apêndice e (2.26), obtém-se

$$\begin{aligned}\frac{d}{dt}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\chi\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} &\leq -2\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\\ +\|\mathbf{z}(\cdot,t)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}&\|D^2\mathbf{z}(\cdot,t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}\\ &\leq -2\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\\ +\|\mathbf{z}(\cdot,t_0)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|D^2\mathbf{z}(\cdot,t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)} &\leq -2\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\\ &+C\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|D^2\mathbf{z}(\cdot,t)\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)},\end{aligned}$$

para algum $C > 0$. Usando a desigualdade de Young (com $p = 4$ e $q = 4/3$), tem-se

$$\begin{aligned}\frac{d}{dt}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\chi\|\mathbf{w}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} &\leq -2\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}\\ &C_\epsilon\|D\mathbf{z}(\cdot,t)\|^4_{L^2(\mathbb{R}^2)} + \epsilon\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

para todo $\epsilon > 0$. Logo,

$$\frac{d}{dt}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq C\|D\mathbf{z}(\cdot,t)\|^4_{L^2(\mathbb{R}^2)}.$$

Aplicando o lema de Grönwall (veja o lema B.9 do apêndice) e usando (2.27)

$$\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq C\exp\Big(\int_0^\infty \|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\Big)\|D\mathbf{z}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \leq C\|D\mathbf{z}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)},$$

para todo $0 \leq s \leq t$. Em outras palavras,

$$\|D\mathbf{z}(\cdot,s)\|^2_{L^2(\mathbb{R}^2)} \geq \eta\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}, \tag{2.28}$$

onde $\eta = 1/C > 0$ e para todo $0 \leq s \leq t$.

Suponha, por absurdo, que (2.25) seja falso. Então, existe $\beta > 0$ e uma sequência $t_n \to \infty$ tal que

$$t_n\|D\mathbf{z}(\cdot,t_n)\|^2_{L^2(\mathbb{R}^2)} > \beta > 0,$$

para todo n. Podemos supor que $t_{n+1} \geq 2t_n$ e usar (2.28). Consequentemente,

$$\begin{aligned}\int_{t_n}^{t_{n+1}} \|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau \geq \eta(t_{n+1}-t_n)\|D\mathbf{z}(\cdot,t_{n+1})\|^2_{L^2(\mathbb{R}^2)} &\geq \eta\beta(t_{n+1}-t_n)/t_{n+1} \\ &= \eta\beta(1-t_n/t_{n+1}) \geq \eta\beta/2.\end{aligned}$$

O que contradiz (2.27) e assim concluí a demonstração de (2.25). □

3 DECAIMENTO EM $L^2(\mathbb{R}^n)$ ($n = 2, 3$)

No presente capítulo, provaremos que a norma L^2 da solução de Leray $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$ de (2.1) e (1.2) decai a zero ao $t \to \infty$, quando $\chi = 0$. Além disso, mostraremos que $\|\mathbf{w}(\cdot, t)\|_{L^2(\mathbb{R}^3)} = o(t^{-1/2})$ e $\|(\boldsymbol{u}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^3)} \to 0$, quando $\chi > 0$. É claro que, em particular, se tem

$$\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{ao} \quad t \to \infty, \tag{3.1}$$

para $n = 2, 3$, em ambos os casos como veremos a seguir. Vamos começar com o caso 3D.

3.1 Caso $n = 3$

Vamos considerar primeiramente o caso em que não se tem viscosidade de vórtice, i.e., $\chi = 0$.

3.1.1 Caso $\chi = 0$

Teorema 3.1. *Seja $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$ solução de Leray do problema (2.1). Se $\chi = 0$, tem-se*

$$\lim_{t \to \infty} \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^3)} \to 0. \tag{3.2}$$

Demonstração. Observe que podemos escrever o sistema (2.1) da seguinte maneira,

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} &= \mu \Delta \boldsymbol{u} + \mathbf{Q}_1(\cdot, t), \\
\frac{\partial \mathbf{w}}{\partial t} &= \gamma \Delta \mathbf{w} + \mathbf{Q}_2 \cdot, t), \\
\frac{\partial \boldsymbol{b}}{\partial t} &= \nu \Delta \boldsymbol{b} + \mathbf{Q}_3(\cdot, t).
\end{aligned}$$

Onde

$$
\begin{aligned}
\mathbf{Q}_1(\cdot,t) &= -\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla \mathsf{p} + \boldsymbol{b}\cdot\nabla\boldsymbol{b},\\
\mathbf{Q}_2(\cdot,t) &= -\boldsymbol{u}\cdot\nabla\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}),\\
\mathbf{Q}_3(\cdot,t) &= -\boldsymbol{u}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}.
\end{aligned}
\tag{3.3}
$$

Pela propriedade de regularidade (2.2), podemos aplicar o princípio de Duhamel para $t_0 > t_*$. Portanto,

$$
\begin{aligned}
\boldsymbol{u}(\cdot,t) &= e^{\mu\Delta(t-t_0)}\boldsymbol{u}(\cdot,t_0) + \int_{t_0}^{t} e^{\mu\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)d\tau,\\
\mathbf{w}(\cdot,t) &= e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-\tau)}\mathbf{Q}_2(\cdot,\tau)d\tau,\\
\boldsymbol{b}(\cdot,t) &= e^{\nu\Delta(t-t_0)}\boldsymbol{b}(\cdot,t_0) + \int_{t_0}^{t} e^{\mu\Delta(t-\tau)}\mathbf{Q}_3(\cdot,\tau)d\tau,
\end{aligned}
$$

para todo $t > t_0$. Tomando a norma L^2, usando a desigualdade triangular e a desigualdade de Minkowski para integrais, tem-se

$$
\begin{aligned}
\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq \|e^{\mu\Delta((t-t_0))}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau,\\
\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq \|e^{\gamma\Delta((t-t_0))}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \|e^{\gamma\Delta(t-\tau)}\mathbf{Q}_2(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau,\\
\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq \|e^{\nu\Delta((t-t_0))}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}\mathbf{Q}_3(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}
\tag{3.4}
$$

Note que, por (C.1) no apêndice, tem-se

$$
\begin{aligned}
\|e^{\mu\Delta((t-t_0))}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} &\to 0,\\
\|e^{\gamma\Delta((t-t_0))}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} &\to 0,\\
\|e^{\nu\Delta((t-t_0))}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} &\to 0,
\end{aligned}
\tag{3.5}
$$

ao $t \to \infty$. Portanto, temos que estimar os termos

$$
\begin{aligned}
&\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau,\\
&\int_{t_0}^{t} \|e^{\gamma\Delta(t-\tau)}\mathbf{Q}_2(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau,\\
&\int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}\mathbf{Q}_3(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}
$$

Vamos começar pelo primeiro. Como

$$\mathbf{Q}_1 + \nabla \mathsf{p} = -\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}$$

e $\nabla \cdot \mathbf{Q}_1 = 0$, temos que $\mathbf{Q}_1 = \mathbb{P}_H[-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}]$, onde $\mathbb{P}_H$ é o projetor de Helmholtz (veja o apêndice D). Logo,

$$\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbb{P}_H[-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}]\|_{L^2(\mathbb{R}^3)} d\tau.$$

Note que, o projetor de Helmholtz comuta com o Heat Kernel (veja o teorema (D.5), no apêndice D). Portanto,

$$\begin{aligned}\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau &\leq \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbb{P}_H[-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}]\|_{L^2(\mathbb{R}^3)} d\tau \\ &\leq \int_{t_0}^{t} \|\mathbb{P}_H\left[e^{\mu\Delta(t-\tau)}(-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b})\right]\|_{L^2(\mathbb{R}^3)} d\tau.\end{aligned}$$

Usando a ortogonalidade em L^2 (veja o lema D.1 no apêndice), tem-se

$$\begin{aligned}\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau &\leq \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}\mathbb{P}_H[-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b}]\|_{L^2(\mathbb{R}^3)} d\tau \\ &\leq \int_{t_0}^{t} \|\mathbb{P}_H\left[e^{\mu\Delta(t-\tau)}(-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b})\right]\|_{L^2(\mathbb{R}^3)} d\tau \\ &\leq \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}(-\boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{b} \cdot \nabla \boldsymbol{b})\|_{L^2(\mathbb{R}^3)} d\tau \\ \leq \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}[\boldsymbol{u}(\cdot,\tau) \cdot \nabla \boldsymbol{u}(\cdot,\tau)]\|_{L^2(\mathbb{R}^3)} d\tau &+ \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}[\boldsymbol{b}(\cdot,\tau) \cdot \nabla \boldsymbol{b}(\cdot,\tau)]\|_{L^2(\mathbb{R}^3)} d\tau.\end{aligned}$$

Vale notar também que, pela desigualdade do lema (C.3) do apêndice, tem-se

$$\|v(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq C\|v(\cdot,0)\|_{L^1(\mathbb{R}^3)}(\alpha t)^{-\frac{3}{4}}, \tag{3.6}$$

onde v é solução do problema de difusão,

$$\begin{aligned} v_t &= \alpha\Delta v \\ v(\cdot,0) &= v_0 \in L^1(\mathbb{R}^3). \end{aligned}$$

Portanto, usando isso, obtém-se

$$\begin{aligned}\int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}[\boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)]\|_{L^2(\mathbb{R}^3)}d\tau + \int_{t_0}^{t} \|e^{\mu\Delta(t-\tau)}[\boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau)]\|_{L^2(\mathbb{R}^3)}d\tau \\ \leq C\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|\boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^1(\mathbb{R}^3)}d\tau \\ +C\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|\boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau)\|_{L^1(\mathbb{R}^3)}d\tau.\end{aligned}$$

Vale notar que, pela desigualdade de Cauchy-Schwarz, tem-se

$$\begin{aligned}\|\boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^1(\mathbb{R}^3)} &= \sum_{i=1}^{3}\int_{\mathbb{R}^3}|\sum_{j=1}^{3}u_jD_ju_i|dx \\ \leq \sum_{i=1}^{3}\int_{\mathbb{R}^3}\sum_{j=1}^{3}|u_jD_ju_i|dx &\leq \int_{\mathbb{R}^3}\sum_{i=1}^{3}(\sum_{j=1}^{3}|u_j|^2)^{\frac{1}{2}}(\sum_{j=1}^{3}|D_ju_i|^2)^{\frac{1}{2}}dx \\ &\leq \int_{\mathbb{R}^3}(\sum_{i=1}^{3}\sum_{j=1}^{3}|u_j|^2)^{\frac{1}{2}}(\sum_{i=1}^{3}\sum_{j=1}^{3}|D_ju_i|^2)^{\frac{1}{2}}dx \\ &\leq \sqrt{3}\left(\int_{\mathbb{R}^3}\sum_{j=1}^{3}|u_j|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\sum_{i=1}^{3}\sum_{j=1}^{3}|D_ju_i|^2\right)^{\frac{1}{2}} \\ &= \sqrt{3}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{3.7}$$

Logo,

$$\begin{aligned}C\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|\boldsymbol{u}(\cdot,\tau)\cdot\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^1(\mathbb{R}^3)}d\tau \\ +C\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|\boldsymbol{b}(\cdot,\tau)\cdot\nabla\boldsymbol{b}(\cdot,\tau)\|_{L^1(\mathbb{R}^3)}d\tau \\ \leq C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ +C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ \leq 2C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.\end{aligned}$$

Pelo teorema (2.4), tem-se que dado $\epsilon > 0$, existe $t_0 > t_*$, tal que

$$t^{1/2}\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon,$$

para todo $t > t_0$. Observe que por (2.4) e pela desigualdade anterior, segue que

$$2C\sqrt{3}\mu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau$$
$$\leq K\mu^{-3/4}\epsilon\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\tau^{-1/2}d\tau.$$

Além disso, supondo $t > 2t_0$.

$$\int_{t_0}^{t}(t-\tau)^{-\frac{3}{4}}\tau^{-1/2}d\tau = \int_{t_0}^{t/2}(t-\tau)^{-\frac{3}{4}}\tau^{-1/2}d\tau + \int_{t/2}^{t}(t-\tau)^{-\frac{3}{4}}\tau^{-1/2}d\tau$$
$$\leq (t/2)^{-3/4}\int_{t_0}^{t/2}\tau^{-1/2}d\tau + (t/2)^{-1/2}\int_{t/2}^{t}(t-\tau)^{-\frac{3}{4}}d\tau$$
$$\leq Kt^{-1/4}.$$

Em vista disso,

$$\int_{t_0}^{t}\|e^{\mu\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \to 0, \tag{3.8}$$

ao $t\to\infty$.

Agora, temos que estimar o termo

$$\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\mathbf{Q}_2(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.$$

Note que, por (3.3) e (3.4),

$$\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\mathbf{Q}_2(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \leq \int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\boldsymbol{u}\cdot\nabla\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau$$
$$+\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.$$

Vamos começar a estimar pelo primeiro termo do lado direito da desigualdade acima. Utilizando o mesmo raciocínio feito em (3.7) anteriormente para mostrar (3.8) e novamente a desigualdade (3.6), obtém-se

$$\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\boldsymbol{u}\cdot\nabla\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \leq K\gamma^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{u}\cdot\nabla\mathbf{w}(\cdot,\tau)\|_{L^1(\mathbb{R})}d\tau$$
$$\leq K\gamma^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau$$
$$\leq K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\int_{t_0}^{t}(t-\tau)^{-3/4}\tau^{-1/2}d\tau$$

e

$$\int_{t_0}^{t}(t-\tau)^{-3/4}\tau^{-1/2}d\tau = \int_{t_0}^{\frac{t}{2}}(t-\tau)^{-3/4}\tau^{-1/2}d\tau + \int_{\frac{t}{2}}^{t}(t-\tau)^{-3/4}\tau^{-1/2}d\tau$$
$$\leq K\left[t^{-3/4}\int_{t_0}^{\frac{t}{2}}\tau^{-1/2}d\tau + t^{-1/2}\int_{\frac{t}{2}}^{t}(t-\tau)^{-3/4}d\tau\right] \leq Kt^{-1/4},$$

para todo $t > 2t_0$. De modo que

$$\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\boldsymbol{u}\cdot\nabla\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \leq K\epsilon\gamma^{-3/4}t^{-1/4}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}.$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty}\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\boldsymbol{u}\cdot\nabla\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau = 0. \tag{3.9}$$

Vamos analisar o próximo termo, i.e.,

$$\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.$$

Para isso vamos utilizar a desigualdade (C.2) do apêndice.

$$\int_{t_0}^{t}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \leq K\gamma^{-1/2}\int_{t_0}^{t}(t-\tau)^{-1/2}\|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau$$
$$\leq K\epsilon\gamma^{-1/2}\int_{t_0}^{t}(t-\tau)^{-1/2}\tau^{-1/2}d\tau$$

e

$$\int_{t_0}^{t}(t-\tau)^{-1/2}\tau^{-1/2}d\tau = \int_{t_0}^{\frac{t}{2}}(t-\tau)^{-1/2}\tau^{-1/2}d\tau + \int_{\frac{t}{2}}^{t}(t-\tau)^{-1/2}\tau^{-1/2}d\tau$$
$$\leq K\left[t^{-1/2}\int_{t_0}^{\frac{t}{2}}\tau^{-1/2}d\tau + t^{-1/2}\int_{\frac{t}{2}}^{t}(t-\tau)^{-1/2}d\tau\right] \leq K.$$

De modo que

$$\int_{t_0}^{t} \|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \leq K\epsilon\gamma^{-1/2}.$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty}\int_{t_0}^{t} \|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau = 0. \tag{3.10}$$

Juntando (3.9) e (3.10), obtém-se

$$\lim_{t\to+\infty}\int_{t_0}^{t} \|e^{\gamma\Delta(t-\tau)}\mathbf{Q}_2(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau = 0. \tag{3.11}$$

Finalmente, estimaremos o termo restante. Por (3.3) e (3.4),

$$\begin{aligned}\int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}\mathbf{Q}_3(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \leq{}& \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ &+ \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}(\boldsymbol{b}\cdot\nabla\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.\end{aligned}$$

Começaremos analisando o primeiro termo do lado direito da desigualdade anterior. Vamos proceder a análise de maneira análoga a feita nos casos anteriores, veja (3.7). Por (3.6) novamente, tem-se

$$\begin{aligned}\int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau &\leq C\nu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|(\boldsymbol{u}\cdot\nabla\boldsymbol{b})(\cdot,\tau)\|_{L^1(\mathbb{R}^3)}d\tau \\ &\leq C\nu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.\end{aligned}$$

Repetindo a mesma análise anterior no segundo termo, resulta

$$\begin{aligned}&\int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}(\boldsymbol{b}\cdot\nabla\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ &\leq C\nu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.\end{aligned}$$

O que implica, pelo teorema (2.4), que dado $\epsilon > 0$ existe t_0 suficientemente grande, tal que

$$\begin{aligned}
&\int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}\mathbf{Q}_3(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq 2C\nu^{-3/4}\int_{t_0}^{t}(t-\tau)^{-3/4}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq K\nu^{-3/4}\epsilon\int_{t_0}^{t}(t-\tau)^{-3/4}\tau^{-1/2} d\tau
\end{aligned}$$

e, como anteriormente,

$$\begin{aligned}
\int_{t_0}^{t}(t-\tau)^{-3/4}\tau^{-1/2}d\tau &= \int_{t_0}^{\frac{t}{2}}(t-\tau)^{-3/4}\tau^{-1/2}d\tau + \int_{\frac{t}{2}}^{t}(t-\tau)^{-3/4}\tau^{-1/2}d\tau \\
&\leq K\left[t^{-3/4}\int_{t_0}^{\frac{t}{2}}\tau^{-1/2}d\tau + t^{-1/2}\int_{\frac{t}{2}}^{t}(t-\tau)^{-3/4}d\tau\right] \leq Kt^{-1/4},
\end{aligned}$$

para todo $t > t_0$. Portanto,

$$\lim_{t\to\infty}\int_{t_0}^{t}\|e^{\nu\Delta(t-\tau)}\mathbf{Q}_3(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau = 0. \tag{3.12}$$

Logo, por (3.5) (3.8),(3.11) e (3.12), tem-se

$$\lim_{t\to\infty}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0.$$

Concluindo a demostração. □

3.1.2 Caso $\chi > 0$

Quando consideramos o caso em que $\chi > 0$, importantes propriedades de decaimento aparecem envolvendo a norma L^2 da velocidade microrrotacional $\mathbf{w}$. Nesta seção, vamos provar que a "energia cinética"associada a tal velocidade decaí para zero a medida que o tempo t passa com uma taxa de $t^{1/2}$. Embora o mesmo não aconteça com os campos $(\boldsymbol{u},\boldsymbol{b})$, isso é interessante sobretudo tendo em vista o resultado assintótico (3.2). Em outras palavras:

Teorema 3.2. *Seja $\mathbf{w}(\cdot,t)$ solução de Leray do problema (2.1). Então,*

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0 \text{ ao } t\to\infty. \tag{3.13}$$

Demonstração. Vamos reescrever a segunda equação do sistema (2.1) como

$$\mathbf{w}_t = \gamma\Delta\mathbf{w} + \mathbf{Q}_2 - 2\chi\mathbf{w}, \tag{3.14}$$

onde

$$\mathbf{Q}_2 := -\boldsymbol{u}\cdot\nabla\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}) + \chi\nabla\wedge\boldsymbol{u}. \tag{3.15}$$

Fazendo a mudança de variável

$$\mathbf{W}(\cdot,t) = e^{2\chi t}\mathbf{w}(\cdot,t)$$

e multiplicando a equação (3.14) por $e^{2\chi t}$, temos

$$\mathbf{W}_t = \gamma\Delta\mathbf{W} + e^{2\chi t}\mathbf{Q}_2(\cdot,t).$$

Aplicando o princípio de Duhamel para $\mathbf{W}(\cdot,t)$ a partir de t_0 grande o suficiente, obtém-se

$$\mathbf{W}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{W}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-\tau)}e^{2\chi\tau}\mathbf{Q}_2(\cdot,\tau)d\tau.$$

Por conseguinte,

$$\begin{aligned}\mathbf{w}(\cdot,t) = e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) - \int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)d\tau \\ + \int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)d\tau + \chi\int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)d\tau.\end{aligned}$$

Multiplicando por $t^{1/2}$, tomando a norma L^2 e usando a desigualdade de Minkowski para integrais, implica que

$$\begin{aligned} t^{\frac{1}{2}}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le\ & t^{\frac{1}{2}}e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} \\ & + t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ & + t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ & + \chi t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau. \end{aligned} \tag{3.16}$$

Dado $\epsilon > 0$ seja t_0 suficientemente grande tal que, pelo teorema (2.4) tem-se

$$t^{1/2}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le \epsilon.$$

para todo $t > 2t_0$. Vamos analisar os quatro termos do lado direito da desigualdade (3.16), começando primeiramente pelo primeiro, i.e.,

$$t^{1/2}e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}.$$

Observe que, por (C.1) do apêndice, é fácil ver que

$$t^{1/2}e^{-2\chi(t-t_0)}\|e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} \to 0 \text{ ao } t\to\infty. \tag{3.17}$$

Vamos analisar os termos restantes de (3.16). Começando por

$$t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.$$

Observe que usando a desigualdade (C.2) descrita no apêndice, resulta

$$\begin{aligned} t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ \leq K\gamma^{-3/4}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\|(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^1(\mathbb{R}^3)}d\tau \\ \leq K\gamma^{-3/4}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ \leq K\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\ \leq K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\tau^{-1/2}d\tau. \end{aligned}$$

Agora, note que

$$\begin{aligned} t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\tau^{-1/2}d\tau = \\ = t^{1/2}\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\tau^{-1/2}d\tau + t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\tau^{-1/2}d\tau \\ \leq t^{1/2}e^{-\chi t}\left(\frac{t}{2}\right)^{-3/4}\int_{t_0}^{\frac{t}{2}}\tau^{-1/2}d\tau + \sqrt{2}t^{1/2}t^{-1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}d\tau \\ \leq K\left(e^{-\chi t}t^{1/4} + (2\chi)^{-1/4}\int_0^\infty e^{-s}s^{-3/4}ds\right) \\ \leq K\left(e^{-\chi t}t^{1/4} + (2\chi)^{-1/4}\Gamma\left(\frac{1}{4}\right)\right), \end{aligned}$$

onde Γ é a famosa função gama de Euler. Ou seja,

$$t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-3/4}\tau^{-1/2}d\tau \leq$$
$$\leq K\epsilon\gamma^{-3/4}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\left[e^{-\chi t}t^{1/4}+(2\chi)^{-1/4}\Gamma\left(\frac{1}{4}\right)\right].$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau = 0. \tag{3.18}$$

Vamos analisar o próximo termo, i.e.,

$$t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.$$

Usando novamente a desigualdade (C.2) descrita no apêndice, tem-se

$$\begin{aligned}
t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\
\leq K\gamma^{-1/2}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-1/2}\|D\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \\
\leq K\epsilon\gamma^{-1/2}t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-1/2}\tau^{-1/2}d\tau
\end{aligned}$$

e, como anteriormente,

$$\begin{aligned}
t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-1/2}\tau^{-1/2}d\tau \leq \\
\leq t^{1/2}\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)}(t-\tau)^{-1/2}\tau^{-1/2}d\tau + t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-1/2}\tau^{-1/2}d\tau \\
\leq t^{1/2}\left(\frac{t}{2}\right)^{-1/2}e^{-\chi t}\int_{t_0}^{\frac{t}{2}}\tau^{-1/2}d\tau + \sqrt{2}t^{1/2}t^{-1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}(t-\tau)^{-1/2}d\tau \\
\leq K\left(e^{-\chi t}t^{1/2}+(2\chi)^{-1/2}\int_0^{\infty} e^{-s}s^{-1/2}ds\right) \\
\leq K\left(e^{-\chi t}t^{1/2}+(2\chi)^{-1/2}\Gamma\left(\frac{1}{2}\right)\right) = K\left(e^{-\chi t}t^{1/2}+(2\chi)^{-1/2}\sqrt{\pi}\right).
\end{aligned}$$

De modo que

$$t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau \leq$$
$$\leq K\epsilon\gamma^{-1/2}\Big[e^{-\chi t}t^{1/2}+(2\chi)^{-1/2}\sqrt{\pi}\Big].$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau = 0. \tag{3.19}$$

Finalmente, vamos analisar o último termo, ou seja,

$$\chi t^{\frac{1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.$$

Utilizando novamente a desigualdade (C.2) descrita no apêndice, obtém-se

$$\begin{aligned}
&\chi t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau\\
&\leq \chi t^{1/2}\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau\\
&\quad+\chi t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau\\
&\leq K\chi\gamma^{-1/2}t^{1/2}e^{-\chi t}\int_{t_0}^{\frac{t}{2}}(t-\tau)^{-1/2}\|\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau+\chi\epsilon t^{1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}\tau^{-1/2}d\tau\\
&\leq K\Big[\chi\gamma^{-1/2}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}t^{1/2}e^{-\chi t}\int_{t_0}^{\frac{t}{2}}(t-\tau)^{-1/2}d\tau+\chi\epsilon t^{1/2}t^{-1/2}\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}d\tau\Big]\\
&\leq K\Big[\chi\gamma^{-1/2}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}e^{-\chi t}+\chi\epsilon\frac{1-e^{-\chi t}}{2\chi}\Big].
\end{aligned}$$

De modo que

$$\chi t^{1/2}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau$$
$$\leq K\Big[\chi\gamma^{-1/2}\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}e^{-\chi t}+\chi\epsilon\frac{1-e^{-\chi t}}{2\chi}\Big].$$

Como $\epsilon > 0$ é arbitrário, temos que

$$\lim_{t\to\infty} \chi t^{1/2} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \| e^{\gamma\Delta(t-\tau)} (\nabla \wedge \boldsymbol{u})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau = 0. \tag{3.20}$$

Juntando (3.16), (3.17), (3.18), (3.19) e (3.20), tem-se

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0,$$

ao $t \to \infty$, o que prova (3.13). □

Tendo isso, podemos agora demonstrar o próximo resultado.

Teorema 3.3. *Sejam $(\boldsymbol{u},\boldsymbol{b})$ soluções do de Leray do problema (2.1). Então,*

$$\|(\boldsymbol{u},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0 \qquad \text{ao} \quad t \to \infty. \tag{3.21}$$

Demonstração. Usando o princípio de Duhamel,

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \|e^{(\mu+\chi)\Delta((t-t_0))}\boldsymbol{u}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-\tau)}\mathbf{Q}_1(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau,$$
$$\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \|e^{\nu\Delta((t-t_0))}\boldsymbol{b}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \|e^{\nu\Delta(t-\tau)}\mathbf{Q}_3(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau.$$

Onde

$$\mathbf{Q}_1(\cdot,t) = -\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla \mathsf{p} + \boldsymbol{b}\cdot\nabla\boldsymbol{b} + \chi\nabla\wedge\mathbf{w},$$
$$\mathbf{Q}_3(\cdot,t) = -\boldsymbol{u}\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}.$$

Note que, por (3.5) e por (3.12), tem-se

$$\|\boldsymbol{b}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0.$$

Além disso, observe que quando $\chi > 0$, o único termo que surge na definição de $\mathbf{Q}_1$ é o termo $\chi\nabla\wedge\mathbf{w}$. Logo, podemos repetir toda a análise feita para concluir (3.8) e analisar somente o termo pendente.

Vamos estimar o termo

$$\int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-\tau)}\chi(\nabla \wedge \mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau.$$

Por (3.13), dado $\epsilon > 0$ existe t_0 suficientemente grande, tal que

$$\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq t^{-1/2}\epsilon,$$

para todo $t > t_0$. Logo,

$$\begin{aligned}
\chi\int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-\tau)}\nabla \wedge \mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau &\leq K\chi(\mu+\chi)^{-1/2}\int_{t_0}^{t}(t-\tau)^{-1/2}\|\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq K\chi\epsilon(\mu+\chi)^{-1/2}\int_{t_0}^{t}(t-\tau)^{-1/2}\tau^{-1/2} d\tau \\
&\leq K\chi\epsilon(\mu+\chi)^{-1/2}.
\end{aligned}$$

Portanto,

$$\lim_{t\to\infty}\chi\int_{t_0}^{t} \|e^{(\mu+\chi)\Delta(t-\tau)}\nabla \wedge \mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau = 0,$$

o que concluí a prova de (3.21). □

3.2 Caso $n = 2$

As únicas diferenças entre os casos 2D e 3D são as estimativas para o *Heat Kernel* descritas no apêndice C (uma vez que elas dependem da dimensão). No entanto, essas mudanças não afetam a demonstração, que segue idêntica ao caso $n = 3$.

4 DECAIMENTO DAS DERIVADAS EM $\mathbb{R}^n$ $(n = 2,3)$

Neste capítulo iremos mostrar que $\|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^n)} = o(t^{-m/2})$ para cada $m \geq 0$ inteiro. Além disso, veremos que a norma L^2 das derivadas de ordem $m \geq 0$ da velocidade microrrotacional $\mathbf{w}$ dacaí ainda mais rápido quando $\chi > 0$, mais precisamente, $\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = o(t^{-\frac{m+1}{2}})$ (veja a seção (4.1.1)). Tal fenômeno não acontece com os campos $(\boldsymbol{u}, \boldsymbol{b})$ o que era de se esperar tendo em vista os resultados do capítulo anterior. Vale ressaltar que por interpolação conseguiremos o seguinte resultado assintótico geral:

$$t^{s/2}\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0 \quad ao \quad t \to \infty,$$

para todo $s \geq 0$. Como consequência disto, por interpolação novamente, para cada $n = 2,3$ e $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)$ solução de Leray de (1.1) e (1.2), obtém-se

$$t^{\gamma(n,m,q)}\|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{as } t \to \infty.$$

Além disso, se $\chi > 0$, então

$$t^{\gamma(n,m,q)+1/2}\|D^m\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0 \;\text{ as }\; t \to \infty,$$

onde $\gamma(n,m,q) = \frac{n}{4} + \frac{m}{2} - \frac{n}{2q}$, para cada $m \geq 0$ (inteiro) e $2 \leq q \leq \infty$.

4.1 Caso $n = 3$

O principal resultado a ser mostrado aqui é o seguinte.

Teorema 4.1. *Dada* $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$ *solução de Leray do problema (2.1), existe* t_0 *suficientemente grande tal que valem as seguintes propriedades*

$$\begin{aligned}
&(t-t_0)^m \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
&+\lambda \int_{t_0}^{t} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
&+2\int_{t_0}^{t} (\tau-t_0)^m \|D^m \nabla\cdot \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\chi \int_{t_0}^{t} (\tau-t_0)^m \|D^m \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
&\leq m \int_{t_0}^{t} (\tau-t_0)^{m-1} \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau,
\end{aligned} \tag{4.1a}$$

$$\int_{t_0}^{\infty} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau < \infty \tag{4.1b}$$

e

$$\lim_{t\to\infty} t^{m/2} \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0, \tag{4.1c}$$

para cada $m \geq 1$ *inteiro. Onde* $\lambda = \min(\mu, \gamma, \nu)$.

Demonstração. Iremos provar os casos $m = 1$ e $m = 2$ mais detalhadamente, o caso $m \geq 3$ será feito de maneira mais sucinta, visto sua análise envolve as mesmas técnicas desenvolvidas no capítulo anterior e nas preliminares. Caso o leitor já esteja familiarizado com estas contas, aconselha-se ir diretamente para a desigualdade (4.5) a frente.

Pelo teorema (2.4), o resultado (4.1c) já foi obtido para o caso $m = 1$. Para derivar (4.1a) e (4.1b) no caso $m = 1$, usaremos o mesmo tipo de raciocínio feito para demonstrar (2.4) e (2.15): derive o sistema (2.5) com relação a D_l, multiplique-o por $2D_l u_i(t-t_0)$, $2D_l w_i(t-t_0)$ e $2D_l b_i(t-t_0)$ respectivamente a primeira, segunda e terceira equação do sistema (2.5). Por fim integre na região $[t_0, t] \times \mathbb{R}^3$. O primeiro termo da primeira equação do sistema (2.5), fica da seguinte forma, integrando por partes no tempo,

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{t_0}^{t} 2(\tau - t_0) D_l u_i D_l \frac{\partial}{\partial \tau} u_i d\tau dx &= \int_{\mathbb{R}^3}\int_{t_0}^{t} (\tau - t_0)\frac{\partial}{\partial \tau}[D_l u_i(\cdot,\tau)^2] d\tau dx \\ &= \int_{\mathbb{R}^3} (t - t_0) D_l u_i(\cdot, t)^2 dx - \int_{\mathbb{R}^3}\int_{t_0}^{t} D_l u_i(\cdot,\tau)^2 d\tau dx.\end{aligned}$$

Somando em $i, l = 1, 2, 3$, o primeiro termo da primeira equação do sistema (2.5) fica da seguinte forma:

$$(t - t_0)\|D\boldsymbol{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} - \int_{t_0}^{t} \|D\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau.$$

Donde segue analogamente para o primeiro termo da segunda e terceira equação do sistema (2.5), respectivamente,

$$(t - t_0)\|D\mathbf{w}(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} - \int_{t_0}^{t} \|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$

e

$$(t - t_0)\|D\boldsymbol{b}(\cdot, t)\|^2_{L^2(\mathbb{R}^3)} - \int_{t_0}^{t} \|D\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau.$$

Logo, como a derivação de (2.15) envolve somente integração por partes no espaço, os mesmos passos feitos em tal derivação podem ser repetidos apenas adicionando-se o termo $(\tau - t_0)$, com exceção dos termos analisados anteriormente, pois envolvem integração por partes no tempo t. Portanto, usando a desigualdade de Sobolev (A.7) descrita no apêndice, tem-se

$$
\begin{aligned}
&(t-t_0)\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+2\lambda\int_{t_0}^{t}(\tau-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\,d\tau\\
&+2\chi\int_{t_0}^{t}(\tau-t_0)\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\int_{t_0}^{t}(\tau-t_0)\|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\,d\tau\\
&\leq\int_{t_0}^{t}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+10\sqrt{3}\int_{t_0}^{t}(\tau-t_0)\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\times\\
&\times\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau\\
&\leq\int_{t_0}^{t}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+10\sqrt{3}K\int_{t_0}^{t}(\tau-t_0)\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\times\\
&\times\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&\leq\int_{t_0}^{t}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+10\sqrt{3}K\|(\boldsymbol{u}_0,\mathbf{w}_0,\boldsymbol{b}_0)\|^{1/2}_{L^2(\mathbb{R}^3)}\int_{t_0}^{t}(\tau-t_0)\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\times\\
&\times\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}
$$

Ou seja,

$$
\begin{aligned}
&(t-t_0)\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+2\lambda\int_{t_0}^{t}(\tau-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\,d\tau\\
&+2\chi\int_{t_0}^{t}(\tau-t_0)\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\int_{t_0}^{t}(\tau-t_0)\|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\,d\tau\\
&\leq\int_{t_0}^{t}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+C\int_{t_0}^{t}(\tau-t_0)\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\times\\
&\times\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau,
\end{aligned}
$$

para algum $C > 0$. Como $t^{1/2}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$, existe t_0 suficientemente grande tal que

$$C^2\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \lambda^2,$$

para todo $t > t_0$. Com isso,

$$\begin{aligned}(t-t_0)\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\ +\lambda\int_{t_0}^t (\tau-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L_2(\mathbb{R}^3)}\,d\tau \\ +2\chi\int_{t_0}^t(\tau-t_0)\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\int_{t_0}^t(\tau-t_0)\|D(\nabla\cdot\mathbf{w}(\cdot,\tau))\|^2_{L^2(\mathbb{R}^3)}\,d\tau \\ \leq \int_{t_0}^t \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.\end{aligned} \tag{4.2}$$

O que prova (4.1a) para $m = 1$. Agora, por (2.4),

$$\int_{t_0}^t \|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau < \infty,$$

o que implica,

$$\int_{t_0}^t(\tau-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\,d\tau < \infty, \tag{4.3}$$

concluindo a prova (4.1b) para $m = 1$. Vamos demonstrar o resultado (4.1) para $m = 2$: derive o sistema (2.5) com relação a x_{l_1} e x_{l_2}, ou seja, aplique $D_{l_1}D_{l_2}$. Para $t_0 > t_*$ (veja (2.2)), multiplique por $2(\tau-t_0)^2D_{l_1}D_{l_2}u_i$, $2(\tau-t_0)^2D_{l_1}D_{l_2}w_i$ e $2(\tau-t_0)^2D_{l_1}D_{l_2}b_i$ a primeira, segunda e terceira equação do sistema (2.5), respectivamente. Integrando na região $[t_0,t]\times\mathbb{R}^3$, tem-se para a primeira equação,

$$\begin{aligned}\int_{t_0}^t\int_{\mathbb{R}^3}(\tau-t_0)^2[(D_{l_1}D_{l_2}u_i)^2]_\tau d\tau dx + 2\int_{t_0}^t\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j=1}^3 D_{l_1}D_{l_2}u_i(D_{l_1}D_{l_2}u_jD_ju_i \\ +D_{l_2}u_jD_{l_1}D_ju_i + D_{l_1}u_jD_{l_2}D_ju_i + u_jD_{l_1}D_{l_2}D_ju_i)dxd\tau \\ +2\int_{t_0}^t\int_{\mathbb{R}^3}(\tau-t_0)^2D_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}D_i\mathsf{p}dxd\tau \\ = 2(\mu+\chi)\int_{t_0}^t\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j=1}^3 D_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}D_jD_ju_idxd\tau \\ +2\chi\int_{t_0}^t\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j,k=1}^3\epsilon_{ijk}D_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}D_jw_kdxd\tau.\end{aligned} \tag{4.4}$$

Vamos analisar cada termo da equação anterior. Começando pelo primeiro. Usando o teorema de Fubini e integrando por partes, obtém-se

$$\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau-t_0)^2[(D_{l_1}D_{l_2}u_i)^2]_\tau d\tau dx = \int_{\mathbb{R}^3}(t-t_0)^2(D_{l_1}D_{l_2}u_i)^2 dx$$
$$-\int_{t_0}^{t}\int_{\mathbb{R}^3}2(\tau-t_0)(D_{l_1}D_{l_2}u_i)^2 d\tau dx.$$

Integrando por partes no segundo termo de (4.4) e por $\nabla\cdot\boldsymbol{u}=0$, tem-se

$$2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j=1}^{3}D_{l_1}D_{l_2}u_i(D_{l_1}D_{l_2}u_jD_ju_i$$
$$+D_{l_2}u_jD_{l_1}D_ju_i+D_{l_1}u_jD_{l_2}D_ju_i+u_jD_{l_1}D_{l_2}D_ju_i)dxd\tau$$
$$=-2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j=1}^{3}u_iD_jD_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}u_jdxd\tau$$
$$-2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j=1}^{3}D_jD_{l_1}D_{l_2}u_iD_{l_2}u_jD_{l_1}u_idxd\tau$$
$$-2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j=1}^{3}D_jD_{l_1}D_{l_2}u_iD_{l_1}u_jD_{l_2}u_idxd\tau$$
$$+2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau-t_0)^2\sum_{j=1}^{3}u_jD_{l_1}D_{l_2}u_iD_{l_1}D_{l_2}D_ju_idxd\tau$$

$$
\begin{aligned}
&= -2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} u_i D_j D_{l_1} D_{l_2} u_i D_{l_1} D_{l_2} u_j dx d\tau \\
&\quad -2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} D_j D_{l_1} D_{l_2} u_i D_{l_2} u_j D_{l_1} u_i dx d\tau \\
&\quad -2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} D_j D_{l_1} D_{l_2} u_i D_{l_1} u_j D_{l_2} u_i dx d\tau \\
&\quad +\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} u_j D_j[(D_{l_1} D_{l_2} u_i)^2] dx d\tau \\
&= -2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} u_i D_j D_{l_1} D_{l_2} u_i D_{l_1} D_{l_2} u_j dx d\tau \\
&\quad -2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} D_j D_{l_1} D_{l_2} u_i D_{l_2} u_j D_{l_1} u_i dx d\tau \\
&\quad -2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} D_j D_{l_1} D_{l_2} u_i D_{l_1} u_j D_{l_2} u_i dx d\tau \\
&\leq 2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} |u_i||D_j D_{l_1} D_{l_2} u_i||D_{l_1} D_{l_2} u_j| dx d\tau \\
&\quad +2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} |D_j D_{l_1} D_{l_2} u_i||D_{l_2} u_j||D_{l_1} u_i| dx d\tau \\
&\quad +2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{j=1}^{3} |D_j D_{l_1} D_{l_2} u_i||D_{l_1} u_j||D_{l_2} u_i| dx d\tau.
\end{aligned}
$$

Portanto, somando em $i, l_1, l_2 = 1, ..., 3$ e usando a desigualdade de Cauchy-Schwarz na soma e na integral de maneira análoga a feita em (2.21), tem-se

$$
\begin{aligned}
&2\int_{t_0}^{t}\int_{\mathbb{R}^3}(\tau - t_0)^2\sum_{i,j,l_1,l_2=1}^{3} D_{l_1} D_{l_2} u_i (D_{l_1} D_{l_2} u_j D_j u_i \\
&\quad + D_{l_2} u_j D_{l_1} D_j u_i + D_{l_1} u_j D_{l_2} D_j u_i + u_j D_{l_1} D_{l_2} D_j u_i) dx d\tau \\
&\leq K\int_{t_0}^{t}(\tau - t_0)^2 \|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\Big\{ \|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \\
&\quad + \|D\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \Big\} d\tau
\end{aligned}
$$

$$\leq K\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u},D^3\mathbf{w},D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\times$$

$$\times\Bigg\{(\|\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u},D^2\mathbf{w},D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}$$

$$+\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\Bigg\}d\tau.$$

Pelas desigualdades de Sobolev (A.8) e (A.9) demonstradas no apêndice e por (2.4), tem-se

$$K\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\Bigg\{\|\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|D^2\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}$$

$$+\|D\boldsymbol{u}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|D\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\Bigg\}d\tau$$

$$\leq K\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\times$$

$$\times\Bigg\{\|\,(\boldsymbol{u},\mathbf{w},\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D^3\boldsymbol{u},D^3\mathbf{w},D^3\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}$$

$$+\|\,(\boldsymbol{u},\mathbf{w},\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D^3\boldsymbol{u},D^3\mathbf{w},D^3\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\Bigg\}d\tau$$

$$\leq K\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\times$$

$$\times\Bigg\{\|\,(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D^3\boldsymbol{u},D^3\mathbf{w},D^3\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}$$

$$+\|\,(D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D^3\boldsymbol{u},D^3\mathbf{w},D^3\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\Bigg\}d\tau,$$

para algum $K>0$. Portanto, pelo teorema (2.4), existe $t_0>t_*$ suficientemente grande tal que

$$\|D\boldsymbol{u},D\mathbf{w},D\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}<\left(\frac{\lambda}{K}\right)^2,$$

para todo $t > t_0$, o que implica por (2.4),

$$K \int_{t_0}^{t} (\tau - t_0)^2 \|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \times$$

$$\times \Big\{ \| (\boldsymbol{u}, \mathbf{w}, \boldsymbol{b}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b}) \|_{L^2(\mathbb{R}^3)}$$

$$+ \| (\boldsymbol{u}, \mathbf{w}, \boldsymbol{b}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b}) \|_{L^2(\mathbb{R}^3)}^{1/2} \| (D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b}) \|_{L^2(\mathbb{R}^3)} \Big\} d\tau$$

$$\leq \lambda \int_{t_0}^{t} (\tau - t_0)^2 \|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau.$$

O terceiro termo de (4.4) envolvendo a pressão p é nulo após somarmos em i e integrarmos por partes, pois $\nabla \cdot \boldsymbol{u} = 0$. O primeiro termo do lado direito da equação (4.4), se torna

$$-2(\mu + \chi) \int_{t_0}^{t} \|D^3\boldsymbol{u}(\cdot,t)\|_{L^2(\mathbb{R}^3)} d\tau$$

e finalmente, de maneira análoga a feita em (2.20), tem-se

$$2\chi \left| \int_{t_0}^{t} \int_{\mathbb{R}^3} \sum_{i,j,k,l_1,l_2=1}^{3} \epsilon_{ijk} D_{l_1} D_{l_2} u_i D_{l_1} D_{l_2} D_j w_k \, dx d\tau \right| \leq \chi \int_{t_0}^{t} \|D^2\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau$$

$$+ \chi \int_{t_0}^{t} \|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau.$$

O mesmo raciocínio pode ser feito na segunda e terceira equação do sistema (2.5). A partir daí, somando tudo tem-se,

$$(t - t_0)^2 \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + 2(\mu + \chi) \int_{t_0}^{t} (\tau - t_0)^2 \|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau$$

$$+ 2\gamma \int_{t_0}^{t} (\tau - t_0)^2 \|D^3\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau + 2\nu \int_{t_0}^{t} (\tau - t_0)^2 \|D^3\boldsymbol{b}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau$$

$$+ 2 \int_{t_0}^{t} (\tau - t_0)^2 \|D^2\nabla \cdot \mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 + 4\chi \int_{t_0}^{t} (\tau - t_0)^2 \|D^2\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2$$

$$\leq 2 \int_{t_0}^{t} (\tau - t_0) \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau$$

$$+ \lambda \int_{t_0}^{t} (\tau - t_0)^2 \|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau$$

$$+ 2\chi \int_{t_0}^{t} \|D^2\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau + 2\chi \int_{t_0}^{t} \|D^3\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau.$$

Logo,

$$
\begin{aligned}
&(t-t_0)^2\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+2\lambda\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+2\int_{t_0}^{t}(\tau-t_0)^2\|D^2\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}+2\chi\int_{t_0}^{t}(\tau-t_0)^2\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\\
&\le (t-t_0)^2\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\mu\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+2\gamma\int_{t_0}^{t}(\tau-t_0)^2\|D^3\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\nu\int_{t_0}^{t}(\tau-t_0)^2\|D^3\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+2\int_{t_0}^{t}(\tau-t_0)^2\|D^2\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}+2\chi\int_{t_0}^{t}(\tau-t_0)^2\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\\
&\le 2\int_{t_0}^{t}(\tau-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+\lambda\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
&(t-t_0)^2\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+\lambda\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+2\int_{t_0}^{t}(\tau-t_0)^2\|D^2\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}+2\chi\int_{t_0}^{t}(\tau-t_0)^2\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\\
&\le 2\int_{t_0}^{t}(\tau-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau,
\end{aligned}
\tag{4.5}
$$

o que prova (4.1a) para o caso $m=2$. Note que por (4.3) e por (4.5), tem-se

$$
\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau<\infty, \tag{4.6}
$$

o que prova (4.1b) para $m=2$. Dado $\epsilon>0$, seja t_0 suficientemente grande tal que por (3.1), tem-se

$$
\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\le\epsilon,
$$

para todo $t\ge t_0$. Logo, por (2.4),

$$
\int_{t_0}^{t}\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\le\epsilon
$$

e por (4.2)

$$\int_{t_0}^{t} (\tau - t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \leq \epsilon.$$

Logo, por (4.5),

$$(t-t_0)^2\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} = \left(\frac{t-t_0}{t}\right)^2 t^2\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \epsilon.$$

Portanto,

$$\lim_{t\to\infty} t\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0, \tag{4.7}$$

pois

$$\left(\frac{t-t_0}{t}\right)^2 \to 1.$$

O que concluí a demonstração do teorema (4.1) para $m = 2$.

Vamos para o próximo caso, i.e., o caso $m = 3$. Repetindo o mesmo raciocínio, derivando três vezes o sistema (2.5) com respeito a x_{l_1}, x_{l_2} e x_{l_3}, multiplicando por $2(\tau - t_0)^3 D_{l_1}D_{l_2}D_{l_3}u_i$ a primeira equação do sistema (2.5), pelo mesmo fator a segunda equação só que com w_i e pelo o mesmo fator só que com b_i a terceira, integrando na região $[t_0,t]\times\mathbb{R}^3$ e usando as mesmas técnicas usadas no caso anterior, tem-se,

$$\begin{aligned}
(t-t_0)^3\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &+ 2\mu\int_{t_0}^{t}(\tau-t_0)^3\|D^4\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\gamma\int_{t_0}^{t}(\tau-t_0)^3\|D^4\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau &+ 2\nu\int_{t_0}^{t}(\tau-t_0)^3\|D^4\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^{t}(\tau-t_0)^3\|D^3\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} &+ 2\chi\int_{t_0}^{t}(\tau-t_0)^3\|D^3\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} \\
&\leq 3\int_{t_0}^{t}(\tau-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+K\int_{t_0}^{t}(\tau-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\times \\
&\times\Big\{\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \\
&+\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\Big\}d\tau,
\end{aligned}$$

usando as desigualdades de Sobolev (A.10) e (A.12), tem-se

$$
\begin{aligned}
(t-t_0)^3\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu\int_{t_0}^t (\tau-t_0)^3\|D^4\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\gamma\int_{t_0}^t (\tau-t_0)^3\|D^4\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\nu\int_{t_0}^t (\tau-t_0)^3\|D^4\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^t (\tau-t_0)^3\|D^3\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t (\tau-t_0)^3\|D^3\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq 3\int_{t_0}^t (\tau-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+K\int_{t_0}^t (\tau-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\times \\
\times\left\{2\|\,(\boldsymbol{u},\mathbf{w},\boldsymbol{b})\,\|^{3/4}_{L^2(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})\,\|^{1/4}_{L^2(\mathbb{R}^3)}\|\,(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\right\}d\tau.
\end{aligned}
$$

Logo, por (3.1) e por (4.7), existe um t_0 suficientemente grande, tal que

$$
\|\,(\boldsymbol{u},\mathbf{w},\boldsymbol{b})\,\|^{3/4}_{L^2(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})\,\|^{1/4}_{L^2(\mathbb{R}^3)} \leq \frac{\lambda}{2K},
$$

para todo $t > t_0$. Logo,

$$
\begin{aligned}
(t-t_0)^3\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \\
+\lambda\int_{t_0}^t (\tau-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^t (\tau-t_0)^3\|D^3\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t (\tau-t_0)^3\|D^3\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq 3\int_{t_0}^t (\tau-t_0)^2\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau,
\end{aligned}
\tag{4.8}
$$

o que prova (4.1a) para o caso $m = 3$. Agora, por (4.6) e por (4.8), tem-se

$$
\int_{t_0}^t (\tau-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau < \infty,
\tag{4.9}
$$

concluindo a prova de (4.1b) para $m = 3$. Note que, dado $\epsilon > 0$, por (4.2), (2.4) e (3.1), existe t_0 suficientemente grande, tal que,

$$
\int_{t_0}^t (\tau-t_0)\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \leq \epsilon.
$$

Então, por (4.5),

$$\int_{t_0}^{t} (\tau - t_0)^2 \|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \leq \epsilon. \tag{4.10}$$

Logo, por (4.8),

$$(t - t_0)^3 \|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq C\epsilon,$$

para alguma constante $C > 0$. O que implica, repetindo o mesmo argumento em (4.7),

$$t^{3/2} \|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty. \tag{4.11}$$

O que concluí a prova de (4.1) para $m = 3$. Vamos para o caso $m = 4$. Analogamente ao caso anterior, tem-se

$$\begin{aligned}
(t - t_0)^4 \|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu \int_{t_0}^{t} (\tau - t_0)^4 \|D^5\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
+2\gamma \int_{t_0}^{t} (\tau - t_0)^4 \|D^5\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\nu \int_{t_0}^{t} (\tau - t_0)^4 \|D^5\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
+2 \int_{t_0}^{t} (\tau - t_0)^4 \|D^4\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau + 2\chi \int_{t_0}^{t} (\tau - t_0)^4 \|D^4\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
\leq 4 \int_{t_0}^{t} (\tau - t_0)^3 \|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau \\
+K \int_{t_0}^{t} (\tau - t_0)^4 \|(D^5\boldsymbol{u}, D^5\mathbf{w}, D^5\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \times \\
\times \Big\{ \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \\
+\|(D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \\
+\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} \Big\} d\tau.
\end{aligned}$$

Usando as desigualdades de Sobolev (A.11) e (A.12) provadas no apêndice, tem-se

$$
\begin{aligned}
&(t-t_0)^4\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\mu\int_{t_0}^{t}(\tau-t_0)^4\|D^5\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+2\gamma\int_{t_0}^{t}(\tau-t_0)^4\|D^5\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\nu\int_{t_0}^{t}(\tau-t_0)^4\|D^5\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+2\int_{t_0}^{t}(\tau-t_0)^4\|D^4\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^4\|D^4\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&\leq 4\int_{t_0}^{t}(\tau-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+K\int_{t_0}^{t}(\tau-t_0)^4\|(D^5\boldsymbol{u}, D^5\mathbf{w}, D^5\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\times\\
&\times\Big\{\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|^{3/4}_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^{1/4}_{L^2(\mathbb{R}^3)}\\
&+\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|^{5/6}_{L^2(\mathbb{R}^3)}\|(D^3\boldsymbol{u}, D^3\mathbf{w}, D^3\boldsymbol{b})(\cdot,\tau)\|^{1/6}_{L^2(\mathbb{R}^3)}\\
&+\|(\boldsymbol{u},\mathbf{w},\boldsymbol{b})(\cdot,\tau)\|^{3/4}_{L^2(\mathbb{R}^3)}\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot,\tau)\|^{1/4}_{L^2(\mathbb{R}^3)}\Big\}d\tau.
\end{aligned}
$$

Pelo mesmo argumento anterior, dado $\epsilon > 0$ existe um $t_0 > t_*$ suficientemente grande tal que por (3.1), (4.7) e (4.11), tem-se,

$$
\begin{aligned}
&(t-t_0)^4\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
&+\lambda\int_{t_0}^{t}(\tau-t_0)^4\|(D^5\boldsymbol{u}, D^5\mathbf{w}, D^5\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&+2\int_{t_0}^{t}(\tau-t_0)^4\|D^4\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^4\|D^4\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
&\leq 4\int_{t_0}^{t}(\tau-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau,
\end{aligned}
\tag{4.12}
$$

provando (4.1a) para o caso $m = 4$. Por (4.9) e (4.12), tem-se,

$$
\int_{t_0}^{t}(\tau-t_0)^4\|(D^5\boldsymbol{u}, D^5\mathbf{w}, D^5\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau < \infty, \tag{4.13}
$$

provando (4.1b) para o caso $m = 4$. Agora, dado $\epsilon > 0$, por (4.10) e (4.8),

$$
\int_{t_0}^{t}(\tau-t_0)^3\|(D^4\boldsymbol{u}, D^4\mathbf{w}, D^4\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau < \epsilon.
$$

Portanto, por (4.12),

$$(t-t_0)^4\|(D^4\boldsymbol{u},D^4\mathbf{w},D^4\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}<\epsilon.$$

para todo $t>t_0$. O que implica, pelo mesmo raciocínio usado anteriormente,

$$t^2\|(D^4\boldsymbol{u},D^4\mathbf{w},D^4\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}\to 0 \qquad \text{ao} \quad t\to\infty. \tag{4.14}$$

Finalmente, por indução em $m\geq 5$ para o caso geral, tem-se

$$\begin{aligned}
(t-t_0)^m\|(D^m\boldsymbol{u},D^m\mathbf{w},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\mu\int_{t_0}^t(\tau-t_0)^m\|D^{m+1}\boldsymbol{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
+2\gamma\int_{t_0}^t(\tau-t_0)^m\|D^{m+1}\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\nu\int_{t_0}^t(\tau-t_0)^m\|D^{m+1}\boldsymbol{b}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
+2\int_{t_0}^t(\tau-t_0)^m\|D^m\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\chi\int_{t_0}^t(\tau-t_0)^m\|D^m\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
\leq m\int_{t_0}^t(\tau-t_0)^{m-1}\|(D^m\boldsymbol{u},D^m\mathbf{w},D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
+K\int_{t_0}^t(\tau-t_0)^m\|(D^{m+1}\boldsymbol{u},D^{m+1}\mathbf{w},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\times\\
\times\sum_{l=0}^{[m/2]}\|(D^l\boldsymbol{u},D^l\mathbf{w},D^l\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|(D^{m-l}\boldsymbol{u},D^{m-l}\mathbf{w},D^{m-l}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau,
\end{aligned}$$

onde $[m/2]$ é a parte inteira de $m/2$. Por (A.12), tem-se

$$\begin{aligned}
(t-t_0)^m\|(D^m\boldsymbol{u},D^m\mathbf{w},D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\\
+\lambda\int_{t_0}^t(\tau-t_0)^m\|(D^{m+1}\boldsymbol{u},D^{m+1}\mathbf{w},D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
+2\int_{t_0}^t(\tau-t_0)^m\|D^m\nabla\cdot\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau+2\chi\int_{t_0}^t(\tau-t_0)^m\|D^m\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
\leq m\int_{t_0}^t(\tau-t_0)^{m-1}\|(D^m\boldsymbol{u},D^m\mathbf{w},D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau,
\end{aligned}$$

aumentando $t_0>t_*$ se necessário e usando a hipótese de indução, i.e., que todas as derivadas de ordem menor ou igual a m vão para zero. O que prova (4.1a). Pelo passo anterior,

$$\int_{t_0}^t(\tau-t_0)^{m-1}\|(D^m\boldsymbol{u},D^m\mathbf{w},D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau<\infty$$

e

$$t^{m-1}\|(D^{m-1}\boldsymbol{u}, D^{m-1}\mathrm{w}, D^{m-1}\boldsymbol{b})\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty.$$

O que implica, repetindo o mesmo raciocínio feito anteriormente, que

$$\int_{t_0}^{\infty} (\tau - t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathrm{w}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^2 d\tau < \infty$$

e

$$\lim_{t\to\infty} t^{m/2}\|(D^m\boldsymbol{u}, D^m\mathrm{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0.$$

O que prova o teorema (4.1). □

Como consequência disto, por interpolação, temos o seguinte resultado:

Corolário 4.2. *Seja $(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)$ solução de Leray do problema (2.1). Então,*

$$t^{s/2}\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \to 0 \qquad \textit{ao} \qquad t \to \infty, \tag{4.15}$$

para todo $s \geq 0$ (real).

Demonstração. Dado $\epsilon > 0$, tome $s > 0$ arbitrário. Então, para qualquer $m > s$, existe t_0^1 tal que, por (3.1),

$$\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon,$$

para todo $t > t_0^1$. Além disso, por (4.1c), existe t_0^2, tal que, para todo $t > t_0^2$

$$t^{\frac{m}{2}}\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^3)} \leq \epsilon,$$

logo, usando o lema (B.5), para $t > \max\{t_0^1, t_0^2\}$ e $\gamma > 0$ (a ser escolhido), tem-se,

$$\begin{aligned} t^{\gamma}\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} &\leq t^{\gamma}\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{s}{m}}\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^3)}^{\frac{s}{m}} \\ &\leq \|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{s}{m}}\left(t^{\gamma\frac{m}{s}}\|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})(\cdot,t)\|_{\dot{H}^m}\right)^{\frac{s}{m}} \leq \epsilon, \end{aligned}$$

se $\gamma\frac{m}{s} = \frac{m}{2}$, portanto, $\gamma = \frac{s}{2}$, para qualquer $s > 0$. O caso $s = 0$ foi provado em (3.1). □

Outra importante consequência do teorema (4.1) é fato de se ter uma taxa de decaimento para todas as normas L^q para todo $2 \leq q \leq \infty$.

Corolário 4.3. *Seja* $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$ *solução de Leray do problema (2.1). Então*

$$t^{\frac{3}{4}-\frac{3}{2q}} \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^q(\mathbb{R}^3)} \to 0 \quad \textit{ao} \quad t \to \infty. \tag{4.16}$$

Demonstração. Por (3.1), dado $\epsilon > 0$, existe t_0^1

$$\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon,$$

para todo $t > t_0^1$. Em particular, por (4.1c), existe $t_0^2 > 0$ tal que

$$t\|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon,$$

para todo $t > t_0^2$ Por (A.4), para $\gamma_1 > 0$ (a ser escolhido) tem-se

$$t^{\gamma_1}\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^3)}^{1/4} \left(t^{\frac{4\gamma}{3}} \|(D^2\boldsymbol{u}, D^2\mathbf{w}, D^2\boldsymbol{b})(\cdot, t)\|_{L^2(\mathbb{R}^3)} \right)^{\frac{3}{4}} \leq \epsilon,$$

se $\frac{4\gamma_1}{3} = 1$, i.e., $\gamma_1 = 3/4$, para todo $t > \max\{t_0^1, t_0^2\}$. Em outras palavras,

$$t^{3/4}\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty. \tag{4.17}$$

Analogamente, aplicando uma simples interpolação (veja o lema (B.3) no apêndice), tem-se

$$t^{\gamma_2}\|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^q(\mathbb{R}^3)} \leq \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{2/q} \left(t^{\frac{\gamma_2}{1-2/q}} \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})\|_{L^\infty(\mathbb{R}^3)} \right)^{1-2/q} \leq \epsilon,$$

se $\frac{\gamma_2}{1-2/q} = \frac{3}{4}$, i.e., $\gamma_2 = 3/4 - 3/2q$. Portanto,

$$t^{\frac{3}{4}-\frac{3}{2q}} \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)\|_{L^q(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty,$$

para cada $2 \leq q \leq \infty$. □

Analogamente, podemos demonstrar o seguinte resultado mais geral.

Corolário 4.4. *Seja* $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$ *solução de Leray do problema (2.1). Então*

$$t^{\gamma(m,q)}\|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{as } t \to \infty,$$

onde $\gamma(m,q) = \frac{3}{4} + \frac{m}{2} - \frac{3}{2q}$, *para cada* $m \geq 0$ *(inteiro) e* $2 \leq q \leq \infty$.

Demonstração. Por (4.1c), dado $\epsilon > 0$, Usando a desigualdade (B.7) descrita no apêndice, tem-se

$$\begin{aligned} &t^{\beta}\|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \\ &\leq Ct^{\beta}\|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|(D^{m+3}\boldsymbol{u}, D^{m+3}\mathbf{w}, D^{m+3}\boldsymbol{b})(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)} \leq \epsilon. \end{aligned}$$

Escolhendo $\beta = m/2 + 3/4$. Usando o lema (B.3) do apêndice, segue o resultado. □

4.1.1 Caso $\chi > 0$

Nesta seção mostraremos o seguinte resultado

Teorema 4.5. *Seja* m *inteiro positivo. Se* $\chi > 0$, *então*

$$\lim_{t\to\infty} t^{\frac{m+1}{2}}\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0.$$

Demonstração. Reescrevendo a segunda equação do sistema (2.1) como

$$\mathbf{w}_t = \gamma\Delta\mathbf{w} + \mathbf{Q}_2 - 2\chi\mathbf{w} \tag{4.18}$$

Onde

$$\mathbf{Q}_2 = -\boldsymbol{u}\cdot\nabla\mathbf{w} + \nabla(\nabla\cdot\mathbf{w}) + \chi\nabla\wedge\boldsymbol{u}$$

Fazendo a mudança de variável,

$$\mathbf{W}(\cdot,t) = e^{2\chi t}\mathbf{w}(\cdot,t)$$

e multiplicando a equação (4.18) por $e^{2\chi t}$, obtemos que

$$\mathbf{W}_t = \gamma\Delta\mathbf{W} + e^{2\chi t}\mathbf{Q}_2.$$

Pelo o que foi estudado nos capítulos anteriores, em particular pelo teorema 4.1, dado $\epsilon > 0$, exite $t_0 > t_*$ suficientemente grande tal que para todo $l \in \mathbb{N}$, $1 \leq l \leq m+2$ vale que

$$\|(D^l\boldsymbol{u}, D^l\mathbf{w}, D^l\boldsymbol{b})(\cdot,t)\| < \epsilon t^{-\frac{l}{2}}, \quad \text{para todo } \ t > t_0.$$

Pelo princípio de Duhamel, temos que

$$\mathbf{W}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{W}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-\tau)}e^{2\chi\tau}\mathbf{Q}_2(\cdot,\tau)d\tau,$$

que pode ser reescrito como

$$\mathbf{w}(\cdot,t) = e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) - \int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)d\tau$$
$$+ \int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)d\tau + \chi\int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,\tau)d\tau.$$

Multiplicando por $t^{\frac{m+1}{2}}$, derivando m vezes, aplicando a norma L^2 e usando a desigualdade de Minkowski para integrais, temos que

$$
\begin{aligned}
t^{\frac{m+1}{2}}\|D^m \mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq k\Bigg[&\underbrace{t^{\frac{m+1}{2}} e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I} \\
&+\underbrace{t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)} D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau}_{II} \\
&+\underbrace{t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)} D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau}_{III} \\
&+\underbrace{t^{\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)} D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau}_{IV}\Bigg].
\end{aligned} \tag{4.19}
$$

Vamos analisar cada termo separadamente.

O termo I representa a solução da equação do calor, com condição inicial $\mathbf{w}(\cdot,t_0)$ e

$$
\lim_{t\to\infty}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0,
$$

cuja prova pode ser encontrada no teorema (C.10) do apêndice.

De modo que

$$
\lim_{t\to\infty} t^{\frac{m+1}{2}} e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{4.20}
$$

Para analisar os demais termos, vamos utilizar a desigualdade (conhecida) de Niremberg-Gagliardo (A.18) descrita no apêndice e a desigualdade de Hölder para soma e integrais. Iremos proceder de maneira mais sucinta, visto que usaremos o mesmo tipo de raciocínio empregado anteriormente. Veja a derivação de desigualdade (2.21), por exemplo. Começando pelo termo II, temos

$$
\begin{aligned}
t^{\frac{m+1}{2}} \int_{t_0}^{t} & e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
\leq kt^{\frac{m+1}{2}} \int_{t_0}^{t} & e^{-2\chi(t-\tau)} \sum_{l=0}^{m} \|D^l \boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^3)} \|D^{m-l+1}\mathbf{w}(\cdot,\tau)\|_{L^4(\mathbb{R}^3)} d\tau \\
& \leq kt^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \sum_{l=0}^{m} \|D^l \boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^{1/4} \|D^{l+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^{3/4} \\
& \qquad \times \|D^{m-l+1}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^{1/4} \|D^{m-l+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}^{3/4} d\tau \\
& \leq kt^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \sum_{l=0}^{m} \epsilon\tau^{-\frac{l}{8}} \epsilon\tau^{-\frac{-3(l+1)}{8}} \epsilon\tau^{-\frac{m-l+1}{8}} \epsilon\tau^{\frac{-3(m-l+2)}{8}} d\tau \\
& \qquad \leq kt^{\frac{m+1}{2}} \epsilon^4 \int_{t_0}^{t} e^{-2\chi(t-\tau)} \tau^{-\frac{2m+5}{4}} d\tau \\
& = kt^{\frac{m+1}{2}} \epsilon^4 \left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)} \tau^{-\frac{2m+5}{4}} d\tau + \int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)} \tau^{-\frac{2m+5}{4}} d\tau \right] \\
\leq k\epsilon^4 \Bigg(t^{\frac{m+1}{2}} e^{-\chi t} \frac{t_0^{-\frac{2m+1}{4}}}{2m+1} & + t^{\frac{m+1}{2}} t^{-\frac{2m+5}{4}} \chi^{-1} \Bigg) \leq k\epsilon^4 \Bigg(t^{\frac{m+1}{2}} e^{-\chi t} + t^{-\frac{3}{4}} \chi^{-1} \Bigg).
\end{aligned}
$$

Como $\epsilon > 0$ é arbitrário, temos que

$$
\lim_{t\to\infty} t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau = 0. \tag{4.21}
$$

Podemos usar o mesmo raciocínio para o termo III,

$$
\begin{aligned}
t^{\frac{m+1}{2}} \int_{t_0}^{t} & e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq k t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq k t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \epsilon \tau^{-\frac{m+2}{2}} d\tau \\
&= k t^{\frac{m+1}{2}} \epsilon \Big[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)} \tau^{-\frac{m+2}{2}} d\tau + \int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)} \tau^{-\frac{m+2}{2}} d\tau \Big] \\
&\leq k\epsilon \Big(t^{\frac{m+1}{2}} e^{-\chi t} \frac{t_0^{-\frac{m}{2}}}{m} + t^{\frac{m+1}{2}} t^{-\frac{m+2}{2}} \chi^{-1} \Big) \leq k\epsilon \Big(t^{\frac{m+1}{2}} e^{-\chi t} + t^{-\frac{1}{2}} \chi^{-1} \Big).
\end{aligned}
$$

Novamente, como $\epsilon > 0$ é arbitrário, temos que

$$
\lim_{t\to\infty} t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau = 0. \tag{4.22}
$$

Usando mais uma vez o mesmo raciocínio para o termo IV,

$$
\begin{aligned}
t^{\frac{m+1}{2}} \int_{t_0}^{t} & e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq k t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
&\leq k t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \epsilon \tau^{-\frac{m+1}{2}} d\tau \\
&= k t^{\frac{m+1}{2}} \epsilon \Big[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)} \tau^{-\frac{m+1}{2}} d\tau + \int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)} \tau^{-\frac{m+1}{2}} d\tau \Big] \\
&\leq k\epsilon \Big(t^{\frac{m+1}{2}} e^{-\chi t} \frac{t_0^{-\frac{m-1}{2}}}{m-1} + t^{\frac{m+1}{2}} t^{-\frac{m+1}{2}} \chi^{-1} \Big) \leq k\epsilon \Big(t^{\frac{m+1}{2}} e^{-\chi t} + \chi^{-1} \Big)
\end{aligned}
$$

Como $\epsilon > 0$ é arbitrário, temos que

$$
\lim_{t\to\infty} t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau = 0. \tag{4.23}
$$

Pelas equações, (4.19), (4.20), (4.21), (4.22) e (4.23), temos que

$$\lim_{t\to\infty} t^{\frac{m+1}{2}}\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} = 0.$$

Q.E.D. $\square$

A partir daí, podemos concluir alguns interessantes resultados por interpolação.

Corolário 4.6. *Seja $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)$ solução de Leray de (2.1) e $\chi > 0$. Então,*

$$t^{\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \to 0 \quad ao \quad t\to\infty,$$

para todo $s \geq 0$ real.

Demonstração. Dado $\epsilon > 0$, tome $s > 0$ arbitrário. Então, para qualquer $m > s$, existe t_0^1 suficientemente grande tal que, por (3.13),

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \epsilon,$$

para todo $t > t_0^1$. Pelo teorema (4.5), existe t_0^2 tal que

$$t^{\frac{m+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^3)} \leq \epsilon.$$

Logo, usando o lema (B.5) do apêndice, para todo $t > \max\{t_0^1, t_0^2\}$ e $\gamma = \gamma_1 + \gamma_2 > 0$ (a ser escolhido), tem-se

$$t^{\gamma}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \left(t^{\frac{\gamma_1}{1-\frac{s}{m}}}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right)^{1-\frac{s}{m}} \left(t^{\frac{\gamma_2}{s/m}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^3)}\right)^{s/m} \leq \epsilon,$$

se $\frac{\gamma_1}{1-\frac{s}{m}} = 1/2$ e $\frac{\gamma_2}{s/m} = \frac{m+1}{2}$. Logo, $\gamma = \gamma_1 + \gamma_2 = \frac{s+1}{2}$. Ou seja,

$$t^{\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \to 0 \quad ao \quad t\to\infty,$$

para cada $s \geq 0$ real. $\square$

Corolário 4.7. *Seja $(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,t)$ solução de Leray de (2.1) e $\chi > 0$. Então,*

$$t^{\frac{5}{4}-\frac{3}{2q}}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^3)} \to 0 \quad ao \quad t\to\infty,$$

para cada $2 \leq q \leq \infty$.

Demonstração. Por (3.13), dado $\epsilon > 0$, existe t_0^1

$$t^{1/2}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le \epsilon,$$

para todo $t > t_0^1$. Em particular, pelo teorema (4.5), existe $t_0^2 > 0$ tal que

$$t^{3/2}\|D^2\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le \epsilon,$$

para todo $t > t_0^2$ Por (A.4), para $\gamma = \gamma_1 + \gamma_2 > 0$ (a ser escolhido) tem-se

$$t^{\gamma}\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \le \left(t^{4\gamma_1}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right)^{1/4}\left(t^{\frac{4\gamma_2}{3}}\|D^2\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right)^{3/4} \le \epsilon^{1/4}\epsilon^{3/4} = \epsilon,$$

se $4\gamma_1 = 1/2$ e $\frac{4\gamma_2}{3} = 3/2$, i.e., $\gamma_1 = 1/8$ e $\gamma_2 = 9/8$, para todo $t > \max\{t_0^1, t_0^2\}$. Em outras palavras,

$$t^{5/4}\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty. \tag{4.24}$$

Analogamente, seja $\beta = \beta_1 + \beta_2$. Basta aplicar uma simples interpolação (veja o lema (B.3) no apêndice), tem-se

$$t^{\beta}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^3)} \le \left(t^{\frac{q}{2}\beta_1}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right)^{2/q}\left(t^{\frac{\beta_2}{1-2/q}}\|\mathbf{w}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}\right)^{1-2/q} \le \epsilon,$$

se $\frac{q}{2}\beta_1 = \frac{1}{2}$ e $\frac{\beta_2}{1-2/q} = 5/4$, i.e., $\beta = 5/4 - 3/2q$. Portanto,

$$t^{\frac{5}{4}-\frac{3}{2q}}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t \to \infty,$$

para cada $2 \le q \le \infty$. $\square$

Da mesma forma que na seção anterior, pode-se mostrar que

$$t^{\frac{5}{4}+\frac{m}{2}-\frac{3}{2q}}\|\mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^3)} \to 0 \quad \textit{ao} \quad t \to \infty,$$

para cada $2 \le q \le \infty$ e $m \ge 0$ inteiro. O que encerra a discussão deste assunto.

4.2 Caso $n = 2$

Para demostrar a propriedade 4.1c no caso bidimensional ($n = 2$), basta usar as desigualdades de Sobolev (A.13-A.16) e o mesmo argumento apresentado anteriormente. Com efeito, por indução, derivando o sistema (1.2) m vezes e multiplicando por $(t-t_0)^m\boldsymbol{u}$, $(t-t_0)^m\mathbf{w}$ e $(t-t_0)^m\boldsymbol{b}$ a primeira, segunda e terceira equação do sistema (1.2) respectivamente, integrando na região $\mathbb{R}^2 \times [t_0, t]$ e usando os mesmos procedimentos usados na seção anterior, tem-se

$$
\begin{aligned}
&(t-t_0)^m \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \\
&+2\lambda \int_{t_0}^{t} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&+2\chi \int_{t_0}^{t} (\tau-t_0)^m \|D^m\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&\leq m \int_{t_0}^{t} (\tau-t_0)^{m-1} \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&+C \int_{t_0}^{t} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} \times \\
&\times \sum_{l=0}^{[m/2]} \|(D^l\boldsymbol{u}, D^l\mathbf{w}, D^l\boldsymbol{b})(\cdot,\tau)\|_{L^\infty(\mathbb{R}^2)} \|(D^{m-l}\boldsymbol{u}, D^{m-l}\mathbf{w}, D^{m-l}\boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} d\tau.
\end{aligned}
$$

Assim, basta usar as desigualdades de Sobolev (mais especificamente (A.16)) para obter

$$
\begin{aligned}
&(t-t_0)^m \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \\
&+2\lambda \int_{t_0}^{t} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&+2\chi \int_{t_0}^{t} (\tau-t_0)^m \|D^m\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&\leq m \int_{t_0}^{t} (\tau-t_0)^{m-1} \|(D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&+C \int_{t_0}^{t} (\tau-t_0)^m \|(D^{m+1}\boldsymbol{u}, D^{m+1}\mathbf{w}, D^{m+1}\boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} \|(\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot,\tau)\|_{L^2(\mathbb{R}^2)} d\tau.
\end{aligned}
$$

Agora, para um t_0 suficientemente grande e por (3.1), tem-se

$$\begin{aligned}
&(t-t_0)^m \|(D^m \boldsymbol{u}, D^m \mathbf{w}, D^m \boldsymbol{b})(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \\
&+\lambda \int_{t_0}^t (\tau-t_0)^m \|(D^{m+1} \boldsymbol{u}, D^{m+1} \mathbf{w}, D^{m+1} \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&+2\chi \int_{t_0}^t (\tau-t_0)^m \|D^m \mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau \\
&\leq m \int_{t_0}^t (\tau-t_0)^{m-1} \|(D^m \boldsymbol{u}, D^m \mathbf{w}, D^m \boldsymbol{b})(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)} d\tau.
\end{aligned}$$

e $t^{m/2}\|(D^m \boldsymbol{u}, D^m \mathbf{w}, D^m \boldsymbol{b})(\cdot,t)\|_{L^2(\mathbb{R}^2)} \to 0$ ao $t \to \infty$.

Como feito na seção anterior, por interpolação segue a demonstração dos resultados em dimensão $n = 2$.

$$t^{\gamma(2,m,q)}\|(D^m \boldsymbol{u}, D^m \mathbf{w}, D^m \boldsymbol{b})(\cdot,t)\|_{L^q(\mathbb{R}^2)} \to 0 \quad \text{as } t \to \infty.$$

Além disso, se $\chi > 0$, então

$$t^{\gamma(2,m,q)+1/2}\|D^m \mathbf{w}(\cdot,t)\|_{L^q(\mathbb{R}^2)} \to 0 \ \text{ as } \ t \to \infty,$$

onde $\gamma(2,m,q) = \frac{1}{2} + \frac{m}{2} - \frac{2}{2q}$, para cada $m \geq 0$ (inteiro) e $2 \leq q \leq \infty$.

5 UMA DESIGUALDADE FUNDAMENTAL PARA AS DERIVADAS EM $\mathbb{R}^n$ $(n = 2, 3)$

Neste capítulo apresentaremos uma desigualdade assintótica para as derivadas de ordem $m \geq 0$ das soluções de Leray do sistema magneto micropolar (2.1). Defina $\lambda := \min\{\mu, \gamma, \nu\}$, $\mathbf{z}(\cdot, t) = (\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$, $D^m\mathbf{z}(\cdot, t) := (D^m\boldsymbol{u}, D^m\mathbf{w}, D^m\boldsymbol{b})(\cdot, t)$ (para cada $m \geq 0$ inteiro) e $\zeta(\cdot, t) := \nabla \cdot \mathbf{w}(\cdot, t)$. Deste modo, descrevemos abaixo o principal resultado a ser mostrado aqui.

Teorema 5.1. *Para cada $\alpha \geq 0$, se*

$$\lambda_0(\alpha) := \limsup_{t\to\infty} t^{\alpha} \|\mathbf{z}(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty. \tag{5.1}$$

Então, para cada $m \geq 1$ inteiro, tem-se

$$\limsup_{t\to\infty} t^{\alpha+\frac{m}{2}} \|D^m\mathbf{z}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha, m) \ \ \lambda^{-m/2} \ \ \lambda_0(\alpha), \tag{5.2}$$

onde,

$$K(\alpha, m) = \min_{\delta>0} \left\{ \delta^{-1/2} \prod_{j=0}^{m} (\alpha + \frac{j}{2} + \delta)^{1/2} \right\} \tag{5.3}$$

e $n = 2, 3$.

Como consequência do teorema anterior, por interpolação pode-se obter resultados em espaços mais gerais como nos capítulos anteriores. Desta forma, os dois teoremas abaixo descrevem a versão da desigualdade fundamental (5.2) em $\dot{H}^s(\mathbb{R}^n)$ e para as derivadas em $L^q(\mathbb{R}^n)$, respectivamente (para cada $2 \leq q \leq \infty$, $s \geq 0$ e $n = 2, 3$).

Teorema 5.2. *Para cada $\alpha \geq 0$, se*

$$\limsup_{t\to\infty} t^{\alpha} \|\mathbf{z}(\cdot, t)\|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty.$$

Então para cada $s \geq 0$ (real), existe $K(\alpha, s) > 0$ constante tal que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}} \|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq K(\alpha,s)\ \lambda_0(\alpha),$$

para cada $n = 2, 3$ e $K(\alpha, s) = \min_{m>s} K(\alpha, m)^{s/m}$.

E, mais geralmente, o resultado abaixo.

Teorema 5.3. *Para cada $\alpha \geq 0$, se*

$$\lim_{t\to\infty} \sup t^{\alpha} \|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty.$$

Então

$$\limsup_{t\to\infty} t^{\kappa(n,m,q)+\alpha} \|D^m \mathbf{z}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq K(\alpha,m,n,q)\ \ \lambda^{-\kappa(n,m,q)}\ \ \lambda_0(\alpha), \tag{5.4}$$

onde $\kappa(n,m,q) = \frac{n}{4} + \frac{m}{2} - \frac{n}{2q}$ e

$$K(\alpha,m,n,q) = K(\alpha,m)^{\frac{1}{q}+\frac{1}{2}}\ K(\alpha,m+n)^{\frac{1}{2}-\frac{1}{q}},$$

para cada $2 \leq q \leq \infty$, $n = 2, 3$ e $m \geq 1$ inteiro.

5.1 Caso $n = 3$

Nesta seção iremos demonstrar (5.2) para o caso tridimensional e algumas consequências interessantes que decorrem de tal propriedade. Como dito na introdução, as contas aqui serão feitas de maneira mais direta tendo em vista que o raciocínio é inteiramente análogo ao desenvolvido, mais detalhadamente, nos capítulos anteriores.

Prova de (5.2)

Seja $t_0 > t_*$ (veja (2.2)) e $\delta > 0$. Fazendo o produto interno de $2(t-t_0)^{2\alpha+\delta}\boldsymbol{u}(\cdot,t)$, $2(t-t_0)^{2\alpha+\delta}\mathbf{w}(\cdot,t)$ e $2(t-t_0)^{2\alpha+\delta}\boldsymbol{b}(\cdot,t)$ com (1.1a), (1.1b) e (1.1c), respectivamente, integrando por partes em $[t_0, t] \times \mathbb{R}^3$ usando o fato de que $\nabla \cdot \boldsymbol{u} = 0 = \nabla \cdot \boldsymbol{b}$ e somando tudo, obtém-se

$$
\begin{aligned}
(t-t_0)^{2\alpha+\delta}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta-1}\|\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \quad t\geq t_0.
\end{aligned}
$$

Agora, dado $0<\epsilon<2$ e escolhendo $t_0>t_*$ grande o suficiente tal que (usando (5.1)) $t^{\alpha}\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\leq(\lambda_0+\epsilon)$, para todo $t>t_0$, tem-se

$$
\begin{aligned}
(t-t_0)^{2\alpha+\delta}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta-1}\tau^{-2\alpha}d\tau.
\end{aligned}
\tag{5.5}
$$

Note que

$$
(\tau-t_0)^{2\alpha+\delta-1}\tau^{-2\alpha}\leq(\tau-t_0)^{\delta-1}, \quad \text{para } \tau\geq t_0.
$$

Aplicando a desigualdade acima em (5.5), temos

$$
\begin{aligned}
(t-t_0)^{2\alpha+\delta}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\int_{t_0}^{t}(\tau-t_0)^{\delta-1}d\tau = (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta}.
\end{aligned}
$$

Ou seja

$$
\begin{aligned}
(t-t_0)^{2\alpha+\delta}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^\delta}{\delta}.
\end{aligned}
$$

Em particular, para todo $t > t_0$,

$$
(t-t_0)^{2\alpha}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \frac{2\alpha+\delta}{\delta}(\lambda_0(\alpha)+\epsilon)^2
$$

e

$$
\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \leq \frac{1}{2\lambda}(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^\delta}{\delta} \tag{5.6}
$$

O próximo passo é derivar (1.1a), (1.1b) e (1.1c) em relação a x_l, multiplicar por $2(t-t_0)^{2\alpha+\delta+1}D_l\boldsymbol{u}(x,t)$, $2(t-t_0)^{2\alpha+\delta+1}D_l\mathbf{w}(x,t)$ e $2(t-t_0)^{2\alpha+\delta+1}D_l\boldsymbol{b}(x,t)$ respectivamente, integrar por partes em $[t_0,t]\times\mathbb{R}^3$ e utilizar que $\nabla\cdot\boldsymbol{u} = 0 = \nabla\cdot\boldsymbol{b}$ para obter

$$
\begin{aligned}
(t-t_0)^{2\alpha+\delta+1}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta+1)\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+C\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|\mathbf{z}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}
\tag{5.7}
$$

Usando a desigualdade (A.7), obtém-se

$$
\begin{aligned}
(t-t_0)^{2\alpha+\delta+1}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta+1)\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+C\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|\mathbf{z}(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,t)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}
$$

Pelo teorema 2.4 e por (2.4), aumentando t_0, se necessário, tem-se

$$\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \frac{\epsilon^2}{C^2}, \quad \text{para todo } t > t_0.$$

Aplicando a desigualdade anterior em (5.7), temos

$$\begin{aligned}
(t-t_0)^{2\alpha+\delta+1}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + (2-\epsilon)\lambda\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta+1)\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}$$

Segue da última desigualdade e de (5.6), que

$$(t-t_0)^{2\alpha+1}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{(2-\epsilon)\delta\lambda}(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2 \tag{5.8}$$

e

$$\begin{aligned}
\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq \frac{1}{\delta((2-\epsilon)\lambda)^2}(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^\delta.
\end{aligned} \tag{5.9}$$

Em um processo análogo ao feito anteriormente, tem-se

$$\begin{aligned}
(t-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+2}\|D^3\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+2\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+2}\|D^2\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
\leq (2\alpha+\delta+2)\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
+C\int_{t_0}^t (\tau-t_0)^{2\alpha+\delta+1}\|D^3\mathbf{z}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\times \\
\times\left\{\|\mathbf{z}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} + \|D\mathbf{z}(\cdot,\tau)\|_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}\right\}d\tau.
\end{aligned}$$

Usando as desigualdades de Sobolev (A.8) e (A.9) do apêndice A, tem-se

$$\begin{aligned}
&(t-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+2}\|D^3\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+2\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+2}\|D^2\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq (2\alpha+\delta+2)\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+C\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+1}\|D^3\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}\|\mathbf{z}(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}(\cdot,\tau)\|^{1/2}_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}$$

Pelo teorema 2.4 e por (2.4), aumentando t_0, se necessário, tem-se

$$\begin{aligned}
&(t-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + (2-\epsilon)\lambda\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+2}\|D^3\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+2\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+2}\|D^2\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+2}\|D^2\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq (2\alpha+\delta+2)\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned} \tag{5.10}$$

Utilizando (5.9) e (5.10), temos

$$(t-t_0)^{2\alpha+2}\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\delta((2-\epsilon)\lambda)^2}(2\alpha+\delta+2)(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2 \tag{5.11}$$

e

$$\begin{aligned}
&\int_{t_0}^t(\tau-t_0)^{2\alpha+\delta+2}\|D^3\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq \frac{1}{\delta((2-\epsilon)\lambda)^3}(2\alpha+\delta+2)(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^\delta.
\end{aligned} \tag{5.12}$$

Procederemos agora por um argumento de **indução**. Para $m \geq 2$ inteiro, vamos supor já demonstradas as desigualdades[1]

$$(t-t_0)^{2\alpha+k}\|D^k\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\delta((2-\epsilon)\lambda)^k}\left\{\prod_{j=0}^{k}(2\alpha+j+\delta)\right\}(\lambda_0(\alpha)+\epsilon)^2 \tag{5.13}$$

[1] Na verdade, a hipótese de indução (5.14) será necessária somente para $k = m-1$: veja (5.18).

e

$$\int_{t_0}^{t} (\tau - t_0)^{2\alpha+k+\delta} \|D^{k+1}\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$
$$\leq \frac{1}{\delta((2-\epsilon)\lambda)^{k+1}} \left\{ \prod_{j=0}^{k} (2\alpha + j + \delta) \right\} (\lambda_0(\alpha) + \epsilon)^2 (t - t_0)^{\delta}, \tag{5.14}$$

para todo $1 \leq k \leq m-1$ inteiro e para todo $t > t_0$.[2]

Assim, queremos mostrar que para cada t_0 suficientemente grande, tem-se

$$(t - t_0)^{2\alpha+m} \|D^m \mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\delta((2-\epsilon)\lambda)^{m}} \left\{ \prod_{j=0}^{m} (2\alpha + j + \delta) \right\} (\lambda_0(\alpha) + \epsilon)^2 \tag{5.15}$$

e

$$\int_{t_0}^{t} (\tau - t_0)^{2\alpha+m+\delta} \|D^{m+1}\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)} d\tau$$
$$\leq \frac{1}{\delta((2-\epsilon)\lambda)^{m+1}} \left\{ \prod_{j=0}^{m} (2\alpha + j + \delta) \right\} (\lambda_0(\alpha) + \epsilon)^2 (t - t_0)^{\delta}, \tag{5.16}$$

para todo $t > t_0$.

Usando o mesmo tipo de argumento empregado nos capítulos anteriores (veja o Teorema 4.1), derivando m vezes as equações (1.1a), (1.1b) e (1.1c) em relação a x_{l_1}, x_{l_2} , ... x_{l_m}, multiplicando por $2(\tau - t_0)^{2\alpha+\delta+m} D_{l_1} D_{l_2} ... D_{l_m} \boldsymbol{u}$, $2(\tau - t_0)^{2\alpha+\delta+m} D_{l_1} D_{l_2} ... D_{l_m} \mathbf{w}$ e $2(\tau - t_0)^{2\alpha+\delta+m} D_{l_1} D_{l_2} ... D_{l_m} \boldsymbol{b}$ respectivamente, integrando por partes em $\mathbb{R}^3 \times (t_0, t)$ e utilizando (A.12), tem-se

[2] Para cada $t_0 > t_{**}$, para algum $t_{**} > t_*$ suficientemente grande (veja (2.2).

$$
\begin{aligned}
&(t-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^{m+1}\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^m\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq (2\alpha+\delta+m)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m-1}\|D^m\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+K_m\sum_{l=1}^{[m/2]}\int_{t_0}^{t}(\tau-t_0)^{2\alpha+m+\delta}\|\mathbf{z}(\cdot,\tau)\|^{1-\theta_{l,m}}_{L^2(\mathbb{R}^3)}\|D^{l+2}\mathbf{z}(\cdot,\tau)\|^{1-\theta_{l,m}}_{L^2(\mathbb{R}^3)}\|D^{m+1}\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau,
\end{aligned}
\tag{5.17}
$$

onde $\theta_{l,m} = \frac{1/2}{\ell+2}$ (para cada $0 \leq l \leq m-3$ inteiro) e K_m é uma constante positiva que depende somente de m. Pelos capítulos anteriores aqui apresentados, sabe-se (veja e.g. (2.4)) que $\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}$ é limitada[3] em $[t_*,\infty)$. Além disso, ainda que tenhamos provado no teorema 4.1, a hipótese de indução (5.13) implica que

$$
\|D^k\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0 \quad \text{ao} \quad t\to\infty.
$$

Logo, escolhendo $t_0 > t_*$ suficientemente grande, a desigualdade (5.17) nos fornece

$$
\begin{aligned}
&(t-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + (2-\epsilon)\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^{m+1}\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^m\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq (2\alpha+\delta+m)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m-1}\|D^m\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau.
\end{aligned}
$$

Usando a hipótese de indução (5.14), para $k = m-1$, tem-se

$$
\begin{aligned}
&(t-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} + (2-\epsilon)\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^{m+1}\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^m\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^m\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau \\
&\leq (2\alpha+\delta+m)\frac{1}{\delta((2-\epsilon)\lambda)^m}\left\{\prod_{j=0}^{m-1}(2\alpha+j+\delta)\right\}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}
\end{aligned}
$$

[3]Na verdade $\|\mathbf{z}(\cdot,t)\|_{L^2}$ vai para zero ao $t\to\infty$ e é limitada em $[0,\infty)$. Veja (2.3) e (3.1).

$$= \frac{1}{\delta((2-\epsilon)\lambda)^m}\left\{\prod_{j=0}^{m}(2\alpha+j+\delta)\right\}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^\delta. \tag{5.18}$$

Isto é,

$$(t-t_0)^{2\alpha+m}\|D^m\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\delta((2-\epsilon)\lambda)^m}\left\{\prod_{j=0}^{m}(2\alpha+j+\delta)\right\}(\lambda_0(\alpha)+\epsilon)^2 \tag{5.19}$$

e

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+m}\|D^{m+1}\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau$$
$$\leq \frac{1}{\delta((2-\epsilon)\lambda)^{m+1}}\left\{\prod_{j=0}^{m}(2\alpha+j+\delta)\right\}(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^\delta.$$

O que conclui a demonstração de (5.15) e (5.16) por indução. Como a desigualdade (5.19) é válida para todo $0<\epsilon<2$, tem-se

$$\begin{aligned}(t-t_0)^{\alpha+m/2}\|D^m\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} &\leq \frac{\delta^{-1/2}}{(2\lambda)^{m/2}}\left\{\prod_{j=0}^{m}(2\alpha+j+\delta)\right\}^{1/2}\lambda_0(\alpha)\\ &= \frac{\delta^{-1/2}2^{\frac{m+1}{2}}}{(2\lambda)^{m/2}}\left\{\prod_{j=0}^{m}(\alpha+j/2+\delta/2)\right\}^{1/2}\lambda_0(\alpha)\\ &= \frac{\delta^{-1/2}2^{\frac{1}{2}}}{\lambda^{m/2}}\left\{\prod_{j=0}^{m}(\alpha+j/2+\delta/2)\right\}^{1/2}\lambda_0(\alpha).\end{aligned}$$

Como a desigualdade anterior é válida para todo $\delta>0$, tem-se

$$(t-t_0)^{\alpha+m/2}\|D^m\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq K(\alpha,m)\ \ \lambda^{-m/2}\ \lambda_0(\alpha)$$

onde

$$K(\alpha,m)=\min_{\delta>0}\left\{\delta^{-\frac{1}{2}}\prod_{j=0}^{m}(\alpha+j/2+\delta)^{\frac{1}{2}}\right\}.$$

Note que podemos escrever

$$t^{\alpha+m/2}\left(\frac{t-t_0}{t}\right)^{\alpha+m/2}\|D^m\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}$$
$$=(t-t_0)^{\alpha+m/2}\|D^m\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq K(\alpha,m)\ \ \lambda^{-m/2}\ \lambda_0(\alpha).$$

Portanto,

$$\limsup_{t\to\infty} t^{\alpha+m/2}\|D^m\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq K(\alpha,m)\ \ \lambda^{-m/2}\ \lambda_0(\alpha),$$

para cada $m \geq 0$ inteiro. O que concluí a demonstração de (5.2) em dimensão $n = 3$.

Corolário 5.4. *Dado $\alpha > 0$, se*

$$\lim_{t\to\infty} \sup t^{\alpha}\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} =: \lambda_0(\alpha) < \infty. \tag{5.20}$$

Então para todo $s \geq 0$ real, existe $K(\alpha,s) > 0$ constante tal que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq K(\alpha,s)\ \lambda_0(\alpha),$$

onde

$$K(\alpha,s) = \min_{m>s}\{K(\alpha,m)^{s/m}\}.$$

Demonstração. Seja $m > s$ inteiro. Pelo lema B.5 do apêndice, temos que

$$\|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{s}{m}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^3)}^{\frac{s}{m}}.$$

Escrevendo $\delta = \delta_1 + \delta_2$, temos que

$$t^{\delta}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \left[t^{\frac{\delta_1}{1-\frac{s}{m}}}\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right]^{1-\frac{s}{m}}\left[t^{\frac{\delta_1}{\frac{s}{m}}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^3)}\right]^{\frac{s}{m}}. \tag{5.21}$$

Escolhendo $\delta_1 = \alpha\big(1-\frac{s}{m}\big)$ e $\delta_2\big(\alpha+\frac{m}{2}\big)\frac{s}{m}$, a equação (5.21) se torna

$$t^{\alpha+\frac{s}{2}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \left[t^{\alpha}\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)}\right]^{1-\frac{s}{m}}\left[t^{\alpha+\frac{m}{2}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^3)}\right]^{\frac{s}{m}}.$$

Logo, pelo Teorema 5.1

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq K(\alpha,s)\ \ \lambda^{-s/2}\ \ \lambda_0(\alpha),$$

onde $K(\alpha,s) = \min_{m>s} K(\alpha,m)^{s/m}$. □

5.2 Caso $n = 2$

Nesta seção, apresentaremos a demonstração resumida de (5.2) para o sistema (1.2), isto é, em dimensão $n = 2$. Considere o sistema (1.2), isto é,

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\mathsf{p} = (\mu+\chi)\,\Delta\boldsymbol{u} + \chi\,\nabla\times\mathbf{w} + \boldsymbol{b}\cdot\nabla\boldsymbol{b}, \tag{5.22a}$$

$$\mathbf{w}_t + \boldsymbol{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \chi\,\nabla\times\,\boldsymbol{u} - 2\,\chi\,\mathbf{w}, \tag{5.22b}$$

$$\boldsymbol{b}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{b} = \nu\Delta\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}, \tag{5.22c}$$

$$\nabla\cdot\boldsymbol{u}(\cdot,t) = \nabla\cdot\boldsymbol{b}(\cdot,t) = 0, \tag{5.22d}$$

Seja $t_0 > 0$ e $\delta > 0$. Fazendo o produto interno de $2(t-t_0)^{2\alpha+\delta}\boldsymbol{u}(\cdot,t)$, $2(t-t_0)^{2\alpha+\delta}\mathbf{w}(\cdot,t)$ e $2(t-t_0)^{2\alpha+\delta}\boldsymbol{b}(\cdot,t)$ com (5.22a), (5.22b) e (5.22c), respectivamente, integrando por partes em $[t_0,t]\times\mathbb{R}^2$ usando o fato de que $\nabla\cdot\boldsymbol{u} = 0 = \nabla\cdot\boldsymbol{b}$ e somando tudo, obtém-se

$$(t-t_0)^{2\alpha+\delta}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau$$
$$+2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau \leq (2\alpha+\delta)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta-1}\|\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau \quad t\geq t_0.$$

Como na seção anterior, dado $0<\epsilon<2$ e escolhendo $t_0 > 0$ grande o suficiente tal que (usando (5.1)) $t^{\alpha}\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq (\lambda_0+\epsilon)$, para todo $t > t_0$, tem-se

$$t-t_0)^{2\alpha+\delta}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau$$
$$+2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau \leq (2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta}.$$

Em particular,

$$(t-t_0)^{2\alpha}\|\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq \frac{2\alpha+\delta}{\delta}(\lambda_0(\alpha)+\epsilon)^2$$

e

$$\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau \leq \frac{1}{2\lambda}(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2\frac{(t-t_0)^{\delta}}{\delta},$$

para todo $t>t_0$.

O próximo passo é derivar (5.22a), (5.22b) e ((5.22c)) em relação a x_l, multiplicar por $2(t-t_0)^{2\alpha+\delta+1}D_l\boldsymbol{u}(x,t)$, $2(t-t_0)^{2\alpha+\delta+1}D_l\mathbf{w}(x,t)$ e $2(t-t_0)^{2\alpha+\delta+1}D_l\boldsymbol{b}(x,t)$ respectivamente, integrar por partes em $[t_0,t]\times\mathbb{R}^2$ e utilizar que $\nabla\cdot\boldsymbol{u}=0=\nabla\cdot\boldsymbol{b}$ para obter

$$\begin{aligned}
(t-t_0)^{2\alpha+\delta+1}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\\
+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|D\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\\
\leq (2\alpha+\delta+1)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^3)}d\tau\\
+C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|\mathbf{z}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\|D\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|D^2\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}d\tau.
\end{aligned}$$

Usando a desigualdade (A.13), obtém-se

$$\begin{aligned}
(t-t_0)^{2\alpha+\delta+1}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} + 2\lambda\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\\
+2\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|D\zeta(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau + 2\chi\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|D\mathbf{w}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\\
\leq (2\alpha+\delta+1)\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta}\|D\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\\
+C\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\|D^2\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)}d\tau.
\end{aligned}$$

Como na seção anterior, aumentando t_0 se necessário e usando o teorema 3.1, tem-se

$$(t-t_0)^{2\alpha+1}\|D\mathbf{z}(\cdot,t)\|^2_{L^2(\mathbb{R}^2)} \leq \frac{1}{(2-\epsilon)\delta\lambda}(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2$$

e

$$\begin{aligned}
\int_{t_0}^{t}(\tau-t_0)^{2\alpha+\delta+1}\|D^2\mathbf{z}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\\
\leq \frac{1}{\delta((2-\epsilon)\lambda)^2}(2\alpha+\delta+1)(2\alpha+\delta)(\lambda_0(\alpha)+\epsilon)^2(t-t_0)^{\delta}.
\end{aligned}$$

Repetindo o mesmo processo de indução feito anteriormente e usando as desigualdades (A.14) e (A.15), segue (5.2) em dimensão $n = 2$. De maneira análoga, por interpolação obtemos o resultado para o espaço $\dot{H}^s(\mathbb{R}^2)$.

Tendo esses resultados, podemos demonstrar (5.4) como nos capítulos anteriores. Observe que pelo Lema B.7 descrito no apêndice e por (5.2), dado $\alpha \geq 0$,

$$\limsup_{t\to\infty} t^{\alpha+\frac{m}{2}+\frac{n}{4}} \|D^m \mathbf{z}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \sqrt{K(\alpha,m)\, K(\alpha,m+n)}\ \lambda^{-\frac{m}{2}-\frac{n}{4}}\, \lambda_0(\alpha),$$

onde $K(\alpha,m)$ é dado por (5.3) e $\lambda = \min\{\mu,\gamma,\nu\}$. A partir daí, basta usar o Lema B.3 para obter (5.4).

6 A DESIGUALDADE FUNDAMENTAL PARA AS DERIVADAS CONSIDERANDO $\chi > 0$

Neste capítulo mostraremos uma versão melhorada da desigualdade fundamental apresentada anteriormente. Tal melhora deve-se ao fato de termos uma taxa mais rápida para o campo microrrotacional $\mathbf{w}$, quando $\chi > 0$. Mais especificamente, os resultados a serem provados aqui são os que seguem.

Defina, por conveniência, $\tilde{\mathbf{z}}(\cdot,t) := (\boldsymbol{u}, \boldsymbol{b})$. Dado $\alpha \geq 0$, seja

$$\tilde{\lambda}_0(\alpha) := \limsup_{t\to\infty} t^{\alpha} \|\tilde{\mathbf{z}}(\cdot,t)\|_{L^2(\mathbb{R}^n)}.$$

Se $\tilde{\lambda}_0(\alpha) < \infty$, então

$$\limsup_{t\to\infty} t^{\alpha+\frac{m}{2}} \|D^m \mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq K(\alpha,m)\lambda^{-\frac{m}{2}} \tilde{\lambda}_0(\alpha). \tag{6.1}$$

Além disso, para o campo $\mathbf{w}$, obtemos

$$\limsup_{t\to\infty} t^{\alpha+\frac{m}{2}+\frac{1}{2}} \|D^m \mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \frac{1}{2} K(\alpha, m+1) \;\; \lambda^{-\frac{m+1}{2}} \tilde{\lambda}_0(\alpha), \tag{6.2}$$

onde $K(\alpha,m)$ é dado em (5.3) e $\lambda = \min\{\mu,\gamma,\nu\}$. Vamos abordar primeiramente o caso 3D.

6.1 Caso $n = 3$

Antes de obtermos os resultados (6.1) e (6.2) em dimensão $n = 3$, será conveniente demonstrar o seguinte resultado. Defina $\lambda := \min\{\mu, \gamma, \nu\}$ e $\mathbf{z}(\cdot, t) := (\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})(\cdot, t)$, $D\mathbf{z}(\cdot, t) := (D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})$ e assim por diante.

Teorema 6.1. *Suponha que $\chi > 0$. Dado $\alpha > 0$, se*

$$\limsup_{t\to\infty} t^{\alpha} \|\mathbf{z}(\cdot, t)\|_{L^2(\mathbb{R}^3)} =: \lambda_0(\alpha) < \infty. \tag{6.3}$$

Então para todo $s \geq 0$ real, existe $K(\alpha, s) > 0$ constante tal que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}} \|\mathbf{w}(\cdot, t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} \left(\frac{K(\alpha, m+1)}{2}\right)^{\frac{s}{m}} \lambda^{-\frac{s}{2}}\, \lambda_0(\alpha).$$

Demonstração. A demonstração deste teorema é semelhante a do Teorema 4.5.

Reescrevendo a equação (1.1b) como

$$w_t = \gamma\Delta\mathbf{w} + \mathbf{Q}(\cdot, t) - 2\chi\mathbf{w} \tag{6.4}$$

Onde

$$\mathbf{Q}(\cdot, t) = -(\boldsymbol{u} \cdot \nabla)\mathbf{w}(\cdot, t) + \nabla(\nabla \cdot \mathbf{w})(\cdot, t) + \chi\nabla \wedge \boldsymbol{u}(\cdot, t).$$

Fazendo a mudança de variável

$$\mathbf{W}(\cdot, t) = e^{2\chi t}\mathbf{w}(\cdot, t)$$

e multiplicando a equação (6.4) por $e^{2\chi t}$, obtemos que

$$\mathbf{W}_t = \gamma\Delta\mathbf{W} + e^{2\chi t}\mathbf{Q}(\cdot,t).$$

Pelo o que foi estudado nos capítulos anteriores, em particular pelo Teorema 4.1, dado $\epsilon > 0$, existe $t_0 > t_*$ suficientemente grande tal que para todo $l \in \mathbb{N}$, $1 \leq l \leq m+2$ vale que

$$\|D^l\mathbf{z}(\cdot,t)\|_{L^2(\mathbb{R}^3)} < \epsilon t^{-\frac{l}{2}} \text{ para todo } t > t_0.$$

Pelo princípio de Duhamel, temos que

$$\mathbf{W}(\cdot,t) = e^{\gamma\Delta(t-t_0)}\mathbf{W}(\cdot,t_0) + \int_{t_0}^{t} e^{\gamma\Delta(t-s)}e^{2\chi s}\mathbf{Q}(\cdot,\tau)d\tau.$$

Que pode ser reescrito como

$$\mathbf{w}(\cdot,t) = e^{-2\chi(t-t_0)}e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0) - \int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}(\boldsymbol{u}\cdot\nabla)\mathbf{w}(\cdot,\tau)d\tau$$
$$+\int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}\nabla(\nabla\cdot\mathbf{w})(\cdot,\tau)d\tau + \chi\int_{t_0}^{t} e^{-2\chi(t-\tau)}e^{\gamma\Delta(t-\tau)}(\nabla\wedge\boldsymbol{u})(\cdot,t)d\tau.$$

Multiplicando por $t^{\alpha+\frac{m+1}{2}}$, derivando m vezes, aplicando a norma L^2 e usando a Desigualdade de Minkowski, temos que

$$
\begin{aligned}
t^{\alpha+\frac{m+1}{2}}\|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq{}& \underbrace{t^{\alpha+\frac{m+1}{2}}e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}}_{I}\\
&+\underbrace{t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau}_{II}\\
&+\underbrace{t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau}_{III}\\
&+\underbrace{\chi t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau}_{IV}.
\end{aligned}
\tag{6.5}
$$

Vamos analisar cada termo separadamente.

O termo I representa a solução da equação do calor, com condição inicial $\mathbf{w}(\cdot,t_0)$ e, pelo Teorema C.1,

$$\lim_{t\to\infty}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0.$$

De modo que

$$\lim_{t\to\infty} t^{\alpha+\frac{m+1}{2}}e^{-2\chi(t-t_0)}\|D^m e^{\gamma\Delta(t-t_0)}\mathbf{w}(\cdot,t_0)\|_{L^2(\mathbb{R}^3)} = 0. \tag{6.6}$$

Para analisar os demais termos, vamos utilizar a desigualdade (C.2) e o Teorema 5.1.

Analisando o termo II temos

$$
\begin{aligned}
&t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau\\
&\leq Ct^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\sum_{l=0}^{m}\|D^l\boldsymbol{u}(\cdot,\tau)\|_{L^4(\mathbb{R}^3)}\|D^{m-l+1}\mathbf{w}(\cdot,\tau)\|_{L^4(\mathbb{R}^3)}d\tau\\
&\leq Ct^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\sum_{l=0}^{m}\|D^l\boldsymbol{u}(\cdot,\tau)\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^{l+1}\boldsymbol{u}(\cdot,\tau)\|^{3/4}_{L^2(\mathbb{R}^3)}\\
&\qquad\times\|D^{m-l+1}\mathbf{w}(\cdot,\tau)\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^{m-l+2}\mathbf{w}(\cdot,\tau)\|^{3/4}_{L^2(\mathbb{R}^3)}d\tau\\
&\leq C\,(\lambda_0(\alpha)+\epsilon)t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\sum_{l=0}^{m}\tau^{-\frac{\alpha}{4}-\frac{l}{8}}\tau^{-\frac{3\alpha}{4}-\frac{-3(l+1)}{8}}\epsilon\tau^{-\frac{m-l+1}{8}}\epsilon\tau^{\frac{-3(m-l+2)}{8}}d\tau\\
&\leq C\,(\lambda_0(\alpha)+\epsilon)t^{\alpha+\frac{m+1}{2}}\epsilon\int_{t_0}^{t} e^{-2\chi(t-\tau)}\tau^{-\alpha}\tau^{-\frac{2m+5}{4}}d\tau\\
&= C\,(\lambda_0(\alpha)+\epsilon)t^{\alpha+\frac{m+1}{2}}\epsilon\left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)}\tau^{-\alpha}\tau^{-\frac{2m+5}{4}}d\tau+\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}\tau^{-\alpha}\tau^{-\frac{2m+5}{4}}d\tau\right]\\
&\leq C\,(\lambda_0(\alpha)+\epsilon)\epsilon\left(t^{\alpha+\frac{m+1}{2}}e^{-\chi t}+t^{-\frac{3}{4}}\chi^{-1}\right).
\end{aligned}
$$

Como $\epsilon>0$ é arbitrário, temos que

$$
\lim_{t\to\infty} t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|e^{\gamma\Delta(t-\tau)}D^m(\boldsymbol{u}\cdot\nabla\mathbf{w})(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau=0. \tag{6.7}
$$

Podemos usar o mesmo raciocínio para o termo III,

$$
\begin{aligned}
&t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-s)}\|e^{\gamma\Delta(t-\tau)}D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau\\
&\leq C\,t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\|D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)}d\tau\\
&\leq C(\lambda_0(\alpha)+\epsilon)t^{\alpha+\frac{m+1}{2}}\int_{t_0}^{t} e^{-2\chi(t-\tau)}\tau^{-\alpha}\tau^{-\frac{m+2}{2}}d\tau\\
&= k(\lambda_0(\alpha)+\epsilon)t^{\alpha+\frac{m+1}{2}}\left[\int_{t_0}^{\frac{t}{2}} e^{-2\chi(t-\tau)}\tau^{-\alpha}\tau^{-\frac{m+2}{2}}d\tau+\int_{\frac{t}{2}}^{t} e^{-2\chi(t-\tau)}\tau^{-\alpha}\tau^{-\frac{m+2}{2}}d\tau\right]\\
&\leq C(\lambda_0(\alpha)+\epsilon)\left(t^{\alpha+\frac{m+1}{2}}e^{-\chi t}+t^{-\frac{1}{2}}\chi^{-1}\right).
\end{aligned}
$$

Novamente, temos que

$$\lim_{t\to\infty} t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^{m+2}\mathbf{w}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau = 0. \tag{6.8}$$

Por último, analisaremos o termo IV. Para isso, vamos fixar $0<\sigma<1$, então

$$\begin{aligned}
\chi t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
\leq \chi\ t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
\leq K(\alpha,m+1)\ \lambda^{-\frac{m+1}{2}}\ \chi\ t^{\alpha+\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} (\lambda_0(\alpha)+\epsilon)\tau^{-\alpha-\frac{m+1}{2}} d\tau \\
= \Xi_{m,\alpha,\epsilon}\ \chi\ t^{\alpha+\frac{m+1}{2}} \left[\int_{t_0}^{\sigma t} e^{-2\chi(t-\tau)} \tau^{-\alpha-\frac{m+1}{2}} d\tau + \int_{\sigma t}^{t} e^{-2\chi(t-\tau)} \tau^{-\alpha-\frac{m+1}{2}} d\tau \right] \\
\leq \Xi_{m,\alpha,\epsilon}\ \chi\ t^{\alpha+\frac{m+1}{2}} \left[e^{-2\chi t(1-\sigma)} \int_{t_0}^{\sigma t} \tau^{-\alpha-\frac{m+1}{2}} d\tau + (\sigma\, t)^{-\alpha-\frac{m+1}{2}} \int_{\sigma t}^{t} e^{-2\chi(t-\tau)} d\tau \right] \\
\leq \Xi_{m,\alpha,\epsilon}\ \chi\ \Bigg[t^{\alpha+\frac{m+1}{2}} e^{-2\chi t(1-\sigma)} \left(2\frac{t_0^{-\alpha-\frac{m-1}{2}} - (\sigma\, t)^{-\alpha-\frac{m-1}{2}}}{2\alpha+m-1} \right) \\
+\sigma^{-\alpha-\frac{m+1}{2}} \left(\frac{1-e^{-2\chi t(1-\sigma)}}{2\chi} \right) \Bigg],
\end{aligned}$$

onde

$$\Xi_{m,\alpha,\epsilon} := K(\alpha,m+1)\ \lambda^{-\frac{m+1}{2}}\ (\lambda_0(\alpha)+\epsilon).$$

Fazendo $\epsilon \to 0$, $t\to\infty$ e $\sigma \to 1$ (nesta ordem), obtemos que

$$\limsup_{t\to\infty} t^{\frac{m+1}{2}} \int_{t_0}^{t} e^{-2\chi(t-\tau)} \|e^{\gamma\Delta(t-\tau)} D^{m+1}\boldsymbol{u}(\cdot,\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq \frac{1}{2} K(\alpha,m+1)\ \lambda^{-\frac{m+1}{2}}\ \lambda_0(\alpha). \tag{6.9}$$

Pelas equações (6.5), (6.6), (6.7), (6.8) e (6.9), temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{m+1}{2}} \|D^m\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{2} K(\alpha,m+1)\ \lambda^{-\frac{m+1}{2}}\ \lambda_0(\alpha).$$

para todo $m > 0$ inteiro.

Fazendo uma interpolação semelhante a do Corolário 5.4, obtemos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}} \|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} \left(\frac{K(\alpha, m+1)}{2} \right)^{\frac{s}{m}} \lambda_0(\alpha).$$

para todo $s > 0$ (real). $\square$

Em posse do Teorema 6.1, podemos enfraquecer a hipótese de que $\lambda_0(\alpha) < \infty$ e obter uma versão mais forte do Teorema 5.1 e do próprio Teorema 6.1. Mais precisamente, obtemos o importante resultado a seguir.

Teorema 6.2. *Suponha que $\chi > 0$. Dado $\alpha \geq 0$, seja $\tilde{\mathbf{z}}(\cdot,t) = (\boldsymbol{u}, \boldsymbol{b})(\cdot,t)$ e*

$$\tilde{\lambda}_0(\alpha) := \limsup_{t\to\infty} t^{\alpha} \|\tilde{\mathbf{z}}(\cdot,t)\|_{L^2(\mathbb{R}^3)}.$$

Se

$$\tilde{\lambda}_0(\alpha) < \infty. \tag{6.10}$$

Então para todo $s \geq 0$ real, temos que

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}} \|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}} \ \lambda^{-\frac{s}{2}} \ \tilde{\lambda}_0(\alpha)$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}} \|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} \left(\frac{K(\alpha, m+1)}{2} \right)^{\frac{s}{m}} \ \lambda^{-\frac{s+1}{2}} \ \tilde{\lambda}_0(\alpha).$$

Demonstração. Se $0 < \alpha \leq 1/2$, pelo Teorema 3.2, temos que $t^{\alpha}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$. Portanto, $\lambda_0(\alpha) < \infty$ e pelos Teoremas 5.1 e 6.1, tem-se

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}} \|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}} \ \lambda^{-\frac{s}{2}} \ \tilde{\lambda}_0(\alpha)$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s}\left(\frac{K(\alpha,m+1)}{2}\right)^{\frac{s}{m}} \lambda^{-\frac{s+1}{2}}\ \tilde{\lambda}_0(\alpha)$$

e o teorema está provado para este caso. Se $\alpha > 1/2$, temos (por hipótese) que $t^{1/2}\|\tilde{\mathbf{z}}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$. Pelos Teoremas 3.2 e 6.1, temos que

$$\limsup_{t\to\infty} t^{\frac{1}{2}+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s}\left(\frac{K(1/2,m+1)}{2}\right)^{\frac{s}{m}} \lambda^{-\frac{s+1}{2}}\ \limsup_{t\to\infty} t^{\frac{1}{2}}\|\tilde{\mathbf{z}}(\cdot,t)\|_{L^2(\mathbb{R}^3)}. \tag{6.11}$$

Assim, se $\alpha \leq 1$, então por (6.11), temos que $t^{\alpha}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$. Portanto, $\lambda_0(\alpha) < \infty$ e pelos Teoremas 5.1 e 6.1,

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}}\ \lambda^{-\frac{s}{2}}\ \tilde{\lambda}_0(\alpha)$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s}\left(\frac{K(\alpha,m+1)}{2}\right)^{\frac{s}{m}} \lambda^{-\frac{s+1}{2}}\ \tilde{\lambda}_0(\alpha)$$

Vamos fazer explicitamente mais um caso.

Se $\alpha > 1$, temos que $t\|\tilde{\mathbf{z}}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$. Por (6.11) e 6.1 Temos que

$$\limsup_{t\to\infty} t^{1+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq K(1,s)\ \lambda^{-\frac{s+1}{2}} \limsup_{t\to\infty} t\|\tilde{\mathbf{z}}(\cdot,t)\|_{L^2(\mathbb{R}^3)}. \tag{6.12}$$

Assim se $\alpha \leq 3/2$, então por (6.12), temos que $t^{\alpha}\|\mathbf{w}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \to 0$. Portanto $\lambda_0(\alpha) < \infty$ e pelos teoremas 5.1 e 6.1,

$$\limsup_{t\to\infty} t^{\alpha+\frac{s}{2}}\|\mathbf{z}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s} K(\alpha,m)^{\frac{s}{m}}\ \lambda^{-\frac{s}{2}}\ \tilde{\lambda}_0(\alpha)$$

e

$$\limsup_{t\to\infty} t^{\alpha+\frac{s+1}{2}}\|\mathbf{w}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^3)} \leq \min_{m>s}\left(\frac{K(\alpha,m+1)}{2}\right)^{\frac{s}{m}} \lambda^{-\frac{s+1}{2}}\ \tilde{\lambda}_0(\alpha)$$

Se $\alpha > 3/2$, repetimos este mesmo raciocínio. Em no máximo $2[\alpha] + 1$ casos, obtemos o resultado desejado. $\square$

6.2 Caso $n = 2$

A demostração dos resultados expostos anteriormente no caso $n = 2$ é similar ao caso 3D. As únicas diferenças está nas estimativas para o *Heat Kernel* que estão descritas no apêndice C. No entanto, isto não afeta a demostração do resultado (da mesma forma que ocorreu nos capítulos anteriores). Deste modo, a prova dos resultados para o caso bidimensional segue idêntica ao caso 3D.

De maneira inteiramente análoga a feita nos capítulos anteriormente desenvolvidos, pode-se mostrar os resultados mais gerais abaixo, por interpolação. Para cada $\alpha \geq 0$, se

$$\lim_{t\to\infty} \sup t^{\alpha} \|\mathbf{z}(\cdot, t)\|_{L^2(\mathbb{R}^n)} =: \tilde{\lambda}_0(\alpha) < \infty.$$

Então

$$\limsup_{t\to\infty} t^{\kappa(n,m,q)+\alpha} \|D^m \mathbf{z}(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq K(\alpha, m, n, q) \ \ \lambda^{-\kappa(n,m,q)} \ \ \tilde{\lambda}_0(\alpha),$$

onde $\kappa(n, m, q) = \frac{n}{4} + \frac{m}{2} - \frac{n}{2q}$ e

$$K(\alpha, m, n, q) = K(\alpha, m)^{\frac{1}{q}+\frac{1}{2}} \ K(\alpha, m + n)^{\frac{1}{2}-\frac{1}{q}},$$

para cada $2 \leq q \leq \infty$, $\ n = 2, 3\ $ e $\ m \geq 1$ inteiro.

Apêndice A O ESPAÇO DE SOBOLEV HOMOGÊNEO $\dot{H}^S$

Nesta seção, daremos uma rápida noção dos espaços que aqui foram considerados, a começar pelo espaço de Lebesgue L^p que é o espaço "trivial"de Sobolev que, como sabemos, é o espaço das funções f tal que, $\int |f|^p dx < \infty$, para $p \in [1, \infty)$. No caso $p = \infty$, a norma usada é a norma do sup, ou seja, $\|u\|_{L^\infty} = \inf\{C \in \mathbb{R}^+ \ : \ |f| < C\}$ e a medida do conjunto $\{x \ : \ |f(x)| > C\}$ é nula. Para um estudo mais construtivo e detalhado desses espaços e uma boa abordagem a respeito da sua estrutura veja [2].

A partir daí, podemos introduzir os espaços de Sobolev $W^{m,p}(\Omega)$ (que também denotaremos ao longo de nosso texto por $H^m(\Omega)$ quando $p = 2$), onde Ω é um aberto em $\mathbb{R}^n$, significando que a função que pertence a esse espaço e todas as suas derivadas espaciais (fracas) de até ordem m pertencem a L^p, onde $m \geq 1$ inteiro e $1 \leq p \leq \infty$. Tendo isso, introduziremos o espaço de Sobolev homogêneo $\dot{H}^m$.

Definição A.1. Os espaços homogêneos de Sobolev $\dot{W}^{m,p}$ (similarmente $\dot{H}^m$) é o espaço das funções m vezes fracamente diferenciáveis f tal que $D^\alpha f \in L^p(\Omega)$, em outras palavras,

$$\|f\|_{\dot{W}^{m,p}(\Omega)} = \left(\sum_{i_1, i_2, \ldots, i_n = 1}^{m} \int_\Omega |D_{i_1} D_{i_2} \ldots D_{i_n} f(x)|^p dx \right)^{1/p} < \infty.$$

Podemos ainda generalizar esse conceito via transformada de Fourier.

Definição A.2. Dado $s > 0$, denotaremos por $H^s(\mathbb{R}^n)$, o espaço das funções f, tais que,

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty,$$

onde $\hat{f}(\xi)$ é a Transformada de Fourier abaixo,

$$\hat{\mathbf{u}}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix.\xi} \mathbf{u}(x) dx.$$

Esses espaços homogêneos preservam boas propriedades de escala, o que motiva seu estudo. Apresentaremos a seguir uma série de resultados que nos auxiliarão para maior compreensão do texto.

Primeiramente, precisaremos das seguintes desigualdades elementares de Sobolev-Nirenberg-Gagliardo (SNG) para funções $\mathrm{u} \in H^2(\mathbb{R}^3)$ quaisquer:

$$\| \mathrm{u} \|_{L^\infty(\mathbb{R}^3)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/4} \| D^2 \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{3/4}, \tag{A.1}$$

veja e.g. { [46], Proposition 2.4, p. 5 }, ou { [42], Teorema 4.5.1, p. 52 }; e

$$\| D\mathrm{u} \|_{L^2(\mathbb{R}^3)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^2 \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/2}, \tag{A.2}$$

facilmente obtida usando a transformada de Fourier. De (A.1), (A.2), obtém-se

$$\|\mathrm{u}\|_{L^\infty(\mathbb{R}^3)} \| D\mathrm{u} \|_{L^2(\mathbb{R}^3)} \leq \| \mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/2} \| D\mathrm{u} \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^2 \mathrm{u} \|_{L^2(\mathbb{R}^3)}, \tag{A.3}$$

Para demonstrar (A.3), basta notar que

$$\begin{aligned}
\| \boldsymbol{u}(\cdot,s) \|_{L^\infty(\mathbb{R}^3)} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} &\leq \| \boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)}^{1/4} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D^2\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)}^{3/4} \\
&= \| D^2\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \Big[\| \boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)}^{1/4} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \| D^2\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)}^{-1/4} \Big] \\
&\leq \| D^2\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)} \Big[\| \boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)}^{1/2} \| D\boldsymbol{u}(\cdot,s) \|_{L^2(\mathbb{R}^3)}^{1/2} \Big].
\end{aligned}$$

No entanto, nosso objetivo é derivar desigualdades do tipo (A.3) para $(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})$. Observe que, por definição (veja 1.7, na introdução), temos

$$\|\boldsymbol{u}\|_{L^q(\mathbb{R}^3)} \leq \|(\boldsymbol{u}, \mathrm{w}, \boldsymbol{b})\|_{L^q(\mathbb{R}^3)}, \text{ para } 1 \leq q \leq \infty.$$

Com isso, por (A.1), tem-se

$$\begin{aligned}
\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)} &\leq \|(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{1/4}\|(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{3/4}\\
\|\mathrm{w}\|_{L^\infty(\mathbb{R}^3)} &\leq \|(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{1/4}\|(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{3/4}\\
\|\boldsymbol{b}\|_{L^\infty(\mathbb{R}^3)} &\leq \|(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{1/4}\|(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{3/4},
\end{aligned}$$

o que resulta

$$\begin{aligned}
\|(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^3)} &= \max\{\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)},\|\mathrm{w}\|_{L^\infty(\mathbb{R}^3)},\|\boldsymbol{b}\|_{L^\infty(\mathbb{R}^3)}\}\\
&\leq \|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)}+\|\mathrm{w}\|_{L^\infty(\mathbb{R}^3)}+\|\boldsymbol{b}\|_{L^\infty(\mathbb{R}^3)}\\
&\leq 3\|(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{1/4}\|(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{3/4},
\end{aligned}$$

ou seja,

$$\|(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\|_{L^\infty(\mathbb{R}^3)} \leq 3\|(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{1/4}\|(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}^{3/4}. \tag{A.4}$$

Analogamente, por (A.2), tem-se

$$\begin{aligned}
\|\, D\boldsymbol{u}\,\|_{L^2(\mathbb{R}^3)} &\leq \|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\,\|\,(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\\
\|\, D\mathrm{w}\,\|_{L^2(\mathbb{R}^3)} &\leq \|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\,\|\,(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\\
\|\, D\boldsymbol{b}\,\|_{L^2(\mathbb{R}^3)} &\leq \|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\,\|\,(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}.
\end{aligned}$$

Donde,

$$\begin{aligned}
\|(D\boldsymbol{u},D\mathrm{w},D\boldsymbol{b})\|_{L^2(\mathbb{R}^3)} &= \left(\|\, D\boldsymbol{u}\,\|_{L^2(\mathbb{R}^3)}^2+\|\, D\mathrm{w}\,\|_{L^2(\mathbb{R}^3)}^2+\|\, D\boldsymbol{b}\,\|_{L^2(\mathbb{R}^3)}^2\right)^{1/2}\\
&\leq \|\, D\boldsymbol{u}\,\|_{L^2(\mathbb{R}^3)}+\|\, D\mathrm{w}\,\|_{L^2(\mathbb{R}^3)}+\|\, D\boldsymbol{b}\,\|_{L^2(\mathbb{R}^3)}\\
&\leq 3\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\,\|\,(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2},
\end{aligned}$$

ou seja,

$$\|(D\boldsymbol{u},D\mathrm{w},D\boldsymbol{b})\|_{L^2(\mathbb{R}^3)} \leq 3\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\,\|\,(D^2\boldsymbol{u},D^2\mathrm{w},D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}. \tag{A.5}$$

Similarente, no caso geral

$$\begin{aligned}
&\|(D^m\boldsymbol{u},D^m\mathrm{w},D^m\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}\\
&\leq 3\|\,(D^{m-1}\boldsymbol{u},D^{m-1}\mathrm{w},D^{m-1}\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}\,\|\,(D^{m+1}\boldsymbol{u},D^{m+1}\mathrm{w},D^{m+1}\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}^{1/2}
\end{aligned} \tag{A.6a}$$

e

$$\begin{aligned}&\|(D^l\boldsymbol{u}, D^l\mathrm{w}, D^l\boldsymbol{b})\|_{L^2(\mathbb{R}^3)}\\ &\le 3\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|^{1-\theta}_{L^2(\mathbb{R}^3)}\,\|\,(D^m\boldsymbol{u}, D^m\mathrm{w}, D^m\boldsymbol{b})\,\|^{\theta}_{L^2(\mathbb{R}^3)},\ \text{onde } \theta = l/m,\end{aligned} \tag{A.6b}$$

para $0 \le l \le m$. O que consequentemente nos fornece o lema a seguir.

Lema A.3.

$$\begin{aligned}&\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|_{L^\infty(\mathbb{R}^3)}\|\,(D\boldsymbol{u}, D\mathrm{w}, D\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\\ &\le\ C\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D\boldsymbol{u}, D\mathrm{w}, D\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathrm{w}, D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{A.7}$$

$$\begin{aligned}&\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|_{L^\infty(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathrm{w}, D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\\ &\le\ C\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D\boldsymbol{u}, D\mathrm{w}, D\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D^3\boldsymbol{u}, D^3\mathrm{w}, D^3\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{A.8}$$

$$\begin{aligned}&\|\,(D\boldsymbol{u}, D\mathrm{w}, D\boldsymbol{b})\,\|_{L^\infty(\mathbb{R}^3)}\|\,(D\boldsymbol{u}, D\mathrm{w}, D\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\\ &\le\ C\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D\boldsymbol{u}, D\mathrm{w}, D\boldsymbol{b})\,\|^{1/2}_{L^2(\mathbb{R}^3)}\|\,(D^3\boldsymbol{u}, D^3\mathrm{w}, D^3\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{A.9}$$

$$\begin{aligned}&\|\,(D\boldsymbol{u}, D\mathrm{w}, D\boldsymbol{b})\,\|_{L^\infty(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathrm{w}, D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\\ &\le\ C\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|^{3/4}_{L^2(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathrm{w}, D^2\boldsymbol{b})\,\|^{1/4}_{L^2(\mathbb{R}^3)}\|\,(D^4\boldsymbol{u}, D^4\mathrm{w}, D^4\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{A.10}$$

$$\begin{aligned}&\|\,(D^2\boldsymbol{u}, D^2\mathrm{w}, D^2\boldsymbol{b})\,\|_{L^\infty(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathrm{w}, D^2\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\\ &\le\ C\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|^{3/4}_{L^2(\mathbb{R}^3)}\|\,(D^2\boldsymbol{u}, D^2\mathrm{w}, D^2\boldsymbol{b})\,\|^{1/4}_{L^2(\mathbb{R}^3)}\|\,(D^5\boldsymbol{u}, D^5\mathrm{w}, D^5\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{A.11}$$

e, para $m \ge 3,\ 0 \le \ell \le m-3$*:*

$$\begin{aligned}&\|\,(D^\ell\boldsymbol{u}, D^\ell\mathrm{w}, D^\ell\boldsymbol{b})\,\|_{L^\infty(\mathbb{R}^3)}\|\,(D^{m-\ell}\boldsymbol{u}, D^{m-\ell}\mathrm{w}, D^{m-\ell}\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)}\\ &\le\ C\|\,(\boldsymbol{u},\mathrm{w},\boldsymbol{b})\,\|^{\frac{\ell+3/2}{\ell+2}}_{L^2(\mathbb{R}^3)}\|\,(D^{\ell+2}\boldsymbol{u}, D^{\ell+2}\mathrm{w}, D^{\ell+2}\boldsymbol{b})\,\|^{\frac{1/2}{\ell+2}}_{L^2(\mathbb{R}^3)}\|\,(D^{m+1}\boldsymbol{u}, D^{m+1}\mathrm{w}, D^{m+1}\boldsymbol{b})\,\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{A.12}$$

Para algum $C > 0$.

Demonstração. Defina $\mathbf{z} := (\boldsymbol{u}, \mathbf{w}, \boldsymbol{b})$, $D\mathbf{z} = (D\boldsymbol{u}, D\mathbf{w}, D\boldsymbol{b})$ e assim por diante. Para mostrar (A.7), observe que, por (A.4) e por (A.5), tem-se

$$\begin{aligned}
\| \mathbf{z} \|_{L^\infty(\mathbb{R}^3)} \| D\mathbf{z} \|_{L^2(\mathbb{R}^3)} &\leq 3\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\| D\mathbf{z} \|_{L^2(\mathbb{R}^3)} \\
&= 3\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\| D\mathbf{z} \|^{1/2}_{L^2(\mathbb{R}^3)}\| D\mathbf{z} \|^{1/2}_{L^2(\mathbb{R}^3)} \\
&\leq 3\sqrt{3}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\| D\mathbf{z} \|^{1/2}_{L^2(\mathbb{R}^3)} \\
&= 3\sqrt{3}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Vamos demonstrar agora (A.8). Por (A.4), (A.6) e (A.7),

$$\begin{aligned}
\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)} \\
&= 3\|\mathbf{z}\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)} \\
&\leq 3\sqrt{3\sqrt{3}}\|\mathbf{z}\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)} \\
&\leq 3\sqrt{3}\sqrt{3\sqrt{3}}\|\mathbf{z}\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)} \\
&= 3\sqrt{3}\sqrt{3\sqrt{3}}\|\mathbf{z}\|^{1/2}_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \\
&\leq 3\sqrt{3}\sqrt{3\sqrt{3}}\sqrt{3\sqrt{3}}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \\
&= 3\sqrt{3}\sqrt{3\sqrt{3}}\sqrt{3\sqrt{3}}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \\
&\leq 3\sqrt{3}\sqrt{3}\sqrt{3\sqrt{3}}\sqrt{3\sqrt{3}}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)} \\
&= 27\sqrt{3}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Vamos demonstrar agora (A.9). Analogamente, usando (A.5), (A.4) para $D\mathbf{z}$ e (A.6), tem-se

$$\begin{aligned}
\|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)} \\
&\leq 3^2\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)} \\
&\leq 3^2\sqrt{3}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)} \\
&= 3^2\sqrt{3}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Usando (A.1) para $D\mathbf{z}$, (A.5) e (A.6), tem-se

$$\begin{aligned}
\|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)} &= \|D\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3\sqrt{3}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\\
&= 3\sqrt{3}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^3\mathbf{z}\|_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\sqrt{3}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&= 3^2\sqrt{3}\|D\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\sqrt{3}\sqrt{3}\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&\leq 3^4\|\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|_{L^2(\mathbb{R}^3)}\\
&= 3^4\|\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|_{L^2(\mathbb{R}^3)},
\end{aligned}$$

o que concluí a prova de (A.10). Para verificar (A.11), procedemos da seguinte maneira: usamos (A.4) para $D^2\mathbf{z}$ e (A.6), o que nos fornece,

$$\begin{aligned}
\|D^2\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\\
&= 3^2\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\|D^4\mathbf{z}\|^{5/4}_{L^2(\mathbb{R}^3)}\\
&\leq 3^2\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^3)}\left(3\|\mathbf{z}\|^{1/5}_{L^2(\mathbb{R}^3)}\|D^5\mathbf{z}\|^{4/5}_{L^2(\mathbb{R}^3)}\right)^{5/4}\\
&= 3^{13/4}\|\mathbf{z}\|^{3/4}_{L^2(\mathbb{R}^3)}\|D^2\mathbf{z}\|^{1/4}_{L^2(\mathbb{R}^3)}\|D^5\mathbf{z}\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Similarmente, no caso geral, dado $m \geq 3$ e $\ell = \{0, 1, \ldots, m-3\}$, tem-se

$$\begin{aligned}
\|D^{\ell}\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^{m-\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)} &\leq 3\|D^{\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|D^{m-\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)} \\
&\leq 3^2\|D^{\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{1/4}\|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{\ell+1}{m+1}}\|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m-\ell}{m+1}} \\
\leq 3^{9/4}\left(\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m+1-\ell}{m+1}}\|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{\ell}{m+1}}\right)^{1/4} & \|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{\ell+1}{m+1}}\|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m-\ell}{m+1}} \\
&= 3^{9/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}\frac{m+3\ell+5}{m+1}}\|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{3/4}\|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m-\frac{3}{4}\ell}{m+1}} \\
= 3^{9/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}\frac{m+3\ell+5}{m+1}}\|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}(1-\beta)}\|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}\beta} & \|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m-\frac{3}{4}\ell}{m+1}} \;(\; \beta \in [0,1] \text{ a ser escolhido }) \\
\leq 3^{9/4}\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m/4+\frac{3\ell}{4}+5/4}{m+1}}\|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}(1-\beta)} & \left(3\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m-\ell-1}{m+1}}\|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{\ell+2}{m+1}}\right)^{\frac{3\beta}{4}}\|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{m-\frac{3}{4}\ell}{m+1}}.
\end{aligned}$$

Logo, devemos escolher β tal que

$$\frac{3\beta}{4}\frac{\ell+2}{m+1} + \frac{m-\frac{3}{4}\ell}{m+1} = 1,$$

i.e.,

$$\beta = \frac{\frac{3\ell}{4}+1}{\frac{3\ell}{4}+3/2} \in (0,1).$$

O que implica,

$$\|D^{\ell}\mathbf{z}\|_{L^\infty(\mathbb{R}^3)}\|D^{m-\ell}\mathbf{z}\|_{L^2(\mathbb{R}^3)} \leq K(\ell)\|\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{\ell+3/2}{\ell+2}}\|D^{\ell+2}\mathbf{z}\|_{L^2(\mathbb{R}^3)}^{\frac{1/2}{\ell+2}}\|D^{m+1}\mathbf{z}\|_{L^2(\mathbb{R}^3)},$$

onde

$$K(\ell) = 3^{9/4}3^{\frac{3}{4}\frac{\frac{3\ell}{4}+1}{\frac{3\ell}{4}+3/2}} = 3^{\frac{3}{4}\frac{\frac{3\ell}{4}+1}{\frac{3\ell}{4}+3/2}+9/4}.$$

O que conclui a demonstração. □

Em dimensão $n = 2$, as desigualdades apresentadas no lema acima ficam mais simples.

Lema A.4.

$$\|\,\mathbf{z}\,\|_{L^\infty(\mathbb{R}^2)}\|\,D\mathbf{z}\,\|_{L^2(\mathbb{R}^2)} \leq\; C\|\,\mathbf{z}\,\|_{L^2(\mathbb{R}^2)}\|\,D^2\mathbf{z}\,\|_{L^2(\mathbb{R}^2)}, \tag{A.13}$$

$$\| \mathbf{z} \|_{L^\infty(\mathbb{R}^2)} \| D^2\mathbf{z} \|_{L^2(\mathbb{R}^2)} \leq C \| \mathbf{z} \|_{L^2(\mathbb{R}^2)} \| D^3\mathbf{z} \|_{L^2(\mathbb{R}^2)}, \tag{A.14}$$

$$\| D\mathbf{z} \|_{L^\infty(\mathbb{R}^2)} \| D\mathbf{z} \|_{L^2(\mathbb{R}^2)} \leq C \| \mathbf{z} \|_{L^2(\mathbb{R}^2)} \| D^3\mathbf{z} \|_{L^2(\mathbb{R}^2)}, \tag{A.15}$$

e, para $m \geq 2$, $0 \leq \ell \leq m-2$:

$$\| D^\ell\mathbf{z} \|_{L^\infty(\mathbb{R}^2)} \| D^{m-\ell}\mathbf{z} \|_{L^2(\mathbb{R}^2)} \leq C \| \mathbf{z} \|_{L^2(\mathbb{R}^2)} \| D^{m+1}\mathbf{z} \|_{L^2(\mathbb{R}^2)}, \tag{A.16}$$

para algum $C > 0$.

Demonstração. As desigualdades acima podem ser obtidas a partir de

$$\|\mathbf{z}\|_{L^\infty(\mathbb{R}^2)} \leq \|\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^2)} \|D^2\mathbf{z}\|^{1/2}_{L^2(\mathbb{R}^2)} \tag{A.17}$$

□

Outra desigualdade (conhecida) de Gagliardo-Nirenberg que será necessária é:

$$\|\mathbf{z}(\cdot,t)\|_{L^4(\mathbb{R}^3)} \leq K \|\mathbf{z}(\cdot,t)\|^{1/4}_{L^2(\mathbb{R}^3)} \|D\mathbf{z}(\cdot,t)\|^{3/4}_{L^2(\mathbb{R}^3)}, \tag{A.18}$$

para algum $K > 0$.

Apêndice B FERRAMENTAS DE ANÁLISE

Faremos aqui, uma pequena reunião de resultados em análise que são úteis para que o leitor tenha um melhor entendimento acerca do que está sendo desenvolvido no texto.

Lema B.1. *Se $f \in L^p(E)$, para $1 \leq p \leq \infty$, então dado $\epsilon > 0$, existe um conjunto $K \subset E$, com $|K| < \infty$, tal que*

$$\|f\|_{L^p(E)} < \|f\|_{L^p(K)} + \epsilon$$

Lema B.2. *Se $f \in L^p(E)$, para $1 \leq p \leq \infty$, então dado $\epsilon > 0$, existe $\delta > 0$ tal que se $H \subset E$ e*

$$\mu(H) < \delta \Rightarrow \int_H |f|^p d\mu < \epsilon$$

Demonstração. Suponha, por absurdo, que existem conjuntos $E_n \subset E$ e $\epsilon > 0$ tal que

$$\mu(E_n) < \frac{1}{2^n} \text{ e } \int_{E_n} |f|^p d\mu \geq \epsilon,$$

para todo $n \in \mathbb{N}$. Seja $F_n = \bigcup_{k=n}^{\infty} E_k$. Logo, $F_{n+1} \subset F_n$, para todo $n \in \mathbb{N}$ e $\mu(F_n) < \frac{1}{2^{n-1}}$, com $\int_{F_n} |f|^p d\mu \geq \epsilon$. Portanto, temos que,

$$\mu(\cap_n F_n) = \lim_n \mu(F_n) = 0$$

e, como a integral é uma medida,

$$\epsilon \leq \lim_n \int_{F_n} |f|^p d\mu = \int_{\cap_n F_n} |f|^p d\mu = 0$$

Absurdo. □

Lema B.3. *Se $f \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ para algum $1 \leq p < \infty$, então $f \in L^q(\mathbb{R}^n)$, para cada $p \leq q \leq \infty$, e, ainda,*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}^{\frac{p}{q}} \|f\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{p}{q}}$$

Lema B.4. *Se $f \in L^1(\mathbb{R}^n)$ e $g \in L^p(\mathbb{R}^n)$ para algum $1 \leq p \leq \infty$, então $f * g \in L^p(\mathbb{R}^n)$ e, além disso,*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}$$

Lema B.5. *Sendo* $f \in L^2(\mathbb{R}^n) \cap \dot{H}^{s_1}(\mathbb{R}^n)$*, para* $s_1 > 0$*, então, tem-se,* $f \in \dot{H}^{s}(\mathbb{R}^n)$*, para cada* $0 < s < s_1$*, com*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{s_1}} \|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}^{\frac{s}{s_1}}$$

Para maiores detalhes sobre essas normas, veja Apêndice A.

Demonstração. De fato, basta notar que, pela desigualdade de Hölder, para $1 \leq p, q \leq \infty$

$$\begin{aligned}\|f\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^{\frac{2}{p}} |\hat{f}(\xi)|^{\frac{2}{q}} d\xi \\ &\leq \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |\xi|^{2sp} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{p}},\end{aligned}$$

logo, deve-se ter,

$$2sp = 2s_1$$

O que implica que,

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{s_1}} \|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}^{\frac{s}{s_1}}$$

□

As demonstrações dos lemas aqui omitidas podem ser encontradas em [16], com exceção do lema B.2. Vale, ainda o caso mais geral:

Lema B.6. *Sendo* $f \in \dot{H}^{s_1}(\mathbb{R}^n) \cap \dot{H}^{s_2}(\mathbb{R}^n)$*, para* $s_1, s_2 > 0$*, então, tem-se,* $f \in \dot{H}^{s}(\mathbb{R}^n)$*, para cada* $s_1 < s < s_2$*, com*

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|_{\dot{H}^{s_1}(\mathbb{R}^n)}^{\theta\frac{s}{s_1}} \|f\|_{\dot{H}^{s_2}(\mathbb{R}^n)}^{(1-\theta)\frac{s}{s_2}}$$

onde $0 \leq \theta \leq 1$*, e* $\frac{1}{s} = \theta\frac{1}{s_1} + (1-\theta)\frac{1}{s_2}$

Demonstração. Observe que, pela desigualdade de Hölder, para $1 \leq p, q \leq \infty$, tem-se,

$$\begin{aligned}
\|f\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\xi|^{\theta 2s}|\xi|^{(1-\theta)2s}|\hat{f}(\xi)|^{\frac{2}{p}}|\hat{f}(\xi)|^{\frac{2}{q}} d\xi \\
&\leq \left(\int_{\mathbb{R}^n} |\xi|^{\theta 2sp}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |\xi|^{(1-\theta)2sq}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{q}},
\end{aligned}$$

logo, deve-se ter $\theta ps = s_1$ e $(1-\theta)qs = s_2$, o que implica que,

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq \|f\|^{\theta\frac{s}{s_1}}_{\dot{H}^{s_1}(\mathbb{R}^n)} \|f\|^{(1-\theta)\frac{s}{s_2}}_{\dot{H}^{s_2}(\mathbb{R}^n)}$$

com $0 \leq \theta \leq 1$, e $\frac{1}{s} = \theta\frac{1}{s_1} + (1-\theta)\frac{1}{s_2}$ □

Lema B.7. *Dada $f \in L^2(\mathbb{R}^n)$, com $D^n f \in L^2(\mathbb{R}^n)$, então $f \in L^\infty(\mathbb{R}^n)$, com*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq K(n)\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)}\|D^n f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)}$$

A demonstração do lema acima encontra-se em [13]. Na verdade, este resultado é um caso particular da seguinte desigualdade geral:

Seja $s > n/2$. *Se* $f \in H^s(\mathbb{R}^n)$, *então*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq K(n,s)\ \|f\|^{1-\frac{n}{2s}}_{L^2(\mathbb{R}^n)}\ \|f\|^{\frac{n}{2s}}_{\dot{H}^s(\mathbb{R}^n)},$$

onde o valor optimal de $K(n,s)$ é dado por

$$K(s,n) = \{4\pi\}^{-\frac{n}{4}} \left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^{-\frac{1}{2}} \left\{\frac{\sin(\frac{\pi n}{2s})}{\frac{\pi n}{2s}}\right\}^{-\frac{1}{2}} \left\{\frac{n}{2s-n}\right\}^{-\frac{n}{4s}} \left\{\frac{2s}{2s-n}\right\}^{\frac{1}{2}}. \tag{B.1}$$

Para a demostração da desigualdade acima, veja [40].

Lema B.8. *Sejam $f \in C^0([t_0,T])$ e $w \in L^1([t_0,T])$, com $f \geq 0$ e $w \geq 0$ tal que*

$$f(t) \leq A + \int_{t_0}^t w(s)f(s)ds,$$

para todo $t \in [t_0,T]$. Então,

$$f(t) \leq Ae^{\int_{t_0}^t w(s)ds},$$

onde A é uma constante positiva.

Demonstração. Note que,

$$\frac{f(t)}{A+\int_{t_0}^{t} w(s)f(s)ds} \leq 1$$

definindo $U(t) = A + \int_{t_0}^{t} w(s)f(s)ds$, tem-se,

$$\frac{U'}{U} \leq w(s)$$

donde, integrando de t_0 a t, segue o resultado. $\square$

Lema B.9. *Seja $f, g : [a, \infty) \to \mathbb{R}$ funções contínuas. Se f é diferenciável em (a, ∞) e satisfaz*

$$f'(t) \leq f(t)g(t),$$

então $f(t) \leq f(a) \exp\left(\int_a^t g(\tau)d\tau\right)$, para todo $t \geq a$.

Apêndice C EQUAÇÃO DO CALOR: PROPRIEDADES DO *HEAT KERNEL*

Nesta seção, o objetivo é provar algumas propriedades conhecidas da solução da equação do calor usando as técnicas desenvolvidas em [23] e [24] para nos motivarmos a atacar sistemas mais complicados.

Considere o problema,

$$(*)\begin{cases}\mathbf{u}_t = \Delta\mathbf{u},\ t > 0,\ x \in \mathbb{R}^n \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 \in L^2(\mathbb{R}^n)\end{cases}$$

Então tem-se o seguinte resultado:

Teorema C.1 (Propriedade de Leray para a equação do Calor)**.** *Dada* $\mathbf{u}(x,t)$ *solução do problema* $(*)$*, então*

$$\lim_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0$$

Provaremos o resultado acima de duas maneiras diferentes, a primeira usando Transformada de Fourier e a segunda usando as técnicas desenvolvidas no presente texto. Posteriormente, compararemos os dois argumentos, explicitando quais as vantagens e desvantagens de cada um.

Demonstração. **Primeiro argumento:**

A maneira mais conhecida de provar esse resultado é usando Transformada de Fourier. Portanto, seja $\hat{\mathbf{u}}$ a transformada de Fourier, definida abaixo,

$$\hat{\mathbf{u}}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix.\xi}\mathbf{u}(x)dx,$$

onde $i = \sqrt{-1}$.

Aplicando a transformada de Fourier no problema $(*)$, obtemos,

$$\hat{\mathbf{u}}_t(\xi, t) = -|\xi|^2 \hat{\mathbf{u}}(\xi, t)$$
$$\hat{\mathbf{u}}(\xi, 0) = \hat{\mathbf{u}}_0(\xi)$$

Resolvendo o problema acima, obtém-se que,

$$\hat{\mathbf{u}}(\xi, t) = e^{-|\xi|^2 t} \hat{\mathbf{u}}_0(\xi)$$

Sabe-se, pelo Teorema de Plancherel, que $\|\mathbf{u}(x,t)\|^2_{L^2(\mathbb{R}^n)} = \|\hat{\mathbf{u}}(\xi,t)\|^2_{L^2(\mathbb{R}^n)}$

Logo

$$\|\mathbf{u}(x,t)\|^2_{L^2(\mathbb{R}^n)} = \|\hat{\mathbf{u}}(\xi,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi,t)|^2 d\xi = \int_{\mathbb{R}^n} e^{-2|\xi|^2 t} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \tag{C.1}$$

Temos que, pelo lema B.2 dado $\epsilon > 0$, existe $\delta > 0$ tal que

$$\int_{|\xi|<\delta} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi < \frac{\epsilon}{2}$$

Note que, $e^{-2|\xi|^2 t} \leq 1$, para todo $t \geq 0$. Observe, também, que dado $\epsilon > 0$, existe $t_0(\epsilon) > 0$, a saber,

$$t_0 = \ln \left(\frac{2\|\mathbf{u_0}\|^2_{L^2(\mathbb{R}^n)}}{\epsilon} \right)^{\frac{1}{2\delta^2}},$$

tal que para todo $t \geq t_0(\epsilon)$, tem-se,

$$e^{-2\delta^2 t} \|\mathbf{u_0}\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2}$$

Portanto, dado $\epsilon > 0$, existe $\delta > 0$ e $t_0 \geq 0$, como definido acima, tal que, para todo $t \geq t_0$, tem-se

$$\int_{\mathbb{R}^n} e^{-2|\xi|^2 t} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi = \int_{|\xi|<\delta} e^{-2|\xi|^2 t} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi + \int_{|\xi|\geq\delta} e^{-2|\xi|^2 t} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \leq$$
$$\int_{|\xi|<\delta} 1.|\hat{\mathbf{u}}_0(\xi)|^2 d\xi + e^{-2\delta^2 t} \int_{|\xi|\geq\delta} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \leq \frac{\epsilon}{2} + e^{-2\delta^2 t} \|\mathbf{u_0}\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

O que implica, por C.1, que

$$\lim_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0.$$

Assim, obtemos o resultado desejado. $\square$

Uma consequência interessante deste argumento é a seguinte afirmação:

Afirmação C.2. *Dado $T > 0$ suficientemente grande e dado $0 < \lambda < 1$, existe $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$, com $\|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} = 1$ tal que*

$$\|u(x,T)\|_{L^2(\mathbb{R}^n)} \geq \lambda$$

Nota. Observe que, a afirmação acima diz, em outras palavras que, a norma L^2 da solução da equação decresce arbitrariamente lenta ao logo do tempo t.

Demonstração. Para construir tal função, faremos via Transformada de Fourier. Definimos, então, a seguinte função,

$$\hat{\mathbf{u}}_0(\xi) = \begin{cases} c(r,n), & |\xi| \leq r \\ 0, & |\xi| > r \end{cases}$$

Onde,

$$c(r,n) = \sqrt{\frac{n}{r^n \omega_n}}, \text{ com } \omega_n = \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$$

Γ é a famosa função Gamma de Euler. Então, seguindo os moldes do argumento anterior, por Plancherel novamente, temos,

$$\|\mathbf{u}(x,T)\|^2_{L^2(\mathbb{R}^n)} = \|\hat{\mathbf{u}}(\xi,T)\|^2_{L^2(\mathbb{R}^n)} = \int_{|\xi|\leq r} e^{-2|\xi|^2 T} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \geq e^{-2r^2T} \int_{|\xi|\leq r} |\hat{\mathbf{u}}_0(\xi)|^2 d\xi \geq \lambda^2,$$

pois dado $0 < \lambda < 1$ fixo, existe $r > 0$ tal que $e^{-r^2T} \geq \lambda$, consequentemente, isso mostra a existência da $\mathbf{u}_0$. $\square$

Portanto uma consequência imediata desse argumento, em outras palavras, é que não podemos dizer o quão rápido $\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}$ vai a zero.

Segundo argumento:

Faremos, agora, outra técnica para demonstrar o teorema C.1 e depois discutiremos as consequências imediatas que tal argumento implicará. Para isso, faz-se necessário o uso do lema a seguir.

Lema C.3. *Dada* $\mathbf{u}(\cdot,\mathbf{t})$ *solução do problema* $(*)$, *com* $\mathbf{u}_0 \in L^1(\mathbb{R}^n)$ *e* $1 \leq p \leq \infty$ *então, tem-se:*

$$\|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^n)} \leq K_n(p)\|\mathbf{u}_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{2}(1-\frac{1}{p})}$$

Demonstração. O resultado é trivial para $p = 1$, basta usar a desigualdade de Young para a convolução provada no lema B.4. Faremos, agora, para $p = \infty$. Note que,

$$|\mathbf{u}(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} |\mathbf{u}_0(y)| dy \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \|\mathbf{u}_0\|_{L^1(\mathbb{R}^n)}$$

Logo,

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{4\pi^{\frac{n}{2}}} \|\mathbf{u}_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{2}}$$

Usando uma simples interpolação, provada no lema B.3, tem-se o resultado desejado. □

Demonstração. Agora, defina,

$$\mathbf{v}_0(x) = \begin{cases} \mathbf{u}_0, & |x| > R_\epsilon \\ 0, & |x| \leq R_\epsilon \end{cases}$$

e, similarmente,

$$\mathbf{w}_0(x) = \begin{cases} 0, & |x| > R_\epsilon \\ \mathbf{u}_0, & |x| \leq R_\epsilon \end{cases}$$

para um certo $R_\epsilon > 0$

Note que,

$$\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{w}_0$$

e, por $\mathbf{u}_0 \in L^2(\mathbb{R}^n)$, tem-se, $\mathbf{w}_0 \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, pois[1]

$$\|\mathbf{w}_0\|_{L^1(\mathbb{R}^n)} = \int_{|\mathbf{w}|>1} |\mathbf{w}_0| dx + \int_{|\mathbf{w}_0|\leq 1} |\mathbf{w}_0| dx \leq \int |\mathbf{w}_0|^2 dx + \frac{\omega_n}{n} R_\epsilon$$

[1] Na verdade, isso vale, não somente para a função $\mathbf{w}_0$ definida acima, mas também, em geral, para qualquer função $\mathbf{w}_0$ com suporte compacto.

Observe que, pelo lema B.2, temos que dado $\epsilon > 0$, existe $R_\epsilon > 0$ suficientemente grande tal que,

$$\|\mathbf{v}_0\|^2_{L^2(\mathbb{R}^n)} = \int_{|x|>R_\epsilon} |\mathbf{u}_0|^2 dx \leq \left(\frac{\epsilon}{2}\right)^2$$

Sabemos que $\mathbf{v}_0$ e $\mathbf{w}_0$, estão em L^2, mas $\mathbf{w}_0 \in L^1(\mathbb{R}^n)$, então pelo lema C.3, temos,

$$\|e^{\Delta t}\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq K_n(2)\|\mathbf{w}_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{4}}$$

Logo, dado $\epsilon > 0$, existe $t_0(\epsilon) > 0$, tal que $\|e^{\Delta t}\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2}$. O que implica que, para $t > t_0(\epsilon)$

$$\|e^{\Delta t}\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} \leq \|e^{\Delta t}\mathbf{v}_0\|_{L^2(\mathbb{R}^n)} + \|e^{\Delta t}\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq \|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)} + \|\mathbf{w}_0\|_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

O que conclui a demonstração do teorema C.1 usando o outro argumento. □

Note que, usamos a linearidade da equação do calor para fechar a prova do teorema C.1 neste último argumento, mas isso não nos impede de atacar equações não lineares, como será mostrado posteriormente. Além disso, usamos que se $\mathbf{u}$ satisfaz $(*)$, então $\|\mathbf{u}\|_{L^p} \leq \|\mathbf{u}_0\|_{L^p}$, para cada[2] $1 \leq p \leq \infty$, basta usar o lema B.4 para ver isso.

Observe que no primeiro argumento, usamos a identidade de Plancherel, que funciona somente para $p = 2$, agora temos o seguinte resultado como consequência imediata de nosso segundo argumento:

Corolário C.4. *Dada $\mathbf{u}(x,t)$ solução do problema $(*)$, com $\mathbf{u}_0 \in L^p(\mathbb{R}^n)$, então*

$$\lim_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{L^p(\mathbb{R}^n)} = 0,$$

para $1 < p < \infty$.

[2]Supondo, é claro, que $\mathbf{u}_0 \in L^p$

Demonstração. Basta usar o segundo argumento usado para demonstrar o teorema C.1 para $1 < p < \infty$. O argumento segue analogamente. $\square$

Provaremos agora a propriedade de Leray do problema $(*)$, para a norma $\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R}^n)}$. Antes, precisaremos demonstrar o seguinte lema,

Lema C.5. *Se $\mathbf{u}(x,t)$ satisfaz o problema $(*)$, com $\|\mathbf{u}_0\| \in L^p(\mathbb{R}^n)$, então,*

$$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R}^n)} \leq \tilde{K}_n(p)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)} t^{-\frac{n}{2p}}$$

Demonstração. Basta, notar que

$$|\mathbf{u}(\cdot,t)| \leq (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{|x-y|^2}{4t}} |\mathbf{u}_0(y)| dy \underbrace{\leq}_{\text{Desig. de Hölder}} \frac{1}{(4\pi t)^{\frac{n}{2p}}} \|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)}$$

Logo, em particular,

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi t)^{\frac{n}{2p}}} \|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)} = \tilde{K}_n(p)\|\mathbf{u}_0\|_{L^p(\mathbb{R}^n)} t^{-\frac{n}{2p}}$$

$\square$

Na verdade, será útil lembrarmos a seguinte estimativa mais geral (bem conhecida):

$$\| D^\alpha [e^{\nu\Delta\tau} \mathrm{u}] \|_{L^2(\mathbb{R}^n)} \leq K(n,m) \| \mathrm{u} \|_{L^r(\mathbb{R}^n)} (\nu\tau)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{|\alpha|}{2}} \qquad \text{(C.2)}$$

para todo $\tau > 0$, e quaisquer α (multi-índice), $1 \leq r \leq 2$, $\mathrm{u} \in L^r(\mathbb{R}^n)$ considerados, $n \geq 1$ arbitrário, e onde $m = |\alpha|$. (Para uma derivação de (C.2), ver e.g. [25, 27, 16].)

Com isso, podemos facilmente provar o seguinte resultado,

Teorema C.6. *Dada $\mathbf{u}$ solução do problema $(*)$, então, tem-se*

$$\lim_{t\to\infty} t^{\frac{n}{4}} \|e^{\Delta t}\mathbf{u}_0\|_{L^\infty(\mathbb{R}^n)} = 0$$

Demonstração. Pelo o lema C.5, temos que

$$\|\mathbf{u}(x,t)\|_{L^\infty(\mathbb{R}^n)} \leq \tilde{K}(n,2)\|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} t^{-\frac{n}{4}}$$

Logo,

$$\|e^{\Delta\frac{t}{2}}[e^{\Delta\frac{t}{2}}\mathbf{u}_0]\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2^{\frac{n}{4}}}\tilde{K}(n,2)\|e^{\Delta\frac{t}{2}}\mathbf{u}_0\|_{L^2(\mathbb{R}^n)} t^{-\frac{n}{4}}$$

Portanto,

$$t^{\frac{n}{4}}\|e^{\Delta\frac{t}{2}}[e^{\Delta\frac{t}{2}}\mathbf{u}_0]\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{2^{\frac{n}{4}}}\tilde{K}(n,2)\|e^{\Delta\frac{t}{2}}\mathbf{u}_0\|_{L^2(\mathbb{R}^n)}$$

O que implica, ao $t \to \infty$, pelo teorema C.1, que

$$\lim_{t\to\infty} t^{\frac{n}{4}}\|e^{\Delta t}\mathbf{u}_0\|_{L^\infty(\mathbb{R}^n)} = \lim_{t\to\infty} t^{\frac{n}{4}}\|e^{\Delta\frac{t}{2}}[e^{\Delta\frac{t}{2}}\mathbf{u}_0]\|_{L^\infty(\mathbb{R}^n)} = 0$$

□

Corolário C.7. *Dada $\mathbf{u}(\cdot,t)$ solução do problema $(*)$, tem-se,*

$$\lim_{t\to\infty} t^{\frac{n}{4}-\frac{n}{2q}}\|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} = 0,$$

para todo $2 \leq q \leq \infty$.

Demonstração. Basta notar que, pelo lema B.3,

$$t^\gamma\|\mathbf{u}(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq \|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{\frac{2}{q}}\left(t^{\frac{\gamma}{1-\frac{2}{q}}}\|\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\right)^{1-\frac{2}{q}}$$

Pelo Teorema C.6 e o Teorema C.1, deve-se ter,

$$\frac{\gamma}{1-\frac{2}{q}} = \frac{n}{4}$$

Logo,

$$\gamma = \frac{n}{4} - \frac{n}{2q}$$

Observe que, aqui há uma melhora do Corolário C.4 . □

Agora proveremos o que chamamos de Problema completo de Leray, para a equação do calor, i.e.,

$$t^{\frac{s}{2}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0,$$

para qualquer $s \geq 0$. Antes, teremos que provar alguns resultados preliminares.

Proposição C.8. *Dada $\mathbf{u}(\cdot,t)$ solução de $(*)$, vale a igualdade de energia:*

$$\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t}\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}d\tau = \|\mathbf{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)},$$

para $0 \leq t_0 < t$.

Demonstração. Observe que o problema $(*)$ está definido em $\mathbb{R}^n$, então, para obtermos a igualdade acima, ao integrarmos por partes, a função deve "decair rápido no infinito". Para formalizarmos este raciocínio, introduziremos uma função auxiliar de corte,

$$\zeta_R(x) := \begin{cases} e^{-\epsilon\sqrt{1+|x|^2}} - e^{-\epsilon\sqrt{1+R^2}}, & |x| < R \\ 0, & |x| \geq R \end{cases},$$

para $R, \epsilon > 0$

Multiplicando a equação do calor em $(*)$ por $2\mathbf{u}\zeta_R$, tem-se,

$$2\mathbf{u}\mathbf{u}_t\zeta_R = 2\mathbf{u}\Delta\mathbf{u}\zeta_R,$$

como $2\mathbf{u}\mathbf{u}_t = \frac{\partial}{\partial t}(\mathbf{u}^2)$, ao integrarmos de t_0 a t e em $\mathbb{R}^n$, obtém-se, pelo Teorema Fundamental do Cálculo e pelo Teorema de Fubini[3], que

$$\int_{|x|<R}\mathbf{u}^2(\cdot,t)\zeta_R dx = \int_{t_0}^{t}\int_{|x|<R} 2\mathbf{u}\Delta\mathbf{u}\zeta_R dx d\tau + \int_{|x|<R}\mathbf{u}^2(\cdot,t_0)\zeta_R dx$$

Fazendo integração por partes no primeiro termo do lado direito da equação acima, como ζ_R se anula em $|x| = R$,

$$\int_{|x|<R}\mathbf{u}^2(\cdot,t)\zeta_R dx + 2\int_{t_0}^{t}\int_{|x|<R}|D\mathbf{u}(\cdot,\tau)|^2\zeta_R dx d\tau =$$
$$-\int_{t_0}^{t}\int_{|x|<R}\langle D\zeta_R, D\mathbf{u}^2(\cdot,\tau)\rangle dx d\tau + \int_{|x|<R}\mathbf{u}^2(\cdot,t_0)\zeta_R dx$$

[3]Para maiores detalhes dos teoremas clássicos de teoria da integração, como os teoremas de Fubini, convergência dominada de Lebesgue, convergência monótona e outros, veja [2]

Integrando por partes, novamente, tem-se,

$$\begin{gathered}\int_{|x|<R}\mathbf{u}^2(\cdot,t)\zeta_R dx + 2\int_{t_0}^{t}\int_{|x|<R}|D\mathbf{u}(\cdot,\tau)|^2\zeta_R dx d\tau = \\ \int_{t_0}^{t}\int_{|x|<R}\Delta\zeta_R\mathbf{u}^2(\cdot,\tau)dxd\tau - \int_{t_0}^{t}\int_{|x|=R}\langle D\zeta_R,\vec{n}\rangle\mathbf{u}^2(\cdot,\tau)d\sigma d\tau + \int_{|x|<R}\mathbf{u}^2(\cdot,t_0)\zeta_R dx\end{gathered} \tag{C.3}$$

ou seja,

$$\begin{gathered}\int_{|x|<R}\mathbf{u}^2(\cdot,t)\zeta_R dx + 2\int_{t_0}^{t}\int_{|x|<R}|D\mathbf{u}(\cdot,\tau)|^2\zeta_R dx d\tau = \\ \int_{t_0}^{t}\int_{|x|<R}\Delta\zeta_R\mathbf{u}^2(\cdot,\tau)dxd\tau + \epsilon\int_{t_0}^{t}\int_{|x|=R}e^{-\epsilon\sqrt{1+R^2}}\sqrt{\frac{R^2}{1+R^2}}\mathbf{u}^2(\cdot,\tau)d\sigma d\tau \\ + \int_{|x|<R}\mathbf{u}^2(\cdot,t_0)\zeta_R dx,\end{gathered}$$

pois $\vec{n} = \frac{x}{R}$ Nossa intenção é fazer $R \to \infty$, no entanto, temos que analisar se o termo de fronteira não irá divergir. Observe que, $\mathbf{u}_0 \in L^2(\mathbb{R}^n) \Rightarrow \mathbf{u}(\cdot,t) \in L^2(\mathbb{R}^n)$, basta usar o lema B.4.

Logo,

$$\int_{t_0}^{t}\int_{\mathbb{R}^n}\mathbf{u}^2 dx d\tau < \infty.$$

Portanto, escrevendo em coordenadas polares e usando o Teorema de Fubini,

$$\begin{aligned}\int_{t_0}^{t}\int_{\mathbb{R}^n}\mathbf{u}^2 dxd\tau &= \int_{t_0}^{t}\int_{0}^{\infty}\int_{|x|=r}\mathbf{u}^2 d\sigma(x)drd\tau = \\ &= \int_{0}^{\infty}\int_{t_0}^{t}\int_{|x|=r}\mathbf{u}^2 d\sigma(x)d\tau dr < \infty.\end{aligned}$$

Logo,

$$\liminf_{r\to\infty}\left[\int_{t_0}^{t}\int_{|x|=r}\mathbf{u}^2 d\sigma(x)d\tau\right] = 0.$$

Então, existe uma subsequência R_k tal que

$$\lim_{R_k \to \infty} \left[\int_{t_0}^{t} \int_{|x|=R_k} \mathbf{u}^2 d\sigma(x) d\tau \right] = 0. \tag{C.4}$$

Como,

$$\epsilon \int_{t_0}^{t} \int_{|x|=R} e^{-\epsilon\sqrt{1+R^2}} \sqrt{\frac{R^2}{1+R^2}} \mathbf{u}^2(\cdot,\tau) d\sigma d\tau \leq \epsilon \int_{t_0}^{t} \int_{|x|=R} \mathbf{u}^2(\cdot,\tau) d\sigma d\tau. \tag{C.5}$$

Portanto, fazendo $R_k \to \infty$ em C.3, usando o Teorema da Convergência Dominada no primeiro e último termo de C.3, usando C.4 e C.5 no termo de fronteira de C.3 e, finalmente, usando o Teorema da Convergência Monótona nos dois termos restantes de C.3, tem-se

$$\int_{\mathbb{R}^n} e^{-\epsilon\sqrt{1+|x|^2}} \mathbf{u}^2(\cdot,t) dx + 2 \int_{t_0}^{t} \int_{\mathbb{R}^n} e^{-\epsilon\sqrt{1+|x|^2}} |D\mathbf{u}(\cdot,\tau)|^2 dx d\tau =$$
$$\int_{t_0}^{t} \int_{\mathbb{R}^n} \Delta(e^{-\epsilon\sqrt{1+|x|^2}}) \mathbf{u}^2(\cdot,\tau) dx d\tau + \int_{\mathbb{R}^n} e^{-\epsilon\sqrt{1+|x|^2}} \mathbf{u}^2(\cdot,t_0) dx,$$

Como

$$|\mathbf{u}^2 \Delta(e^{-\epsilon\sqrt{1+|x|^2}})| \leq \epsilon \mathbf{u}^2,$$

supondo $0 < \epsilon \leq 1$, sem perda de generalidade. Então, tomando $\epsilon \to 0$, usando o Teorema da Convergência Dominada de Lebesgue no termo que contém o laplaciano da igualdade acima e usando o Teorema da Convergência Monótona no termos restantes, obtemos:

$$\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2 \int_{t_0}^{t} \|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} d\tau = \|\mathbf{u}(\cdot,t_0)\|^2_{L^2(\mathbb{R}^n)}.$$

□

Corolário C.9. *Vale, também, as igualdades,*

$$(t-t_0)\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2 \int_{t_0}^{t} (\tau - t_0) \|D^2\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau = \int_{t_0}^{t} \|D\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau,$$

$$
\begin{aligned}
(t-t_0)^2\|D^2\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t}(\tau-t_0)^2\|D^3\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
= \int_{t_0}^{t}(\tau-t_0)\|D^2\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau,
\end{aligned}
$$

E assim, sucessivamente,

$$
\begin{aligned}
(t-t_0)^m\|D^m\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t}(\tau-t_0)^m\|D^{m+1}\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau \\
= \int_{t_0}^{t}(\tau-t_0)^{m-1}\|D^m\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau
\end{aligned}
$$

para $m \geq 1$ inteiro.

Demonstração. Vamos introduzir, novamente, uma função de corte. Seja $\zeta \in C^2(\mathbb{R}^n)$ tal que,

$$
\zeta(x) := \begin{cases} 1, & |x| < 1 \\ \Phi(x), & 1 \leq |x| < 2 \\ 0, & |x| \geq 2 \end{cases},
$$

onde $\Phi : \mathbb{R}^n \to \mathbb{R}$ é uma função qualquer, obviamente, de classe C^2. Agora, defina:

$$
\zeta_R = \zeta\left(\frac{|x|}{R}\right),
$$

para $R > 0$.

Derivando a equação do problema $(*)$ com relação a x_i, i.e., aplicando o operador D_i e multiplicando por $2(t-t_0)\zeta_R D_i\mathbf{u}$, tem-se,

$$
2(t-t_0)\zeta_R D_i\mathbf{u}(D_i\mathbf{u})_t = 2(t-t_0)\zeta_R D_i\mathbf{u}\Delta(D_i\mathbf{u}).
$$

Integrando em $[t_0, t] \times \mathbb{R}^n$, tem-se

$$
\int_{|x|<2R}\int_{t_0}^{t}(\tau-t_0)\zeta_R (D_i\mathbf{u})^2{}_t d\tau dx = \int_{|x|<2R}\int_{t_0}^{t} 2(\tau-t_0)\zeta_R D_i\mathbf{u}\Delta(D_i\mathbf{u})d\tau dx.
$$

Integrando por partes no lado esquerdo, a expressão acima fica,

$$(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx = \int_{|x|<2R}\int_{t_0}^{t}2(\tau-t_0)\zeta_R D_i\mathbf{u}\Delta(D_i\mathbf{u})d\tau dx$$
$$+\int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2d\tau dx.$$

Agora, integrando por partes em $\mathbb{R}^n$, no primeiro termo do lado direito, tem-se,

$$\begin{aligned}(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx &= -\int_{|x|<2R}\langle\nabla\zeta_R,\nabla(D_i\mathbf{u})^2\rangle dx\\ -\int_{|x|<2R}2\zeta_R|\nabla(D_i\mathbf{u})|^2dx &+ \underbrace{\int_{|x|=2R}2\zeta_R D_i\mathbf{u}\nabla(D_i\mathbf{u})d\sigma(x)}_{=0}\\ &+\int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2d\tau dx\end{aligned}$$

Integrando por partes, novamente, em $\mathbb{R}^n$ no primeiro termo do lado direito da equação acima, tem-se

$$\begin{aligned}(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx &+ \int_{|x|<2R}2\zeta_R|\nabla(D_i\mathbf{u})|^2dx\\ = \int_{|x|<2R}\Delta\zeta_R(D_i\mathbf{u})^2dx &- \int_{|x|=2R}\langle\nabla\zeta_R,\vec{n}\rangle(D_i\mathbf{u})^2d\sigma(x)\\ &+\int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2d\tau dx\end{aligned}\tag{C.6}$$

Observe que, $\nabla\zeta_R = 0$, para todo $|x| > 2R$. Sabe-se que, em particular, $\zeta_R \in C^1$, logo,

$$\nabla\zeta_R\Big|_{|x|=2R} = \lim_{|x|\to 2R^+}\nabla\zeta_R = 0$$

Portanto, a equação C.6, fica,

$$\begin{aligned}(t-t_0)\int_{|x|<2R}\zeta_R(D_i\mathbf{u}(x,t))^2dx &+ \int_{|x|<2R}2\zeta_R|\nabla(D_i\mathbf{u})|^2dx\\ = \int_{|x|<2R}\Delta\zeta_R(D_i\mathbf{u})^2dx &+ \int_{|x|<2R}\int_{t_0}^{t}\zeta_R(D_i\mathbf{u}(x,\tau))^2d\tau dx\end{aligned}\tag{C.7}$$

Note que, como $\zeta \in C^2$, o laplaciano de ζ, atinge o máximo em $|x| \leq 2R$. Portanto,

$$|\Delta\zeta_R(D_i\mathbf{u})^2| < \frac{M}{R^2}|D_i\mathbf{u}|^2,$$

em $|x| < 2R$. Onde $M > 0$ é o máximo de $\Delta\zeta$. Além disso, pela proposição C.8,

$$\int_{\mathbb{R}^n} |D_i\mathbf{u}|^2 dx < \infty$$

Logo, ao $R \to \infty$, podemos usar o teorema da convergência dominada no primeiro termo do lado direito da equação C.7, no primeiro termo do lado esquerdo e no último termo do lado direito(usando a proposição C.8) e convergência monótona no termo restante, a equação se torna, somando em i,

$$(t - t_0)\|D\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_{t_0}^{t}(\tau - t_0)\|D^2\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau = \int_{t_0}^{t}\|D\mathbf{u}(\cdot,\tau)\|^2_{L^2(\mathbb{R}^n)}d\tau,$$

Para demonstrar as igualdades seguintes, basta aplicar o operador

$$2\zeta_R(t - t_0)^2 D_iD_j\mathbf{u}D_iD_j$$

em ambos o lados na equação do calor definida no problema $(*)$ e usar o mesmo tipo de argumento feito anteriormente. Fazendo de forma indutiva, i.e., aplicado o operador

$$2\zeta_R(t - t_0)^m D_{i1}D_{i2}\dots D_{im}\mathbf{u}D_{i1}D_{i2}\dots D_{im}$$

em ambos os lados da equação do calor definida no problema $(*)$ e usando os mesmos argumentos feitos acima, conclui-se a demonstração, para $m \geq 1$ inteiro. □

Tendo isso, é fácil obter o seguinte resultado:

Teorema C.10. *Dada* $\mathbf{u}(\cdot,t)$ *solução do problema* $(*)$*, então, vale que*

$$\lim_{t\to\infty} t^{\frac{m}{2}}\|D^m\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)} = 0,$$

Em outras palavras,

$$\lim_{t\to\infty} t^{\frac{m}{2}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)} = 0,$$

para $m \geq 0$ *inteiro.*

Demonstração. Pelo teorema C.1, dado $\epsilon > 0$, existe $t_0 > 0$, tal que,

$$\|\mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \epsilon,$$

para todo $t \geq t_0$. Portanto, pela proposição C.8, tem-se

$$\int_{t_0}^{t} \|D\mathbf{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \frac{\epsilon^2}{2} \tag{C.8}$$

Pelo corolário C.9, obtemos o seguinte

$$(t - t_0)\|D\mathbf{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} = t\left(\frac{t - t_0}{t}\right)\|D\mathbf{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon^2}{2}$$

Logo,

$$t\|D\mathbf{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} \leq \left(\frac{t}{t - t_0}\right)\frac{\epsilon^2}{2}$$

Observe que $(\frac{t}{t-t_0}) \leq 2$, para $t \geq 2t_0$.

$$t^{\frac{1}{2}}\|D\mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon$$

O que prova o Teorema, para $m = 1$

Por C.8 e pelo corolário C.9, tem-se,

$$\int_{t_0}^{t} (\tau - t_0)\|D^2\mathbf{u}(\cdot, \tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \frac{\epsilon^2}{2}$$

Usando a segunda equação do corolário C.9, tem-se,

$$(t - t_0)^2\|D^2\mathbf{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} = t^2\left(\frac{t - t_0}{t}\right)^2\|D^2\mathbf{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\epsilon^2}{2}$$

Portanto, como $(\frac{t}{t-t_0})^2 \leq 2$, para $t \geq \sqrt{2}t_0$,

$$t\|D^2\mathbf{u}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq \epsilon$$

O que prova o Teorema, para $m = 2$. Repetindo o argumento sucessivamente de forma análoga, obtém-se o resultado desejado, para $m \geq 1$ inteiro. O resultado para $m = 0$ foi demonstrado no teorema C.1. $\square$

Usando o lema de interpolação B.5 tem-se o seguinte resultado:

Corolário C.11. *Para qualquer $s > 0$ tem-se,*

$$t^{\frac{s}{2}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \to 0, \ \textit{ao } t \to \infty$$

Demonstração. Dado $\epsilon > 0$, tome $s > 0$ arbitrário. Então, para qualquer $m > s$, existe t_* tal que, pelo Teorema C.1,

$$\|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}} \leq \sqrt{\epsilon},$$

para todo $t > t_*$. Além disso, pelo Teorema C.10, existe t_{**}, tal que, para todo $t > t_{**}$

$$t^{\frac{m}{2}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)} \leq \left(\sqrt{\epsilon}\right)^{\frac{m}{s}},$$

logo, usando o lema B.5, para $t > \max\{t_*, t_{**}\}$ e $\gamma > 0$(a ser escolhido), tem-se,

$$t^{\gamma}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^s(\mathbb{R}^n)} \leq t^{\gamma}\|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^m(\mathbb{R}^n)}^{\frac{s}{m}}$$

$$\leq \|\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{1-\frac{s}{m}}\left(t^{\gamma\frac{m}{s}}\|\mathbf{u}(\cdot,t)\|_{\dot{H}^m}\right)^{\frac{s}{m}} \leq \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon,$$

se $\gamma\frac{m}{s} = \frac{m}{2}$, portanto, $\gamma = \frac{s}{2}$, para qualquer $s > 0$. □

Observe que usando a desigualdade de Sobolev do lema B.7, temos o seguinte resultado:

Corolário C.12. *Dada* $\mathbf{u}$ *solução de* $(*)$, *tem-se*

$$t^{\frac{m}{2}+\frac{n}{4}}\|D^m\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \to 0 \ \textit{ao } t \to \infty,$$

para qualquer $m \geq 0$ inteiro.

Demonstração. Basta notar que pelo lema B.7,

$$t^{2\gamma}\|D^m\mathbf{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^2 \leq (K(n))^2 t^{2\gamma}\|D^m\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}\|D^{m+n}\mathbf{u}(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

Pelo teorema C.10, $2\gamma = \frac{m}{2} + \frac{m+n}{2}$, o que implica,

$$\gamma = \frac{m}{2} + \frac{n}{4}$$

$\square$

O que encerra a discussão desse tema para a equação do calor.

Apêndice D O PROJETOR DE HELMHOLTZ

Lema D.1. *[15, III, p. 104] Seja* $f = (f_1, f_2, ..., f_n) \in L^q(\mathbb{R}^n)$ *com* $n \geq 2$, $1 < q < \infty$. *Então existem* $p \in L^q_{loc}(\mathbb{R}^n)$ *com* $\nabla p \in L^q(\mathbb{R}^n)$ *e* $f_0 \in L^q_\sigma(\mathbb{R}^n)$*, unicamente determinados, tais que*

$$f = f_0 + \nabla p$$

e

$$\|f_0\|_{L^q(\mathbb{R}^n)} + \|\nabla p\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}$$

Definição D.2. Seja $f \in L^q(\mathbb{R}^n)$, considere a decomposição $f = f_0 + \nabla p$ como no lema (D.1), definimos o Projetor de Helmholtz $\mathbb{P}_h : L^q(\mathbb{R}^n) \to L^q_\sigma(\mathbb{R}^n)$ como $\mathbb{P}_h[f] := f_0$. $\mathbb{P}_h$ é linear, limitado e idempotente (veja [37], página 126).

Teorema D.3. *Seja* $1 < q_0 < \infty$ *e* $1 < q_1 < \infty$. *Sendo* $\boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$*, com* $D_j\boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$. *Então* $D_j\mathbb{P}_h[\boldsymbol{u}] \in L^{q_1}_\sigma(\mathbb{R}^n)$ *e*

$$D_j\mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[D_j\boldsymbol{u}]$$

Demonstração. Vamos dividir a demonstração deste Teorema em três partes. Primeiro vamos supor que $\boldsymbol{u} \in C_0^\infty(\mathbb{R}^n)$, provado este caso, vamos supor que $\boldsymbol{u}$ tem suporte compacto, e finalmente vamos provar para o caso geral $u \in L^{q_0}(\mathbb{R}^n)$.

Caso I: $\boldsymbol{u} \in C_0^\infty(\mathbb{R}^n)$.

Seja $1 \leq j \leq n$, usando as propriedades do Projetor de Helmholtz, existem p e q tais que

$$\boldsymbol{u} = \mathbb{P}_h[\boldsymbol{u}] + \nabla p \tag{D.1}$$

e

$$D_j\boldsymbol{u} = \mathbb{P}_h[D_j\boldsymbol{u}] + \nabla q. \tag{D.2}$$

Derivando (D.1), temos que

$$D_j\boldsymbol{u} = D_j\mathbb{P}_h[\boldsymbol{u}] + \nabla D_j p. \tag{D.3}$$

Aplicando o divergente nas equações (D.2) e (D.3), e subtraindo (D.2) de (D.3), obtemos que

$$\Delta D_j p - \Delta q = \Delta[(D_j p) - q] = 0. \tag{D.4}$$

Como $(D_j p) - q$ é harmônica e limitada, temos pelo Teorema de Liouville que $(D_j p) - q = 0$, isto é, $D_j p = q$. Subtraindo (D.2) de (D.3), temos que

$$\mathbb{P}_h[D_j\boldsymbol{u}] = D_j\mathbb{P}_h[\boldsymbol{u}].$$

Caso II: *supp* $\boldsymbol{u} \subset B_R$.

Seja $f \in C_0^\infty$, $f \geq 0$ tal que

$$\int_{\mathbb{R}^n} f(x)dx = 1$$

Dado $\epsilon > 0$, defina

$$\phi_\epsilon := f_\epsilon * \boldsymbol{u}, \text{ onde } \quad f_\epsilon(x) = \epsilon^{-n} f(\frac{x}{\epsilon})$$

Temos que $\phi_\epsilon \to \boldsymbol{u}$ ao $\epsilon \to 0$ em $L^{q_0}(\mathbb{R}^n)$, portanto $\phi_\epsilon \to \boldsymbol{u}$ em $\mathcal{D}'(\mathbb{R}^n)$.

Além disso, $D_j\phi_\epsilon = f_\epsilon * D_j\boldsymbol{u} \to D_j\boldsymbol{u}$ ao $\epsilon \to 0$ em $L^{q_1}(\mathbb{R}^n)$ de modo que $D_j\phi_\epsilon \to D_j\boldsymbol{u}$ em $\mathcal{D}'(\mathbb{R}^n)$.

Como o Projetor de Helmholtz é contínuo, vale que

$$\mathbb{P}_h[\phi_\epsilon] \to \mathbb{P}_h[\boldsymbol{u}] \quad \text{em} \;\; L^{q_0}(\mathbb{R}^n)$$

e portanto

$$D_j\mathbb{P}_h[\phi_\epsilon] \to D_j\mathbb{P}_h[\boldsymbol{u}] \quad \text{em} \;\; \mathcal{D}'(\mathbb{R}^n).$$

Temos também que

$$\mathbb{P}_h[D_j\phi_\epsilon] \to \mathbb{P}_h[D_j\boldsymbol{u}] \quad \text{em} \;\; \mathcal{D}'(\mathbb{R}^n).$$

Como $\mathbb{P}_h[D_j\phi_\epsilon] = D_j\mathbb{P}_h[\phi_\epsilon]$ (já que $\phi_\epsilon \in C_0^\infty(\mathbb{R}^n)$), pela unicidade do limite que

$$D_j\mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[D_j\boldsymbol{u}].$$

Caso III: $u \in L^{q_0}(\mathbb{R}^n)$.

Seja $f \in C_0^\infty(\mathbb{R}^n)$ tal que f é não crescente, $f(0) = 1$ e $f(1) = 0$. Seja f_R definida por

$$f_R(x) = \begin{cases} 1, & |x| < R \\ f(|x| - R), & R < |x| < R+1 \\ 0, & |x| \geq R+1 \end{cases}$$

Observe que $\boldsymbol{u} f_R \to \boldsymbol{u}$ em $L^{q_0}(\mathbb{R}^n)$ ao $R \to \infty$. Como $\mathbb{P}_h$ é contínuo, vale que $D_j \mathbb{P}_h[\boldsymbol{u} f_R] \to D_j \mathbb{P}_h[\boldsymbol{u}]$ em $\mathcal{D}'(\mathbb{R}^n)$. Temos também, pelo caso II, que

$$\mathbb{P}_h[D_j(\boldsymbol{u} f_R)] \to D_j \mathbb{P}_h[\boldsymbol{u}] \quad \text{em} \quad \mathcal{D}'(\mathbb{R}^n). \tag{D.5}$$

Por outro lado

$$D_j(\boldsymbol{u} f_R) = (D_j f_R)\boldsymbol{u} + f_R D_j \boldsymbol{u},$$

$$\lim_{R\to\infty} (D_j f_R)\boldsymbol{u} = 0 \quad \text{em} \quad L^{q_0}(\mathbb{R}^n)$$

e

$$\lim_{R\to\infty} f_R D_j \boldsymbol{u} = D_j \boldsymbol{u} \quad \text{em} \quad L^{q_1}(\mathbb{R}^n).$$

Note que $D_j(\boldsymbol{u} f_R) = (D_j f_R)\boldsymbol{u} + f_R D_j \boldsymbol{u}$, $(D_j f_R)\boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$ e $f_R D_j \boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$. Temos por Hölder, que se $q_0 < q_1$, então $f_R D_j \boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$ e se $q_0 > q_1$, então $(D_j f_R)\boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$, de modo que exite r ($r = q_0$ ou r$= q_1$) tal que $D_j(\boldsymbol{u} f_R) \in L^r(\mathbb{R}^n)$.

Como $\mathbb{P}_h$ é contínuo, temos que

$$\lim_{R\to\infty} \mathbb{P}_h[D_j(\boldsymbol{u} f_R)] = \lim_{R\to\infty} \mathbb{P}_h[(D_j f_R)\boldsymbol{u}] + \lim_{R\to\infty} \mathbb{P}_h[f_R D_j \boldsymbol{u}] = 0 + \mathbb{P}_h[D_j \boldsymbol{u}] \quad \text{em} \quad L^r(\mathbb{R}^n) \tag{D.6}$$

Pelas equações (D.5) e (D.6), e pela unicidade do limite,concluímos que

$$D_j \mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[D_j \boldsymbol{u}].$$

$\square$

Corolário D.4. *Seja* $1 < q_0 < \infty$ *e* $1 < q_1 < \infty$. *Sendo* $\boldsymbol{u} \in L^{q_0}(\mathbb{R}^n)$, *com* $D_j\boldsymbol{u} \in L^{q_1}(\mathbb{R}^n)$. *Então*

$$\Delta\mathbb{P}_h[\boldsymbol{u}] = \mathbb{P}_h[\Delta\boldsymbol{u}]$$

Demonstração. Basta aplicar o teorema (D.3) duas vezes.

$$\Delta\mathbb{P}_h[\boldsymbol{u}] = \sum_{j=1}^{3} D_j D_j \mathbb{P}_h[\boldsymbol{u}] = \sum_{j=1}^{3} D_j \mathbb{P}_h[D_j\boldsymbol{u}] = \sum_{j=1}^{3} \mathbb{P}_h[D_j D_j\boldsymbol{u}] = \mathbb{P}_h[\sum_{j=1}^{3} D_j D_j\boldsymbol{u}] = \mathbb{P}_h[\Delta\boldsymbol{u}].$$

$\square$

Teorema D.5. *Seja* $u \in L^q(\mathbb{R}^n)$, $1 < q < \infty$. *Então*

$$\mathbb{P}_h[e^{\nu\Delta t}u] = e^{\nu\Delta t}[\mathbb{P}_h u]$$

Demonstração. Defina v e w como

$$v(\cdot, t) := e^{\nu\Delta t}u$$
$$w(\cdot, t) := e^{\nu\Delta t}\mathbb{P}_h u.$$

v e w satisfazem, respectivamente, as seguintes equações do calor

$$v_t = \nu\Delta v,\ v(\cdot, t) \in C^0([0, \infty), L^q(\mathbb{R}^n))$$
$$v(\cdot, 0) = u,$$

$$
\begin{aligned}
&w_t = \nu\Delta w,\ w(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n)) \\
&w(\cdot,0) = \mathbb{P}_h u.
\end{aligned} \tag{D.7}
$$

Vamos mostrar que $w(\cdot,t) = \mathbb{P}_h[v(\cdot,t)]$. Nossa estratégia será mostrar que $\mathbb{P}_h[v(\cdot,t)]$ é solução de (D.7) e que (D.7) tem solução única.

Definindo $z(\cdot,t) := \mathbb{P}_h[v(\cdot,t)]$, como $v(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n))$ e $\mathbb{P}_h : L^q(\mathbb{R}^n) \to L^q_\sigma(\mathbb{R}^n)$ é contínuo, segue que $z(\cdot,t) \in C^0([0,\infty), L^q(\mathbb{R}^n))$ e, em particular, $z(\cdot,t) \to \mathbb{P}_h u$ em $L^q(\mathbb{R}^n)$ ao $t \to 0$.

Como v é solução de uma equação do calor com condição inicial em $L^q(\mathbb{R}^n)$, vale que $Dv \in L^q(\mathbb{R}^n)$. Pelo Corolário (D.4), temos que $\Delta\mathbb{P}_h[v] = \mathbb{P}_h[\Delta v]$.

Por outro lado

$$
\begin{aligned}
\|z_t(\cdot,t) - \mathbb{P}_h[v_t(\cdot,t)]\|_{L^q(\mathbb{R}^n)} &= \left\| \lim_{h\to 0} \frac{z(\cdot,t+h) - z(\cdot,t)}{h} - \mathbb{P}_h[v_t(\cdot,t)] \right\|_{L^q(\mathbb{R}^n)} \\
&= \left\| \lim_{h\to 0} \mathbb{P}_h\left[\frac{v(\cdot,t+h) - v(\cdot,t)}{h} - v_t(\cdot,t) \right] \right\|_{L^q(\mathbb{R}^n)} \\
&= \left\| \mathbb{P}_h\left[\lim_{h\to 0} \frac{v(\cdot,t+h) - v(\cdot,t)}{h} - v_t(\cdot,t) \right] \right\|_{L^q(\mathbb{R}^n)} \\
&= \|\mathbb{P}_h[0]\|_{L^q(\mathbb{R}^n)} = 0.
\end{aligned}
$$

De modo que $z_t = \mathbb{P}_h[v_t]$. Portanto

$$
\Delta z = \Delta\mathbb{P}_h[v] = \mathbb{P}_h[\Delta v] = \mathbb{P}_h[v_t] = z_t.
$$

z então satisfaz a equação (D.7).

$\square$

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