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Matrix Representations for Integer Partitions:
Some Consequences and a New Approach

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"Para ser grande, sê inteiro: nada
Teu exagera ou exclui.
Sê todo em cada coisa. Põe quanto és
No mínimo que fazes.
Assim em cada lago a lua toda
Brilha, porque alta vive."
Fernando Pessoa

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Abstract

The present work is dedicated to a study on some consequences and behaviors of the two-line matrix representations for some sets of integer partitions and mock theta functions. In the first part of the text, we classify the partitions generated by six different mock theta functions, according to the sum of the second line of their associated matrices, and present some closed formulas and identities concerning those partitions. We also define the family of mock theta functions $\{f_*^m(q)\}_{m\geq 1}$, inspired by what we have called the unsigned version of function $f_1(q)$. We are able to give analogous matrix representations for all of the functions $f_*^m(q)$, which lead to interesting results concerning the partitions generated by them. Part II of the text deals with a new approach that generates a different set of integer partitions. Its definition is based on a path through the $\mathbb{Z}^2$ lattice, connecting the line $x + y = n$ to the origin, which is determined by the two-line matrix representation for different sets of partitions of n. The new partitions have only distinct odd parts with some particular restrictions. This process of getting new partitions, which has been called the *Path Procedure*, is applied to unrestricted partitions, to partitions counted by 1^{st} and 2^{nd} Rogers-Ramanujan Identities, and to those generated by mock theta functions $f_*^5(q)$ and $T_1(-q)$.

Keywords: Integer Partitions, Mock Theta Functions, Partition Identities, Matrix Representation

Resumo

O presente trabalho dedica-se ao estudo de algumas consequências da representação matricial para conjuntos de partições de inteiros e funções mock theta. Na primeira parte do texto, classificamos as partições geradas por seis diferentes funções mock theta, de acordo com a soma das entradas da segunda linha das matrizes associadas, e apresentamos algumas fórmulas fechadas e identidades para essas partições. Definimos também a família $\{f_*^m(q)\}_{m\geq 1}$ de funções mock theta, inspiradas pelo que chamamos de versão sem sinal da função $f_1(q)$. Fornecemos uma representação matricial análoga para as funções $f_*^m(q)$, o que leva a resultados interessantes a respeito das partições geradas por elas. A parte II do texto trata de uma nova abordagem que gera um conjunto diferente de partições de inteiros. A definição desse conjunto baseia-se na construção de um caminho sobre o reticulado $\mathbb{Z}^2$, determinado pela representação matricial para diferentes conjuntos de partições de n, e que liga a reta $x + y = n$ à origem. As novas partições possuem apenas partes ímpares distintas, com algumas restições particulares. Esse processo de construção de novas partições, chamado de *Path Procedure*, é aplicado a partições irrestritas, bem como para partições contadas pelas 1^a e 2^a Identidades de Rogers-Ramanujan e funções mock theta $f_*^5(q)$ e $T_1(-q)$.

Palavras-chave: Partição de inteiros, Funções Mock Theta, Identidades de Partições, Representação Matricial

Contents

Introduction

The Theory of Partitions is a many-sided field whose range and applicability pass through many branches of mathematics. The first important discoveries were made in the eighteenth century by L. Euler, who set the foundations of the theory and proved important theorems, such as the famous Euler Identity, which states that "*the number of partitions of an integer n into distinct parts equals the number of partitions of n into odd parts*".

Many other names can be mentioned as contributors to the initial development of the theory, such as Gauss, Jacobi, Hardy, and Rademacher, but surely the indian name Srinivasa Ramanujan is the one which got the greatest popularity. Historical backgrounds concerning Ramanujan's life and work and its range along time can be found in [AB05], [AB09], [AB12], and [AB13], to name a few classical references (a nice and easy reading is [Ono10] by K. Ono).

Integers Partitions are the object of study of the theory of partitions. The most relevant information to understand the basis of the theory are condensed in books like *Integer Partitions*, by George Andrews and Kimmo Eriksson [AE04], and *The Theory of Partitions*, also by G. Andrews [And98], the last one being a great reference which interrelates both combinatorial and analytic aspects of the theory of partitions.

As an example of the analytic aspects of the theory we have the Mock Theta Functions. They were introduced by Ramanujan shortly before his early death in a letter sent to Hardy, in 1920. At that time, what Ramanujan meant for a mock theta function was not very clear ([AH91]). However, nowadays these functions have been largely explored and many applications, as for example in modular forms, have appeared (see [Duk14], [Zag09], [Zwe01], and [Zwe08]).

Besides modular forms, mock theta functions have an interesting interpretation when seen as generating functions for integer partitions. For example, the partition function $p(n)$ has the well known generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

However, the values of $p(n)$ can also be seen as coefficients of Ramanujan's third order mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

The coefficients of $f(q)$ are related to Dyson's rank of a partition, defined in [Dys44], which motivated a conjecture by G. E. Andrews [And66] on the values of these coefficients.

Proved in 2006 by K. Bringmann and K. Ono [BO06], Andrews' conjecture turned out to be true. Not only the coefficients of Mock Theta function $f(q)$ were determined, but also a new formula for $p(n)$ became known. Latterly the formula for $p(n)$ was greatly improved by J. H. Bruinier and K. Ono [BO13], showing it can be written as a finite sum of algebraic numbers.

In a work of 2013 [BSS13], E. Brietzke et al. presented a combinatorial interpretation as two-line matrices for many Mock Theta functions, and consequently for many different types of integer partitions. A few years before, in [SMR11], three distinct matrix representations for unrestricted partitions were given, one of them completely describing the conjugated partition. The bijective proofs between the set of partitions and the set of two-line matrices can be found in [SMR11] and [BSS10].

Motivated by these ideas, this thesis is dedicated to study some consequences and behaviors of the two-line matrix representations of a variety of integer partitions sets and mock theta functions. The text is separated into two independent and self-contained parts. Standard notations and definitions used in both parts compose Chapter 1, the Preliminaries chapter.

Part I is inspired by the matrix representations for what we have called the unsigned version of some mock theta functions, in particular function $f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n}$, that is,

$$f_1^*(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}.$$

Its matrix representation is described in [BSS13], in the table on page 240. We enunciate it as the following theorem.

Theorem 0.0.1 ([BSS13], page 241)**.** *The coefficient of q^n in the expansion of $f_1^*(q)$ is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{1}$$

with non-negative integer entries satisfying $c_s = 2$, $c_t = 2 + c_{t+1} + d_{t+1}$, $\forall t < s$, and $n = \sum c_t + \sum d_t$.

When seen as generating function for integer partitions, $f_1^*(q)$ counts the partitions of n containing all parts from 1 to some s, with no gaps, and multiplicity at least two. This means that the number of partitions of n counted by $f_1^*(q)$ equals the number of matrices of type (1) described in Theorem 0.0.1.

Partition identities and closed formulas concerning $f_1^*(q)$ and some other mock theta functions can be found in Chapter 2, dedicated to a work done by the author in partnership with A. Bagatini and A. Wagner [BMW17].

In Chapter 3 we define a collection of Mock Theta functions inspired by the definition of $f_1^*(q)$. We call these functions $f_*^m(q)$, with $m \geq 1$, and write them as

$$f_*^m(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}. \tag{2}$$

For a fixed $m \geq 1$, the general term

$$\frac{q^{m(1+2+3+\cdots+s)}}{(1-q)(1-q^2)\cdots(1-q^s)}$$

generates the partitions of n containing at least m parts equal to each one of the numbers $1, 2, 3, \ldots, s$, with no gaps. By conjugation, this general term also generates the partitions of n into exactly s parts, with smallest part $\lambda_s \geq m$ and with difference between consecutive parts $\lambda_t - \lambda_{t+1} \geq m$.

Remark 0.0.2. *$f_*^m(q)$ with $m = 2$ is the mock theta function $f_1^*(q)$. In the present work we deal with general aspects of function $f_*^m(q)$, for any $m \geq 1$. For more specific details about $f_*^2(q) = f_1^*(q)$ see [BMW17].*

In Section 3.2 we present the matrix representation for integer partitions counted by $f_*^m(q)$ and the next sections are dedicated to a collection of results derived from this representation, concerning the integer partitions given by the generating functions $f_*^m(q)$. The particular case of $f_*^m(q)$ with $m = 1$ is treated separately from the others in Section 3.7, since its behavior is a little bit peculiar.

Part II of this text is composed of Chapter 4, in which we define a new way of looking to the two-line matrix representation introduced in [SMR11]. Similar meaning has already been given by M. Alegri et al. [ABSS11] for matrices representing plane partitions.

Since every matrix described in [SMR11] has its entries summing n, we consider the entries of each matrix as a guidance for a lattice path in $\mathbb{Z}^2$ from de line $x + y = n$ to the origin $(0, 0)$. In each matrix like (1), the entries c_i determine the moves along the y direction, and the entries d_i determine the moves along the x direction. Starting at the point $P = (\sum_{i=1}^{s} d_i, \sum_{i=1}^{s} c_i)$, the path consists of shifting c_i units down followed by d_i units to the left, for every i from s to 1.

Each path defined above is then associated to a partition of another integer, and these partitions are the object of study of Part II. To describe in details the construction of these new partitions, through what we have called the *Path Procedure*, escapes the purpose of this introduction. The complete description is given in Section 4.2.

This procedure of getting partitions from a two-line matrix can be applied to any set of integer partitions or mock theta function whose two-line matrix representation is known. In fact, Chapter 4 shows the consequences of this procedure when applied not only to unrestricted integer partitions, but also to the ones counted by the Rogers-Ramanujan Identities and the mock theta functions $f_5^*(q)$, from Chapter 3, and $T_1(-q)$.

Finally, the appendices gather some auxiliary information for this thesis. Important tables for consultation during the reading of Chapter 3 can be found in Appendix A, while some extra information concerning the Path Procedure and its partitions into distinct odd parts can be found in Appendix B, where an alternative and equivalent way of looking to these partitions is defined.

CHAPTER 1

Preliminaries

1.1 Introduction

In this chapter we rapidly present some standard notations and definitions from the Theory of Partitions, and a few more we adopt throughout this text. When necessary, any change along the forthcoming chapters will be informed before being used.

It is worth highlighting that the theory of partitions deals with non-negative integers. Therefore, except for the q in generating functions, the variables that will appear in the following chapters are always integers, and there is no meaning in supposing they could be any other real number. Thus we choose to omit specifications like $\mathbb{Z}$ or $\mathbb{N}$ for them; we only specify the range of the variables when they cannot be any non-negative integer.

1.2 General aspects of the Theory of Partitions

A partition of an integer n is a non-ordered finite sequence of positive integers whose sum is n. Formally speaking, the standard notation for a partition is given in the following definition.

Definition 1.2.1. Given $n \in \mathbb{Z}_+$, a *partition* of n is a list $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, with $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq s-1$, such that $\sum_{i=1}^{s} \lambda_i = n$. Also, each λ_i is called a *part* of the partition.

Although λ usually follows a nonincreasing order, there is no loss in changing the order of its parts. Along this work the order of the parts will not be necessarily preserved for purposes of simplification, specially in proofs, and we may write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ both with the same meaning.

Definition 1.2.2. Let $P(n)$ denote the set of partitions of an integer n. We write $p(n)$ to denote the number of elements of $P(n)$, that is, the number of partitions of n. Then $|P(n)| = p(n)$.

Definition 1.2.3. $p(n)$ is called the *partition function.*

Example 1.2.4. $P(4) = \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}$ *and* $p(4) = 5$.

Remark 1.2.5. *Clearly,* $P(0) = \{\emptyset\}$*. By convention, we adopt* $p(0) = 1$*, and we may interpret the only partition of* 0 *as the empty partition, which is not included in* $P(n)$ *for* $n > 0$*.*

We may also be interested in partitions of n which satisfy special conditions, that is, a subset of $P(n)$.

Definition 1.2.6. Let $P_A^B(n, C)$ denote the set of partitions of n subject to conditions A, B, and C. In particular, $P_d(n)$ denotes the set of partitions of n into distinct parts. Also, $|P_A^B(n, C)| = p_A^B(n, C)$, the number of partitions of n subject to conditions A, B, and C.

Example 1.2.7. *In Chapter 3 we define* $P_{[s]}^{m[s]}(n, k)$ *as the set of partitions of* n *into parts ranging from* 1 *to* s*, with no gaps and multiplicity* m*, and* k *other parts from* 1 *to* s *(see Definition 3.2.2 for more details).*

Remark 1.2.8. *As there is no restriction in the parts of the partitions counted by* $p(n)$*, these partitions are also called* unrestricted partitions *of* n*.*

Partitions can be represented graphically by what is called the Ferrers graph.

Definition 1.2.9. The *Ferrers graph* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n is a list of n dots disposed in s rows, with the i^{th} row having λ_i dots, for $1 \leq i \leq s$.

Example 1.2.10. *The Ferrers graph of the partition* $(7,4,2,1)$ *of* 14 *is*

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & & & \\
\bullet & & & & & &
\end{array}$$

An usual operation with partitions becomes clearer when explained using its Ferrers graph.

Definition 1.2.11. Given the Ferrers graph of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n, the *conjugation* of λ consists of changing its rows by its columns. The resulting partition is called the *conjugated partition of* λ and is denoted by $\overline{\lambda}$.

Example 1.2.12. *The conjugation of* $(7,4,2,1)$ *gives the Ferrers graph*

$$\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & & \\
\bullet & \bullet & & \\
\bullet & & & \\
\bullet & & & \\
\bullet & & &
\end{array}$$

which corresponds to the partition $\overline{\lambda} = (4,3,2,2,1,1,1)$.

Definition 1.2.13. The *Durfee square* of a partition of n is the largest square that fits inside the Ferrers graph of the partition. It is always located in the upper left-hand corner of the graph.

A partition whose Durfee square has side k has its k^{th} part greater than or equal to k while the $(k+1)^{th}$ part is less than or equal to k.

Example 1.2.14. *The Durfee square of the partition* $(7,4,2,1)$ *has side of size* 2.

$$\begin{array}{ccccccc}
\circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \bullet & \bullet & & & \\
\bullet & \bullet & & & & & \\
\bullet & & & & & &
\end{array}$$

An important and very useful tool of the theory of partitions are the so called *generating functions*.

Definition 1.2.15. Given a sequence $(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, a_3 \ldots)$, the *generating function* $A(q)$ for $(a_n)_{n=0}^{\infty}$ is the power series

$$A(q) = \sum_{n=0}^{\infty} a_n q^n.$$

When considering $(a_n)_{n=0}^{\infty}$ as some sequence $(p_A^B(n, C))_{n=0}^{\infty}$ we may have a power series whose coefficient of q^n equals the number of partitions of n subject to conditions A, B, and C. These generating functions become very useful when they can be expressed as a product.

Example 1.2.16. *The unrestricted integer partitions have the well-known generating function*

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1.1}$$

It is easy to be convinced of this fact if we note that each factor in the infinite product may be rewritten as

$$\frac{1}{(1-q^n)} = 1 + q^n + q^{2n} + q^{3n} + q^{4n} + \cdots ,$$

where the exponent of each term q^{kn} gives the contribution of k parts equal to n to some partition λ.

Example 1.2.17. *Another noted generating function is the one for integer partitions into distinct parts. Each part can contribute at most once in any partition. So,*

$$\sum_{n=0}^{\infty} p_d(n)q^n = \prod_{n=1}^{\infty} (1+q^n). \tag{1.2}$$

Infinite products like the ones that appear in (1.1) and (1.2) may be written in a simpler way if the following definition is adopted.

Definition 1.2.18. Given $a, q \neq 0$, we define

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k) \tag{1.3}$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{n\geq 0}(1-aq^n), \quad |q| < 1. \tag{1.4}$$

The generating functions (1.1) and (1.2) can be rewritten respectively as

$$\sum_{n=0}^{\infty} p(n)q^n = (q;q)_\infty^{-1}$$

and

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty.$$

Remark 1.2.19. *All of the infinite series and products above make sense once their convergence can be proved. We do not exhibit the arguments here, but a precise and concise explanation can be found in Appendix A of* [AE04].

1.3 Additional notations

Finally, the following definitions are not exclusive from the theory of partitions, but are used in many moments along this text.

Definition 1.3.1. $[n]$ denotes the set of positive integers less then or equal to n. That is, $[n] = \{1, 2, 3, \ldots, n\}$.

Definition 1.3.2. Given $\frac{a}{b} \in \mathbb{Q}$, $\left\lfloor \frac{a}{b} \right\rfloor$ denotes the integer part of $\frac{a}{b}$. That is, for $k \in \mathbb{Z}$,

$$k \leq \frac{a}{b} < k+1 \Longrightarrow \left\lfloor \frac{a}{b} \right\rfloor = k.$$

Definition 1.3.3. Given $\frac{a}{b} \in \mathbb{Q}$, $\left\{ \frac{a}{b} \right\}$ denotes the nearest integer to $\frac{a}{b}$. That is, for $k \in \mathbb{Z}$,

$$k \leq \frac{a}{b} \leq k + \frac{1}{2} \Longrightarrow \left\{ \frac{a}{b} \right\} = \left\lfloor \frac{a}{b} \right\rfloor = k$$

and

$$k + \frac{1}{2} < \frac{a}{b} \leq k+1 \Longrightarrow \left\{ \frac{a}{b} \right\} = \left\lfloor \frac{a}{b} \right\rfloor + 1 = k+1.$$

Example 1.3.4. *In* [AE04]*, Chapter 6, the symbols* $\left\lfloor \frac{a}{b} \right\rfloor$ *and* $\left\{ \frac{a}{b} \right\}$ *appear when calculating the number of partitions of n with each part less than or equal to 2 and 3, respectively. More precisely,*

$$p(n, \textit{each part} \leq 2) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \qquad ([\text{AE04}], \text{p.}56) \tag{1.5}$$

and

$$p(n, \textit{each part} \leq 3) = \left\{ \frac{(n+3)^2}{12} \right\} \qquad ([\text{AE04}], \text{p.}58). \tag{1.6}$$

Part I

CHAPTER 2

The mock theta function $f_1^*(q)$

2.1 Introduction

This chapter is dedicated to an initial work done in partnership with Alessandro Bagatini and Adriana Wagner, based on the matrix representations for some mock theta functions, given by Brietzke et al. [BSS13]. These representations originated a lot of partition identities, some of them registered in the paper below, published in the Bulletin of the Brazilian Mathematical Society, New Series (see [BMW17]).

The paper deals with six different mock theta functions, but it is important to highlight Section 4, which discusses specifically the case of function $f_1(q)$ that motivated the work done in Chapter 3 of this thesis.

2.2 Identities for Partitions Generated by the Unsigned Versions of Some Mock Theta Functions

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Identities for Partitions Generated by the Unsigned Versions of Some Mock Theta Functions

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Abstract Considering the unsigned version of the mock theta functions $\phi(q)$, $\psi(q)$, $f_0(q)$, $F_0(q)$, $f_1(q)$ and $F_1(q)$, they represent partitions subject to some rules. Based on a two-line matrix representation, we can classify them according to the sum of second line and derive some results such as closed formulas and identities for other types of partitions.

Keywords Mock Theta functions · Theory of partitions · Combinatorial interpretations

Mathematics Subject Classification Primary 11P81; Secondary 05A19

1 Introduction

A new combinatorial way to look at partitions of integers was introduced in Santos et al. (2011). The authors established two identities that relate unrestricted partitions of n to two-line matrices whose elements add up n. In Andrade et al. (2016) (preprint),

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the authors interpreted those matrices and observed that in one of them the elements of the second row show the appearance of each part in the associated partition. In the other one, the entries of the second row give a complete description of the conjugate partition.

Despite the fact it worked for unrestricted partitions, it is possible to adapt the entries of the matrices to count other types of partitions, overpartitions and plane partitions, as done in Alegri et al. (2011). A similar representation was also useful to describe the coefficients of some mock theta functions, first considering their unsigned version and setting a weight to the partitions generated by their general terms. For many mock theta functions, in Brietzke et al. (2013) we find a characterization of those matrices, which helps us to evaluate the coefficients of each function.

In this work we study especially the unsigned versions of six mock theta functions: $\phi(q)$, $\psi(q)$, $f_0(q)$, $F_0(q)$, $f_1(q)$ and $F_1(q)$. Considering their matrix representation and observing the second rows, it allows us to find great information about the parts of the partitions the functions generate. Classifying these matrices according to the sum of their second line and organizing these numbers in a table, we get some partitions identities. Sometimes the table suggests closed formulas to count partitions or identities that relate them to other kind of partitions.

Each section of this paper is related to one of the mock theta functions mentioned before, and all proofs we present are given in a combinatorial way. First of all, we set some notations and definitions to guide the reader throughout the paper.

As usual, we denote a partition of an integer n by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_i \geq \lambda_{i+1}$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. However, in most of our proofs the order of the parts does not matter, and we ignore it. Also, when denoting by $p_A(N, B)$ the number of partitions of N subject to certain conditions A and B, we denote by $P_A(N, B)$ the set of partitions counted by $p_A(N, B)$. So, $|P_A(N, B)| = p_A(N, B)$.

2 Mock Theta Function $\phi(q)$

We start by considering the unsigned version of the mock theta function $\phi(q)$ of order 3,

$$\phi^*(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2; q^2)_n}. \tag{1}$$

Its general term

$$\frac{q^{1+3+5+\cdots+(2s-1)}}{(1-q^2)(1-q^4)\cdots(1-q^{2s})}$$

generates the partitions of n containing each one of the odd numbers $1, 3, 5, \ldots, 2s-1$ as part of multiplicity 1 and any number of even parts less than or equal to $2s$. Also, note that it generates the partitions of n into distinct odd parts, which can be seen by considering the Ferrers graph of a partition as the merging of a triangular graph of parts $2s-1, \ldots, 3, 1$ with pairs of columns of size s, at most. Moreover, by equation (3.4) in Andrews and Eriksson (2004), Eq. (1) is also the generating function for self-conjugate partitions.

In Brietzke et al. (2013), the following combinatorial interpretation for this function in terms of two-line matrices is given.

Theorem 2.1 *The coefficient of q^n in the expansion of* (1) *is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{2}$$

with non-negative integer entries whose sum is n, satisfying

$$\begin{aligned} c_s &= 1; \quad d_t \geq 0; \quad d_t \equiv 0 \pmod 2; \\ c_t &= 2 + c_{t+1} + d_{t+1}, \quad \forall t < s. \end{aligned}$$

Example 2.2 The matrix

$$A = \begin{pmatrix} 15 & 11 & 3 & 1 \\ 4 & 2 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$
$$+3\begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

represents the partition λ containing a part equal to each one of the numbers 7, 5, 3, and 1, plus two parts equal to 2, one part equal to 4, and three parts equal to 6, i.e., $\lambda = (7, 6, 6, 6, 5, 4, 3, 2, 2, 1)$.

The second row of each matrix in Theorem 2.1 describes the even parts of the partition associated to it. In order to know how many even parts the partition has, we have to sum all $\frac{d_i}{2} = e_i$, for $i = 1, 2, \ldots, s$.

Definition 2.3 Let $p_\phi(n, k)$ be the number of partitions of n into distinct odd parts, ranging from 1 to $2s - 1$, with no gaps, and k even parts less than or equal $2s$.

For a fixed n, we classify its partitions of type described in Definition 2.3 according to the sum of the elements of the second row of the matrix associated to them. By counting the appearance of each number in these sums, we organize the data on a table, which is presented next. The entry in line n and column $n - j$ is the number of times j appears as sum on the entries of the second row in type (2) matrices.

As we can see in the table, below certain cell the columns become constant and equal to 0. Those specific cells are always equal to 1, which is proved next.

Proposition 2.4 *For all $n \geq 1$ and $i \geq 0$, we have*

(i) $p_\phi(2n - 1, n - 1) = 1$;
(ii) $p_\phi(2n + i, n + i) = 0$.

Proof For item (i), observe that the partition $(1, \underbrace{2, 2, \ldots, 2}_{n-1\ times})$ of $2n - 1$ is the only one that satisfies the conditions we need. To prove item (ii), suppose we have a partition

Table 1 Table from the characterization given by Theorem 2.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																								
2	0	0																							
3	0	1	0																						
4	0	0	0	1																					
5	0	0	1	0	0																				
6	0	0	0	0	1	0																			
7	0	0	0	1	0	0	0																		
8	0	0	0	0	0	1	1	0																	
9	0	0	0	0	1	0	0	0	1																
10	0	0	0	0	0	0	1	1	0	0															
11	0	0	0	0	0	1	0	0	0	1	0														
12	0	0	0	0	0	0	0	1	1	1	0	0													
13	0	0	0	0	0	0	1	0	0	0	1	1	0												
14	0	0	0	0	0	0	0	0	1	1	1	0	0	0											
15	0	0	0	0	0	0	0	1	0	0	0	1	1	1	0										
16	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1									
17	0	0	0	0	0	0	0	0	1	0	0	0	1	1	2	0	0								
18	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	1	0							
19	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	2	1	0	0						
20	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	0					
21	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	2	2	1	0	0				
22	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	1	1	1	0			
23	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	2	2	2	0	0	0		
24	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	2	1	0	
25	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	2	2	3	1	0	0	1

$(1, 3, \ldots, 2s-1, 2l_1, 2l_2, \ldots, 2l_{n+i})$ counted by $p_\phi(2n+i, n+i)$. For any $s \geq 1$,

$$\begin{aligned} &1 + 3 + \cdots + 2s - 1 + 2l_1 + 2l_2 + \cdots + 2l_{n+i} \\ &\quad \geq 1 + 3 + \cdots + 2s - 1 + 2 \cdot (n+i) > 2n + i, \end{aligned}$$

which is a contradiction. □

The next two theorems are also inspired by Table 1, by looking to its third and fourth diagonals, respectively.

Theorem 2.5 *For all $n \geq 1$ we have*:

(i) $p_\phi(4n^2 + 4n, 2) = p_\phi(4n^2 + 4n + 2, 2) = p_\phi(4n^2 + 4n + 4, 2) = n$;
(ii) $p_\phi(4n^2 + 8n + 1, 2) = p_\phi(4n^2 + 8n + 3, 2) = n$;
(iii) $p_\phi(4n^2 + 1, 2) = n$.

Proof We prove item (i), being (ii) and (iii) analogous. First of all, note that any partition of $4n^2 + 4n$, $4n^2 + 4n + 2$ or $4n^2 + 4n + 4$ with two even parts must have $4n - 1$ as the largest odd part. So, let $(1, 3, 5, \ldots, 4n-1, 2i, 2j)$ be a partition of $4n^2 + 4n$, with $1 \leq i \leq j \leq 2n$. To get a partition of $4n^2 + 4n + 2$ and a partition of $4n^2 + 4n + 4$ we take, respectively,

$$(1, 3, 5, \ldots, 4n-1, 2i, 2j+2) \quad \text{and} \quad (1, 3, 5, \ldots, 4n-1, 2i+2, 2j+2).$$

Conversely, given $(1, 3, \ldots, 4n-1, 2l, 2t)$ a partition of $4n^2 + 4n + 4$, we have $l + t = 2n + 2$ with $1 \leq l \leq t \leq 2n$. So, both l and t have to be greater than or equal to 2 and the smallest even part of any partition of $4n^2 + 4n + 4$ has to be 4. Hence, the map is invertible.

Finally, the number $p_\phi(4n^2+4n+4, 2)$ is the number of solutions of $l+t = 2n+2$, with $l, t \geq 2$, which is $\lfloor \frac{2n+2}{2} \rfloor - 1 = n$. □

Theorem 2.6 *For all $n \geq 1$*:

(i) $p_\phi(n^2, 3) = p_\phi(n^2 - 1, 3) = p(n-1, 3)$;
(ii) $p_\phi(n^2 + 2n + 2, 3) = p_\phi(n^2 + 2n - 1, 3) = p(n+1, 3)$;
(iii) $p_\phi(n^2 + 2n + 3, 3) = p_\phi(n^2 + 2n - 2, 3) = p(n-1, 3)$;
(iv) $p_\phi(n^2 + 4n + 8, 3) = p_\phi(n^2 + 4n - 1, 3) = p(n-1, 3)$.

Proof We prove only the first item, the others being easily adaptable.

Note that the largest odd part of any partition of n^2 and $n^2 - 1$ must be, respectively, $2n - 5$ and $2n - 3$. So, let $(1, 3, 5, \ldots, 2n-5, 2i, 2j, 2k)$ be a partition of n^2. To get a partition of $n^2 - 1$, we map every odd part $2s - 1$ into the part $2(n-s) - 1$ and every even part $2s$ into $2(n-s-1)$. After that, add one part of size 1. Indeed, as $2i + 2j + 2k = 4n - 4$, we get

$$\begin{aligned} &2(n-1) - 1 + \cdots + 2 \cdot 2 - 1 + 2(n-i-1) \\ &\quad +2(n-j-1) + 2(n-k-1) + 1 = n^2 - 1. \end{aligned}$$

For the second equality, from any partition of $n^2 - 1$ we remove the odd parts and divide by 2 every even part. Observe that this is possible, once $2(n-i-1)+2(n-j-1)+2(n-k-1) = 2n-2$. □

Example 2.7 Considering $n = 10$ in identity (i) of Theorem 2.6, we get the partitions below:

$P_\phi(100, 3)$	$P_\phi(99, 3)$	$P(9, 3)$
(**16**, **16**, 15, 13, 11, 9, 7, 5, **4**, 3, 1)	(17, 15, **14**, 13, 11, 9, 7, 5, 3, **2**, **2**, 1)	(7, 1, 1)
(**16**, 15, **14**, 13, 11, 9, 7, **6**, 5, 3, 1)	(17, 15, 13, **12**, 11, 9, 7, 5, **4**, 3, **2**, 1)	(6, 2, 1)
(**16**, 15, 13, **12**, 11, 9, **8**, 7, 5, 3, 1)	(17, 15, 13, 11, **10**, 9, 7, **6**, 5, 3, **2**, 1)	(5, 3, 1)
(**16**, 15, 13, 11, **10**, **10**, 9, 7, 5, 3, 1)	(17, 15, 13, 11, 9, **8**, **8**, 7, 5, 3, **2**, 1)	(4, 4, 1)
(15, **14**, **14**, 13, 11, 9, **8**, 7, 5, 3, 1)	(17, 15, 13, 11, **10**, 9, 7, 5, **4**, **4**, 3, 1)	(5, 2, 2)
(15, **14**, 13, **12**, 11, **10**, 9, 7, 5, 3, 1)	(17, 15, 13, 11, 9, **8**, 7, **6**, 5, **4**, 3, 1)	(4, 3, 2)
(15, 13, **12**, **12**, **12**, 11, 9, 7, 5, 3, 1)	(17, 15, 13, 11, 9, 7, **6**, **6**, **6**, 5, 3, 1)	(3, 3, 3)

In order to prove an identity brought forth by the fifth diagonal of Table 1, we present the next Lemma.

Lemma 2.8 *Let $p_e(4n-8, 4, \lambda_2 = \lambda_3)$ be the number of partitions of $4n-8$ into four even parts smaller than or equal to $2n-4$ and such that $\lambda_2 = \lambda_3$. We have $p_e(4n-8, 4, \lambda_2 = \lambda_3) = p(n-1, 3)$.*

Proof We separate the proof according to the parity of n.

If n is even, we set a bijection that takes the equal parts λ_2 and λ_3 from a partition counted by $p_e(4n-8, 4, \lambda_2 = \lambda_3)$ into the smallest part of a partition counted by $p(n-1, 3)$.

- If $\lambda_2 = \lambda_3 = n + 2k - 2$, for $k \geq 0$, take
$$(\lambda_1, n+2k-2, n+2k-2, \lambda_4) \quad \text{into} \quad \left(\frac{\lambda_1 - 4k}{2}, \frac{\lambda_4 + 4k}{2}, 2k+1\right).$$
- If $\lambda_2 = \lambda_3 = n - 2k$, for $k \geq 2$, take
$$(\lambda_1, n-2k, n-2k, \lambda_4) \quad \text{into} \quad \left(\frac{\lambda_1 - 8k + 10}{2}, \frac{\lambda_4}{2}, 2k-2\right).$$
Note that conditions $\lambda_4 \leq n - 2k \leq \lambda_1 \leq 2n-4$ and $\lambda_1 + \lambda_4 = 2n - 8 + 4k$ lead us to $\frac{\lambda_4}{2} \geq 2k - 2$.

For odd n, the bijection is similar to the previous one.

- If $\lambda_2 = \lambda_3 = n + 2k - 3$, for $k \geq 1$, take
$$(\lambda_1, n+2k-3, n+2k-3, \lambda_4) \quad \text{into} \quad \left(\frac{\lambda_1 - 4k + 2}{2}, \frac{\lambda_4 + 4k - 2}{2}, 2k+2\right).$$
- If $\lambda_2 = \lambda_3 = n - 2k - 1$, for $k \geq 1$, take
$$(\lambda_1, n-2k-1, n-2k-1, \lambda_4) \quad \text{into} \quad \left(\frac{\lambda_1 - 8k + 6}{2}, \frac{\lambda_4}{2}, 2k-1\right).$$

Note that conditions $\lambda_4 \le n-2k-1 \le \lambda_1 \le 2n-4$ and $\lambda_1+\lambda_4 = 2n-6+4k$ lead us to $\frac{\lambda_4}{2} \ge 2k-1$. $\square$

Example 2.9 We illustrate the bijection above for $n = 12$ and $n = 13$.

$P_e(40, 4, \lambda_2 = \lambda_3)$		$P(11, 3)$
(18, 10, 10, 2)	$\left(\frac{18-4\cdot 0}{2}, \frac{2+4\cdot 0}{2}, 1\right)$	(9, 1, 1)
(16, 10, 10, 4)	$\left(\frac{16-4\cdot 0}{2}, \frac{4+4\cdot 0}{2}, 1\right)$	(8, 2, 1)
(14, 10, 10, 6)	$\left(\frac{14-4\cdot 0}{2}, \frac{6+4\cdot 0}{2}, 1\right)$	(7, 3, 1)
(12, 10, 10, 8)	$\left(\frac{12-4\cdot 0}{2}, \frac{8+4\cdot 0}{2}, 1\right)$	(6, 4, 1)
(10, 10, 10, 10)	$\left(\frac{10-4\cdot 0}{2}, \frac{10+4\cdot 0}{2}, 1\right)$	(5, 5, 1)
(20, 8, 8, 4)	$\left(\frac{20-8\cdot 2+10}{2}, \frac{4}{2}, 2\right)$	(7, 2, 2)
(18, 8, 8, 6)	$\left(\frac{18-8\cdot 2+10}{2}, \frac{6}{2}, 2\right)$	(6, 3, 2)
(16, 8, 8, 8)	$\left(\frac{16-8\cdot 2+10}{2}, \frac{8}{2}, 2\right)$	(5, 4, 2)
(14, 12, 12, 2)	$\left(\frac{14-4\cdot 1}{2}, \frac{2+4\cdot 1}{2}, 3\right)$	(5, 3, 3)
(12, 12, 12, 4)	$\left(\frac{12-4\cdot 1}{2}, \frac{4+4\cdot 1}{2}, 3\right)$	(4, 4, 3)

$P_e(44, 4, \lambda_2 = \lambda_3)$		$P(12, 3)$
(22, 10, 10, 2)	$\left(\frac{22-8\cdot 1+6}{2}, \frac{2}{2}, 1\right)$	(10, 1, 1)
(20, 10, 10, 4)	$\left(\frac{20-8\cdot 1+6}{2}, \frac{4}{2}, 1\right)$	(9, 2, 1)
(18, 10, 10, 6)	$\left(\frac{18-8\cdot 1+6}{2}, \frac{6}{2}, 1\right)$	(8, 3, 1)
(16, 10, 10, 8)	$\left(\frac{16-8\cdot 1+6}{2}, \frac{8}{2}, 1\right)$	(7, 4, 1)
(14, 10, 10, 10)	$\left(\frac{14-8\cdot 1+6}{2}, \frac{10}{2}, 1\right)$	(6, 5, 1)
(18, 12, 12, 2)	$\left(\frac{18-4\cdot 1+2}{2}, \frac{2+4\cdot 1-2}{2}, 2\right)$	(8, 2, 2)
(16, 12, 12, 4)	$\left(\frac{16-4\cdot 1+2}{2}, \frac{4+4\cdot 1-2}{2}, 2\right)$	(7, 3, 2)
(14, 12, 12, 6)	$\left(\frac{14-4\cdot 1+2}{2}, \frac{6+4\cdot 1-2}{2}, 2\right)$	(6, 4, 2)
(12, 12, 12, 8)	$\left(\frac{12-4\cdot 1+2}{2}, \frac{8+4\cdot 1-2}{2}, 2\right)$	(5, 4, 2)
(22, 8, 8, 6)	$\left(\frac{22-8\cdot 2+6}{2}, \frac{6}{2}, 3\right)$	(6, 3, 3)
(20, 8, 8, 8)	$\left(\frac{20-8\cdot 2+6}{2}, \frac{8}{2}, 3\right)$	(5, 4, 3)
(14, 14, 14, 2)	$\left(\frac{14-4\cdot 2+2}{2}, \frac{2+4\cdot 2-2}{2}, 4\right)$	(4, 4, 4)

Theorem 2.10 *For all $n \ge 1$, we have*

$$p_\phi(n^2-4, 4) = \sum_{i=1}^{n} p(i-1, 3). \tag{3}$$

Proof By proving that

$$p_\phi(n^2-4,4)-p_\phi((n-1)^2-4,4)=p(n-1,3),$$

equality (3) follows by induction.

So, given a partition counted by $p_\phi(n^2-4,4)$, its largest odd part is $2n-5$. Then, the four even parts must be less than or equal to $2n-4$ and satisfy

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=4n-8.$$

There is a natural bijection between the partitions counted by $p_\phi((n-1)^2-4,4)$ and those counted by $p_\phi(n^2-4,4)$ whose even parts satisfy $\lambda_2>\lambda_3$, that is, we add a part $2n-5$ and increase by 2 the two largest even parts. The remaining partitions are those in which the central parts are equal. By Lemma 2.8, there are $p(n-1,3)$ of them. □

Example 2.11 The bijection between partitions of $(n-1)^2-4$ and n^2-4 described above is illustrated in the next table for $n=8$.

$P_\phi(60,4)$	$P_\phi(45,4)$
$(\mathbf{12},11,9,\mathbf{8},7,5,3,\mathbf{2},\mathbf{2},1)$	$(\mathbf{10},9,7,\mathbf{6},5,3,\mathbf{2},\mathbf{2},1)$
$(\mathbf{12},11,9,7,\mathbf{6},5,\mathbf{4},3,\mathbf{2},1)$	$(\mathbf{10},9,7,5,\mathbf{4},\mathbf{4},3,\mathbf{2},1)$
$(11,\mathbf{10},\mathbf{10},9,7,5,3,\mathbf{2},\mathbf{2},1)$	$(9,\mathbf{8},\mathbf{8},7,5,3,\mathbf{2},\mathbf{2},1)$
$(11,\mathbf{10},9,\mathbf{8},7,5,\mathbf{4},3,\mathbf{2},1)$	$(9,\mathbf{8},7,\mathbf{6},5,\mathbf{4},3,\mathbf{2},1)$
$(11,\mathbf{10},9,7,\mathbf{6},5,\mathbf{4},\mathbf{4},3,1)$	$(9,\mathbf{8},7,5,\mathbf{4},\mathbf{4},\mathbf{4},3,1)$
$(11,9,\mathbf{8},\mathbf{8},7,\mathbf{6},5,3,\mathbf{2},1)$	$(9,7,\mathbf{6},\mathbf{6},\mathbf{6},5,3,\mathbf{2},1)$
$(11,9,\mathbf{8},\mathbf{8},7,5,\mathbf{4},\mathbf{4},3,1)$	$(9,7,\mathbf{6},\mathbf{6},5,\mathbf{4},\mathbf{4},3,1)$
$(\mathbf{12},11,9,7,5,\mathbf{4},\mathbf{4},\mathbf{4},3,1)$	
$(11,\mathbf{10},9,7,\mathbf{6},\mathbf{6},5,3,\mathbf{2},1)$	
$(11,9,\mathbf{8},7,\mathbf{6},\mathbf{6},5,\mathbf{4},3,1)$	
$(11,9,7,\mathbf{6},\mathbf{6},\mathbf{6},\mathbf{6},5,3,1)$	

3 Mock Theta Function $\psi(q)$

Consider the unsigned version of the mock theta function $\psi(q)$ of order 3,

$$\psi^*(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_n}. \tag{4}$$

Its general term

$$\frac{q^{1+3+5+\cdots+(2s-1)}}{(1-q)(1-q^3)\cdots(1-q^{2s-1})}$$

generates the partitions of n containing each one of the odd numbers $1,3,5,\ldots,2s-1$ as part of multiplicity at least 1, i.e., partitions of n into odd parts with no gaps.

The following combinatorial interpretation for this function is presented in Brietzke et al. (2013).

Theorem 3.1 *The coefficient of q^n in the expansion of* (4) *is equal to the number of elements in the set of matrices of the form* (2) *with non-negative integer entries whose sum is n, satisfying*

$$\begin{aligned} c_s &= 1; \quad d_t \geq 0; \\ c_t &= 2 + c_{t+1} + 2d_{t+1}, \quad \forall t < s. \end{aligned}$$

Example 3.2 The matrix

$$A = \begin{pmatrix} 13 & 11 & 5 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

represents the partition λ containing one part equal to each one of 1, 3, 5, and 7 plus one additional copy of 7, two copies of 5, and one copy of 1, i.e., $\lambda = (7, 7, 5, 5, 5, 3, 1, 1)$.

Given a partition as described above, the second row of the matrix associated to it describes how many parts there are besides the minimum number of parts s. In order to have this number, we just add all d_i, for $i = 1, 2, \ldots, s$.

Definition 3.3 Let the *excess of a part* λ_i related to a partition λ be the number of times the part λ_i appears more than once. Notation: $x(\lambda_i)$. We call the *excess of a partition* λ the sum of the excesses of all parts. Notation: $x(\lambda) = \sum_i x(\lambda_i)$.

Example 3.4 Consider

$$\lambda = (15, 13, 11, 9, \mathbf{9}, 7, 5, 3, 1, \mathbf{1}, \mathbf{1}),$$

a partition of 75 into odd parts, with no gaps. We have $x(1) = 2$, $x(9) = 1$, $x(3) = x(5) = x(7) = x(11) = x(13) = x(15) = 0$ and $x(\lambda) = 3$.

Definition 3.5 Let $p_\psi(n, k)$ be the number of partitions of n into odd parts, ranging from 1 to $2s - 1$, with no gaps, and excess k.

We build a table (Table 2) according to Theorem 3.1 in the same way we did for $\phi^*(q)$.

By observing the table, it is possible to see that from certain cells on the columns become constant. This turns into a result described as follows.

Theorem 3.6 *For all $i \geq 2$ and $n \geq 3\lfloor \frac{i}{2} \rfloor - 2$, we have*

$$p_\psi(n, n - i) = o_{d,1}(i),$$

where $o_{d,1}(i)$ is the number of partitions of i into distinct odd parts, having 1 as a part.

Table 2 Table from the characterization given by Theorem 3.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																								
2	1	0																							
3	1	0	0																						
4	1	0	0	1																					
5	1	0	0	1	0																				
6	1	0	0	1	0	0																			
7	1	0	0	1	0	1	0																		
8	1	0	0	1	0	1	0	0																	
9	1	0	0	1	0	1	0	0	1																
10	1	0	0	1	0	1	0	1	1	0															
11	1	0	0	1	0	1	0	1	1	0	0														
12	1	0	0	1	0	1	0	1	1	0	1	0													
13	1	0	0	1	0	1	0	1	1	1	1	0	0												
14	1	0	0	1	0	1	0	1	1	1	1	0	1	0											
15	1	0	0	1	0	1	0	1	1	1	1	0	2	0	0										
16	1	0	0	1	0	1	0	1	1	1	1	1	2	0	0	1									
17	1	0	0	1	0	1	0	1	1	1	1	1	2	0	1	1	0								
18	1	0	0	1	0	1	0	1	1	1	1	1	2	0	2	1	0	0							
19	1	0	0	1	0	1	0	1	1	1	1	1	2	1	2	1	1	1	0						
20	1	0	0	1	0	1	0	1	1	1	1	1	2	1	2	1	2	1	0	0					
21	1	0	0	1	0	1	0	1	1	1	1	1	2	1	2	1	3	1	0	1	0				
22	1	0	0	1	0	1	0	1	1	1	1	1	2	1	2	2	3	1	1	2	0	0			
23	1	0	0	1	0	1	0	1	1	1	1	1	2	1	2	2	3	1	2	2	0	1	0		
24	1	0	0	1	0	1	0	1	1	1	1	1	2	1	2	2	3	1	3	2	1	2	0	0	
25	1	0	0	1	0	1	0	1	1	1	1	1	2	1	2	2	3	2	3	2	2	3	0	0	1

Proof Let us consider a partition counted by $p_\psi(n, n-i)$. Then, a possible way to write n is

$$n = 1 + 1 \cdot x(1) + 3 + 3 \cdot x(3) + \cdots + (2s-1) + (2s-1) \cdot x(2s-1),$$

where $x(1) + x(3) + \cdots + x(2s-1) = n - i$. As this partition has s different parts, we set a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ counted by $o_{d,1}(i)$, also into s parts, by the following rule.

$$\begin{aligned}
\lambda_s &= 1\\
\lambda_{s-1} &= 3 + 2 \cdot x(2s-1)\\
\lambda_{s-2} &= 5 + 2 \cdot x(2s-1) + 2 \cdot x(2(s-1)-1)\\
&\vdots\\
\lambda_k &= (2(s-k+1)-1) + 2 \cdot x(2s-1) + \cdots + 2 \cdot x(2(k+1)-1)\\
&\vdots\\
\lambda_1 &= (2s-1) + 2 \cdot x(2s-1) + \cdots + 2 \cdot x(3).
\end{aligned}$$

□

Example 3.7 For $n = 22$ and $i = 17$, we have

$P_\psi(22, 5)$		$O_{d,1}(17)$
(5, 5, 5, 3, 1, 1, 1, 1)	$x(5) = 2, x(3) = 0, x(1) = 3$	(9, 7, 1)
(5, 5, 3, 3, 3, 1, 1, 1)	$x(5) = 1, x(3) = 2, x(1) = 2$	(11, 5, 1)
(5, 3, 3, 3, 3, 3, 1, 1)	$x(5) = 0, x(3) = 4, x(1) = 1$	(13, 3, 1)

The following theorem is related to partitions with 3 excesses. We prove the first statement only, the others being similar.

Theorem 3.8 *For all $n \geq 3$ we have*:

(i) $p_\psi(n^2-5, 3) = p_\psi(n^2-2, 3) = p(n-3, \leq 3)$;
(ii) $p_\psi(n^2+2n-2, 3) = p_\psi(n^2+2n-3, 3) = p(n-3, \leq 3)$;
(iii) $p_\psi(n^2+4n+3, 3) = p_\psi(n^2+4n-2, 3) = p(n-3, \leq 3)$;
(iv) $p_\psi(n^2+6n+10, 3) = p_\psi(n^2+6n+1, 3) = p(n-3, \leq 3)$.

Proof Given a partition counted by $p_\psi(n^2-5, 3)$, its largest part is $2n-5$. So,

$$n^2 - 5 = (n-2)^2 + \sum_{\substack{i=1\\ i\ odd}}^{2n-5} x(i) \cdot i \Rightarrow 4n - 9 = \sum_{\substack{i=1\\ i\ odd}}^{2n-5} x(i) \cdot i.$$

Consider the sequence of all excesses of the partition

$$(x(1), x(3), \ldots, x(k), \ldots, x(2n-5)).$$

We invert this sequence by changing k by $2n-k-4$. Now, consider the partition

$$(2n-3)+(2n-5)(1+x(1))+\cdots+k\cdot(1+x(2n-k-4)) \\ +\cdots+1\cdot(1+x(2n-5)).$$

We need to prove that this sums n^2-2 and belongs to the set of partitions counted by $p_\psi(n^2-2,3)$. Once $\sum x(i)=3$, the number of non-zero excess remains the same. And indeed, by rearranging the terms, the sum of the parts of the new partition is

$$= (2n-3)+\sum_{\substack{i=1\\ i\ odd}}^{2n-5}(1+x(i))\cdot(2n-i-4)$$

$$= (n-1)^2+(2n-4)\cdot\underbrace{\sum_{\substack{i=1\\ i\ odd}}^{2n-5}x(i)}_{3}-\underbrace{\sum_{\substack{i=1\\ i\ odd}}^{2n-5}x(i)\cdot i}_{4n-9} \;=\; n^2-2.$$

In order to prove the second equality, as the largest part of a partition counted by $p_\psi(n^2-2,3)$ is $2n-3$, its three odd excesses $2i-1, 2j-1, 2k-1$ have to sum $2n-3$. Writing $(i-1, j-1, k-1)$ we have a partition of $n-3$ into up to 3 parts. □

Example 3.9 In order to illustrate the bijection described above, consider $n=8$.

$P_\psi(59,3)$	$P_\psi(62,3)$	$P(5,\le 3)$
(11, **11**, **11**, 9, 7, 5, 3, 1, **1**)	(13, 11, **11**, 9, 7, 5, 3, **1**, **1**)	(5, 0, 0)
(11, **11**, 9, **9**, 7, 5, 3, **3**, 1)	(13, 11, 9, **9**, 7, 5, 3, **3**, 1, **1**)	(4, 1, 0)
(11, **11**, 9, 7, **7**, 5, **5**, 3, 1)	(13, 11, 9, 7, **7**, 5, **5**, 3, 1, **1**)	(3, 2, 0)
(11, 9, **9**, **9**, 7, 5, **5**, 3, 1)	(13, 11, 9, 7, **7**, 5, 3, **3**, **3**, 1)	(3, 1, 1)
(11, 9, **9**, 7, **7**, **7**, 5, 3, 1)	(13, 11, 9, 7, 5, **5**, **5**, 3, **3**, 1)	(2, 2, 1)

The following theorem sets a relation between partitions generated by functions $\phi^*(q)$ and $\psi^*(q)$.

Theorem 3.10 *For all $n\ge 2$, we have*

(i) $p_\psi(n^2-3,3)=p_\phi(n^2-1,3)$;
(ii) $p_\psi(n^2-5,3)=p_\phi(n^2-2,3)=p(n,3)$;
(iii) $p_\psi(n^2-1,3)=p_\phi(n^2-3,3)$;
(iv) $p_\psi(n^2-8,3)=p_\phi(n^2-5,3)=p(n-3,3)$.

Proof We prove item (i) by a bijection, valid analogously to the other items.

Let $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ be a partition counted by $p_\psi(n^2-3,3)$ and consider $2s-1$ its largest part. We map each excess $2i-1$ into $2(n-i-1)$ and add one part $2s+1$. □

Example 3.11 Taking $n = 10$, we have:

$P_\psi(97, 3)$	$P_\phi(99, 3)$
(15, **15**, **15**, 13, 11, 9, 7, 5, 3, **3**, 1)	(17, 15, **14**, 13, 11, 9, 7, 5, 3, **2**, **2**, 1)
(15, **15**, 13, **13**, 11, 9, 7, 5, **5**, 3, 1)	(17, 15, 13, **12**, 11, 9, 7, 5, **4**, 3, **2**, 1)
(15, **15**, 13, 11, **11**, 9, 7, **7**, 5, 3, 1)	(17, 15, 13, 11, **10**, 9, 7, **6**, 5, 3, **2**, 1)
(15, **15**, 13, 11, 9, **9**, **9**, 7, 5, 3, 1)	(17, 15, 13, 11, 9, **8**, **8**, 7, 5, 3, **2**, 1)
(15, 13, **13**, **13**, 11, 9, 7, **7**, 5, 3, 1)	(17, 15, 13, 11, **10**, 9, 7, 5, **4**, **4**, 3, 1)
(15, 13, **13**, 11, **11**, 9, **9**, 7, 5, 3, 1)	(17, 15, 13, 11, 9, **8**, 7, **6**, 5, **4**, 3, 1)
(15, 13, 11, **11**, **11**, **11**, 9, 7, 5, 3, 1)	(17, 15, 13, 11, 9, 7, **6**, **6**, **6**, 5, 3, 1)

4 Mock Theta Function $f_1(q)$

Consider the unsigned version of the mock theta function $f_1(q)$ of order 5

$$f_1^*(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}. \tag{5}$$

Its general term

$$\frac{q^{2(1+2+3+\cdots+s)}}{(1-q)(1-q^2)\cdots(1-q^s)},$$

generates the partitions of n containing all parts from 1 to s, with no gaps and multiplicity at least two. By conjugation, this general term also generates the partitions of n into exactly s parts such that the smallest part $\lambda_s \geq 2$ and the difference between consecutive parts is $\lambda_t - \lambda_{t+1} \geq 2$.

The next theorem shows a combinatorial interpretation for function (5), given in Brietzke et al. (2013).

Theorem 4.1 *The coefficient of q^n in the expansion of (5) is equal to the number of elements in the set of matrices of the form (2) with non-negative integer entries whose sum is n, satisfying*

$$\begin{aligned} c_s &= 2; \quad d_t \geq 0; \\ c_t &= 2 + c_{t+1} + d_{t+1}, \quad \forall t < s. \end{aligned}$$

Given a partition as described above, the second row of the matrix associated to it describes how many parts there are besides the two copies of each part from 1 to s that necessarily appear in the partition.

Definition 4.2 Let $p_{f_1}(n, k)$ be the number of partitions of n into parts ranging from 1 to s, with no gaps and multiplicity 2, and k other parts from 1 to s.

As we have done before, we build a table (Table 3) for function $f_1(q)$, which leads us to some similar results.

Table 3 Table from the characterization given by Theorem 4.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	0																								
2	0	1																							
3	0	1	0																						
4	0	1	0	0																					
5	0	1	0	0	0																				
6	0	1	0	0	0	1																			
7	0	1	0	0	0	1	0																		
8	0	1	0	0	0	1	1	0																	
9	0	1	0	0	0	1	1	0	0																
10	0	1	0	0	0	1	1	1	0	0															
11	0	1	0	0	0	1	1	1	0	0	0														
12	0	1	0	0	0	1	1	1	1	0	0	1													
13	0	1	0	0	0	1	1	1	1	0	0	1	0												
14	0	1	0	0	0	1	1	1	1	1	0	1	1	0											
15	0	1	0	0	0	1	1	1	1	1	0	1	1	1	0										
16	0	1	0	0	0	1	1	1	1	1	1	1	1	2	0	0									
17	0	1	0	0	0	1	1	1	1	1	1	1	1	2	1	0	0								
18	0	1	0	0	0	1	1	1	1	1	1	2	1	2	2	1	0	0							
19	0	1	0	0	0	1	1	1	1	1	1	2	1	2	2	2	0	0	0						
20	0	1	0	0	0	1	1	1	1	1	1	2	2	2	2	3	1	0	0	1					
21	0	1	0	0	0	1	1	1	1	1	1	2	2	2	2	3	2	1	0	1	0				
22	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	3	3	2	0	1	1	0			
23	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	3	3	3	1	1	1	1	0		
24	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	3	3	4	2	2	1	2	1	0	
25	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	3	3	4	3	3	1	2	2	0	0

Theorem 4.3 *For all $n \geq 3$ we have*

$$p_{f_1}(n^2-3,3) = p(n-3,3).$$

Proof The largest part of any partition counted by $p_{f_1}(n^2-3,3)$ has to be $n-1$. So, as $n^2-3 = 2\cdot(n-1)+\cdots+2\cdot 1+r+s+t$, with $1 \leq t \leq s \leq r \leq n-1$, we get $r+s+t = n-3$. The number of solutions of this equation is clearly $p(n-3,3)$. □

Theorem 4.4 *For all $n \geq 1$ we have*

(i) $p_{f_1}(4n^2+n+i,3) = p_{f_1}(4n^2+n-i,3)$, *for* $0 \leq i \leq n+2$;
(ii) $p_{f_1}(4n^2+5n+2+i,3) = p_{f_1}(4n^2+5n+2-i,3)$, *for* $0 \leq i \leq n+2$;
(iii) $p_{f_1}(4n^2+n,3) = \left\lfloor \dfrac{n^2+1}{2} \right\rfloor$;
(iv) $p_{f_1}(4n^2+5n+i,3) = T_n$, *for* $i = 0,1,2,3$.

Proof (i) The largest part of any partition counted by $p_{f_1}(4n^2+n+i,3)$ is $2n-1$. Let $\lambda = (1,1,2,2,\ldots,2n-1,2n-1,r,s,t)$ be a partition of $4n^2+n+i$ with $1 \leq r \leq s \leq t \leq 2n-1$, from which we get $r+s+t = 3n+i$. Set $\mu = (1,1,2,2,\ldots,2n-1,2n-1,2n-r,2n-s,2n-t)$, a partition of $4n^2+n-i$ and note that it belongs to the set counted by $p_{f_1}(4n^2+n-i,3)$.

(ii) The same map from item (i) works in this case, with the only difference that the largest part is $2n$.

(iii) Similar to (i) we need to count the number of solutions of $r+s+t = 3n$ with $1 \leq r \leq s \leq t \leq 2n-1$. With no restrictions, this number is $p(3n,3)$, which by Andrews and Eriksson (2004) is

$$\left\{\frac{(3n+3)^2}{12}\right\} - \left\lfloor\frac{3n}{2}\right\rfloor - 1 = \left\lfloor\frac{n^2+1}{2}\right\rfloor + \left\lfloor\frac{n^2}{4}\right\rfloor.$$

Now we have to eliminate those solutions that do not satisfy $1 \leq r \leq s \leq t \leq 2n-1$. Clearly only t can be greater than $2n-1$. In this case, if $t = 2n+i$ with $0 \leq i \leq n+2$, the number of solutions of $r+s = n-i$ is equal to $\left\lfloor\frac{n-i}{2}\right\rfloor$. Hence, the number of solutions we do not have to consider is

$$\sum_{i=0}^{n-2}\left\lfloor\frac{n-i}{2}\right\rfloor = \sum_{i=2}^{n}\left\lfloor\frac{i}{2}\right\rfloor = \left\lfloor\frac{n^2}{4}\right\rfloor.$$

(iv) The same bijection built in the first item proves that $P_{f_1}(4n^2+5n+i,3)$ has the same cardinality if $i = 0$ and $i = 3$ and also if $i = 1$ and $i = 2$. By following the argument in (iii), we can adapt the proof and get identity (iv). □

Theorem 4.5 *For all $n \geq 2$ and $0 \leq i \leq 3$ we have*

$$p_{f_1}(2T_n+i,4) = p_{f_1}(2T_n-i,4).$$

Proof Any partition counted by $p_{f_1}(2T_n \pm i, 4)$ has largest part $n - 1$. By taking k_1, k_2, k_3, k_4 the excesses of a partition counted by $p_{f_1}(2T_n + i, 4)$, we map them into $n - k_1, n - k_2, n - k_3, n - k_4$. □

5 Mock Theta Function $F_1(q)$

Consider the unsigned version of the mock theta function $F_1(q)$ of order 5

$$F_1^*(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q; q^2)_{n+1}}. \tag{6}$$

Its general term

$$\frac{q^{2(2+4+6+\cdots+2s)}}{(1-q)(1-q^3)\cdots(1-q^{2s+1})},$$

generates the partitions of n into even parts ranging from 2 to $2s$ with no gaps and multiplicity 2, and any number of odd parts less than or equal to $2s + 1$.

As for $f_1^*(q)$, in Brietzke et al. (2013) the following interpretation for function $F_1^*(q)$ is given.

Theorem 5.1 *The coefficient of q^n in the expansion of (6) is equal to the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s+1} \\ d_1 & d_2 & \cdots & d_{s+1} \end{pmatrix}, \tag{7}$$

with non-negative integer entries whose sum is n, satisfying

$$\begin{aligned} c_{s+1} &= 0; \quad d_t \geq 0; \\ c_t &= 4 + c_{t+1} + 2d_{t+1}, \quad \forall t < s+1. \end{aligned}$$

Given a matrix from Theorem 5.1, its second row describes the odd parts from 1 to $2s + 1$ of the partition associated do it. To know how many of these parts the partition has, we have to sum the d_i, for $i = 1, 2, \ldots, s + 1$.

Definition 5.2 Let $p_{F_1}(n, k)$ be the number of partitions of n into even parts ranging from 2 to $2s$ with no gaps and multiplicity 2, and k other odd parts less than or equal to $2s + 1$.

As we have done before, we build a table (Table 4) that classifies the partitions generated by the unsigned version of the mock theta function $F_1(q)$, according to the sum of the second line of its related matrix representation.

The next results follow from observations in Table 4.

Table 4 Table from the characterization given by Theorem 5.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	0																								
2	0	0																							
3	0	0	0																						
4	0	0	0	1																					
5	0	0	0	1	0																				
6	0	0	0	1	0	0																			
7	0	0	0	1	0	1	0																		
8	0	0	0	1	0	1	0	0																	
9	0	0	0	1	0	1	0	0	0																
10	0	0	0	1	0	1	0	1	0	0															
11	0	0	0	1	0	1	0	1	0	0	0														
12	0	0	0	1	0	1	0	1	0	0	0	1													
13	0	0	0	1	0	1	0	1	0	1	0	1	0												
14	0	0	0	1	0	1	0	1	0	1	0	1	0	0											
15	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0										
16	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	0									
17	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	1	0								
18	0	0	0	1	0	1	0	1	0	1	0	2	0	1	0	2	0	0							
19	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	0	0						
20	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	1	0	0					
21	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	2	0	2	0	0	0				
22	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	1	0	0			
23	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	2	0	0	0		
24	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	2	0	3	0	0	0	1	
25	0	0	0	1	0	1	0	1	0	1	0	2	0	2	0	3	0	3	0	3	0	1	0	1	0

Theorem 5.3 *For all $n \geq 1$ and $1 \leq i \leq 2n$ we have*

(i) $p_{F_1}(8n^2 \pm (2i-1), 2) = 0$;
(ii) $p_{F_1}(8n^2 \pm 2i, 2) = n - \left\lfloor \dfrac{i}{2} \right\rfloor$;
(iii) $p_{F_1}(8n^2 - 8n + 1 \pm (2i-1), 2) = 0$;
(iv) $p_{F_1}(8n^2 - 8n + 1 \pm 2i, 2) = n - \left\lfloor \dfrac{i+1}{2} \right\rfloor$.

Proof We prove items (i) and (ii). Items (iii) and (iv) have respectively analogous proofs.

(i) If there was a partition counted by $p_{F_1}(8n^2 \pm (2i-1), 2)$, its largest even part of multiplicity 2 would be $4n-2$. Both odd parts r and s would satisfy $r + s = 4n \pm (2i-1)$ with $1 \leq r \leq s \leq 4n-1$, which is not possible.
(ii) The largest even part of multiplicity 2 of a partition counted by $p_{F_1}(8n^2 \pm 2i, 2)$ is also $4n-2$. The odd parts r and s satisfy $r+s = 4n \pm 2i$ with $1 \leq r \leq s \leq 4n-1$. Writing $r = 2k-1$ and $t = 2m-1$, it turns into $k + m = 2n \pm i + 1$, with $1 \leq k \leq m \leq 2n$.
It is easy to see that $k + m = 2n - i + 1$ has $n - \left\lfloor \dfrac{i}{2} \right\rfloor$ solutions.
For equation $k + m = 2n + i + 1$, the number of solutions, without counting the order of parts, is $\left\lfloor \dfrac{2n+i+1}{2} \right\rfloor$. Considering the condition $1 \leq k \leq m \leq 2n$, we need to eliminate the solutions where $m > 2n$, whose amount is i. So we get $\left\lfloor \dfrac{2n+i+1}{2} \right\rfloor - i = n - \left\lfloor \dfrac{i}{2} \right\rfloor$. □

Similar to the previous theorem, we get the following one, whose proof is analogous.

Theorem 5.4 *For all $n \geq 1$ we have*

(i) $p_{F_1}(8n^2 - 6n + 1 \pm (2i-1), 3) = 0$, *if* $0 \leq i \leq n$;
(ii) $p_{F_1}(8n^2 - 6n + 1 + 2i, 3) = p_{F_1}(8n^2 - 6n + 1 - 2i, 3)$, *if* $0 \leq i \leq n$.
(iii) $p_{F_1}(8n^2 - 6n + 1, 3) = \left\lfloor \dfrac{n^2+1}{2} \right\rfloor$;
(iv) $p_{F_1}(8n^2 + 2n - 3 + i, 3) = T_n$ *with* $i = 0, 2, 4, 6$.

As it occurs for the previous table, we can observe in Table 4 that its columns become constant below certain cells. Looking at the sequence of these fixed numbers, it is the same as the sequence of the number os partitions into parts congruent to $\pm 2 \pmod 5$. This result is sattled in the next theorem.

Theorem 5.5 *For all $n \geq 1$ and $i \leq 0$ we have*

$$p_{F_1}(3n + 1 + i, n - 1 + i) = p(n + 1|\textit{parts congruent to } \pm 2 \pmod 5).$$

Proof We present the prove for $i = 0$. For $i \geq 0$, it is possible to set a bijection between $P_{F_1}(3n+1+i, n-1+i)$ and $p_{F_1}(3n+1, n-1)$ by removing (conversely, adding) i parts of size1.

We denote by $P^*(n+1)$ the set of partitions of $n+1$ in which every part from 1 to s appears at least twice. So, we can build a bijection between sets $P_{F_1}(3n+1, n-1)$ and $P^*(n+1)$ by decreasing 1 from every odd part of a partition counted by $p_{F_1}(3n+1, n-1)$ and then dividing all parts by 2. Clearly, the reverse map is possible.

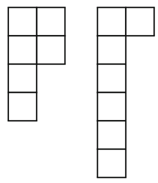
Facing the 2nd Rogers–Ramanujan Identity, we are going to prove that $p^*(n+1)$ is equal to the number of partitions of $n+1$ into 2-distinct parts, greater than or equal to 2. Consider the following steps. □

- Given a partition in $P^*(n+1)$, split it into two new ones, the first one made of two copies of each part from 1 to s, and the other one with the remaining parts.
- In the first partition, merge equal parts $1, 2, \ldots, s$, getting $(2, 4, \ldots, 2s)$. From the second one, take its conjugate. Once that its parts are smaller than or equal to s, now it has at most s parts.
- Get both partitions together side-by-side.

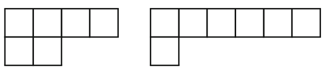
Example 5.6 For $n = 12$, the partition $(2, 2, 4, 4, 5, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1) \in P_{F_1}(37, 11)$ leads to $(10, 3)$, a partition of 13 into 2-distinct parts greater than or equal to 2.

$$(2,2,4,4,5,3,3,3,3,3,1,1,1,1,1) \longrightarrow (2,2,4,4,4,2,2,2,2,2)$$

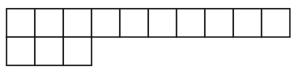
$$\longrightarrow (1,1,2,2,2,1,1,1,1,1) \longrightarrow$$



$\longrightarrow$



$\longrightarrow$



6 Mock Theta Function $f_0(q)$

Consider the unsigned version of the mock theta function $f_0(q)$ of order 5,

$$f_0^*(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}. \tag{8}$$

Its general term

$$\frac{q^{1+3+5+\cdots+(2s-1)}}{(1-q)(1-q^2)\cdots(1-q^s)},$$

generates the partitions of n containing each one of the odd numbers $1, 3, 5, \ldots, 2s-1$ as part of multiplicity 1 and any number of parts less than or equal to s. It also generates superdistinct partitions (partitions where $\lambda_i - \lambda_{i+1} \geq 2$) of n into exactly s parts. This can be seen by adding an unrestricted partition into at most s parts to the right hand side of the triangular partition $1 + 3 + \ldots + (2s - 1)$.

A combinatorial interpretation for this function in terms of two-line matrices is again presented in Brietzke et al. (2013).

Theorem 6.1 *The coefficient of q^n in the expansion of (8) is equal to the number of elements in the set of matrices of the form (2) with non-negative integer entries whose sum is n, satisfying*

$$c_s = 1; \quad d_t \geq 0;$$
$$c_t = 2 + c_{t+1} + d_{t+1}, \quad \forall t < s.$$

The second row of a matrix from Theorem 6.1 describes the parts from 1 to s, besides the odd parts from 1 to $2s - 1$, of the partition associated to it. To know how many parts from 1 to s the partition has, we sum the d_i, for $i = 1, 2, \ldots, s$.

Definition 6.2 Let $p_{f_0}(n, k)$ be the number of partitions of n into distinct odd parts, ranging from 1 to $2s - 1$ with no gaps, and k other parts less than or equal to s.

Again we build a table (Table 5) according to the matrix representation of partitions generated by function $f_0(q)$.

Theorem 6.3 *For all $n \geq 1$ we have*

$$p_{f_0}(n^2, 3) = p(n - 2, \leq 3).$$

Proof The largest part of any partition counted by $p_{f_0}(n^2, 3)$ must be $2n - 3$. The three parts $r \leq s \leq t \leq n - 1$ must satisfy $r + s + t = 2n - 1$.

Considering $\mu = (n - 1 - t, n - 1 - s, n - 1 - r)$, we get a partition of $n - 2$. This process can be easily inverted. □

Example 6.4 Considering $n = 7$ we get the partitions below:

$P_{f_0}(49, 3)$	(r, s, t)	$P(5, \leq 3)$
(11, 9, 7, **6**, **6**, 5, 3, 1, **1**)	(6, 6, 1)	(5)
(11, 9, 7, **6**, 5, **5**, 3, **2**, 1)	(6, 5, 2)	(4, 1)
(11, 9, 7, **6**, 5, **4**, 3, **3**, 1)	(6, 4, 3)	(3, 2)
(11, 9, 7, 5, **5**, **5**, 3, **3**, 1)	(5, 5, 3)	(3, 1, 1)
(11, 9, 7, 5, **5**, **4**, **4**, 3, 1)	(5, 4, 4)	(2, 2, 1)

In order to prove the next theorem, we need the following lemma.

Lemma 6.5 *For all $n \geq 1$ set*

$$A_n = \left\{(r, s, t)\ ;\ r + s + t = 3n + 3, 1 \leq t \leq s \leq r \leq 2n \text{ and } (r = 2n \text{ or } t = 1)\right\}.$$

We have $|A_n| = n$.

Table 5 Table from the characterization given by Theorem 6.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																								
2	1	0																							
3	1	0	0																						
4	1	0	0	1																					
5	1	0	0	1	0																				
6	1	0	0	1	1	0																			
7	1	0	0	1	1	0	0																		
8	1	0	0	1	1	1	0	0																	
9	1	0	0	1	1	1	0	0	1																
10	1	0	0	1	1	1	1	0	1	0															
11	1	0	0	1	1	1	1	0	1	1	0														
12	1	0	0	1	1	1	1	1	1	1	1	0													
13	1	0	0	1	1	1	1	1	1	1	2	0	0												
14	1	0	0	1	1	1	1	1	2	1	2	1	0	0											
15	1	0	0	1	1	1	1	1	2	1	2	2	1	0	0										
16	1	0	0	1	1	1	1	1	2	2	2	2	2	0	0	1									
17	1	0	0	1	1	1	1	1	2	2	2	2	3	1	0	1	0								
18	1	0	0	1	1	1	1	1	2	2	3	2	3	2	1	1	1	0							
19	1	0	0	1	1	1	1	1	2	2	3	2	3	3	2	1	1	1	0						
20	1	0	0	1	1	1	1	1	2	2	3	3	3	3	3	2	1	2	1	0					
21	1	0	0	1	1	1	1	1	2	2	3	3	3	3	4	3	2	2	2	0	0				
22	1	0	0	1	1	1	1	1	2	2	3	3	4	3	4	4	3	2	3	2	0	0			
23	1	0	0	1	1	1	1	1	2	2	3	3	4	3	4	5	4	3	3	3	1	0	0		
24	1	0	0	1	1	1	1	1	2	2	3	3	4	4	4	5	5	4	4	4	3	1	0	0	
25	1	0	0	1	1	1	1	1	2	2	3	3	4	4	4	5	6	5	5	4	4	3	0	0	1

Proof First, consider $n = 1$ and note that the identity holds. To prove it for $n > 1$, we separate the proof in two cases:

(i) If $r = 2n$, then $s+t = n+3$, which has $\lfloor \frac{n+3}{2} \rfloor$ solutions satisfying the conditions we need.
(ii) If $t = 1$, then $r + s = 3n + 2$. As $r \leq 2n$, then $s \geq n + 2$. This gives us $\lfloor \frac{n}{2} \rfloor$ different solutions, without counting the order, as usual. Now, note that the solution where $r = 2n$ and $t = 1$ at the same time has been counted in both cases. So, the number of elements of the set A_n is

$$\left\lfloor \frac{n+3}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - 1 = n.$$

□

Theorem 6.6 *For all $n \geq 1$ we have*

$$p_{f_0}(4n^2 + 3n + 3, 3) = T_n.$$

Proof The largest part of any partition counted by $p_{f_0}(4n^2+3n+3, 3)$ must be $4n-1$. The three parts less than or equal to $2n$ must satisfy $r + s + t = 3n + 3$. This means that

$$p_{f_0}(4n^2 + 3n + 3, 3) = |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 1 \leq t \leq s \leq r \leq 2n\}|,$$

which is the same as

$$\begin{aligned} p_{f_0}(4n^2 + 3n + 3, 3) &= |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 2 \leq t \leq s \leq r \leq 2n - 1\}| \\ &+ |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 1 \leq t \leq s \leq r \leq 2n \text{ and } (r = 2n \text{ or } t = 1)\}| \end{aligned}$$

By Lemma 6.5, this turns into

$$\begin{aligned} p_{f_0}(4n^2 + 3n + 3, 3) = |\{(r, s, t)\ ;\ r + s + t = 3n + 3, 2 \leq t \\ \leq s \leq r \leq 2n - 1\}| + n, \end{aligned}$$

and what is left to prove is that

$$|\{(r, s, t)\ ;\ r + s + t = 3n + 3, 2 \leq t \leq s \leq r \leq 2n - 1\}| = p_{f_0}(4n^2 - 5n + 4, 3).$$

This can be done by building a bijection.

Given a partition counted by $p_{f_0}(4n^2-5n+4, 3)$, consider the parts r, s, t satisfying $r+s+t = 3n$, with $1 \leq t \leq s \leq r \leq 2n-2$. Note that the triple $(r+1, s+1, t+1)$ satisfies $2 \leq t + 1 \leq s + 1 \leq r + 1 \leq 2n - 1$.

The result follows by induction. □

Example 6.7 In the following table, we show the partitions of $P_{f_0}(4n^2 + 3n + 3, 3)$, for $n = 1, 2, 3$, according to Theorem 6.6.

$P_{f_0}(10, 3)$	$P_{f_0}(25, 3)$	$P_{f_0}(48, 3)$
$(3, \mathbf{2}, \mathbf{2}, \mathbf{2}, 1)$	$(7, 5, 3, \mathbf{3}, \mathbf{3}, \mathbf{3}, 1)$	$(11, 9, 7, 5, \mathbf{4}, \mathbf{4}, \mathbf{4}, 3, 1)$
	$(7, 5, \mathbf{4}, 3, \mathbf{3}, \mathbf{2}, 1)$	$(11, 9, 7, 5, \mathbf{5}, \mathbf{4}, 3, \mathbf{3}, 1)$
	$(7, 5, \mathbf{4}, \mathbf{4}, 3, 1, \mathbf{1})$	$(11, 9, 7, 5, \mathbf{5}, \mathbf{5}, 3, \mathbf{2}, 1)$
		$(11, 9, 7, \mathbf{6}, 5, \mathbf{5}, 3, 1, \mathbf{1})$
		$(11, 9, 7, \mathbf{6}, 5, \mathbf{4}, 3, \mathbf{2}, 1)$
		$(11, 9, 7, \mathbf{6}, 5, 3, \mathbf{3}, \mathbf{3}, 1)$

As before, the columns in Table 5 become constant below certain entry. The sequence of this fixed numbers is the same as the sequence of balanced partitions, which are those whose smallest part equals the number of parts.

Theorem 6.8 *For all $n \geq 2$ and $i \geq 0$ we have*

$$p_{f_0}(2n + i, n - 2 + i) = p_b(n + 2),$$

where $p_b(n)$ is the number of balanced partitions of n, i.e., the number of partitions of n where the smallest part equals the number of parts.

Proof Starting with $i = 0$, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition counted by $p_b(n + 2)$. Note that $\lambda_k = k$ and the Ferrers graph of this partition has Durfee square of size k. Considering the k^2 points of the Durfee square, write them as the partition $(2k - 1, 2k - 3, \ldots, 3, 1)$.

Observe that, as $n \geq 2$, we have $k \geq 2$, and as $n + 2 \geq k^2 + \lambda_1 - k$, then $\lambda_1 - k \leq n + 2 - k^2 \leq n - 2$. Denoting by $\lambda^{(1)}$ the partition on the right hand side of the Durfee square, consider $\overline{\lambda^{(1)}}$ the conjugated partition of $\lambda^{(1)}$ and add to its left hand side the partition $(\underbrace{1, 1, \ldots, 1}_{n-2})$. As the number of parts of $\lambda^{(1)}$ is less than k (because $\lambda_k = k$), each part of this new partition we built is less than or equal to k.

By joining this partition to the partition $(2k - 1, 2k - 3, \ldots, 3, 1)$, we get $2n$ partitioned into k odd parts plus $n - 2$ other parts less than or equal to k. In other words, it is a partition counted by $p_{f_0}(2n, n - 2)$. The inverse map is easy to build.

If $i \geq 1$, a partition counted by $p_{f_0}(2n + i, n - 2 + i)$ must have i parts 1. Clearly, a bijection between $P_{f_0}(2n + i, n - 2 + i)$ and $P_b(n + 2)$ removes (conversely, adds) i parts of size 1. □

7 Mock Theta Function $F_0(q)$

Consider the unsigned version of the mock theta function $F_0(q)$ of order 5

$$F_0^*(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q^2)_n}. \tag{9}$$

Its general term

$$\frac{q^{2(1+3+5+\cdots+(2s-1))}}{(1 - q)(1 - q^3) \cdots (1 - q^{2s-1})},$$

generates the partitions of n into odd parts, with no gaps, such that each part has multiplicity at least two and the largest part is $2s - 1$.

This function also has a combinatorial interpretation, find in Brietzke et al. (2013).

Theorem 7.1 *The coefficient of* q^n *in the expansion of* (9) *is equal to the number of elements in the set of matrices of the form* (2) *with non-negative integer entries whose sum is n, satisfying*

$$\begin{aligned} c_s &= 2; \quad d_t \geq 0; \\ c_t &= 4 + c_{t+1} + 2d_{t+1}, \quad \forall t < s. \end{aligned}$$

The entries on the second row of the matrices above describe how many parts from 1 to $2s-1$ appear more than twice in the partition associated to each matrix. To know this quantity, we have to sum these entries.

Definition 7.2 Let $p_{F_0}(n, k)$ be the number of partitions of n into odd parts from 1 to $2s - 1$, with no gaps, each one with multiplicity at least two, where k indicates how many parts appear more than twice.

A simple identity relates mock theta function $F_0(q)$ with $f_0(q)$, and summarizes a lot of information.

Theorem 7.3 *For all* $n \geq 1$ *and* $0 \leq i \leq n$, *we have*

$$p_{F_0}(2n + i, i) = p_{f_0}(n + i, i). \tag{10}$$

Proof Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $2n + i$ counted by $p_{F_0}(2n + i, i)$, with largest odd part $2s - 1$. Note that the exceeding parts sum $2n + i - 2s^2$. As there are at least two copies of each odd part in λ, we remove one copy of each of them, getting $2n + i - s^2$. Each exceeding part λ_j, which is odd, we replace by $\mu_j = \frac{\lambda_j + 1}{2}$, getting a part between 1 and s.

The reverse map is easy to get and we have identity (10). □

As consequence of Theorem 7.3, we get the following corollary.

Corollary 7.4 *For* $n \geq 1$, *we have the following identities*:

(i) $p_{F_0}(2n^2, 0) = p_{f_0}(n^2, 0) = 1$;
(ii) $p_{F_0}(2(n+1)^2 + 2i + 1, 1) = p_{f_0}((n+1)^2 + i + 1, 1) = 1$, $0 \leq i \leq n$;
(iii) $p_{F_0}(2n^2 + 2n \pm 2i, 2) = p_{f_0}(n^2 + n + 1 \pm i, 2)$;
(iv) $p_{F_0}(2n^2 - 3, 3) = p_{f_0}(n^2, 3) = p(n - 2, \leq 3)$;
(v) $p_{F_0}(8n^2 + 6n + 3, 3) = p_{f_0}(4n^2 + 3n + 3, 3) = T_n$;
(vi) $p_{F_0}(3n + 5 + i, n - 1 + i) = p_{f_0}(2n + 2 + i, n - 1 + i) = p_b(n + 3)$, $i \geq 0$.

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CHAPTER 3

The mock theta function $f_*^m(q)$

3.1 Introduction

In this chapter we define a collection of mock theta functions inspired by the definition of function $f_1^*(q)$ from Chapter 2. This collection of functions, which we call $f_*^m(q)$ for $m \geq 1$, generates integer partitions that can also be interpreted combinatorially as a set of matrices. The results of this chapter have already been submitted for publication (see [Mat17]).

Given any $m \geq 1$, let us consider the mock theta function

$$f_*^m(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}. \tag{3.1}$$

For a fixed $m \geq 1$, the general term

$$\frac{q^{m(1+2+3+\cdots+s)}}{(1-q)(1-q^2)\cdots(1-q^s)}$$

generates the partitions of n containing at least m parts equal to each one of the numbers $1, 2, 3, \ldots, s$, with no gaps. By conjugation, this general term also generates the partitions of n into exactly s parts, with smallest part $\lambda_s \geq m$ and with difference between consecutive parts $\lambda_t - \lambda_{t+1} \geq m$.

Remark 3.1.1. *$f_*^m(q)$ with $m = 2$ is the mock theta function $f_1^*(q)$ from Chapter 2. In the present chapter we deal with more general aspects of function $f_*^m(q)$, for any $m \geq 1$. For more specific details about $f_*^2(q) = f_1^*(q)$ see Chapter 2.*

In the following pages we present the matrix representation for integer partitions counted by $f_*^m(q)$ and a collection of results derived from this representation.

3.2 The family $\{f_*^m(q)\}_{m\geq 1}$

Let us consider a partition of n counted by the mock theta function

$$f_*^m(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n} \tag{3.2}$$

for some fixed $m \geq 1$. Remind that such a partition has each part from 1 to s with no gaps and multiplicity at least m. So, n can be written as

$$n = (m+d_s)\cdot s + (m+d_{s-1})\cdot(s-1) + \cdots + (m+d_2)\cdot 2 + (m+d_1)\cdot 1,$$

with $d_t \geq 0$ for all $1 \leq t \leq s$.

By rearranging these numbers, we may have n as the sum of the entries of the matrix

$$A = \begin{pmatrix} s\cdot m + d_2 + d_3 + \cdots + d_s & (s-1)\cdot m + d_3 + \cdots + d_s & \cdots & 2m + d_s & m \\ d_1 & d_2 & \cdots & d_{s-1} & d_s \end{pmatrix},$$

which allows us to state the following theorem.

Theorem 3.2.1. *The coefficient of q^n in the expansion of function (3.2) equals the number of elements in the set of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{3.3}$$

with non-negative integer entries, satisfying $c_s = m$, $c_t = m + c_{t+1} + d_{t+1}$ for all $t < s$, and $n = \sum c_t + \sum d_t$.

Given any partition generated by function $f_*^m(q)$, the second row of its associated matrix informs us how many "extra" parts the partition has, besides the m copies of each part from 1 to s. This then motivates the following definition.

Definition 3.2.2. Let $P_{[s]}^{m[s]}(n,k)$ be the set of partitions of n into parts ranging from 1 to s, with no gaps and multiplicity m, and k other parts from 1 to s. Also, $|P_{[s]}^{m[s]}(n,k)| = p_{[s]}^{m[s]}(n,k)$.

Remark 3.2.3. *Noting that function $f_1^*(q)$ is the same as $f_*^2(q)$, from now on, instead of using $p_{f_1}(n,k)$ as in Chapter 2, we use $p_{[s]}^{2[s]}(n,k)$.*

Motivated by Definition 3.2.2, given a fixed m, for each n we classify its partitions according to the sum of the entries in the second row of the associated matrix. For different values of m, we count the appearance of each number in these sums and organize the data in tables. Excerpts from the tables obtained for $m = 1, 2, 3, 4, 5$ and 6 are presented in Appendix A (Tables 28, 29, 30, 31, 32 and 33).

In any of those tables the entry in line n and column $n-j$ is the number of times j appears as sum of the entries of the second row in type (3.3) matrices. That is, how many partitions of n have j extra parts, besides the m copies of each integer from 1 to s.

Tables 29, 30, 31, 32 and 33 and other ones that can be easily obtained for other values of m, which are omitted from this text, have interesting values of $p_{[s]}^{m[s]}(n,k)$, for different fixed values of k. In order to refer to these values in a simpler way, we adopt the following definition.

Definition 3.2.4. Given $m \geq 2$ and $k \geq 0$ we call the sequence of $p_{[s]}^{m[s]}(n,k)$ for $n \geq 0$ the k^{th} *diagonal* of function $f_*^m(q)$ table.

Remark 3.2.5. *The particular case of $f_*^m(q)$ with $m = 1$ does not follow the expected regularity of $f_*^m(q)$ for $m \geq 2$. Therefore, the case of $m = 1$ will be treated separately in Section 3.7.*

Remark 3.2.6.

(i) Note that the k^{th} diagonal makes sense only for $n \geq k$. So, from now on we omit $p_{[s]}^{m[s]}(0,k)$ for any $k \geq 1$ and assume $p_{[s]}^{m[s]}(n,k) = 0$ for $1 \leq n < k$ ($p_{[s]}^{m[s]}(0,0) = 1$, by definition).

(ii) Moreover, the results for any k^{th} diagonal are valid for functions $f_*^m(q)$ whenever $m \geq k$.

Some facts about the zero and the first diagonals are easily observed and set in the following proposition.

Proposition 3.2.7. *Given $m \geq 2$, we have*

$$(i)\ p_{[s]}^{m[s]}(n,0) = \begin{cases} 1, \text{ if } n = m \cdot T_j \text{ , for some } j \geq 0 \\ 0, \text{ otherwise} \end{cases}$$

and

$$\textit{(ii)}\ p_{[s]}^{m[s]}(n,1) = \begin{cases} 1, \textit{ if } n = m \cdot T_j + i\ , \textit{ for some } j \geq 1 \textit{ and every } 1 \leq i \leq j \\ 0, \textit{ otherwise.} \end{cases}$$

In fact, Proposition 3.2.7 gives us a complete characterization of the zero and the first diagonals of the table of function $f_*^m(q)$, for any value of m.

Similar results were obtained in order to characterize other diagonals of the tables. In the next sections we will show the sequence of values of those diagonals for $1 \leq n \leq 200$, clearly many more then those shown in Tables 29, 30, 31, 32 and 33, and present the results derived from the analysis of these sequences.

3.3 The 2^{nd} diagonal

Table 1 below shows us the sequence of values contained in the 2^{nd} diagonal, that is, the values of $p_{[s]}^{m[s]}(n,2)$, for $1 \leq n \leq 200$ counted by functions $f_*^2(q)$, $f_*^3(q)$, and $f_*^4(q)$.

Observe that each sequence contained in Table 1 has a lot of zeros in very regular positions. Also, between two lists of zeros, greater integers appear in such a way that each list of integers is symmetrical, always starting and ending at 1, and increasing and decreasing by at most 1 unit.

Moreover, note that each list of integers grater than zero has odd size. In some cases the central term is different from the others, while in other cases the central term is the same as its adjacent terms.

In the following results we formalize all of the previous observations, getting a complete characterization of the 2^{nd} diagonal, for any n and any $m \geq 2$.

Proposition 3.3.1. *Given $m \geq 2$, for all $n \geq 1$ and $1 \leq i \leq (m-2)n+3$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2 - i, 2\right) = 0.$$

Proof. Let us suppose we could partition $\dfrac{m(n^2+n)}{2} + 2 - i$ into m copies of each part from 1 to some j and two more parts less than or equal to j. So,

$f_*^m(q)$	$p_{[s]}^{m[s]}(n,2)$
$f_*^2(q)$	$(0,0,0,1,0,0,0,1,1,1,0,0,0,1,1,2,1,1,0,0,0,1,1,2,2,2,1,1,0,0,0,1,1,2,2,3,2,2,$ $1,1,0,0,0,1,1,2,2,3,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,3,3,2,2,1,1,0,0,0,1,1,2,$ $2,3,3,4,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,5,4,4,3,3,2,2,1,1,0,0,0,1,1,2,$ $2,3,3,4,4,5,5,5,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,4,5,5,6,5,5,4,4,3,3,2,2,$ $1,1,0,0,0,1,1,2,2,3,3,4,4,5,5,6,6,6,5,5,4,4,3,3,2,2,1,1,0,0,0,1,1,2,2,3,3,4,$ $4,5,5,6,6,7,6,6,5,5,\ldots)$
$f_*^3(q)$	$(0,0,0,0,1,0,0,0,0,0,1,1,1,0,0,0,0,0,0,1,1,2,1,1,0,0,0,0,0,0,0,1,1,2,2,2,1,1,$ $0,0,0,0,0,0,0,0,1,1,2,2,3,2,2,1,1,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,3,2,2,1,1,0,$ $0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,$ $3,4,4,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,5,4,4,3,3,2,2,1,$ $1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,5,5,5,4,4,3,3,2,2,1,1,0,0,0,0,0,$ $0,0,0,0,0,0,0,0,0,1,\ldots)$
$f_*^4(q)$	$(0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,1,1,2,1,1,0,0,0,0,0,0,0,0,$ $0,0,0,1,1,2,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,2,2,1,1,0,0,0,0,0,0,$ $0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,$ $1,2,2,3,3,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,$ $4,4,3,3,2,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,3,3,4,4,5,$ $4,4,3,3,2,2,1,1,0,0,\ldots)$

Table 1: Values contained in the 2^{nd} diagonals of the tables of functions $f_*^2(q)$, $f_*^3(q)$, and $f_*^4(q)$.

we would write

$$\begin{aligned}\frac{m(n^2+n)}{2}+2-i &= \underbrace{j+\cdots+j}_{m}+\cdots+\underbrace{2+\cdots+2}_{m}+\underbrace{1+\ldots+1}_{m}+r+s \\ &= \frac{m(j^2+j)}{2}+r+s\end{aligned}$$

for some $j \geq 1$ and $1 \leq s \leq r \leq j$. Note that j has to be less than n, and so we can make the following estimations:

$$\frac{m(n^2+n)}{2}+2-i = \frac{m(j^2+j)}{2}+r+s \leq \frac{m((n-1)^2+(n-1))}{2}+2(n-1),$$

which is equivalent to

$$mn+2-i \leq 2n-2.$$

On the other hand,

$$mn+2-i \geq mn+2-(m-2)n-3,$$

and so

$$mn+2-(m-2)n-3 \leq 2n-2$$

or

$$1 \leq 0,$$

which is an absurd.

As we cannot write $\frac{m(n^2+n)}{2}+2-i = \underbrace{j+\cdots+j}_{m}+\cdots+\underbrace{1+\ldots+1}_{m}+r+s$, this means $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2-i, 2\right) = 0.$

□

The symmetry of the list of integers between the zeros is described in the next proposition. In order to prove it we need the following lemma, which will also be useful in further sections.

Lemma 3.3.2. *Given $m \geq k \geq 2$, for all $n \geq 2$ and $1 \leq t \leq n-1$ we have*

$$\frac{m(2nt-t^2+t)}{2} > k(n-1). \tag{3.4}$$

Proof. First of all, as $m \geq k \geq 2$ let us write $m = k + j$, with $j \geq 0$. We prove inequality (3.4) by induction on $n \geq 2$. For $n = 2$ we have only $t = 1$, which implies

$$\frac{m(2nt - t^2 + t)}{2} = 2m = 2k + 2j > 2k - k = k \cdot n - k = k(n-1),$$

and so (3.4) is true.

By supposing (3.4) is true for some $n = b \geq 2$ and all $1 \leq t \leq b - 1$, let us prove it also holds for $b + 1$, with $1 \leq t \leq b$.

$$\frac{m(2(b+1)t - t^2 + t)}{2} = \frac{m(2bt - t^2 + t)}{2} + mt.$$

For $1 \leq t \leq b - 1$, by induction hypothesis we have

$$\frac{m(2bt - t^2 + t)}{2} + mt > k(b-1) + mt \geq k(b-1) + k = k((b+1) - 1).$$

For $t = b$ we have

$$\frac{m(2bt - t^2 + t)}{2} + mt \geq \frac{m(b^2 + b)}{2} + kb > kb = k((b+1) - 1).$$

Therefore, by induction we have (3.4) valid for all $n \geq 2$ and $1 \leq t \leq n - 1$. □

The particular case of Lemma 3.3.2 with $k = 2$ is used in the following proposition.

Proposition 3.3.3. *Given $m \geq 2$, for all $n \geq 1$ and $0 \leq i \leq n - 1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2 + i, 2\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2n - i, 2\right).$$

Proof. We begin by claiming that the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2 + i, 2\right)$ is exactly n. Indeed, if the greatest part were larger than n, let us say $n + t$ with $t \geq 1$, we would have

$$m \cdot (n+t) + m \cdot (n+t-1) + \cdots + m \cdot 1 + r + s = \frac{m(n^2+n)}{2} + 2 + i, \tag{3.5}$$

with $1 \leq s \leq r \leq n + t$. By doing some estimations and using Lemma 3.3.2 with $k = 2$ we get (3.5) equivalent to

$$r + s = 2 + i - \frac{m(2nt + t + t^2)}{2} \leq 0,$$

contradicting the fact that $1 \leq s \leq r$.

Moreover, for $n > 1$ the greatest part cannot be smaller than n either. Indeed, if it were $n - t$ with $t \geq 1$, we would have

$$m \cdot (n-t) + m \cdot (n-t-1) + \cdots + m \cdot 1 + r + s = \frac{m(n^2+n)}{2} + 2 + i, \tag{3.6}$$

with $1 \leq s \leq r \leq n - t$. According to Lemma 3.3.2 with $k = 2$, equation (3.6) is equivalent to

$$r + s = \frac{m(2nt - t^2 + t)}{2} + 2 + i > 2n - 2. \tag{3.7}$$

However, as $s \leq r \leq n-t$ we have $r+s \leq 2(n-t) \leq 2n-2$, and so inequality (3.7) is an absurd.

Therefore, the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2 + i, 2\right)$ has to be n. Analogous arguments allow us to conclude that the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2n - i, 2\right)$ is also exactly n.

Then, given λ a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2 + i, 2\right)$, we have

$$\lambda = (\underbrace{n, \ldots, n}_{m}, \ldots, \underbrace{2, \ldots, 2}_{m}, \underbrace{1, \ldots, 1}_{m}, r, s),$$

with $1 \leq s \leq r \leq n$. So,

$$m \cdot n + m \cdot (n-1) + \cdots + m \cdot 2 + m \cdot 1 + r + s = \frac{m(n^2+n)}{2} + 2 + i,$$

and therefore

$$r + s = 2 + i.$$

By writing $\mu = (\underbrace{n, \ldots, n}_{m}, \ldots, \underbrace{2, \ldots, 2}_{m}, \underbrace{1, \ldots, 1}_{m}, n+1-r, n+1-s)$ we get a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 2n - i, 2\right)$ as

$$\begin{aligned} m \cdot n + \cdots + m \cdot 2 + m \cdot 1 + n + 1 - r + n + 1 - s \\ &= \frac{m(n^2+n)}{2} + 2n + 2 - (r+s) \\ &= \frac{m(n^2+n)}{2} + 2n + 2 - (2+i) \\ &= \frac{m(n^2+n)}{2} + 2n - i. \end{aligned}$$

Easily we can build the reverse map, getting

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2+i,2\right)=p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+2n-i,2\right).$$

□

We illustrate Proposition 3.3.3 by making $m=4$, $n=9$, and $i=7$ in the following example.

Example 3.3.4. *The number of partitions counted by* $p_{[s]}^{4[s]}(189,2)$ *is the same as* $p_{[s]}^{4[s]}(191,2)$*, as shown in Table 2.*

$P_{[s]}^{4[s]}(189,2)$	$P_{[s]}^{4[s]}(191,2)$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{8},\mathbf{1})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{9},\mathbf{2})$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{7},\mathbf{2})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{8},\mathbf{3})$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{6},\mathbf{3})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{7},\mathbf{4})$
$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{5},\mathbf{4})$	$(9,9,9,9,8,8,8,8,\dots,1,1,1,1,\mathbf{6},\mathbf{5})$

Table 2: Table for Example 3.3.4

In order to have the 2^{nd} diagonal completely described, what remains to be shown is the exact value of each term in the 2^{nd} diagonal of any function $f_*^m(q)$. This result is given next and has a simple demonstration.

Proposition 3.3.5. *Given* $m\geq 2$*, for all* $n\geq 1$ *and* $0\leq i\leq n-1$ *we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+1-i,2\right)=\left\lfloor\frac{n+1-i}{2}\right\rfloor.$$

Proof. By observing that $2+i$ and $n+1-i$ both range from 2 to $n+1$ if $0\leq i\leq n-1$, Proposition 3.3.3 gives us that the greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+1-i,2\right)$ is n. So we have

$$\frac{m(n^2+n)}{2}+n+1-i=m\cdot n+\dots+m\cdot 2+m\cdot 1+r+s,$$

for $1\leq s\leq r\leq n$, which can be rewritten as

$$r+s=n+1-i. \tag{3.8}$$

According to (1.5), the number of partitions of $n+1-i$ into exactly two parts, that is, the number of solutions of (3.8) satisfying $1 \leq s \leq r \leq n$ is $\left\lfloor \dfrac{n+1-i}{2} \right\rfloor$ (see [AE04]).

$\square$

Now we may observe that, as it happens with $\dfrac{m(n^2+n)}{2}+2+i$ and $\dfrac{m(n^2+n)}{2}+n+1-i$, also $\dfrac{m(n^2+n)}{2}+2n-i$ and $\dfrac{m(n^2+n)}{2}+n+1+i$ both range through the same values for $0 \leq i \leq n-1$. Therefore, by joining Propositions 3.3.3 and 3.3.5 we get the following corollary.

Corollary 3.3.6. *Given $m \geq 2$, for all $n \geq 1$ and $0 \leq i \leq n-1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+1\pm i, 2\right) = \left\lfloor \frac{n+1-i}{2} \right\rfloor.$$

Example 3.3.7. *For $m=4$, $n=6$ and $0 \leq i \leq 5$ we have the partitions shown in Table 3.*

Now we have the 2^{nd} diagonal of function $f_*^m(q)$, for any $m \geq 2$, completely described.

3.4 The 3^{rd} diagonal

The sequences of values of $p_{[s]}^{m[s]}(n,3)$ for $1 \leq n \leq 200$, contained in the 3^{rd} diagonal of the tables of functions $f_*^3(q)$, $f_*^4(q)$, and $f_*^5(q)$, are shown in Table 4 below.

The zeros in any of the sequences contained in Table 4 are described in the next result, whose proof is analogous to the one of Proposition 3.3.1 and, therefore, omitted.

Proposition 3.4.1. *Given $m \geq 3$, for all $n \geq 1$ and $1 \leq i \leq (m-3)n+5$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+3-i, 3\right) = 0.$$

Also, non-zero values constitute again finite symmetrical lists of integers, which get longer as n increases. The proof of the next proposition follows the same direction as the one for Proposition 3.3.3, by building an analogous bijection and using the particular case of Lemma 3.3.2 with $k=3$.

$91 \pm i$	$P_{[s]}^{4[s]}(91 \pm i, 2)$	$p_{[s]}^{4[s]}(91 \pm i, 2)$
86	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{1},\mathbf{1})$	1
87	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{2},\mathbf{1})$	1
88	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{3},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{2},\mathbf{2})$	2
89	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{3},\mathbf{2})$	2
90	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{2})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{3},\mathbf{3})$	3
91	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{1})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{2})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{3})$	3
92	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{2})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{3})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{4},\mathbf{4})$	3
93	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{3})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{4})$	2
94	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{4})$ $(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{5},\mathbf{5})$	2
95	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{5})$	1
96	$(6,6,6,6,5,5,5,5,\ldots,1,1,1,1,\mathbf{6},\mathbf{6})$	1

Table 3: Table for Example 3.3.7

$f_*^m(q)$	$p_{[s]}^{m[s]}(n,3)$
$f_*^3(q)$	$(0,0,0,0,0,1,0,0,0,0,0,1,1,1,1,0,0,0,0,0,1,1,2,2,2,1,1,0,0,0,0,0,1,1,2,3,3,3,$ $3,2,1,1,0,0,0,0,0,1,1,2,3,4,4,5,4,4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,6,6,6,6,5,$ $4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,7,7,8,8,8,7,7,5,4,3,2,1,1,0,0,0,0,0,1,1,2,3,$ $4,5,7,8,9,10,10,10,10,9,8,7,5,4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,7,8,10,11,12,$ $12,13,12,12,11,10,8,7,5,4,3,2,1,1,0,0,0,0,0,1,1,2,3,4,5,7,8,10,12,13,14,15,$ $15,15,15,14,13,12,10,8,7,5,4,3,2,1,1,0,0,0,0,0,\ldots)$
$f_*^4(q)$	$(0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,1,1,2,2,2,1,1,0,0,0,0,0,$ $0,0,0,0,1,1,2,3,3,3,3,2,1,1,0,0,0,0,0,0,0,0,0,1,1,2,3,4,4,5,4,4,3,2,1,1,0,$ $0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,6,6,6,6,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,$ $1,1,2,3,4,5,7,7,8,8,8,7,7,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,$ $7,8,9,10,10,10,10,9,8,7,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,$ $7,8,10,11,12,12,13,12,12,11,10,8,\ldots)$
$f_*^5(q)$	$(0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,1,1,2,2,2,1,$ $1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,3,3,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$ $0,1,1,2,3,4,4,5,4,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,6,$ $6,6,6,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,7,7,8,8,$ $8,7,7,5,4,3,2,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,2,3,4,5,7,8,$ $9,10,10,10,10,9,8,7,5,4,\ldots)$

Table 4: Values contained in the 3^{rd} diagonals of the tables of functions $f_*^3(q)$, $f_*^4(q)$, and $f_*^5(q)$.

Proposition 3.4.2. *Given $m \geq 3$, for all $n \geq 1$ and $0 \leq i \leq \left\lfloor \frac{3n-1}{2} \right\rfloor - 1$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3 + i, 3\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3n - i, 3\right).$$

Proof. The greatest part of any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3 + i, 3\right)$ is n. In fact, as done in Proposition 3.3.3, some estimations and Lemma 3.3.2 with $k = 3$ let us conclude that the greatest part cannot be larger nor smaller than n, and the same for any partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3n - i, 3\right)$.

Then, given λ a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3 + i, 3\right)$, we have

$$\lambda = (\underbrace{n, \ldots, n}_{m}, \ldots, \underbrace{2, \ldots, 2}_{m}, \underbrace{1, \ldots, 1}_{m}, r, s, t),$$

with $1 \leq t \leq s \leq r \leq n$. So,

$$m \cdot n + m \cdot (n-1) + \cdots + m \cdot 2 + m \cdot 1 + r + s + t = \frac{m(n^2+n)}{2} + 3 + i,$$

and therefore

$$r + s + t = 3 + i.$$

By writing $\mu = (\underbrace{n, \ldots, n}_{m}, \ldots, \underbrace{2, \ldots, 2}_{m}, \underbrace{1, \ldots, 1}_{m}, n+1-r, n+1-s, n+1-t)$, we get a partition counted by $p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3n - i, 3\right)$, since

$$\begin{aligned} & m \cdot n + \cdots + m \cdot 2 + m \cdot 1 + n + 1 - r + n + 1 - s + n + 1 - t \\ = & \frac{m(n^2+n)}{2} + 3n + 3 - (r + s + t) \\ = & \frac{m(n^2+n)}{2} + 3n + 3 - (3 + i) \\ = & \frac{m(n^2+n)}{2} + 3n - i. \end{aligned}$$

Easily we can build the reverse map, getting

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3 + i, 3\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + 3n - i, 3\right).$$

□

Example 3.4.3. *For $m = 3$, $n = 5$ and $i = 4$ we have the same number of elements in $P_{[s]}^{3[s]}(52, 3)$ as in $P_{[s]}^{3[s]}(56, 3)$, as shown in Table 5.*

$P_{[s]}^{3[s]}(52, 3)$	$P_{[s]}^{3[s]}(56, 3)$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{5},\mathbf{1},\mathbf{1})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{1},\mathbf{5},\mathbf{5})$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{4},\mathbf{2},\mathbf{1})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{2},\mathbf{4},\mathbf{5})$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{3},\mathbf{5})$
$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2})$	$(5,5,5,4,4,4,3,3,3,2,2,2,1,1,1,\mathbf{3},\mathbf{4},\mathbf{4})$

Table 5: Table for Example 3.4.3

The values described in Proposition 3.4.2 compose a sequence which has a combinatorial interpretation in terms of another type of partitions, easier to count. This follows in the theorem below.

Theorem 3.4.4. *Given $m \geq 3$, for all $n \geq 1$ and $j \geq 1$ we have*

$$\begin{aligned} p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + n + 2, 3\right) &= p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2} + n + 2, 3\right) \\ &= p(n-1, \text{at most } 3 \text{ parts}). \end{aligned}$$

Proof. First of all, note that $\dfrac{m(n^2+n)}{2} + n + 2 = \dfrac{m(n^2+n)}{2} + 3 + (n-1)$. Since $0 \leq n-1 \leq \left\lfloor \dfrac{3n-1}{2} \right\rfloor - 1$, we already know from Proposition 3.4.2 that a partition counted by $p_{[s]}^{m[s]}\left(\dfrac{m(n^2+n)}{2} + n + 2, 3\right)$ has n as its greatest part. So, by writing

$$m \cdot n + \cdots + m \cdot 2 + m \cdot 1 + r + s + t = \frac{m(n^2+n)}{2} + n + 2,$$

with $1 \leq t \leq s \leq r \leq n$, we get

$$r + s + t = n + 2.$$

By decreasing r, s and t in one unit we get $r' = r - 1$, $s' = s - 1$ and $t' = t - 1$. Therefore $r' + s' + t' = r + s + t - 3 = n - 1$, with $0 \leq t' \leq s' \leq r' \leq n - 1$. So, (r', s', t') is a partition of $n - 1$ into at most 3 parts. The reverse map is clear.

Finally, the first equality of the theorem is easy to see since, by Proposition 3.4.2 again, any partition counted by $p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2}+n+2,3\right)$, or also by $p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2}+3+(n-1),3\right)$, has $n+j$ as its greatest part.

$\square$

Example 3.4.5. *For $m=3$, $n=5$ and $j=4$, we have the partitions shown in Table 6.*

$P_{[s]}^{3[s]}(52,3)$	$P_{[s]}^{3[s]}(142,3)$	$P(4, \text{at most } 3 \text{ parts})$
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{5},\mathbf{1},\mathbf{1})$	$(9,9,9,\cdots,1,1,1,\mathbf{5},\mathbf{1},\mathbf{1})$	(4)
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{4},\mathbf{2},\mathbf{1})$	$(9,9,9,\cdots,1,1,1,\mathbf{4},\mathbf{2},\mathbf{1})$	$(3,1)$
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1})$	$(9,9,9,\cdots,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1})$	$(2,2)$
$(5,5,5,\cdots,2,2,2,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2})$	$(9,9,9,\cdots,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2})$	$(2,1,1)$

Table 6: Table for Example 3.4.5

In order to have the 3^{rd} diagonal of any table of function $f_*^m(q)$ completely described, for any $m \geq 2$, we need some simple identities which we enunciate as a lemma. The demonstrations are simple but have extensive calculation.

Lemma 3.4.6.

(i) For all $j \geq 1$,

$$p(3j+1, \textit{exactly } 3 \textit{ parts}) = \frac{j(j+1)}{2} + \left\lfloor \frac{j^2}{4} \right\rfloor; \tag{3.9}$$

(ii) For all $j \geq 2$,

$$\sum_{i=1}^{j-1} \left\lfloor \frac{j+1-i}{2} \right\rfloor = \left\lfloor \frac{j^2}{4} \right\rfloor; \tag{3.10}$$

(iii) For all $j \geq 1$,

$$p(3j, \textit{exactly } 3 \textit{ parts}) = \left\lfloor \frac{j^2+1}{2} \right\rfloor + \left\lfloor \frac{j^2}{4} \right\rfloor. \tag{3.11}$$

Proof of Lemma 3.4.6, (i). According to [AE04], the number of partitions of n into exactly 3 parts is $\left\{\frac{(n+3)^2}{12}\right\} - \left\lfloor\frac{n}{2}\right\rfloor - 1$. By writing $3j+1$ in place of n we get statement (3.9) by proving the following one:

$$\left\{\frac{(3j+4)^2}{12}\right\} - \left\lfloor\frac{3j+1}{2}\right\rfloor - 1 = \frac{j(j+1)}{2} + \left\lfloor\frac{j^2}{4}\right\rfloor. \tag{3.12}$$

In order to prove (3.12) we write j in four different ways, according to its congruence modulus 4. First of all, note that

$$\left\{\frac{(3j+4)^2}{12}\right\} = \left\{\frac{9j^2+24j+16}{12}\right\} = \left\{\frac{3j^2}{4} + 2j + \frac{4}{3}\right\} = 2j + \left\{\frac{3j^2}{4} + \frac{4}{3}\right\},$$

and we may write (3.12) as

$$2j + \left\{\frac{3j^2}{4} + \frac{4}{3}\right\} - \left\lfloor\frac{3j+1}{2}\right\rfloor - 1 = \frac{j(j+1)}{2} + \left\lfloor\frac{j^2}{4}\right\rfloor. \tag{3.13}$$

We analyse separately each side of equation (3.13), and for any value of j modulus 4 we conclude that the equality is true.

□

Proof of Lemma 3.4.6, (ii). We prove statement (3.10) by induction over j. Note that for $j=2$ we have

$$\sum_{i=1}^{2-1}\left\lfloor\frac{2+1-i}{2}\right\rfloor = \left\lfloor\frac{2+1-1}{2}\right\rfloor = 1 = \left\lfloor\frac{2^2}{4}\right\rfloor.$$

By supposing for certain $j = b \geq 2$ that we have

$$\sum_{i=1}^{b-1}\left\lfloor\frac{b+1-i}{2}\right\rfloor = \left\lfloor\frac{b^2}{4}\right\rfloor,$$

let us prove that

$$\sum_{i=1}^{(b+1)-1}\left\lfloor\frac{(b+1)+1-i}{2}\right\rfloor = \left\lfloor\frac{(b+1)^2}{4}\right\rfloor$$

or, which is the same, that

$$\sum_{i=1}^{b}\left\lfloor\frac{b+2-i}{2}\right\rfloor = \left\lfloor\frac{b^2+2b+1}{4}\right\rfloor. \tag{3.14}$$

First of all, note that if i and b have the same parity, then $\left\lfloor \frac{b+2-i}{2} \right\rfloor = \left\lfloor \frac{b+1-i}{2} \right\rfloor + 1$. And if i and b have different parities, then $\left\lfloor \frac{b+2-i}{2} \right\rfloor = \left\lfloor \frac{b+1-i}{2} \right\rfloor$.

If b is even, saying $b = 2k$, then half of the values of i in the sum on the left hand side of (3.14) are even and the other half are odd. So we have

$$\sum_{i=1}^{b} \left\lfloor \frac{b+2-i}{2} \right\rfloor = \sum_{i=1}^{b} \left\lfloor \frac{b+1-i}{2} \right\rfloor + k = \sum_{i=1}^{b-1} \left\lfloor \frac{b+1-i}{2} \right\rfloor + k,$$

which by induction hypothesis equals

$$\left\lfloor \frac{b^2}{4} \right\rfloor + \frac{b}{2} = \left(\frac{2k}{2}\right)^2 + \frac{2k}{2} = \frac{4k^2+4k}{4} = \left\lfloor \frac{b^2+2b}{4} \right\rfloor = \left\lfloor \frac{b^2+2b+1}{4} \right\rfloor,$$

as we wanted.

If b is odd, saying $b = 2k+1$, then $k+1$ of the values of i in the sum on the left hand side of (3.14) are odd and k of those values are even. So we have

$$\sum_{i=1}^{b} \left\lfloor \frac{b+2-i}{2} \right\rfloor = \sum_{i=1}^{b} \left\lfloor \frac{b+1-i}{2} \right\rfloor + k + 1 = \sum_{i=1}^{b-1} \left\lfloor \frac{b+1-i}{2} \right\rfloor + k + 1,$$

which by induction hypothesis equals

$$\begin{aligned} \left\lfloor \frac{b^2}{4} \right\rfloor + k + 1 &= \left\lfloor \frac{4k^2+4k+1}{4} \right\rfloor + k + 1 = k^2 + 2k + 1 \\ &= \left(\frac{b-1}{2}\right)^2 + b = \frac{b^2+2b+1}{4} = \left\lfloor \frac{b^2+2b+1}{4} \right\rfloor, \end{aligned}$$

as we needed.

So, by induction statement (3.10) is proved.

□

Proof of Lemma 3.4.6, (iii). By using again the expression for the number of partitions of n into exactly 3 parts ([AE04]), with $3j$ in place of n we get what we need by proving the following statement:

$$\left\{ \frac{(3j+3)^2}{12} \right\} - \left\lfloor \frac{3j}{2} \right\rfloor - 1 = \left\lfloor \frac{j^2+1}{2} \right\rfloor + \left\lfloor \frac{j^2}{4} \right\rfloor. \tag{3.15}$$

In order to prove (3.15) we write j in four different ways, according to its congruence modulus 4. First of all, note that

$$\left\{\frac{(3j+3)^2}{12}\right\} = \left\{\frac{9j^2+18j+9}{12}\right\} = \left\{\frac{3j^2+3}{4} + \frac{3j}{2}\right\}$$

and we rewrite (3.15) as

$$\left\{\frac{3j^2+3}{4} + \frac{3j}{2}\right\} - \left\lfloor\frac{3j}{2}\right\rfloor - 1 = \left\lfloor\frac{j^2+1}{2}\right\rfloor + \left\lfloor\frac{j^2}{4}\right\rfloor. \tag{3.16}$$

As done in the proof of item (i), we analyse separately each side of equation (3.16), and for any value of j modulus 4 we conclude that the equality is true.

□

From Lemma 3.4.6 we finalize the characterization of the sequence of values $p_{[s]}^{m[s]}(n, 3)$ by setting the three final theorems of this section.

By considering only the non-zero integers of the 3^{rd} diagonal, the following theorem deals with the central terms of the lists of non-zero integers in even positions, that is, the central terms of the $2j^{th}$ lists of non-zero integers, for any $j \geq 1$.

Definition 3.4.7. We define $T_j := \dfrac{j(j+1)}{2}$, the j^{th} triangular number.

Theorem 3.4.8. *Given $m \geq 3$, for all even $n \geq 2$, let us say $n = 2j$ with $j \geq 1$, we have*

$$\begin{aligned} p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor - 1, 3\right) &= T_j \\ = p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor - 2, 3\right). \end{aligned} \tag{3.17}$$

Before the proof, observe that, according to Proposition 3.4.2, for all $j \geq 1$ we have

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor - 1, 3\right) = p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor, 3\right)$$

and

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor - 2, 3\right) = p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor\frac{3\cdot 2j-1}{2}\right\rfloor + 1, 3\right).$$

So, Theorem 3.4.8 says that these four numbers are actually all the same and equal to the j^{th} triangular number T_j. That is, the 3^{rd} diagonal of the table of any function $f_*^m(q)$ has constant subsequences of size four located in lines

$$n = \frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor \frac{3\cdot 2j-1}{2} \right\rfloor - 2,\ n = \frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor \frac{3\cdot 2j-1}{2} \right\rfloor - 1,$$

$$n = \frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor \frac{3\cdot 2j-1}{2} \right\rfloor, \text{ and } n = \frac{m((2j)^2+2j)}{2} + 3 + \left\lfloor \frac{3\cdot 2j-1}{2} \right\rfloor + 1,$$

each of these four terms equal to T_j.

Proof of Theorem 3.4.8. Noting that $\left\lfloor \frac{3\cdot 2j-1}{2} \right\rfloor = 3j-1$, we can rewrite statement (3.17) as

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3j+1, 3\right) = T_j = p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3j, 3\right).$$

According to Proposition 3.4.2, with $n = 2j$ and respectively $i = 3j-2$ and $i = 3j-3$, the greatest part of any partition counted by

$$p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3j+1, 3\right) \text{ and } p_{[s]}^{m[s]}\left(\frac{m((2j)^2+2j)}{2} + 3j, 3\right)$$

is $2j$. So, partitioning $\frac{m((2j)^2+2j)}{2}+3j+1$ and $\frac{m((2j)^2+2j)}{2}+3j$ according to our rules is the same as partitioning $3j+1$ and $3j$ into three parts less than or equal to $2j$. That is,

$$r+s+t = 3j+1 \tag{3.18}$$

and

$$r+s+t = 3j, \tag{3.19}$$

with $1 \leq t \leq s \leq r \leq 2j$.

Clearly, the number of solutions of (3.18) with no restriction on parts is $p(3j+1, \textit{exactly } 3 \textit{ parts})$, which by item (i) of Lemma 3.4.6 is $\frac{j(j+1)}{2} + \left\lfloor \frac{j^2}{4} \right\rfloor$.

Now, we eliminate the solutions of $r+s+t = 3j+1$ that do not satisfy $1 \leq t \leq s \leq r \leq 2j$, which are those where $r > 2j$, or $r = 2j+i$ with $1 \leq i \leq j-1$.

For each value of i, we have to eliminate the solutions of $s + t = j + 1 - i$, which we already know are in number of $\left\lfloor \frac{j+1-i}{2} \right\rfloor$. From item (ii) of Lemma 3.4.6, the total amount of solutions we have to eliminate is $\sum_{i=1}^{j-1} \left\lfloor \frac{j+1-i}{2} \right\rfloor = \left\lfloor \frac{j^2}{4} \right\rfloor$. Observe that item (ii) of Lemma 3.4.6 is valid only for $j \geq 2$. However, the case with $j = 1$ has no solution to be eliminated, because $r > 2$ never occurs in equation 3.18, since $r, s, t \geq 1$.

Then, the number of solutions of $r + s + t = 3j + 1$ with the restriction $1 \leq t \leq s \leq r \leq 2j$ is

$$\frac{j(j+1)}{2} + \left\lfloor \frac{j^2}{4} \right\rfloor - \left\lfloor \frac{j^2}{4} \right\rfloor = \frac{j(j+1)}{2} = T_j.$$

The same proof is adaptable for equation (3.19) by using item (iii) of Lemma 3.4.6, and rearranging the indexes of the sum in item (ii). Therefore, equality (3.17) is proved and Theorem 3.4.8 is valid.

$\square$

In an analogous way as done in Theorem 3.4.8, the next two theorems characterize the central terms of the lists of non-zero integers in odd positions. The proofs of both theorems use items (ii) and (iii) of Lemma 3.4.6 with some replacements of j, and the formula for unrestricted partitions with exactly 3 parts, available in [AE04], also necessary for proving items (i) and (iii) of Lemma 3.4.6.

Theorem 3.4.9. *Given $m \geq 3$, for all $n \equiv 1 \pmod 4$, let us say $n = 4j + 1$ with $j \geq 0$, we have*

$$\begin{aligned} p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2 + (4j+1))}{2} + 3 + \left\lfloor \frac{3 \cdot (4j+1) - 1}{2} \right\rfloor - 1, 3\right) &= j^2 + (j+1)^2 \\ = p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2 + (4j+1))}{2} + 3 + \left\lfloor \frac{3 \cdot (4j+1) - 1}{2} \right\rfloor - 2, 3\right) &+ 1. \end{aligned} \tag{3.20}$$

Proof. Note that $\left\lfloor \frac{3 \cdot (4j+1) - 1}{2} \right\rfloor = 6j + 1$, and we can rewrite statement (3.20) as

$$p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2 + (4j+1))}{2} + 6j + 3, 3\right) = j^2 + (j+1)^2$$

$$= p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2+(4j+1))}{2}+6j+2,3\right)+1.$$

According to Proposition 3.4.2 the greatest part of any partition counted by

$$p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2+(4j+1))}{2}+6j+3,3\right)$$

and

$$p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2+(4j+1))}{2}+6j+2,3\right)$$

is $4j+1$. So, partitioning

$$\frac{m((4j+1)^2+(4j+1))}{2}+6j+3 \text{ and } \frac{m((4j+1)^2+(4j+1))}{2}+6j+2$$

according to our rules is the same as partitioning $6j+3$ and $6j+2$ into three parts less than or equal to $4j+1$. That is,

$$r+s+t=6j+3 \tag{3.21}$$

and

$$r+s+t=6j+2, \tag{3.22}$$

with $1 \leq t \leq s \leq r \leq 4j+1$.

Clearly, the number of solutions of (3.21) with no restriction is $p(6j+3, \textit{exactly } 3 \textit{ parts})$ which, by item (iii) of Lemma 3.4.6 with $2j+1$ in place of j, is

$$\left\lfloor\frac{(2j+1)^2+1}{2}\right\rfloor+\left\lfloor\frac{(2j+1)^2}{4}\right\rfloor=\frac{(2j+1)^2+1}{2}+\left\lfloor\frac{(2j+1)^2}{4}\right\rfloor.$$

Now we eliminate the solutions of $r+s+t=6j+3$ that do not satisfy $1 \leq t \leq s \leq r \leq 4j+1$, which are those with $r > 4j+1$, or $r=4j+1+i$ with $1 \leq i \leq 2j$. For each value of i, we have to eliminate the solutions of $s+t=2j+2-i$, which are $\left\lfloor\frac{2j+2-i}{2}\right\rfloor$ solutions. From item (ii) of Lemma 3.4.6 with $2j+1$ in place of j, the total amount of solutions we have to eliminate is

$$\sum_{i=1}^{2j}\left\lfloor\frac{2j+2-i}{2}\right\rfloor=\left\lfloor\frac{(2j+1)^2}{4}\right\rfloor.$$

Observe that the case with $i=0$ has no solution to be eliminated.

So, the number of solutions of (3.21) with $1 \leq t \leq s \leq r \leq 4j+1$ is

$$\frac{(2j+1)^2+1}{2}+\left\lfloor\frac{(2j+1)^2}{4}\right\rfloor-\left\lfloor\frac{(2j+1)^2}{4}\right\rfloor=\frac{(2j+1)^2+1}{2}=j^2+(j+1)^2,$$

and we have

$$p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2+(4j+1))}{2}+3+\left\lfloor\frac{3\cdot(4j+1)-1}{2}\right\rfloor-1,3\right)+1=j^2+(j+1)^2.$$

Moreover, the number of solutions of (3.22) with no restriction is $p(6j+2, \textit{exactly } 3 \textit{ parts})$, which according to [AE04] is $\left\{\frac{(6j+5)^2}{12}\right\}-\left\lfloor\frac{6j+2}{2}\right\rfloor-1$, or also, with simple calculation, $3j^2+2j$.

Now we eliminate the solutions of $r+s+t=6j+2$ that do not satisfy $1\le t\le s\le r\le 4j+1$, which are those with $r>4j+1$, or $r=4j+1+i$ with $1\le i\le 2j-1$. For each value of i, we have to eliminate the solutions of $s+t=2j+1-i$, which are in number of $\left\lfloor\frac{2j+1-i}{2}\right\rfloor$. Again from Lemma 3.4.6, item (ii), with $2j$ in place of j, the total amount of solutions we have to eliminate is

$$\sum_{i=1}^{2j-1}\left\lfloor\frac{2j+1-i}{2}\right\rfloor=\left\lfloor\frac{(2j)^2}{4}\right\rfloor=j^2.$$

And again in this case if $i=0$ there is nothing to be eliminated.

Then, the number of solutions of (3.22) with $1\le t\le s\le r\le 4j+1$ is

$$3j^2+2j-j^2=2j^2+2j=j^2+(j+1)^2-1.$$

Hence, we have

$$p_{[s]}^{m[s]}\left(\frac{m((4j+1)^2+(4j+1))}{2}+3+\left\lfloor\frac{3\cdot(4j+1)-1}{2}\right\rfloor-2,3\right)+1=j^2+(j+1)^2.$$

$\square$

Theorem 3.4.10. *Given $m\ge 3$, for all $n\equiv 3 \pmod 4$, let us say $n=4j+3$ with $j\ge 0$, we have*

$$p_{[s]}^{m[s]}\left(\frac{m((4j+3)^2+(4j+3))}{2}+3+\left\lfloor\frac{3\cdot(4j+3)-1}{2}\right\rfloor-1,3\right)=2(j+1)^2$$
$$=p_{[s]}^{m[s]}\left(\frac{m((4j+3)^2+(4j+3))}{2}+3+\left\lfloor\frac{3\cdot(4j+3)-1}{2}\right\rfloor-2,3\right).$$

$\square$

There is no need to show the proof of this last theorem. The ideas are analogous to those from Theorem 3.4.9.

3.5 The 4^{th} diagonal

Most of the results from this section are similar to those presented in previous sections, as well as their proofs. Therefore we choose to exhibit only the proofs that are essentially different.

First of all, let us highlight in Table 7 some values of $p_{[s]}^{m[s]}(n,4)$ for $1 \leq n \leq 200$, contained in the 4^{th} diagonal of the tables of functions $f_*^4(q)$, $f_*^5(q)$, and $f_*^6(q)$.

$f_*^m(q)$	$p_{[s]}^{m[s]}(n,4)$
$f_*^4(q)$	$(0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 2, 3, 2, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 5, 7, 7, 8, 7, 7, 5, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 8, 9, 11, 11, 12, 11, 11, 9, 8, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 9, 10, 13, 14, 16, 16, 18, 16, 16, 14, 13, 10, 9, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 9, 11, 14, 16, 19, 20, 23, 23, 24, 23, 23, 20, 19, 16, 14, 11, 9, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 9, 11, 15, 17, 21, 23, 27, 28, 31, 31, 33, 31, 31, 28, 27, 23, 21, 17, 15, 11, 9, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, \ldots)$
$f_*^5(q)$	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 2, 3, 2, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 5, 7, 7, 8, 7, 7, 5, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 8, 9, 11, 11, 12, 11, 11, 9, 8, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 9, 10, 13, 14, 16, 16, 18, 16, 16, 14, 13, 10, 9, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 9, 11, 14, 16, 19, 20, 23, 23, 24, 23, 23, 20, 19, 16, 14, 11, 9, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots)$
$f_*^6(q)$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 2, 3, 2, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 5, 7, 7, 8, 7, 7, 5, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 8, 9, 11, 11, 12, 11, 11, 9, 8, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 6, 9, 10, 13, 14, 16, 16, 18, 16, 16, 14, 13, 10, 9, 6, 5, 3, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, \ldots)$

Table 7: Values contained in the 4^{th} diagonals of the tables of functions $f_*^4(q)$, $f_*^5(q)$, and $f_*^6(q)$.

The following results and examples characterize the position of zeros and non-zero integers contained in the 4^{th} diagonals, and also provide the exact values contained in those list of non-zero integers, based on a set of integer partitions easier to be counted.

Proposition 3.5.1. *Given $m \geq 4$, for all $n \geq 1$ and $1 \leq i \leq (m-4)n+7$*

we have
$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+4-i,4\right)=0.$$

Non-zero lists of values in the 4^{th} diagonal also obey a symmetry.

Proposition 3.5.2. *Given $m \geq 4$, for all $n \geq 1$ and $0 \leq i \leq 2n-2$ we have*
$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+4+i,4\right)=p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+4n-i,4\right).$$

Example 3.5.3. *For $m=5$, $n=4$ and $i=5$, we have the same number of elements in $P_{[s]}^{4[s]}(59,4)$ as in $P_{[s]}^{4[s]}(61,4)$, as shown in Table 8.*

$P_{[s]}^{4[s]}(59,4)$	$P_{[s]}^{4[s]}(61,4)$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{1})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{3},\mathbf{1})$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{4},\mathbf{2},\mathbf{1})$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{2})$
$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2})$	$(4,4,4,4,4,3,\ldots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{2})$

Table 8: Table for Example 3.5.3

The following theorem describes the identity in Proposition 3.5.2 in terms of a simpler type of partitions whose exact value is easier to find. Again its demonstration follows the lines of those analogous results from Sections 3.3 and 3.4.

Theorem 3.5.4. *Given $m \geq 4$, for all $n \geq 1$ and $j \geq 1$ we have*
$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2}+n+3,4\right)=p_{[s]}^{m[s]}\left(\frac{m((n+j)^2+(n+j))}{2}+n+3,4\right)$$
$$=p(n-1,\textit{at most } 4 \textit{ parts}).$$

Example 3.5.5. *For $m=5$, $n=6$ and $j=4$, we have the partitions shown in Table 9.*

$P^{5[s]}_{[s]}(114,4)$	$P^{5[s]}_{[s]}(284,4)$	$P(5, \text{at most 4 parts})$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{6},\mathbf{1},\mathbf{1},\mathbf{1})$	(5)
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{5},\mathbf{2},\mathbf{1},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{5},\mathbf{2},\mathbf{1},\mathbf{1})$	$(4,1)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1})$	$(3,2)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{1})$	$(3,1,1)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1})$	$(2,2,1)$
$(6,6,6,6,6,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2})$	$(10,10,10,10,10,\cdots,1,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2})$	$(2,1,1,1)$

Table 9: Table for Example 3.5.5

Recalling Definition 1.3.1, the notation $[n] := \{1,2,3,\dots,n-1,n\}$ appears in the next theorem. It deals with the particular case of Proposition 3.5.2, with $i = 2n-2$, characterizing the central term of any list of non-zero integers in the sequence of values of $p^{m[s]}_{[s]}(n,4)$.

Theorem 3.5.6. *Given $m \geq 4$, for all $n \geq 1$ we have*

$$p^{m[s]}_{[s]}\left(\frac{m(n^2+n)}{2}+2n+2,4\right) = p(2n+8, \textit{exactly } 4 \textit{ distinct parts in } [n+3]).$$

Proof. First observe that $\dfrac{m(n^2+n)}{2}+2n+2 = \dfrac{m(n^2+n)}{2}+4n-(2n-2)$.

Now, according to Proposition 3.5.2, the greatest part of any partition counted by $p^{m[s]}_{[s]}\left(\dfrac{m(n^2+n)}{2}+4n-(2n-2),4\right)$ is n. So we may write

$$r+s+t+u = 2n+2,$$

with $1 \leq u \leq t \leq s \leq r \leq n$.

By making

$$r' = r+3,$$

$$s' = s+2,$$

$$t' = t+1 \text{ and}$$

$$u' = u,$$

note that $1 \leq u' < t' < s' < r' \leq n+3$. Therefore, r', s', t' and u' are distinct, they belong to $[n+3]$, and

$$\begin{aligned} r'+s'+t'+u' &= r+3+s+2+t+1+u \\ &= 2n+2+6 \\ &= 2n+8. \end{aligned}$$

So we have $\mu = (r', s', t', u') \in P(2n{+}8, \text{exactly } 4 \text{ distinct parts in } [n+3])$. The reverse map is simple to build.

□

Example 3.5.7. *For $m = 4$ and $n = 1, 2, 3$, we have the partitions shown in Table 10.*

$n=1$	$P^{4[s]}_{[s]}(8,4)$	$P(10, \text{exactly } 4 \text{ distinct parts in } [4])$
	$(1,1,1,1,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(4,3,2,1)$
$n=2$	$P^{4[s]}_{[s]}(18,4)$	$P(12, \text{exactly } 4 \text{ distinct parts in } [5])$
	$(2,2,2,2,1,1,1,1,\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	$(5,4,2,1)$
$n=3$	$P^{4[s]}_{[s]}(32,4)$	$P(14, \text{exactly } 4 \text{ distinct parts in } [6])$
	$(3,3,3,3,2,2,2,2,1,1,1,1,\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1})$	$(6,5,2,1)$
	$(3,3,3,3,2,2,2,2,1,1,1,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{1})$	$(6,4,3,1)$
	$(3,3,3,3,2,2,2,2,1,1,1,1,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$	$(5,4,3,2)$

Table 10: Table for Example 3.5.7

3.6 Some considerations about the k^{th} diagonals for $k \geq 5$

Some results of the previous sections involving partitions counted by $p^{m[s]}_{[s]}(n,k)$ were very similar for $k = 2, 3,$ and 4. At this point of the text, we may already believe that those facts might be extensible for other values of k, or maybe even any value of $k \geq 2$.

Indeed, a very simple fact that can be observed anywhere, in every table of functions $f^m_*(q)$, is that every k^{th}-diagonal seems to be formed by non-constant symmetrical list of integers, besides a few zeros between these lists. Both these results were presented in the previous sections of this chapter and can be generalized for any value of k, as it is set below. The proofs are similar to those from the previous sections and completely adaptable, thus we chose to omit them.

Theorem 3.6.1. *Given $k \geq 2$ and $m \geq k$, for all $n \geq 1$ and $1 \leq i \leq$*

$(m-k)n + 2k - 1$ *we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + k - i, k\right) = 0.$$

Theorem 3.6.2. *Given $k \geq 2$ and $m \geq k$, for all $n \geq 1$ and $0 \leq i \leq \left\lfloor \frac{k(n-1)}{2} \right\rfloor$ we have*

$$p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + k + i, k\right) = p_{[s]}^{m[s]}\left(\frac{m(n^2+n)}{2} + kn - i, k\right)$$

3.7 The particular case of $m = 1$

By making $m = 1$ in (3.1) we get the function

$$f_*^1(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n^2+n}{2}}}{(q;q)_n}.$$

Remark 3.7.1. *Do not mix up functions $f_*^1(q)$ and $f_1^*(q)$. The first one is $f_*^m(q)$ with $m = 1$ and the second is $f_*^m(q)$ with $m = 2$ from Chapter 2.*

Function $f_*^1(q)$ table (see Table 28, page 133) has some peculiar sequences in its diagonals. That is, its diagonals seem not to behave in the same way as the ones from the tables of other functions $f_*^m(q)$. Therefore, instead of looking to its diagonals, an easy fact to observe in Table 28 is that the values of its columns become fixed below certain cells. This sequence of fixed values can be interpreted in terms of partitions into distinct parts, as described by the following theorem.

Theorem 3.7.2. *For all $n \geq 0$ and $i \geq 0$ we have*

$$p_{[s]}^{1[s]}(2n+3+i, n+i) = p_d(n+3, \text{with } 1 \text{ as a part}). \tag{3.23}$$

Proof. We begin by establishing a bijection between sets $P_{[s]}^{1[s]}(2n+3+i, n+i)$ and $P_d(n+3, \text{with } 1 \text{ as a part})$ for $i = 0$.

Let us consider the Ferrers graph of a partition counted by $p_{[s]}^{1[s]}(2n+3, n)$. Since it has all the parts from 1 to s with no gaps, we can separate, from the top to the bottom, a triangle of size s. Let us call it T_s.

We rearrange the remaining dots of the graph by aligning them horizontally to the top. Note that there are exactly n remaining parts less than or equal to s. We subtract one unit from each one of these n parts and conjugate what is left, getting a partition into at most $s-1$ parts less than or equal to n.

By joining this new partition aside to the triangle T_s, we get a partition of $n+3$ into distinct parts having 1 as a part.

Conversely, given a partition counted by $p_d(n+3,\ \textit{with } 1 \textit{ as a part})$, we separate the greatest possible triangle of its Ferrers graph, saying, T_j. Then we rearrange the remaining dots vertically on the left, having a partition V. We conjugate V, and add n dots to it as it follows: create enough parts of size 1 in such a way as to have a total amount of n parts; distribute the remaining dots one by one through the parts, starting by the greatest one and noting that this one has to be at most of size j.

Looking to V by columns, we slide its first column j units down; its second column $j-1$ units down; and so on. In general, we slide the k^{th} column $j-k+1$ units down so that the triangle T_j fits on the left hand side of V. By joining T_j and V aside, we get a partition of $2n+3$ into parts ranging from 1 to s with no gaps and multiplicity 1, and n other parts from 1 to s.

In order to have the result for every $i \geq 0$, note that an easy bijection between $P^{1[s]}_{[s]}(2n+3, n)$ and $P^{1[s]}_{[s]}(2n+3+i, n+i)$ adds (conversely, subtracts) i parts of size 1.

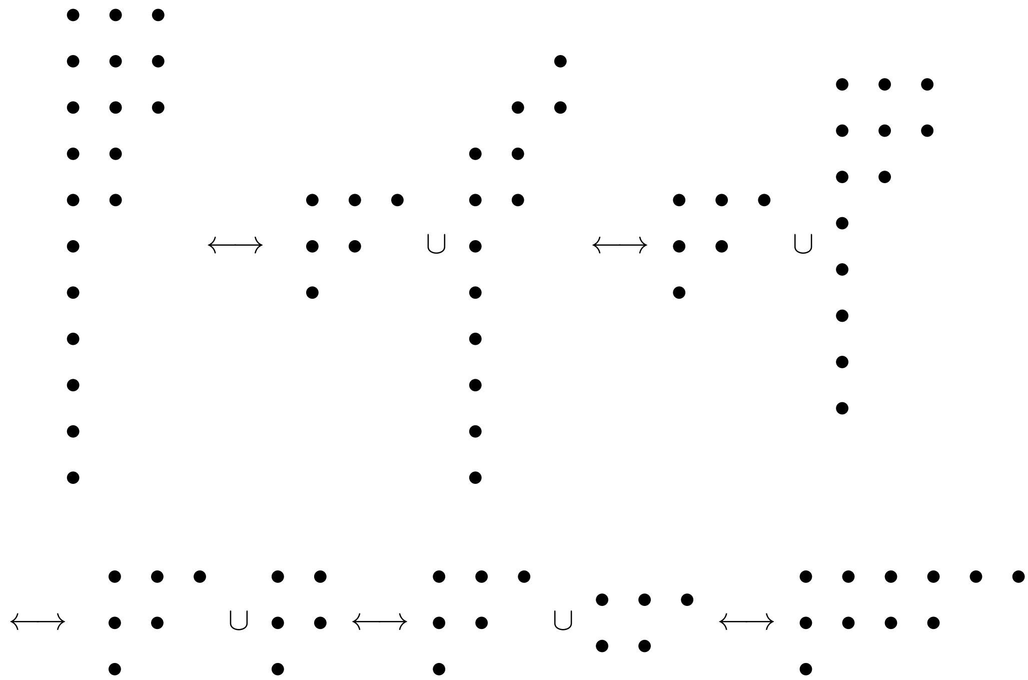
$\square$

Example 3.7.3. *For $n = 8$, $i = 4$ and $i = 0$, we have the partitions shown in Table 11.*

$P^{1[s]}_{[s]}(23, 12)$	$P^{1[s]}_{[s]}(19, 8)$	$P_d(11, \text{with 1 as a part})$
$(4,3,2,1,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(4,3,2,1,\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(5,3,2,1)$
$(3,2,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(3,2,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(6,4,1)$
$(3,2,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(3,2,1,\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(7,3,1)$
$(3,2,1,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(3,2,1,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$	$(8,2,1)$
$(2,1,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	$(2,1,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$	$(10,1)$

Table 11: Table for Example 3.7.3

In order to illustrate the bijection described in the proof of Theorem 3.7.2 we take the partition $(3,2,1,\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \in P^{1[s]}_{[s]}(19,8)$ and describe the map step by step until we get the partition $(6,4,1) \in P_d(11, \textit{with } 1 \textit{ as a part})$.



Remark 3.7.4. *Observe that statement (3.23) in Theorem 3.7.2 can also be written as:*

$$p^{1[s]}_{[s]}(2n+3+i, n+i) = p_d(n+2, \text{ with no part of size } 1).$$

3.8 Final words

According to the information contained in this chapter, in every table, for any value of $m \geq 2$, the results we proved tell us exactly the number of partitions of n counted by $p^{m[s]}_{[s]}(n,k)$, for $k = 2, 3, 4$.

The k^{th} diagonal of any table generated by function $f^m_*(q) = \sum_{n=0}^{\infty} \dfrac{q^{\frac{m(n^2+n)}{2}}}{(q;q)_n}$ alternates a sequence of zeros with a symmetrically increasing and decreasing list of non-vanishing numbers. For the special cases of $k = 2, 3,$ and 4, these lists may be counted in an easier way when seen as other type of partitions. The influence of m in such a list is restricted to the size of the sequence of zeros, while, given a fixed k, the non-vanishing sequences are the same in all tables.

Every value of n can be written in some way given by the propositions and theorems of this thesis. The actual number of partitions of such n into parts ranging from 1 to s, with no gaps and multiplicity m, and k other parts from 1 to s, may depend on the combination of few results.

Part II

CHAPTER 4

A new approach for two-line matrix representations

4.1 Introduction

In this part of the work we deal with the matrix representation for integer partitions in a different way. As it has been pointed out in [SMR11], each two-line matrix representation for the partitions of n can be associated to a lattice path through the Cartesian plane, connecting the origin $(0,0)$ to the line $x+y=n$. Although a description of a possible path is rapidly done in [SMR11], in this thesis we explore this approach more deeply.

In the following pages we describe the lattice paths induced by the two-line matrix representation and associate different sets of integer partitions to them. In particular, we deal with unrestricted partitions, partitions counted by the 1^{st} and 2^{nd} Rogers-Ramanujan Identities, and those generated by the Mock Theta Functions $f_*^5(q) = \sum_{n=0}^{\infty} \dfrac{q^{\frac{5(n^2+n)}{2}}}{(q;q)_n}$, studied in Chapter 3, and $T_1^*(q) = \sum_{n=0}^{\infty} \dfrac{q^{n(n+1)}(-q^2,q^2)_n}{(q,q^2)_{n+1}}$. The main results contained in this chapter have already been accepted for publication at the Bulletin of the Brazilian Mathematical Society, New Series (see [SM18]).

4.2 The Path Procedure

Let us consider the unrestricted integer partitions, which have at least three different two-line matrix representation. We choose the following one.

Theorem 4.2.1 (Theorem 4.1, [SMR11]). *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix}, \tag{4.1}$$

where $c_s = 0$, $c_t = c_{t+1} + d_{t+1}$, and the sum of all entries is equal to n.

Observe that for every matrix the sum of the entries of each column gives the respective part of the original partition. That is, the k^{th} part of the associated partition of n is equal to $c_k + d_k$. For $n = 6$ we have the following.

Example 4.2.2. *For $n = 6$ we have $p(6) = 11$, and so there are 11 matrices satisfying Theorem 4.2.1, as it can be seen in Table 12.*

$P(6)$	Matrix of type (4.1)	$P(6)$	Matrix of type (4.1)
$(1,1,1,1,1,1)$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$(3,3)$	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$
$(2,1,1,1,1)$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$(4,1,1)$	$\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$
$(2,2,1,1)$	$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	$(4,2)$	$\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$
$(2,2,2)$	$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$(5,1)$	$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$
$(3,1,1,1)$	$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$	(6)	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$
$(3,2,1)$	$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$		

Table 12: Table for Example 4.2.2

We associate each matrix of Theorem 4.2.1 (and thus each partition of n) to a path built through the Cartesian plane, connecting the point $P = (\sum_{i=1}^{s} d_i, \sum_{i=1}^{s} c_i)$ in the line $x + y = n$ to the origin $(0,0)$. We choose the second line of the matrix to be associated to the x-axis, and the first line to be associated to the y-axis.

The path consists of shifting c_s units down, d_s units to the left, then c_{s-1} units down, d_{s-1} units to the left, and so on, ending with d_1 units to the left. So we create a path which connects the following points:

$$P = \left(\sum_{i=1}^{s} d_i, \sum_{i=1}^{s} c_i\right) \to \left(\sum_{i=1}^{s} d_i, \sum_{i=1}^{s-1} c_i\right) \to \left(\sum_{i=1}^{s-1} d_i, \sum_{i=1}^{s-1} c_i\right) \to$$

$$\to \left(\sum_{i=1}^{s-1} d_i, \sum_{i=1}^{s-2} c_i\right) \to \left(\sum_{i=1}^{s-2} d_i, \sum_{i=1}^{s-2} c_i\right) \to \ldots \to$$

$$\to (d_1 + d_2, c_1 + c_2) \to (d_1 + d_2, c_1) \to (d_1, c_1) \to (d_1, 0) \to (0, 0).$$

Example 4.2.3. *For $n = 6$ we take the matrix*

$$M = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

associated to the partition $(2, 2, 1, 1)$. The path associated to M connects the points $(2, 4)$, $(1, 4)$, $(1, 3)$, $(1, 2)$, $(0, 2)$, and $(0, 0)$, as shown in Figure 4.1 below.

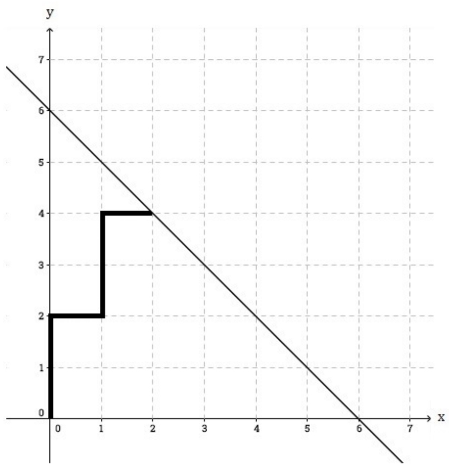


Figure 4.1: Illustration for Example 4.2.3

Remark 4.2.4. *Note that* $\left(\sum_{i=1}^{s} d_i, \sum_{i=1}^{s-1} c_i\right) = P$, *since* $c_s = 0$.

Remark 4.2.5. *According to the conditions satisfied by the entries of each matrix of type (4.1), note that every down move is necessarily at least as large as the previous left move.*

Now we reflect the path through the line $x + y = n$ and create the parts of a new partition by taking hooks of the following sizes.

$$2(n - d_1) - 1;$$

$$2(n - d_1 - 1) - 1;$$

$$\dots$$

$$2(n - d_1 - (c_1 - 1)) - 1;$$
$$2(n - d_1 - d_2 - c_1) - 1;$$
$$2(n - d_1 - d_2 - c_1 - 1) - 1;$$

$$\dots$$

$$2(n - d_1 - d_2 - c_1 - (c_2 - 1)) - 1;$$

$$\dots$$

$$2(n - d_1 - \cdots - d_k - c_1 - \ldots - c_{k-1}) - 1;$$
$$2(n - d_1 - \cdots - d_k - c_1 - \ldots - c_{k-1} - 1) - 1;$$

$$\dots$$

$$2(n - d_1 - \cdots - d_k - c_1 - \ldots - c_{k-1} - (c_k - 1)) - 1;$$

$$\dots$$

$$2(n - d_1 - \cdots - d_s - c_1 - \ldots - c_{s-1} + 1) - 1.$$

Based on the construction above, we get a partition of some integer m into distinct odd parts greater than 1 and less than or equal to $2n - 1$, since the matrix representation of an original unrestricted partition of n has entry $d_1 \geq 0$. Therefore, $m \leq n^2 - 1$.

Example 4.2.6. *For the matrix*

$$M = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

associated to the partition $(2, 2, 1, 1)$ *of* $n = 6$*, the hooks given by the reflection of the path through the line* $x + y = 6$ *provide the parts* $11, 9, 5,$ *and* 3*, that is, the partition* $\mu = (11, 9, 5, 3)$ *of* $m = 28$*. Figure 4.2 below helps to understand the process.*

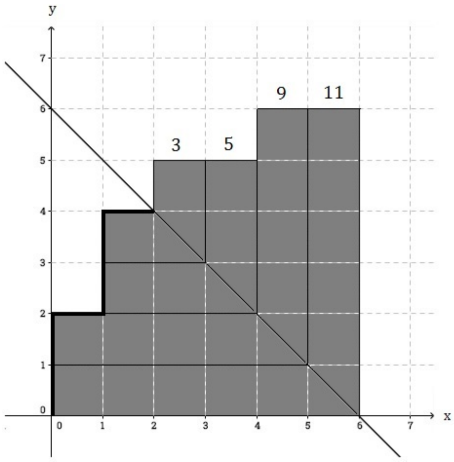


Figure 4.2: Illustration for Example 4.2.6

Matrix for $n = 6$	Partition into distinct odd parts	Matrix for $n = 6$	Partition into distinct odd parts
$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$(11, 9, 7, 5, 3)$	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	$(11, 9, 7)$
$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$(9, 7, 5, 3)$	$\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$	$(5, 3)$
$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	$(11, 9, 5, 3)$	$\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$	$(7, 5)$
$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$(11, 9, 7, 5)$	$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$	(3)
$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$	$(7, 5, 3)$	$\begin{pmatrix} 0 \\ 6 \end{pmatrix}$	$\emptyset$
$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$(9, 7, 3)$		

Table 13: Table for Example 4.2.7

Example 4.2.7. *By considering all the matrices in Example 4.9.3 we get the partitions into distinct odd parts contained in Table 13.*

It is clear that each matrix associated to a different partition of n generates a different path from the line $x+y = n$ to the origin $(0, 0)$, and therefore a different partition into distinct odd parts. However, there are matrices associated to partitions of different integers n that induce different paths but same hooks, generating the same partition into distinct odd parts. This fact is illustrated in the next example.

Example 4.2.8. *Let us take the partition* $(4, 1, 1)$ *of* 6 *and the partition* $(2, 1, 1)$ *of* 4*. The matrices associated to them are, respectively,*

$$\begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The paths that these matrices origin are different, although both paths induce the same hooks and, therefore, the same partition into distinct odd parts. Figure 4.3 helps to understand this example.

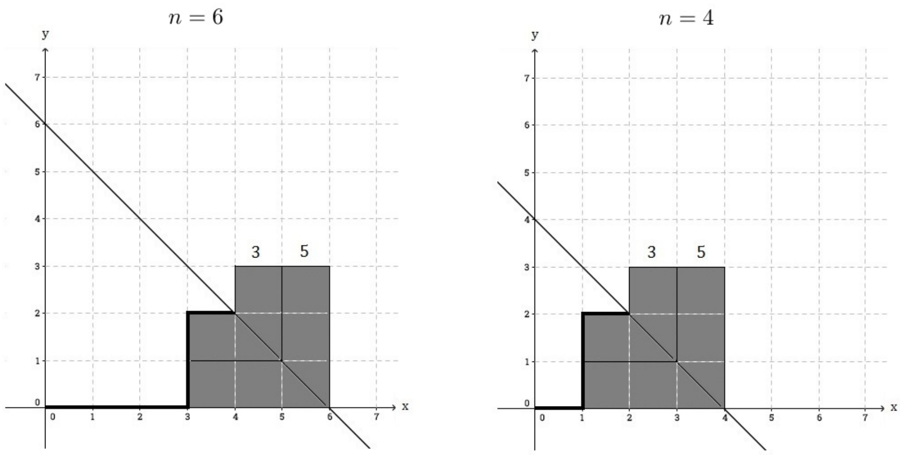


Figure 4.3: Illustration for Example 4.2.8

The fact in Example 4.2.8 occurs precisely because of the entry d_1 in both matrices. Since a step of size d_1 lies over the x-axis, it doesn't produce any hook and, therefore, doesn't contribute with any odd part to our new partition. In fact, once a partition of some m into distinct odd parts generated by the matrix representation of some partition of n appears for the first time, it will continue to appear for any larger value of n, since n always has a partition whose matrix representation only changes the entry d_1 from the matrix associated to a partition of $n - 1$.

In order to simplify our writing, we give a name to this process of building new partitions into distinct odd parts from the usual partitions of n. This is set in the following definition.

Definition 4.2.9. We call the process described above the *Path Procedure.* More precisely, from now on, we use the denomination Path Procedure when referring to the process of building partitions into distinct odd parts from the two-line matrix representation of a specific set of integer partitions. The construction consists of connecting the point $P = (\sum_{i=1}^{s} d_i, \sum_{i=1}^{s} c_i)$ in the line $x + y = n$ to the origin $(0, 0)$ by shifting successively c_i units down and d_i units to the left, for i from s to 1, reflecting the path through the line $x + y = n$, and taking hooks of odd sizes which will constitute the parts of the new partition.

The previous considerations motivate questions like:

Question 1. Which integers are being partitioned into distinct odd parts by the Path Procedure?

Question 2. Given a partition into distinct odd parts, is it generated by an unrestricted partition, according to our construction?

Question 3. In case of an affirmative answer to Question 2, how to recover the matrix which induced the given partition into distinct odd parts?

Remark 4.2.10. *Question 3 might be inaccurate, since we have seen that the matrix which induced a given partition into distinct odd parts isn't in fact unique. However, except for the entry d_1, the matrix we are looking for is well determined.*

We intend to answer the questions above in the following sections.

4.3 Recovering a matrix

We start with some general observations which will lead us to a map from some specific partitions into distinct odd parts to the matrix representation from Theorem 4.2.1.

Remark 4.3.1. *The j^{th} odd integer is $2j - 1$.*

Remark 4.3.2. *Given a partition into distinct odd parts $\lambda = (2\lambda_i+1, 2\lambda_{i-1}+1, \ldots, 2\lambda_2 + 1, 2\lambda_1 + 1)$, with $\lambda_j > \lambda_{j-1}$, observe that there are $\lambda_j - \lambda_{j-1} - 1$ distinct odd integers between $2\lambda_j + 1$ and $2\lambda_{j-1} + 1$.*

Let λ be a partition obtained from the matrix representation of an unrestricted partition of n, and call its smallest part $2\lambda_1 + 1$. According to the Path Procedure, the hook generated by the path from the line $x+y = n$ to the origin $(0,0)$ which gave origin to the smallest part $2\lambda_1+1$ was obtained from a left move of λ_1 units. So, $c_s = 0$ and $d_s = \lambda_1$. Necessarily, $c_{s-1} = \lambda_1$, which means that a down move of λ_1 units generates a sequence of λ_1 consecutive odd parts, which are $2\lambda_1+1, 2(\lambda_1+1)+1, 2(\lambda_1+2)+1, \ldots, 2(2\lambda_1-1)+1$.

Now there are two possibilities:

(a) If $2(2\lambda_1)+1$ is a part of the partition λ, it means that it was generated by a down move, with no left move before it. In this case, $d_{s-1} = 0$ and $c_{s-2} = \lambda_1$, which means that again there is a sequence of λ_1 consecutive odd parts $2(2\lambda_1)+1, 2(2\lambda_1+1)+1, \ldots, 2(3\lambda_1-1)-1$.

(b) If $2(2\lambda_1)+1$ is not a part of the partition λ, then a left move is allowed. In this case, let us call $2\lambda_2+1$ the first part that appears in λ after $2(2\lambda_1-1)+1$. So $d_{s-1} = \lambda_2-(2\lambda_1-1)-1 = \lambda_2-2\lambda_1$ and $c_{s-2} = \lambda_2-\lambda_1$.

By repeating an analogous argument until the last part of the partition λ, we obtain the values of c_j for all $j \geq 1$, and d_j for all $j \geq 2$. The value of d_1 is the size of the last left move of the path. Observe that it does not generate any odd part, and so it can be as large as we want. In other words, the entry d_1 of the matrix does not affect the size of the partition into distinct odd parts; it only affects the size of n in the original unrestricted partition.

Now, having a matrix representation which originated the partition into distinct odd parts (which turns out to be precisely the representation given by Theorem 4.2.1), by summing its columns we get the original unrestricted partition of n that induced the partition into distinct odd parts.

Example 4.3.3. *Let us take the partition* $(11,9,5,3)$. *By considering all the positive odd integers we see that, before the part* 3, *the odd integer* 1 *is omitted. This means that the path between* $x+y=n$ *and* $(0,0)$ *starts with a left move of size* 1, *which implies* $d_s = 1$. *Consequently,* $c_{s-1} = 1$. *This implies a down move of size 1, which generates the part* 3.

As the next part is the consecutive odd number 5, *we have one more down move of size* 1, *with no left move in between. This means that* $d_{s-1} = 0$ *and* $c_{s-2} = 1$.

After that, the part 7 *is omitted because of a left move of size* 1, *which means that* $d_{s-2} = 1$. *Necessarily* $c_{s-3} = 2$, *and so we have a down move of size* 2, *which generates the last two parts,* 11 *and* 9.

By this construction, $s-3=1$ *and so* $s=4$. *The entry* d_1 *has the size of the last left move necessary to reach the origin* $(0,0)$.

As it is illustrated in Example 4.3.3, the first line of the matrix associated to some partition of n expresses the size of the sequences of consecutive odd parts less than or equal to $2n-1$ that appear in the partition. The second line expresses the size of the sequence of consecutive odd numbers that are not parts of the partition.

By recursion it is easy to note that $c_i = \sum_{j=i+1}^{s} d_j$, and so we can conclude that, given an increasing finite sequence of distinct odd integers, it is a partition generated by a matrix, according to Theorem 4.2.1, if the size of any subsequence of consecutive odd numbers is either exactly the number of smaller odd integers that were omitted before the subsequence started or, in case there is some entry $d_i = 0$, a greater multiple of it. Also, note that the part 1 does never appear but the smallest part may be any odd integer greater than or equal to 3.

4.4 The Path Procedure applied to unrestricted partitions

In this section we set some results obtained from the Path Procedure applied to unrestricted partitions of n.

Just to keep how we got here in mind, let us recall what we understand as the Path Procedure: given any value of n, we consider the matrix representation of its unrestricted partitions, according to Theroem 4.2.1, and associate each one of them to a path through the Cartesian plane, connecting the line $x+y=n$ to the origin $(0,0)$. Then, by reflecting this path through the line $x+y=n$, we build hooks of odd sizes which constitute distinct odd parts. By adding up these new parts we get a partition of some integer $m < n^2$ into distinct odd parts, whose size of any subsequence of consecutive odd numbers that are parts of the partition is either exactly the number of smaller odd integers that were omitted before the subsequence started or a greater multiple of it.

For each value of n we count how many times each integer m appears in the construction described above. This data is organized in squares of size $n \times n$ (or tables with n^2 cells), as illustrated in Figures 4.4, 4.5, and 4.6 below.

The figures illustrate the distribution of frequencies of partitions of m in squares of size $n \times n$, induced by the partitions of n. Each cell contains how

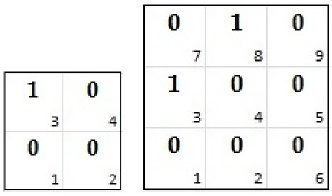


Figure 4.4: $n \times n$ squares for $n = 2$ and 3

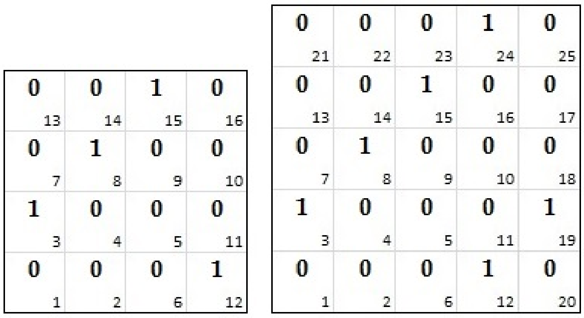


Figure 4.5: $n \times n$ squares for $n = 4$ and 5

many partitions of m (m is indicated in the right down side of the cell) are generated by the Path Procedure applied to partitions of n.

Another representation for the distribution of frequencies is presented in Figure 4.7. We organize in columns the same frequencies contained in the cells of the figures above. We show the case for $n = 8$.

Remark 4.4.1. *Observe that the column for* $m = 48$ *has height* 2*. This happens because the number* 48 *is generated by two different partitions into distinct odd parts greater than* 1 *and less than or equal to* $2 \cdot 8 - 1 = 15$*. That is, the partitions* $(13, 11, 9, 7, 5, 3)$ *and* $(15, 13, 11, 9)$*, coming from the original partitions of* 8*,* $(2, 1, 1, 1, 1, 1, 1)$ *and* $(4, 4)$*, respectively.*

Observe that, as n gets larger, the number of partitions of $m < n^2$ increases quickly. That is, the number of partitions of $m < n^2$ grows faster than n. So, the frequency columns can get very high. However, these frequencies clearly cannot grow indefinitely; as n gets larger, new distinct odd parts are allowed but they will not be used in every partition. At some point, every column reaches a limited height.

By considering all the remarks and restrictions in the process of getting new partitions, the Path Procedure motivates the following definition.

Definition 4.4.2. Let $P_{od}(m)$ be the set of partitions of m into distinct odd parts greater than 1 whose size of any subsequence of consecutive odd integers

0 (421)	0 (422)	1 (423)	1 (424)	0 (425)	0 (426)	1 (427)	0 (428)	0 (429)	0 (430)	1 (431)	1 (432)	0 (433)	0 (434)	1 (435)	0 (436)	0 (437)	0 (438)	0 (439)	1 (440)	0 (441)
0 (381)	0 (382)	5 (383)	5 (384)	0 (385)	0 (386)	5 (387)	4 (388)	0 (389)	0 (390)	2 (391)	6 (392)	0 (393)	0 (394)	4 (395)	5 (396)	0 (397)	0 (398)	3 (399)	3 (400)	0 (401)
7 (343)	7 (344)	0 (345)	0 (346)	8 (347)	9 (348)	0 (349)	0 (350)	7 (351)	7 (352)	0 (353)	0 (354)	6 (355)	4 (356)	0 (357)	0 (358)	7 (359)	6 (360)	0 (361)	0 (362)	0 (402)
8 (307)	6 (308)	0 (309)	0 (310)	5 (311)	10 (312)	0 (313)	0 (314)	9 (315)	8 (316)	0 (317)	0 (318)	8 (319)	7 (320)	0 (321)	0 (322)	7 (323)	8 (324)	0 (325)	7 (363)	2 (403)
0 (273)	0 (274)	4 (275)	6 (276)	0 (277)	0 (278)	8 (279)	5 (280)	0 (281)	0 (282)	7 (283)	6 (284)	0 (285)	0 (286)	6 (287)	8 (288)	0 (289)	0 (290)	0 (326)	6 (364)	2 (404)
0 (241)	0 (242)	7 (243)	4 (244)	0 (245)	0 (246)	4 (247)	5 (248)	0 (249)	0 (250)	4 (251)	8 (252)	0 (253)	0 (254)	6 (255)	5 (256)	0 (257)	9 (291)	7 (327)	0 (365)	0 (405)
5 (211)	2 (212)	0 (213)	0 (214)	3 (215)	5 (216)	0 (217)	0 (218)	5 (219)	4 (220)	0 (221)	0 (222)	4 (223)	3 (224)	0 (225)	0 (226)	0 (258)	6 (292)	8 (328)	0 (366)	0 (406)
4 (183)	4 (184)	0 (185)	0 (186)	3 (187)	3 (188)	0 (189)	0 (190)	3 (191)	6 (192)	0 (193)	0 (194)	4 (195)	3 (196)	0 (197)	5 (227)	6 (259)	0 (293)	0 (329)	5 (367)	3 (407)
0 (157)	0 (158)	3 (159)	2 (160)	0 (161)	0 (162)	4 (163)	2 (164)	0 (165)	0 (166)	3 (167)	4 (168)	0 (169)	0 (170)	0 (198)	4 (228)	5 (260)	0 (294)	0 (330)	6 (368)	4 (408)
0 (133)	0 (134)	2 (135)	2 (136)	0 (137)	0 (138)	2 (139)	3 (140)	0 (141)	0 (142)	2 (143)	4 (144)	0 (145)	4 (171)	2 (199)	0 (229)	0 (261)	7 (295)	8 (331)	0 (369)	0 (409)
2 (111)	1 (112)	0 (113)	0 (114)	2 (115)	1 (116)	0 (117)	0 (118)	1 (119)	2 (120)	0 (121)	0 (122)	0 (146)	3 (172)	3 (200)	0 (230)	0 (262)	6 (296)	8 (332)	0 (370)	0 (410)
1 (91)	2 (92)	0 (93)	0 (94)	0 (95)	2 (96)	0 (97)	0 (98)	2 (99)	1 (100)	0 (101)	4 (123)	4 (147)	0 (173)	0 (201)	5 (231)	4 (263)	0 (297)	0 (333)	6 (371)	2 (411)
0 (73)	0 (74)	3 (75)	1 (76)	0 (77)	0 (78)	1 (79)	1 (80)	0 (81)	0 (82)	0 (102)	2 (124)	2 (148)	0 (174)	0 (202)	5 (232)	7 (264)	0 (298)	0 (334)	5 (372)	2 (412)
0 (57)	0 (58)	1 (59)	2 (60)	0 (61)	0 (62)	1 (63)	1 (64)	0 (65)	1 (83)	2 (103)	0 (125)	0 (149)	2 (175)	3 (203)	0 (233)	0 (265)	5 (299)	5 (335)	0 (373)	0 (413)
1 (43)	0 (44)	0 (45)	0 (46)	1 (47)	2 (48)	0 (49)	0 (50)	0 (66)	1 (84)	2 (104)	0 (126)	0 (150)	2 (176)	6 (204)	0 (234)	0 (266)	8 (300)	9 (336)	0 (374)	0 (414)
0 (31)	1 (32)	0 (33)	0 (34)	1 (35)	1 (36)	0 (37)	1 (51)	1 (67)	0 (85)	0 (105)	2 (127)	2 (151)	0 (177)	0 (205)	5 (235)	7 (267)	0 (301)	0 (337)	7 (375)	1 (415)
0 (21)	0 (22)	0 (23)	1 (24)	0 (25)	0 (26)	0 (38)	1 (52)	0 (68)	0 (86)	0 (106)	2 (128)	2 (152)	0 (178)	0 (206)	5 (236)	6 (268)	0 (302)	0 (338)	5 (376)	1 (416)
0 (13)	0 (14)	1 (15)	0 (16)	0 (17)	1 (27)	1 (39)	0 (53)	0 (69)	2 (87)	2 (107)	0 (129)	0 (153)	3 (179)	5 (207)	0 (237)	0 (269)	8 (303)	8 (339)	0 (377)	0 (417)
0 (7)	1 (8)	0 (9)	0 (10)	0 (18)	1 (28)	0 (40)	0 (54)	0 (70)	2 (88)	3 (108)	0 (130)	0 (154)	3 (180)	5 (208)	0 (238)	0 (270)	6 (304)	6 (340)	0 (378)	0 (418)
1 (3)	0 (4)	0 (5)	0 (11)	1 (19)	0 (29)	0 (41)	0 (55)	1 (71)	0 (89)	0 (109)	1 (131)	2 (155)	0 (181)	0 (209)	1 (239)	1 (271)	0 (305)	0 (341)	6 (379)	2 (419)
0 (1)	0 (2)	0 (6)	1 (12)	0 (20)	0 (30)	0 (42)	1 (56)	2 (72)	0 (90)	0 (110)	3 (132)	3 (156)	0 (182)	0 (210)	5 (240)	5 (272)	0 (306)	0 (342)	5 (380)	2 (420)

Figure 4.6: $n \times n$ square for $n = 21$

0 (57)	0 (58)	0 (59)	1 (60)	0 (61)	0 (62)	1 (63)	0 (64)
1 (43)	0 (44)	0 (45)	0 (46)	1 (47)	2 (48)	0 (49)	0 (50)
0 (31)	1 (32)	0 (33)	0 (34)	1 (35)	1 (36)	0 (37)	1 (51)
0 (21)	0 (22)	0 (23)	1 (24)	0 (25)	0 (26)	0 (38)	1 (52)
0 (13)	0 (14)	1 (15)	0 (16)	0 (17)	1 (27)	1 (39)	0 (53)
0 (7)	1 (8)	0 (9)	0 (10)	0 (18)	1 (28)	0 (40)	0 (54)
1 (3)	0 (4)	0 (5)	0 (11)	1 (19)	0 (29)	0 (41)	0 (55)
0 (1)	0 (2)	0 (6)	1 (12)	0 (20)	0 (30)	0 (42)	1 (56)

is either exactly the number of smaller odd integers that were omitted before the subsequence started or a greater multiple of it. Also, $\left|P_{od}(m)\right| = p_{od}(m)$.

Remark 4.4.3. *The index "od" stands for odd and distinct.*

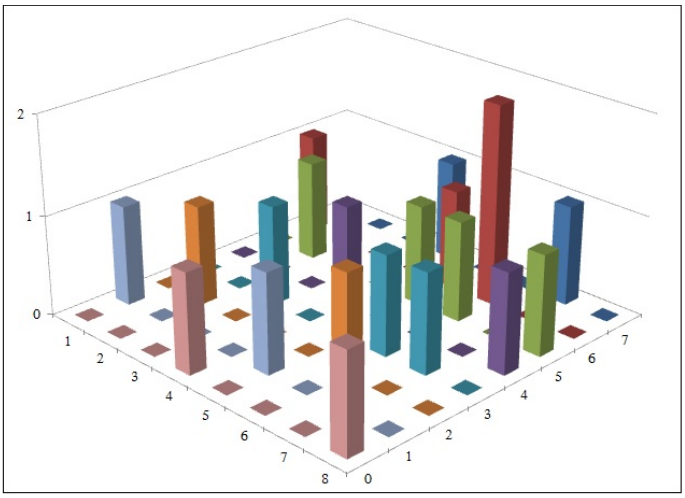


Figure 4.7: $n \times n$ square for $n = 8$ and its representation by columns

Remark 4.4.4. *If no odd integer is omitted after some subsequence of parts, we assume the number of omitted parts is zero, and the size of the following subsequence of parts will have the same size of the previous one. Note that the integer* 1 *is never a part, so the first subsequence of omitted parts has at least size* 1*.*

As an example, we take the cell of $m = 232$ in Figure 4.6, and so the greatest part allowed in any of its partitions is $2 \cdot 21 - 1 = 41$ (this is important for determining the value of the entry d_1 of the associated matrices). As this cell has the number 5 in it, this means $p_{od}(m) = 5$.

Example 4.4.5. *For $n = 21$ and $m = 232$ we have $p_{od}(m) = 5$, as it can be seen in Table 14.*

$P(21)$	Matrix representation	$P_{od}(232)$
$(7,2,2,1,1,1,1,1,1,1,1,1,1)$	$\begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$(31,29,27,25,21,19,17,15,13,11,9,7,5,3)$
$(9,5,3,3,1)$	$\begin{pmatrix} 5 & 3 & 3 & 1 & 0 \\ 4 & 2 & 0 & 2 & 1 \end{pmatrix}$	$(33,31,29,27,25,19,17,15,13,11,9,3)$
$(9,5,4,1,1,1)$	$\begin{pmatrix} 5 & 4 & 1 & 1 & 1 & 0 \\ 4 & 1 & 3 & 0 & 0 & 1 \end{pmatrix}$	$(33,31,29,27,25,21,19,17,15,7,5,3)$
$(9,6,2,1,1,1,1)$	$\begin{pmatrix} 6 & 2 & 1 & 1 & 1 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$(35,33,31,29,27,25,15,13,9,7,5,3)$
$(11,8,1,1)$	$\begin{pmatrix} 8 & 1 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{pmatrix}$	$(35,33,31,29,27,25,23,21,5,3)$

Table 14: Table for Example 4.4.5

With all the considerations made until now, we can assure that both sets of unrestricted partitions of n and of partitions defined by Definition 4.4.2 have the same cardinality.

In the following pages we set some results that characterize the numbers contained in squares like the ones in Figures 4.4, 4.5, 4.7, and 4.6. We begin with a characterization of the numbers m that appear with frequency zero in our squares.

Proposition 4.4.6. *For all $n \geq 0$ we have*

$$p_{od}(4n+1) = 0 = p_{od}(4n+2).$$

Proof. We prove that $p_{od}(4n+1) = 0$. The other equality has an analogous proof.

Let us suppose, by absurd, that $p_{od}(4n+1) \neq 0$. So, we would like to partition $4n+1$ into distinct odd parts greater than or equal to 3. For these parts to sum $4n+1$, there are the following possibilities:

(i) a number $\equiv 1 \pmod 4$ of parts, all of them $\equiv 1 \pmod 4$;

(ii) a number $\equiv 3 \pmod 4$ of parts, all of them $\equiv 3 \pmod 4$;

(iii) j parts $\equiv 1 \pmod 4$ and a number $\equiv j-1 \pmod 4$ of parts $\equiv 3 \pmod 4$;

(iv) j parts $\equiv 3 \pmod 4$ and a number $\equiv j-3 \pmod 4$ of parts $\equiv 1 \pmod 4$.

Let us begin with case (i), supposing we could partition $4n+1$ as

$$\lambda = \big(2(2k_j+1)-1, 2(2k_{j-1}+1)-1, \ldots, 2(2k_2+1)-1, 2(2k_1+1)-1\big),$$

with $j \equiv 1 \pmod 4$ and $k_i > k_{i-1}$.

In this case, observe that, as 1 cannot be a part of λ, the smallest odd part that can appear is 5. So, $d_s \geq 2$ and then $c_{s-1} = c_s + d_s = d_s \geq 2$. However, every c_i should be 1, since there are no consecutive odd parts in λ. As this is a contradiction, case (i) can never occur.

With an analogous argument, case (ii) is also not possible: if we could partition $4n+1$ as

$$\lambda = \big(2(2k_j+2)-1, 2(2k_{j-2}+2)-1, \ldots, 2(2k_2+2)-1, 2(2k_1+2)-1\big),$$

with $j \equiv 3 \pmod 4$ and $k_i > k_{i-1}$, observe that again there would not be consecutive odd parts in λ, which means $d_i \neq 0\ \forall i$ and $c_i = 1\ \forall i \neq s$. But then we get $c_{s-1} \geq 2$, and so this case cannot occur either.

In case (iii) observe that in a partition with j parts $\equiv 1 \pmod 4$ and a number $\equiv j-1 \pmod 4$ of parts $\equiv 3 \pmod 4$, essentially two configurations are possible: either all the parts are non-consecutive odd integers or there is at least a subsequence of two consecutive odd parts.

In the first configuration, with no consecutive odd parts, the same argument used in cases (i) and (ii) is valid: every $c_i\ \forall i \neq s$ should be 1 when they are actually not. So this configuration is not possible.

If there is a subsequence of consecutive odd parts, let us say it has size r and suppose it is the first subsequence of consecutive odd parts of the partition, when reading from the smallest to the largest part.

If this subsequence does not contain the smallest parts of the partition, again we have the problem of existing a t such that c_t must be 1 when it is actually not. So, let us suppose the r consecutive odd parts are the smallest parts of the partition, saying $2k_r - 1, 2(k_r - 1) - 1, \ldots, 2(k_r - r + 1) - 1$. In this situation, $c_s = 0$, $d_s = k_r - r$, and $c_{s-1} = k_r - r$.

If $k_r - r > r$ we have a contradiction. If $k_r - r < r$, then $d_{s-1} = 0$ and $c_{s-2} = k_r - r$. By repeating this argument (which has an end, since the sequence $2k_r - 1, 2(k_r - 1) - 1, \ldots, 2(k_r - r + 1) - 1$ is finite), we will find a t such that c_t must be 1 when it is actually not. If $k_r - r = r$, then $d_{s-1} > 0$ and $c_{s-2} \geq r$, and we may start a new subsequence of consecutive odd parts. By analysing the parity of $k_r - r$, r, and the others c_i and d_i, we conclude that no configuration allows j parts $\equiv 1 \pmod 4$ and a number $\equiv j - 1 \pmod 4$ of parts $\equiv 3 \pmod 4$. So, case (iii) does not happen either.

By noting that in case (iv) we may use the same argument as in case (iii), we conclude that $p_{od}(4n + 1) = 0$.

In order to show that $p_{od}(4n + 2) = 0$, we just observe that for distinct odd parts greater than or equal to 3 to sum $4n + 2$, there are the following possibilities:

(i) a number $\equiv 2 \pmod 4$ of parts, all of them $\equiv 1 \pmod 4$;

(ii) a number $\equiv 2 \pmod 4$ of parts, all of them $\equiv 3 \pmod 4$;

(iii) j parts $\equiv 1 \pmod 4$ and a number $\equiv j - 2 \pmod 4$ of parts $\equiv 3 \pmod 4$;

(*iv*) j parts $\equiv 3 \pmod 4$ and a number $\equiv j-2 \pmod 4$ of parts $\equiv 1 \pmod 4$.

The arguments in each case are the same we have already used and we omit them. So, $p_{od}(4n+1) = 0 = p_{od}(4n+2)$.

□

In order to proof the next results we have to set some definitions. First, recall that the smallest odd part of any partition generated by the Path Procedure has to be greater than 1, since the matrix representation for unrestricted partitions of n has entry $c_s = 0$, which means that the path from the line $x + y = n$ to the origin always starts with a left move of size d_s.

The size of the smallest part of our new partitions is determined by the size of d_s, which we call *the first missing subsequence* of the partition. We understand *the first missing subsequence* as the first sequence of consecutive odd integers that do not appear as parts of the partition, which are $2k_1 - 1, \ldots, 3, 1$. Let us call $d_s = k_1$.

After the first missing subsequence of k_1 consecutive odd integers we have *the first subsequence of consecutive odd parts* that compose the partition. Its size is determined by the entry c_{s-1} of the matrix. As $c_{s-1} = c_s + d_s = 0 + k_1 = k_1$, this means that the parts of the first sequence are $2(2k_1) - 1, \ldots, 2(k_1+2) - 1, 2(k_1+1) - 1$.

Some examples of partitions that have only the first missing subsequence and after it exactly one subsequence of consecutive odd parts are

$$(3), (7, 5), (11, 9, 7), (15, 13, 11, 9),$$

which are, respectively, partitions of 3, 12, 27, and 48 generated by the Path Procedure applied to unrestricted partitions of $n = 8$. In the general case, those partitions into odd parts have to be generated by a partition of n into two parts, since its matrix representation has two columns, which means a path with only one down move of size $c_1 = k_1$. The following result gives a general characterization of partitions like those.

Proposition 4.4.7. *The Path Procedure applied to the unrestricted partitions of n into exactly two parts generates partitions of $m = 3k_1^2$, with $1 \leq k_1 \leq \left\lfloor \dfrac{n}{2} \right\rfloor$, those being precisely all of the numbers whose partition has only the first missing subsequence and after it exactly one subsequence of consecutive odd parts.*

Proof. The parts of the first subsequence are $2(2k_1) - 1, \ldots, 2(k_1 + 2) - 1, 2(k_1 + 1) - 1$. As our partition has no other part, we may write

$$\begin{aligned} m &= 2(2k_1) - 1 + \cdots + 2(k_1 + 2) - 1 + 2(k_1 + 1) - 1 \\ &= \frac{(2(2k_1) - 1 + 2(k_1 + 1) - 1)(k_1)}{2} \\ &= 3k_1^2. \end{aligned}$$

Clearly k_1 has to be at most $\left\lfloor \frac{n}{2} \right\rfloor$, otherwise the greatest part $2(2k_1) - 1$ would exceed $2n - 1$.

□

Unrestricted partitions of n with more than two parts have a matrix representation into more than two columns, which means that each one of its entries c_t generates a different sequence of consecutive odd parts.

We call *the second missing subsequence* the second sequence of $d_{s-1} = k_2 \geq 0$ consecutive odd integers that do not appear as parts of the partition. They are $2(2k_1 + k_2) - 1, \ldots, 2(2k_1 + 2) - 1, 2(2k_1 + 1) - 1$. Observe that k_2 can actually be equal to 0: its size is determined by $\lambda_{s-1} = c_{s-1} + d_{s-1}$, where λ_{s-1} is part of an unrestricted partition of n, and if $\lambda_{s-1} = \lambda_s$ this means $d_{s-1} = 0$.

After the second missing subsequence of k_2 consecutive odd integers we have *the second subsequence of consecutive odd parts* that compose the partition, determined by the size of the entry c_{s-2} of the matrix. As $c_{s-2} = c_{s-1} + d_{s-1} = k_1 + k_2$, the parts of the second subsequence are $2(3k_1 + 2k_2) - 1, \ldots, 2(2k_1 + k_2 + 2) - 1, 2(2k_1 + k_2 + 1) - 1$. For example, $n = 8$ generates the partitions

$$(5, 3), (9, 7, 3), (11, 9, 7, 5), (13, 11, 9, 3), (15, 13, 11, 7, 5),$$

which are, respectively, partitions of 8, 19, 32, 36, and 51. A general characterization of partitions like those is given next.

Proposition 4.4.8. *The Path Procedure applied to the unrestricted partitions of n into exactly three parts generates partitions of $m = 8k_1^2 + k_2(8k_1 + 3k_2)$, with $1 \leq k_1 \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $0 \leq k_2 \leq \left\lfloor \frac{n - 3k_1}{2} \right\rfloor$.*

Proof. A partition having the first and the second subsequences of parts can be written as

$$\begin{aligned} m &= 2(3k_1+2k_2)-1+\cdots+2(2k_1+k_2+1)-1 \\ &+2(2k_1)-1+\cdots+2(k_1+1)-1 \\ &= \frac{(2(3k_1+2k_2)-1+2(2k_1+k_2+1)-1)(k_1+k_2)}{2} \\ &+ \frac{(2(2k_1)-1+2(k_1+1)-1)(k_1)}{2} \\ &= (5k_1+3k_2)(k_1+k_2)+3k_1^2 \\ &= 8k_1^2+k_2(8k_1+3k_2). \end{aligned}$$

The limitation for k_2 is obtained by observing that the greatest part $2(3k_1+2k_2)-1$ cannot exceed $2n-1$. So,

$$2(3k_1+2k_2)-1 \le 2n-1 \Longrightarrow 3k_1+2k_2 \le n \overset{k_2\in\mathbb{N}}{\Longrightarrow} k_2 \le \left\lfloor \frac{n-3k_1}{2} \right\rfloor.$$

When $k_2 = 0$ the second missing subsequence does not exist, and so the two subsequences of consecutive parts are actually seen as only one subsequence.

□

Now let us consider a partition into distinct odd parts having any number of *missing subsequences* and *subsequences of consecutive odd parts*, saying t and $t+1$, respectively. We call $k_1, k_2, \ldots, k_t$ the sizes of the missing subsequences and, consequently, the subsequence after a missing subsequence of size k_i has size $k_1+k_2+\cdots+k_i$.

Definition 4.4.9. (i) The sequence of d_{s-i+1} consecutive odd integers is called the i^{th} *missing subsequence* of the partition. We call $d_{s-i+1} = k_i$.

(ii) The sequence of c_{s-i} consecutive odd integers is called the i^{th} *subsequence of parts* of the partition, and has size $k_1+k_2+\cdots+k_i$.

The following lemma establishes the limits for each k_i.

Lemma 4.4.10. *The i^{th} missing subsequence of a partition into distinct odd parts, whose parts derive from the Path Procedure applied to unrestricted partitions of n, is at most*

$$\frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1})}{2}.$$

Proof. The sequence of all odd integers, those from the missing subsequences and from the subsequences of parts, cannot exceed $2n-1$. So, for example, when there are 3 missing subsequences and 3 subsequences of parts, it is necessary that

$$k_1 + k_2 + k_3 + (k_1) + (k_1 + k_2) + (k_1 + k_2 + k_3) \leq n,$$

which implies

$$k_3 \leq \frac{n - 4k_1 - 3k_2}{2}.$$

If there are i missing subsequences and i subsequences of parts, it is necessary that the sum of the sizes of the i missing subsequences and of the sizes of the i subsequences does not exceed n. This means

$$k_1 + k_2 + \cdots + k_i + (k_1) + (k_1 + k_2) + \cdots + (k_1 + k_2 + \cdots + k_i) \leq n,$$

which implies

$$(i+1)k_1 + ik_2 + \cdots + 4k_{i-2} + 3k_{i-1} + 2k_i \leq n.$$

So,

$$k_i \leq \frac{n - ((i+1)k_1 + ik_2 + \cdots + 4k_{i-2} + 3k_{i-1})}{2}.$$

□

Remark 4.4.11. *The number of missing subsequences depends on the size of each missing subsequence. We can have from just one to even $n-1$ missing subsequences. The first case is the one with $k_1 = \left\lfloor \frac{n}{2} \right\rfloor$; the second one is the case with $k_1 = 1$ and $k_i = 0$ for $2 \leq i \leq n-1$.*

Now we can extend our construction to a more general characterization of the numbers partitioned into distinct odd parts, whose parts derive from the Path Procedure applied to unrestricted partitions of n.

Theorem 4.4.12. *The partitions into distinct odd parts induced by the Path Procedure applied to unrestricted partitions of n are all of the form*

$$\sum_{\substack{i=1 \\ 1 \leq k_1 \leq \frac{n}{2} \\ i>1,\, 0 \leq k_i \leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1})}{2}}}^{t} [(t+2-i)^2 - 1]k_i^2$$
$$+ \sum_{\substack{i=1 \\ 1 \leq k_1 \leq \frac{n}{2} \\ i>1,\, 0 \leq k_i \leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1})}{2}}}^{t-1} k_i \sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1}, \tag{4.2}$$

where $1 \leq t \leq n-1$.

Proof. First of all, let us rewrite the expression (4.2) by expanding the sums.

$$
\begin{aligned}
& \sum_{\substack{i=1 \\ 1\leq k_1\leq \frac{n}{2} \\ i>1,\, 0\leq k_i\leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1})}{2}}}^{t} [(t+2-i)^2-1]k_i^2 \\
& + \sum_{\substack{i=1 \\ 1\leq k_1\leq \frac{n}{2} \\ i>1,\, 0\leq k_i\leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1})}{2}}}^{t-1} k_i \sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1} \\
= \; & ((t+1)^2-1)k_1^2 + (t^2-1)k_2^2 + \cdots + 8k_{t-1}^2 + 3k_t^2 \\
& + k_1(2(t+2)k_t + 2(2(t+2))k_{t-1} + \cdots + (t-2)(2(t+2))k_3 \\
& + (t-1)(2(t+2))k_2) \\
& + k_2(2(t+1)k_t + 2(2(t+1))k_{t-1} + \cdots + (t-2)(2(t+1))k_3) \\
& + \cdots \\
& + k_{t-3}(12k_t + 24k_{t-1} + 36k_{t-2}) \\
& + k_{t-2}(10k_t + 20k_{t-1}) \\
& + k_{t-1}(8k_t) \qquad (4.3)
\end{aligned}
$$

By considering a partition generated by the Path Procedure induced by the unrestricted partitions of n, suppose it has t subsequences of consecutive odd parts. Let us call k_1, k_2, $\ldots$, k_t the sizes of the missing subsequences. So, the sizes of the subsequences of consecutive odd parts are k_1, k_1+k_2, $k_1+k_2+k_3$, $\ldots$, $k_1+k_2+\cdots+k_t$ and the actual partition we are considering has the following subsequences of consecutive odd parts:

$$2((t+1)k_1+\cdots+3k_{t-1}+2k_t)-1,\ldots,2(tk_1+\cdots+2k_{t-1}+k_t+2)-1,$$
$$2(tk_1+\cdots+2k_{t-1}+k_t+1)-1;$$
$$2(tk_1+\cdots+3k_{t-2}+2k_{t-1})-1,\ldots,2((t-1)k_1+\cdots+2k_{t-2}+k_{t-1}+2)-1,$$
$$2((t-1)k_1+\cdots+2k_{t-2}+k_{t-1}+1)-1;$$
$$\cdots$$
$$2(4k_1+3k_2+2k_3)-1,\ldots 2(3k_1+2k_2+k_3+2)-1,2(3k_1+2k_2+k_3+1)-1;$$
$$2(3k_1+2k_2)-1,\ldots,2(2k_1+k_2+2)-1,2(2k_1+k_2+1)-1;$$

and

$$2(2k_1)-1,\ldots,2(k_1+2)-1,2(k_1+1)-1.$$

The sum of those parts equals

$$
\begin{aligned}
& \frac{(2((t+1)k_1+\cdots+3k_{t-1}+2k_t)-1}{2} \\
& + \frac{2(tk_1+\cdots+2k_{t-1}+k_t+1)-1)(k_1+k_2+\cdots+k_t)}{2} \\
& + \frac{(2(tk_1+\cdots+3k_{t-2}+2k_{t-1})-1}{2} \\
& + \frac{2((t-1)k_1+\cdots+2k_{t-2}+k_{t-1}+1)-1)(k_1+k_2+\cdots+k_{t-1})}{2} \\
& + \cdots \\
& + \frac{(2(4k_1+3k_2+2k_3)-1+2(3k_1+2k_2+k_3+1)-1)(k_1+k_2+k_3)}{2} \\
& + \frac{(2(3k_1+2k_2)-1+2(2k_1+k_2+1)-1)(k_1+k_2)}{2} \\
& + \frac{(2(2k_1)-1+2(k_1+1)-1)(k_1)}{2} \\
= \ & ((2t+1)k_1+\cdots+5k_{t-1}+3k_t)(k_1+k_2+\cdots+k_t) \\
& + ((2t-1)k_1+\cdots+5k_{t-2}+3k_{t-1})(k_1+k_2+\cdots+k_{t-1}) \\
& + \cdots \\
& + (7k_1+5k_2+3k_3)(k_1+k_2+k_3) \\
& + (5k_1+3k_2)(k_1+k_2) \\
& + (3k_1)(k_1).
\end{aligned}
$$

By rearranging the terms in the sum above we get expression (4.3), which proves the theorem.

□

4.5 The Path Procedure applied to partitions counted by the 1^{st} Rogers-Ramanujan Identity

Motivated by what we did in the previous sections, let us consider now the matrix representation for the partitions of n into 2-distinct parts, that is, partitions of n where the difference between parts is at least two. These partitions are counted by the right-hand side of the 1^{st} Rogers-Ramanujan Identity,

$$p(n|\text{parts} \equiv 1 \text{ or } 4 \pmod 5) = p(n|\text{2-distinct parts}).$$

Theorem 4.5.1 (Corollary 3.2, [SMR11]). *The number of partitions of n where the difference between parts is at least two is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix}, \tag{4.4}$$

where $c_s = 1$, $c_t = 2 + c_{t+1} + d_{t+1}$, and the sum of all entries is equal to n.

For every matrix the sum of the entries of each column gives the respective part of the original partition. That is, the k^{th} part is equal to $c_k + d_k$. As an example, we take $n = 6$ and show its partitions into 2-distinct parts with their associated matrices.

Example 4.5.2. *For $n = 6$ there are 3 partitions into 2-distinct parts, and so there are 3 matrices satisfying Theorem 4.5.1, as we can see in Table 15.*

$P(6\vert$ 2-distinct parts)	Matrix of type (4.4)
$(4, 2)$	$\begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}$
$(5, 1)$	$\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$
(6)	$\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

Table 15: Table for Example 4.5.2

As we have done in the previous sections, now we apply the Path Procedure to the matrices from Theorem 4.5.1, producing partitions into distinct odd parts.

Example 4.5.3. *The Path Procedure applied to the partitions of 6 from Example 4.5.2 generates the partitions into distinct odd parts contained in Table 16.*

Remark 4.5.4. *Note that now every integer partition generated by the Path Procedure has a part of size 1. This is easy to see since $c_s = 1$, which means that the first move in the path from $P = (\sum_{i=1}^{s} d_i, \sum_{i=1}^{s} c_i)$ to $(0, 0)$ is an exact down shift of size one.*

$P(6\|$ 2-distinct parts)	Matrix of type (4.4)	Partition into distinct odd parts
$(4,2)$	$\begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}$	$(11,9,7,5,1)$
$(5,1)$	$\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$	$(7,5,3,1)$
(6)	$\begin{pmatrix} 1 \\ 5 \end{pmatrix}$	(1)

Table 16: Table for Example 4.5.3

Figure 4.8 below illustrates the distribution of frequencies of partitions of m in a square of size 20×20, induced by the partitions of 20, according to Theorem 4.5.1. Each cell contains how many partitions of m (indicated in the right down side of the cell) are generated by the matrix representation of the partitions of 20.

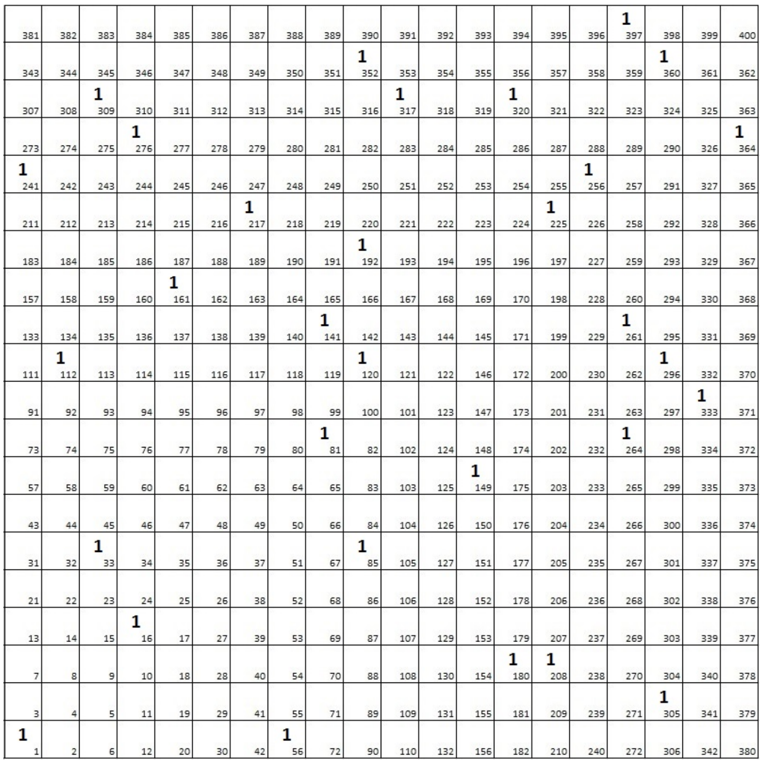


Figure 4.8: $n \times n$ square for $n = 20$

Similar to Definition 4.4.2, the Path Procedure applied to the matrices of Theorem 4.5.1 motivates the following one.

Definition 4.5.5. We call $P_{R(od)}(m)$ the set of partitions of m into distinct odd parts, always having 1 as their smallest part, and whose size of any subsequence of consecutive odd integers equals the size of the previous subsequence of consecutive odd integers that were omitted before the subsequence started plus the size of the previous subsequence of consecutive odd parts plus 2. Also, $\left|P_{R(od)}(m)\right| = p_{R(od)}(m)$.

Remark 4.5.6. *The letter R in the index refers to the 1^{st} Rogers-Ramanujan Identity.*

Remark 4.5.7. *If the part 3 appears after the part 1, this means that no odd integer was omitted. In this case, the next subsequence of odd parts will have size $1+2=3$. If again after the first four odd parts the next one is exactly the fifth odd integer, then the next subsequence of odd parts will have size $3+2=5$. In general, if no odd integer is omitted after some subsequence of parts, we assume the number of omitted parts is zero, and the size of the following subsequence of odd parts will be the size of the previous subsequence of consecutive odd parts plus 2.*

Proposition 4.5.8. *The hooks induced by the order 2×2 matrices of Theorem 4.5.1 associated to the partitions of some n constitute partitions of $3k^2+14k+16$, with $0 \le k \le \left\lfloor \dfrac{n}{2} \right\rfloor - 2$.*

Proof. Let λ_2 be the smallest part of some partition of n into 2-distinct parts whose associated matrix, according to Theorem 4.5.1, has order 2×2. Note that this means that $n = \lambda_1 + \lambda_2$, with $\lambda_1 \ge \lambda_2 + 2$. So, $1 \le \lambda_2 \le \left\lfloor \dfrac{n}{2} \right\rfloor - 1$. By making $k = \lambda_2 - 1$, we have $0 \le k \le \left\lfloor \dfrac{n}{2} \right\rfloor - 2$.

The path induced by the matrix begins with a down move of size 1, then a left move of size k, followed by a down move of size $2+1+k = k+3$, and ending with a left move of size $n-1-k-(k+3) = n-2k-4$.

When reflected through the line $x+y=n$ the path generates hooks of sizes $1, 2(k+2)-1, 2(k+3)-1, \ldots, 2(k+2+k+1)-1, 2(k+2+k+2)-1$. As the path ends with a left move of size $n-2k-4$, it lies on the x-axis and does not generate a hook.

So the new partition into odd parts is

$$(2(2k+4)-1, 2(2k+3)-1, \ldots, 2(k+3)-1, 2(k+2)-1, 1),$$

a partition of $3k^2+14k+16$.

□

Example 4.5.9. *For $n = 11$ there are 4 partitions into 2-distinct parts whose associated matrix has order 2×2, as shown in Table 17.*

$P(11\|$ 2-distinct parts)	Matrix of type (4.4)	k	$P_{R(od)}((3k+8)(k+2))$
$(7,4)$	$\begin{pmatrix} 6 & 1 \\ 1 & 3 \end{pmatrix}$	3	$(19,17,15,13,11,9,1)$
$(8,3)$	$\begin{pmatrix} 5 & 1 \\ 3 & 2 \end{pmatrix}$	2	$(15,13,11,9,7,1)$
$(9,2)$	$\begin{pmatrix} 4 & 1 \\ 5 & 1 \end{pmatrix}$	1	$(11,9,7,5,1)$
$(10,1)$	$\begin{pmatrix} 3 & 1 \\ 7 & 0 \end{pmatrix}$	0	$(7,5,3,1)$

Table 17: Table for Example 4.5.9

Proposition 4.5.10. *The hooks induced by the order 2×3 matrices of Theorem 4.5.1 with $d_2 = 0$ or 1, associated to the partitions of some n, constitute partitions of $m = \dfrac{4k^2+44k+117+3(-1)^k}{2}$, with $1 \le k \le \left\lfloor \dfrac{2n}{3} \right\rfloor - 5$. Moreover, these partitions appear for the first time when $n = \left\lfloor \dfrac{3k}{2} \right\rfloor + 8$.*

Proof. Let us separate the proof in two cases: (i) odd k; and (ii) even k. In fact, odd values of k are related to $d_2 = 0$ and even values of k to $d_2 = 1$.

(i) If k is odd, let us call $k = 2j-1$ with $j \ge 1$, and so $j = \dfrac{k+1}{2}$.

For odd k, $m = \dfrac{4k^2+44k+117+3(-1)^k}{2} = 2k^2+22k+57 = 8j^2 + 36j+37$, which can be partitioned as

$$\lambda = \big(2(3j+6)-1, 2(3j+5)-1, \ldots, 2(j+1)-1, 1\big).$$

Observe that $3j+6 = \left\lfloor \dfrac{3(2j-1)}{2} \right\rfloor + 8 = \left\lfloor \dfrac{3k}{2} \right\rfloor + 8$, and so the greatest part of λ equals $2\left(\left\lfloor \dfrac{3k}{2} \right\rfloor + 8\right) - 1$. Clearly, this partition is only possible for $n \ge \left\lfloor \dfrac{3k}{2} \right\rfloor + 8$. Also, for each $n \ge \left\lfloor \dfrac{3k}{2} \right\rfloor + 8$, partitions like λ are possible only for certain values of k:

$$\begin{aligned} n \geq \left\lfloor \frac{3k}{2} \right\rfloor + 8 &\implies \left\lfloor \frac{3(2j-1)}{2} \right\rfloor \leq n-8 \\ &\implies 3j-2 \leq n-8 \\ &\implies j \leq \frac{n-6}{3} \\ &\implies \frac{k+1}{2} \leq \frac{n-6}{3} \\ &\implies k \leq \frac{2(n-6)}{3} - 1, \end{aligned}$$

and as $k \in \mathbb{Z}$, $k \leq \left\lfloor \frac{2n-15}{3} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor - 5$.

It is not difficult to verify that the matrix associated to the partition above is

$$M_{(i)} = \begin{pmatrix} j+4 & j+2 & 1 \\ d_1 & 0 & j-1 \end{pmatrix}.$$

Recalling that $\left\lfloor \frac{k}{2} \right\rfloor = j-1$, we have

$$M_{(i)} = \begin{pmatrix} \left\lfloor \frac{k}{2} \right\rfloor + 5 & \left\lfloor \frac{k}{2} \right\rfloor + 3 & 1 \\ d_1 & 0 & \left\lfloor \frac{k}{2} \right\rfloor \end{pmatrix},$$

where $d_1 = n - \left(3\left\lfloor \frac{k}{2} \right\rfloor + 9\right) = n - \left(\left\lfloor \frac{3k}{2} \right\rfloor + 8\right) \geq 0$.

(ii) If k is even, let us call $k = 2j$ with $j \geq 1$, and so $j = \frac{k}{2}$.

For even k, $m = \frac{4k^2 + 44k + 117 + 3(-1)^k}{2} = 2k^2 + 22k + 60 = 8j^2 + 44j + 60$, which can be partitioned as

$$\lambda = \big(2(3j+8)-1, 2(3j+7)-1, \ldots, 2(2j+4)-1, 2(2j+2)-1+\ldots+2(j+1)-1+1\big).$$

Observe that $3j+8 = \left\lfloor \frac{3(2j)}{2} \right\rfloor + 8 = \left\lfloor \frac{3k}{2} \right\rfloor + 8$, and so the greatest part of λ equals $2\left(\left\lfloor \frac{3k}{2} \right\rfloor + 8\right) - 1$. Clearly, this partition is possible only for $n \geq \left\lfloor \frac{3k}{2} \right\rfloor + 8$. Also, for each $n \geq \left\lfloor \frac{3k}{2} \right\rfloor + 8$, partitions like λ are possible only for certain values of k:

$$\begin{aligned} n \geq \left\lfloor \frac{3k}{2} \right\rfloor + 8 \implies & \left\lfloor \frac{3(2j)}{2} \right\rfloor \leq n-8 \\ \implies & 3j \leq n-8 \\ \implies & j \leq \frac{n-8}{3} \\ \implies & \frac{k}{2} \leq \frac{n-6}{3} \\ \implies & k \leq \frac{2(n-8)}{3}, \end{aligned}$$

and as $k \in \mathbb{Z}$, $k \leq \left\lfloor \frac{2n-16}{3} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor - 5$.

The matrix associated to the partition above is

$$M_{(ii)} = \begin{pmatrix} j+5 & j+2 & 1 \\ d_1 & 1 & j-1 \end{pmatrix}.$$

Recalling $\left\lfloor \frac{k}{2} \right\rfloor = \frac{k}{2} = j$, we have

$$M_{(ii)} = \begin{pmatrix} \left\lfloor \frac{k}{2} \right\rfloor + 5 & \left\lfloor \frac{k}{2} \right\rfloor + 2 & 1 \\ d_1 & 1 & \left\lfloor \frac{k}{2} \right\rfloor - 1 \end{pmatrix},$$

where $d_1 = n - \left(3\left\lfloor \frac{k}{2} \right\rfloor + 8\right) = n - \left(\left\lfloor \frac{3k}{2} \right\rfloor + 8\right)$.

$\square$

Example 4.5.11. *For $n = 14$ there are 4 partitions into 2-distinct parts whose associated matrix has order 2×3 and entry $d_2 = 0$ or 1, as we can see in Table 18.*

Proposition 4.5.10 can be generalized in such a way that an unique result characterizes all the matrices of order 2×3, not only those with entry $d_2 = 0$ or 1. This is what the following theorem describes.

Theorem 4.5.12. *The matrices of order 2×3 from Theorem 4.5.1, associated to the partitions of some n, have the form*

$$\begin{pmatrix} k+t+5 & t+3 & 1 \\ n-9-2k-3t & k & t \end{pmatrix}, \tag{4.5}$$

$P(14\|$ 2-distinct parts)	Matrix of type (4.4)	k	$m = \dfrac{4k^2+44k+117+3(-1)^k}{2}$	$P_{R(od)}(m)$
$(10,3,1)$	$\begin{pmatrix} 5 & 3 & 1 \\ 5 & 0 & 0 \end{pmatrix}$	1	81	$(17,15,13,11,9,7,5,3,1)$
$(9,4,1)$	$\begin{pmatrix} 6 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix}$	2	112	$(21,19,17,15,13,11,7,5,3,1)$
$(8,4,2)$	$\begin{pmatrix} 6 & 4 & 1 \\ 2 & 0 & 1 \end{pmatrix}$	3	141	$(23,21,19,17,15,13,11,9,7,5,1)$
$(7,5,2)$	$\begin{pmatrix} 7 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	4	180	$(27,25,23,21,19,17,15,11,9,7,5,1)$

Table 18: Table for Example 4.5.11

with $0 \leq t \leq \left\lfloor \dfrac{n-9}{3} \right\rfloor$ *and* $0 \leq k \leq \left\lfloor \dfrac{n-9-3t}{2} \right\rfloor$.

Moreover, these matrices induce partitions into distinct odd parts of type

$$\begin{aligned} \mu &= (2(3t+2k+9)-1, \ldots, 2(2t+k+6)-1, 2(2t+k+5)-1, \\ &\quad 2(2t+4)-1, \ldots, 2(t+3)-1, 2(t+2)-1, 1). \end{aligned} \tag{4.6}$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition of n into 2-distinct parts. So, $\lambda_1 \geq \lambda_2 + 2$ and $\lambda_2 \geq \lambda_3 + 2$. Let us call $\lambda_2 = \lambda_3 + 2 + k$ and $\lambda_3 = 1 + t$, with $k, t \geq 0$. Then

$$\lambda_2 = \lambda_3 + 2 + k = k + t + 3$$

and

$$\lambda_1 = n - \lambda_2 - \lambda_3 = n - (k+t+3) - (1+t) = n - k - 2t - 4.$$

It is easy to see that λ can be written as a matrix of type (4.5). And from the Path Procedure we clearly get the partition μ (4.6). The limitations for t and k easily follow from $d_1 = n - 9 - 2k - 3t$ being greater than or equal to 0.

□

In a similar way as done in the previous section, now we characterize a more general type of partitions into distinct odd parts generated by the Path Procedure applied to the partitions of n into 2-distinct parts.

Recall that the smallest odd part of any of these partitions is always 1, since the matrix representation of every partition of n into 2-distinct parts has entry $c_s = 1$. This means that the path from the line $x + y = n$ to the origin starts with a down move of size 1.

In this case, *the first subsequence of consecutive odd parts* is composed exactly of one part of size 1. After it, we have *the first missing subsequence* of the partition, determined by the size of d_s, which we call $k_1 - 1$, with $k_1 \geq 1$. The first missing subsequence of odd integers is

$$2k_1 - 1, \ldots, 5, 3.$$

After *the first missing subsequence* of $k_1 - 1$ consecutive odd integers we have *the second subsequence of consecutive odd parts* that compose the partition. Its size is determined by the entry c_{s-1} of the matrix. As $c_{s-1} = 2 + c_s + d_s = 2 + 1 + k_1 - 1 = k_1 + 2$, this means that the parts of the second subsequence are

$$2(2k_1 + 2) - 1, \ldots, 2(k_1 + 2) - 1, 2(k_1 + 1) - 1.$$

Some examples of partitions that have a part 1 followed by the first missing subsequence and after it exactly one subsequence of consecutive odd parts are

$$(7, 5, 3, 1), (11, 9, 7, 5, 1), (15, 13, 11, 9, 7, 1), (19, 17, 15, 13, 11, 9, 1),$$

and

$$(23, 21, 19, 17, 15, 13, 11, 1),$$

which are, respectively, partitions of 16, 33, 56, 85, and 120 generated by the Path Procedure applied to partitions of $n = 13$ into 2-distinct parts.

The following result gives a general characterization of which numbers are partitioned into distinct odd parts, its partition having one part of size 1 followed by the first missing subsequence and after it exactly one subsequence of consecutive odd parts.

Proposition 4.5.13. *The Path Procedure applied to the partitions of* n *into* 2*-distinct parts generates partitions of* $m = 3k_1^2 + 8k_1 + 5$*, with* $1 \leq k_1 \leq \left\lfloor \dfrac{n-2}{2} \right\rfloor$*, those being precisely all of the numbers whose partition has two subsequences of consecutive odd parts.*

Proof. As mentioned before, the first subsequence of consecutive parts is always composed by a unique part of size 1. After it, the first missing subsequence, which might exist or not, has size $d_s = k_1 - 1$. And the parts of the second subsequence are $2(2k_1+2)-1,\dots,2(k_1+2)-1, 2(k_1+1)-1$. So we may write m as

$$\begin{aligned} m &= 2(2k_1+2)-1+\cdots+2(k_1+2)-1+2(k_1+1)-1+1 \\ &= \frac{(2(2k_1+2)-1+2(k_1+1)-1)(k_1+2)}{2}+1 \\ &= (3k_1+2)(k_1+2)+1 \\ &= 3k_1^2+8k_1+5. \end{aligned}$$

So we get $\mu = (2(2k_1+2)-1,\dots,2(k_1+2)-1, 2(k_1+1)-1, 1)$ a partition of $m = 3k_1^2+8k_1+5$. Clearly k_1 has to be at most $\dfrac{n-2}{2}$, otherwise the greatest part $2(2k_1+2)-1$ would exceed $2n-1$. And k_1 has to be greater than or equal to 1 so we can have $d_1 = k_1 - 1 = 0$.

□

As it happens with unrestricted partitions, partitions of n into 2-distinct parts with more than two parts have a matrix representation into more than two columns, which means that each one of its c_t generates a different subsequence of consecutive odd parts.

We call *the second missing subsequence* the sequence of $d_{s-1} = k_2 \geq 0$ consecutive odd integers that do not appear as parts of the partition, which are

$$2(2k_1+k_2+2)-1,\dots,2(2k_1+4)-1, 2(2k_1+3)-1.$$

Again, k_2 can actually be equal to 0. Its size is determined by $\lambda_{s-1} = c_{s-1}+d_{s-1}$, where λ_{s-1} is part of a partition of n into 2-distinct parts, and if $\lambda_{s-1} = \lambda_s + 2$ this means $d_{s-1} = 0$.

After the second missing subsequence of k_2 consecutive odd integers we have *the third subsequence of consecutive odd parts* that compose the partition, determined by the size of entry c_{s-2} of the matrix. As $c_{s-2} = 2 + c_{s-1} + d_{s-1} = 2+2+k_1+k_2 = k_1+k_2+4$, the parts of the second subsequence are

$$2(3k_1+2k_2+6)-1,\dots,2(2k_1+k_2+2+2)-1, 2(2k_1+k_2+2+1)-1.$$

For example, $n = 13$ generates the partitions

$$(17,15,13,11,9,7,5,3,1), (21,19,17,15,13,11,7,5,3,1),$$

$$(23,21,19,17,15,13,11,9,7,5,1), \text{ and } (25,23,21,19,17,15,13,7,5,3,1),$$

which are, respectively, partitions of 81, 112, 141, and 149.

A general characterization of which numbers are partitioned into distinct odd parts, its partition having one part of size 1, the first and second missing subsequences and after each one of them a subsequence of consecutive odd parts, is given next.

Proposition 4.5.14. *The Path Procedure applied to partitions of* n *into* 2*-distinct parts, having exactly three parts generates partitions of* $m = 8k_1^2 + 3k_2^2 + 8k_1k_2 + 36k_1 + 20k_2 + 37$*, with* $1 \leq k_1 \leq \left\lfloor \dfrac{n-2}{2} \right\rfloor$ *and* $0 \leq k_2 \leq \left\lfloor \dfrac{n-3k_1-6}{2} \right\rfloor$.

Proof. We may write

$$\begin{aligned} m &= 2(3k_1+2k_2+6)-1+\cdots+2(2k_1+k_2+2+1)-1+2(2k_1+2)-1 \\ &\quad +\cdots+2(k_1+1)-1+1 \\ &= \frac{(2(3k_1+2k_2+6)-1+2(2k_1+k_2+2+1)-1)(k_1+k_2+4)}{2} \\ &\quad +\frac{(2(2k_1+2)-1+2(k_1+1)-1)(k_1+2)}{2}+1 \\ &= (5k_1+3k_2+8)(k_1+k_2+4)+(3k_1+2)(k_1+2)+1 \\ &= 8k_1^2+3k_2^2+8k_1k_2+36k_1+20k_2+37. \end{aligned}$$

Then we get $\mu = (2(3k_1+2k_2+6)-1, \ldots, 2(2k_1+k_2+2+1)-1, 2(2k_1+2)-1, \ldots, 2(k_1+1)-1, 1)$ a partition of $m = 8k_1^2+3k_2^2+8k_1k_2+36k_1+20k_2+37$. The limitation for k_2 is obtained by observing that the greatest part $2(3k_1+2k_2+6)-1$ cannot exceed $2n-1$. So,

$$2(3k_1+2k_2+6)-1 \leq 2n-1 \Longrightarrow 3k_1+2k_2+6 \leq n \Longrightarrow k_2 \leq \frac{n-3k_1-6}{2}.$$

□

Now let us consider a partition into distinct odd parts having a number t of *missing subsequences* and $t+1$ *subsequences of consecutive odd parts*. We call $k_1-1, k_2, \ldots, k_t$ the sizes of the missing subsequences and, consequently, the subsequence after a missing subsequence of size k_i has size $k_1+k_2+\cdots+k_i+2i$. The following lemma establishes the limits for each k_i.

Lemma 4.5.15. *The* i^{th} *missing subsequence of a partition into distinct odd parts, whose parts derive from the Path Procedure applied to partitions of* n *into* 2*-distinct parts, is at most*

$$\frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}.$$

Proof. The sequence of all odd integers, those from the missing subsequences and from the subsequences of parts, cannot exceed $2n-1$. So, for example, when there are 3 missing subsequences and 4 subsequences of parts, it is necessary that

$$(k_1-1)+k_2+k_3+1+(k_1+2)+(k_1+k_2+4)+(k_1+k_2+k_3+6) \leq n,$$

which implies

$$k_3 \leq \frac{n-4k_1-3k_2+12}{2}.$$

If there are i missing subsequences and $i+1$ subsequences of parts, it is necessary that the sum of the sizes of the missing subsequences and of the sizes of the subsequences does not exceed n. This means

$$(k_1-1)+k_2+\cdots+k_i+1+(k_1+2)+(k_1+k_2+4)+\cdots+(k_1+k_2+\cdots+k_i+2i) \leq n,$$

which implies

$$(i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+2k_i+i(i+1) \leq n.$$

So,

$$k_i \leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}.$$

□

Now we can extend our construction to a more general characterization of the numbers partitioned into distinct odd parts, whose parts derive from the Path Procedure applied to partitions of n into 2-distinct parts.

Theorem 4.5.16. *The partitions into distinct odd parts induced by the Path Procedure applied to partitions of n into 2-distinct parts are all of the form*

$$\begin{aligned}
&\sum_{\substack{i=1 \\ 1\leq k_1\leq \frac{n-2}{2} \\ i>1,\, 0\leq k_i\leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t} [(t+2-i)^2-1]k_i^2 \\
&+ \sum_{\substack{i=1 \\ 1\leq k_1\leq \frac{n-2}{2} \\ i>1,\, 0\leq k_i\leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t-1} k_i \sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1} \\
&+ \sum_{i=1}^{t} k_i\big(2(t-i+1)t^2+6(t-i+1)t-2(t-i)(t-i+1)\big) \\
&+ [t(t+1)]^2+1, \qquad (4.7)
\end{aligned}$$

where $1 \leq t \leq n-1$.

Proof. First of all, let us rewrite the expression (4.7) by expanding the sums.

$$\begin{aligned}
& \sum_{\substack{i=1 \\ 1\le k_1\le \frac{n-2}{2} \\ i>1,\, 0\le k_i\le \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t} [(t+2-i)^2-1]k_i^2 \\
& + \sum_{\substack{i=1 \\ 1\le k_1\le \frac{n-2}{2} \\ i>1,\, 0\le k_i\le \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t-1} k_i \sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1} \\
& + \sum_{i=1}^{t} k_i\big(2(t-i+1)t^2+6(t-i+1)t-2(t-i)(t-i+1)\big) \\
& + [t(t+1)]^2+1 \\
= \; & ((t+1)^2-1)k_1^2+(t^2-1)k_2^2+\cdots+8k_{t-1}^2+3k_t^2 \\
& + k_1(2(t+2)k_t+2(2(t+2))k_{t-1}+\cdots+(t-2)(2(t+2))k_3 \\
& + (t-1)(2(t+2))k_2) \\
& + k_2(2(t+1)k_t+2(2(t+1))k_{t-1}+\cdots+(t-2)(2(t+1))k_3) \\
& + \cdots \\
& + k_{t-3}(12k_t+24k_{t-1}+36k_{t-2}) \\
& + k_{t-2}(10k_t+20k_{t-1}) \\
& + k_{t-1}(8k_t) \\
& + k_1((2t)t^2+(6t)t-2(t-1)t) \\
& + k_2(2(t-1)t^2+6(t-1)t-2(t-2)(t-1)) \\
& + \cdots \\
& + k_{t-1}(4t^2+12t-4) \\
& + k_t(2t^2+6t) \\
& +[t(t+1)]^2+1
\end{aligned} \tag{4.8}$$

Let us consider a partition generated by the Path Procedure and induced by the partitions of n into 2-distinct parts, and suppose it has t subsequences of missing parts. Let us call their sizes k_1-1, k_2, $\ldots$, k_t. So, the sizes of the subsequences of consecutive odd parts are 1, k_1+2, k_1+k_2+4, $k_1+k_2+k_3+6$, $\ldots$, $k_1+k_2+\cdots+k_t+2t$ and the actual partition we are considering has the following subsequences of consecutive odd parts:

$2((t+1)k_1+\cdots+3k_{t-1}+2k_t+(t+1)t)-1, \ldots, 2(tk_1+\cdots+k_t+t(t-1)+2)-1,$

$$2(tk_1 + \cdots + 2k_{t-1} + k_t + t(t-1) + 1) - 1;$$

$$2(tk_1 + \cdots + 3k_{t-2} + 2k_{t-1} + t(t-1)) - 1, \ldots, 2((t-1)k_1 + \cdots + 2k_{t-2} + k_{t-1} + (t-1)(t-2) + 2) - 1,$$

$$2((t-1)k_1 + \cdots + 2k_{t-2} + k_{t-1} + (t-1)(t-2) + 1) - 1;$$

$$\cdots$$

$$2(4k_1+3k_2+2k_3+12)-1, \ldots, 2(3k_1+2k_2+k_3+8)-1, 2(3k_1+2k_2+k_3+7)-1;$$

$$2(3k_1 + 2k_2 + 6) - 1, \ldots, 2(2k_1 + k_2 + 2 + 2) - 1, 2(2k_1 + k_2 + 2 + 1) - 1;$$

$$2(2k_1 + 2) - 1, \ldots, 2(k_1 + 2) - 1, 2(k_1 + 1) - 1;$$

and

$$1.$$

The sum of those parts equals

$$\begin{aligned}
& 2((t+1)k_1 + \cdots + 2k_t + (t+1)t) - 1 + 2(tk_1 + \cdots + k_t + t(t-1) + 1) - 1 \\
& \cdot \frac{k_1 + \cdots + k_t + 2t}{2} \\
& + 2(tk_1 + \cdots + 2k_{t-1} + t(t-1)) - 1 + 2((t-1)k_1 + \cdots + k_{t-1} + (t-1)(t-2) + 1) - 1 \\
& \cdot \frac{k_1 + \cdots + k_{t-1} + 2(t-1)}{2} \\
& + \cdots \\
& + \frac{(2(4k_1 + 3k_2 + 2k_3 + 12) - 1 + 2(3k_1 + 2k_2 + k_3 + 6 + 1) - 1)(k_1 + k_2 + k_3 + 6)}{2} \\
& + \frac{(2(3k_1 + 2k_2 + 6) - 1 + 2(2k_1 + k_2 + 2 + 1) - 1)(k_1 + k_2 + 4)}{2} \\
& + \frac{(2(2k_1 + 2) - 1 + 2(k_1 + 1) - 1)(k_1 + 2)}{2} \\
& + 1 \\
= \; & ((2t+1)k_1 + \cdots + 5k_{t-1} + 3k_t + 2t^2)(k_1 + k_2 + \cdots + k_t + 2t) \\
& + ((2t-1)k_1 + \cdots + 5k_{t-2} + 3k_{t-1} + 2(t-1)^2)(k_1 + k_2 + \cdots + k_{t-1} + 2(t-1)) \\
& + \cdots \\
& + (7k_1 + 5k_2 + 3k_3 + 18)(k_1 + k_2 + k_3 + 6) \\
& + (5k_1 + 3k_2 + 8)(k_1 + k_2 + 4) \\
& + (3k_1 + 2)(k_1 + 2) \\
& + 1.
\end{aligned}$$

By rearranging the terms in the sum above we get the expression (4.8), which proves the theorem.

□

4.6 The Path Procedure applied to partitions counted by the 2^{nd} Rogers-Ramanujan Identity

In a similar way as done in Section 4.5, also in [SMR11] there is a result which characterizes a matrix representation for partitions into 2-distinct parts, greater than 1. These partitions are counted by the right-hand side of the 2^{nd} Rogers-Ramanujan Identity,

$$p(n|\text{parts} \equiv 2 \text{ or } 3 \pmod 5) = p(n|\text{2-distinct parts} > 1).$$

Theorem 4.6.1 (Corollary 3.4, [SMR11])**.** *The number of partitions of n where the difference between parts is at least two and each part is greater than one is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix}, \tag{4.9}$$

where $c_s = 2$, $c_t = 2 + c_{t+1} + d_{t+1}$, and the sum of all entries is equal to n.

Again the sum of the entries of each column gives the respective part of the original partition. That is, the k^{th} part is equal to $c_k + d_k$. As an example, we take $n = 8$ and show its partitions into 2-distinct parts greater than 1 with their associated matrices.

Example 4.6.2. *For $n = 8$ there are 3 partitions into 2-distinct parts greater than 1, and so there are 3 matrices satisfying Theorem 4.6.1, as shown in Table 19.*

$P(8\vert$ 2-distinct parts $> 1)$	Matrix of type (4.9)
$(5,3)$	$\begin{pmatrix} 5 & 2 \\ 0 & 1 \end{pmatrix}$
$(6,2)$	$\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$
(8)	$\begin{pmatrix} 2 \\ 6 \end{pmatrix}$

Table 19: Table for Example 4.6.2

The same Path Procedure is now applied to the matrices from Theorem 4.6.1, generating new integer partitions into distinct odd parts.

Example 4.6.3. *The partitions of* 8 *from Example 4.6.2 generate the partitions into distinct odd parts contained in Table 20.*

$P(8\vert$ 2-distinct parts $> 1)$	Matrix of type (4.9)	Partition into distinct odd parts
$(5,3)$	$\begin{pmatrix} 5 & 2 \\ 0 & 1 \end{pmatrix}$	$(15,13,11,9,7,3,1)$
$(6,2)$	$\begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$	$(11,9,7,5,3,1)$
(8)	$\begin{pmatrix} 2 \\ 6 \end{pmatrix}$	$(3,1)$

Table 20: Table for Example 4.6.3

Remark 4.6.4. *Note that in the present case every integer partition generated by the Path Procedure has a part of size* 1 *and a part of size* 3. *This is easy to see since* $c_s = 2$, *which means that the first move in the path from* $P = (\sum_{i=1}^{s} d_i, \sum_{i=1}^{s} c_i)$ *to* $(0,0)$ *is an exact down shift of size two, generating the first two odd integers as parts of the new partition.*

Figure 4.9 below illustrates the distribution of frequencies of partitions of m in a square of size 20×20, induced by the partitions of 20, according to Theorem 4.6.1. Each cell contains how many partitions of m (indicated in the right down side of the cell) are generated by the matrix representation of the partitions of 20.

Again motivated by the Path Procedure applied to the matrices of Theorem 4.6.1 we have the following definition.

Definition 4.6.5. We call $P_{R_2(od)}(m)$ the set of partitions of m into distinct odd parts, always having 1 and 3 as their smallest parts, and whose size of any subsequence of consecutive odd integers equals the size of the previous subsequence of consecutive odd integers that were omitted before the subsequence started plus the size of the previous subsequence of consecutive odd parts plus 2. Also, $\left|P_{R_2(od)}(m)\right| = p_{R_2(od)}(m)$.

Remark 4.6.6. *The index* R_2 *refers to the* 2^{nd} *Rogers-Ramanujan Identity.*

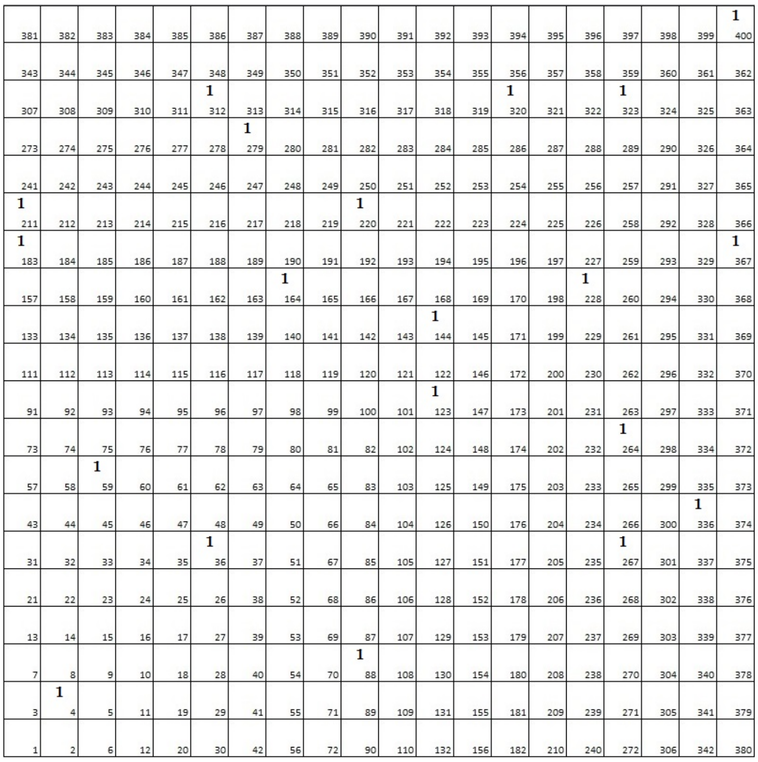


Figure 4.9: $n \times n$ square for $n = 20$

Remark 4.6.7. *If no odd integer is omitted after some subsequence of parts, we assume the number of omitted parts is zero, and the size of the following subsequence of odd parts will be the size of the previous subsequence of consecutive odd parts plus 2.*

Proposition 4.6.8. *The hooks induced by the order 2×2 matrices of Theorem 4.6.1 associated to the partitions of some n constitute partitions of $3k^2 + 20k + 36$, with $0 \leq k \leq \left\lfloor \dfrac{n}{2} \right\rfloor - 3$.*

Proof. Let λ_2 be the smallest part of some partition of n into 2-distinct parts greater than 1 whose associated matrix, according to Theorem 4.6.1, has order 2×2. Note that this means that $n = \lambda_1 + \lambda_2$, with $\lambda_2 \geq 2$ and $\lambda_1 \geq \lambda_2 + 2$. So, $2 \leq \lambda_2 \leq \left\lfloor \dfrac{n}{2} \right\rfloor - 1$. By making $k = \lambda_2 - 2$, we have $0 \leq k \leq \left\lfloor \dfrac{n}{2} \right\rfloor - 3$.

The path induced by the matrix begins with a down move of size 2, then a left move of size k, followed by a down move of size $2 + 2 + k = k + 4$, and ending with a left move of size $n - 2 - k - (k + 4) = n - 2k - 6$. Indeed, any matrix of order 2×2 has the form

$$M = \begin{pmatrix} k+4 & 2 \\ n-(2k+6) & k \end{pmatrix}$$

When reflected through the line $x + y = n$ the path generates hooks of sizes $1, 3, 2(k+3)-1, 2(k+4)-1, \ldots, 2(2k+6)-1$. As the path ends with a left move of size $n - 2k - 6$, it lies on the x-axis and does not generate a hook. So the new partition into odd parts is

$$(2(2k+6)-1, \ldots, 2(k+3)-1, 3, 1),$$

a partition of $3k^2 + 20k + 36$.

$\square$

Example 4.6.9. *For $n = 13$ there are 4 partitions into 2-distinct parts greater than 1 whose associated matrix has order 2, as shown in Table 21.*

$P(13\vert$ 2-distinct parts $> 1)$	Matrix of type (4.9)	k	$P_{R_2(od)}(3k^2+20k+36)$
$(8,5)$	$\begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix}$	3	$(23,21,19,17,15,13,11,3,1)$
$(9,4)$	$\begin{pmatrix} 6 & 2 \\ 3 & 2 \end{pmatrix}$	2	$(19,17,15,13,11,9,3,1)$
$(10,3)$	$\begin{pmatrix} 5 & 2 \\ 5 & 1 \end{pmatrix}$	1	$(15,13,11,9,7,3,1)$
$(11,2)$	$\begin{pmatrix} 4 & 2 \\ 7 & 0 \end{pmatrix}$	0	$(11,9,7,5,3,1)$

Table 21: Table for Example 4.6.9

Proposition 4.6.10. *The Path Procedure applied to the order 2×3 matrices of Theorem 4.6.1 with $d_2 = 0$, associated to the partitions of some n, generates partitions of $m = 8k^2 + 68k + 144$, with $0 \le k \le \left\lfloor \frac{n}{3} \right\rfloor - 4$. Moreover, these partitions appear for the first time when $n = 3k + 12$.*

Proof. Easily we can see that $m = 8k^2 + 68k + 144$ can be partitioned as

$$\lambda = (2(3k+12)-1, 2(3k+11)-1, \ldots, 2(k+3)-1, 3, 1).$$

As the greatest part of λ is $2(3k+12)-1$, clearly this partition is possible only for $n \geq 3k+12$. Also, for each $n \geq 3k+12$, partitions like λ are possible for certain values of k:

$$n \geq 3k+12 \implies k \leq \frac{n-12}{3},$$

and as $k \in \mathbb{Z}$, $k \leq \left\lfloor \frac{n-12}{3} \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor - 4$.

The matrix associated to the partition λ is

$$M = \begin{pmatrix} k+6 & k+4 & 2 \\ d_1 & 0 & k \end{pmatrix},$$

where $d_1 = n - (3k+12)$.

□

Example 4.6.11. *For $n = 25$ there are 5 partitions into 2-distinct parts greater than 1 whose associated matrix has order 2×3 and entry $d_2 = 0$, as shown in Table 22.*

$P(25 \mid$ 2-distinct parts $> 1)$	Matrix of type (4.9)	k	$m = 8k^2 + 68k + 144$	$P_{R_2(od)}(m)$
$(19, 4, 2)$	$\begin{pmatrix} 6 & 4 & 2 \\ 13 & 0 & 0 \end{pmatrix}$	0	144	$(23, 21, 19, 17, 15, 13, 11, 9, 7, 5, 3, 1)$
$(17, 5, 3)$	$\begin{pmatrix} 7 & 5 & 2 \\ 10 & 0 & 1 \end{pmatrix}$	1	220	$(29, 27, 25, 23, 21, 19, 17, 15, 13, 11, 9, 7, 3, 1)$
$(15, 6, 4)$	$\begin{pmatrix} 8 & 6 & 2 \\ 7 & 0 & 2 \end{pmatrix}$	2	312	$(35, 33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 13, 11, 9, 3, 1)$
$(13, 7, 5)$	$\begin{pmatrix} 9 & 7 & 2 \\ 4 & 0 & 3 \end{pmatrix}$	3	420	$(41, 39, 37, 35, 33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 13, 11, 3, 1)$
$(11, 8, 6)$	$\begin{pmatrix} 10 & 8 & 2 \\ 1 & 0 & 4 \end{pmatrix}$	4	544	$(47, 45, 43, 41, 39, 37, 35, 33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 13, 3, 1)$

Table 22: Table for Example 4.6.11

A similar result describes the order 2×3 matrices with $d_2 = 1$.

Proposition 4.6.12. *The Path Procedure applied to the order 2×3 matrices of Theorem 4.6.1 with $d_2 = 1$, associated to the partitions of some n, generates partitions of $m = 8k^2 + 76k + 183$, with $0 \leq k \leq \left\lfloor \frac{n-2}{3} \right\rfloor - 4$. Moreover, these partitions appear for the first time when $n = 3k + 14$.*

Proof. $m = 8k^2 + 76k + 183$ can be partitioned as

$$\lambda = (2(3k+14)-1, \ldots, 2(2k+8)-1, 2(2k+6)-1, \ldots, 2(k+3)-1, 3, 1).$$

As the greatest part of λ is $2(3k+14)-1$, clearly this partition is possible only for $n \geq 3k+14$. Also, for each $n \geq 3k+14$, partitions like λ are possible for $k \leq \dfrac{n-14}{3}$. As $k \in \mathbb{Z}$, $k \leq \left\lfloor \dfrac{n-14}{3} \right\rfloor$.

The matrix associated to λ is

$$M = \begin{pmatrix} k+7 & k+4 & 2 \\ d_1 & 1 & k \end{pmatrix},$$

where $d_1 = n - (3k+14)$.

□

Example 4.6.13. *For $n = 25$ there are 4 partitions into 2-distinct parts greater than 1 whose associated matrix has order 2×3 and entry $d_2 = 1$, as shown in Table 23.*

$P(25\|$ 2-distinct parts $> 1)$	Matrix of type (4.9)	k	$m = 8k^2 + 76k + 183$	$P_{R_2(od)}(m)$
$(18, 5, 2)$	$\begin{pmatrix} 7 & 4 & 2 \\ 11 & 1 & 0 \end{pmatrix}$	0	183	$(27, 25, 23, 21, 19, 17, 15, 11, 9, 7, 5, 3, 1)$
$(16, 6, 3)$	$\begin{pmatrix} 8 & 5 & 2 \\ 8 & 1 & 1 \end{pmatrix}$	1	267	$(33, 31, 29, 27, 25, 23, 21, 19, 15, 13, 11, 9, 7, 3, 1)$
$(14, 7, 4)$	$\begin{pmatrix} 9 & 6 & 2 \\ 5 & 1 & 2 \end{pmatrix}$	2	367	$(39, 37, 35, 33, 31, 29, 27, 25, 23, 19, 17, 15, 13, 11, 9, 3, 1)$
$(12, 8, 5)$	$\begin{pmatrix} 10 & 7 & 2 \\ 2 & 1 & 3 \end{pmatrix}$	3	483	$(45, 43, 41, 39, 37, 35, 33, 31, 29, 27, 23, 21, 19, 17, 15, 13, 11, 3, 1)$

Table 23: Table for Example 4.6.13

The Path Procedure applied to the partitions of n into 2-distinct parts greater than 1 differs a little bit from the one applied to partitions of n into 2-distinct parts of any size, studied in Section 4.5.

Recall that in the present case the smallest odd parts of any of these partitions are always 1 and 3, since the matrix representation of every partition of n into 2-distinct parts greater than 1 has entry $c_s = 2$. This means that the path from the line $x + y = n$ to the origin starts with a down move of size 2.

In this case, *the first subsequence of consecutive odd parts* is composed of one part of size 1 and one part of size 3. After it, we have *the first missing subsequence* of the partition, determined by the size of d_s, which we call $k_1 - 2$, with $k_1 \geq 2$. The first missing subsequence of parts is

$$2k_1 - 1, \ldots, 7, 5.$$

After *the first missing subsequence* of $k_1 - 2$ consecutive odd integers we have *the second subsequence of consecutive odd parts* that compose the partition. Its size is determined by the entry c_{s-1} of the matrix. As $c_{s-1} = 2 + c_s + d_s = 2 + 2 + k_1 - 2 = k_1 + 2$, this means that the parts of the second subsequence are

$$2(2k_1 + 2) - 1, \ldots, 2(k_1 + 2) - 1, 2(k_1 + 1) - 1.$$

Some examples of partitions that have a part 1 and a part 3 followed by the first missing subsequence and after it exactly one subsequence of consecutive odd parts are

$$(11, 9, 7, 5, 3, 1), (15, 13, 11, 9, 7, 3, 1), (19, 17, 15, 13, 11, 9, 3, 1),$$

$$\text{and } (23, 21, 19, 17, 15, 13, 11, 3, 1),$$

partitions of 36, 59, 88, and 123, respectively, generated by the Path Procedure applied to partitions of $n = 13$ into 2-distinct parts greater than 1.

The following result gives a general characterization of which numbers are partitioned into distinct odd parts, its partition having one part of size 1 and one part of size 3, followed by the first missing subsequence and after it exactly one subsequence of consecutive odd parts.

Proposition 4.6.14. *The Path Procedure applied to the partitions of n having exactly three 2-distinct parts greater than 1 generates partitions of $m = 3k_1^2 + 8k_1 + 8$, with $2 \leq k_1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor$, those being precisely all of the numbers whose partition has two subsequences of consecutive odd parts.*

Proof. As m has two subsequences of consecutive parts, we may write

$$\begin{aligned} m &= 2(2k_1 + 2) - 1 + \cdots + 2(k_1 + 2) - 1 + 2(k_1 + 1) - 1 + 3 + 1 \\ &= \frac{(2(2k_1 + 2) - 1 + 2(k_1 + 1) - 1)(k_1 + 2)}{2} + 3 + 1 \\ &= (3k_1 + 2)(k_1 + 2) + 3 + 1 \\ &= 3k_1^2 + 8k_1 + 8. \end{aligned}$$

So we get $\mu = (2(2k_1 + 2) - 1, \ldots, 2(k_1 + 2) - 1, 2(k_1 + 1) - 1, 3, 1)$ a partition of $m = 3k_1^2 + 8k_1 + 8$. Clearly k_1 has to be at most $\left\lfloor \frac{n-2}{2} \right\rfloor$, otherwise the greatest part $2(2k_1 + 2) - 1$ would exceed $2n - 1$.

□

As it happens in the cases explored in previous sections, partitions of n into 2-distinct parts greater than 1 with more than two parts have a matrix representation into more than two columns, which means that each one of its c_t generates a different sequence of consecutive odd parts.

We call *the second missing subsequence* the sequence of $d_{s-1} = k_2 \geq 0$ consecutive odd integers that do not appear as parts of the partition, which are

$$2(2k_1 + k_2 + 2) - 1, \ldots, 2(2k_1 + 4) - 1, 2(2k_1 + 3) - 1.$$

Again, k_2 can actually be equal to 0. Its size is determined by $\lambda_{s-1} = c_{s-1} + d_{s-1}$, where λ_{s-1} is part of a partition of n into 2-distinct parts greater than 1, and if $\lambda_{s-1} = \lambda_s + 2$ this means $d_{s-1} = 0$.

After the second missing subsequence of k_2 consecutive odd integers we have *the third subsequence of consecutive odd parts* that compose the partition, determined by the size of entry c_{s-2} of the matrix. As $c_{s-2} = 2 + c_{s-1} + d_{s-1} = 2 + k_1 + 2 + k_2 = k_1 + k_2 + 4$, the parts of the second subsequence are

$$2(3k_1 + 2k_2 + 6) - 1, \ldots, 2(2k_1 + k_2 + 2 + 2) - 1, 2(2k_1 + k_2 + 2 + 1) - 1.$$

For example, $n = 14$ generates the partitions

$$(23, 21, 19, 17, 15, 13, 11, 9, 7, 5, 3, 1) \text{ and } (27, 25, 23, 21, 19, 17, 15, 11, 9, 7, 5, 3, 1),$$

which are, respectively, partitions of 144 and 183.

A general characterization of which numbers are partitioned into distinct odd parts, its partition having one part of size 1, one part of size 3, the first and second missing subsequences and after each one of them a subsequence of consecutive odd parts, is given next.

Proposition 4.6.15. *The Path Procedure applied to the partitions of n into 2-distinct parts, greater than 1, having exactly three parts generates partitions of $m = 8k_1^2 + 3k_2^2 + 8k_1k_2 + 36k_1 + 20k_2 + 40$, with $2 \leq k_1 \leq \left\lfloor \dfrac{n-2}{2} \right\rfloor$ and $0 \leq k_2 \leq \left\lfloor \dfrac{n - 3k_1 - 6}{2} \right\rfloor$.*

Proof. m can be written as

$$\begin{aligned} m &= 2(3k_1 + 2k_2 + 6) - 1 + \cdots + 2(2k_1 + k_2 + 2 + 1) - 1 + 2(2k_1 + 2) - 1 + \\ &\quad + \cdots + 2(k_1 + 1) - 1 + 3 + 1 \\ &= \frac{(2(3k_1 + 2k_2 + 6) - 1 + 2(2k_1 + k_2 + 2 + 1) - 1)(k_1 + k_2 + 4)}{2} \\ &\quad + \frac{(2(2k_1 + 2) - 1 + 2(k_1 + 1) - 1)(k_1 + 2)}{2} + 3 + 1 \\ &= (5k_1 + 3k_2 + 8)(k_1 + k_2 + 4) + (3k_1 + 2)(k_1 + 2) + 4 \\ &= 8k_1^2 + 3k_2^2 + 8k_1k_2 + 36k_1 + 20k_2 + 40. \end{aligned}$$

So, $\mu = (2(3k_1 + 2k_2 + 6) - 1, \ldots, 2(2k_1 + k_2 + 2 + 1) - 1, 2(2k_1 + 2) - 1, \ldots, 2(k_1+1)-1, 3, 1)$ is a partition of m. The limitation for k_2 is obtained by observing that the greatest part $2(3k_1+2k_2+6)-1$ cannot exceed $2n-1$. □

Now let us consider a partition into distinct odd parts having t *missing subsequences* and $t+1$ *subsequences of consecutive odd parts*. We call $k_1-2, k_2, \ldots, k_t$ the sizes of the missing subsequences and, consequently, the subsequence after a missing subsequence of size k_i has size $k_1+k_2+\cdots+k_i+2i$. The following lemma establishes the limits for each k_i.

Lemma 4.6.16. *The i^{th} missing subsequence of a partition into distinct odd parts, whose parts derive from the Path Procedure applied to partitions of n into 2-distinct parts greater than 1, is at most*

$$\frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}.$$

Proof. The sequence of all odd integers, those from the missing subsequences and from the subsequences of odd parts, cannot exceed $2n-1$. So, for example, when there are 3 missing subsequences and 4 subsequences of parts, it is necessary that

$$(k_1-2)+k_2+k_3+2+(k_1+2)+(k_1+k_2+4)+(k_1+k_2+k_3+6) \leq n,$$

which implies

$$k_3 \leq \frac{n-4k_1-3k_2+12}{2}.$$

If there are i missing subsequences and $i+1$ subsequences of parts, it is necessary that the sum of the sizes of these subsequences does not exceed n. This means

$$(k_1-2)+k_2+\cdots+k_i+2+(k_1+2)+(k_1+k_2+4)+\cdots+(k_1+k_2+\cdots+k_i+2i) \leq n,$$

which implies

$$(i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+2k_i+i(i+1) \leq n.$$

So,

$$k_i \leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}.$$

□

Now we can extend our construction to a more general characterization of the numbers partitioned into distinct odd parts, whose parts derive from the Path Procedure applied to partitions of n into 2-distinct parts greater than 1.

Theorem 4.6.17. *The partitions into distinct odd parts induced by the Path Procedure applied to partitions of n into 2-distinct parts greater than 1, are all of the form*

$$\begin{aligned}
&\sum_{\substack{i=1\\ 2\le k_1\le \frac{n-2}{2}\\ i>1,\, 0\le k_i\le \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t} [(t+2-i)^2-1]k_i^2\\
&+\sum_{\substack{i=1\\ 2\le k_1\le \frac{n-2}{2}\\ i>1,\, 0\le k_i\le \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t} k_i\sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1}\\
&+\sum_{i=1}^{t} k_i\big(2(t-i+1)t^2+6(t-i+1)t-2(t-i)(t-i+1)\big)\\
&+[t(t+1)]^2+4,
\end{aligned} \tag{4.10}$$

where $1\le t\le n-1$.

Proof. First of all, let us rewrite the expression (4.10) by expanding the sums.

$$\begin{aligned}
&\sum_{\substack{i=1\\ 2\le k_1\le \frac{n-2}{2}\\ i>1,\, 0\le k_i\le \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t} [(t+2-i)^2-1]k_i^2\\
&+\sum_{\substack{i=1\\ 2\le k_1\le \frac{n-2}{2}\\ i>1,\, 0\le k_i\le \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+i(i+1))}{2}}}^{t-1} k_i\sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1}\\
&+\sum_{i=1}^{t} k_i\big(2(t-i+1)t^2+6(t-i+1)t-2(t-i)(t-i+1)\big)\\
&+[t(t+1)]^2+4
\end{aligned}$$

$$\begin{aligned}
= \quad & ((t+1)^2-1)k_1^2 + (t^2-1)k_2^2 + \cdots + 8k_{t-1}^2 + 3k_t^2 \\
& + k_1(2(t+2)k_t + 2(2(t+2))k_{t-1} + \cdots + (t-1)(2(t+2))k_2) \\
& + k_2(2(t+1)k_t + 2(2(t+1))k_{t-1} + \cdots + (t-2)(2(t+1))k_3) \\
& + \cdots \\
& + k_{t-3}(12k_t + 24k_{t-1} + 36k_{t-2}) \\
& + k_{t-2}(10k_t + 20k_{t-1}) \\
& + k_{t-1}(8k_t) \\
& + k_1((2t)t^2 + (6t)t - 2(t-1)t) \\
& + k_2(2(t-1)t^2 + 6(t-1)t - 2(t-2)(t-1)) \\
& + \cdots \\
& + k_{t-1}(4t^2 + 12t - 4) \\
& + k_t(2t^2 + 6t) \\
& + [t(t+1)]^2 + 4 \qquad (4.11)
\end{aligned}$$

Let us consider a partition generated by the Path Procedure induced by the partitions of n into 2-distinct parts greater than 1, and suppose it has t subsequences of missing parts. Let us call their sizes $k_1 - 2$, k_2, $\ldots$, k_t. So, the sizes of the subsequences of consecutive odd parts are 2, k_1+2, k_1+k_2+4, $k_1+k_2+k_3+6$, $\ldots$, $k_1+k_2+\cdots+k_t+2t$ and the actual partition we are considering has the following subsequences of consecutive odd parts:

$$2((t+1)k_1+\cdots+3k_{t-1}+2k_t+(t+1)t)-1, \ldots, 2(tk_1+\cdots+k_t+t(t-1)+2)-1,$$

$$2(tk_1+\cdots+2k_{t-1}+k_t+t(t-1)+1)-1;$$

$$2(tk_1+\cdots+3k_{t-2}+2k_{t-1}+t(t-1))-1, \ldots, 2((t-1)k_1+\cdots+2k_{t-2}+k_{t-1}+(t-1)(t-2)+2)-1,$$

$$2((t-1)k_1+\cdots+2k_{t-2}+k_{t-1}+(t-1)(t-2)+1)-1;$$

$$\cdots$$

$$2(4k_1+3k_2+2k_3+12)-1, \ldots, 2(3k_1+2k_2+k_3+6+2)-1, 2(3k_1+2k_2+k_3+6+1)-1;$$

$$2(3k_1+2k_2+6)-1, \ldots, 2(2k_1+k_2+2+2)-1, 2(2k_1+k_2+2+1)-1;$$

$$2(2k_1+2)-1, \ldots, 2(k_1+2)-1, 2(k_1+1)-1;$$

and

$$3, 1.$$

The sum of those parts equals

$$
\begin{aligned}
& \frac{(2((t+1)k_1+\cdots+3k_{t-1}+2k_t+(t+1)t)-1}{2} \\
& + \frac{2(tk_1+\cdots+2k_{t-1}+k_t+t(t-1)+1)-1)(k_1+k_2+\cdots+k_t+2t)}{2} \\
& + \frac{(2(tk_1+\cdots+3k_{t-2}+2k_{t-1}+t(t-1))-1}{2} \\
& + (2((t-1)k_1+\cdots+2k_{t-2}+k_{t-1}+(t-1)(t-2)+1)-1) \\
& \frac{k_1+k_2+\cdots+k_{t-1}+2(t-1)}{2} \\
& + \cdots \\
& + (2(4k_1+3k_2+2k_3+12)-1+2(3k_1+2k_2+k_3+6+1)-1) \\
& \frac{(k_1+k_2+k_3+6)}{2} \\
& + \frac{(2(3k_1+2k_2+6)-1+2(2k_1+k_2+2+1)-1)(k_1+k_2+4)}{2} \\
& + \frac{(2(2k_1+2)-1+2(k_1+1)-1)(k_1+2)}{2} \\
& + 4 \\
= \;& ((2t+1)k_1+\cdots+5k_{t-1}+3k_t+2t^2)(k_1+k_2+\cdots+k_t+2t) \\
& + ((2t-1)k_1+\cdots+3k_{t-1}+2(t-1)^2)(k_1+k_2+\cdots+k_{t-1}+2(t-1)) \\
& + \cdots \\
& + (7k_1+5k_2+3k_3+18)(k_1+k_2+k_3+6) \\
& + (5k_1+3k_2+8)(k_1+k_2+4) \\
& + (3k_1+2)(k_1+2) \\
& + 4.
\end{aligned}
$$

By rearranging the terms in the sum above we get the expression (4.11), which proves the theorem.

□

Remark 4.6.18. *Observe that the matrix representation for partitions of n into 2-distinct parts greater than 1 is exactly the same as the one for partitions generated by mock theta function*

$$f_1^*(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}.$$

The general term

$$\frac{q^{2(1+2+\cdots+s)}}{(1-q)(1-q^2)\cdots(1-q^s)}$$

counts the partitions of n containing at least 2 copies of each part from 1 to s. By conjugation, this is the same as counting the partitions of n into 2-distinct parts greater than 1.

4.7 The Path Procedure applied to partitions counted by the mock theta function $f_*^5(q)$

With simple modifications, it is not difficult to enunciate analogous results for the mock theta functions $f_*^m(q)$ from Chapter 3. In this section we take $m = 5$ as an example.

Let us recall

$$f_*^5(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{5(n^2+n)}{2}}}{(q;q)_n},$$

whose general term

$$\frac{q^{5(1+2+\cdots+s)}}{(1-q)(1-q^2)\cdots(1-q^s)}$$

generates the partitions of n into 5-distinct parts greater than 4. According to Theorem 3.2.1, its matrix representation has the form

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix}, \tag{4.12}$$

where $c_s = 5$, $c_t = 5 + c_{t+1} + d_{t+1}$, and the sum of all entries is equal to n.

The Path Procedure applied to the partitions of n into 5-distinct parts greater than 4 induces partitions into distinct odd parts always having 9, 7, 5, 3, and 1 as the smallest parts. This is due to the matrix representation (4.12) above which has entry $c_s = 5$. This means that the path from the line $x + y = n$ to the origin starts with a down move of size 5.

In this case, *the first subsequence of consecutive odd parts* is composed exactly of parts $9, 7, 5, 3,$ and 1. After it, we have *the first missing subsequence* of the partition, determined by the size of d_s, which we call k_1-5, with $k_1 \geq 5$. The first missing subsequence of parts is

$$2k_1 - 1, \ldots, 13, 11.$$

After *the first missing subsequence* of $k_1 - 5$ consecutive odd integers we have *the second subsequence of consecutive odd parts* that compose the partition. Its size is determined by the entry c_{s-1} of the matrix. As $c_{s-1} = 5 + c_s + d_s = 5 + 5 + k_1 - 5 = k_1 + 5$, this means that the parts of the second subsequence are

$$2(2k_1 + 5) - 1, \ldots, 2(k_1 + 2) - 1, 2(k_1 + 1) - 1.$$

Some examples of partitions that have parts $9, 7, 5, 3$, and 1, followed by the first missing subsequence, and after it exactly one subsequence of consecutive odd parts are

$$(29, 27, 25, 23, 21, 19, 17, 15, 13, 11, 9, 7, 5, 3, 1),$$

$$(33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 13, 9, 7, 5, 3, 1),$$

$$\text{and } (37, 35, 33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 9, 7, 5, 3, 1),$$

which are, respectively, partitions of 225, 278, and 337 generated by the partitions of $n = 19$.

The following result gives a general characterization of which numbers are partitioned into distinct odd parts, its partition having the parts $9, 7, 5, 3$, and 1, followed by the first missing subsequence, and after it exactly one subsequence of consecutive odd parts.

Proposition 4.7.1. *The Path Procedure applied to the partitions of n having exactly two 5-distinct parts greater than 4 generates partitions of $m = 3k_1^2 + 20k_1 + 50$, with $5 \leq k_1 \leq \left\lfloor \frac{n-5}{2} \right\rfloor$, those being precisely all of the numbers whose partition has parts $9, 7, 5, 3$, and 1, one subsequence of missing parts, and after it exactly one subsequence of consecutive odd parts.*

Proof. A partition having only the two first subsequences of consecutive odd parts has the parts $2(2k_1+5)-1, \ldots, 2(k_1+2)-1, 2(k_1+1)-1, 9, 7, 5, 3, 1$. Noting that $m = 3k_1^2 + 20k_1 + 50$ can be written as

$$\begin{aligned} m &= 2(2k_1+5) - 1 + \cdots + 2(k_1+1) - 1 + 9 + 7 + 5 + 3 + 1 \\ &= \frac{(2(2k_1+5) - 1 + 2(k_1+1) - 1)(k_1+5)}{2} + 9 + 7 + 5 + 3 + 1 \\ &= (3k_1+5)(k_1+5) + 25 \\ &= 3k_1^2 + 20k_1 + 50, \end{aligned}$$

then $\mu = (2(2k_1+5)-1, \ldots, 2(k_1+2)-1, 2(k_1+1)-1, 9, 7, 5, 3, 1)$ is a partition of m having exactly two subsequences of consecutive odd parts.

Clearly k_1 has to be at most $\left\lfloor \dfrac{n-5}{2} \right\rfloor$, otherwise the greatest part $2(2k_1+5)-1$ would exceed $2n-1$.

□

The second missing subsequence of $d_{s-1} = k_2 \geq 0$ consecutive odd integers that do not appear as parts of the partition is

$$2(2k_1+k_2+5)-1, \ldots, 2(2k_1+7)-1, 2(2k_1+6)-1.$$

k_2 can actually be equal to 0. Its size is determined by $\lambda_{s-1} = c_{s-1} + d_{s-1}$, where λ_{s-1} is part of a partition of n into 5-distinct parts greater than 4, and if $\lambda_{s-1} = \lambda_s + 5$ this means $d_{s-1} = 0$.

After the second missing subsequence of k_2 consecutive odd integers we have *the third subsequence of consecutive odd parts* that compose the partition, determined by the size of entry c_{s-2} of the matrix. As $c_{s-2} = 5 + c_{s-1} + d_{s-1} = 5 + k_1 + 5 + k_2 = k_1 + k_2 + 10$, the parts of the second subsequence are

$$2(3k_1+2k_2+15)-1, \ldots, 2(2k_1+k_2+5+2)-1, 2(2k_1+k_2+5+1)-1.$$

For example, $n = 30$ generates the partition

$$(59, 57, 55, 53, 51, 49, 47, 45, 43, 41, 39, 37, 35, 33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 13, 11, 9, 7, 5, 3, 1)$$

of 900.

A general characterization of which numbers are partitioned into distinct odd parts, its partition having the parts $9, 7, 5, 3$, and 1, the first and second missing subsequences and after each one of them a subsequence of consecutive odd parts, is given next.

Proposition 4.7.2. *The Path Procedure applied to the partitions of n into 5-distinct parts greater than 4, having exactly three parts, generates partitions of $m = 8k_1^2 + 3k_2^2 + 8k_1k_2 + 90k_1 + 50k_2 + 250$, with $1 \leq k_1 \leq \left\lfloor \dfrac{n-5}{2} \right\rfloor$ and $0 \leq k_2 \leq \left\lfloor \dfrac{n-3k_1-15}{2} \right\rfloor$.*

Proof. A partition having the first, second, and third subsequences of parts can be written as

$$\begin{aligned}
m &= 2(3k_1+2k_2+15)-1+\cdots+2(2k_1+k_2+5+1)-1+2(2k_1+5)-1 \\
&\quad +\cdots+2(k_1+1)-1+9+7+5+3+1 \\
&= \frac{(2(3k_1+2k_2+15)-1+2(2k_1+k_2+5+1)-1)(k_1+k_2+10)}{2} \\
&\quad + \frac{(2(2k_1+5)-1+2(k_1+1)-1)(k_1+5)}{2}+9+7+5+3+1 \\
&= (5k_1+3k_2+20)(k_1+k_2+10)+(3k_1+5)(k_1+5)+25 \\
&= 8k_1^2+3k_2^2+8k_1k_2+90k_1+50k_2+250,
\end{aligned}$$

getting $\mu = (2(3k_1+2k_2+15)-1,\ldots,2(2k_1+k_2+5+1)-1, 2(2k_1+5)-1,\ldots,2(k_1+1)-1,9,7,5,3,1)$ a partition of $8k_1^2+3k_2^2+8k_1k_2+90k_1+50k_2+250$. The limitation for k_2 is obtained by observing that the sequence of all odd integers, those from the missing subsequences and from the subsequences of parts, cannot exceed $2n-1$. So,

$$k_1+k_2+(k_1+5)+(k_1+k_2+10) \leq n \overset{k_2\in\mathbb{N}}{\Longrightarrow} k_2 \leq \left\lfloor \frac{n-3k_1-15}{2} \right\rfloor.$$

□

Now let us consider a partition into distinct odd parts having t *missing subsequences* and $t+1$ *subsequences of consecutive odd parts*. As before, we call $k_1, k_2, \ldots, k_t$ the sizes of the missing subsequences and, consequently, the subsequence after a missing subsequence of size k_i has size $k_1+k_2+\cdots+k_i+5i$. The following lemma establishes the limits for each k_i.

Lemma 4.7.3. *The i^{th} missing subsequence of a partition into distinct odd parts, whose parts derive from the Path Procedure applied to partitions of n into 5-distinct parts greater than 4, is at most*

$$\frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+\frac{5i(i+1)}{2})}{2}.$$

Proof. The sequence of all odd integers, those from the missing subsequences and from the subsequences of parts, cannot exceed $2n-1$. So, for example, when there are 3 missing subsequences and 4 subsequences of parts, it is necessary that

$$(k_1-5)+k_2+k_3+5+(k_1+5)+(k_1+k_2+10)+(k_1+k_2+k_3+15) \leq n,$$

which implies

$$k_3 \leq \frac{n - 4k_1 - 3k_2 + 30}{2}.$$

If there are i missing subsequences and $i+1$ subsequences of parts, it is necessary that the sum of the sizes of the missing subsequences and of the sizes of the subsequences does not exceed n. This means

$$(k_1-5)+k_2+\cdots+k_i+5+(k_1+5)+(k_1+k_2+10)+\cdots+(k_1+k_2+\cdots+k_i+5i) \leq n,$$

which implies

$$(i+1)k_1 + ik_2 + \cdots + 4k_{i-2} + 3k_{i-1} + 2k_i + \frac{5i(i+1)}{2} \leq n,$$

and so,

$$k_i \leq \frac{n - ((i+1)k_1 + ik_2 + \cdots + 4k_{i-2} + 3k_{i-1} + \frac{5i(i+1)}{2})}{2}.$$

$\square$

Lemma 4.7.3 allows us to extend our construction to a more general characterization of the numbers partitioned into distinct odd parts, whose parts derive from the Path Procedure applied to partitions of n into 5-distinct parts greater than 4.

Theorem 4.7.4. *The partitions into distinct odd parts induced by the Path Procedure applied to partitions of n into 5-distinct parts greater than 4, are all of the form*

$$\begin{aligned}
&\sum_{\substack{i=1\\ 5\leq k_1\leq \frac{n-5}{2}\\ i>1,\, 0\leq k_i\leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+\frac{5i(i+1)}{2})}{2}}}^{t} [(t+2-i)^2-1]k_i^2\\
&+\sum_{\substack{i=1\\ 5\leq k_1\leq \frac{n-5}{2}\\ i>1,\, 0\leq k_i\leq \frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+\frac{5i(i+1)}{2})}{2}}}^{t-1} k_i\sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1}\\
&+\sum_{i=1}^{t} k_i\big(5(t-i+1)t^2+15(t-i+1)t-5(t-i)(t-i+1)\big)\\
&+\frac{25[t(t+1)]^2}{4}+25,
\end{aligned}\tag{4.13}$$

where $1 \leq t \leq n-1$.

Proof. First of all, let us rewrite the expression (4.13) by expanding the sums.

$$
\begin{aligned}
&\sum_{\substack{i=1\\5\le k_1\le\frac{n-5}{2}\\ i>1,\,0\le k_i\le\frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+\frac{5i(i+1)}{2})}{2}}}^{t} [(t+2-i)^2-1]k_i^2\\
&+\sum_{\substack{i=1\\5\le k_1\le\frac{n-5}{2}\\ i>1,\,0\le k_i\le\frac{n-((i+1)k_1+ik_2+\cdots+4k_{i-2}+3k_{i-1}+\frac{5i(i+1)}{2})}{2}}}^{t-1} k_i\sum_{j=1}^{t-i} j(2(t-i+3))k_{t-j+1}\\
&+\sum_{i=1}^{t} k_i\big(5(t-i+1)t^2+15(t-i+1)t-5(t-i)(t-i+1)\big)\\
&+\frac{25[t(t+1)]^2}{4}+25\\
=\ &((t+1)^2-1)k_1^2+(t^2-1)k_2^2+\cdots+8k_{t-1}^2+3k_t^2\\
&+k_1(2(t+2)k_t+2(2(t+2))k_{t-1}+\cdots+(t-2)(2(t+2))k_3\\
&+(t-1)(2(t+2))k_2)\\
&+k_2(2(t+1)k_t+2(2(t+1))k_{t-1}+\cdots+(t-2)(2(t+1))k_3)\\
&+\cdots\\
&+k_{t-3}(12k_t+24k_{t-1}+36k_{t-2})\\
&+k_{t-2}(10k_t+20k_{t-1})\\
&+k_{t-1}(8k_t)\\
&+k_1((2t)t^2+(6t)t-2(t-1)t)\\
&+k_2(2(t-1)t^2+6(t-1)t-2(t-2)(t-1))\\
&+\cdots\\
&+k_{t-1}(4t^2+12t-4)\\
&+k_t(2t^2+6t)\\
&+\frac{25[t(t+1)]^2}{4}+25
\end{aligned}
\tag{4.14}
$$

Let us consider a partition generated by the Path Procedure induced by the partitions of n into 5-distinct parts greater than 4, and suppose it has t subsequences of missing parts. Let us call their sizes k_1-5, k_2, …, k_t. So, the sizes of the subsequences of consecutive odd parts are 5, k_1+5, k_1+k_2+10, $k_1+k_2+k_3+15$, …, $k_1+k_2+\cdots+k_t+5t$, and the actual partition we are considering has the following subsequences of consecutive odd parts:

$$2((t+1)k_1+\cdots+3k_{t-1}+2k_t+\tfrac{5(t+1)t}{2})-1,\ldots,2(tk_1+\cdots+2k_{t-1}+k_t+\tfrac{5t(t-1)}{2}+2)-1,$$
$$2(tk_1+\cdots+2k_{t-1}+k_t+\tfrac{5t(t-1)}{2}+1)-1;$$

$2(tk_1 + \cdots + 3k_{t-2} + 2k_{t-1} + \frac{5t(t-1)}{2}) - 1, \ldots, 2((t-1)k_1 + \cdots + 2k_{t-2} + k_{t-1} + \frac{5(t-1)(t-2)}{2} + 2) - 1,$

$$2((t-1)k_1 + \cdots + 2k_{t-2} + k_{t-1} + \tfrac{5(t-1)(t-2)}{2} + 1) - 1;$$

$$\cdots$$

$$2(4k_1+3k_2+2k_3+30)-1, \ldots, 2(3k_1+2k_2+k_3+15+2)-1, 2(3k_1+2k_2+k_3+15+1)-1;$$

$$2(3k_1 + 2k_2 + 15) - 1, \ldots, 2(2k_1 + k_2 + 5 + 2) - 1, 2(2k_1 + k_2 + 5 + 1) - 1;$$

$$2(2k_1 + 5) - 1, \ldots, 2(k_1 + 2) - 1, 2(k_1 + 1) - 1;$$

and

$$9, 7, 5, 3, 1.$$

The sum of those parts equals

$$\begin{aligned}
&(2((t+1)k_1 + \cdots + 2k_t + \frac{5(t+1)t}{2}) - 1 + 2(tk_1 + \cdots + k_t + \frac{5t(t-1)}{2} + 1) - 1) \\
&\cdot \frac{(k_1 + \cdots + k_t + 5t)}{2} \\
&+ (2(tk_1 + \cdots + 2k_{t-1} + \frac{5t(t-1)}{2}) - 1 + 2((t-1)k_1 + \cdots + k_{t-1} + \frac{5(t-1)(t-2)}{2} + 1) - 1) \\
&\cdot \frac{(k_1 + \cdots + k_{t-1} + 5(t-1))}{2} \\
&+ \cdots \\
&+ (2(4k_1 + 3k_2 + 2k_3 + 30) - 1 + 2(3k_1 + 2k_2 + k_3 + 15 + 1) - 1) \\
&\cdot \frac{(k_1 + k_2 + k_3 + 15)}{2} \\
&+ \frac{(2(3k_1 + 2k_2 + 15) - 1 + 2(2k_1 + k_2 + 5 + 1) - 1)(k_1 + k_2 + 10)}{2} \\
&+ \frac{(2(2k_1 + 5) - 1 + 2(k_1 + 1) - 1)(k_1 + 5)}{2} \\
&+ 25
\end{aligned}$$

$$\begin{aligned}
= \;& ((2t+1)k_1 + \cdots + 5k_{t-1} + 3k_t + 5t^2)(k_1 + k_2 + \cdots + k_t + 5t) \\
&+ ((2t-1)k_1 + \cdots + 3k_{t-1} + 5(t-1)^2)(k_1 + k_2 + \cdots + k_{t-1} + 5(t-1)) \\
&+ \cdots \\
&+ (7k_1 + 5k_2 + 3k_3 + 45)(k_1 + k_2 + k_3 + 15) \\
&+ (5k_1 + 3k_2 + 20)(k_1 + k_2 + 10) \\
&+ (3k_1 + 5)(k_1 + 5) \\
&+ 25.
\end{aligned}$$

By rearranging the terms in the sum above we get the expression (4.14), which proves the theorem.

□

4.8 The Path Procedure applied to partitions counted by the mock theta function $T_1(-q)$

In this section we describe which partitions into distinct odd parts are obtained from the Path Procedure applied to the partitions generated by mock theta function $T_1(-q)$, that is, the unsigned version of $T_1(q)$ (see [BSS13]).

So, let us consider the mock theta function

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2, q^2)_n}{(-q, q^2)_{n+1}}.$$

Its unsigned version

$$T_1^*(q) := T_1(-q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2, q^2)_n}{(q, q^2)_{n+1}}$$

has general term

$$\frac{q^{2+4+\cdots+2s}(1+q^2)(1+q^4)\cdots(1+q^{2s})}{(1-q)(1-q^3)\cdots(1-q^{2s+1})},$$

which counts the partitions of n containing one or two parts equal to each one of the even numbers from 2 to $2s$, and any number of odd parts less than or equal to $2s+1$.

Example 4.8.1. *The partitions of* 10 *counted by* $T_1^*(q)$ *are*

$$(1,1,1,1,1,1,1,1,1,1), (2,1,1,1,1,1,1,1,1), (2,2,1,1,1,1,1,1),$$

$$(3,2,1,1,1,1,1), (3,2,2,1,1,1)(3,3,2,1,1), (3,3,2,2),$$

$$(4,2,1,1,1,1), (4,2,2,1,1), (4,3,2,1), \textit{ and } (4,4,2).$$

According to [BSS13], the partitions generated by $T_1^*(q)$ can also be expressed in terms of two-line matrices. We describe this matrix representation in the following theorem.

Theorem 4.8.2 (Table p. 243, [BSS13])**.** *The number of partitions of n generated by $T_1^*(q)$ is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s & c_{s+1} \\ d_1 & d_2 & \cdots & d_s & d_{s+1} \end{pmatrix}, \tag{4.15}$$

where $c_{s+1} = 0$, $d_t \geq 0$, $c_t = j_t + c_{t+1} + 2d_{t+1}$, with $j_t \in \{2, 4\}$, and the sum of all entries is equal to n.

Proof. If we write

$$n = 2\cdot(1+j_1)+4\cdot(1+j_2)+\cdots+2s\cdot(1+j_s)+1\cdot d_1+3\cdot d_2+\cdots+(2s+1)\cdot d_{s+1},$$

with $j_t \in \{0, 1\}$ and $d_t \geq 0$, we can easily organize this sum in a two-line matrix like

$$\begin{pmatrix} \sum_{t=1}^{s} 2(1+j_t) + \sum_{t=2}^{s+1} 2d_t & \cdots & 2(1+j_s)+2d_{s+1} & 0 \\ d_1 & \cdots & d_s & d_{s+1} \end{pmatrix},$$

whose entries satisfy exactly the conditions we needed.

□

Remark 4.8.3. *Note that the number of columns in the matrix representation* (4.15) *equals the number of different even parts plus one. Differently from the representations in previous sections, where the number of columns was the same as the number of parts of the partitions, in the present case we can say that the first line of the matrix counts the even parts and the second line counts the odd parts. Since this matrix representation is a bit different from the others, most of the analogous results couldn't be proved.*

Example 4.8.4. *Table 24 gives the matrix representations of the partitions of* 10*, given in Example 4.8.1.*

Now we apply the Path Procedure to the set of matrices from Theorem 4.8.2 and create partitions of integers m into distinct odd parts.

Example 4.8.5. *The partitions of* 10 *from Example 4.8.4 generate the partitions into distinct odd parts given in Table 25.*

Remark 4.8.6. *In every matrix like* (4.15) *we have $c_{s+1} = 0$, which means that the entry which determines the first move in the path from $x + y = n$ to $(0, 0)$ is d_{s+1}. Although, as d_{s+1} may be zero, in this case the first move is determined by the entry c_s. So, if $d_{s+1} > 0$, the partition has $2d_{s+1} + 1$ as its smallest part, and if $d_{s+1} = 0$, the smallest part of the partition is* 1*.*

Partition of 10 generated by $T_1^*(q)$	Matrix of type (4.15)	Partition of 10 generated by $T_1^*(q)$	Matrix of type (4.15)
$(1,1,1,1,1,1,1,1,1,1)$	$\begin{pmatrix} 0 \\ 10 \end{pmatrix}$	$(3,3,2,2)$	$\begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$
$(2,1,1,1,1,1,1,1,1)$	$\begin{pmatrix} 2 & 0 \\ 8 & 0 \end{pmatrix}$	$(4,2,1,1,1,1)$	$\begin{pmatrix} 4 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$
$(2,2,1,1,1,1,1,1)$	$\begin{pmatrix} 4 & 0 \\ 6 & 0 \end{pmatrix}$	$(4,2,2,1,1)$	$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$
$(3,2,1,1,1,1,1)$	$\begin{pmatrix} 4 & 0 \\ 5 & 1 \end{pmatrix}$	$(4,3,2,1)$	$\begin{pmatrix} 6 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
$(3,2,2,1,1,1)$	$\begin{pmatrix} 6 & 0 \\ 3 & 1 \end{pmatrix}$	$(4,4,2)$	$\begin{pmatrix} 6 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(3,3,2,1,1)$	$\begin{pmatrix} 6 & 0 \\ 2 & 2 \end{pmatrix}$		

Table 24: Table for Example 4.8.4

Figure 4.10 below illustrates the distribution of frequencies of partitions of m into distinct parts in a square of size 20×20, induced by the partitions of 20, according to Theorem 4.8.2. Each cell contains how many partitions of m (indicated in the right down side of the cell) are generated by the matrix representation of the partitions of 20.

When applied to the matrices of Theorem 4.8.2, the Path Procedure motivates the next definition.

Definition 4.8.7. We call $P_{T(od)}(m)$ the set of partitions of m into distinct odd parts greater than or equal to 1 whose size of any subsequence of consecutive odd integers equals two times the size of the previous subsequence of consecutive odd integers that were omitted before the subsequence started plus the size of the previous subsequence of consecutive odd parts plus 2 or 4. Also, $\left|P_{T(od)}(m)\right| = p_{T(od)}(m)$.

Remark 4.8.8. *If no odd integer is omitted after some subsequence of parts, we assume the number of omitted parts is zero, and the size of the following subsequence of odd parts will be the size of the previous subsequence of consecutive odd parts plus* 2 *or* 4*.*

Now we describe what kind of partitions into distinct odd parts may be obtained from the partitions generated by $T_1^*(q)$.

First we recall Remark 4.8.6, which says that if a partition of n generated by the function $T_1^*(q)$ has an even part as its greatest one, then the

Partition of 10 generated by $T_1^*(q)$	Matrix of type (4.15)	Partition into distinct odd parts
$(1,1,1,1,1,1,1,1,1,1)$	$\begin{pmatrix} 0 \\ 10 \end{pmatrix}$	$\emptyset$
$(2,1,1,1,1,1,1,1,1)$	$\begin{pmatrix} 2 & 0 \\ 8 & 0 \end{pmatrix}$	$(3,1)$
$(2,2,1,1,1,1,1,1)$	$\begin{pmatrix} 4 & 0 \\ 6 & 0 \end{pmatrix}$	$(7,5,3,1)$
$(3,2,1,1,1,1,1)$	$\begin{pmatrix} 4 & 0 \\ 5 & 1 \end{pmatrix}$	$(9,7,5,3)$
$(3,2,2,1,1,1)$	$\begin{pmatrix} 6 & 0 \\ 3 & 1 \end{pmatrix}$	$(13,11,9,7,5,3)$
$(3,3,2,1,1)$	$\begin{pmatrix} 6 & 0 \\ 2 & 2 \end{pmatrix}$	$(15,13,11,9,7,5)$
$(3,3,2,2)$	$\begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$	$(19,17,15,13,11,9,7,5)$
$(4,2,1,1,1,1)$	$\begin{pmatrix} 4 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$	$(11,9,7,5,3,1)$
$(4,2,2,1,1)$	$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$	$(15,13,11,9,7,5,3,1)$
$(4,3,2,1)$	$\begin{pmatrix} 6 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$(17,15,13,11,9,7,3,1)$
$(4,4,2)$	$\begin{pmatrix} 6 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$(19,17,15,13,11,9,7,5,3,1)$

Table 25: Table for Example 4.8.5

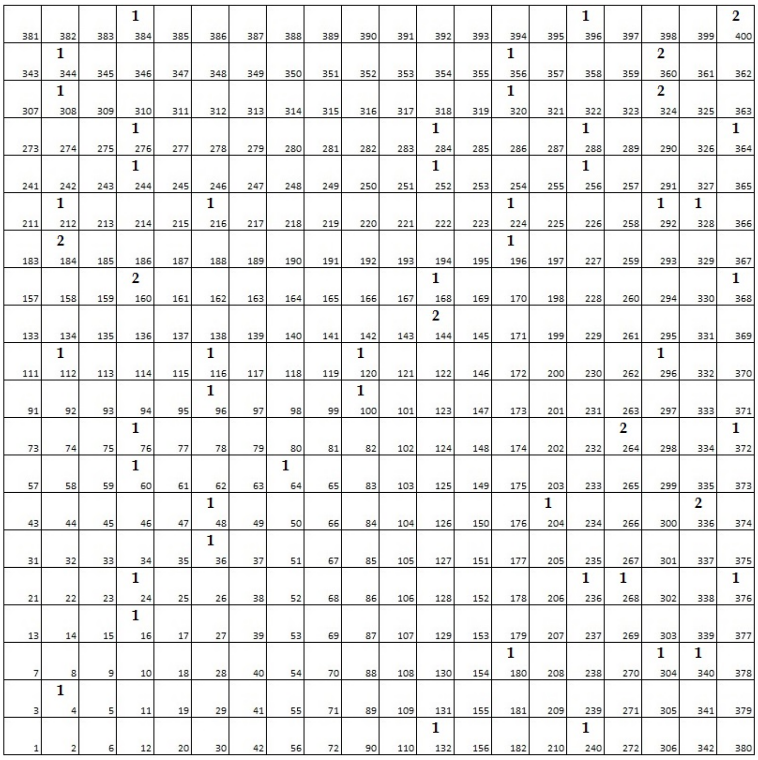


Figure 4.10: $n \times n$ square for $n = 20$

corresponding partition into distinct odd parts via Path Procedure has 1 as its smallest part. On the other hand, if the greatest part of a partition of n generated by the function $T_1^*(q)$ is odd, then the smallest part of the corresponding partition into distinct odd parts via Path Procedure is $2d_{s+1} + 1$ (see for example partitions $(2, 2, 1, 1, 1, 1, 1, 1)$ and $(3, 3, 2, 2)$ of 10 in Example 4.8.5).

Therefore, as the smallest part of a generated partition into distinct odd parts may be any integer greater than or equal to 1, we have *the first missing subsequence* of $k_1 \geq 0$ consecutive odd integers. After it, *the first subsequence of parts* of the partition has size $j_1 + 2k_1$, with $j_1 \in \{2, 4\}$. That is, the sequence of parts $2(k_1+1)-1, 2(k_1+2)-1, \cdots 2(3k_1)-1, 2(3k_1+1)-1, 2(3k_1+2)-1$ and maybe $2(3k_1+3)-1$ and $2(3k_1+4)-1$.

It is not difficult to choose between $j_1 = 2$ or $j_1 = 4$ when, after the first subsequence of parts, *the second missing subsequence* has size $k_2 > 0$. If $k_2 = 0$, we need to verify the size of *the second subsequence of parts*, which is $j_2 + (j_1 + 2k_1) + 2k_2$, with $j_2 \in \{2, 4\}$.

This process goes on until the end of the sequence of different odd parts of the partition, paying attention to the fact that the last entry d_1 depends on the number n, whose original partition induced the partition into distinct odd parts.

Some numbers that appear as partitions into distinct odd parts, induced by the Path Procedure applied to the function $T_1^*(q)$, are described in the following result.

Proposition 4.8.9. *The Path Procedure applied to partitions generated by the mock theta function $T_1^*(q)$ induces partitions of $(2k)^2$ into distinct odd parts.*

Proof. We may partition $(2k)^2$ as

$$(2(2k) - 1, 2(2k - 1) - 1, \ldots, 3, 1),$$

that is, in $2k$ consecutive odd parts. This means that all the missing subsequences have size zero, and the second line of the matrix which originated the partition of $(2k)^2$ has all its entries equal to zero (except possibly for d_1, which does not generate any part). So, the partition of $(2k)^2$ is determined by the first line of the matrix, which has even decreasing entries from c_1 to c_{s+1} whose difference between consecutive entries equals 2 or 4.

If $2k = t(t + 1)$ for some $t \in \mathbb{N}$, we can write

$$2k = t(t + 1) = 2t + (2t - 2) + \cdots + 4 + 2,$$

and we get $2t$, $2t - 2$, ..., 4, 2, and 0 as the entries c_1, c_2, ..., c_s, and c_{s+1}.

If $2k = t(t + 1) + 2l$ for $1 \leq l \leq t$, we take the values of c_1, c_2, ..., c_s, and c_{s+1} from the previous case and add 2 to the first l entries c_i.

So we may write

$$\begin{aligned} 2k &= t(t + 1) + 2l = 2(t + 1) + (2(t + 1) - 2) + \cdots + (2(t + 1) - 2(l - 1)) \\ &\quad + (2(t + 1) - 2(l + 1)) + \cdots + 4 + 2, \end{aligned}$$

and we get $2(t + 1)$, $2(t + 1) - 2$, ..., $2(t + 1) - 2(l - 1)$,$2(t + 1) - 2(l + 1)$, ..., 4, 2, and 0 as the entries c_1, c_2, ..., c_s, and c_{s+1}.

□

Example 4.8.10. *Take the partitions* $(2, 1, 1, 1, 1, 1, 1, 1, 1)$*,* $(2, 2, 1, 1, 1, 1, 1, 1)$*,* $(4, 2, 1, 1, 1, 1)$*,* $(4, 2, 2, 1, 1)$*, and* $(4, 4, 2)$ *of* 10 *from Example 4.8.5. They generate partitions of* 2^2*,* 4^2*,* 6^2*,* 8^2*, and* 10^2*, respectively.*

4.9 The Path Procedure applied to different matrix representations for unrestricted partitions

Besides the representation given by Theorem 4.2.1 in Section 4.4, unrestricted integer partitions have at least two more matrix representations, also given in [SMR11].

Theorem 4.9.1 (Theorem 4.3, [SMR11]). *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix}, \tag{4.16}$$

where $d_t \neq 0$, $c_t \geq 1 + c_{t+1} + d_{t+1}$, and the sum of all entries is equal to n.

Theorem 4.9.2 (Corollary 4.5, [SMR11]). *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_s \\ d_1 & d_2 & d_3 & \cdots & d_s \end{pmatrix}, \tag{4.17}$$

where $c_s \neq 0$, $c_t \geq 2 + c_{t+1} + d_{t+1}$, and the sum of all entries is equal to n.

The bijective proofs of both theorems can be found in [BSS10]. Differently from the first matrix representation, studied in Section 4.4, the number s of columns in matrices (4.16) and (4.17) equals the size of the side of the Durfee square of the associated partition. As an example, we take $n = 5$ and show all of its partitions with their associated matrices of types (4.16) and (4.17).

Example 4.9.3. *For $n = 5$ we have $p(5) = 7$, and so there are 7 matrices satisfying Theorems 4.9.1 and 4.9.2, as shown in Table 26.*

We now apply the Path Procedure to the matrices associated to the partitions of n, then generating partitions of $m \leq n^2$ into distinct odd parts.

Example 4.9.4. *The matrices associated to the partitions of $n = 6$ from Example 4.9.3 generate the partitions into distinct odd parts contained in Table 27.*

Theorem 4.9.5. *The Path Procedure applied to the matrices of Theorem 4.9.1 generates partitions of $m = j^2 - k^2$, for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $2k + 1 \leq j \leq n - 1$.*

$P(5)$	Matrix of type (4.16)	Matrix of type (4.17)
$(1,1,1,1,1)$	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$
$(2,1,1,1)$	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$(2,2,1)$	$\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$
$(3,1,1)$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$(3,2)$	$\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$
$(4,1)$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$
(5)	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 0 \end{pmatrix}$

Table 26: Table for Example 4.9.3

$P(5)$	Matrix of type (4.16)	Partition into distinct odd parts	Matrix of type (4.17)	Partition into distinct odd parts
$(1,1,1,1,1)$	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$	$\emptyset$	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$	(1)
$(2,1,1,1)$	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$	(1)	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	$(3,1)$
$(2,2,1)$	$\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$	$(5,3)$	$\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$	$(7,5,3,1)$
$(3,1,1)$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	$(3,1)$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	$(5,3,1)$
$(3,2)$	$\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$	$(7,5,3)$	$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$	$(9,7,5,3,1)$
$(4,1)$	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	$(5,3,1)$	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$(7,5,3,1)$
(5)	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$(7,5,3,1)$	$\begin{pmatrix} 5 \\ 0 \end{pmatrix}$	$(9,7,5,3,1)$

Table 27: Table for Example 4.9.4

Proof. Note that

$$\begin{aligned} j^2 - k^2 &= (j+k)(j-k) \\ &= \frac{(2(k+1)-1+2j-1)(j-(k+1)+1)}{2} \\ &= (2(k+1)-1)+(2(k+2)-1)+\cdots+(2(j-1)-1)+(2j-1). \end{aligned}$$

If seen as a partition, the sequence $((2j-1),(2(j-1)-1),\ldots,(2(k+2)-1),(2(k+1)-1))$ is generated by the Path Procedure applied to a matrix of order 2×2, having entries $c_2 = 0$, $d_2 = k$, $c_1 = j-(k+1)+1 = j = k$, and $d_1 = n - c_2 - d_2 - c_1 = n - j$. Observe that

$$c_1 \geq 1 + c_2 + d_2 \Longrightarrow j - k \geq 1 + k \Longrightarrow j \geq 2k+1$$

and

$$d_1 \neq 0 \Longrightarrow n - j \geq 1 \Longrightarrow n - 1 \geq j.$$

So we get the limitation $2k+1 \leq j \leq n-1$.

We get the limitations for k by observing that, if we had $k > \left\lfloor \frac{n}{2} \right\rfloor - 1$, then

$$k > \left\lfloor \frac{n}{2} \right\rfloor - 1 \Longrightarrow k \geq \left\lfloor \frac{n}{2} \right\rfloor \Longrightarrow j \geq 2\left\lfloor \frac{n}{2} \right\rfloor + 1 \geq n,$$

which is a contradiction.

□

Theorem 4.9.6. *The Path Procedure applied to the matrices of Theorem 4.9.2 generates partitions of* $m = 1 + j^2 - k^2$, *for* $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ *and* $2k+2 \leq j \leq n$.

Proof. Note that

$$\begin{aligned} 1 + j^2 - k^2 &= 1 + (j+k)(j-k) \\ &= 1 + \frac{(2(k+1)-1+2j-1)(j-(k+1)+1)}{2} \\ &= 1 + (2(k+1)-1)+(2(k+2)-1)+\cdots+(2(j-1)-1)+(2j-1). \end{aligned}$$

If seen as a partition, the sequence $((2j-1),(2(j-1)-1)\ldots,(2(k+2)-1),(2(k+1)-1),1)$ is generated by the Path Procedure applied to a matrix of order 2×2, having entries $c_2 = 1$, $d_2 = k-1$, $c_1 = j-(k+1)+1 = j-k$, and $d_1 = n - c_2 - d_2 - c_1 = n - j$. Observe that

$$c_1 \geq 2 + c_2 + d_2 \Longrightarrow j - k \geq 2 + 1 + k - 1 \Longrightarrow j \geq 2k+2$$

and

$$d_1 \geq 0 \Longrightarrow n - j \geq 0 \Longrightarrow n \geq j.$$

So we get the limitation $2k + 2 \leq j \leq n$.

Moreover,

$$d_2 \geq 0 \Longrightarrow k - 1 \geq 0 \Longrightarrow k \geq 1$$

and if we had $k > \left\lfloor \frac{n}{2} \right\rfloor - 1$, then

$$k > \left\lfloor \frac{n}{2} \right\rfloor - 1 \Longrightarrow k \geq \left\lfloor \frac{n}{2} \right\rfloor \Longrightarrow j \geq 2\left\lfloor \frac{n}{2} \right\rfloor + 1 \geq n,$$

which is a contradiction. So, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

□

With analogous arguments, we can set a more general result, as it follows.

Theorem 4.9.7. *The Path Procedure applied to the order 2×2 matrices of Theorem 4.9.1 generates precisely partitions of $m = t^2 + j^2 - k^2$, for $0 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor - 2$, $t + 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $2k + 1 \leq j \leq n - 1$.*

Proof. Note that

$$\begin{aligned} t^2 + j^2 - k^2 &= t^2 + (j + k)(j - k) \\ &= 1 + 3 + \cdots + 2t - 1 + \frac{(2(k+1) - 1 + 2j - 1)(j - (k+1) + 1)}{2} \\ &= 1 + 3 + \cdots + 2t - 1 + (2(k+1) - 1) + (2(k+2) - 1) \\ &\quad + \cdots + (2j - 1). \end{aligned}$$

If seen as a partition, the sequence $((2j-1), \ldots, (2(k+2)-1), (2(k+1)-1), 2t-1\ldots, 3, 1)$ is generated by the Path Procedure applied to a matrix of order 2×2, having entries $c_2 = t$, $d_2 = k - t$, $c_1 = j - k$, and $d_1 = n - j$. Clearly the limits set for t, k, and j satisfy the conditions we need, but it has to be explained why are these limitations precisely the exact ranges for t, k, and j.

Let us suppose $t > \left\lfloor \frac{n}{2} \right\rfloor - 2$, saying $t = \left\lfloor \frac{n}{2} \right\rfloor - 1 + r$, with $r \geq 0$. As $k \geq t + 1$, then $k \geq \left\lfloor \frac{n}{2} \right\rfloor + r$, which implies $j \geq 2\left\lfloor \frac{n}{2} \right\rfloor + 2r + 1 \geq n$. But then the entry d_1 can only be 0, which contradicts the conditions of Theroem 4.9.1.

Moreover, as we are dealing with matrices of order 2×2, the original partition of n associated to each matrix has Durfee square of side 2. So, the generating function for these partitions of n is

$$\begin{aligned}
\sum_{n=0}^{\infty} a(n)q^n &= \frac{q^4}{(1-q)^2(1-q^2)^2} \\
&= -\frac{1}{8(1+q)} + \frac{1}{16(1+q)^2} - \frac{1}{8(1-q)} + \frac{11}{16(1-q)^2} - \frac{3}{4(1-q)^3} \\
&\quad + \frac{1}{4(1-q)^4} \\
&= \sum_{n=0}^{\infty} \left[-\frac{1}{8}(-1)^n + \frac{1}{16}(-1)^n(n+1) - \frac{1}{8} + \frac{11}{16}(n+1) - \frac{3}{4}\binom{n+2}{2} \right. \\
&\quad \left. + \frac{1}{4}\binom{n+3}{3} \right] q^n \\
&= \sum_{n=0}^{\infty} \frac{1}{48}(n-1)(3(-1)^n - 3 - 4n + 2n^2)q^n. \qquad (4.18)
\end{aligned}$$

It turns out that the coefficient $a(n)$ of q^n in expression (4.18) is exactly

$$\sum_{t=0}^{\lfloor \frac{n}{2} \rfloor - 2} \sum_{k=t+1}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{j=2k+1}^{n-1} 1,$$

which is $\dfrac{(n+1)(n-1)(n-3)}{24}$ for odd n, and $\dfrac{n(n-1)(n-2)}{24}$ for even n. □

An analogous result is valid for the 2×2 matrices of Theorem 4.9.2.

Theorem 4.9.8. *The Path Procedure applied to the order 2×2 matrices of Theorem 4.9.2 generates precisely partitions of $m = t^2 + j^2 - k^2$, for $1 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, $t \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and $2k + 2 \leq j \leq n$.*

Proof. The result is true once we note that a bijection between the matrices of Theorems 4.9.1 and 4.9.2 makes

$$\begin{pmatrix} j-k & t \\ n-j & k-t \end{pmatrix} \longleftrightarrow \begin{pmatrix} j-k+1 & t+1 \\ n-j-1 & k-t-1 \end{pmatrix}$$

□

4.10 Final words

Once we have a bijection between some set of partitions and some set of matrices, it may be possible to apply the Path Procedure to any type of integer partitions. Therefore, the Path Procedure turns out being a promising road in the study of integer partitions and partition identities, as already the two-line matrix representation for different sets of partitions is.

Clearly the results registered in the previous sections of this chapter do not cover all of the possibilities. Many more results may be conjectured by observing the partitions into distinct odd parts generated by the Path Procedure applied to each one of the different sets of partitions.

APPENDIX A

Important Tables

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
0	1																																													
1	0	1																																												
2	0	1	0																																											
3	0	1	0	1																																										
4	0	1	0	1	0																																									
5	0	1	0	1	1	0																																								
6	0	1	0	1	1	0	1																																							
7	0	1	0	1	1	1	1	0																																						
8	0	1	0	1	1	1	1	1	0																																					
9	0	1	0	1	1	1	1	1	1	0																																				
10	0	1	0	1	1	1	2	1	2	0	1																																			
11	0	1	0	1	1	1	2	2	2	1	1	0																																		
12	0	1	0	1	1	1	2	2	2	2	2	1	0																																	
13	0	1	0	1	1	1	2	2	3	2	3	1	1	0																																
14	0	1	0	1	1	1	2	2	3	2	4	2	2	1	0																															
15	0	1	0	1	1	1	2	2	3	3	4	3	3	2	0	1																														
16	0	1	0	1	1	1	2	2	3	3	4	4	4	3	2	1	0																													
17	0	1	0	1	1	1	2	2	3	3	5	4	5	4	3	2	1	0																												
18	0	1	0	1	1	1	2	2	3	3	5	4	6	5	5	4	2	1	0																											
19	0	1	0	1	1	1	2	2	3	3	5	5	6	6	6	5	4	2	1	0																										
20	0	1	0	1	1	1	2	2	3	3	5	5	6	7	7	7	6	4	2	1	0																									
21	0	1	0	1	1	1	2	2	3	3	5	5	7	7	8	8	8	6	4	3	0	1																								
22	0	1	0	1	1	1	2	2	3	3	5	5	7	7	9	9	10	8	7	5	2	1	0																							
23	0	1	0	1	1	1	2	2	3	3	5	5	7	8	9	10	11	10	9	8	4	3	1	0																						
24	0	1	0	1	1	1	2	2	3	3	5	5	7	8	9	11	12	12	12	11	7	6	2	1	0																					
25	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	13	13	14	13	11	9	5	3	1	0																				
26	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	11	14	14	16	16	14	13	9	6	2	1	0																			
27	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	14	15	17	18	17	17	13	10	6	3	1	0																		
28	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	14	16	18	20	20	20	18	15	10	6	3	0	1																	
29	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	19	21	22	23	22	20	15	12	6	3	1	0																
30	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	16	20	22	24	26	26	25	21	18	11	7	3	1	0															
31	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	20	23	25	28	29	29	27	24	18	13	7	3	1	0														
32	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	20	24	26	30	32	33	32	31	25	20	13	7	3	1	0													
33	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	24	27	31	34	36	37	37	32	29	21	14	8	3	1	0												
34	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	24	28	32	36	39	41	42	40	37	30	23	14	8	3	1	0											
35	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	28	33	37	41	44	47	46	45	40	33	24	16	8	3	1	0										
36	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	28	34	38	43	47	51	52	53	49	44	35	26	16	8	4	0	1									
37	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	34	39	44	49	54	57	59	58	55	47	39	27	17	9	3	1	0								
38	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	34	40	45	51	57	61	65	66	65	59	53	40	29	18	8	4	1	0							
39	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	40	46	52	59	64	70	73	74	71	67	55	45	31	18	10	3	1	0						
40	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	40	47	53	61	67	74	79	82	82	80	71	61	47	32	20	9	4	1	0					
41	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	47	54	62	69	77	84	89	91	93	86	79	66	50	35	20	10	3	1	0				
42	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	47	55	63	71	80	88	95	100	104	100	97	85	71	54	36	21	10	4	1	0			
43	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	55	64	72	82	91	100	107	113	114	113	105	93	77	57	39	21	11	3	1	0		
44	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	55	65	73	84	94	104	113	122	125	128	124	115	101	82	61	39	24	10	4	1	0	
45	0	1	0	1	1	1	2	2	3	3	5	5	7	8	10	12	15	17	21	25	29	35	41	48	56	65	74	85	96	107	118	129	135	142	141	137	126	110	88	65	43	23	11	4	0	1

Table 28: Values of $p^{1[s]}_{[s]}(n,k)$ in line n and column $n-k$ given by the generating function $f^1_*(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n^2+n}{2}}}{(q;q)_n}$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
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6	0	0	1	0	0	0	1																																							
7	0	0	1	0	0	0	1	0																																						
8	0	0	1	0	0	0	1	1	0																																					
9	0	0	1	0	0	0	1	1	0	0																																				
10	0	0	1	0	0	0	1	1	1	0	0																																			
11	0	0	1	0	0	0	1	1	1	0	0	0																																		
12	0	0	1	0	0	0	1	1	1	1	0	0	1																																	
13	0	0	1	0	0	0	1	1	1	1	0	0	1	0																																
14	0	0	1	0	0	0	1	1	1	1	1	0	1	1	0																															
15	0	0	1	0	0	0	1	1	1	1	1	0	1	1	1	0																														
16	0	0	1	0	0	0	1	1	1	1	1	1	1	1	2	0	0																													
17	0	0	1	0	0	0	1	1	1	1	1	1	1	1	2	1	0	0																												
18	0	0	1	0	0	0	1	1	1	1	1	1	2	1	2	2	1	0	0																											
19	0	0	1	0	0	0	1	1	1	1	1	1	2	1	2	2	2	0	0	0																										
20	0	0	1	0	0	0	1	1	1	1	1	1	2	2	2	2	3	1	0	0	1																									
21	0	0	1	0	0	0	1	1	1	1	1	1	2	2	2	2	3	2	1	0	1	0																								
22	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	3	3	2	0	1	1	0																							
23	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	2	3	3	3	1	1	1	1	0																						
24	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	3	3	4	2	2	1	2	1	0																					
25	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	3	3	4	3	3	1	2	2	0	0																				
26	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	3	4	4	4	2	2	3	2	0	0																			
27	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	3	4	4	5	3	3	3	3	1	0	0																		
28	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	4	4	6	4	4	3	4	3	1	0	0																	
29	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	4	4	6	5	5	4	4	4	3	0	0	0																
30	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	4	6	6	6	5	5	5	5	2	0	0	1															
31	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	4	6	6	7	6	6	5	6	4	1	0	1	0														
32	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	6	6	8	7	7	6	7	6	4	1	1	1	0													
33	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	6	6	8	8	8	7	8	7	6	3	1	1	1	0												
34	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	6	8	9	9	8	9	8	8	6	3	1	2	1	0											
35	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	6	8	9	10	9	10	9	9	8	6	2	2	2	1	0										
36	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	8	9	11	10	11	10	11	10	9	5	3	3	3	0	0									
37	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	8	9	11	11	12	11	12	11	11	8	5	3	4	2	0	0								
38	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	9	11	12	13	12	13	13	13	11	9	5	5	4	2	0	0							
39	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	9	11	12	14	13	14	14	15	13	12	8	6	5	5	1	0	0						
40	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	10	11	12	15	14	15	15	17	15	15	12	9	7	7	4	1	0	0					
41	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	10	11	12	15	15	16	16	18	17	17	15	13	9	8	7	4	0	0	0				
42	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	10	12	12	15	16	17	17	19	19	20	18	17	13	11	9	8	3	0	0	1			
43	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	10	12	12	15	16	18	18	20	20	22	20	20	17	14	11	11	7	2	0	1	0		
44	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	10	12	13	15	16	19	19	21	21	24	23	23	21	19	15	14	11	7	1	1	1	0	
45	0	0	1	0	0	0	1	1	1	1	1	1	2	2	3	3	4	4	5	5	7	7	9	10	12	13	15	16	19	20	22	22	25	25	26	24	23	19	17	14	12	5	2	1	1	0

Table 29: Values of $p_{[s]}^{2[s]}(n,k)$ in line n and column $n-k$ given by the generating function $f_*^2(q) = f_{1^*}(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
0	1																																													
1	0	0																																												
2	0	0	0																																											
3	0	0	0	1																																										
4	0	0	0	1	0																																									
5	0	0	0	1	0	0																																								
6	0	0	0	1	0	0	0																																							
7	0	0	0	1	0	0	0	0																																						
8	0	0	0	1	0	0	0	0	0																																					
9	0	0	0	1	0	0	0	0	0	1																																				
10	0	0	0	1	0	0	0	0	0	1	0																																			
11	0	0	0	1	0	0	0	0	0	1	1	0																																		
12	0	0	0	1	0	0	0	0	0	1	1	0	0																																	
13	0	0	0	1	0	0	0	0	0	1	1	1	0	0																																
14	0	0	0	1	0	0	0	0	0	1	1	1	0	0	0																															
15	0	0	0	1	0	0	0	0	0	1	1	1	1	0	0	0																														
16	0	0	0	1	0	0	0	0	0	1	1	1	1	0	0	0	0																													
17	0	0	0	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0																												
18	0	0	0	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	1																											
19	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	0																										
20	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	0	0	0	1	1	0																									
21	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	0	0	1	1	1	0																								
22	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	0	0	1	1	2	0	0																							
23	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	1	1	2	1	0	0																						
24	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	1	1	2	2	1	0	0																					
25	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	2	2	2	0	0	0																				
26	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	2	2	3	1	0	0	0																			
27	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	1	2	2	3	2	1	0	0	0																		
28	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	1	2	2	3	3	2	0	0	0	0																	
29	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	3	1	0	0	0	0																
30	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4	2	1	0	0	0	1															
31	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	3	2	0	0	0	1	0														
32	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	3	1	0	0	1	1	0													
33	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	3	3	4	4	4	2	1	0	1	1	1	0												
34	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	3	3	4	4	5	3	2	0	1	1	2	1	0											
35	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	3	4	4	5	4	3	1	1	1	2	2	0	0										
36	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	3	4	4	5	5	4	2	2	1	2	3	2	0	0									
37	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	4	4	5	5	5	3	3	1	2	3	3	1	0	0								
38	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	4	4	5	5	6	4	4	2	2	3	4	3	1	0	0							
39	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	5	4	5	5	6	5	5	3	3	3	4	4	3	0	0	0						
40	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	5	4	5	5	6	6	6	4	4	3	4	5	5	2	0	0	0					
41	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	5	5	5	5	6	6	7	5	5	4	4	5	6	4	1	0	0	0				
42	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	5	5	5	5	6	6	8	6	6	5	5	5	7	6	4	1	0	0	0			
43	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	5	5	6	5	6	6	8	7	7	6	6	5	7	7	6	3	0	0	0	0		
44	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	5	5	6	5	6	6	8	8	8	7	7	6	7	8	8	6	2	0	0	0	0	
45	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	5	5	6	6	6	6	8	8	9	8	8	7	8	8	9	8	5	1	0	0	0	1

Table 30: Values of $p^{3[s]}_{[s]}(n,k)$ in line n and column $n-k$ given by the generating function $f^3_*(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{3(n^2+n)}{2}}}{(q;q)_n}$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
0	1																																													
1	0	0																																												
2	0	0	0																																											
3	0	0	0	0																																										
4	0	0	0	0	1																																									
5	0	0	0	0	1	0																																								
6	0	0	0	0	1	0	0																																							
7	0	0	0	0	1	0	0	0																																						
8	0	0	0	0	1	0	0	0	0																																					
9	0	0	0	0	1	0	0	0	0	0																																				
10	0	0	0	0	1	0	0	0	0	0	0																																			
11	0	0	0	0	1	0	0	0	0	0	0	0																																		
12	0	0	0	0	1	0	0	0	0	0	0	0	1																																	
13	0	0	0	0	1	0	0	0	0	0	0	0	1	0																																
14	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0																															
15	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0																														
16	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0																													
17	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	0	0	0																												
18	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0	0																											
19	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0																										
20	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0																									
21	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0																								
22	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0																							
23	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0																						
24	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	1																					
25	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	1	0																				
26	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	0																			
27	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	0																		
28	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	0	0																	
29	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	1	0	0																
30	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	1	1	2	2	1	0	0															
31	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	1	1	2	2	2	0	0	0														
32	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	2	2	3	1	0	0	0													
33	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	2	2	3	2	1	0	0	0												
34	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	2	0	0	0	0											
35	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	3	1	0	0	0	0										
36	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	3	3	4	2	1	0	0	0	0									
37	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	3	3	4	3	2	0	0	0	0	0								
38	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4	4	3	1	0	0	0	0	0							
39	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	3	3	4	4	4	2	1	0	0	0	0	0						
40	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	3	2	0	0	0	0	0	1					
41	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	4	3	1	0	0	0	0	1	0				
42	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	4	2	1	0	0	0	1	1	0			
43	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	5	3	2	0	0	0	1	1	1	0		
44	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	6	4	3	1	0	0	1	1	2	1	0	
45	0	0	0	0	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	2	3	3	4	4	5	5	6	5	4	2	1	0	1	1	2	2	0	0

Table 31: Values of $p^{4[s]}_{[s]}(n,k)$ in line n and column $n-k$ given by the generating function $f^4_*(q) = \sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(q;q)_n}$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
0	1																																													
1	0	0																																												
2	0	0	0																																											
3	0	0	0	1																																										
4	0	0	0	0	0																																									
5	0	0	0	0	0	1																																								
6	0	0	0	0	0	1	0																																							
7	0	0	0	0	0	1	0	0																																						
8	0	0	0	0	0	1	0	0	0																																					
9	0	0	0	0	0	1	0	0	0	0																																				
10	0	0	0	0	0	1	0	0	0	0	0																																			
11	0	0	0	0	0	1	0	0	0	0	0	0																																		
12	0	0	0	0	0	1	0	0	0	0	0	0	0																																	
13	0	0	0	0	0	1	0	0	0	0	0	0	0	0																																
14	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0																															
15	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1																														
16	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0																													
17	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0																												
18	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	0	0																											
19	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	0																										
20	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0																									
21	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0																								
22	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0																							
23	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0																						
24	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0																					
25	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0																				
26	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0																			
27	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0																		
28	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0																	
29	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0																
30	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1															
31	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	0														
32	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	0													
33	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	0												
34	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	2	0	0											
35	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	2	1	0	0										
36	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	2	2	1	0	0									
37	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	2	2	0	0	0								
38	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	2	2	3	1	0	0	0							
39	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	2	2	3	2	1	0	0	0						
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41	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	2	2	3	3	3	1	0	0	0	0				
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43	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	4	3	2	0	0	0	0	0		
44	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	3	4	4	3	1	0	0	0	0	0	
45	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	2	2	3	3	4	4	4	2	1	0	0	0	0	0

Table 32: Values of $p_{[s]}^{5[s]}(n,k)$ in line n and column $n-k$ given by the generating function $f_*^5(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{5(n^2+n)}{2}}}{(q;q)_n}$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
0	1																																													
1	0	0																																												
2	0	0	0																																											
3	0	0	0	0																																										
4	0	0	0	0	0																																									
5	0	0	0	0	0	0																																								
6	0	0	0	0	0	0	1																																							
7	0	0	0	0	0	0	1	0																																						
8	0	0	0	0	0	0	1	0	0																																					
9	0	0	0	0	0	0	1	0	0	0																																				
10	0	0	0	0	0	0	1	0	0	0	0																																			
11	0	0	0	0	0	0	1	0	0	0	0	0																																		
12	0	0	0	0	0	0	1	0	0	0	0	0	0																																	
13	0	0	0	0	0	0	1	0	0	0	0	0	0	0																																
14	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0																															
15	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0																														
16	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0																													
17	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0																												
18	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1																											
19	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0																										
20	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0																									
21	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0																								
22	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0																							
23	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0																						
24	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0																					
25	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0																				
26	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0																			
27	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0																		
28	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0																	
29	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0																
30	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0															
31	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0														
32	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0													
33	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0												
34	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0											
35	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0										
36	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1									
37	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	0								
38	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	0							
39	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	0						
40	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	2	0	0					
41	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	2	1	0	0				
42	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	2	2	1	0	0			
43	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	2	2	2	0	0	0		
44	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	2	2	3	1	0	0	0	
45	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	2	2	3	2	1	0	0	0

Table 33: Values of $p_{[s]}^{6[s]}(n,k)$ in line n and column $n-k$ given by the generating function $f_*^6(q) = \sum_{n=0}^{\infty} \frac{q^{3(n^2+n)}}{(q;q)_n}$.

APPENDIX B

An alternative interpretation for integer partitions

B.1 Introduction

By considering the unrestricted integer partitions, recall that the Path Procedure induces partitions into distinct odd parts, where the size of every sequence of consecutive odd parts is equal to the number of odd integers that were omitted before this sequence started, or a greater multiple of it. Our objective is to associate each partition into distinct odd parts generated by the Path Procedure to a sequence of positive integers.

B.2 Plane Partitions

Definition B.2.1. A *plane partition* π of n is a left-justified array of positive integers $(\pi_{i,j})_{i,j\geq 1}$ such that $\pi_{i,j} \geq \pi_{i,j+1}$ and $\pi_{i,j} \geq \pi_{i+1,j}\, \forall i,j \geq 1$, and $\sum_{i,j\geq 1} \pi_{i,j} = n$.

Observe that every partition into distinct odd parts $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ generated by the Path Procedure can be seen as a plane partition if we pile up its parts, from the largest to the smallest one, as symmetric hooks, with the largest part on the bottom of the pile and the smallest one at the top of the pile, having a common central axis.

Example B.2.2. *The partition $\lambda = (11, 9, 5, 3)$ can be represented as a plane partition like in Figure B.1. Its associated plane partition is*

$$\begin{array}{cccccc} 4 & 4 & 3 & 2 & 2 & 1 \\ 4 & & & & & \\ 3 & & & & & \\ 2 & & & & & \\ 2 & & & & & \\ 1 & & & & & \end{array}$$

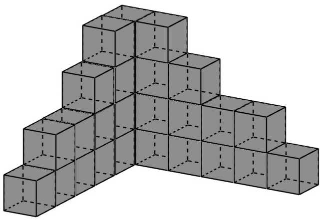

Figure B.1: Illustration for Example B.2.2

As the hooks are symmetric and have a common central axis, we can eliminate one branch of each hook and the central axis, and work only with the remaining branches, which constitute a pile of distinct parts, not necessarily odd anymore.

B.3 Converting partitions into sequences

Let us look to the heights of the columns of the remaining partition and list these heights as a finite sequence $(\eta)_{i=1}^{k} = (n_1, n_2, n_3, \dots, n_k)$ of positive integers. Observe that, according to our construction, consecutive columns may differ from at most one unit and there is necessarily one column of height 1. Therefore, the sequence of consecutive terms associated to the heights of the columns may have repetitions and ends with $n_k = 1$.

Example B.3.1. *When taking the columns from partition $\lambda = (11, 9, 5, 3)$ we get Figure B.2, which can be associated to the sequence of heights $(\eta)_{i=1}^{5} = (4, 3, 2, 2, 1)$.*

Now, let us consider a finite nonincreasing sequence $(\eta)_{i=1}^{k} = (n_1, n_2, \dots, 1)$ of positive integers such that $n_{i+1} \in \{n_i, n_i - 1\}$. If there are repetitions in this sequence, we mark every repeated term, except its last appearance. Note

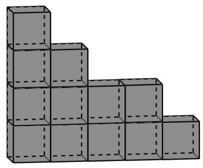

Figure B.2: Illustration for Example B.3.1

that each repeated term means that an odd integer was omitted in the original partition obtained from the Path Procedure. For example, the sequence from Example B.3.1 turns to $(\eta)_{i=1}^{5} = (4, 3, \mathbf{2}, 2, 1)$.

In order to identify which partition λ into distinct odd parts is associated to the sequence $(\eta)_{i=1}^{k}$, some observations have to be made:

- As λ never has 1 as a part, the number of omitted parts of λ equals the number of marked terms of $(\eta)_{i=1}^{k}$ plus 1.

- As the sequences of unmarked consecutive terms of $(\eta)_{i=1}^{k}$ indicate the sequences of consecutive odd parts of λ, we need to verify the size of these sequences of unmarked consecutive terms. According to the rule obeyed by the partitions into distinct odd parts generated by the Path Procedure, $(\eta)_{i=1}^{k}$ has to satisfy the following: each sequence of decreasing unmarked consecutive terms after a sequence of repeated terms (marked terms) has to have the same size as the number of previous repetitions plus 1, or a greater multiple of it.

- The largest term of $(\eta)_{i=1}^{k}$, that is, the highest column of the diagram, indicates how many parts the original partition λ has. So, if the sequence $(\eta)_{i=1}^{k}$ has $n_1 = j$ as its largest term (which also means that the sequence has j different terms), then λ has j distinct odd parts;

- The number of terms of $(\eta)_{i=1}^{k}$, that is, the number of columns of the diagram, determines the size of the largest part of the partition λ. So, if the sequence has k terms, counting repetitions, the original partition has largest part $\lambda_1 = 2 \cdot k + 1$;

Example B.3.2. *By looking to the sequence $(\eta)_{i=1}^{5} = (4, 3, \boldsymbol{2}, 2, 1)$, we observe that the sequence $4, 3$ has size 2, a multiple of 1, which is the number of marked terms until then (none) plus 1; the sequence $2, 1$ has size 2, exactly the number of marked terms until then plus 1. Therefore, $(\eta)_{i=1}^{5} = (4, 3, \boldsymbol{2}, 2, 1)$*

generates a partition λ, according to the Path Procedure. Moreover, as $n_1 = 4$, then λ has 4 distinct odd parts; and as $k = 5$, the largest part of λ is $2 \cdot 5 + 1 = 11$. So,

$$n_5 = 1 \Longrightarrow \lambda_1 = 11;$$

$$n_4 = 2 \Longrightarrow \lambda_2 = 9;$$

$$n_3 = \boldsymbol{2} \Longrightarrow 7 \textit{ is omitted};$$

$$n_2 = 3 \Longrightarrow \lambda_3 = 5;$$

$$n_1 = 4 \Longrightarrow \lambda_4 = 3.$$

As it was supposed to be, according to Example B.3.1, $\lambda = (11, 9, 5, 3)$.

There are some finite sequence $(\eta)_{i=1}^k = (n_1, n_2, n_3, \dots, 1)$ of decreasing positive integers such that $n_{i+1} \in \{n_i, n_i - 1\}$ that do not generate a partition into distinct odd parts as the ones induced by the Path Procedure. The following example shows some possible and impossible sequences $(\eta)_{i=1}^k$ and, in the first case, the associated partitions into distinct odd parts.

Example B.3.3.

Possible $(\eta)_{i=1}^k$	Impossible $(\eta)_{i=1}^k$
$(\eta)_{i=1}^{13} = (11, 10, 9, 8, \mathbf{7}, 7, 6, 5, 4, \mathbf{3}, 3, 2, 1)$ associated to the partition $\lambda = (27, 25, 23, 19, 17, 15, 13, 9, 7, 5, 3)$	$(\eta)_{i=1}^{13} = (11, 10, 9, 8, 7, \mathbf{6}, 6, 5, 4, 3, \mathbf{2}, 2, 1)$
$(\eta)_{i=1}^{10} = (\mathbf{7}, \mathbf{7}, 7, 6, 5, \mathbf{4}, 4, 3, 2, 1)$ associated to the partition $\lambda = (21, 19, 17, 15, 11, 9, 7)$	$(\eta)_{i=1}^{10} = (7, \mathbf{6}, \mathbf{6}, 6, 5, 4, \mathbf{3}, 3, 2, 1)$
$(\eta)_{i=1}^{4} = (3, \mathbf{2}, 2, 1)$ associated to the partition $\lambda = (9, 7, 3)$	$(\eta)_{i=1}^{4} = (\mathbf{3}, 3, 2, 1)$

Table 34: Examples of possible and impossible sequences $(\eta)_{i=1}^k$

Clearly we can find nonincreasing finite sequences of positive integers to be associated to other types of partitions into distinct odd parts. As long as we can apply the Path Procedure to a set of two-line matrices, the induced distinct odd parts can be arranged as a symmetric plane partition and, therefore, be associated to a sequence of positive integers.

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