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CONTINUITY OF LYAPUNOV EXPONENTS FOR COCYCLES WITH INVARIANT HOLONOMIES

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ABSTRACT. We prove a conjecture of Viana which states that Lyapunov exponents vary continuously when restricted to $GL(2,\mathbb{R})$-valued cocycles over a subshift of finite type which admit invariant holonomies that depend continuously on the cocycle.

1. INTRODUCTION

Consider an invertible measure preserving transformation $f:(X,\mu)\to(X,\mu)$ of a standard probability space. For simplicity, assume μ to be ergodic. Given a measurable function $A\colon X\to GL(d,\mathbb{R})$ we define the linear cocycle over f by the dynamically defined products

$$(1)\qquad A^n(x)=\begin{cases} A(f^{n-1}(x))\cdots A(f(x))A(x) & \text{if } n>0\\ Id & \text{if } n=0\\ (A^{-n}(f^n(x)))^{-1}=A(f^n(x))^{-1}\cdots A(f^{-1}(x))^{-1} & \text{if } n<0.\end{cases}$$

Under certain integrability hypotheses (for instance, if the range of A is bounded), Oseledets theorem guarantees the existence of numbers $\lambda_1>\ldots>\lambda_k$, called the *Lyapunov exponents*, and a decomposition $\mathbb{R}^d=E^1_x\oplus\ldots\oplus E^k_x$, called the *Oseledets splitting*, into vector subspaces depending measurably on x such that for almost every x

$$A(x)E^i_x=E^i_{f(x)} \text{ and } \lambda_i=\lim_{n\to\pm\infty}\frac{1}{n}\log\|A^n(x)v\|$$

for every non-zero $v\in E^i_x$ and $1\le i\le k$.

Lyapunov exponents arrive naturally in the study smooth dynamics. Indeed, given a diffeomorphism of a manifold that preserves a probability measure, the derivative determines a natural cocycle associated to the system. The corresponding Lyapunov exponents play a central role in the modern study of dynamical systems. For instance, given a C^2 diffeomorphism preserving a measure

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with negative exponents, Pesin constructed stable manifolds through almost every point [26]. Moreover, Lyapunov exponents are deeply connected with the entropy of smooth dynamical systems and the geometry measures as shown by the entropy formulas of Ruelle [28], Pesin [27], and Ledrappier–Young [20, 21].

In the present paper, we are interested in the continuity properties of Lyapunov exponents as one varies the cocycle and the underlying measure while keeping the base dynamics constant. Our base dynamics will be a subshift of finite type or, more generally, a hyperbolic set and our measures will always be taken to be measures admitting a local product structure. As a corollary of our main result, we obtain continuity of Lyapunov exponents for fiber-bunched cocycles in the space of Hölder continuous cocycles, giving an affirmative answer to a conjecture [31, Conjecture 10.12] of Viana (see Sections 2 and 3 for precise definitions and statements):

THEOREM 1.1. *Lyapunov exponents vary continuously when restricted to the subset of fiber-bunched elements $A\colon M \to GL(2,\mathbb{R})$ of the space $H^r(M)$.*

In general, one can not expect to obtain continuity of Lyapunov exponents in the space of Hölder cocycles without any extra assumption. Indeed, in [6], Bocker and Viana presented an example of a Hölder-continuous, $SL(2,\mathbb{R})$-valued cocycle with non-zero Lyapunov exponents which is approximated in the Hölder topology by cocycles with zero Lyapunov exponents. Recently the third author has refined the Bocker-Viana construction to build a family of examples of discontinuity of Lyapunov exponents in the Hölder topology which are arbitrarily close to being fiber-bunched [12]. Thus, Theorem 1.1 is sharp for this family.

The technique employed by Bocker and Viana to construct their example is a refinement of a technique used by Bochi [4, 5] to prove the Bochi–Mañé theorem. This theorem implies that, in the space of continuous cocycles over aperiodic base dynamics, the only continuity points for Lyapunov exponents of $SL(2,\mathbb{R})$-valued cocycles are those which are (uniformly) hyperbolic and those with zero exponents. Thus, discontinuity of Lyapunov exponents is typical if one only assumes continuous variation of the cocycle.

The main dynamical feature exhibited by fiber-bunched cocycles is the existence of a continuous family of invariant holonomies. These holonomies moreover vary continuously with the cocycle. This is the main geometric property we exploit to establish the continuity of Lyapunov exponents. Our main theorem below states that Lyapunov exponents depend continuously on the cocycle and on the underlying measure if we restrict ourselves to families of cocycles admitting invariant holonomies and to families of invariant measures with local product structure and "well behaved" Jacobians.

Even though discontinuity of Lyapunov exponents is a quite common feature as pointed out above, there are some contexts where continuity has been previously established. For instance, Furstenberg and Kifer [14, 17] established continuity of the largest Lyapunov exponent for i.i.d. random matrices under certain irreducibility conditions. In the same setting, but under assumption of strong irreducibility and a certain contraction property, Le Page [22] showed local Hölder continuity and even smoothness of Lyapunov exponents. Duarte and Klein [13] derived Hölder continuity of the Lyapunov exponents for a class irreducible Markov cocycles. In certain cases one can obtain real-analyticity of the Lyapunov exponents [29, 25]. Continuity has also been shown in the context of Schrödinger cocycles by Bourgain and Jitomirskaya [9, 10]. More recently, Bocker and Viana [6] and Malheiro and Viana [23] proved continuity of Lyapunov exponents for random products of 2-dimensional matrices in the Bernoulli and Markov settings. Our result extends the results of [6] and [23]. In higher dimensions, continuous dependence of all Lyapunov exponents for i.i.d. random products of matrices in $GL(d,\mathbb{R})$ was announced by Avila, Eskin, and Viana [2].

2. DEFINITIONS AND STATEMENT OF MAIN THEOREM

2.1. **Subshifts of finite type.** Let $Q = (q_{ij})_{1\le i,j\le \ell}$ be an $\ell\times\ell$ matrix with $q_{ij}\in\{0,1\}$. The *subshift of finite type* associated to the matrix Q is the subset of the bi-infinite sequences $\{1,\dots,\ell\}^{\mathbb{Z}}$ satisfying

$$\hat\Sigma = \{(x_n)_{n\in\mathbb{Z}} \colon q_{x_n x_{n+1}} = 1 \text{ for all } n\in\mathbb{Z}\}.$$

We require that each row and column of Q contains at least one nonzero entry. We let $\hat f\colon \hat\Sigma\to\hat\Sigma$ be the left-shift map defined by $\hat f(x_n)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$. We will always assume that $\hat f$ is topologically transitive on $\hat\Sigma$. We let

$$\Sigma^u = \{(x_n)_{n\ge 0} : q_{x_n x_{n+1}} = 1 \text{ for all } n\ge 0\},$$

$$\Sigma^s = \{(x_n)_{n\le 0} : q_{x_n x_{n+1}} = 1 \text{ for all } n\le -1\}.$$

We have projections $P^u\colon\hat\Sigma\to\Sigma^u$ and $P^s\colon\hat\Sigma\to\Sigma^s$ obtained by dropping all of the negative coordinates and all of the positive coordinates, respectively, of a sequence in $\hat\Sigma$. We let f_s and f_u denote the right and left shifts on Σ^s and Σ^u, respectively.

We define the *local stable set* of $\hat x\in\hat\Sigma$ to be

$$W^s_{\mathrm{loc}}(\hat x) = \{(y_n)_{n\in\mathbb{Z}}\in\hat\Sigma : x_n = y_n \text{ for all } n\ge 0\},$$

and the *local unstable set* to be

$$W^u_{\mathrm{loc}}(\hat x) = \{(y_n)_{n\in\mathbb{Z}}\in\hat\Sigma : x_n = y_n \text{ for all } n\le 0\}.$$

We think of open subsets of Σ^s and Σ^u, respectively, as parametrizations of the local stable and unstable sets. We define

$$\Omega^s = \{(\hat x,\hat y)\in\hat\Sigma\times\hat\Sigma : \hat y\in W^s_{\mathrm{loc}}(\hat x)\},$$

$$\Omega^u = \{(\hat x,\hat y)\in\hat\Sigma\times\hat\Sigma : \hat y\in W^u_{\mathrm{loc}}(\hat x)\}.$$

Then Ω^s and Ω^u can be expressed locally as the product of a cylinder in $\hat\Sigma$ with a cylinder in Σ^s and Σ^u, respectively. For $x\in\Sigma^u$ we define $W^s_{\mathrm{loc}}(x) = (P^u)^{-1}(x)$ and for $y\in\Sigma^s$ we write $W^u_{\mathrm{loc}}(y) = (P^s)^{-1}(y)$. Observe that if $\hat x\in\hat\Sigma$, then $W^s_{\mathrm{loc}}(\hat x) = W^s_{\mathrm{loc}}(P^u(\hat x))$. In this way, we also think of Σ^u as a parametrization of the space of local stable sets.

Each $\theta \in (0,1)$ gives rise to a metric on $\hat{\Sigma}$,

$$d_\theta(\hat{x},\hat{y}) = \theta^{N(\hat{x},\hat{y})}, \text{ where } N(\hat{x},\hat{y}) = \max\{N \geq 0; x_n = y_n \text{ for all } |n| < N\}.$$

These metrics are all Hölder equivalent to one another and thus each defines the same topology on $\hat{\Sigma}$.

For $m \in \mathbb{Z}$ and $a_0,\ldots,a_k \in \{1,\ldots,\ell\}$, we define the cylinder notation

$$[m; a_0,\ldots,a_k] = \{\hat{x} \in \hat{\Sigma} : x_{m+i} = a_i, 0 \leq i \leq k\}.$$

2.2. **Stable and unstable holonomies.** A d-dimensional *linear cocycle* $\hat{A}$ over $\hat{f}$ is a map $\hat{A} \colon \hat{\Sigma} \to GL(d,\mathbb{R})$.

DEFINITION 2.1. A *stable holonomy* for a linear cocycle $\hat{A}$ over $\hat{f}$ is a collection of linear maps $H^{s,\hat{A}}_{\hat{x}\hat{y}} \in GL(d,\mathbb{R})$ defined for $\hat{y} \in W^s_{\mathrm{loc}}(\hat{x})$ which satisfy the following properties,

- $H^{s,\hat{A}}_{\hat{y}\hat{z}} = H^{s,\hat{A}}_{\hat{x}\hat{z}} H^{s,\hat{A}}_{\hat{y}\hat{x}}$ and $H^{s,\hat{A}}_{\hat{x}\hat{x}} = Id$;
- $H^{s,\hat{A}}_{\hat{f}(\hat{y})\hat{f}(\hat{z})} = \hat{A}(\hat{z}) H^{s,\hat{A}}_{\hat{y}\hat{z}} \hat{A}(\hat{y})^{-1}$;
- The map $\Omega^s \to GL(d,\mathbb{R})$ given by $(\hat{x},\hat{y}) \mapsto H^{s,\hat{A}}_{\hat{x}\hat{y}}$ is continuous.

By replacing $\hat{f}$ and $\hat{A}$ with the inverse cocycle $\hat{A}^{-1}$ over $\hat{f}^{-1}$, we get an analogous definition of unstable holonomies $H^{u,\hat{A}}_{\hat{x}\hat{y}}$ for $\hat{y} \in W^u_{\mathrm{loc}}(\hat{x})$.

Stable and unstable holonomies for linear cocycles are not unique in general, even if the cocycle is locally constant (see [16]). To circumvent this issue we define a *cocycle with holonomies* to be a triple $(\hat{A}, H^{s,\hat{A}}, H^{u,\hat{A}})$ where $\hat{A}$ is a linear cocycle over $\hat{f}$ and $H^{s,\hat{A}}$ and $H^{u,\hat{A}}$ are a stable and unstable holonomy for $\hat{A}$, respectively. We let $\mathscr{H}$ denote the space of all cocycles with holonomies, endowed with the subspace topology given by the inclusion

$$\mathscr{H} \hookrightarrow C^0(\hat{\Sigma}, GL(d,\mathbb{R})) \times C^0(\Omega^s, GL(d,\mathbb{R})) \times C^0(\Omega^u, GL(d,\mathbb{R})),$$

where $\mathscr{H}$ is cut out by the linear equations in Definition 2.1 and these spaces of maps have the uniform topology. This means that a sequence of cocycles with holonomies $\{(\hat{A}_n, H^{s,\hat{A}_n}, H^{u,\hat{A}_n})\}_{n\in\mathbb{N}}$ converges to $(\hat{A}, H^{s,\hat{A}}, H^{u,\hat{A}})$ if $\hat{A}_n \to \hat{A}$ uniformly and the stable and unstable holonomies converge uniformly on local stable and unstable sets, respectively.

DEFINITION 2.2. A sequence of linear cocycles $\{\hat{A}_n\}_{n\in\mathbb{N}}$ over $\hat{f}$ *converges uniformly with holonomies* to a linear cocycle $\hat{A}$ if for each n there is a triple

$$(\hat{A}_n, H^{s,\hat{A}_n}, H^{u,\hat{A}_n}) \in \mathscr{H}$$

such that this sequence converges in $\mathscr{H}$ to a triple $(\hat{A}, H^{s,\hat{A}}, H^{u,\hat{A}})$ defining a stable and unstable holonomy for $\hat{A}$.

REMARK 2.3. If $\hat{A}$ is α-Hölder continuous and α-fiber-bunched (see Definition 3.1) or locally constant, there is a canonical stable holonomy for $\hat{A}$ defined by the formula

$$H^{s,\hat{A}}_{\hat{x}\hat{y}} = \lim_{n\to\infty} \hat{A}^n(\hat{y})^{-1} \hat{A}^n(\hat{x}), \;\; \hat{y} \in W^s_{\mathrm{loc}}(\hat{x}).$$

Our definition of stable and unstable holonomies is more general and does not imply that the sequence on the right converges.

2.3. **Product structure of measures.** For an $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{\Sigma}$ we let $\mu^u = P^u_*\hat{\mu}$ and $\mu^s = P^s_*\hat{\mu}$. The map $[0;i] \to P^s([0;i]) \times P^u([0;i])$ induced by $\hat{x} \to (P^s(\hat{x}), P^u(\hat{x}))$ is a homeomorphism. We say that an $\hat{f}$-invariant measure $\hat{\mu}$ on $\hat{\Sigma}$ has *local product structure* if there is a positive continuous function $\psi : \hat{\Sigma} \to (0,\infty)$ such that the restriction is of the form

$$d\hat{\mu}|_{[0;i]} = \psi \ d(\mu^s|_{P^s([0;i])} \times \mu^u|_{P^u([0;i])}).$$

That is, for $A \subset [0;i]$, we have

$$\mu(A) = \int_A \psi d(\mu^s|_{P^s([0;i])} \times \mu^u|_{P^u([0;i])}).$$

Observe that since $\mu^u = P^u_*\hat{\mu}$, we have that $\int_{W^s_{\text{loc}}(x)} \psi(\hat{x}) d\mu^s(\hat{x}) = 1$ on every local stable set $W^s_{\text{loc}}(x)$.

A *Jacobian* of the measure μ^u *with respect to the dynamics* f_u is the measurable function $J_{\mu^u} f_u$ such that

$$d\left((f_u)_*(\mu^u|_{[0;i]})\right)(f(y)) = (J_{\mu^u} f_u(y))^{-1} \ d\mu^u(f(y)).$$

A Jacobian of μ^s with respect to f_s is defined similarly.

Lemmas 2.4 and 2.5 below give consequences of the existence of local product structure for μ which are well known, see for instance [8, Lemmas 2.1, 2.2]. We reproduce the proofs here to indicate explicitly how the local product structure of $\hat{\mu}$ is used; in particular, we emphasize in the proofs that the Jacobians and disintegrations constructed depend continuously on the function ψ which gives the local product structure of $\hat{\mu}$ via explicit formulas.

LEMMA 2.4. *Assume $\hat{\mu}$ has local product structure. Then the measure μ^u admits a continuous positive Jacobian $J_{\mu^u} f_u$ with respect to the map f_u. Similarly μ^s admits a continuous positive Jacobian $J_{\mu^s} f_s$ with respect to the map f_s.*

Proof. Let $y \in \Sigma^u$ and D be any measurable set containing y and contained in a cylinder $[0;i,j]$. Thus, by definition,

$$\mu^u(f_u(D)) = \hat{\mu}((P^u)^{-1}(f_u(D))) = \int_{\{x \in f_u(D)\}} \psi(x,z) d\mu^u(x) d\mu^s(z)$$

and moreover,

$$\mu^u(D) = \hat{\mu}((P^u)^{-1}(D)) = \hat{\mu}(\hat{f}((P^u)^{-1}(D))) = \int_{\{x \in f_u(D), z_{-1}=i\}} \psi(x,z) d\mu^u(x) d\mu^s(z),$$

where in the second equality we have used the $\hat{f}$-invariance of $\hat{\mu}$. Now, letting D shrink to $\{y\}$ we have

$$\frac{\mu^u(f_u(D))}{\mu^u(D)} \to \frac{1}{\int_{\{z_{-1}=i\}} \psi(f_u(y),z) d\mu^s(z)}.$$

Defining $J_{\mu^u} f_u(y) := \dfrac{1}{\int_{\{z_{-1}=i\}} \psi(f_u(y), z)\, d\mu^s(z)}$, which is clearly positive and continuous, we get the desired result. An analogous proof replacing f_u by f_s shows that f_s admits a continuous positive Jacobian $J_{\mu^s} f_s$ with respect to μ^s. $\square$

Given $x, y \in \Sigma^u$ in the same cylinder $P^u([0;i])$, we define the *unstable holonomy map* $h_{x,y}\colon W^s_{\mathrm{loc}}(x) \to W^s_{\mathrm{loc}}(y)$, by assigning to each $\hat{x} \in W^s_{\mathrm{loc}}(x)$ the unique $\hat{y} = h_{x,y}(\hat{x}) \in W^s_{\mathrm{loc}}(y)$ with $\hat{y} \in W^u_{\mathrm{loc}}(\hat{x})$.

The partition of $(\Sigma, \hat{\mu})$ into local stable sets is a measurable partition and thus induces a disintegration into a family of conditional measures $\{\hat{\mu}_x\}_{x \in \Sigma^u}$ with each $\hat{\mu}_x$ supported on $W^s_{\mathrm{loc}}(x)$. All such families agree up to null sets.

Using the local product structure of the measure $\hat{\mu}$ we have the following.

LEMMA 2.5. *Assume $\hat{\mu}$ has local product structure. Then the measure $\hat{\mu}$ has a disintegration into conditional measures $\{\hat{\mu}_x\}_{x\in\Sigma^u}$ that vary continuously with x in the weak-$*$ topology. In fact, for every $x, y \in \Sigma^u$ in the same cylinder $[0;i]$,*

$$h_{x,y}\colon (W^s_{\mathrm{loc}}(x), \hat{\mu}_x)) \to (W^s_{\mathrm{loc}}(y), \hat{\mu}_y)$$

is absolutely continuous, with Jacobian $R_{x,y}$ depending continuously on (x, y).

Proof. For each $i \in \{1, \dots, \ell\}$, the local product structure of $\hat{\mu}$ allows us to express $\hat{\mu}|_{[0;i]}$ as $\psi \cdot (\mu^s|_{P^s([0;i])} \times \mu^u|_{P^u([0;i])})$ for a positive continuous function ψ. We have $\int_{W^s_{\mathrm{loc}}(x)} \psi(\hat{x})\, d\mu^s(\hat{x}) = 1$ on every local stable set and thus $\hat{\mu}_x = \psi(\hat{x})\mu^s$ and $R_{x,y}(\hat{x}) = \dfrac{\psi(h_{x,y}(\hat{x}))}{\psi(\hat{x})}$ define a disintegration of $\hat{\mu}$ and a Jacobian for $h_{x,y}$ as we want. $\square$

REMARK 2.6. Observe that, with the above disintegration of $\hat{\mu}$ and the Jacobians given in Lemma 2.4, we have that

$$\hat{\mu}_{f^n_u(y)} \,|_{\hat{f}^n(W^s_{\mathrm{loc}}(y))} = \frac{1}{J_{\mu^u} f^n_u(y)} \hat{f}^n_* \hat{\mu}_y$$

for every $y \in \Sigma^u$.

In order to state the main theorem we need to formulate a notion of convergence of probability measures on $\hat{\Sigma}$ which is stronger than weak-$*$-convergence. We say that a sequence of $\hat{f}$-invariant probability measures $\{\hat{\mu}_k\}_{k\in\mathbb{N}}$ with local product structure converges to an $\hat{f}$-invariant measure $\hat{\mu}$ with local product structure if $\hat{\mu}_k$ converges to $\hat{\mu}$ in the weak-$*$ topology on probability measures on $\hat{\Sigma}$ and the positive continuous functions ψ_k defining the local product structure of $\hat{\mu}_k$ converge uniformly to the function ψ defining the local product structure of $\hat{\mu}$. Uniform convergence of ψ_k to ψ together with the weak-$*$ convergence of $\hat{\mu}_k$ implies that the sequences of stable and unstable Jacobians $\{J_{\mu^u_k} f_u\}_{k\in\mathbb{N}}$ and $\{J_{\mu^s_k} f_s\}_{k\in\mathbb{N}}$ converge uniformly to $J_{\mu^u} f_u$ and $J_{\mu^s} f_s$, respectively. For furthermore, the conditional measures $\hat{\mu}^k_x$ of $\hat{\mu}_k$ along Σ^u converge uniformly to the conditional measures of $\hat{\mu}$ along Σ^u; that is, identifying all local unstable sets $W^u_{\mathrm{loc}}(\hat{x})$ for $\hat{x} \in [0;i]$ with a 0-cylinder in Σ^u, the functions $\hat{x} \mapsto \hat{\mu}^k_x$ converge uniformly to $\hat{x} \mapsto \hat{\mu}_x$.

As a shorthand for this notion of convergence we will say that "$\hat{\mu}_k$ converges to $\hat{\mu}$ as in Section 2.3". A useful criterion for checking this notion of convergence as well as the existence of local product structure is given in the next lemma.

LEMMA 2.7. *Let $\hat{\mu}$ be an ergodic, fully supported probability measure on $\hat{\Sigma}$. Suppose that the projected measure $\mu^u = P^u_*\hat{\mu}$ admits a positive β-Hölder continuous Jacobian $J_{\mu^u} f_u$ in the β-Hölder norm with respect to f_u. Then $\hat{\mu}$ has local product structure given by a function $\psi\colon \hat{\Sigma} \to (0,\infty)$ and moreover ψ depends continuously on $J_{\mu^u} f_u$ in the β-Hölder norm.*

Proof. The assertion that $\hat{\mu}$ admits local product structure follows from [8, Lemmas 2.4, 2.6] since the Jacobian $J_{\mu^u} f_u$ is assumed to be Hölder continuous. To establish that ψ depends continuously on $J_{\mu^u} f_u$, we recall the formula for ψ derived in the course of the proof.

Fix points $z_i \in P^u([0;i])$ for $1 \le i \le \ell$. The construction of the local product structure in [8] gives the following formula for ψ: for $\hat{x} \in [0;i]$,

$$\psi(\hat{x}) = \lim_{n\to\infty} \frac{J_{\mu^u} f_u(P^u(\hat{f}^n(h_{x,z_i}(\hat{x}))))}{J_{\mu^u} f_u(P^u(\hat{f}^n(\hat{x})))}$$

if we identify μ^u with the measure $\hat{\mu}_{z_i}$ from Lemma 2.5. A standard argument using distortion estimates shows that the limit on the right side exists and depends continuously on the function $J_{\mu^u} f_u$ in the β-Hölder topology, see [8, Lemma 2.4] or the arguments at the beginning of the proof of Lemma 3.2. □

As a consequence, if $\hat{\mu}_k \to \hat{\mu}$ is a sequence of measures converging in the weak-* topology all of which are ergodic, fully supported, and have local product structure, and moreover the Jacobians $J_{\mu^u_k} f_u$ are β-Hölder continuous and converge in the β-Hölder norm to $J_{\mu^u} f_u$, then $\hat{\mu}_k$ converges to $\hat{\mu}$ as in Section 2.3.

2.4. **Main theorem.** For a continuous cocycle $\hat{A}$ over $\hat{f}$ and an $\hat{f}$-invariant probability measure $\hat{\mu}$ on $\hat{\Sigma}$, it follows by the Kingman Sub-Additive Ergodic Theorem ([18]) that

$$\lambda_+(\hat{A},\hat{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|\hat{A}^n(\hat{x})\|$$

and

$$\lambda_-(\hat{A},\hat{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|(\hat{A}^n(\hat{x}))^{-1}\|^{-1}$$

are well-defined at $\hat{\mu}$ almost every point $\hat{x} \in \hat{\Sigma}$. These are the (extremal) *Lyapunov exponents* of $\hat{A}$. These functions are $\hat{f}$-invariant and hence, if $\hat{\mu}$ is ergodic with respect to $\hat{f}$, these functions are constant $\hat{\mu}$-a.e. In this case, we define $\lambda_+(\hat{A},\hat{\mu})$ and $\lambda_-(\hat{A},\hat{\mu})$ to be the $\hat{\mu}$-a.e. constant values of the extremal Lyapunov exponents.

The main theorem of the paper is a criterion for joint continuity of the Lyapunov exponents $\lambda_+(\hat{A},\hat{\mu})$ and $\lambda_-(\hat{A},\hat{\mu})$ in the cocycle $\hat{A}$ and the measure $\hat{\mu}$ in the case when $\hat{A}$ is 2-dimensional.

THEOREM 2.8. *Let $\{\hat{A}_n\}_{n\in\mathbb{N}}$ be a sequence of 2-dimensional linear cocycles over $\hat{f}$ converging uniformly with holonomies to a cocycle $\hat{A}$ and $\{\hat{\mu}_n\}_{n\in\mathbb{N}}$ a sequence of fully supported, ergodic, $\hat{f}$-invariant probability measures converging as in Section 2.3 to an ergodic, $\hat{f}$-invariant measure $\hat{\mu}$ with local product structure and full support. Then $\lambda_+(\hat{A}_n,\hat{\mu}_n)\to\lambda_+(\hat{A},\hat{\mu})$ and $\lambda_-(\hat{A}_n,\hat{\mu}_n)\to\lambda_-(\hat{A},\hat{\mu})$.*

Theorem 2.8 provides an affirmative answer to [31, Conjecture 10.13]. The proof of Theorem 2.8 begins in Section 4. We collect some corollaries of Theorem 2.8 in Section 3 below.

3. COROLLARIES

In this section we demonstrate how to apply Theorem 2.8 to prove continuity of the Lyapunov exponents for certain classes of 2-dimensional linear cocycles over hyperbolic systems. We fix a $\theta\in(0,1)$ and for $\alpha>0$ we let $C^\alpha(\hat{\Sigma},GL(d,\mathbb{R}))$ be the space of α-Hölder continuous linear cocycles over the shift with respect to the metric d_θ on $\hat{\Sigma}$. $C^\alpha(\hat{\Sigma},GL(d,\mathbb{R}))$ is a Banach space with the α-Hölder norm

$$\|\hat{A}\|_\alpha=\sup_{\hat{x}\in\hat{\Sigma}}\|\hat{A}(\hat{x})\|+\sup_{\hat{x}\neq\hat{y}\in\hat{\Sigma}}\frac{\|\hat{A}(\hat{x})-\hat{A}(\hat{y})\|}{d_\theta(\hat{x},\hat{y})^\alpha}.$$

DEFINITION 3.1. A linear cocycle $\hat{A}\colon\hat{\Sigma}\to GL(d,\mathbb{R})$ is *α-fiber-bunched* if $\hat{A}\in C^\alpha(\hat{\Sigma},GL(d,\mathbb{R}))$ and there is an $N>0$ such that

$$\|\hat{A}^N(\hat{x})\|\cdot\|(\hat{A}^N(\hat{x}))^{-1}\|^{-1}\cdot\theta^{-N\alpha}<1$$

for every $\hat{x}\in\hat{\Sigma}$.

The set of α-fiber-bunched cocycles is open in $C^\alpha(\hat{\Sigma},GL(d,\mathbb{R}))$.

For each Hölder continuous potential $\varphi\colon\hat{\Sigma}\to\mathbb{R}$ we may associate a unique *equilibrium state* $\hat{\mu}_\varphi$ which is an ergodic, fully supported probability measure on $\hat{\Sigma}$ with local product structure [11, 24]. The following lemma shows that Hölder-convergence of potentials implies convergence of equilibrium states as in Section 2.3.

LEMMA 3.2. *If $\varphi_k\to\varphi$ in $C^\beta(\hat{\Sigma},\mathbb{R})$ for some $\beta>0$ then $\hat{\mu}_{\varphi_k}$ converges to $\hat{\mu}_\varphi$ as in Section 2.3.*

Proof. We recall some well-known facts about equilibrium states which can be found in [11]. We first note that it suffices to prove the claim when the functions φ_k are constant on the local stable sets of $\hat{f}$. For $\hat{x},\hat{y}\in[0;i]$ for some $1\le i\le\ell$ we let $h^s_{\hat{x},\hat{y}}$ denote the stable holonomy from $W^u_{\mathrm{loc}}(\hat{x})$ to $W^u_{\mathrm{loc}}(\hat{y})$ which assigns to each $\hat{z}\in W^u_{\mathrm{loc}}(\hat{x})$ the unique point $h^s_{\hat{x},\hat{y}}(\hat{z})\in W^s_{\mathrm{loc}}(\hat{z})\cap W^u_{\mathrm{loc}}(\hat{y})$. Now fix ℓ points $\hat{z}_1,\dots,\hat{z}_\ell$ such that $\hat{z}_i\in[0;i]$. We then define for $\hat{x}\in[0;i]$,

$$\psi^u_k(\hat{x})=\sum_{j=0}^{\infty}\varphi_k(\hat{f}^j(\hat{x}))-\varphi_k(\hat{f}^j(h^s_{\hat{x},\hat{z}_i}))$$

and define $\varphi^u_k(\hat{x})=\varphi_k(h^s_{\hat{x},\hat{z}_i}(\hat{x}))$. These functions then satisfy the equation

$$\varphi^u_k=\varphi_k+\psi^u_k\circ\hat{f}-\psi^u_k$$

which implies that φ_k^u is cohomologous to φ_k. Furthermore φ_k^u is constant on local stable sets and thus descends to a continuous function on Σ^u for each k. Since cohomologous potentials define the same equilibrium state we get that $\hat{\mu}_{\varphi_k^u} = \hat{\mu}_{\varphi_k}$ for all k. Each function ψ_k^u is β-Hölder continuous and thus so is φ_k^u, and as $k \to \infty$ convergence of φ_k to φ in $C^\beta(\hat{\Sigma}, \mathbb{R})$ implies convergence of ψ_k^u to the corresponding function ψ_k for φ in $C^\beta(\hat{\Sigma}, \mathbb{R})$, and thus φ_k^u converges to φ^u in $C^\beta(\hat{\Sigma}, \mathbb{R})$.

Thus we may assume that φ_k and φ are constant on local stable sets of $\hat{f}$, and hence they descend to Hölder continuous functions on Σ^u. Recall that the transfer operator $T_\varphi \colon C^0(\Sigma^u, \mathbb{C}) \to C^0(\Sigma^u, \mathbb{C})$ associated to φ on Σ^u is defined on continuous functions $g \colon \Sigma^u \to \mathbb{C}$ by

$$T_\varphi g(x) = \sum_{y \in f_u^{-1}(x)} e^{\varphi(y)} g(y).$$

By the Ruelle-Perron-Frobenius theorem, T_φ acts with a spectral gap on the Banach space $C^\beta(\Sigma^u, \mathbb{C})$[11]. Let ν_φ^u be the dominant eigenvector for the adjoint action of T_φ on probability measures and $\zeta_\varphi^u \in C^\beta(\Sigma^u, \mathbb{C})$ the strictly positive dominant eigenvector for T_φ which satisfies $\int_{\Sigma^u} \zeta_\varphi^u \, d\nu_\varphi = 1$ and has eigenvalue e^P, where P is the topological pressure of φ. Then μ_φ^u is given by $\mu_\varphi^u = \zeta_\varphi^u \nu_\varphi^u$ and the Jacobian of f_u with respect to μ_φ^u is $e^{P-\varphi} \frac{\zeta_\varphi}{\zeta_\varphi \circ f_u}$.

Since T_φ depends continuously on $\varphi \in C^\beta(\hat{\Sigma}, \mathbb{R})$, we conclude from the spectral gap property that the dominant eigenvector ζ_φ^u and its eigenvalue e^P depend continuously on φ in the β-Hölder norm and similarly the dominant eigenvector ν_φ^u for the adjoint depends continuously on φ in the weak-$*$ topology on probability measures on $\hat{\Sigma}$. Consequently convergence of φ_k to φ implies weak-$*$ convergence of $\mu_{\varphi_k}^u$ to μ_φ^u and convergence in the β-Hölder norm of $J_{\mu_{\varphi_k}^u} f_u$ to $J_{\mu_\varphi^u} f_u$. By Lemma 2.7 and the subsequent remark, we conclude that $\hat{\mu}_{\varphi_k}$ converges to $\hat{\mu}$ as in Section 2.3. □

COROLLARY 3.3. *For each $\alpha, \beta > 0$, the Lyapunov exponents*

$$\lambda_\pm \colon C^\alpha(\hat{\Sigma}, GL(2,\mathbb{R})) \times C^\beta(\hat{\Sigma}, \mathbb{R}) \to \mathbb{R}$$
$$(\hat{A}, \varphi) \to \lambda_\pm(\hat{A}, \hat{\mu}_\varphi)$$

are continuous when restricted to $\hat{A} \in C^\alpha(\hat{\Sigma}, GL(2,\mathbb{R}))$ which are α-fiber-bunched.

Proof. For fiber-bunched cocycles stable and unstable holonomies exist and moreover they vary continuously with respect to the cocycle in the α-Hölder topology (see [7] and [30]). Lemma 3.2 implies that if φ_k converges to φ in $C^\beta(\hat{\Sigma}, \mathbb{R})$ then the corresponding equilibrium states $\hat{\mu}_{\varphi_k}$ converge to $\hat{\mu}_\varphi$ as in Section 2.3. These two statements together then imply the corollary by Theorem 2.8. □

Continuous dependence of holonomies in the space of α-fiber-bunched cocycles may actually be shown under slightly weaker hypotheses than convergence in the Hölder topology. It suffices to assume that the linear cocycle A is

α-fiber-bunched, A_k converges to A in the C^0 topology, and each A_k and A are α-Hölder continuous with uniformly bounded Hölder constant. We refer the interested reader again to [7] and [30] for further details.

For our second application we give an example of how to use Markov partitions to prove continuity of the Lyapunov exponents for cocycles over other hyperbolic systems besides subshifts of finite type. Let M be a closed Riemannian manifold. Let $f: M \to M$ be an Anosov diffeomorphism, meaning that there is a Df-invariant splitting $TM = E^s \oplus E^u$ and constants $C > 0$, $0 < \nu < 1$ such that

$$\|Df^n|E^s\| \le C\nu^n, \ \|Df^{-n}|E^u\| \le C\nu^n, \ n \ge 1.$$

For $0 < \alpha \le 1$ we say that f is a $C^{1+\alpha}$ diffeomorphism of M if Df is α-Hölder continuous. We write $\mathrm{Diff}^{1+\alpha}(M)$ for the space of $C^{1+\alpha}$ diffeomorphisms of M, equipped with the topology of uniform convergence for f together with α-Hölder convergence for the derivative Df. For a $C^{1+\alpha}$ Anosov diffeomorphism f, the stable and unstable bundles E^s and E^u are each β-Hölder continuous for some $\beta > 0$. In analogy to Definition 3.1 we say that the derivative cocycle $Df|E^u$ is *fiber-bunched* if there is an $N > 0$ such that

$$\|Df_x^N|E_x^u\| \cdot \|(Df_x^N|E_x^u)^{-1}\| \cdot \max\{\|Df_x^N|E_x^s\|^\beta, \|(Df_x^N|E_x^u)^{-1}\|^\beta\} < 1.$$

As in the case of a subshift of finite type, for each Hölder continuous potential $\varphi: M \to \mathbb{R}$ we have an equilibrium state μ_φ which is a fully supported ergodic invariant probability measure for f. The two most important equilibrium states for f are the measure of maximal entropy (given by the potential $\varphi \equiv 0$) and the SRB measure characterized by having absolutely continuous conditional measures on the unstable leaves of f (given by $\varphi(x) = -\log(|\det(Df_x|E_x^u)|)$) which coincides with volume if f is volume-preserving. To emphasize the dependence of E^u on f we will write $E^{u,f}$ for the unstable bundle associated to f.

COROLLARY 3.4. *Let $f: M \to M$ be a transitive $C^{1+\alpha}$ Anosov diffeomorphism for some $\alpha > 0$ and $\varphi: M \to \mathbb{R}$ a Hölder continuous potential. If $\dim E^u = 2$ and $Df|E^u$ is fiber-bunched then f is a continuity point for the Lyapunov exponents $\lambda_\pm(Df|E^{u,f}, \mu_\varphi)$ as a function of $f \in \mathrm{Diff}^{1+\alpha}(M)$ and $\varphi \in C^\beta(M, \mathbb{R})$.*

Proof. Let f_k be a sequence of $C^{1+\alpha}$-diffeomorphisms converging in $\mathrm{Diff}^{1+\alpha}(M)$ to f. For large enough k, f_k is also an Anosov diffeomorphism, and moreover by structural stability there is a unique Hölder continuous homeomorphism $g_k: M \to M$ close to the identity such that $g_k \circ f_k = f \circ g_k$. Let $G_k: E^{u,f_k} \to E^{u,f}$ be a homeomorphism covering the homeomorphism $g_k: M \to M$ which is linear on the fibers $G_k(x): E_x^{u,f_k} \to E_{g_k(x)}^{u,f}$ and such that G_k converges uniformly to the identity map on $E^{u,f}$ as $k \to \infty$. For k sufficiently large we may take $G_k(x)$ to be the orthogonal projection of the plane E_x^{u,f_k} onto $E_{g_k(x)}^{u,f}$.

Since $Df|E^{u,f}$ is fiber-bunched and fiber bunching of $Df|E^u$ is an open condition in $\mathrm{Diff}^{1+\alpha}(M)$ we conclude that the cocycles $Df_k|E^{u,f_k}$ all admit stable and unstable holonomies $H^{s,k}$ and $H^{u,k}$ along the stable and unstable manifolds of f_k and moreover that these stable and unstable holonomies converge locally uniformly to the stable and unstable holonomies H^s and H^u of $Df|E^u$ as the local stable and unstable manifolds of f_k converge uniformly to those of f (see [7] and [30]). We then define for each k a new cocycle A_k on the vector bundle $E^{u,f}$ by

$$A_k(x) = G_k(f_k(g_k^{-1}(x))) \circ Df_k(g_k^{-1}(x))|E^{u,f_k} \circ G_k^{-1}(g_k^{-1}(x))$$

which admits stable and unstable holonomies

$$\widetilde{H}^{*,k}_{xy} = G_k(g_k^{-1}(y)) \circ H^{*,k}_{g_k^{-1}(x)g_k^{-1}(y)} \circ G_k^{-1}(g_k^{-1}(x))$$

for $y \in W^*_f(x)$, $* = s,u$ and W^*_f being the stable and unstable manifolds of f. Since G_k converges uniformly to the identity on $E^{u,f}$ we conclude that A_k converges to $Df|E^{u,f}$ uniformly and further that the stable and unstable holonomies of A_k converge uniformly to those of $Df|E^{u,f}$.

The diffeomorphism f admits a Markov partition and thus there is a subshift of finite type $\hat{f}\colon \hat{\Sigma} \to \hat{\Sigma}$ and a topological semiconjugacy $h\colon \hat{\Sigma} \to M$ such that $h \circ \hat{f} = f \circ h$ [11]. By refining the Markov partition if necessary, we can assume that the image of each cylinder $[0;j]$ of $\hat{\Sigma}$ under h in M is contained inside of an open set on which the bundle $E^{u,f}$ is trivializable. For $\hat{x} \in [0;j]$ let $L_j(\hat{x})\colon \mathbb{R}^2 \to E^{u,f}_{h(\hat{x})}$ be the linear map associated to a fixed trivialization of $E^{u,f}$ over $h([0;j])$. We can then extend h to a continuous surjection $L\colon \hat{\Sigma} \times \mathbb{R}^2 \to E^{u,f}$,

$$L(\hat{x}, v) = [L_j(\hat{x})](v)$$

which is a linear isomorphism on each of the fibers. We then define new linear cocycles $\hat{A}_k\colon \hat{\Sigma} \to GL(2,\mathbb{R})$ by

$$\hat{A}_k(\hat{x}) = L^{-1}(\hat{f}(\hat{x})) \circ A_k(h(\hat{x})) \circ L(\hat{x})$$

which admit stable and unstable holonomies

$$\hat{H}^{*,k}_{\hat{x}\hat{y}} = L^{-1}(\hat{y}) \circ \widetilde{H}^{*,k}_{h(\hat{x})h(\hat{y})} \circ L(\hat{x})$$

for $y \in W^*_{\mathrm{loc}}(x)$, $* = s,u$. It is again clear that $\hat{A}_k$ converges to $\hat{A}$ uniformly and that the new stable and unstable holonomies $\hat{H}^{*,k}$ for $\hat{A}_k$ converge uniformly to those for $\hat{A}$.

Let $\nu_k = (g_k)_*\mu_{\varphi_k}$. ν_k is the equilibrium state for f associated to the potential $\varphi_k \circ g_k$ and thus is a fully supported ergodic f-invariant measure with local product structure on M. Let

$$\Omega = \{x \in M : \#h^{-1}(x) > 1\}.$$

Ω is a null set for any equilibrium state associated to a Hölder continuous potential [11]. Hence we can lift ν_k to an $\hat{f}$-invariant measure $\hat{\nu}_k$ on $\hat{\Sigma}$ such that $h_*\hat{\nu}_k = \nu_k$. Furthermore $\hat{\nu}_k$ is the equilibrium state associated to the potential $\psi_k = \varphi_k \circ g_k \circ h$ on $\hat{\Sigma}$. As $k \to \infty$, ψ_k converges in a Hölder norm to $\psi = \varphi \circ h$. It follows from Lemma 3.2 that $\hat{\nu}_k$ converges to $\hat{\nu}$ as in Section 2.3.

Hence by the criterion of the Theorem 2.8 we get that $\lambda_+(\hat{A}_k, \hat{\nu}_k) \to \lambda_+(\hat{A}, \hat{\nu})$ and the same statement for λ_-. By construction the map $h\colon (\hat{\Sigma}, \hat{\nu}_k) \to (M, \nu_k)$ is a measurable isomorphism, and the same holds with $\hat{\nu}_k$ and ν_k replaced by $\hat{\nu}$

and ν. Then by construction the map $L\colon \hat{\Sigma}\times\mathbb{R}^2 \to E^{u,f}$ gives a measurable conjugacy between the cocycles $\hat{A}_k$ and A_k. It follows that $\lambda_+(\hat{A}_k,\hat{\nu}_k) = \lambda_+(A_k,\nu_k)$ and $\lambda_+(\hat{A},\hat{\nu}) = \lambda_+(A,\nu) = \lambda_+(Df|E^{u,f},\mu_\varphi)$. The map $g_k\colon (M,\varphi_k) \to (M,\nu_k)$ is also a measurable isomorphism by construction and G_k gives a measurable conjugacy from $Df_k|E^{u,f_k}$ to A_k over this isomorphism. Hence we conclude that $\lambda_+(A_k,\nu_k) = \lambda_+(Df_k|E^{u,f_k},\mu_{\varphi_k})$ for each k, which completes the proof. $\square$

By replacing f with f^{-1} we obtain the same corollary for $Df|E^s$ instead, provided that $\dim E^s = 2$.

REMARK 3.5. The conclusions of Corollary 3.4 can be extended to 2-dimensional cocycles over maps $f\colon X\to X$ which are hyperbolic homeomorphisms (see [1]) with X a compact metric space. This includes the derivative cocycle of a diffeomorphism $f\colon M\to M$ over a hyperbolic set Λ for f. Corollary 3.4 can also be extended to the case of Anosov flows with 2-dimensional unstable bundle by using the fact that an Anosov flow is topologically semiconjugate via a Markov partition to a suspension flow over a subshift of finite type and then inducing on a transverse section to reduce to the case of a subshift of finite type.

4. PRELIMINARY RESULTS

The rest of the paper is devoted to the proof of Theorem 2.8. From now on $\hat{\mu}$ will denote an ergodic $\hat{f}$-invariant measure with local product structure and full support on $\hat{\Sigma}$. In this section we prove some preliminary results.

4.1. **Projective cocycles.** Let $\mathbb{P}^1$ be the 1-dimensional real projective space of lines in $\mathbb{R}^2$. Given a one-dimensional subspace $U\subset\mathbb{R}^2$ we consider both $U\subset\mathbb{R}^2$ and $U\in\mathbb{P}^1$. Given a non-zero vector $v\in\mathbb{R}^2$ we abuse notation and consider $v\in\mathbb{P}^1$ by identifying v with its linear span. Given $T\in GL(2,\mathbb{R})$ we write $\mathbb{P}T\colon\mathbb{P}^1\to\mathbb{P}^1$ for the induced projective map.

Consider a cocycle $\hat{A}\colon\hat{\Sigma}\to GL(2,\mathbb{R})$. The *projective cocycle* associated to $\hat{A}$ and $\hat{f}$ is the map $\hat{F}_{\hat{A}}\colon\hat{\Sigma}\times\mathbb{P}^1\to\hat{\Sigma}\times\mathbb{P}^1$ given by

$$\hat{F}_{\hat{A}}(\hat{x},v) = (\hat{f}(\hat{x}),\mathbb{P}\hat{A}(\hat{x})v).$$

4.2. **s- and u-states.** Let $\hat{m}$ be a probability measure on $\hat{\Sigma}\times\mathbb{P}^1$ projecting to $\hat{\mu}$; that is, $\hat{\pi}_*\hat{m} = \hat{\mu}$ where $\hat{\pi}\colon\hat{\Sigma}\times\mathbb{P}^1\to\hat{\Sigma}$ is the canonical projection. A *disintegration* of $\hat{m}$ along the fibers is a measurable family $\{\hat{m}_{\hat{x}} : \hat{x}\in\hat{\Sigma}\}$ of probabilities on $\mathbb{P}^1$ satisfying

$$\hat{m}(D) = \int_{\hat{\Sigma}} \hat{m}_{\hat{x}}(\{v : (\hat{x},v)\in D\})\; d\hat{\mu}(\hat{x})$$

for any measurable set $D\subset\hat{\Sigma}\times\mathbb{P}^1$. Observe that $\hat{m}$ is $\hat{F}_{\hat{A}}$-invariant if and only if $\mathbb{P}\hat{A}(\hat{x})_*\hat{m}_{\hat{x}} = \hat{m}_{\hat{f}(\hat{x})}$ for $\hat{\mu}$-almost every $\hat{x}\in\hat{\Sigma}$.

Following [1] we say that a disintegration $\{\hat{m}_{\hat{x}} : \hat{x}\in\hat{\Sigma}\}$ of an $\hat{F}_{\hat{A}}$-invariant probability measure $\hat{m}$ projecting to $\hat{\mu}$ is *essentially s-invariant* with respect to a stable holonomy $H^{s,\hat{A}}$ for $\hat{A}$ if there is a full measure subset E of $\hat{\Sigma}$ such that $\hat{x}, \hat{y} \in E$ and $\hat{y} \in W^s_{\text{loc}}(\hat{x})$ implies that

$$(H^{s,\hat{A}}_{\hat{x}\hat{y}})_* \hat{m}_{\hat{x}} = \hat{m}_{\hat{y}}$$

We define the notion of an *essentially u-invariant* disintegration similarly. An $\hat{F}_{\hat{A}}$-invariant probability measure $\hat{m}$ projecting to $\hat{\mu}$ is called an *s-state* with respect to a stable holonomy $H^{s,\hat{A}}$ if it admits some disintegration which is essentially s-invariant. We will always assume that the subset E is *s-saturated*, meaning that if $\hat{x} \in E$ then $W^s_{\text{loc}}(\hat{x}) \subset E$. This can always be done by modifying the disintegration of $\hat{m}$ on a $\hat{\mu}$-null set. We define *u-states* similarly.

An $\hat{F}_{\hat{A}}$-invariant probability measure $\hat{m}$ is an *su-state* if it is simultaneously an s-state and a u-state. The main property of su-states is the following.

PROPOSITION 4.1. *Assume that $\hat{\mu}$ is fully supported and has local product structure. If $\hat{m}$ is an su-state then it admits a disintegration for which the conditional probabilities $\hat{m}_{\hat{x}}$ depend continuously on $\hat{x}$ and are both s-invariant and u-invariant.*

For a proof of this proposition see [1, Proposition 4.8].

Given a cocycle with holonomies, there is always at least one s- and one u-state. On the other hand, su-states impose some rigidity on the system as exhibited by Proposition 4.1 and as such need not always exist. However, here is one situation in which su-states are guaranteed to exist. As stated here, this follows from the main result in [19] and has been extended to more general settings in [1].

THEOREM 4.2 (Invariance Principle)**.** *Let $\hat{A}: \hat{\Sigma} \to SL(2,\mathbb{R})$ be a cocycle admitting stable and unstable holonomies and assume that $\hat{\mu}$ is an ergodic $\hat{f}$-invariant probability measure with local product structure. If $\lambda_+(\hat{A},\hat{\mu}) = \lambda_-(\hat{A},\hat{\mu}) = 0$ then any $\hat{F}_{\hat{A}}$-invariant probability measure projecting to $\hat{\mu}$ is an su-state.*

In the sequel, we will be interested in sequences of s- and u-states projecting to different base measures and invariant under different projective cocycles and corresponding holonomies. The next lemma gives a criterion for an accumulation point of such a sequence to be an s- or u-state for the limiting cocycle.

LEMMA 4.3. *Let $\hat{A}_k: \hat{\Sigma} \to GL(2,\mathbb{R})$ be a sequence of linear cocycles with holonomies and suppose that $\hat{A}_k$ converges to $\hat{A}$ uniformly with holonomies. For each k let $\hat{m}_k$ be an s-state for $\hat{A}_k$ with respect to the stable holonomies $H^{s,\hat{A}_k}$ of $\hat{A}_k$ and projecting to a fully supported $\hat{f}$-invariant probability measure $\hat{\mu}_k$ with local product structure. Suppose that the sequence $\hat{\mu}_k$ converges to $\hat{\mu}$ as in Section 2.3 and that $\hat{m}_k \to \hat{m}$ in the weak-$*$ topology. Then $\hat{m}$ is an s-state with respect to the stable holonomies $H^{s,\hat{A}}$ for $\hat{A}$ which projects to $\hat{\mu}$. The same holds with unstable holonomies and u-states replacing stable holonomies and s-states.*

Proof. We will prove the statement for s-states. The statement for u-states then follows by considering the inverse cocycle $\hat{A}^{-1}$ over $\hat{f}^{-1}$.

We begin by defining continuous changes of coordinates which make each $\hat{A}_k$ and $\hat{A}$ constant on local stable manifolds. For each k let $\{\hat{m}^k_{\hat{x}}\}_{\hat{x}\in\hat{\Sigma}}$ be a disintegration of $\hat{m}_k$ along the $\mathbb{P}^1$ fibers. Each of these conditional measures is defined on a $\hat{\mu}_k$-full measure set $E_k \subset \hat{\Sigma}$ which we may assume to be s-saturated, since these measures are s-states, and we may assume these conditional measures are invariant under stable holonomy on E_k. We may also assume that the sets E_k are $\hat{f}$-invariant.

Fix ℓ points $\hat{z}_1,\dots,\hat{z}_\ell$ with $\hat{z}_i \in [0;i]$. For $\hat{x}\in[0;i]$, let $g(\hat{x})$ be the unique point in the intersection $W^u_{\mathrm{loc}}(\hat{z}_i)\cap W^s_{\mathrm{loc}}(\hat{x})$. Note that $g(\hat{x}) = g(\hat{y})$ if $\hat{y}\in W^s_{\mathrm{loc}}(\hat{x})$. Define

$$\widetilde{A}_k(\hat{x}) = \hat{A}_k(g(\hat{x}))$$

for each k, and similarly define $\widetilde{A}(\hat{x}) = \hat{A}(g\hat{(x)})$. By construction each $\widetilde{A}_k$ is constant along local stable sets and furthermore and $\widetilde{A}_k \to \widetilde{A}$ uniformly.

Consider the map $\Psi_k\colon \hat{\Sigma}\to GL(2,\mathbb{R})$ given by

$$\Psi_k\colon \hat{x}\mapsto H^{s,\hat{A}_k}_{\hat{x}g(\hat{x})}.$$

We have that

$$\tilde{A}_k(x)\circ\Psi_k(x) = \Psi_k(f(x))\circ\hat{A}_k(x).$$

In particularly, $\tilde{A}_k$ and $\hat{A}_k$ are continuously cohomologous via Ψ_k. Since the stable holonomies $H^{s,\hat{A}_k}$ converge uniformly to $H^{s,\hat{A}}$, we also have Ψ_k converges uniformly to $\Psi_0\colon \hat{x}\mapsto H^{s,\hat{A}}_{\hat{x}g(\hat{x})}$. In particular, up to the continuous change of coordinates Ψ_k and Ψ_0, we may assume that $\hat{A}_k$ and $\hat{A}$ are constant on local stable sets.

Define $\hat{\nu}^k_{\hat{x}} = (H^{s,\hat{A}_k}_{\hat{x}g(\hat{x})})_*\hat{m}^k_{\hat{x}}$ and let $\hat{\nu}_k$ be the probability measure on $\hat{\Sigma}\times\mathbb{P}^1$ projecting to $\hat{\mu}_k$ with this disintegration along the $\mathbb{P}^1$ fibers. $\hat{\nu}_k$ is $\hat{F}_{\widetilde{A}_k}$-invariant and since the linear maps $H^{s,\hat{A}_k}_{\hat{x}g(\hat{x})}$ depend continuously on $\hat{x}$, we conclude that $\hat{\nu}_k$ converges in the weak-$*$ topology to a measure $\hat{\nu}$ with disintegration $\hat{\nu}_{\hat{x}} = (H^{s,\hat{A}}_{\hat{x}g(\hat{x})})_*\hat{m}_{\hat{x}}$. To prove that $\hat{m}$ is an s-state it thus suffices to show that for $\hat{\mu}$-a.e. pair of points $\hat{x}$ and $\hat{y}$ with $\hat{y}\in W^s_{\mathrm{loc}}(\hat{x})$ we have $\hat{\nu}_{\hat{x}} = \hat{\nu}_{\hat{y}}$, because if $\hat{y}\in W^s_{\mathrm{loc}}(\hat{x})$ for some $\hat{x}$ in the intersection of this full measure subset with E, we then have

$$(H^{s,\hat{A}}_{\hat{x}g(\hat{x})})_*\hat{m}_{\hat{x}} = \hat{\nu}_{\hat{x}} = \hat{\nu}_{\hat{y}} = (H^{s,\hat{A}}_{\hat{y}g(\hat{y})})_*\hat{m}_{\hat{y}}$$

and therefore

$$\hat{m}_{\hat{y}} = \left(H^{s,\hat{A}}_{g(\hat{y})\hat{y}}\circ H^{s,\hat{A}}_{\hat{x}g(\hat{x})}\right)_*\hat{m}_{\hat{x}} = \left(H^{s,\hat{A}}_{g(\hat{x})\hat{y}}\circ H^{s,\hat{A}}_{\hat{x}g(\hat{x})}\right)_*\hat{m}_{\hat{x}} = \left(H^{s,\hat{A}}_{\hat{x}\hat{y}}\right)_*\hat{m}_{\hat{x}}$$

where we used $g(\hat{x}) = g(\hat{y})$ in the second line. Since the measures $\hat{\nu}_k$ are s-states we have $\hat{\nu}^k_{\hat{y}} = \hat{m}^k_{g(\hat{x})}$ for every $\hat{y}\in W^s_{\mathrm{loc}}(\hat{x})$, so the disintegrations of the measures $\hat{\nu}_k$ are constant on $\hat{\mu}_k$-a.e. local stable sets.

Since $\tilde{A}_k\colon\hat{\Sigma}\to GL(2,\mathbb{R})$ is constant along local stable sets, there are continuous maps $A_k\colon\Sigma^u\to GL(2,\mathbb{R})$ such that $A_k\circ P^u = \widetilde{A}_k$ and such that $A_k\to A$ uniformly, where A is defined by $A\circ P^u = \widetilde{A}$. Let ν_k, ν be the images of the measures $\hat{\nu}_k$, $\hat{\nu}$ under the projection $P^u \times Id\colon \hat{\Sigma} \times \mathbb{P}^1 \to \Sigma^u \times \mathbb{P}^1$. The disintegration $\{\hat{\nu}^k_{\hat{x}}\}_{\hat{x}\in\hat{\Sigma}}$ descends under this projection to a disintegration $\{\nu^k_x\}_{x\in\Sigma^u}$ with the property that for μ^u_k-a.e. x,

$$A_k(x)_*\nu^k_x = \nu^k_{f_u(x)}.$$

We first show that $A(x)_*\nu_x = \nu_{f_u(x)}$ for μ^u-a.e. $x \in \Sigma^u$. Let η be the probability measure on $\Sigma^u \times \mathbb{P}^1$ with disintegration $\{A^{-1}(x)_*\nu_{f_u(x)}\}_{x\in\Sigma^u}$. It suffices for this claim to prove that $\eta = \nu$, since the disintegration of ν along the $\mathbb{P}^1$ fibers is unique up to μ^u-null sets. Let $\varphi\colon \Sigma^u \times \mathbb{P}^1 \to \mathbb{R}$ be a continuous function and define

$$\Phi(x) = \int_{\mathbb{P}^1} \varphi(x, A^{-1}(x)v)\, d\nu_{f_u(x)}(v).$$

Since μ^u is f_u-invariant and admits a positive Jacobian $J_{\mu^u} f_u$ with respect to f_u,

$$\begin{aligned}\int_{\Sigma^u} \Phi(x)\, d\mu^u(x) &= \int_{\Sigma^u} \left(\sum_{y\in f_u^{-1}(x)} \frac{1}{J_{\mu^u} f_u(y)} \Phi(y) \right) d\mu^u(x)\\ &= \int_{\Sigma^u}\int_{\mathbb{P}^1} \sum_{y\in f_u^{-1}(x)} \frac{1}{J_{\mu^u} f_u(y)} \varphi(y, A^{-1}(y)v)\, d\nu_x(v)\, d\mu^u(x)\\ &= \int_{\Sigma^u\times\mathbb{P}^1} \sum_{y\in f_u^{-1}(x)} \frac{1}{J_{\mu^u} f_u(y)} \varphi(y, A^{-1}(y)v)\, d\nu(x,v).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\Sigma^u} \Phi(x)\, d\mu^u(x) &= \int_{\Sigma^u}\int_{\mathbb{P}^1} \varphi(x, A^{-1}(x)v)\, d\nu_{f_u(x)}(v)\, d\mu^u(x)\\ &= \int_{\Sigma^u\times\mathbb{P}^1} \varphi(x,v)\, d\eta(x,v).\end{aligned}$$

Hence it suffices to show that for every continuous map $\varphi\colon \Sigma^u \times \mathbb{P}^1 \to \mathbb{R}$ we have

$$\int \varphi\, d\nu = \int \sum_{y\in f_u^{-1}(x)} \frac{1}{J_{\mu^u} f_u(y)} \varphi(y, A^{-1}(y)v)\, d\nu(x,v).$$

But for each k we know that for μ^u_k-a.e. $x \in \Sigma^u$ we have $A_k^{-1}(x)_*\nu^k_{f_u(x)} = \nu^k_x$. The same calculation as above shows that the above equality holds with appropriate modifications for ν_k, i.e.,

$$\int \varphi\, d\nu_k = \int \sum_{y\in f_u^{-1}(x)} \frac{1}{J_{\mu^u_k} f_u(y)} \varphi(y, A_k^{-1}(y)v)\, d\nu_k(x,v).$$

By assumption, ν_k converges to ν in the weak-$*$ topology, $A_k^{-1} \to A^{-1}$ uniformly, and $J_{\mu^u_k} f_u \to J_{\mu^u} f_u$ uniformly. It follows that this equality holds in the limit $k \to \infty$, and hence that $A(x)_*\nu_x = \nu_{f_u(x)}$ for μ^u-a.e. x.

The disintegration of the measure $\hat{\nu}$ along the $\mathbb{P}^1$ fibers of $\hat{\Sigma} \times \mathbb{P}^1$ can be recovered from the disintegration of ν along the $\mathbb{P}^1$ fibers of $\Sigma^u \times \mathbb{P}^1$ by the formula

$$\hat{\nu}_{\hat{x}} = \lim_{n\to\infty} A^n(P^u(\hat{f}^{-n}(\hat{x})))_*\nu_{P^u(\hat{f}^{-n}(\hat{x}))}$$

(see Lemma 3.4 of [1]) for $\hat{\mu}$-a.e. $\hat{x}$. But we have just shown that

$$A^n(P^u(\hat{f}^{-n}(\hat{x})))_*\nu_{P^u(\hat{f}^{-n}(\hat{x}))} = \nu_{P^u(\hat{x})}$$

for every n. Hence we conclude that

$$\hat{\nu}_{\hat{x}} = \nu_{P^u(\hat{x})}$$

and thus $\hat{\nu}_{\hat{x}} = \hat{\nu}_{\hat{y}}$ for $\hat{y} \in W^s_{\mathrm{loc}}(\hat{x})$. □

4.3. **Continuity of conditional measures.** From now on we will write Σ, f, P and μ for Σ^u, f_u, P^u, and μ^u, respectively. Moreover, from the proof of Lemma 4.3 it follows that an arbitrary sequence of cocycles $\{\hat{A}_k\}_{k\in\mathbb{N}}$ converging uniformly with holonomies to a cocycle $\hat{A}$ may be straightened out using the stable holonomies so that each $\hat{A}_k$ and $\hat{A}$ are constant on local stable sets and the property of uniform convergence is preserved. Moreover, the straightened out cocycles still admit u-holonomies and the u-holonomies also converge uniformly.

Consider such a cocycle $\hat{A}$ that has been straightened out along stable holonomies. We write $A\colon \Sigma \to GL(2,\mathbb{R})$ for the continuous map defined by $\hat{A} = A \circ P$. In particular, $A(x) = \hat{A}(\hat{x})$ for every $\hat{x} \in W^s_{\mathrm{loc}}(x)$.

4.3.1. *Measures induced from a u-state.* In the sequel, we will be primarily interested in families of measure on $\Sigma \times \mathbb{P}^1$ induced from measures on $\hat{\Sigma} \times \mathbb{P}^1$ with certain dynamical properties. The measures on $\Sigma \times \mathbb{P}^1$ will in turn have certain geometric properties that we describe here.

DEFINITION 4.4. A probability measure m on $\Sigma \times \mathbb{P}^1$ is said to be *induced from a u-state* if there exists

- a cocycle $\hat{A}\colon \hat{\Sigma} \to GL(2,\mathbb{R})$ that is constant along local stable sets and admits a continuous family of unstable holonomies $H^{u,\hat{A}}$,
- a fully supported measure $\hat{\mu}$ on $\hat{\Sigma}$ with local product structure,
- and an $\hat{F}_{\hat{A}}$-invariant measure $\hat{m}$ on $\hat{\Sigma} \times \mathbb{P}^1$ projecting to $\hat{\mu}$ such that $\hat{m}$ is a u-state for the holonomies $H^{u,\hat{A}}$ with $m = (P \times Id)_*\hat{m}$.

Note that such an m is necessarily F_A-invariant, where A is such that $\hat{A} = A \circ P$ as above.

4.3.2. *Continuity of the disintegration of measures induced from u-states.* They key geometric fact we exploit in the remainder of the paper is that every measure m induced from a u-state admits a disintegration into a *continuous* family of conditional measures $\{m_x : x \in \Sigma\}$. The continuity properties of the conditional measures of m were first established in [8]; in this section we establish additional equicontinuity properties of the conditional measures over families of linear cocycles for which unstable holonomies exist and vary continuously.

We retain all notation from Definition 4.4. Observe that if $m = (P \times Id)_*\hat{m}$ and $\{\hat{m}_{\hat{x}} : \hat{x} \in \hat{\Sigma}\}$ is a disintegration of $\hat{m}$ along the fibers $\{\hat{\pi}^{-1}(\hat{x}); \hat{x} \in \hat{\Sigma}\}$, then for

$x \in \Sigma$

$$m_x = \int_{W^s_{\mathrm{loc}}(x)} \hat{m}_{\hat{x}} \, d\hat{\mu}_x(\hat{x}) \tag{2}$$

is a disintegration of m relative to $\{\pi^{-1}(x); x \in \Sigma\}$, where $\pi \colon \Sigma \times \mathbb{P}^1 \to \Sigma$ is the canonical projection.

PROPOSITION 4.5. *Any probability measure m induced from a u-state admits a disintegration into conditional measures $\{m_x\}_{x\in\Sigma}$ that are defined for every $x \in \Sigma$ and vary continuously with x in the weak-$*$ topology.*

Proof. Let $\hat{m}$ be a u-state such that $(P \times Id)_* \hat{m} = m$ and $\{\hat{\mu}_x\}_{x\in\Sigma}$ a disintegration of $\hat{\mu}$ as in Lemma 2.5. Take a disintegration $(\hat{m}_{\hat{x}})_{\hat{x}\in\hat{\Sigma}}$ of $\hat{m}$ such that for $\hat{\mu}$-a.e. $\hat{x} \in \hat{\Sigma}$,

$$(H^{u,\hat{A}}_{\hat{x}\hat{y}})_* \hat{m}_{\hat{x}} = \hat{m}_{\hat{y}} \text{ for every } \hat{y} \in W^u_{\mathrm{loc}}(\hat{x})$$

and let $\{m_x\}_{x\in\Sigma}$ be the disintegration of m as in (2).

Let $g \colon \mathbb{P}^1 \to \mathbb{R}$ be continuous and consider $x, y \in \Sigma$ in the same cylinder $[0; i]$. Then, changing variables $\hat{y} = h_{x,y}(\hat{x})$ we get that

$$\begin{aligned}\int_{\mathbb{P}^1} g\,dm_y &= \int_{W^s_{\mathrm{loc}}(y)} \int_{\mathbb{P}^1} g\,d\hat{m}_{\hat{y}} d\hat{\mu}_y(\hat{y}) \\ &= \int_{W^s_{\mathrm{loc}}(x)} \left(\int_{\mathbb{P}^1} g \circ H^{u,\hat{A}}_{\hat{x}\hat{y}} d\hat{m}_{\hat{x}} \right) R_{x,y}(\hat{x}) d\hat{\mu}_x(\hat{x})\end{aligned}$$

since $\hat{m}$ is a u-state. Thus,

$$\left| \int g\,dm_y - \int g\,dm_x \right| \le \int_{W^s_{\mathrm{loc}}(x)} \int_{\mathbb{P}^1} \left| R_{x,y}(\hat{x}) \cdot g \circ H^{u,\hat{A}}_{\hat{x}\hat{y}} - g \right| d\hat{m}_{\hat{x}} d\hat{\mu}_x(\hat{x}).$$

From the continuity properties of unstable holonomies (see Definition 2.1) we have that $\|H^{u,\hat{A}}_{\hat{x}\hat{y}} - Id\|$ is uniformly close to zero whenever x and y are close. Moreover, Lemma 2.5 implies that $\|R_{x,y} - 1\|_{L^\infty}$ is also close to zero whenever x and y are close. Therefore, given $\varepsilon > 0$ there exist $\gamma > 0$ such that $d(x, y) < \gamma$ implies $\|R_{x,y}(\hat{x}) \cdot g \circ H^{u,\hat{A}}_{\hat{x}\hat{y}} - g\|_{L^\infty} < \varepsilon$ and thus $|\int g\,dm_y - \int g\,dm_x| < \varepsilon$ as we want. □

REMARK 4.6. A probability measure m in $\Sigma \times \mathbb{P}^1$ is F_A-invariant if and only if

$$\sum_{y \in f^{-1}(x)} \frac{1}{J_\mu f(y)} A(y)_* m_y = m_x \tag{3}$$

for μ almost every $x \in \Sigma$ and any disintegration $\{m_x\}_{x\in\Sigma}$. When m is induced from a u-state and $\{m_x\}_{x\in\Sigma}$ is the the continuous family of conditional measures above then (3) holds for *every* $x \in \Sigma$.

We recall the setting of Lemma 4.3. Let $\hat{\mu}_k$ be a family of fully supported measures on $\hat{\Sigma}$ with product structure. Assume $\hat{\mu}_k$ converges as in Section 2.3 to a fully supported measure $\hat{\mu}$ with product structure. In particular, the family of Jacobians $R^k_{x,y}$ associated to the disintegration of $\hat{\mu}_k$ given by Lemma 2.5 converge uniformly to the Jacobians $R_{x,y}$ of $\hat{\mu}$.

For each k let $\hat{A}_k$ be a cocycle that is constant along stable sets, and suppose $\hat{A}_k \to \hat{A}$ uniformly. Moreover assume $\hat{A}_k$ and $\hat{A}$ admit (unstable) holonomies and that $H^{u,\hat{A}_k}$ converges to $H^{u,\hat{A}}$ as in Section 2.2. For each k, let m_k be a measure on $\Sigma \times \mathbb{P}^1$ induced by a u-state $\hat{m}_k$ for the holonomies $H^{u,\hat{A}_k}$ and projecting to μ_k. Assume that $\hat{m}_k$ converges in the weak-$*$ topology to $\hat{m}$. From Lemma 4.3 we have that $\hat{m}$ is a u-state for the holonomies $H^{u,\hat{A}}$ and projects to $\hat{\mu}$. Let $m = (P \times Id)_* \hat{m}$ be the measure induced by the u-state $\hat{m}$.

Observing that all the convergences above are uniform and following the same lines as in the proof of the previous proposition we get

PROPOSITION 4.7. *The measures m_k and m admit disintegrations into conditional measures $\{m_x^k\}_{x\in\Sigma}$ and $\{m_x\}_{x\in\Sigma}$, respectively, which are defined for every $x \in \Sigma$ and such that the family $\{\{m_x^k\}_{x\in\Sigma}, \{m_x\}_{x\in\Sigma}\}_k$ is equicontinuous. More precisely, for every continuous function $g\colon \mathbb{P}^1 \to \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d_\theta(x, y) < \delta$ implies*

$$\left|\int g\,dm_x - \int g\,dm_y\right| < \varepsilon \quad \textit{and} \quad \left|\int g\,dm_x^k - \int g\,dm_y^k\right| < \varepsilon$$

for every $k \in \mathbb{N}$.

Let $\{m_x^k\}_{x\in\Sigma}$ and $\{m_x\}_{x\in\Sigma}$ be the continuous family of conditional measures constructed above.

LEMMA 4.8. *For any $x \in \Sigma$, $m_x^k \to m_x$. Moreover, the convergence is uniform in x.*

Proof. Let $g\colon \mathbb{P}^1 \to \mathbb{R}$ be continuous and $\varepsilon > 0$. By Proposition 4.7, there exists $\delta > 0$ such that, if $d_\theta(x, y) \le \delta$ then

$$\left|\int_{\mathbb{P}^1} g\,dm_x - \int_{\mathbb{P}^1} g\,dm_y\right| < \frac{\varepsilon}{10} \quad \text{and} \quad \left|\int_{\mathbb{P}^1} g\,dm_x^k - \int_{\mathbb{P}^1} g\,dm_y^k\right| < \frac{\varepsilon}{10} \tag{4}$$

for every $k \in \mathbb{N}$. Cover Σ with finitely many clopen sets V_i with $\operatorname{diam}(V_i) < \delta$. As m_k converges to m there exists $k_0 \in \mathbb{N}$ such that

$$\left|\int_{V_i}\left(\int_{\mathbb{P}^1} g\,dm_x^k\right)d\mu_k(x) - \int_{V_i}\left(\int_{\mathbb{P}^1} g\,dm_x\right)d\mu(x)\right| < \frac{\varepsilon\mu(V_i)}{10} \tag{5}$$

and, with $M = \max\{1, \max|g|\}$,

$$\left|1 - \frac{\mu_k(V_i)}{\mu(V_i)}\right| \le \frac{\varepsilon}{10M}$$

for every $k \ge k_0$ and each V_i.

Given $x \in \Sigma$ take V_i with $x \in V_i$. Then

$$\begin{aligned}
\left|\int_{\mathbb{P}^1} g\, dm_x^k - \int_{\mathbb{P}^1} g\, dm_x\right| &= \frac{1}{\mu(V_i)}\left|\int_{V_i}\left(\int_{\mathbb{P}^1} g\, dm_x^k\right) d\mu(y) - \int_{V_i}\left(\int_{\mathbb{P}^1} g\, dm_x\right) d\mu(y)\right| \\
&\le \frac{1}{\mu(V_i)}\left|\int_{V_i}\int_{\mathbb{P}^1} g\, dm_x^k d\mu(y) - \int_{V_i}\int_{\mathbb{P}^1} g\, dm_x^k d\mu_k(y)\right| \\
&\quad + \frac{1}{\mu(V_i)}\left|\int_{V_i}\int_{\mathbb{P}^1} g\, dm_x^k d\mu_k(y) - \int_{V_i}\int_{\mathbb{P}^1} g\, dm_y^k d\mu_k(y)\right| \\
&\quad + \frac{1}{\mu(V_i)}\left|\int_{V_i}\int_{\mathbb{P}^1} g\, dm_y^k d\mu_k(y) - \int_{V_i}\int_{\mathbb{P}^1} g\, dm_y d\mu(y)\right| \\
&\quad + \frac{1}{\mu(V_i)}\left|\int_{V_i}\int_{\mathbb{P}^1} g\, dm_y d\mu(y) - \int_{V_i}\int_{\mathbb{P}^1} g\, dm_x d\mu(y)\right| \\
&\le M\left|1 - \frac{\mu_k(V_i)}{\mu(V_i)}\right| \\
&\quad + \frac{1}{\mu(V_i)}\left(\int_{V_i}\left|\int_{\mathbb{P}^1} g\, dm_x^k - \int_{\mathbb{P}^1} g\, dm_y^k\right| d\mu_k(y)\right) \\
&\quad + \frac{1}{\mu(V_i)}\left|\int_{V_i}\int_{\mathbb{P}^1} g\, dm_y^k d\mu_k(y) - \int_{V_i}\int_{\mathbb{P}^1} g\, dm_y d\mu(y)\right| \\
&\quad + \frac{1}{\mu(V_i)}\left(\int_{V_i}\left|\int_{\mathbb{P}^1} g\, dm_y - \int_{\mathbb{P}^1} g\, dm_x\right| d\mu(y)\right) \\
&\le \frac{\varepsilon}{10} + \left(1 + \frac{\varepsilon}{10M}\right)\frac{\varepsilon}{10} + \frac{\varepsilon}{10} + \frac{\varepsilon}{10} \\
&\le \frac{5\varepsilon}{10}.
\end{aligned}$$

□

5. REDUCTIONS IN THE PROOF OF THEOREM 2.8

We begin the proof of Theorem 2.8. We start by observing that it suffices to prove continuity for cocycles taking values in the group of matrices with determinant ± 1. We write $SL^{\pm}(2,\mathbb{R})$ for this group. By continuity of $\hat{A}$ and compactness of $\hat{\Sigma}$, the function $s(\hat{x}) = \operatorname{sgn}(\det(A(x)))$ is continuous on $\hat{\Sigma}$. Given $\hat{A} : \hat{\Sigma} \to GL(2,\mathbb{R})$ consider $g_{\hat{A}} : \hat{\Sigma} \to \mathbb{R}$ defined by $g_{\hat{A}}(\hat{x}) = s(\hat{x})(|\det \hat{A}(\hat{x})|)^{\frac{1}{2}}$ and $\hat{B} : \hat{\Sigma} \to SL^{\pm}(2,\mathbb{R})$ such that $\hat{A}(\hat{x}) = g_{\hat{A}}(\hat{x})\hat{B}(\hat{x})$. Thus, since

$$\lambda^{\pm}(\hat{A},\hat{\mu}) = \lambda^{\pm}(\hat{B},\hat{\mu}) + \int \log|g_{\hat{A}}(\hat{x})|\, d\hat{\mu}(\hat{x}),$$

and $g_{\hat{A}_k} \to g_{\hat{A}}$ uniformly, we get that $\lambda^{\pm}(\hat{A}_k,\hat{\mu}_k) \to \lambda^{\pm}(\hat{A},\hat{\mu})$ if and only if

$$\lambda^{\pm}(\hat{B}_k,\hat{\mu}_k) \to \lambda^{\pm}(\hat{B},\hat{\mu}),$$

where $\hat{B}_k$ is defined analogously to $\hat{B}$ for $\hat{A}_k$. Moreover,

$$\lambda^{+}(\hat{A},\hat{\mu}) = \lambda^{-}(\hat{A},\hat{\mu}) \Longleftrightarrow \lambda^{+}(\hat{B},\hat{\mu}) = 0 = \lambda^{-}(\hat{B},\hat{\mu}).$$

From now on, we will assume that our cocycles always take values in $SL^{\pm}(2,\mathbb{R})$.

The proof of Theorem 2.8 is by contradiction. Suppose $(\hat{A}, \hat{\mu}, H^{s,\hat{A}}, H^{u,\hat{A}})$ and $(\hat{A}_k, \hat{\mu}_k, H^{s,\hat{A}_k}, H^{u,\hat{A}_k})$ are as in Theorem 2.8. Moreover, suppose for the purposes of contradiction that

$$\lambda_+(\hat{A}_k, \hat{\mu}_k) \not\to \lambda_+(\hat{A}, \hat{\mu}). \tag{6}$$

We then also have $\lambda_-(\hat{A}_k, \hat{\mu}_k) \not\to \lambda_-(\hat{A}, \hat{\mu})$.

5.1. **Characterization of discontinuity points.** From [31, Lemma 9.1] we have that the functions $(\hat{B}, \hat{\nu}) \mapsto \lambda_+(\hat{B}, \hat{\nu})$ and $(\hat{B}, \hat{\nu}) \mapsto \lambda_-(\hat{B}, \hat{\nu})$ are, respectively, upper- and lower-semicontinuous with respect to the topology of uniform convergence on continuous cocycles $\hat{B}$ and weak-$*$ convergence in $\hat{\nu}$. Thus, assuming (6) we may assume $\lambda_-(\hat{A}, \hat{\mu}) < 0 < \lambda_+(\hat{A}, \hat{\mu})$.

Let $\mathbb{R}^2 = E^{s,\hat{A}}_{\hat{x}} \oplus E^{u,\hat{A}}_{\hat{x}}$ be the Oseledets decomposition associated to $\hat{A}$ at the point $\hat{x} \in \hat{\Sigma}$. Consider the measures on $\hat{\Sigma} \times \mathbb{P}^1$ defined by

$$\hat{m}^s = \int_{\hat{\Sigma}} \delta_{(\hat{x}, E^{s,\hat{A}}_{\hat{x}})}\, d\hat{\mu}(\hat{x}) \quad \text{and} \quad \hat{m}^u = \int_{\hat{\Sigma}} \delta_{(\hat{x}, E^{u,\hat{A}}_{\hat{x}})}\, d\hat{\mu}(\hat{x}).$$

By construction, $\hat{m}^s$ and $\hat{m}^u$ are $\hat{F}_{\hat{A}}$-invariant probability measures with projections $\hat{\mu}$. Moreover, $\hat{m}^s$ is an s-state (with disintegration $\{\delta_{E^{s,\hat{A}}_{\hat{x}}}\}_{\hat{x} \in \hat{\Sigma}}$) and $\hat{m}^u$ is a u-state. By the Birkhoff ergodic theorem

$$\lambda_-(\hat{A}, \hat{\mu}) = \int_{\hat{\Sigma} \times \mathbb{P}^1} \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m}^s(\hat{x}, v)$$

and

$$\lambda_+(\hat{A}, \hat{\mu}) = \int_{\hat{\Sigma} \times \mathbb{P}^1} \Phi_{\hat{A}}(\hat{x}, v)\, d\hat{m}^u(\hat{x}, v)$$

where

$$\Phi_{\hat{A}}(\hat{x}, v) = \log\left(\frac{\|\hat{A}(\hat{x})(v)\|}{\|v\|}\right).$$

By the (non-uniform) hyperbolicity of $(\hat{A}, \hat{\mu})$ we have the following.

CLAIM 5.1. *Let $\hat{m}$ be a probability measure on $\hat{\Sigma} \times \mathbb{P}^1$ projecting to $\hat{\mu}$. Then, $\hat{m}$ is $\hat{F}_{\hat{A}}$-invariant if and only if it is a convex combination of $\hat{m}^s$ and $\hat{m}^u$: $\hat{m} = \alpha \hat{m}^s + \beta \hat{m}^u$ where α and β are constant.*

Indeed, one only has to note that every compact subset of $\mathbb{P}^1$ disjoint from $\{E^u, E^s\}$ accumulates on E^u in the future and on E^s in the past. That α and β are constant (independent of $\hat{x} \in \hat{\Sigma}$) follows from ergodicity.

We now prove the key characterization of discontinuity points for extremal Lyapunov exponents. The proof is well known but is included here for completeness.

PROPOSITION 5.2. *If $(\hat{A}, \hat{\mu})$ is as in (6), then every $\hat{F}_{\hat{A}}$-invariant probability measure $\hat{m}$ on $\hat{\Sigma} \times \mathbb{P}^1$ projecting to $\hat{\mu}$ is an su-state for $\hat{F}_{\hat{A}}$.*

Proof. By Claim 5.1, it suffices to show the measures $\hat{m}^s$ and $\hat{m}^u$ are su-states.

By the upper semi-continuity of $\lambda_+(\cdot,\cdot)$, passing to a subsequence we may assume $\lim_{k\to\infty}\lambda_+(\hat{A}_k,\hat{\mu}_k) < \lambda_+(\hat{A},\hat{\mu})$. For each k, there exists an ergodic, $\hat{F}_{\hat{A}_k}$-invariant probability measure $\hat{m}_k$, projecting to $\hat{\mu}_k$, which is a u-state for $H^{u,\hat{A}_k}$, and such that

$$\lambda_+(\hat{A}_k,\hat{\mu}_k) = \int_{\hat{\Sigma}\times\mathbb{P}^1}\Phi_{\hat{A}_k}\,d\hat{m}_k.$$

Indeed, if $\lambda_+(\hat{A}_k,\hat{\mu}_k)\neq 0$ we can take

$$\hat{m}_k = \int_{\hat{\Sigma}}\delta_{(\hat{x},E^{u,\hat{A}_k}_{\hat{x}})}\,d\hat{\mu}_k(\hat{x})$$

as above. If $\lambda_+(\hat{A}_k,\hat{\mu}_k)=0$ then (as $\hat{A}_k\in SL^{\pm}(2,\mathbb{R})$) we have $\lambda_-(\hat{A}_k,\hat{\mu}_k)=0$ and by Theorem 4.2, any $\hat{F}_{\hat{A}_k}$-invariant probability measure $\hat{m}_k$, projecting to $\hat{\mu}_k$ is a su-state; moreover for any such measure $\int_{\hat{\Sigma}\times\mathbb{P}^1}\Phi_{\hat{A}_k}\,d\hat{m}_k=0$.

Consequently

$$\lim_{k\to\infty}\int_{\hat{\Sigma}\times\mathbb{P}^1}\Phi_{\hat{A}_k}\;d\hat{m}_k < \lambda_+(\hat{A},\hat{\mu}).$$

Taking subsequences again, we may assume that $(\hat{m}_k)_k$ converges to a $\hat{F}_{\hat{A}}$-invariant probability measure $\hat{m}$. By Lemma 4.3, $\hat{m}$ is a u-state for $H^{u,\hat{A}}$. Now, by Claim 5.1,

$$\hat{m} = \alpha\hat{m}^s + \beta\hat{m}^u$$

for some constants $\alpha,\beta\in[0,1]$. By uniform convergence of $\Phi_{\hat{A}_k}\to\Phi_{\hat{A}}$ and weak-$*$ convergence of $\hat{m}_k\to\hat{m}$ we have

$$\int_{\hat{\Sigma}\times\mathbb{P}^1}\Phi_{\hat{A}}\;d\hat{m} = \lim_{k\to\infty}\int_{\hat{\Sigma}\times\mathbb{P}^1}\Phi_{\hat{A}_k}\;d\hat{m}_k < \lambda_+(\hat{A},\hat{\mu})$$

hence $\hat{m}\neq\hat{m}^u$. It follows that $\alpha\neq 0$ and

$$\hat{m}^s = \tfrac{1}{\alpha}\left(\hat{m}-\beta\hat{m}^u\right)$$

is a u-state for $H^{u,\hat{A}}$. Similarly, $\hat{m}^u$ is an s-state for $H^{s,\hat{A}}$. In particular, $\hat{m}^s$ and $\hat{m}^u$ are su-states. Claim 5.1 completes the proof. □

5.2. **Final reductions and standing notation.** As discussed in the proof of Lemma 4.3, the family of invariant stable holonomies defines a continuous change of linear coordinates on the fibers $\{x\}\times\mathbb{P}^1$ that makes the cocycle constant along local stable sets of $\hat{f}$. The convergence of the cocycles $\hat{A}_k\to\hat{A}$ is not affected by this coordinate change. Moreover, the straightened out cocycles admit unstable holonomies with the appropriate convergence and have the same Lyapunov exponents. We assume for the remainder we have straightened out the cocycles in (6) along their respective stable holonomies. Following the notation introduced in Section 4.3, let $A, A_k\colon\Sigma\to SL^{\pm}(2,\mathbb{R})$ be such that $\hat{A}=A\circ P$ and $\hat{A}_k = A_k\circ P$ where $P\colon\hat{\Sigma}\to\Sigma$ is the natural projection.

We assume for the remainder that

$$\lambda_+(\hat{A}_k,\hat{\mu}_k)\not\to\lambda_+(\hat{A},\hat{\mu})$$

and fix a sequence of ergodic u-states $\hat{m}_k$ as in the proof of Proposition 5.2. We assume $\hat{m}_k$ converges to some measure $\hat{m}$. From (the proof of) Proposition 5.2, we have that $\hat{m} = \alpha \hat{m}^s + \beta \hat{m}^u$ and, moreover that $\hat{m}^s$ and $\hat{m}^u$ are su-states.

From Proposition 4.1, it follows that there are continuous functions $\sigma^{s/u}\colon \hat{\Sigma} \to \mathbb{P}^1$ such that $E^{s/u,\hat{A}}_{\hat{x}} = \sigma^{s/u}(\hat{x})$. Using $\sigma^{s/u}$, we perform a final continuous change of coordinates, that is projective in each fiber, such that for $\hat{x} \in \hat{\Sigma}$

$$\sigma^s(\hat{x}) = [1:0] := q, \text{ and } \sigma^u(\hat{x}) = [0:1] := p.$$

In particular, after this coordinate change the projective cocycle $\mathbb{P}\hat{A}(y)$ leaves q and p invariant for every y. Note that the change of coordinate is constant on local stable sets so the cocycle $\hat{A}$ is still of the form $\hat{A} = A \circ P$ for some $A\colon \Sigma \to SL(2,\mathbb{R})$. Note that in order to define this coordinate change, we rely on the fact that the limiting measure μ is fully supported.

We take $m_k := (P \times Id)_* \hat{m}_k$ and similarly take $m := (P \times Id)_* \hat{m}$, $m^s := (P \times Id)_* \hat{m}^s$, $m^u := (P \times Id)_* \hat{m}^u$. Each of the above measures is induced by a u-state on $\hat{\Sigma} \times \mathbb{P}^1$ and hence induces a continuous family of conditional measures. Since the measures $\hat{m}^u_k$ are ergodic for each k, the projected measures m_k are ergodic.

Let $\{m^k_x\}$ and $\{m_x\}$ denote a continuous family of conditional measure for m_k and m, respectively, given by Proposition 4.5. Observe that, for every $x \in \Sigma$, $m_x = \alpha\delta_q + \beta\delta_p$ where $\alpha, \beta \in (0,1)$. We split the proof of Theorem 2.8 into two cases. In Section 6 we consider the case that for infinitely many k there is a $x \in \Sigma$ such that the conditional measure m^k_x has an atom. In Section 7 we consider the case that the measures m^k_x are non-atomic for every x and infinitely many k. Passing to subsequences, we can assume that either the measures m^k_x are non-atomic for all x and k or contains an atom for some x and all k. In both cases, we derive a contradiction showing that $(\hat{A}, \hat{\mu})$ can not satisfy (6).

6. CASE 1: THE MEASURES m^k_x ARE ATOMIC

In this section we will deduce a contradiction to (6) under the assumption that for every $k \in \mathbb{N}$ there is some $x \in \Sigma$ such that the conditional measure m^k_x contains an atom. We first claim that m^k_x contains an atom for *every* $x \in \Sigma$ which, by ergodicity, implies that the measures m^k_x are all finitely supported. The proofs of Lemmas 6.1 and 6.2 given below are not new; to the best of our knowledge they first appear as a consequence of [8, Lemmas 5.2, 5.3]. We reproduce the proofs here for completeness.

For each k, consider

$$\gamma^k_0 := \sup\left\{ m^k_x(v) : x \in \Sigma, v \in \mathbb{P}^1 \right\}.$$

By hypothesis, $\gamma^k_0 > 0$ for all k.

LEMMA 6.1. *For each $x \in \Sigma$ there exists $v^k_x \in \mathbb{P}^1$ such that $m^k_x(v^k_x) = \gamma^k_0$. Moreover, $m^k_x(v^k_x) = \gamma^k_0$ if and only if $m^k_y(A_k(y)^{-1}(v^k_x)) = \gamma^k_0$ for all $y \in f^{-1}(x)$.*

Proof. Consider $\Gamma^k_0 := \{x \in \Sigma : m^k_x(v) = \gamma^k_0$ for some $v \in \mathbb{P}^1\}$. We argue that this is a non-empty closed set. Indeed, let $\{x_j\}_{j\in\mathbb{N}} \subset \Sigma$ and $\{v_j\}_{j\in\mathbb{N}} \subset \mathbb{P}^1$ be sequences such that $m^k_{x_j}(v_j) \xrightarrow{j\to\infty} \gamma^k_0$. Restricting to a subsequence we may assume that $\{x_j\}_{j\in\mathbb{N}}$ converges to some $x\in\Sigma$ and $\{v_j\}_{j\in\mathbb{N}}$ converges to some $v\in\mathbb{P}^1$. Now, as $x\mapsto m^k_x$ is continuous, for each $\varepsilon>0$ we have that

$$\gamma^k_0 \le \limsup_{j\to\infty} m^k_{x_j}(\overline{B(v,\varepsilon)}) \le m^k_x(\overline{B(v,\varepsilon)}).$$

Thus $m^k_x(v)\ge\gamma^k_0$ and hence $m^k_x(v)=\gamma^k_0$. It follows that Γ^k_0 is non-empty and closed.

By Remark 4.6,

$$m^k_x(v) = \sum_{y\in f^{-1}(x)} \frac{1}{J_{\mu_k} f(y)} m^k_y(A_k(y)^{-1}(v))$$

for all $x\in\Sigma$ and $v\in\mathbb{P}^1$. As $\sum_{y\in f^{-1}(x)} \frac{1}{J_{\mu_k} f(y)} = 1$ for every $x\in\Sigma$, it follows that $m^k_x(v)=\gamma^k_0$ if and only if $m_y(A_k(y)^{-1}(v))=\gamma^k_0$ for every $y\in f^{-1}(x)$. In particular, $f^{-1}(\Gamma^k_0)\subset\Gamma^k_0$. Since f is transitive, Σ is the unique non-empty, closed, backwards-invariant subset of Σ. Hence $\Gamma^k_0=\Sigma$. □

We show that points realizing the maximal atomic mass of m^k_x have the same property for the measure $\hat m_k$.

LEMMA 6.2. *Given $x\in\Sigma$ and $v\in\mathbb{P}^1$ we have that $\hat m^k_{\hat x}(v)\le\gamma^k_0$ for $\hat\mu^k_x$ almost every $\hat x\in W^s_{\mathrm{loc}}(x)$. Consequently, $m^k_x(v^k_x)=\gamma^k_0$ if and only if $\hat m^k_{\hat x}(v^k_x)=\gamma^k_0$ for $\hat\mu^k_x$ almost every $\hat x\in W^s_{\mathrm{loc}}(x)$.*

Proof. Suppose that there exist $v\in\mathbb{P}^1$, $x\in\Sigma$, $\gamma_1>\gamma^k_0$ and a subset $Z\subset W^s_{\mathrm{loc}}(x)$ with positive $\hat\mu^k_x$-measure such that $\hat m^k_{\hat x}(v)\ge\gamma_1$ for every $\hat x\in Z$. For any $n\ge0$ let us consider the partition of $W^s_{\mathrm{loc}}(x)$ given by

$$\{\hat f^n(W^s_{\mathrm{loc}}(y)) : y\in f^{-n}(x)\}.$$

Observe that the diameter of this partition goes to zero when n goes to infinity. Therefore, given $\varepsilon>0$ we can find $n\ge1$ and $y\in f^{-n}(x)$ such that

$$\hat\mu^k_x(Z\cap\hat f^n(W^s_{\mathrm{loc}}(y))) > (1-\varepsilon)\hat\mu^k_x(\hat f^n(W^s_{\mathrm{loc}}(y))). \tag{7}$$

Take $\varepsilon>0$ so that $(1-\varepsilon)\gamma_1>\gamma^k_0$. As $\hat m_k$ is an $F_{\hat A_k}$-invariant measure we have

$$\hat m^k_{\hat y}(A^n_k(y)^{-1}(v)) = A^n_k(y)_*\hat m^k_{\hat y}(v) = \hat A^n_k(\hat y)_*\hat m^k_{\hat y}(v) = \hat m^k_{\hat f^n(\hat y)}(v)\ge\gamma_1$$

for almost every $\hat y\in\hat f^{-n}(Z)\cap W^s_{\mathrm{loc}}(y)$. By Remark 2.6,

$$\begin{aligned}\hat\mu^k_y(\hat f^{-n}(Z)\cap W^s_{\mathrm{loc}}(y)) &= J_{\mu_k} f^n(y)\hat\mu^k_x(Z\cap\hat f^n(W^s_{\mathrm{loc}}(y)))\\ &\ge (1-\varepsilon)J_{\mu_k} f^n(y)\hat\mu^k_x(\hat f^n(W^s_{\mathrm{loc}}(y))) = (1-\varepsilon)\end{aligned}$$

and it follows that

$$m^k_y(A^n_k(y)^{-1}(v)) = \int_{W^s_{\mathrm{loc}}(y)} \hat m^k_{\hat y}(A^n_k(y)^{-1}(v))\,d\hat\mu^k_y(\hat y) \ge (1-\varepsilon)\gamma_1>\gamma^k_0$$

contradicting the definition of γ^k_0. □

Note that we have shown that if m_x^k contains an atom for some $x \in \Sigma$, then m_x^k contains an atom of mass γ_0^k for every $x \in \Sigma$. By ergodicity of the measure m_k, it follows that m_x^k is finitely supported for every $x \in \Sigma$ and, moreover, that every atom of m_x^k has mass γ_0^k. To derive a contradiction to (6), we further divide the case that m_x^k contains atoms into two subcases.

6.1. **Case A: Positive Lyapunov exponents.** Passing to a subsequence, assume that $\lambda_+(\hat{A}_k, \hat{\mu}_k) > 0$ for every $k \in \mathbb{N}$.

Recall Lemma 6.1. Given $x \in \Sigma$, let $v_x^k \in \mathbb{P}^1$ be such that such that $m_x^k(v_x^k) = \gamma_0^k > 0$ for all $x \in \Sigma$. From Lemma 6.2 we have $\hat{m}_{\hat{x}}^k(v_x^k) = \gamma_0^k$ for $\hat{\mu}_x$ almost every $\hat{x} \in W^s_{\mathrm{loc}}(x)$. But, as we assume $\lambda_+(\hat{A}_k, \hat{\mu}_k) > 0$, it follows from the definition of $\hat{m}_k$ (see the proof of Proposition 5.2) that $\hat{m}_{\hat{x}}^k = \delta_{E_{\hat{x}}^{u,k}}$. Consequently, $\gamma_0^k = 1$ and $v_x^k = E_{\hat{x}}^{u,k}$ for $\hat{\mu}_x$ almost every $\hat{x} \in W^s_{\mathrm{loc}}(x)$.

It then follows from Lemma 4.8 that $m_x^k = \delta_{v_x^k}$ converges to $m_x = \alpha\delta_q + \beta\delta_p$ for every $x \in \Sigma$. Since $\alpha, \beta \in (0,1)$ and $p \neq q$ this gives a contradiction.

6.2. **Case B: Zero Lyapunov exponents.** We now suppose $\lambda_+(\hat{A}_k, \hat{\mu}_k) = 0$ for every $k \in \mathbb{N}$.

First note that, as $\lambda_+(\hat{A}_k, \hat{\mu}_k) = 0 = \lambda_-(\hat{A}_k, \hat{\mu}_k)$ for every $k \in \mathbb{N}$, by Theorem 4.2, each measure $\hat{m}_k$ is an su-state. By Proposition 4.1 we may find an su-invariant disintegration into a continuous family of conditional measures $\{\hat{m}_{\hat{x}}^k\}_{\hat{x}\in\hat{\Sigma}}$. As the stable holonomies are trivial, we have that $m_x^k = \hat{m}_{\hat{x}}^k = \hat{m}_{\hat{z}}^k$ for every $\hat{x}, \hat{z} \in W^s_{\mathrm{loc}}(x)$ and all $x \in \Sigma$. By Proposition 5.2 the same property holds for the disintegrations of $\hat{m}$ and m.

In particular this allows to identify m_x^k and m_y^k via unstable holonomies, for x, y in the cylinder $[0; i]$ for $1 \le i \le \ell$.

CLAIM 6.3. *For each $1 \le i \le \ell$ and every $x, y \in [0; i]$, $\hat{x} \in W^s_{\mathrm{loc}}(x)$, and $\hat{y} = W^s_{\mathrm{loc}}(y) \cap W^u_{\mathrm{loc}}(\hat{x})$ we have*

$$m_y = \left(H^{u,\hat{A}}_{\hat{x}\hat{y}}\right)_* m_x \quad \text{and} \quad m_y^k = \left(H^{u,k}_{\hat{x}\hat{y}}\right)_* m_x^k.$$

Let $\{V_x^k\}_{x\in\Sigma}$ be the family of finite subsets of $\mathbb{P}^1$ given by

$$V_x^k := \{v \in \mathbb{P}^1 : m_x^k(v) = \gamma_0^k\}.$$

Note that $m_x^k(V_x^k) = 1$. Moreover, combining Lemma 6.1 and the previous claim we have

CLAIM 6.4. *For $x, y \in \Sigma$, and $k \in \mathbb{N}$,*

1. *card$(V_x^k) =$ card(V_y^k),*
2. *$A_k(x)(V_x^k) = V^k_{f(x)}$, and*
3. *$V_y^k = H^{u,k}_{\hat{x}\hat{y}}(V_x^k)$ for any $\hat{x} \in W^s_{\mathrm{loc}}(x)$ and $\hat{y} \in W^s_{\mathrm{loc}}(y) \cap W^u_{\mathrm{loc}}(\hat{x})$.*

We now bound the number of atoms appearing in the measure m_x^k.

LEMMA 6.5. *For every $x \in \Sigma$ we have that card$(V_x^k) \le 2$ for k sufficiently large.*

Proof. As $\mathrm{card}(V_x^k)$ is defined for every x and is moreover constant, it is enough to prove that $\mathrm{card}(V_x^k) \le 2$ for some $x \in \Sigma$.

We claim there is a periodic point $x \in \Sigma$ with period ℓ such that $A^\ell(x) := A(f^{\ell-1}(x))\dots A(x)$ is hyperbolic. Indeed, recall that the cocycle $A(x)$ preserves the coordinate axes and is thus of the form

$$A(x) = \begin{pmatrix} \gamma(x) & 0 \\ 0 & \pm\gamma(x)^{-1} \end{pmatrix}.$$

If follows that the logarithm of the eigenvalues of $A^\ell(x)$ for any such periodic point x are

$$\frac{1}{\ell}\sum_{j=0}^{\ell-1} \log|\gamma^{\pm 1}(f^j(x))|.$$

If the logarithm of the eigenvalues of $A^\ell(x)$ vanished for every periodic point x then, as measures concentrated on periodic orbits are dense in the set of all f-invariant measures, it follows that $\int \log|\gamma(x)|\, d\mu'(x) = 0$ for every f-invariant measure μ'. It follows that the Lyapunov exponents of the cocycle vanish for every f-invariant measure μ' contradicting our assumption on the measure μ. The matrix $A^\ell(x)$ is thus hyperbolic for some periodic point x and, as the set of hyperbolic matrices is open, for k sufficiently large $A_k^\ell(x)$ is also hyperbolic. Therefore, as $A_k^\ell(x)(V_x^k) = V_x^k$ and V_x^k is finite, it follows that $\mathrm{card}(V_x^k) \le 2$. □

Let $V_x^k = \{v_x^k\}$ or $V_x^k = \{v_x^k, w_x^k\}$ depending on the cardinality of V_x^k. As m_k is ergodic, either

$$m_k = \begin{cases} \int \delta_{(x,v_x^k)}\, d\mu_k(x) & \mathrm{card}(V_x^k) = 1 \\ \int \frac{1}{2}\delta_{(x,v_x^k)} + \frac{1}{2}\delta_{(x,w_x^k)}\, d\mu_k(x) & \mathrm{card}(V_x^k) = 2. \end{cases}$$

As before, we write $\Phi_{A_k} : \Sigma \times \mathbb{P}^1 \to \mathbb{R}$ for

$$\Phi_{A_k}(x,v) = \log\left(\frac{\|A_k(x)(v)\|}{\|v\|}\right).$$

Recalling that the cocycle $\hat{A}_k$ is constant along local stable sets and recalling the definition of the measure m_k we have

$$\lambda_+(\hat{A}_k, \hat{\mu}_k) = \int_{\hat{\Sigma}\times\mathbb{P}^1} \Phi_{\hat{A}_k}\, d\hat{m}_k = \int_{\Sigma\times\mathbb{P}^1} \Phi_{A_k}\, dm_k. \tag{8}$$

In particular

$$0 = \lambda_+(\hat{A}_k, \hat{\mu}_k) = \begin{cases} \int \Phi_{A_k}(x, v_x^k)\, d\mu_k(x) & \mathrm{card}(V_x^k) = 1 \\ \int \frac{1}{2}\Phi_{A_k}(x, v_x^k) + \frac{1}{2}\Phi_{A_k}(x, w_x^k)\, d\mu_k(x) & \mathrm{card}(V_x^k) = 2. \end{cases} \tag{9}$$

We now consider two subcases depending on the cardinality of V_x^k.

6.2.1. *Subcase: card*$(V_x^k) = 1$ *for every* $k \in \mathbb{N}$. Passing to a subsequence, suppose that $\operatorname{card}(V_x^k) = 1$ for every $k \in \mathbb{N}$. For every $x \in \Sigma$, let $V_x^k = \{v_x^k\}$.

As in the previous case, from Lemma 4.8 we have that $m_x^k = \delta_{v_x^k}$ converges to $m_x = \alpha\delta_q + \beta\delta_p$ for every $x \in \Sigma$. Again, since $\alpha, \beta \in (0,1)$ and $p \neq q$ this gives a contradiction.

6.2.2. *Subcase: card*$(V_x^k) = 2$ *for every* $k \in \mathbb{N}$. If $\operatorname{card}(V_x^k) = 2$ for every k we may take $V_y^k = \{v_y^k, w_y^k\}$ with

$$v_y^k = H_{\hat{x}\hat{y}}^{u,k}(v_x^k) \quad \text{and} \quad w_y^k = H_{\hat{x}\hat{y}}^{u,k}(w_x^k) \tag{10}$$

for every x and y in the same cylinder. Moreover,

$$A_k(y)\left(\left\{v_y^k, w_y^k\right\}\right) = \left\{v_{f(y)}^k, w_{f(y)}^k\right\} \tag{11}$$

for every $y \in \Sigma$.

Fix $x \in \Sigma$. Passing to subsequences suppose that v_x^k converges to v_0 and w_x^k converges to w_0 in $\mathbb{P}^1$. Then, by (10), it follows that

$$v_y^k = H_{\hat{x}\hat{y}}^{u,k}(v_x^k) \xrightarrow{k\to\infty} H_{\hat{x}\hat{y}}^{u,\hat{A}}(v_0) := v_y$$

and

$$w_y^k = H_{\hat{x}\hat{y}}^{u,k}(w_x^k) \xrightarrow{k\to\infty} H_{\hat{x}\hat{y}}^{u,\hat{A}}(w_0) := w_y$$

for every y in the same cylinder as x. Invoking (11) and (10) it follows that v_y^k converges to some v_y and w_y^k converges to some w_y in $\mathbb{P}^1$ for *every* $y \in \Sigma$. Moreover, $y \mapsto v_y$ and $y \mapsto w_y$ are continuous and

$$A(y)\left(\left\{v_y, w_y\right\}\right) = \left\{v_{f(y)}, w_{f(y)}\right\} \tag{12}$$

and

$$v_z = H_{\hat{y}\hat{z}}^{u,\hat{A}}(v_y) \quad \text{and} \quad w_z = H_{\hat{y}\hat{z}}^{u,\hat{A}}(w_y)$$

for every $y, z \in \Sigma$ in the same cylinder.

The functions $y \mapsto v_y$ and $y \mapsto w_y$ are continuous and defined everywhere. From (12) the graphs of $y \mapsto v_y$ and $y \mapsto w_y$ together form a closed, F_A-invariant subset of $\Sigma \times \mathbb{P}^1$. Hence (by an argument similar to the proof of Claim 5.1), the (non-uniform) hyperbolicity of the cocycle A implies that either $v_y = q$ for every $y \in \Sigma$ or $v_y = p$ for every $y \in \Sigma$. Similarly, we have that $w_y = q$ for every $y \in \Sigma$ or $w_y = p$ for every $y \in \Sigma$.

Sub-subcase: $v_0 = w_0$. Suppose first that $w_0 = v_0 = q$. Then $w_y = v_y = q$ for every $y \in \Sigma$ and from (9)

$$\begin{aligned} 0 &= \int \tfrac{1}{2}\Phi_{A_k}(x, v_x^k) + \tfrac{1}{2}\Phi_{A_k}(x, w_x^k)\; d\mu(x) \\ &\to \int_\Sigma \Phi_A(x, q)\; d\mu(x) = \int \Phi_A\; dm^s = \lambda_-(\hat{A}, \hat{\mu}) \end{aligned}$$

which is a contradiction since $\lambda_-(\hat{A},\hat{\mu})<0$. Similarly, if $w_y=v_y=p$ for every y then

$$0=\int \tfrac{1}{2}\Phi_{A_k}(x,v_x^k)+\tfrac{1}{2}\Phi_{A_k}(x,w_x^k)\ d\mu(x)$$
$$\to \int \Phi_A(x,p)\ d\mu(x)=\int \Phi_A\ dm^u=\lambda_+(\hat{A},\hat{\mu})$$

contradicting that $\lambda_+(\hat{A},\hat{\mu})>0$.

Sub-subcase: $v_0\neq w_0$. If $v_0\neq w_0$ then (by ergodicity) without loss of generality we may assume $v_y=q$ and $w_y=p$ for all $y\in\Sigma$. However, as we assumed m_k to be ergodic, for any k we have that $A_k(y)v_y^k=w_{f(y)}^k$ for some $y\in\Sigma$. Having taken k sufficiently large, it follows from the uniform convergence of $A_k\to A$, $v_y^k\to v_y$ and $w_y^k\to w_y$ that $A(y)v_y=w_{f(y)}$ and hence $A(y)(p)=q$, a contradiction.

7. CASE 2: THE MEASURES m_x^k ARE NON-ATOMIC

We now derive a contradiction to (6) under the assumption that the measures m_x^k are non-atomic for every $x\in\Sigma$ and every k. To arrive at a contradiction, we introduce the tools of couplings and (additive) energies: $\{m_x^k\}_{x\in\Sigma}$ being non-atomic implies that the family of measures $\{m_x^k|_{U_x}\}$ obtained from $\{m_x^k\}$ by restriction to a suitable family of sets $\{U_x\}$ admit a family of symmetric self-couplings with finite energy. Taking advantage of the fact that the stable space is a repeller for the action of the cocycle A on $\mathbb{P}^1$, we are able to build a new family of symmetric self-couplings of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ with energy strictly smaller by a definite factor coming from the rate of expansion of A at the stable space. We can then iterate this procedure to construct a symmetric self-coupling of $\{m_x^k|_{U_x}\}$ with negative energy, arriving at a contradiction. This approach follows the main ideas in [31, Chapter 10] though we deviate slightly from [31] by using *additive Margulis functions* introduced in [2]. The contradiction is given by Proposition 7.6.

Recall we changed coordinates on the cocycle A so that $A(y)$ fixes $q=[1:0]$ and $p=[0:1]$ for every $y\in\Sigma$. Recall the sequence of measures m_k projecting to μ_k and admitting continuous family of conditional measures $\{m_x^k\}_{x\in\Sigma}$. Moreover m_k converges to the measure $\mu\times(\alpha\delta_q+\beta\delta_p)$ and the conditional measures m_x^k converge uniformly to $\alpha\delta_q+\beta\delta_p$. Recall we assume $\alpha>0$. We write $\mathbb{P}A\colon\Sigma\to\mathrm{Diff}^\infty(\mathbb{P}^1)$, $x\mapsto\mathbb{P}A(x)$ for the projective cocycle. Similarly, write $\mathbb{P}A_k$ for the projectivized cocycle of A_k.

7.1. q is an expanding point.

We begin by recalling that, given $B\in GL(2,\mathbb{R})$ and $v\in\mathbb{P}^1$, the derivative at the point v of the action of $\mathbb{P}B$ in the projective space is given explicitly by

$$D_v\mathbb{P}B(\dot{v})=\frac{\mathrm{proj}_{B(v)}B(\dot{v})}{\|B(v)\|}\quad\text{for every }\dot{v}\in T_v\mathbb{P}^1=\{v\}^\perp$$

where $\mathrm{proj}_v : w \to w - v\frac{\langle w,v\rangle}{\langle v,v\rangle}$ denotes the orthogonal projection to the hyperplane orthogonal to v.

CLAIM 7.1. *For almost every $x \in \Sigma$ we have*

$$\lim_{n\to\infty} \log\left(\left\| D_q\mathbb{P}A^n(x)\right\|^{1/n}\right) = \lambda_+(\hat{A},\hat{\mu}) - \lambda_-(\hat{A},\hat{\mu}) =: c > 0.$$

Proof. Recall p, q are orthogonal and preserved by the cocycle A.

Then for $v \in T_q\mathbb{P}^1 = \{q\}^\perp$ we have $v \in \mathrm{span}(p)$. Thus if $\|v\| = 1$ we have $\|A^n(y)(v)\| = \|A^n(y)(p)\|$. Projecting back to $T_q\mathbb{P}^1$ we have

$$\|D_q\mathbb{P}A^n(y)(v)\| = \frac{\|A^n(y)(p)\|}{\|A^n(y)(q)\|}.$$

The claim then follows from the pointwise ergodic theorem. $\square$

CLAIM 7.2. *We may select $N \in \mathbb{N}$ such that*

$$\int_\Sigma \log\left(\|D_q\mathbb{P}A^N(x)\|\right) d\mu(x) > 6.$$

Proof. We have $\lim_{n\to\infty} \frac{1}{n}\log\left(\|D_q\mathbb{P}A^n(x)\|\right) \to c > 0$ almost everywhere. Moreover, as $\frac{1}{n}\log\left(\|D_q\mathbb{P}A^n(x)\|\right)$ is bounded above and below uniformly in x and n, by dominated convergence we have

$$\lim_{n\to\infty} \frac{1}{n}\int \log\left(\|D_q\mathbb{P}A^n(x)\|\right) d\mu(x) \to c > 0. \qquad \square$$

Fix such an N for the remainder. We define

$$\kappa(x) := \log\left(\|D_q\mathbb{P}A^N(x)\|\right).$$

As $\kappa\colon \Sigma \to \mathbb{R}$ is a continuous function, for all sufficiently large k we have

$$\int \kappa(x)\ d\mu_k(x) > 4. \tag{13}$$

7.2. **Couplings and energy.** Let d be the distance on $\mathbb{P}^1$ defined by the angle between two directions. We assume d is normalized so that $\mathbb{P}^1$ has diameter 1.

Consider a Borel probability measure μ' on Σ and a μ'-measurable family $\{\nu_x\}_{x\in\Sigma}$ of finite Borel measures on $\mathbb{P}^1$. The measures ν_x are not assumed to be probabilities nor are they assumed to have the same mass. For $j \in \{1,2\}$, let $\pi_j : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection on the j-th factor. For $x \in \Sigma$, let ξ_x be a measure on $\mathbb{P}^1 \times \mathbb{P}^1$. We say a parameterized family of (positive) measures $\{\xi_x\}_{x\in\Sigma}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is a (measurable) *family of symmetric self-couplings* of $\{\nu_x\}_{x\in\Sigma}$ if

1. $x \mapsto \xi_x$ is μ'-measurable,
2. $(\pi_j)_*\xi_x = \nu_x$ for $j \in \{1,2\}$, and
3. $\iota_*\xi_x = \xi_x$ where $\iota\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the involution $\iota\colon (u,v) \to (v,u)$.

We note that we always have one family of symmetric self-couplings constructed by taking for every x the product measure

$$\xi_x = \frac{1}{\|\nu_x\|}\nu_x \times \nu_x$$

for all x with $\|\nu_x\| \neq 0$, where $\|\nu_x\| := \nu_x(\mathbb{P}^1)$ denotes the mass of the measure ν_x.

We define a function $\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{R}$ by

$$\varphi(u, v) = -\log d(u, v).$$

Note that φ is non-negative. In the language of [2], the function φ is an *additive Margulis function* and its properties will be used to deduce the contradiction in Proposition 7.6 below. For a family of symmetric self-couplings $\{\xi_x\}_{x\in\Sigma}$ of $\{\nu_x\}_{x\in\Sigma}$ as above we define the (additive) *energy* of $\{\xi_x\}_{x\in\Sigma}$ to be

$$\int_\Sigma \int_{\mathbb{P}^1\times\mathbb{P}^1} \varphi(u, v)\ d\xi_x(u, v) d\mu'(x).$$

7.3. **Choice of parameters.** To establish a contradiction to (6) we select a number of parameters that will be fixed for the remainder. Recall the N fixed above and the function κ.

1. Let $U_0 \subset \mathbb{P}^1$ be an open ball centered at q with $p \notin \overline{U}_0$.
2. Let $U_1 \subset \overline{U_1} \subset U_0$ be an open neighborhood of q such that for every $x \in \Sigma$ and every sufficiently large k we have
   (a) $\mathbb{P}A_k^N(x)(\overline{U_1}) \subset U_0$;
   (b) $\overline{U_1} \subset \mathbb{P}A_k^N(x)(U_0)$;
   (c) for every $u, v \in U_1$

$$d(\mathbb{P}A_k^N(x)(u), \mathbb{P}A_k^N(x)(v)) \geq e^{-\alpha} e^{\kappa(x)} d(u, v). \tag{14}$$

From (14) it follows for every $u, v \in U_1$, $x \in \Sigma$ and k sufficiently large that

$$\varphi(\mathbb{P}A_k^N(x)(u), \mathbb{P}A_k^N(x)(v)) \leq \varphi(u, v) - \kappa(x) + \alpha. \tag{15}$$

3. Fix $q \in U_4 \subset \overline{U_4} \subset U_3 \subset \overline{U_3} \subset U_2 \subset \overline{U_2} \subset U_1$ such that
   (a) each U_j is an open set;
   (b) $\mathbb{P}A_k^N(y)(\overline{U}_4) \subset U_3$ for every $y \in \Sigma$ and k sufficiently large.
4. By compactness of Σ and uniform convergence of A_k to A, we may select $M_1 > 1$ so that for all $x \in \Sigma, u \in \mathbb{P}^1$ and k sufficiently large,
$$-M_1 < \log(\|D_u \mathbb{P}A_k^N(x)\|) < M_1.$$
   Note in particular that $|\kappa(x)| \leq M_1$.
5. Take $M_2 > 1$ to be the maximum of
$$\sup\{\varphi(u, v) : u \in U_3, v \in U_2^c\} \text{ and } \sup\{\varphi(u, v) : u \in U_2, v \in U_1^c\}.$$
6. Fix $0 < \delta < 1 - \alpha$ with $100\delta M_1 M_2 < \alpha$.
7. For k sufficiently large, we have for every $x \in \Sigma$ that
   (a) $m_x^k(U_4) > \alpha - \delta$;
   (b) $m_x^k(U_0) < \alpha + \delta$.
8. For the remainder, fix k sufficiently large so that all estimates above (including (13)) hold.
9. Given our k fixed above define $\rho\colon \Sigma \to [0, 1)$ so that
$$m_x^k(B(q, \rho(x))) = \alpha + \delta.$$

Observe that as m_x^k is assumed to have no atoms and as the measures m_x^k vary continuously in x, the function ρ is continuous. We write

$$U_x := B(q, \rho(x)).$$

Note that the choices above ensure that $U_0 \subset U_x$.

7.4. **Constructing finite energy families of symmetric self-couplings.** For the remainder of this section we work exclusively with the k fixed above. We will work primarily with the family of measures $\{m_x^k|_{U_x}\}$. Recall that the measure $m_x^k|_{U_x}$ is defined for every $x \in \Sigma$. Moreover, the dependence on x is continuous. Below, we will define a number of families of symmetric self-couplings $\{\xi_x\}_{x\in\Sigma}$ of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$. For every such family $\{\xi_x\}_{x\in\Sigma}$, the measure ξ_x will be defined for every $x \in \Sigma$. We first construct a family of symmetric self-couplings of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ with finite energy.

From the continuity and non-atomicity of the conditional measures m_x^k we obtain the following.

CLAIM 7.3. *There is an $r > 0$ so that for every $x \in \Sigma$ and $u \in \mathbb{P}^1$*

$$m_x^k(B(u, 3r)) < \frac{\alpha+\delta}{10}.$$

Using the above claim we have the following lemma.

LEMMA 7.4. *There exists a family of symmetric self-couplings $\{\xi_x\}_{x\in\Sigma}$ of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ with finite energy.*

Proof. Let $\{\xi_x\}_{x\in\Sigma}$ be any family of symmetric self-couplings of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$. Let $\{v_i\}_{i=1,\dots,m} \subset \mathbb{P}^1$ be such that $\mathbb{P}^1 = \bigcup_{i=1}^m B(v_i, r)$. Define ξ_x^1 by

$$\xi_x^1 := \xi_x - \xi_x|_{B(v_1,r)\times B(v_1,r)} - \theta_x \xi_x|_{B(v_1,3r)^c\times B(v_1,3r)^c} + \zeta_x + \iota_*\zeta_x,$$

where

$$\theta_x = \frac{\xi_x\,(B(v_1,r)\times B(v_1,r))}{\xi_x\,(B(v_1,3r)^c\times B(v_1,3r)^c)}$$

and

$$\zeta_x := \frac{1}{\xi_x\left((B(v_1,r))^2\right)}\left((\pi_1)_*\xi_x|_{B(v_1,r)^2} \times (\pi_1)_*\theta_x\xi_x|_{(B(v_1,3r)^c)^2}\right).$$

As

$$\begin{aligned}
\xi_x(B(v_1,r)\times B(v_1,r)) &\le \xi_x(B(v_1,3r)\times B(v_1,3r))\\
&= m_x^k|_{U_x}(B(v_1,3r)) - \xi_x(B(v_1,3r)\times B(v_1,3r)^c)\\
&< m_x^k|_{U_x}(B(v_1,3r)^c) - \xi_x(B(v_1,3r)^c\times B(v_1,3r))\\
&= \xi_x(B(v_1,2r)^c\times B(v_1,3r)^c)
\end{aligned}$$

we have that $\theta_x < 1$, hence ξ_x^1 is a (positive) measure. Moreover, ξ_x^1 is clearly symmetric and we check that $(\pi_1)_*\xi_x^1 = m_x^k|_{U_x}$.

The family $\{\xi_x^1\}_{x\in\Sigma}$ depends measurably on x and satisfies

$$\xi_x^1(B(v_1,r)\times B(v_1,r)) = 0.$$

Moreover, we have

$$\xi_x^1(B(v_i,r)\times B(v_i,r)) \le \xi_x(B(v_i,r)\times B(v_i,r))$$

for all $1 \le i \le m$. Indeed, ζ_x is supported on $B(v_1,r)\times B(v_1,3r)^c$. But for any $v \in \mathbb{P}^1$,

$$\big(B(v,r)\times B(v,r)\big)\cap\big(B(v_1,r)\times B(v_1,3r)^c\big)=\varnothing$$

and thus $\zeta_x(B(v_i,r)\times B(v_i,r))=0$ for each i. Thus with $S = B(v_i,r)\times B(v_i,r)$ we have

$$\xi_x^1(S)=\left(\xi_x-\xi_x|_{B(v_1,r)\times B(v_1,r)}-\theta_x\xi_x|_{B(v_1,3r)^c\times B(v_1,3r)^c}\right)(S).$$

Iterating the above construction yields a measurable family of symmetric self-couplings $\{\xi_x^m\}_{x\in\Sigma}$ of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$, defined for every $x \in \Sigma$, with

$$\xi_x^\ell(B(v_i,r)\times B(v_i,r))=0$$

for every $1 \le i \le m$. Taking $0 < r_0 < r$ to be the Lebesgue number of the cover $\{B(v_i,r)\}_{i=1}^l$ we have

$$\xi_x^m(B(u,r_0)\times B(u,r_0))=0$$

for every $u \in \mathbb{P}^1$ and $x \in \Sigma$. Then $\int \varphi \, d\xi_x^m \le -\log(r_0)$ for every $x \in \Sigma$. □

We introduce the first of many modifications we perform on our families of symmetric self-couplings. The construction of the family $\{\dot{\xi}_x\}_{x\in\Sigma}$ is formally very similar to the construction of $\{\xi_x^1\}_{x\in\Sigma}$ in Lemma 7.4.

LEMMA 7.5. *Let $\{\xi_x\}_{x\in\Sigma}$ be a family of symmetric self-couplings of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ with finite energy. Then there exists a family of symmetric self-couplings $\{\dot{\xi}_x\}_{x\in\Sigma}$ of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ with $\dot{\xi}_x(U_2^c\times U_2^c)=0$ and such that for every $x\in\Sigma$*

$$\int \varphi \, d\dot{\xi}_x \le \int \varphi \, d\xi_x + 4\delta M_2.$$

Proof. Let $\nu_x := (\pi_1)_*\xi_x|_{U_2^c\times U_2^c}$ and $\eta_x = (\pi_1)_*\xi_x|_{U_3\times U_3}$. Define

$$\dot{\xi}_x := \xi_x - \xi_x|_{U_2^c\times U_2^c} - \frac{\|\nu_x\|}{\|\eta_x\|}\xi_x|_{U_3\times U_3} + \frac{1}{\|\eta_x\|}\left(\nu_x\times\eta_x+\eta_x\times\nu_x\right).$$

We have that $\dot{\xi}_x(U_2^c\times U_2^c)=0$, $(\pi_j)_*(\dot{\xi}_x)=m_x^k|_{U_x}$ for both $j=\{1,2\}$, and that $\dot{\xi}_x$ is symmetric. Moreover, we have

$$\|\eta_x\| \ge \|\xi_x\| - 2\xi_x((U_x\setminus U_3)\times\mathbb{P}^1) = (\alpha+\delta)-4\delta.$$

and

$$\|\nu_x\| = \|\xi_x|_{U_2^c\times U_2^c}\| \le \xi_x(U_2^c\times\mathbb{P}^1) \le 2\delta.$$

It follows for every x that $\|\nu_x\| \le \|\eta_x\|$ and hence $\dot{\xi}_x$ is a (positive) measure.

Note that $\nu_x \times \eta_x$ is supported on $U_2^c \times U_3$. It follows that

$$\begin{aligned}\int \varphi\, d\dot{\xi}_x &\le \int \varphi\, d\xi_x + \frac{1}{\|\eta_x\|}\int \varphi\, d(\nu_x \times \eta_x + \eta_x \times \nu_x)\\ &\le \int \varphi\, d\xi_x + \frac{2}{\|\eta_x\|}\int \varphi\, d(\nu_x \times \eta_x)\\ &\le \int \varphi\, d\xi_x + \frac{2}{\|\eta_x\|}\int M_2\, d(\nu_x \times \eta_x)\\ &\le \int \varphi\, d\xi_x + \frac{2M_2\|\eta_x\|\|\nu_x\|}{\|\eta_x\|}.\end{aligned}$$

As $\|\nu_x\| \le 2\delta$ the claim follows. □

7.5. **Key proposition.** We are now in position to state the key proposition that establishes the contradiction to (6) in the case when the measures m_x^k are non-atomic. To prove it we exploit the fact that q is an expanding point for the projective cocycle $\mathbb{P}A_k^N$ (recall (15)) and the invariance of m_k with respect to F_{A_k} (recall Remark 4.6).

PROPOSITION 7.6. *Let $\{\xi_x\}_{x\in\Sigma}$ be a family of symmetric self-couplings of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ with finite energy. Then, there exists a family of symmetric self-couplings $\{\ddot{\xi}_x\}_{x\in\Sigma}$ of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ such that*

$$\iint \varphi\, d\ddot{\xi}_x d\mu_k(x) \le \iint \varphi\, d\xi_x d\mu_k(x) - \alpha.$$

As φ is a non-negative function, by recursive applications of Proposition 7.6 we arrive at a contradiction.

The proof of Proposition 7.6 is as follows: acting on a symmetric self-coupling $\{\xi_x\}_{x\in\Sigma}$ by the diagonal action of F_{A_k} creates a new measure $\{\hat{\xi}_x\}_{x\in\Sigma}$ on $\Sigma\times\mathbb{P}^1\times\mathbb{P}^1$ whose energy is strictly smaller than that of $\{\xi_x\}_{x\in\Sigma}$. However, the new family of measures $\{\xi_x\}_{x\in\Sigma}$ is not a self-coupling so we correct this to create a new symmetric self-coupling whose energy is still strictly smaller than that of $\{\xi_x\}_{x\in\Sigma}$.

To start the proof of Proposition 7.6, given $\{\xi_x\}_{x\in\Sigma}$ let $\{\dot{\xi}_x\}_{x\in\Sigma}$ be the family of symmetric self-couplings constructed in Lemma 7.5. For each $x \in \Sigma$ define

$$\hat{\xi}_x = \sum_{y\in f^{-N}(x)} \frac{1}{J_{\mu_k} f^N(y)} \left(\mathbb{P}A_k^N(y) \times \mathbb{P}A_k^N(y)\right)_* \dot{\xi}_y.$$

The restriction of $\hat{\xi}_x$ to $U_x \times U_x$ is not necessarily a self-coupling of $m_x^k|_{U_x}$. Write η_x for the projection $\eta_x := (\pi_1)_*\left(\hat{\xi}_x|_{U_x\times U_x}\right)$. Below, we estimate the defect between η_x and $m_x^k|_{U_x}$.

Write $g(y) = \dfrac{1}{J_{\mu_k} f^N(y)}$. Recall that for any $x \in \Sigma$

$$\sum_{y\in f^{-N}(x)} g(y)\mathbb{P}A_k^N(y)_* m_y^k = m_x^k.$$

Define two families of measures on $\mathbb{P}^1$ by

- $I_x := \left(\sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* \left(m_y^k|_{U_y^c} \right) \right) \Big|_{U_x}$, and
- $O_x := (\pi_1)_* \left(\hat{\xi}_x|_{U_x \times U_x^c} \right)$.

The families $\{I_x\}_{x\in\Sigma}$ and $\{O_x\}_{x\in\Sigma}$ are measurable.

LEMMA 7.7. *We have*

$$m_x^k|_{U_x} = \eta_x + I_x + O_x.$$

Moreover, for every $x \in \Sigma$ we have $\|O_x\| \le \|I_x\| \le 2\delta$ and supp$(I_x) \subset U_1^c$.

Proof. We have

$$m_x^k|_{U_x} = \left(\sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* m_y^k \right) \Big|_{U_x}$$

$$= \left(\sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* (m_y^k|_{U_y^c}) \right) \Big|_{U_x} \tag{16}$$

$$+ \left(\sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* (m_y^k|_{U_y}) \right) \Big|_{U_x}. \tag{17}$$

The term (16) is precisely I_x. The term (17) is

$$\left((\pi_1)_* \hat{\xi}_x \right)|_{U_x} = (\pi_1)_* (\hat{\xi}_x|_{U_x \times U_x}) + O_x$$

hence we obtain

$$m_x^k|_{U_x} = \eta_x + I_x + O_x.$$

To bound $\|I_x\|$ note that for any measurable set $B \subset \mathbb{P}^1$ we have $I_x(B) \le m_x^k(B)$. Moreover, I_x is supported on

$$\bigcup_{y \in f^{-N}(x)} \mathbb{P}A_k^N(y)(U_y^c) \subset \bigcup_{y \in f^{-N}(x)} \mathbb{P}A_k^N(y)(U_0^c) \subset U_1^c.$$

Thus $\|I_x\| \le m_x^k(U_x \setminus U_1) \le 2\delta$.

To derive the bound on $\|O_x\|$ first note that

$$m_x^k(U_x) = \sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* \left(m_y^k|_{U_y} \right)(U_x) + \sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* \left(m_y^k|_{U_y^c} \right)(U_x).$$

Similarly,

$$\begin{aligned}
&\sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* \left(m_y^k|_{U_y} \right)(U_x) + \sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* \left(m_y^k|_{U_y} \right)(U_x^c) \\
&= \sum_{y \in f^{-N}(x)} g(y) \mathbb{P}A_k^N(y)_* \left(m_y^k|_{U_y} \right)(\mathbb{P}^1) \\
&= \sum_{y \in f^{-N}(x)} g(y) \left(m_y^k(U_y) \right) \\
&= \alpha + \delta = m_x^k(U_x),
\end{aligned}$$

where the final equality follows from the choice of open sets U_y. Combining the above we have for any $x \in \Sigma$ that

$$\sum_{y \in f^{-N}(x)} g(y) \mathbb{P} A_k^N(y)_* \left(m_y^k|_{U_y^c}\right)(U_x) = \sum_{y \in f^{-N}(x)} g(y) \mathbb{P} A_k^N(y)_* \left(m_y^k|_{U_y}\right)(U_x^c). \tag{18}$$

The left hand side of (18) is I_x. The right hand side of (18) is

$$(\pi_2)_* (\hat{\xi}|_{\mathbb{P}^1 \times U_x^c}).$$

Then

$$\|I_x\| = \|(\pi_2)_*(\hat{\xi}|_{\mathbb{P}^1 \times U_x^c})\| \geq \|(\pi_2)_*(\hat{\xi}|_{U_x \times U_x^c})\| = \|O_x\|$$

and the lemma follows. □

7.6. **Proof of Proposition 7.6.** We conclude this section with the proof of Proposition 7.6.

Proof of Proposition 7.6. Recall the notation and constructions above (see also Figure 1 below). Define measurable families of measures

- $\theta_x := (\pi_1)_* \left(\hat{\xi}_x|_{U_3 \times U_3}\right)$;
- $\lambda_x := \left(1 - \dfrac{\|O_x|_{U_2}\|}{\|I_x\|}\right) I_x + O_x^k|_{U_2^c}$.

For $x \in \Sigma$ we define a new measure on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$\begin{aligned}\ddot{\xi}_x := \hat{\xi}_x|_{U_x \times U_x} - \frac{\|\lambda_x\|}{\|\theta_x\|} \hat{\xi}_x|_{U_3 \times U_3} + \frac{1}{\|I_x\|}\left(O_x|_{U_2} \times I_x + I_x \times O_x|_{U_2}\right) \\ + \frac{1}{\|\theta_x\|}(\lambda_x \times \theta_x + \theta_x \times \lambda_x).\end{aligned}$$

The family $\{\ddot{\xi}_x\}_{x \in \Sigma}$ is measurable.

As $\|O_x\| \leq \|I_x\|$, we have that λ_x is a (positive) measure. Moreover, we have

$$\|\theta_x\| \geq \|\hat{\xi}_x|_{U_3 \times \mathbb{P}^1}\| - \|\hat{\xi}_x|_{U_3 \times U_3^c}\| \geq m_x^k(U_3) - m_x^k(U_x \setminus U_3) \geq \alpha - 3\delta.$$

As

$$\|\lambda_x\| \leq 4\delta \leq \alpha - 3\delta \leq \|\theta_x\|$$

we have that $\ddot{\xi}_x$ is a (positive) measure. Also $\ddot{\xi}_x$ is clearly symmetric.

Let $D \subset \mathbb{P}^1$ be a measurable set. Then,

$$\begin{aligned}(\pi_1)_* \ddot{\xi}_x(D) &= \ddot{\xi}_x(D \times \mathbb{P}^1) \\ &= \eta_x(D) - \frac{\|\lambda_x\|}{\|\theta_x\|}\theta_x(D) + O_x(D \cap U_2) + \frac{1}{\|I_x\|} I_x(D) O_x(U_2) \\ &\quad + \lambda_x(D) + \frac{\|\lambda_x\|}{\|\theta_x\|}\theta_x(D) \\ &= \eta_x(D) + O_x(D \cap U_2) + \frac{1}{\|I_x\|} I_x(D) O_x(U_2) + \lambda_x(D) \\ &= \eta_x(D) + O_x|_{U_2}(D) + \frac{\|O_x|_{U_2}\|}{\|I_x\|} I_x(D) + \left(1 - \frac{\|O_x|_{U_2}\|}{\|I_x\|}\right) I_x(D) + O_x|_{U_2^c}(D) \\ &= \eta_x(D) + O_x(D) + I_x(D) = m_x^k|_{U_x}(D)\end{aligned}$$

hence the family $\{\ddot{\xi}_x\}_{x\in\Sigma}$ is a family of symmetric self-couplings of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$.

From the definition of $\ddot{\xi}_x$ we have

$$\begin{aligned}\iint \varphi\, d\ddot{\xi}_x d\mu_k(x) \le &\iint \varphi\, d\hat{\xi}_x|_{U_x\times U_x}\, d\mu_k(x)\\ &+\int \frac{1}{\|I_x\|}\int \varphi\, d\left(O_x|_{U_2}\times I_x + I_x\times O_x|_{U_2}\right) d\mu_k(x)\\ &+\int \frac{1}{\|\theta_x\|}\int \varphi\, d\left(\lambda_x\times\theta_x+\theta_x\times\lambda_x\right)\, d\mu_k(x).\end{aligned}$$

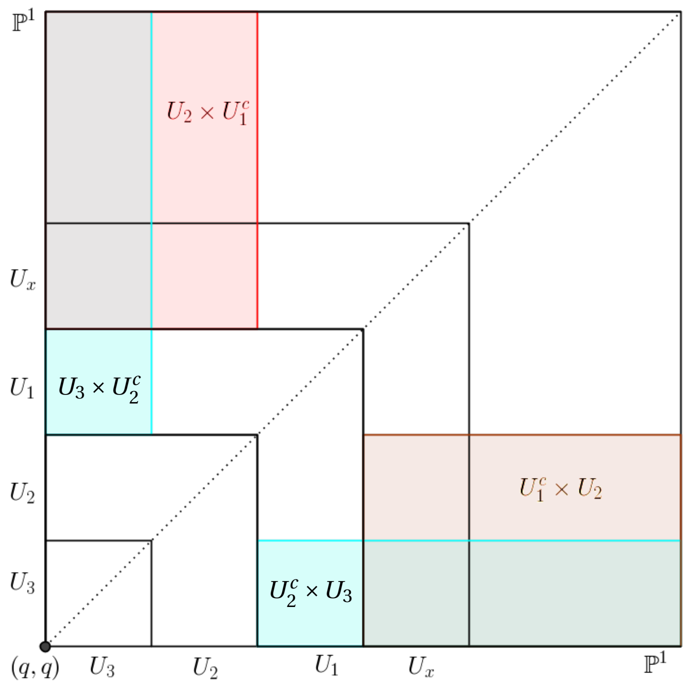

FIGURE 1. Mass away from the diagonal

Since $\mathrm{supp}(I_x)\subset U_1^c$ and $\mathrm{supp}(O_x|_{U_2})\subset U_2$ we have

$$\begin{aligned}\int \frac{1}{\|I_x\|}&\int \varphi\, d\left(O_x|_{U_2}\times I_x + I_x\times O_x|_{U_2}\right) d\mu_k(x)\\ &=\int \frac{2}{\|I_x\|}\int \varphi\, d\left(O_x|_{U_2}\times I_x\right) d\mu_k(x)\\ &\le\int \frac{2}{\|I_x\|}\int M_2\, d\left(O_x|_{U_2}\times I_x\right) d\mu_k(x)\\ &\le\int \frac{2M_2\|I_x\|O_x(U_2)}{\|I_x\|}\, d\mu_k(x)\le 4\delta M_2.\end{aligned}$$

Similarly, $\operatorname{supp}(\theta_x) \subset U_3$ and $\operatorname{supp}(\lambda_x) \subset U_2^c$ hence

$$\begin{aligned}
\int \frac{1}{\|\theta_x\|} \int \varphi \; d(\lambda_x \times \theta_x + \theta_x \times \lambda_x)\, d\mu_k(x) \\
&\leq \int \frac{2}{\|\theta_x\|} \int \varphi \; d(\lambda_x \times \theta_x)\, d\mu_k(x) \\
&\leq \int \frac{2}{\|\theta_x\|} \int M_2 \; d(\lambda_x \times \theta_x)\, d\mu_k(x) \\
&\leq \int \frac{2M_2 \|\theta_x\| \|\lambda_x\|}{\|\theta_x\|} \; d\mu_k(x) \leq 8\delta M_2.
\end{aligned}$$

Moreover, from the construction of $\ddot{\xi}_x$ and recalling that $\dot{\xi}_y(U_2^c \times U_2^c) = 0$ for all y we have

$$\begin{aligned}
\iint \varphi \; d\hat{\xi}_x|_{U_x \times U_x} \; d\mu_k(x) &\leq \int \int \varphi \; d\hat{\xi}_x \; d\mu_k(x) \\
&= \int \sum_{y \in f^{-N}(x)} g(y) \left(\int_{U_y \times U_y} \varphi\left(\mathbb{P}A_k^N(y)(u), \mathbb{P}A_k^N(y)(v)\right) d\dot{\xi}_y \right) d\mu_k(x) \\
&\leq \int \sum_{y \in f^{-N}(x)} g(y) \left(\int_{U_1 \times U_1} \varphi\left(\mathbb{P}A_k^N(y)(u), \mathbb{P}A_k^N(y)(v)\right) d\dot{\xi}_y \right) d\mu_k(x) \\
&\quad + \int \sum_{y \in f^{-N}(x)} g(y) \left(\int_{(U_2 \times U_1^c) \cup (U_1^c \times U_2)} \varphi\left(\mathbb{P}A_k^N(y)(u), \mathbb{P}A_k^N(y)(v)\right) d\dot{\xi}_y \right) d\mu_k(x).
\end{aligned}$$

Recalling the choice of M_1, M_2 and δ we have

$$\begin{aligned}
\int \sum_{y \in f^{-N}(x)} g(y) \left(\int_{(U_2 \times U_1^c) \cup (U_1^c \times U_2)} \varphi\left(\mathbb{P}A_k^N(y)(u), \mathbb{P}A_k^N(y)(v)\right) \; d\dot{\xi}_y \right) d\mu_k(x) \\
&\leq 2 \int \sum_{y \in f^{-N}(x)} g(y) \left(\int_{U_2 \times U_1^c} M_1 \varphi(u, v) \; d\dot{\xi}_y \right) d\mu_k(x) \\
&\leq 2M_1 M_2 \int \dot{\xi}_x(U_1^c \times U_2) \; d\mu_k(x) \\
&\leq 2M_1 M_2 \int m_x^k|_{U_x}(U_1^c) \; d\mu_k(x) \leq 4\delta M_1 M_2.
\end{aligned}$$

On the other hand, it follows from (15) and Lemma 7.5 that

$$\begin{aligned}
\int \sum_{y \in f^{-N}(x)} g(y) \left(\int_{U_1 \times U_1} \varphi\left(\mathbb{P}A_k^N(y)(u), \mathbb{P}A_k^N(y)(v)\right) d\dot{\xi}_y \right) d\mu_k(x) \\
&\leq \iint_{U_1 \times U_1} \varphi(u, v) \; d\dot{\xi}_x d\mu_k(x) - \int \dot{\xi}_x(U_1 \times U_1) \kappa(x) \; d\mu_k(x) + \alpha \\
&\leq \iint_{\mathbb{P}^1 \times \mathbb{P}^1} \varphi(u, v) \; d\dot{\xi}_x d\mu_k(x) - \int \dot{\xi}_x(U_1 \times U_1) \kappa(x) \; d\mu_k(x) + \alpha \\
&\leq \iint_{\mathbb{P}^1 \times \mathbb{P}^1} \varphi(u, v) \; d\xi_x d\mu_k(x) + 4\delta M_2 - \int \dot{\xi}_x(U_1 \times U_1) \kappa(x) \; d\mu_k(x) + \alpha.
\end{aligned}$$

Note that for any family of symmetric self-couplings $\{\dot{\xi}_x\}_{x\in\Sigma}$ of $\{m_x^k|_{U_x}\}_{x\in\Sigma}$ we have

$$\alpha - 3\delta \le \dot{\xi}_x(U_1\times\mathbb{P}^1) - \dot{\xi}_x(U_1\times(U_x\setminus U_1)) = \dot{\xi}_x(U_1\times U_1) \le \dot{\xi}_x(U_1\times\mathbb{P}^1) \le \alpha+\delta.$$

Writing $\kappa^+(x)$ and $\kappa^-(x)$, respectively, for the positive and negative parts of $\kappa(x)$ we have that

$$\begin{aligned}
&\int \dot{\xi}_x(U_1\times U_1)\kappa(x)\, d\mu_k(x)\\
&\ge (\alpha-3\delta)\int \kappa^+(x) d\mu_k(x) - (\alpha+\delta)\int \kappa^-(x)\, d\mu_k(x)\\
&= (\alpha-3\delta)\int \kappa(x)\, d\mu_k(x) - 4\delta\int \kappa^-(x)\, d\mu_k(x)\\
&\ge (\alpha-3\delta)4 - 4\delta M_1 \ge 4\alpha - 16\delta M_1 > 3\alpha.
\end{aligned}$$

We therefore have

$$\begin{aligned}
&\iint \varphi\, d\ddot{\xi}_x d\mu_k(x)\\
&\quad\le \iint \varphi\, d\xi_x d\mu_k(x) + 4\delta M_2 - 3\alpha + 4\delta M_1 M_2 + 4\delta M_2 + 8\delta M_2 + \alpha\\
&\quad\le \iint \varphi\, d\xi_x d\mu_k(x) - \alpha. \qquad \square
\end{aligned}$$

This completes the proof of Proposition 7.6. Combined with the results of Section 6 this completes the proof of Theorem 2.8.

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