Universidade Federal do Rio Grande do Sul
Instituto de Matemática

# Kinematic Dynamo
# in
# Random Flows

Mark Thompson

Cadernos de Matemática e Estatística
Série A, nọ 18, DEZ/90
Porto Alegre, dezembro de 1990

# KINEMATIC DYNAMO IN RANDOM FLOWS

**M. Thompson***

Instituto de Matemática
UFRGS
9500 Bento Gonçalves
91 500 Porto Alegre RS

*Research supported by the CNPq under grant 300147-80

## Abstract

The existence of a fast dynamo effect is shown for the kinematic equation of magneto-hydrodynamics with stochastic dependence in the random flows.



Key words: Fast dynamo; kinematic equation;
magneto-hydrodynamics; Lyapunov indices;
large deviation theory

## RESUMO

Dinamo cinemático em fluxos aleatórios.

A existência de um efeito de um dínamo rápido é demonstrado para a equação cinemática de magneto-hidrodinâmico com dependência estocástica nos fluxos aleatórios.



Palavras Chaves: dínamo rápido; equação cinemática; magneto hidro-dinâmico; índices de Lyapunov; deviações grandes.

## PREFÁCIO

O presente trabalho é uma versão extensamente revisada de um trabalho apresentado no 22º Seminário Brasileiro de Análise, e faz parte de uma série de trabalhos sobre problemas correlatos com o problema do dínamo.

Agradeço a excelente digitação da Srª Rosaura M. Pinheiro.

# 1. Formulation of the dynamo problem and basic results
## 1.1. Introduction and Preliminaries

The idea of a hydromagnetic dynamo was first put forward by Larmor in 1919 in a report to the British Association for Advancent of Science in an attempt to explain the existence of the Earth 's and the Sun 's magnetic fields. Roughly speaking it was proposed that motion of a conducting fluid or gas could lead to the enhancement of an initially weak magnetic in the absence of external electromagnetic forces. The linear version of this problem in the absence of the Lorentz back reaction may simply be considered to be that of the induction equation of magneto hydrodynamics with a prescribed divergence free velocity field:

$$\frac{\partial B}{\partial t} = \nu_m \Delta B - v.\nabla B + \left(\frac{\partial v}{\partial x}\right) B, \quad divB = 0$$

$B(x,0) = B_0(x)$ , where $B$ is the magnetic field and $\nu_m = \ell\, v_o\, R_m^{-1}$, where $R_m$ the magnetic Reynolds number and $\ell$ and $v_o$ are the characteristic scale and amplitude of the velocity field. In order to get some idea of the magnitudes involved, in the convective shells of the Sun, $R_m \sim 10^8$ ; in the liquid core of the Earth, $\nu_m \sim 6.10^{-3}$ and in the Super Phoenix power plants, $\nu_m \sim 1.2.10^{-2}$. It is therefore clear that a relatively small or small magnetic diffusivity is a feature of many physical and technological problems. Of key importance is some energy or mean energy growth rate $g^{\nu_m}(2)$ of the solution $B$

$$g^{\nu_m}(2) = \overline{\lim_{t\to\infty}} \frac{log \,|||\, B \,|||^2}{t}$$

where $|||\ |||$ is simply the $L^2(R^3)$ norm in case $v$ is deterministic and some averaged $L^2$-norm is the case of stochastic $v$. The Russian physicists Zeldovich and Ruzmaiken, were led to classify kinematic dynamos into fast and slow dynamos according to whether $g^{\nu_m}(2)$ tends or not to a strictly positive limit as $\nu_m \to 0$. The case of steady flow has attracted a great deal of interest since the seminal work of Arnold et al in [7] where it was suggested that dynamo growth should be related to the positive Lyapunov exponent in chaotic flow, although it should be said that the seed of this idea is present in the physics literature for much longer (see [10], [11], [49]).

More recent work on these lines is described in the tecnical report from the Laboratory for Plasma Research, Maryland, [23] (see also [21], [22], [24], [25], [26] for the work of this group) and important mathematical contributions have been made by Collet in [12].

The case of fluid flows with an extrinsic stochastic dependence has also received considerable analysis principally envolving a multiple scale analysis of a slowly varying magnetic field (see the book by Moffatt [45] for a discussion of the Steinbeck, Krauser, Rädler tecniques as well as the papers [42], [43], [44]). On the other some interesting results have been obtained by various Russian mathematicians and physicists in the case of non stationary fluid flows in the limit of small magnetic diffusivity using extensions of the Oseledec theory of Lyapunov exponents to stochastic flows. This analysis is easier than the case of stationary fluid flows giving rise to chaotic dynamics as from our point of view it may be reformulated in terms of the theory of degenerate diffusions and their support properties.

The present paper is concerned with problems related to the kinematic dynamo equations. In $n$-dimensions, $n \geq 3$, the equation equivalent to the kinematic equations for magneto- hydrodynamics may be taken in the form:

$$\frac{\partial y^\epsilon}{\partial t} = \frac{\epsilon^2}{2} \Delta_x y^\epsilon - v(x, \xi(t)). \nabla_x y^\epsilon + \left(\frac{\partial v}{\partial x}\right) (x, \xi(t)) y^\epsilon \qquad (1.1)a$$

$$div\ v = 0,\ x \in R^n, \quad t > 0, \qquad (1.1)b$$

$$y^\epsilon|_{t=0} = y_0(x),\ div\ y_0 = 0,\ x \in R^n \qquad (1.1)c$$

(1.1)a $\xi(t)(\sigma)$ is assumed to be an ergodic diffusion, $\sigma \in (\Omega, P)$, a probability space, with an associated system of measure preseving transformations $\theta_t$ and state space $\mathcal{M}$, where $\mathcal{M}$ is a compact connected $m_0$-dimensional manifold. The diffusion $\xi$ is assumed to be strongly elliptic with invariant measure $\rho$ and generated by the stochastic differential equation.

$$d\xi = d\xi(t) = X_0(\xi)dt + \sum_{i=1}^{r} X_i(\xi) o\ d\ B_i\ (t)(\sigma), \quad \xi(0) = \xi_0 \qquad (1.2)$$

where the $X_i(\xi)$ are smooth vector fields in $\mu$ and $o\ d\ B_i$ indicates the Fisk-Stratonovich differential with respect to Bronnian motion (see [31]). We set $H_\xi = \frac{\varepsilon^2}{2} \Delta_x - v.\nabla_x$.

The limiting case $\varepsilon = 0$ leads to the related system of equations

$$\frac{\partial y^0}{\partial t} = -v.\nabla y^0 + \left(\frac{\partial v}{\partial x}\right) y^0 \qquad (1.3)a$$

$$div\ v = 0, \quad x \in R^n, \quad t > 0 \qquad (1.3)b$$

$$y^0|_{t=0} = y_0(x),\ div\ y_0 = 0,\ x \in R^n, \qquad (1.3)c$$

As before this implies that $div\ y_0 = 0$, $x \in R^n$, $t > 0$.

Let $\gamma(t)x$ be the random flow generated by the dynamical system

$$\frac{d\ \gamma(t)x}{dt} = v(\gamma(t)x,\ \xi(t)(\sigma)) \qquad (1.4)$$

$$\gamma(0) = x$$

so that $\gamma(s+1)(\sigma)x = \gamma(t, \theta_s, \sigma)_o\gamma(s,\sigma)x$, the cocycle property (see [4]).

It is well known that (1.3) has the Lagrangian solution given by

$$y^0(\gamma(t)x, t) = \left(\frac{\partial\gamma(t)x}{\partial x}\right) y_0(x) \qquad (1.5)$$

Setting $\Theta\ (x,t) = \left(\frac{\partial\gamma(t)x}{\partial x}\right)$ we have

$$\frac{\partial\Theta}{\partial t} = \left(\frac{\partial v}{\partial x}\right)(\gamma(t)x)\Theta\ , \qquad (1.6)$$

$$\Theta\ |_{t=0} = I$$

We set $U = \left(\frac{\partial v}{\partial x}\right)$ and $W(t) = U(\gamma(t)x, \xi(t))$, $\qquad (1.7)$

and note that by (1.1)b trace $W = 0$ or $det\ \Theta(t) = 1$

This property of unimodularity is extremely important in the subsequent analysis.

The usual kind of asymptotic analysis as given by Aronson in [8] or by probabilistic methods as in the book by Freidlin and Wentzell [28] tells us that

$$y^\varepsilon(\gamma(t)x, t) = y^0(\gamma(t)x, t) + 0(\varepsilon^2 t)^{\frac{1}{2}})$$

as $\varepsilon \downarrow 0$ for $t \in [0,T]$ using the Lipschitz continuity of first order deviatives of $v$. However, we are interested in a rather more delicate question, the description of $E \int_{R^n} |y^\varepsilon(.,t)|^2\, dx$ for large time $t$, with hopefully, estimates uniform for small $\varepsilon > 0$. Consequently, our asymptotics are different from those of [8], in that we first consider $\varepsilon$ fixed and obtain the asymptotic behaviour for large $t$ only then taking the limit $\varepsilon \to 0$.

A crucial question becomes how to describe the behaviour of $\Theta(t)y_0(x)$ for large $t$, that is to say the case of the perfect conductivity limit. Here, a circle of ideas is involved related to the multiplicative ergodic theorem and its ramifications, in this paper we are particularly concerned with results about mean Lyapunov indices (see [1], [2], [5], [6]).

Now Has 'minski in his work on perturbations of linear systems by White Noise popularised a well known tecnique for analysing the growth properties of expressions such as (1.5). In fact, setting

$$\frac{dz}{dt} = W(t)z,$$

$$z(0) = y_0(x)$$

we see that $z(t) = y^0(\gamma_x(t), t)$.
Set $\theta = z/\| z \|$, $z \neq 0$, $\theta \in P^{n-1}$, where $P^{n-1}$ is the usual $n-1$-dimensional projective space obtained by identifying elements $\theta$ and $-\theta$ on the $n-1$-dimensional sphere, so that

$$\frac{d\theta}{dt} = (W - \theta^t\, W\theta)\theta, \quad \theta(0) = \theta_0\ , \qquad (1.8)a$$

where $\theta^*$ denotes the transpose of $\theta$,

$$|\, z \,| = |\, z(0) \,|\ \exp \int_0^t (\theta^*\, W\theta)(\tau) d\tau \quad . \qquad (1.8)b$$

Accordingly, we may use (1.2), (1.4), (1.8)a to introduce into the problem an auxiliary diffusion

$$\eta(t) = (\gamma(t), \xi(t), \theta(t)) \text{ on the state space}$$
$$S = R^n \times \mathcal{M} \times P^{n-1},\ s = (x, m, \theta),$$

which although degenerate, under additional conditions discussed in section 1.2, may be assumed to be hypoelliptic in the sense of Hörmander (see [31] and [34]). The work of Arnold and Kliemann in as given in [1], [2] does not immediately or fully apply to the case under consideration due to the non-compactness of $S$.

In section 2, making a more immediate use of large deviation theory following the work of Donsker and Varadhan (see [15], [16]) we give a reformulation of certain results of Arnold and Kliemann, requiring additional hypotheses given in section 1.2. Further in general we suppose that the following hypotheses are satisfied.

C1a $v$ and its first order derivatives are Lipschitz continuous with Lipschitz constant $M$.

C1b The initial data satisfy $y_0(x) \neq 0$ a.e and $y_0 \in L^2((1+|x|)^{2k})dx)$, for some $k > 0$.

With regard to these hypotheses C1a is used to guarantee the global existence of $\gamma(t)x$ as well as in the proof of a technical result of Miyahara (see [40], [41]). The hypotheses C1b are used in Lemma 3.4 and Theorem 3.5 dealing with the $\varepsilon \downarrow 0$ limit of $y^\varepsilon$. Under conditions C1a, C1b, control hypotheses C2, C2′ and additional Lyapunov and spectral hypotheses (essentially) C3*, C4′, C5, we establish that

$$\underline{lim}_{\varepsilon \to 0} \underline{lim}_{t \to \infty} \; t^{-1} \, log(E \,||\, y^\varepsilon(.,t) \,||_2^2) \;\geq\; 2g(1) > 0$$

in Theorem 3.5, where $g(p)$ is the p-th mean Lyapunov index (see section 2 and [1]) associated with the limit problem $\varepsilon = 0$.

This in a slightly weakened sense establishes the existence of a "fast" dynamo in the sense of [48].

In the case of linear stochastic velocity fields we use a different hypothesis on the initial data:

C1c: The initial data $y_0(x)$ satisfy $y_0(x) \neq a.e$ and

$$y_0 \in L^2(R^n) \; ; \; ||y_0(x) - y_0(y)|| \;\leq\; (K(|x|)) + K(|y|))||x-y||$$

where

$$\int_{R^n} ||x||^{n+2} \, K(||x||)^2 \, dx < \infty \, .$$

Then certain results of Aronson [8] together with the original large deviation results given in [3], under the conditions

$$g'(2) \ < \ \frac{g(2)}{2} + \frac{\sigma_0^2}{4\,\widehat{g}''(0),} \,,$$

where

$$0 < \sigma_0 < \frac{n\,\widehat{\ell}}{2(n+1)},$$

establish the existence of a fast dynamo independently in Theorem 3.2. Here $g(p)$ refer to the mean Lyapunov exponent of order $p$ associated to - $W^{\bullet}$ and $\widehat{\ell}$ is the largest Lyapunov exponent associated to $\Theta^{-1}$.

### 1.2 The existence of invariant measures

In the present section we need to recall some results related to the control theoretic characterization of the support properties of diffusion given by Stroock and Varadhan and extended by various authors including Kunita [36].

We set $Z = \begin{pmatrix} X_0 \\ v \end{pmatrix}$ and write $h(x,m,\theta) = (W(x,m) - \theta^t W(x,m)\theta)\theta$, finally setting $Y = \begin{pmatrix} Z \\ h \end{pmatrix}$. We continue to write $X_i$ for $\begin{pmatrix} X_i \\ 0 \end{pmatrix}$, $i = 1,\ldots,r$. Let $B_0 = LA(X_1,\ldots,X_r)$ and $B = LA(Z, X_1,\ldots,X_r)$ be the Lie algebras generated by the elements indicated and $I$ be the ideal generated by $(X_1,\ldots,X_r)$ in $B$. The process $\varsigma(t) = (\gamma(t), \xi(t))$ has an infinitesimal generator $L_0 = G + v.\nabla_x$, where $G = \frac{1}{2}\sum_{i=1}^{r} X_1^2 + X_0$.

The system (1.2), (1.4) may be written as

$$d\varsigma = Z dt \ + \sum_{i=1}^{r} X_i\ (\varsigma)\ o\ dB_i$$

Similar definitions hold for the stocastic process $\eta(t)$ with infinitesimal generator $L = L_0 + h.\nabla_\theta$.

In the following we use results formulated in terms of notions of stochastic control theory, first introduced by Stroock and Varadhan and extended in a number of important papers. We refer the reader to the book by Ikeda and Watanabe [31] for the basic theory and for extensions to degenerate diffusions to the papers by Kunita [36], Ichihara and Kunita [30] and

Kliemann [33]. We use the language and notation of these papers without further comment.

Following Kunita in [36] we assume that:

$$C2 \quad dim\ I(x,m) = m_0 + n \text{ for all } (x,m) \in R^n \times M$$

and that $B_0$ is locally finitely generated,

$$[B_0,\ I] \subset B_0 \text{ for all } (x,m) \in R^n \times M .$$

We observe that these conditions may be weakened (see Theorem 5.8 [36] ).

Then by Theorem 5.3 of [36] we have that $\varsigma(t)$ gives rise to the above control system which is strongly completely controllable. This condition has the important consequence (see [36] the remark following Proposition 5.2) that $dim\ B' = m_0 + n + 1$ here $dim\ B_0(m) = m_0$ and $B' = LA\left(Z + \frac{\partial}{\partial t},\ X_1, \ldots, X_r\right)$ and, hence, that $\frac{\partial}{\partial t} + L_0$ is hypoelliptic and the transition function $p_t(x, m, .)$ has a smooth density with respect to the natural Lebesque measure on $R^n \times M$. We summarise a number of technical results in the following sequence of remarks.

**Remark 1**

Note that the formal adjoints $L_0^+$ and $L_0^+ - \frac{\partial}{\partial t}$ are hypoelliptic under the proceding condition (see [22], the remark following equation 4.4).

**Remark 2**

Let $B(R^n \times M)$ and $C(R^n \times M)$ be the Banach spaces of bounded measurable functions over $R^n \times M$ and the continuous bounded functions over $R^n \times M$ respectively. Let $P_t$ be the semi-group generated by $\varsigma(t)$ on $B(S)$ . Then $P_t$ is a strongly Feller continuous semi-group, which is to say that it maps $B(R^n \times M)$ into $C(R^n \times M)$ for each $t > 0$ (see [16]).

**Remark 3**

If there exists an invariant measure $\nu$ for the process $\varsigma$ condition C2 guarantees that it is unique with support $R^n \times M$ (see [5], [36] ).

Now obviously strong recurrence conditions have to be imposed in order to obtain an invariant measure and, indeed, we suppose following the work of Miyahara ( [29], [30] ) that there exists a Lyapunov $V$ satisfying:

C3a . $V(x,m)$ has continuous second order derivatives and satisfies the estimates

$$c_3\,|x|^{p_0} + \alpha_2 \geq V(x,m) \geq c_1\,|x|^{p_0} + \alpha_1 \quad with$$
$$c_1,\ p_0,\ \alpha_1 > 0$$

*C3b* $L_0\,V(x,m) \leq -c_2\,V(x,m) + \beta\,, c_2,\ \beta > 0\,,$ *for all* $x,\ m \in R^n \times \mathcal{M}$

C3c There exists $\hat{V}(x,m)$ with continuous second order derivatives such that

$$\{(x,m)\ :\ \hat{V}(x,m) \geq \ell^*\}$$

is a compact set for each $\ell^* > -\infty$ and a sequence $u_k(x,m) \in D(L_0)$ satisfying:

(1) $u_k(x,m) \geq 1$ all $k$ and all $(x,m) \in R^n \times \mathcal{M}$

(2) for each compact set $B \subset R^n \times \mathcal{M}$

$$\sup_{(x,m)\,\in\,B}\ \sup_k\ u_k(x,m) < \infty$$

(3) for each $(x,m)$

$$\lim_{k\to\infty} \frac{L_0\ u_k}{u_k} = \hat{V}(x,m)$$

(4) for some $A < \infty$

$$\sup_{k,x,m} \frac{L_0 u_k}{u_k} \leq A$$

C3a is assumed in this unnecessarily strong form in order to simplify the definition of the opeator $L_V$ introduced in C4.

Also note that we assume that the functions $V, \tilde{V}$ do not depend on $\theta$ in order to simplify the subsequent arguments. Including such dependence would lead only to some inessential modification of constants.

We remark that C3 implies that the process $\varsigma(t)$ is exponentially $p_0 - th$ ultimately bounded with constant c2 (see Theorem 3.1 [31]), in that

$$E_{x,m} \, |\varsigma_x \, (t)|^{\, p_0} \; \leq \; K + e^{-c_2 t} \, |x|^{\, p_0} \qquad (1.9)$$

We also have the following technical lemma.

Note that as a consequence of (1.9) strictly $n-1$ dimensional velocity fields are excluded. Indeed, suppose that $v_n \; (x, \; \xi(t)(\sigma)) \; \equiv \; 0$ . Then we conclude that $(\gamma \; o \; x)_n \; \equiv \; x_n$ . It follows from (1.9) that

$$| \; x_n \; |^{p_0} \; \leq \; E_{x,m} \; |\varsigma(t)|^p \; \leq \; K + Ce^{-c_2 t} \; |x|^{p_0}$$

and letting $t \; \rightarrow \; \infty$ we have $|x_n|^{p_0} \; \leq \; K$ for all $x_n$ , a contradiction.

This is a desirable property as generally dynamo action is impossible with plane two dimensional motion, see the book by Moffatt [ 45 ] (section 6.8) and also [ 48 ].

Lemma 1 (Miyahara)

**Under hypothesis C3 there exists an invariant measure $\nu$ such that**

$$\int_{R^n \; \times \; \mathcal{M}} V(x,m) \; \nu \; (dx, dm) \; < \; \infty \, .$$

By Remark 3 the measure $\nu$ is the unique ergodic measure with marginal $\rho$ . Now consider the process $\eta(t)$ with the transition probability $p(t, s_0, ds)$ and associated semi-group $T_t$. In the next Lemma we consider the process $\varsigma(t)$ with initial measure $\nu$ . Note that $R^n \; \times \; \mathcal{M}$ is a separable complete metric space (in fact, we have already used this property in (19).

Then we have the following result due to Crauel [13].

Lemma 1,2 (Crauel).

**The stochastic process $\eta(t)$ has an invariant measure $\mu$ with marginal distribution $\nu$ on $R^n \; \times \; \mathcal{M}$.**

We are able to impose a hypothesis analogous to C2, C3 namely:

C2′, C3′ The hypotheses equivalent to C2, C3 involving the vector fields $Y$ in place of $X$ and $L$ in place of $L_0$ hold.

As previously by Lemma 1.2 and Remark 3 it follows that $\mu$ is unique and that there exists a smooth density $\phi(s)$ such that $\mu(ds) = \phi(s)m(ds)$, $\phi \; > \; 0$ on $S$ . $L$ generates a strongly Feller continuous semi-group $T_t$

under C2′. The hypotheses C2′, C2 obviously impose regularity hypotheses on the vector fields $v$ additional to those imposed in C1a although they are assumed to hold without bounds.

In the following, we set $Q(x, m, \theta) = \theta^t W(x, m)\theta$. Now it follows from (1.8)b that

$$\begin{aligned} \lim_{t\to\infty} \frac{log\,|z|}{t} &= \lim_{t\to\infty} t^{-1} \int_0^t \theta^t W\theta dz \\ &= \int_S Q(x, m, \theta)\ \mu\ (dx, dm, d\theta) \\ &= \ell\ , \quad a.e \end{aligned} \qquad (1.10)$$

applying the ergodic theorem for stationary stochastic processes. It is shown that $\ell$ coincides with the top Lyapunov index as determined by the multiplicative ergodic theorem (see [2]).

In order for us to show that the dynamo effect exists we have to establish that $\ell > 0$ and this leads us to consider the properties of the mean Lyapunov indices $g(p)$ and to considerations of the theory of large deviations taken up in the next section.

## 2. Large deviation theory and mean Lyapunov indices

For general references to large deviation theory the reader is referred to the basic papers by Donsker and Varadhan [15], [16] and the books by Ellis [19], Freidlin and Wentzell [28] and Stroock [51].

Let $\widetilde{M}$ be the space of probability measures on $S$. For each $t > 0$, $\eta \in \tilde{\Omega}$ (the space of continuous functions on $(0, \infty)$ taking values in $S$) and Borel set $A \subset S$ let

$$L_t\ (\eta, A) = \frac{1}{t} \int_0^t \chi_A\ (\eta\ (\tau))d\tau\ .$$

For each $t > 0$ and each $\eta$ , $L_t\ (\eta, .) \in \widetilde{M}$. For each $s \in S$ and $t > 0$ let $Q_{s,\,t}$ be the measure on $\widetilde{M}$ induced by $L_t$ from $P_s$ :

$$Q_{s,\,t} = P_s\ L_t^{-1}$$

or

$$Q_{s,\,t}\ (B) = P_s(\eta \in \hat{\eta}\ :\ L_t(\eta, .) \in B)\ , B \subset \widetilde{M}\ .$$

Define $I(\tilde{\mu})$ , $\tilde{\mu} \in \widetilde{M}$ by

$$I(\tilde{\mu}) = - \inf_{\substack{u > 0 \\ u \in D(L)}} \int_S \left(\frac{Lu}{u}\right) (s) \, \tilde{\mu} \, (ds).$$

Now there are certain technical difficulties in applying large deviation theory in the non-compact case due to the nonuniformity in $s$ of the limit

$$\lim_{t \to \infty} \frac{1}{t} \log E_s \{\exp -t < -Q, L_t(\eta, .) >\}$$
$$= - \inf_{\tilde{\mu} \in \tilde{M}} [- < Q, \tilde{\mu} > + I(\tilde{\mu})]$$

with respect to $s$ , where $< , >$ indicates the $(C, C^*)$ duality. Indeed, a number of technical conditions have to be satisfied in order to eliminate this difficulty, including the hypotheses H1 to H5 listed in [16] (section 8). In the following we summarise the arguments involved in establishing that indeed these conditions are satisfied under the hypotheses introduced in section 1. First let us observe that since the process $\eta(t)$ gives rise to associated control problems which are strongly completely controllable (see section 1) we have $p(t, s, ds') = p_t(s, s')m(ds')$ by Theorem 3 of [30] (where $m(ds')$ indicates local Lebesgue measure) yielding H1 of [16]. Further, identifying the $G$ of [16] with $G \equiv S$, we have on setting $U^{\varpi}(s, A) = \int_0^{\infty} e^{-\varpi t} p(t, s, A)dt$ , $\varpi > 0$, that $U^1(s, A) << \mu(A) << m(A)$ by Theorem 5.2 of [5]. Since $U^{\varpi}(s, A) = 0$ , $\varpi > 0$, if and only if $U^1(s, A) = 0$ we see that hypothessis H4 of [6] is satisfied while $U^{\varpi}(s, A) = \int_0^{\infty} e^{-\varpi t} (T_t X_A)(s)$ is continuous by the strong Feller property, so that H5 is satisfied. Let $\hat{C}(S) = \{f \in C(S) : f(s) \to o\}$ outside of compacts exhausting $S$). By Lemma 5.11 of [16], every diffusion is normal and stochastically continuous so that by Lemma 2.11 of [18] the semi-group $T_t$ is continuous on $\hat{C}(S)$. Accordingly we have in the notation of [16] that $\hat{C}(S) \subset B_o(S)$ and by Lemma 2.10 of [16], $p(t, s, ds')$ is uniformly stochastically continuous on compacts. This last property is easily seen to imply that $\hat{C}(S) \subset B_{00}(S)$ (see [16] for notation) this establishes H2 . The hypothesis H3 is simply a consequence of the fact that $C(S) = C(S, C) = C(S, C, \beta)$ (see [18] and the remark in Donsker and Varadhan [15] just after the statement of Theorem 4 in their introduction for both the substance of the observation as well as for the notation utilised). Finally hypothesis C3′ c is exactly the unlabelled condition introduced by Donsker and Varadhan in [16] (section 7 before lemma 7.1, which, for convenience we call H6.

Let us set $z(\theta) = \Theta(\eta)\theta$ , $\theta = \frac{x}{|x|}$.

Definition.

We define the mean Lyapunov index $g_{\tilde{\mu}}\ (p)$ , $p \in R$ , with respect to a measure $\tilde{\mu}$ on $S$ by

$$g_{\mu^*}\ (p) = \overline{\lim_{t \to \infty}}\ t^{-1}\ log\ E_{\mu^*}\ \ (E_{x,m,\theta}\ \ |z(\theta)|^p)$$

Lemma 2.1

**Under the hypotheses** C1, C2, C2′, C3′ , $g_{\mu^*}\ (p) = g\ (p)$ **where**

$$g\ (p) = \sup_{\tilde{\mu} \in \tilde{M}} \left\{ p\ <\ Q,\ \tilde{\mu}\ > + \inf_{\substack{u\ >\ 0 \\ n\ \in\ D(L)}} \int_S \left(\frac{Lu}{u}\right)\ \ (s)\ \tilde{\mu}\ (ds) \right\}$$

Proof.

Under the hypotheses indicated we have shown that the hypotheses H1 to H6 required in the theory of Donsker and Varadhan are satisfied.

As previously remarked

$$E_s\ |z\ (\theta)|^p\ =\ E_s\ (\exp\ -t\ (-p\ Q\ (L_t\ (\eta,.)))\ .$$

Under conditions H1 to H6 given above it has been proved by Stroock, for example, in [51] Theorem 8.12 8.6 that the uniform upper bound holds:

$$\overline{\lim_{t \to \infty}}\ t\ \log \sup_{s\ \in\ M}\ Q_{t,s}\ \ (\mathcal{M})\ \leq\ -\inf_{\tilde{\mu}\ \in\ \tilde{M}}\ I\ (\tilde{\mu})\ .$$

As consequence we have the estinate for all $\varepsilon'\ >\ 0$

$$Q_{t,s}\ (\mathcal{M})\ \leq\ \exp\ (-\inf_{\tilde{\mu}\ \in\ \tilde{M}}\ I(\tilde{\mu})\ t\ (1 + \varepsilon')$$

uniformly for $t\ \geq\ T(\varepsilon')$ and hence,

$$E_s\ |z|^p\ \leq\ \exp\ t\ \sup_{\tilde{\mu}\ \in\ \tilde{M}}\ (p\ < Q, \tilde{\mu} >\ -I(\tilde{\mu}))(1 + \varepsilon')\ \ for\ \ t\ \geq\ T(\varepsilon')\ \ (2.1)$$

following arguments of Varadhan given is Theorem 2.6 of [54]

Then clearly we have

$$E_{\mu^*}\ E_s\ |z|^p \ \leq\ \exp\ t \sup_{\tilde{\mu}\in\tilde{\mathcal{M}}} (p < Q, \tilde{\mu} > -I(\tilde{\mu})\ (1+\varepsilon')\ \ for\ \ t \geq T(\varepsilon')\ .$$

Hence, it follows that

$$\overline{\lim_{t\to\infty}}\ t^{-}\ log\ E_{\mu^*}\ E_s\ |z|^p \ \leq\ g(p)\ .$$

Another result of Donsker and Varadhan given in [16] (Theorem 8.1) yields that

$$\underline{\lim_{t\to\infty}}\ t^{-1}\ \log\ Q_{t,s}\ (\mathcal{M}) \ \geq\ -\inf_{\tilde{\mu}\in\tilde{\mathcal{M}}}\ I(\tilde{\mu})$$

uniformly for $s$ in compact subsets of $S$.
This yields that

$$\underline{\lim}_{t\to\infty}\ t^{-1}\ log\ E_s\ |z|^p \ \geq\ -\inf_{\tilde{\mu}\in\tilde{\mathcal{M}}} (-p\ < Q, \tilde{\mu} > \ +I\ (\tilde{\mu})) = g(p)$$

by Theorem 2.6 of [4] uniformly for $s$ in compact subsets of $S$.
Choosing $R$ sufficiently large we may ensure that

$$\mu^*(C)\ >\ \frac{1}{2}\ \mu^*\ (S)\ where\ C = B(0,r)\ \times\ \mathcal{M}\ \times\ P^{n-1}$$

is a compact subset of $S$.
Thus it follows that

$$E_{\mu^*} E_s |z|^p \ \geq\ |z|^p\ \geq\ \int_C\ E_s\ |z|^p\ \mu^*\ (ds)$$
$$\geq\ \frac{1}{2}\ \mu^*\ (S)\ \ \exp\ \ t\ g(p)\ (1+\varepsilon')$$

uniformly for $t\ \geq\ T(\varepsilon')$.
It follows that

$$\underline{\lim}_{t\to\infty}\ t^{-1}\ log\ E_{\mu^*}\ E_s\ |z|^p \ \geq\ g(p)$$

giving the result.
<u>Remark 4</u>.

(a) Note that setting $\theta_0(x) = y_o(x) \ / \ ||y_0(x) \ \neq \ 0$ then

$$
\begin{aligned}
& E \int_{R^n} E_{x,m} \ ||\Theta \ y_0(x)|| \ ^p \ k(x)dx \\
= \quad & E \int_{R^n} k(x) \ ||y_0(x)|| \ ^p \ E_{x,m} \ ||z(\theta_0)|| \ ^p \\
= \quad & \int_{R^n \times \mu} k(x)||y_0(x)|| \ ^p \ dx \ \rho(dm) \ E_{x,m,\theta_0} \ ||z(\theta_0)|| \ ^p \\
= \quad & \exp \ g(p) \ \{1 + o(1)\} \ ,
\end{aligned}
$$

by an argument identical to the preceding provided that

$$
\int_{R^n} k(x) \ ||y_0(x)|| \ ^p \ dx \ < \ \infty \, .
$$

(b) Similarly,

$$
\begin{aligned}
& E_\mu \ |\Theta \ y_0(x)| \ ^p = \ E_\mu \ E_{x,\mu}|| \ \Theta y_0(x)|| \ ^p \\
= \quad & \int_S \mu(ds) \ ||y_0(x)|| \ ^p \ E_{x,m,\theta_0} \ (x) \ ||z(\theta_0)|| \ ^p \\
= \quad & \exp \ g(p) \ t\{1 + o(1)\}
\end{aligned}
$$

provided that

$$
\begin{aligned}
\int_S \mu(ds) \ ||y_0(x)|| \ ^p = \quad & \int_S \phi(s) \ m(ds) \ ||y_0(x)|| \ ^p \\
= \quad & \int_S V(x,m) \ \varphi(s)m(ds) \ \frac{||y_0(x)||^p}{V(x,m)} \\
& < \ \infty
\end{aligned}
$$

if $||y_0(x)|| \ ^p \ \leq$ const $( \ ||x|| \ ^{p_0} \ + 1)$

In particular, if $y_0(x) \equiv$ constant $\neq 0$, this is always the case.

We assume the following technical hypotheses, motivated by a result given in the paper by Kotani [35] (Theorem 1 and Remark 2). It should be noted that Kotani uses a compacity condition related to well known ergodicity results (see [17] Lemma 8.8.2 and Theorem 8.8.8).

Let $D_\alpha$ denote the punctured open disc $\{\varsigma \ \in \ C \ : 0 \ < \ |\varsigma| \ < \ \alpha\}$.

C4 (a) The operator $L_V = V^{-1} LV$ has an isolated eigenvalue 0 with corresponding eigenfunction $\psi(0) = V^{-1}$ in $C(S)$ and $D_\alpha \subset \rho(L_V)$, for some $\alpha > 0$.

(b) Further we assume that $L_V$ generates a $C_0$-semi-group on $C(S)$.

Let us set $L_V(p) = V^{-1}(L + pQ)V$, for $p \in R^1$.

Lemma 2.2

**Under the additional hypothesis C4 there exists an eigenvalue $\ell(p)$ and an eigenfunction $\psi(p)$ *in* $C(S)$ with $\ell(p), \psi(p)$ holomorphic in $p$ for $|p| < \frac{\alpha}{2M}$.**

Proof

We set

$$R(\varsigma) = R(\varsigma, L_V(p)) = (L_V(p) - \varsigma)^{-1}$$

and

$$R_0(\varsigma) = (L_V - \varsigma)^{-1}$$

and

$$P(p) = \frac{-1}{2\pi i} \int_P (L_V(p) - \varsigma)^{-1}$$

where

$$\Gamma = \{\varsigma : |\varsigma| = \frac{\alpha}{2}\}$$

Let us now define $\psi(p) = P(p)\psi(0)$ so that by an elementary calculation $L_V(p)\psi(p) = \ell(p)\psi(p)$. Since $||Qu|| \leq sup|Q|\,||u|| \leq M\,||u||$ and by the second Neumann resolvent formula we have

$$R(\varsigma) = R_0(\varsigma)\{1 + pR_0(\varsigma)\}^{-1}$$

(see [32] Chapter 7) it follows that

$$\begin{aligned} ||R(\varsigma) - R_0(\varsigma)|| &\leq ||R_0(\varsigma)||(1 + pQR_0(\varsigma)^{-1} - I|| \\ &\leq \frac{2}{\alpha} \sum_{=\phi}^{\infty} |p|^r M^r \left(\frac{2}{\alpha}\right)^r \\ &\leq \left(\frac{2}{\alpha}\right)^2 \frac{|p|\,M}{1 - \frac{2}{\alpha}|p|\,M} \quad provided \ \frac{2\,|p|\,M}{\alpha} < 1 \end{aligned}$$

It follows that

$$\begin{aligned} ||\psi\ (p) - \psi\ (0)|| &\leq ||(P\ (p) - P\ (0))\ \psi\ (0)|| \\ &\leq ||\frac{1}{2\ \pi\ i} \int_{\Gamma} (R\ (\varsigma) - R_0\ (\varsigma))\ \psi\ (0)\ ds|| \\ &\leq \frac{\left(\frac{2}{\alpha}\right)^2\ |p|\ M}{1 - \frac{2}{\alpha}\ |p|\ M}\ ||\ V^{-1}\ ||\ \ , \ provided\ \ |p|\ <\ \frac{\alpha}{2\ M} \end{aligned}$$

In fact, by the theory of holomorphic perturbations of type A ( [32]; 375) $\mu(p)$ and $\ell(p)$ are analytic for $p$ in this range and

$$\ell(p) = \ p\ < Q, \phi >\ -p^2\ < QTQ^{-1}\ ;\ V\ \phi >\ +0\ (p^2)$$

when $T$ is the reduced resolvent of $L_V$ and higher order approximations are available.

Corollary

$$g\ (p) = \ell\ (p)\ \ is\ analytic\ for\ \ |p|\ <\ \frac{\alpha}{2\ M}\ (1 + \frac{2}{\alpha}\ ||V^{-1}||_\infty ||V\phi||_{L^1})^{-1} = \delta'$$

Proof

First we note that by Remark 4(b) if we choose $y_0\ \equiv\ 1$

$$g(p) = \lim_{t\ \to\ \infty}\ t^{-1}\ log\ E_\mu\ |z|^p$$

Since $L_v(p)$ generates a $C_0$-semigroup in $C(S)$ by the theory of holomorphi perturbation of type A we have

$$g(p) = \lim_{t\ \to\ \infty}\ E_\mu\ log\ < P(p)V^{-1}\ , V\phi >\ e^{\ell\ (p)\ t}\ >,$$

by C4, for $|p|\ <\ \frac{\alpha}{2\ M}$

Also by the previous estimates

$$\begin{aligned} |\ <\ (P(P) - P(0))\ V^{-1},\ V\phi\ >| &\leq ||(\mu(p) - \mu(0))\ ||\ ||V\phi||_{L^1} \\ &\leq \left(\frac{2}{\alpha}\right)^2\ \frac{|p|\ M\ ||V^{-1}||_\infty ||V\phi||_{L^1}}{1 - \frac{2}{\alpha}\ |p|\ M}\ , \end{aligned}$$

It follows that

$$| < P(P)V^{-1} , V\phi > | \geq 1 - \frac{\left(\frac{2}{\alpha}\right)^2 \ |p| \ M \ ||V||_\infty ||V\phi||_{L^1}}{1 - \frac{2}{\alpha} \ |p| \ M} > 0$$

in case

$$1 > \left( \frac{2 \ M}{\alpha} + \left(\frac{2}{\alpha}\right)^2 \ M \ ||V^{-1}||_\infty ||V\phi||_{L^1} \right) \ |p|$$

or in case

$$|p| < \frac{\alpha}{2 \ M} \left( 1 + \frac{2}{\alpha} \ ||V^{-1}||_\infty ||V\phi||_{L^1} \right)^{-1}$$

The results follows immediately. Note that in this calculation we really need to use the initial law $\mu$.

We want to establish a result analogous to that given in Theorem 2 of [1]. Unfortunately due to the use of a Lyapunov type hypothesis there is an asymmetry in our results. There is however a very nice trick used in [4] (section 5) which serves the same purpose for unimodular matrices.

Lemma 2.3

**Under hypotheses** C1, C2, C2′, C3′ $g(-n) = 0$.

Proof

First, let us note that the map of $S^n$ into $S^n$ given by $k \rightarrow \dfrac{z(k)}{||z(k)||}$ has Jacobian $||z(k)||^{-n}$. It follows that as $det \ \Theta(t) = 1$ , we have

$$\omega_n = \int_{S^n} do = \int_{S^n} ||z(k)|| \ ^{-n} do \ ,$$

identically in $x, m \in R^n \times \mathcal{M}$ considering $z(k)$ as a functional of the stochastic process $\varsigma(t)$ starting at $x, m$.

Taking the average with respect to $E_\nu$ we have

$$\omega_n = \int_{S^n} E_\nu \ ||z(k)|| \ ^{-n} \ do = \int_{P^{n-1}} E_\nu \ ||z(k)|| \ ^{-n} d\hat{o}$$

identifying $k$ with $-k$ and letting $d\hat{o}$ denote the Haar measure on $P^{n-1}$.

Now define the measure $\hat{\mu} = \nu(dx, dm)d\hat{\sigma}$ on $S$ and as in Remark 4(a) and the Proof of Lemma 2.1, note that one has the lower bound

$$\omega_n \geq \frac{1}{2} \hat{\mu}(S) \exp t\, g(-n) \{1 - \varepsilon\}$$

uniformly for $t \geq T(\varepsilon)$.

The result follows immediately.

Remark 5

It is easily checked that $g(p)$ is convex in $p$, that $|g(p)| \leq M$ and $\frac{g(p)}{p}$ is increasing in $p$ (see [1], [2] and Remark 4b). By Lemma 2.2 and its corollary we see that $g(p)$ is analytic in the neighbourhood of the origin, so that $g'(0) = \ell$ and since $g(0) = 0$ we have $g(p) \geq \ell p$, $p \in R'$. For small $p$, we have $g(p) = \sum_{s=1}^{\infty} g_s\, p^s$. Let $s_0$ be the first index such that $g_{s_0} \neq 0$. Then $g(p) = g_{s_0}\, p^{s_0} \{1 + 0(p)\}$. The fact that $g(-n) = g(0)$ and convexity yield that $s_0 = 1$ and $g_{s_0} = \ell > 0$ so that $g(p) = \ell p\{1 + O(p)\}$. The alternative is that $g_s = 0$ for all $s$, hence, we have the dichotomy that either $g(p) \sim \ell p, \ell > 0$, for small $p$, or $g(p) \equiv 0$ in a neighbourhood of the origin and hence for $|p| < \delta'$. The second case is ruled out by the Lemma which follows. In fact, the Lemma shows that in the first case there is an initial interval on which $g(p)$ is strictly convex since $g(p) \not\equiv \ell p$ in a neighbourhood of the origin.

Obviously, we have $\lim_{p \to -\infty} \frac{g(p)}{p} = \gamma_- < \infty$ and

$$\lim_{p \to \infty} \frac{g(p)}{p} = \gamma_+$$

Let $I(r)$ denote the Legendre-Fenchel transform of $g(p)$:

$$I(r) = \sup_{p \in R^1} (rp - g(p)),\ r \in R^1.$$

If $\gamma_- < \gamma_+$ and $g(p)$ is strictly convex it is known that $I(r)$ is differentiable on $(\gamma_-, \gamma_+)$ (see [19] Theorem VI 5.6) while it is readily seen that $I(\ell) = 0$, $I'(\ell) = 0$. It follows that $I(r) = \frac{(\ell - r)^2}{2g''(0)} + 0(\ell - r)^2)$

by Taylor 's theorem. The strict convexity of $g(p)$ in case $\ell > 0$, would follow immediately from the analyticity of $g(p)$ , however, we only have available the much weaker corollary to lemma 2.2.

It is convenient to assume the following strengthened from of the hypotheses C3 which simplifies the calculation in the following lemma, namely:

C3′ Suppose that there exists a Lyapunov function $V(x,m)$ with continuous second order derivatives on $R^n \times M$ satisfying

$$(a) \quad c_3 \, |x|^{p_0} + \alpha_2 \geq V(x,m) \geq c_1 \, |x|^{p_0} + \alpha_1 \;, c_1, p_0, \alpha_1 > 0$$

and

$$(b) \quad LV \leq -c_2 V + \beta \;, x, m \in R^n \times M$$

where

$$c_2, \; \beta > 0 \; and \; c_2 > \beta \, / \, \alpha_1$$

(c) C3c is unchanged.

<u>Lemma 2.4</u>

**Under hypotheses** C1a, C2, C2′, C3″, C4 **we have** $g(p) \not\equiv \ell p$ **for** $p$ **in a neighbourhood of the origin.**

<u>Proof</u>

Indeed, suppose that

$$V^{-1} (L + pQ) \, V\psi \, (p) \equiv \ell p \psi \, (p)$$

in a neighbourhood of the origin.

Then it follows that

$$\begin{aligned} < \phi \,, V^{-1} \, LV\psi \, (p) > \quad &= p(\ell < \phi, \; \psi \, (p) > - < \phi, Q\psi \, (p) >) \\ &= 0(p) \; . \end{aligned}$$

Noting that

$$V^{-1} \, LV\psi = V^{-1} \, (LV)\psi + L\psi + \sum_i \frac{X_i \, V X_i \psi}{V}$$

we see that

$$< \phi, V^{-1} \, (LV) \, \psi(p) > + \sum_i < \phi, \frac{X_i \, V X_i \psi}{V} >= 0(p) \; .$$

By Lemma 2.3 $\psi(p) = V^{-1} + 0(p)$ in $C(S)$ and $\phi > 0$. Hence,

$$< \phi, V^{-1} (LV) V^{-1} > - \sum_i < \phi, \frac{(X_i V)^2}{V^3} >= 0(p) ,$$

using hypoellipticity [34].

By hypothesis C3′ $V^{-1} LV \le -c_2 + \frac{\beta}{V}$ so that $0 < \frac{1}{2}\left(-c_2 + \frac{\beta}{\alpha_1}\right)$ for small $p$, which is a contradiction.

Corollary 1

$$\gamma_- < \gamma_+ .$$

Proof

This is an immediate consequence of the dichotomy observed in Remark 5.

Corollary 2

**Under conditions** C1a, C2, C2′, C3″, C4 **we have** $\ell > 0$.

Proof

This is immediate from Remark 5.

Remark 6

As in the paper by Le Page [39] (Theorem 2), Lemma 2.1 and the Corollary to Lemma 2.2 and Lemma 2.4 and Corollaries imply that a large deviation principle holds in the form

$$\overline{\lim_{t\to\infty}}\, t^{-1} \, log P\{| \, t^{-1} \, log \frac{|z|}{|z(0)|} - \ell| > \sigma_0\} = c(\sigma_0) < 0 ,$$

for $\sigma_0$ sufficiently small, uniformly for $(x, \theta_0)$ belonging to compact subsets. In fact, to be more precise, using the notation of [19] if we set

$$a_t = t \uparrow \infty \;\; and \;\; W_t = log \frac{|z(t, z_0)|}{|z_0|} ,$$

Lemma 2.1 has shown that

$$\lim_{t\to\infty} c_t(p) = \lim_{t\to\infty} t^{-1} \, log \, E(\exp \, p \, W_t) ,$$

uniformly for $(x, \theta_o)$ in compact subsets.

$I(r)$ satisfies the properties of an entropy function (according to definition II.3.1 pf [19]). The compactness of the level sets of $I(r)$ follows from Corollary 1 to Lemma 2.4 above. The other properties have already been remarked. It follows that we may apply the upper bound of Theorem II.6.1 of [19], namely:

$$\overline{\lim_{t\to\infty}}\ log\ P\left\{\frac{1}{t}\ log\ \frac{|z(t,z_0)|}{z_0}\ \in\ (\ell\ -\ \sigma_0,\ \ell\ +\ \sigma_0)^c\right\}$$
$$\leq\ -\inf_{r\ \in\ (\ell\ -\ \sigma_0,\ \ell\ +\ \sigma_0\ )^c} I(r)$$

uniformly for $(x,\theta_0)$ in compact subsets.

Taking $\sigma_0$ sufficiently small we may suppose that $g(p)$ is holomorphic in $p$ for

$$|p|\ <\ 2\left(\frac{8(k+1)\ !\ \sigma_0}{|g_{k+1}|}\right)^{1/k} = 2a\ .$$

Then

$$\begin{aligned} I\ (\ell+r)\ &\geq\ \sup_{|p|\ \leq\ a} \{pr-g(p)\}\ ,\ 0\ \leq\ |r|\ \leq\ 2\sigma_0 \\ &=\ \sup_{|p|\ \leq\ a} \{pr-\frac{g_{k+1}}{k+1!}\ p^{k+1}\ \{1+0(p)\}\} \\ &\geq\ \frac{|g_{k+1}|a^{k+1}}{4(k+1)\ !}\ ,\ if\ \ 2a\sigma_0\ \leq\ \frac{|g_{k+1}|a^{k+1}}{4(k+1)\ !}\ , \end{aligned}$$

which is satisfied choosing $a = \left(\frac{8(k+1)\ !\ \sigma_0}{|g_{k+1}|}\right)^{1/k}$ and $\sigma_0$ sufficiently small. It follows that

$$\begin{aligned} -\inf_{r\ \in\ (\ell-\sigma_0,\ \ell+\sigma_0)^c} I(r)\ &\leq\ -\inf_{\sigma_0\ \leq\ |r|\ \leq\ 2\sigma_0} I(\ell+r) \\ &\leq\ -2\left(\frac{8(k+1)\ !}{|g_{k+1}|}\right)^{1/k}\ \sigma_0^{1+\frac{1}{k}}\ \equiv\ c(\sigma_0)\ . \end{aligned}$$

## 3. Magnetic induction type equations with small parameter

### 3.1 Probabilistic representation of solutions

Associated with the equation (1.8) is the following backward Cauchy problem

$$\frac{\partial Y^\epsilon}{\partial t} + \frac{\varepsilon^2}{2} \Delta Y^\epsilon - v(x, \hat{\xi}(t)\ (\sigma)) \ . \ \nabla_x Y^\epsilon + U(x, \hat{\xi}(t)\ (\sigma))Y^\epsilon = 0 \ ,$$

$$Y^\epsilon(x,T) = y_0(x) \ , \ 0 \leq t \leq T \ , \qquad (3.1)$$

where $\hat{\xi}(t)\ (\sigma) = \xi(T-t)\ (\sigma)$ , for $0 \leq t \leq T$.

Clearly, for $t$ in the range $0 \leq t \leq T$ , the solutions of the two problems are related by $Y^\epsilon\ (x,t) = y^\epsilon(x, T-t)$.

Now associated with the equations (3.1) is the stochastic differential equation (see [9], [50]):

$$\alpha_s\ (t) = I \quad -\varepsilon^{-1} \int_s^t \alpha_s\ (u)\ v\ (B(u,\omega)\ ,\ \hat{\xi}(u)\ (\sigma))\ dB\ (u,\omega)$$
$$+ \int_s^t \alpha_s\ (u)\ U\ (B(u,\omega)\ , \hat{\xi}(u)\ (\sigma) du\ , \qquad (3.2)$$

where $B(u,\omega)$ is $n$-dimensional Brownian motion and the first integral in (3.2) is interpreted in terms of the Ito stochastic calculus (see, for example 27], [29]).

Formally, as remarked in [50], (3.2) has a solution which is given in terms of a stochastic product integral and from (3.2) the inverse to $\alpha_s(t)$ may be constructed as the solution $\alpha_s^*$ to

$$\alpha_s^*(t) = I + \varepsilon^{-1} \quad \int_s^t v(B(u,\omega)\ ,\ \hat{\xi}(u)\ (\sigma))\ dB\ (u)\ (\omega)\ \alpha_s^*\ (u)$$
$$- \quad \int_s^t U(B(u,\omega)\ , \hat{\xi}(u)\ (\sigma))\ \alpha_s^*(u)\ du\ . \qquad (3.3)$$

(see [14] Chapter 3 Theorem 4,2).

The sum rule ([14]) suggests that the solution of (3.3) is given by

$$\alpha_s^*(t) = \ \exp\left(\int_s^t \varepsilon^{-1}\ \hat{v}.dB\right)\ \Theta\ (s,t)(-\hat{U}) \quad (3.4)$$

writing $\hat{U}\ (u)$ for $U(B(u,\omega)\ , \hat{\xi}(u)\ (\sigma))$ and $\hat{v}\ (u)$ for $v(B(u,\omega)\ , \hat{\xi}(u)\ (\sigma))$ and where $\Theta(s,t)\ (-\hat{U})$ denotes the solution of $\frac{d\Theta}{dz} = -\hat{U}(t)\ \Theta$ , $\Theta(s) = I$ we write $\Theta(0,t) = \Theta(t)$.

Finally, using the diagonality of $\exp\left(\int_s^t -\varepsilon^{-1}\ \hat{v}.dB\right)$, we conclude from (3.4) that

$$\alpha_s(t) = \exp\left(-\int_s^t \varepsilon^{-1}\ \hat{v}.dB\right)\ \Theta(s,t)(-\hat{U})^{-1} \qquad (3.5)$$

Using the Cameron-Martin-Girsanov formula (see [27] and [29]) in (3.5) we conclude that

$$\alpha_s(t) = \exp(-\varepsilon^{-1} \int_s^t \hat{v}.dB + \frac{1}{2}\varepsilon^{-2} \int_s^t \hat{v}^{\,2}\, du\ \Theta(s,t)\ (-\hat{U})^{-1} \qquad (3.6)$$

The basic representation formula proved in [50] by Stroock (a version of the Feymann-Kac 's formula for a parabolic system) states that

$$Y^{\,\epsilon}(x,s) = E_{x,s}\ (\alpha_s(\tau)\ ,\tau)\ \ ,\ \ 0 < s < \tau < T\ ,$$

Where $E_{x,s}$ is taken with respect to the measure generated by the process $\varepsilon B = X$.

Choosing $s = 0$ and $\tau = T$ in the above formula we obtain

$$\begin{aligned} y^{\,\epsilon}(x,T) &= E_{x,s}\ (\alpha_0\ (T) y_0\ (X(T)) \\ &= E_x\ (\exp(-\varepsilon^{-1} \int_0^T \hat{v}.dB + \frac{1}{2}\ \varepsilon^{-2} \int_0^T \hat{v}^2 du)) \\ &\times \Theta(T)(-\hat{U})^{-1}\ y_0(X(T))) \qquad (3.7) \end{aligned}$$

an expression holding for all $T < \infty$.

Once again using the formula of Cameron-Martin-Girsanov and a simple formula for product integrals we see that

$$y^{\,\epsilon}(x,t) = E_{x,t}\ (\Theta(t)(U(X(.),\xi(.))y_0\ (x(0))\ , \qquad (3.8)$$

where now in (3.8) the expectation is taken with respect to the process

$$X(t)\ (\omega) = x + \varepsilon B(t-u) + \int_t^u v(X(\theta),\ \xi(\theta))d\theta\ .$$

The derivation of (3.8) is easily rigorised using Ito 's stochastic calculus. we recall that $X$ converges to the classical path $\hat{\gamma}_x(t)$ which is the solution of

$$\frac{d\hat{\gamma}}{dt} \equiv v(\hat{\gamma}_x, t) \quad , \qquad (3.9)$$

$$\hat{\gamma}_x(T) = x \quad ,$$

(see [28] and [20]). Hence, it follows that

$$\hat{\gamma}_x(t) = \gamma(t - T \ , \theta_{T-t} \ \sigma)x \ , 0 \ \leq \ t \ \leq \ T \ .$$

Note that a related derivation on (3.8) is given in [20] for the stochastic vorticity equation while an alternative informal proof for the induction equation of magneto-hydrodynamics is given in [47]

From (3.9) it follows from results of Aronson given in [8] that

$$y^\varepsilon(x,t) \ \sim \ (0_t \ o \ \Theta \ (. \ ,t) \ y_0(.))(x)$$

as

$$\varepsilon \ \downarrow \ 0 \ , \qquad (3.10)$$

where $0_t$ is the classical propagator defined on continuous functions (matrices) over $R^n$ by $(0_t f) \ (x) = f(\gamma(-t)(\theta_t \ \sigma)(x)$ . Moreover, we note that in the dynamo problem we are fundamentally interested in the behaviour for large times providing the link with large deviation theory (for an intuitive physical discussion of these questions, see [49] (17.6) and also the papers by Cocke [10], [11]).

3.2 **Linear stochastic velocity fields**

Before proceeding to the analysis of the problem posed in (3, 8) it may be useful to look at an example treated in [46], [48] as a kind of test case, the linear stochastic velocity field, where we may take over without further ado the large deviation theory analysis as formulated by Arnold and Kliemann and other collaborators in a series of papers [1], [2], [3], [4], [5].

We deal with the equations (1.1) where $v$ is a linear velocity field $v = C(\xi(t))x$ . We assume that $\xi(t)$ is a stationary ergodic diffusion process on a smooth compact connected Riemannian manifold $M$ of dimension $m_0$ , solving

$$d\xi = X_0(\xi)dt + \sum_{j=1}^{r} \ X_j \ (\xi) \ odB_j$$

as before.

Since $div \ v = 0$ we have trace $C(\xi(t)) = 0$ and also $\left(\dfrac{\partial v_i}{\partial x_j}\right) = C_{ij} \ (\xi(t))$. Additionally, we assume that the Lie algebra conditions of [2] are satisfied and that the system corresponding to (1.2), (1.5), (1.8)a with $W \equiv C(\xi_t)$ is exactly controllable with invariant probability $\nu$ (that is to say with support $\mathcal{M} \times R^{n-1}$ ) . In this section we work under the hypothesis that the inital data satisfy C1c:

$$y_0 \in L^2 \,,\ ||y_0(y) - y_0(x)|| \le (K(||x||) + K(||y||))\ ||x-y||$$

where $\int_{R^n} ||x||^{n+2} K(||x||)^2\, dx < \infty$ .

Let then $X(t)$ be the stochastic process generated by

$$dX = C(\xi(t))X dt + \varepsilon dW \quad . \qquad (3.11)$$

Then following our discussion in section 3.1 we know that the solution of (11) is given by

$$y^{\epsilon}(x,t) = E_{x,t}\left(\Theta(t) y_0(X(0))\right) .$$

In [8] Aronson establishes estimates on the heat kernel associated with the operator

$$\frac{\varepsilon^2}{2} \sum_{ij=1}^{n} a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=i}^{n} a_i(x,t) \frac{\partial}{\partial x_i} + b(x,t) = H_\epsilon \,.$$

by constructing parametrices linearised around the paths of the associated dynamical system in the limit $\varepsilon \to 0$ . In the notation we use in the paper the approximating operator $\widetilde{H}_{\epsilon,y}$ to $H_\epsilon$ has the form

$$\frac{\varepsilon^2}{2}\Delta_x - \left(v(\gamma(t)y, \xi(t)) + \frac{\partial v}{\partial x}(\gamma(t)y, \xi(t))\ .\ (x - \gamma(t)y).\ \nabla_x\right.$$

for general $v$ . However, in the case of linear stochastic velocity fields the operator $H_\epsilon$ is already linearised and may be rewritten

$$H_\epsilon \equiv \frac{\varepsilon^2}{2}\Delta_x - (C(\xi_t)\Theta(y,t) - C(\xi(t))(x - \Theta(y,t)).\nabla_x$$

It follows that the construction of the parametrix $G$ given in section 4 of [8] via the method of Fourier transforms yields in fact the heat kernel $\Gamma$ of $H_\epsilon$ . Using the notation of section 1 we set

$$C_y(t) = \int_0^t \Theta(y,t)(U^{-1})\ (\Theta(y,t)(U^{-1*})\ du$$

and

$$F_y(t) = \Theta(y,t)^{-1})^{*}\ C_y(t)^{-1}\Theta(y,t)^{-1}\ .$$

In the case of linear stochastic velocity fields we have, in fact,

$$\Theta(y,t)\ \equiv\ \Theta(C(\xi))\ .$$

Note that by trace $C(\xi(u)) = 0$ we have $\det\ \Theta(t) = 1$ .

Then from the construction of Aronson gives in section 4 of [8] the parametrix $G(\equiv\ \Gamma)$ associated with $\frac{\partial}{\partial t} - H_\varepsilon$ is given by $G(x,t,y,0) =$

$$2^{-n}\ \pi^{\frac{-n}{2}} \left(\frac{\varepsilon^2}{2}\right)^{-n/2} (det\ C_y(t))^{1/2}\ \exp\ \frac{-(x-\gamma(t)y.F_y(t)(x-\gamma(t)y)}{2\varepsilon^2} \quad (3.12)$$

In this special case we may state part of Theorem 1 [8] in a form more useful for our stochastic calculations.

Lemma 3.1 (Aronson)

**For linear stochastic velocity fields the Green 's function $\Gamma$ is given by $\Gamma(x,t;y,0) =$**

$$2^{-n}\ \pi^{\frac{-n}{2}} \left(\frac{\varepsilon^2}{2}\right)^{-n/2} (det\ C(t))^{-1/2})\ \exp\ \frac{-(\gamma(t)^{-1}x-y).C(t)^{-1}(\gamma(t)^{-1}x-y)}{2\varepsilon^2}$$

Proof

From the equality (3.12) making the specializations indicated above and noting that

$$x-\gamma(t)y = \Theta(t)(\gamma(t)^{-1}x-y)$$

we have

$$(x-\gamma(t)y\ ,F(t)\ x-y(t)y) = (\gamma(t)^{-1}x-y\ ,C(t)^{-1}\ (\gamma(t)^{-1}x-y))\ .$$

giving the result

For $\theta\ \in\ \Omega$ , the unit sphere in $R^n$ , we set $\hat{z}(\theta) = \Theta(t)^{-1}\ \theta$ and let $\hat{g}(p)$ denote the mean $p$-th Lyapunov index associated with $\hat{z}$ . If the Lyapunov exponents of $z(\theta)$ are ordered $\ell_1 < \ell_2 < \ldots < \ell_r = \ell$, note that by the fact that trace $W = 0$ we have $\Sigma\ \ell_j = 0$ so that $\ell_1 < 0$ since $\ell > 0$ . Let the Lyapunov indices of $\hat{z}$ be $\ell_1 <\ .\ < \hat{\ell}_r = \hat{\ell}$

then note that $\hat{\ell}_k = -\ell_{r-k+1}$ , $k = 1, \ldots r$ (see the paper by Ledrappier [38] Proposition 4.1) and, hence, $\hat{\ell} = \hat{\ell}_r \equiv -\ell_1 > 0$ .

Further, it is known that a uniform large deviation result holds for $\hat{z}(\theta)$, in the sense that

$$\sup_{\theta \in \Omega} P\{ \mid t^{-1} \log \| \hat{z}(\theta) \| - \hat{\ell} \mid > \sigma_0 \} \leq \exp\left(\frac{-\sigma_0^2}{2\,\hat{g}''(0)} t\right)$$

(see [3] Proposition 3.1 and Corollary 3.4).

We introduce the new norm $||| \ |||^2 = E \| \ \|_2^2$ , where $\| \ \|_2$ denotes the $L^2$-norm on $R^n$ , together with the measure $\hat{\mu}$ on $R^n \times \mathcal{M}$ given by $\hat{\mu}(dx, dm) = \rho(dm) \| y_0(x) \|^2 dx$.

By hypothesis C1c, $\hat{\mu}$ is a finite measure.

We have from (3.8) that

$$\begin{aligned} y^\epsilon(\gamma(t)x) &= \Theta(t)y_0(x) + E_{\gamma(t)x,t}(\Theta(t)(y_0(X(0)) - y_0(x)) \\ &= I_1(x) + I_2(x) . \end{aligned} \tag{3.13}$$

First observe that

$$||| I_1 |||^2 = \int_{R^n \times \mathcal{M}} E_{\theta_0(x),m} \| z(\theta_0) \|^2 \| y_0(x) \|^2 \rho(dm)dx ,$$

where

$$\theta_0(x) = y_0(x) \| y_0(x) \|^{-1}$$

when

$$y_0(x) \neq 0$$

$$\begin{aligned} &= E_{\hat{\mu}} E_{\theta_0,m} \| z(\theta_0) \|^2 \\ &\sim \exp g(2)t \quad \| y_0 \|_2^2 \end{aligned} \tag{3.14}$$

using Theorem 1 of (1) and an observation analogous to that of Remark 4.

On the other hand, we have from Lemma 3.1

$$| I_2(x) | \leq \int_{R^n} \Gamma(x, t; y, 0)$$

$$\times \| \Theta \| \, \|y_0(y) - y_0(x) \| \, dy$$

$$\leq 2^{\frac{-n}{2}} \pi^{\frac{-n}{2}} \int_{R^n} \exp\left(\frac{-\chi^2}{2}\right) \| \Theta \| \, \| y_0(\gamma(t)^{-1} x + \varepsilon C(t)^{\frac{1}{2}} \chi) - y_0(\gamma(t)^{-1} x) \| d\chi$$

Further, we see that

$$\begin{aligned} \| C(t)^{\frac{1}{2}} \chi \|^2 &= \int_0^t \| \Theta(u)^{-1*} \chi \|^2 \, du \\ &\geq e^{-2M} \| \Theta(t)^{-1*} \|^2 \| \chi \|^2 \\ &\geq e^{-2M} \| \Theta(t)^{-1} \|^2 \| \chi \|^2 \end{aligned}$$

by an elementary calculation using the Lipschitz bound on the first order derivatives of $v$ and (1.8)b.

We seek to use the "swirling" and stretching properties as expressed by the presence of $\Theta(t)^{-1}$ and $\Theta(t)$ in the above integral in order to show that fact $||| I_2 |||^2$ is negligible.

Consider the sets

$$\Sigma_x = \{ \sigma \ : \ \exp(\hat{\ell} - \sigma_0)t \leq \| \hat{z}(\theta) \| \leq \exp(\hat{\ell} + \sigma_0)t \ : \ \theta = x / \| x \| \}$$

and set

$$\Sigma_{\chi, x} = \Sigma_\chi \cap \Sigma_x$$

Let us examine the region in $R^n \times \Sigma_{\chi,x}$ where

$$\varepsilon \| C(t)^{\frac{1}{2}} \chi \| < \| \frac{1}{2} \| \gamma(t)^{-1} x \| .$$

Then clearly one must have

$$e^{-M} \varepsilon \exp(\hat{\ell} - \sigma_0)t \| \chi \| < \frac{1}{2} \exp(\hat{\ell} + \sigma_0)t \| x \|$$

or

$$\| \chi \| < \frac{\exp(2 \sigma_0 t)}{2 \varepsilon \exp M} \| x \|$$

Also, in this region we have $\| I_2(x) \|^2 \leq 0( \| \Theta \|^2 G_1)$ where

$$G_1 = \int_{|x| < \frac{\exp(2\sigma_0 t) |x|}{2\varepsilon \exp M}} e^{\frac{-\chi^2}{2}} \frac{1}{2} \| \gamma(t)^{-1} x \|^2 K \left( \frac{1}{2} \| \gamma(t)^{-1} x \| \right)^2 d\chi$$

and, hence the estimate $||| I_2(.) |||^2 \leq 0(E((|| \Theta ||^2 G_2)$

$$= 0\left(\varepsilon^{-n} \exp \left(g(2) - 2n\left(\hat{\ell} - 2 \left(\frac{n+1}{n}\right)\sigma_0\right)\right)t\right),$$

where $G_2$ is given by

$$G_2 = \int_{R^n} \frac{\exp(2n\ \sigma_o\ t)}{\varepsilon^n} \exp\ 2(\hat{\ell} + \sigma_o)t\ ||x||^{n+2}\ K\left(\frac{1}{2}||\gamma(t)^{-1}x||\right)^2 dx$$

In the alternative case, where we have

$$\varepsilon\ ||\ C(t)^{\frac{1}{2}}\ \chi\ || > \frac{1}{2}\ ||\gamma(t)^{-1}\ x\ ||$$

one must have

$$||\ \chi\ || \geq \frac{\exp\ (2\ \sigma_0\ t)}{2\ \varepsilon\ \exp\ M}\ ||\ x\ ||\ ,$$

from which it follows that

$$||\ I_2(x)\ ||\ \leq\ 0\left(\ ||\ \Theta\ ||^2\ \exp\ \left(\frac{-\exp(4\ \sigma_0\ t)\ |x|^2}{16\ \varepsilon^2\ \exp\ 2M}\right)\right)$$

and, hence, that

$$|||\ I_2(.)\ |||^2\ \leq\ 0(\exp(g(2) - 2n\ \sigma_0)t)$$

The remaining case is where $\sigma\ \in \sum_{\chi,x}^{c}$ with uniform probability

$$\leq \exp\left(\frac{-\sigma_0^2\ t}{2\ \hat{g}''\ (0)}\right).$$

We set $\Delta = \gamma(t)^{\frac{1}{2}}x + C(t)^{\frac{1}{2}}\chi$.
Hence, we see that

$$||I_2(x)||\ \leq\ 0\ \left(\chi_{\Sigma_{\chi,x}^c}\ ||\Theta|| \int_{R^n} e^{\frac{-\chi^2}{2}}||\ y_0\ (\Delta)|| + ||\ y_0(\gamma(t)^{-1}x)||\ d\chi\right)$$

so that

$$|||I_2\ (.)|||^2\ \leq 0\left(\int_{R^n \times R^n} dx d\chi\ e^{\frac{-\chi^2}{2}}\ E(||y_0\ (\Delta)||^2 + ||y_0(\gamma(t)^{-1}x)||^2)\chi_{\Sigma_{\chi,x}^c}||\ \Theta||^2\right)$$

$$\leq 0 \int_{R^n \times R^n} dx\, d\chi\, e^{\frac{-\chi^2}{2}}\, E(P(\Sigma^c_{\chi,x})^{\frac{1}{q}}\, E(\|\Theta\|^{2p})^{\frac{1}{p}}$$

$$(\, E \,\|\, y_0(\Delta)\|^{2r})^{\frac{1}{r}} + +(E \,\|\, y_0(\gamma(t)^{-1}\, x)\|^{2r}\,)^{\frac{1}{r}}\,,)$$

using Hölder 's inequality, with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ ,

$$\leq 0( \int_{R^n \times R^n} dx d\chi\, e^{\frac{-\chi^2}{2}}\, \|\, y_0(x)\,\|^2$$

$$P\,(\Sigma^c_{\chi,x})^{\frac{1}{q}}\,(E\,\|\Theta\|^{2p}\,)^{\frac{1}{p}}\,)$$

where we have used the unimodularity of $x \to \gamma(t)x$ , and with bounds independent of $r$,

$$\leq 0 \left( \int_{R^n \times R^n} dx d\,\chi\, e^{\frac{-\chi^2}{2}}\, \|y_0(x)\|^2 \right)$$

$$\times \exp\left( -\frac{\sigma_o^2}{2q\, \hat{g}''\,(o)} t \right)\, \exp\left( \frac{g\,(2p)}{p} t \right)\,,$$

by the uniform large deviation estimate, where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$,

$$=0\, \left( \exp\left( \frac{g\,(2p)}{p} - \frac{p-1}{2p}\, \frac{\sigma_0^2}{\hat{g}''\,(0)} \right)^t \right)\,,$$

letting $r \to \infty$.

The final result which we obtain is therefore

$$\begin{aligned} \|\| \, I_2(.)\, \|\|^2 = \quad & 0\, \left( \varepsilon^{-n}\, \exp\left( g(2) - 2n \left( \hat{\ell} - \frac{2(n+1)}{n}\, \sigma_o \right) \right) t \right) \\ & +0\, \left(\exp(g(2) - 2n\sigma_o)t\, \right) \\ & +0\, \left( \exp\, \left( \frac{g(2p)}{p} - \frac{p-1}{2p}\, \frac{\sigma_o^2}{\hat{g}''\,(0)} \right) t \right) \; as\; t \to \infty\,. \end{aligned}$$

It follows that if we choose $0 < \sigma_0 < \frac{n\,\hat{\ell}}{2(n+1)}$ and $p > 1$ such that $\frac{g(2p)}{p} - \frac{p-1}{2p}\,\frac{\sigma_0^2}{\hat{g}''\,(0)} < g(2)$ , then it is an immediate consequence of this last estimate that

$$\lim_{t \to \infty} t^{-1} \log ||| y^\epsilon(.) |||^2 = g(2) .$$

By analyticity, we see that

$$g(2p) - g(2) \sim 2(p-1)g'(2)$$

and by convexity

$$g'(2) \geq \frac{g(2)}{2} .$$

It follows that for $p$ close to $1$

$$g(2p) - \frac{p-1}{2} \frac{\sigma_0^2}{\hat{g}''(0)} < pg(2) + (p-1)g(2)$$

$$if \quad g(2) + 2(p-1)g'(2) - \frac{(p-1)\,\sigma_0^2}{2\hat{g}''(0)} < g(2) + (p-1)g(2)$$

or if $g'(2) < \dfrac{g(2)}{2} + \dfrac{\sigma_0^2}{4\hat{g}''(0)}$.

Hence, we have established the following theorem.

Theorem 3.2

**Suppose that the inital data satisfy condition** C1c **and that the linear stochastic velocity field satisfies the Lie algebra and control conditions** of [1] **together with the semi-group condition of** [2] (Theorem 3.1 ii ) **and the condition** $g'(2) < \dfrac{g(2)}{2} + \dfrac{\sigma_0^2}{4\hat{g}''(0)}$ **where** $0 < \sigma_0 < \dfrac{n\hat{\ell}}{2(n+1)}$ **then we have**

$$\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} t^{-1} \log ||| y^\epsilon |||^2 = g(2) > 0$$

This establishes the existence of a fast dynamo in the sense of [48] . The typical line stretching and swirling phenomena discussed in the physics literature is clearly exhibited here (see the papers by Cocke [10], [11] and also the book by Parker [49] ).

**3.3 Fast dynamo behaviour for nonlinear velocity fields**

We first deal with the case of the weak diffusion limit.

Lemma 3.3

**Suppose that the hypotheses** C1a, b, C2, C2′, C3′, C4 **hold. Then**

$$\lim_{t \to \infty} t^{-1} \log ||| y^0(.\,,t) |||^2 = g(2) > 0 .$$

Proof

By (1.4) and the unimodularity of the diffeomorphism $x \to \gamma(t)x$ we have

$$\begin{aligned} ||| y^0(.\,,t) |||^2 &= ||| y^0(\gamma(t).\,,t) |||^2 \\ &= ||| \Theta(t) y_0(.) |||^2 . \end{aligned}$$

Then as in the calculations leading to (3.14) we observe that

$$E \int_{R^n} dx \, || \Theta(t) y_0(x) ||^2 dx = \int_{R^n \times M} E_{\theta_0(x)} \, || z(t,\theta_0) ||^2 \, || y_0(x) ||^2 dx \rho(dm)$$

and applying the method of Lemma 2.1 (see Remark 4) we conclude that

$$|| \Theta(t) y_0(.) |||^2 = \exp g(2)t \, \{1 + o(1)\} \text{ as } t \to \infty ,$$

giving the result.

The question of the existence of fast dynamos in the sense of [48] is related to the stablity of this result in the presence of a small diffusivity term.

Let $E_W$ denote the expectation with respect to Wiener measure on paths in $R^n$ beginning at the origin at time 0.

Associated to the original evolution equation (1.1)a and the stochastic process $\xi(t)$ is the stochastic differential system

$$\begin{aligned} dX^\epsilon(t) &= \varepsilon \, dB + v(X^\epsilon(u), \; \xi(u))dt \, , X^\epsilon(0) = x \, , & (3.15)a \\ d\xi(t) &= X_j \, o dB_j + X_0 dt \, , \; \xi(0) = m \, , & (3.15)b \\ d\theta(t) &= h(X^\epsilon, \; \xi, \; \theta)dt \, , \; \theta(0) = \theta_o \, , & (3.15)c \, . \end{aligned}$$

carring over the notation introduces in Section 1.

We set $\varsigma^\epsilon = (X^\epsilon, \xi)$ and $\eta^\epsilon = (X^\epsilon, \xi, \theta)$. If we need to indicate a different starting point we use the notation $\varsigma^\epsilon_{x,m,t}$, $\eta^\epsilon_{s,t}$. We denote by $\Theta^\epsilon$ the matrix solution of the differential equation

$$\frac{d\Theta^\epsilon}{du} = U(\varsigma^\epsilon(u)) \, \Theta^\epsilon = W^\epsilon(u) \, \Theta^\epsilon \, , \; \Theta^\epsilon(0) = I \, ,$$

and by $\Theta^{\varepsilon}_{x,m,t}$ the solution of

$$\frac{d\Theta^{\varepsilon}(u)}{du} = U(\varsigma^{\varepsilon}_{x,m,t}(u)) \ , \ \Theta^{\varepsilon}(u) = I \ .$$

Similar definitions hold for $\Theta_{x,m,\theta,t}$ and $Q^{\varepsilon}$.

Associated with the system of stochastic differential equations (3.15)a, b, c is the infinitesimal generator

$$L^{\varepsilon} = \frac{1}{2}\,\varepsilon^2\,\Delta_x + v.\nabla_x + \frac{1}{2}\,\Sigma\,X_j^2 + X_0 + h\ .\ \nabla_\theta$$

acting on $C(S)$ and generating a strongly continuous Feller semi-group as before. We set $L^{\varepsilon}_V = V^{-1}\,L^{\varepsilon}\,V$.

C3*

C3a holds and C3″ b, c are strengthened here to C3* b $L^{\varepsilon}\,V(x,m) \leq -c_2\,V(x,m)+\beta\ ,\ c_2,\beta\ >\ 0$ for all $(x,m,\theta)\ \in\ S$ and all $\varepsilon\ ,0\ <\ \varepsilon\ \leq\ \varepsilon_\theta$ , for some small $\varepsilon_0$.

C3* c There exists a twice continuously differentiable functiuon $\hat{V}\ ,\hat{V}\ \geq\ 1$, and a sequence $u_k\ \in\ D(L_\varepsilon)$ such that C3c continues to hold with respect to $L_\varepsilon$

There now holds an analogous version of Lemma 1 and we guarantee the existence of an invariant measure (unique)

$$\mu^{\varepsilon} = \phi^{\varepsilon}\ m(ds)\ ,\varepsilon\ >\ 0\ ,\ s\ \in\ S\ ,L^{*}\ \phi^{\varepsilon} = 0$$

and

$$\int_S V(x,m)\ \mu^{\varepsilon}(ds)\ <\ \infty$$

We define $g_{\mu^{\varepsilon}}(p)$ analogously to the definition given before Lemma 2.1 substuting $z(\theta)$ by $z^{\varepsilon}(\theta)$ where

$$z^{\varepsilon}(\theta) = \Theta^{\varepsilon}\ \theta\ .$$

Then we have the following analogue to Lemma 2.1 , namely:

<u>Lemma 2.1</u>

Under the hypotheses C1, C2, C2′, C3* the conclusions of Lemma 2.1 continue to hold with $g(p)$ replaced by $g^{\varepsilon}(p)$ and $L$ by $L^{\varepsilon}$ in the statement of the Lemma

Remark 4′

Analogous remarks to Remark 4 continue to hold.

We strengthen C4 to the following

C4′ C4 holds with the spectral gap $\alpha$ supposed to satisfy

$$\frac{\alpha}{2m}\left(1+\frac{2}{\alpha}\,\|\,V^{-1}\,\|_\infty\|\,V\,\phi\,\|_{L^1\,(S)}\right)^{-1} > 2\,.$$

Additionally we need the following hypothesis. We set $\Delta_V = V^{-1}\,\Delta V$.

C5. Suppose that $D(\Delta_V)\,\cap\,D(L_V)$ is a core for $L_V$ and that

$$\Gamma = \left\{\varsigma\;:\;\|\,\varsigma\,\| = \frac{\alpha}{2}\right\}\;C\;\Delta_s\,,$$

the region of strong convergence for $L_V$ in the sense of Kato [32] chapter §2.1 , while

$$\|\,R(\varsigma\,,L_V^\varepsilon)\;\Delta_V\;R(\varsigma,L_V)\,\|\;\le\;C\,,$$

uniformly in $0\;<\;\varepsilon\;\le\;\varepsilon_0\,,\varsigma\;\in\;\Gamma$.

With regard to this last hypothesis, let us recall that the hypoellipticity implies locally the loss of half a power of a derivative (see [34]) so that hypothesis C5 may be regarded as a globalised version of this result. A more precise analytic version of this result would be rather difficult, however see the papers by Takanobu [51], [53] for probabilistic results involving the use of the Malliavin calculus together with the references cited there.

Using C5, by the resolvent identity we have

$$R(\varsigma,L_V^\varepsilon)\;-\;R(\varsigma,L_V)\;=\frac{-\varepsilon^2}{2}\;R(\varsigma,L_V^\varepsilon)\;\Delta_V\;R(\varsigma,L_V)$$

and hence

$$\|\,R(\varsigma,L_V^\varepsilon)\;-\;R(\varsigma,L_V)\,\|\;\le\;\frac{\varepsilon^2}{2}\,C\,,\;\varsigma\;\in\;\Gamma\,.$$

Arguing as in Lemma 2.2 we see that

$$\|\,\phi^\varepsilon\;-\;\phi\,\|\;\le\;0(\varepsilon^2)\;\;as\;\;\varepsilon\;\downarrow\;0$$

and, further,

$$g^\varepsilon(p) = g(p) + 0(\varepsilon^2)\;,$$

for

$$|p| < \frac{\alpha}{2M} \left(1 + \frac{2}{\alpha} \|V^{-1}\|_\infty \|V\phi\|_{L^1(S)}\right)^{-1}$$

taking $\varepsilon$ sufficiently small.

Let $X^{x,t}$ be the stochastic process associated with the infinitesimal generator

$$H^{\varepsilon^*} = \frac{\varepsilon^2}{2} \Delta_x + v(x, \xi(t)(\sigma)) .\nabla_x ,$$

beginning at $x$ at time $t$ . Let $\theta_t(\sigma)$ be the usual transformation operators on paths associated with the process (see the book by Dynkin [16] section 3.5). Then from results, given by Kunita in [37] (Chapter II, section 4, Theorem 4.4 and section 6 Theorem 6.1) and setting $X(t) = X^{x,0}(t)$ there exists a stochastic diffeomorphism $x \rightarrow X(t)^{-1} x$ and we may regard $X^{x,t}(0) = X(t)^{-1}x = X(-t, \theta_t(\sigma))x$, a result which we use in the proof of Theorem 3.5.

In this section we work under the hypothesis C1b on the initial data: $y_0 \neq 0$ a.e and $y_0 \in L^2 \left((1+|x|)^{2k} dx\right)$

We first need to establish an additional technical lemma which is a variant of the result given in Lemma 2.1

<u>Lemma 3.4</u>

**Under conditions** C1a, b , C2, C2′, C3″ **if** $R_0 < \infty$ **and**

$$\int_{B(0, R_0)} \| y_0(x) \|^2 dx \neq 0$$

**then**

$$\varliminf_{t \to \infty} t^{-1} \log E \int_{B(0, R_0)} dx \, |E_{x,m} \, \Theta^\varepsilon \, y_0(x) \,|^2 \geq 2g^\varepsilon(1) .$$

<u>Proof</u>

We set $\theta_0(x) = y_0(x) \,/\, \| y_0(x) \|$ , $y_0(x) \neq 0$ .

We first note that

$$\begin{aligned} \theta = \quad & \theta_0 + \int_0^t h\,(\eta^\varepsilon)\, du \\ = \quad & \theta_0 + t \left(t^{-1} \int_0^t h\,(\eta^\varepsilon)\, du\right) \\ = \quad & \theta_0 + t < h,\, L_t\,(\eta^\varepsilon) > . \end{aligned}$$

If follows that we have to deal with the asymptotics of the integral

$$E \int_{B(0,\ R_0)} dx\, |E_{x,m,\theta_0(x)}\,(\exp(- < Q,\ L_t\,(\eta^\varepsilon)\ > t)(\theta_0 + t\ <\ h, L_t\,(\eta^\varepsilon))\ |^2 .$$

Now by an argument similar to Lemma 4.2 of [54] the supremum in

$$\sup_{\tilde{\mu}\,\in\,\tilde{M}} (< Q,\ \tilde{\mu}\ > -I\ (\tilde{\mu}))$$

is attained at some $\mu^*\ \in\ \widetilde{M}$, so that by Theorem 4.1 of [54], since $<\ h,\ \tilde{\mu}\ >$ is continuous in $\tilde{\mu}$, it follows that the above integral is asymptotic to

$$E \int_{B(0,\ R_0)} dx\ |\ \theta_0 + <\ h,\ \mu^*\ > t\ |^2\ |\ y_0(x)\ |^2\ \exp\ 2g^\varepsilon(1)t\ \{1 + o(1)\ \}$$

as $t\ \to\ \infty$,

$$\geq\ const\ \exp\ 2g^\varepsilon\ (1)t\ \{1 + o(1)\ \}$$

as $t\ \to\ \infty$. It follows that

$$\underline{\lim}_{t\,\to\,\infty}\ t^{-1}\ \log\ E \int_{B(0,\ R_0)} dx\ |E_{x,m}\ (\Theta^\varepsilon\ y_0(x))|^2\ \geq\ 2g^\varepsilon(1)\ .$$

<u>Theorem 3.5</u>
**Suppose that conditions** C1a, b, C2, C2′, C3*, C4′, C5 **hold. Then**

$$2g(1)\ \{1 + o_\varepsilon(1)\ \}\quad \leq\ \underline{\lim}_{t\,\to\,\infty}\ t^{-1}\ \log\ |||\ y^\varepsilon(.\ ,t)\ |||^2$$

$$\leq \overline{\lim}_{t\,\to\,\infty} t^{-1}\ \log\ |||\ y^\varepsilon(.\ ,t)\ |||^2\ \leq\ g(2)\ \{1 + o_\varepsilon(1)\}\ ,$$

**uniformly in** $\varepsilon$ , $0\ <\ \varepsilon\ \leq\ \varepsilon_0$ , **for** $\varepsilon$ **sufficiently small.**

<u>Proof</u>
First let us note that from (3.8).

$$|||\ y^\varepsilon(.\ ,t)\ |||^2\ \leq\quad E \int_{R^n} dx\ E_W\ \{\frac{dP^{x,t}}{dW}$$
$$||\ \Theta(t)(X^{x,t}(.)\ ,\xi(.))\ ||^2\ ||\ y_0(X^{x,t}(0))\ ||^2\ \}$$

where the Radon-Nikodym derivative $\frac{dP^{x,t}}{dW}$ exists by the Cameron-Martin-Girsanov formula where the explicit formula involved depends functionally on $X(u) o X(t)^{-1} x$ and we have used Schwarz 's inequality,

$$\leq E\, E_W \int_{R^n} dx \frac{dP^{x,t}}{dW} \| \Theta(t)(X^{x,t}(.)\, , \xi(.) \|^2 \| y_0(X^{x,t}(0)) \|^2$$

using Fubini 's theorem.

Now making the unimodular transformation $x \to X(t)x$ , we see that

$$\leq E\, E_W \int_{R^n} dx \frac{dP^x}{dW} \| \Theta(t)(X(t)\,(.)x,\ \xi(.) \|^2 \| y_0(x) \|^2$$
$$\leq E \int_{R^n} E_x \{ \| \Theta(X(t)\,(.)x,\ \xi(.) \|^2 \} \| y_0(x) \|^2$$
$$\leq E \int_{R^n} E_{x,m} \{ \| \Theta^\epsilon \|^2 \| y_0(x) \|^2 \}$$
$$\leq \exp\ g^\epsilon(2)t\ \{1 + o(1)\ \}\ as\ t \to \infty$$

as in Lemma 2.1

$$\leq \exp\ g(2)t\ \{1 + o_\epsilon(1)\}\ \{1 + o(1)\}\ as\ t \to \infty$$

uniformly in $\varepsilon, 0 < \varepsilon \leq \varepsilon_0$ .

Hence, we conclude that

$$\overline{\lim_{t \to \infty}}\ t^{-1} \log \||| y^\epsilon(.,\ ,t) \|||^2 \leq g(2)\ \{1 + o_\epsilon(1)\}\ .$$

The lower bound is rather more complicated.

First let us observe that

$$\||| y^\epsilon(.,\ ,t) \|||^2 \geq E \int_{B(0,\ R)} dx \| E_{x,t}\ (\Theta(t)(X(.)\, , \xi(.))y_0(X(0)) \|^2$$

so that by Jensen 's inequality

$$\geq | B(0,\ R) |^{-1} \| E \int_{B(0,\ R)} dx\ E_{x,t}\ (\Theta(t)(X(.)\, , \xi(.))y_0(X(0)) \|^2$$

$$\geq | B(0,\ R) |^{-1} \| E\, E_W \int_{B(0,\ R)} dx \frac{dP^{x,t}}{dW} \Theta(t)(X^{x,t}(.)\, , \xi(.)y_0(X^{x,t}(0)) \|^2$$

$$\geq \frac{1}{2}(B(0,\ R))^{-1} \| E\, E_W \int_{R^n} dx \frac{dP^{x,t}}{dW}\, \Theta(t)(X^{x,t}(.)\ , \xi(.) y_0(X^{x,t}(0)) \|^2\}$$

$$- \mid B(0,\ R) \mid^{-1} \| E\, E_W \int_{B(0,\ R)^c} dx\, \frac{dP^{x,t}}{dW}\, \Theta(t)(X^{x,t}(.)\ , \xi(.)) y_0(X^{x,t}(0)) \|^2 ,$$

using the inequality $\mid a-(-b) \mid^2 \geq \frac{a^2}{2} - b^2$ ,

$$\geq \frac{1}{2} \mid B(0,\ R)^{-1} \| \| E\, E_W \int_{R^n} dx\, \frac{dP^{x,t}}{dW}\, \Theta(t)(X^{x,t}(.)\ , \xi(.)) y_0(X^{x,t}(0)) \|^2$$

$$- \mid B(0,\ R)^{-1}\, 0(R^{-2k}) \| E\, E_W \int_{R^n} dx\, \frac{dP^{x,t}}{dW}\, \Theta(t)(X^{x,t}(.)\ , \xi(.))$$

$$\times (1 + |X^{x,t}(0)|)^k\, y_0(X^{x,t}(0)) \|^2$$

Now by the unimodularity of the stochastic diffeomorphism $x \rightarrow X(t)x$, it follows that

$$\||\, y^\epsilon(.\ ,t) \||^2 \geq \frac{1}{2} \mid B(0,\ R) \mid^{-1} \| E \int_{R^n} dx\, E_{x,m}\, (\Theta^\epsilon\, y_0(x) \|^2$$

$$- \mid B(0,\ R) \mid^{-1} 0(R^{-2k})\, E \int_{R^n} dx\, E_{x,m}\, ( \| |\Theta^\epsilon(t) \|(1 + |x|)^k\, |y_0(x))^2$$

Then for $R_0 < \infty$ with $\int_{B(0,\ R_0)} \mid y_0(x) \mid^2 dx > 0$ we have

$$\||\, y^\epsilon(.\ ,t) \||^2 \geq \frac{1}{2} \mid B(0,\ R) \mid^{-1} \| E \int_{(B(0,R_0)} dx\, E_{x,m}\, (\Theta^\epsilon\, y_0(x)) \|^2$$

$$- \mid B(0,\ R_0)|^{-1}\, 0(R^{-2k})\, E \int_{R^n} (1 + |x|)^{2k} \| y_0(x) \|^2\, (E_{x,m}\, (\| \Theta^\epsilon(t) \|))^2$$

Now by Lemma 3.4 and Lemma 2.1′ and Remark 4′ , for $\delta > 0$ , there exists $T(\delta)$ such that

$$\||\, y^\epsilon(.,\ t) \||^2 \geq \frac{1}{2} \mid B(0,\ R) \mid^{-1} \exp\, (2g^\epsilon(1)t\, (1 - \frac{\delta}{2}))$$

$$- \mid B(0,\ R) \mid^{-1} 0(R^{-2k})\ \exp\ (2g^{\epsilon}(1)t\ (1 + \frac{\delta}{2}))$$

for $t \geq T(\delta)$ . given that

$$\int_{R^n} (1 + |x|)^{2k} \ \| \ y_0(x) \ \|^2 \ dx \ < \ \infty \, .$$

Let us choose

$$R = \exp\ \left( \frac{5\ g^{\epsilon}(1)\ \delta}{k} t \right) \quad ,$$

from which it follows that

$$||| \ y^{\epsilon}(.,\ t) \ |||^2 \ \geq \ const\ \exp\ (2g^{\epsilon}(1)t\ (1 - \left( \frac{5n}{2k} + \frac{1}{2} \right) \delta)$$

for $t \geq T(\delta)$ , uniformly in $0 < \varepsilon \leq \varepsilon_0$ .

However, $\delta > 0$ is arbitrary which establishes that

$$\underline{\lim}_{t \to \infty} \ t^{-1} \log \ ||| \ y^{\epsilon}(.,\ t) \ |||^2 \ \geq \ 2g^{\epsilon}(1)$$

and the result follows immediately.

It should be noted that the lower bound here is crude and a further large deviation analysis on time steps large with $\varepsilon$ but small with respect to $t$ has to be performed enabling are to use the analysis of [8] in order to improve this estimate as in Section 3.2. (in a compact setting this actually carried out by Collet in [12]). However, the lack of uniformity of $\underline{\lim}$ bounds on noncompact sets is a complicating factor. The above theorem does show that a "fast" dynamo type effect exists although we do not have precise asymptotics.

Finally, let us remark that if we consider a model where there is an inbuilt compacteness it is possible to establish results under less restrictive conditions on the velocity fields. This is the case, for example, in laminar dynamo theory, where $u = 0$ , $\nabla \wedge B = 0$ outside of a bounded region $\Omega$ (see [45] 6.1, 6.11 ).

**ACKOWLEDGEMENTS**

The author is grateful to Professors L. Arnold and W. Kliemann for various profitable conversations , to Professors M. E. Vares, A. Lopes and J. Ruiz Claeyssen for some useful comments and references to the literature and is especially grateful to the referee for his critical and helpful comments.

## References

1. L. Arnold, A formula connecting sample and moment stability of linear stochastic systems, S.I.A.M. J. Appl. Math; 44, 1984, 1 793-802.

2. L. Arnold and W. Kliemann, Lyapunov exponents of linear stochastic systems, Report n$^{o}$ 93, Forschungsschwerpünkt Dynamische Systeme, Universität Bremen, June 1983.

3. L. Arnold and Kliemann, Large deviations of linear stochastic differential equations, in Stochastic Differential Systems, ed. H.J. Engelbert and W. Schmidt, Lecture Notes in Control and Information Science, 96, Springer-Verlag, Berlin, 1987, 117-151.

4. L. Arnold, W. Kliemann and E. Oeljeklaus, Lyapunov exponents of linear stochastic systems, in Lyapunov exponents, Lectures Notes in Mathematics 1186, Springer-Verlag, Berlin, 129- 159.

5. L. Arnold and W. Kliemann, On unique ergodicity for degenerate diffusions, Stochastics 21, 1987, 41-61.

6. L. Arnold, E. Oeljeklaus, E. Pardoux, Almost sure and moment stability for linear Ito equations in Lyapunov Exponents, Lecture Notes in Mathematics, 1186, Springer-Verlag, Berlin, 1986, 129-159.

7. V.I. Arnold, Ya B. Zel'dovich and D.O. Sokolov. A magnetic field in a stationary flow with stretching in Riemann space, Sov. Phys. JETP, 45(6), 1982, 1083-1086.

8. D.G. Aronson, The fundamental solution of a linear parabolic equation containing a small parameter, I11. J. Math; 3, 1959, 580-619.

9. D.G. Babbitt, Wiener integral representations for certain semi-groups wich are infinitesimal generators with matrix coefficients, J. Math. Mech; 19, 1970, 1051-1067.

10. W.J. Cocke, Turbulent hydrodynamic line stretching: consequence of isotropy, Phys. Fluids, 12, 1969, 2488-2492

11. W.J. Cocke, turbulent hydrodynamic line stretching: the radom walk limit, Phys. Fluids, 14, 1971, 1624-1628.

12. P. Collet, Fast dynamo for some Anosov flows, preprint 1990.

13. H. Crauel, Lyapunov numbers of Markov solutions of linear stochastic systems, Stochastic, 14, 1984, 11-28.

14. J.D. Dollard and C.N. Friedman, Product Integration, Addison-Wesley, Reading, Massachussetts, 1979.

15. M.D. Donsker and S.R.S. Varadhan, Asymptotic evaluation of certain Markov Process expectations for large time, I; Comm. Pure Appl. Math; 27, 1975, 1-47.

16. M.D. Donsker and S.R.S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time III, Comm. Appl. Math; 29, 1976, 389-461.

17. N. Dunford and J.T. Schwartz, Linear operators I, Interscience, New York, 1967.

18. E.B. Dynkin, Markov Processes, I, Band 121, Grundlehren Math. wissenschaften, Springer-Verlag, Berlin, 1965.

19. R. Ellis, Entropy, Large Deviations and Statistical Mechanics, Springer-Verlag, Berlin, 1985.

20. R. Esposito, R. Marra, M. Pulvirenti and C. Sciarretta, A. Stochastic Lagrangean Picture for the Three Dimensional Navier- Stokes Equations, Comm. in Part. Diff. $Eq^{\underline{s}}$, 13, 1988, 1601-1610.

21. J.M. Finn, Chaotic Generation of Magnetic Fields - The Fast Dynamo Problem, Plasma preprint UMLPR 90-031, Laboratory for Plasma Research, University of Maryland.

22. J.M. Finn, J.D. Hanson, I. Kan and E. Ott, Do Steady Fast Magnetic Dynamos Exist?, Physics Rev. Letters, 62, 1988, 2965- 2968.

23. J.M. Finn, J.D. Hanson, I. Kan and E. Ott, Steady Fast Dynamo Flows, Plasma preprint UMLPR 90-015, Laboratory for Plasma Research, University of Maryland.

24. J.M. Finn and E. Ott, Chaotic Flows and Fast Magnetic Dynamos, Phys. Fluids 31(10), 1988, 2992-3011.

25. J.M. Finn and E. Ott, Chaotic Flows and Magneto Dynamos, Physics Rev. Letters, 1988, 760-763.

26. J.M. Finn and E. Ott, The Fast Kinematic Magnetic Dynamo and the Dissipationless Limit, Phys. Fluids B, 1990, 916-926.

27. M. Freidlin, Functional Integration and Partial Differential Equations, Princeton University Press, Princeton, N.J., 1985.

28. M. Freidlin and A.D. Wentzell, Random Pertubations of Dynamical Systems, Springer-Verlag, Berlin, 1984.

29. T. Hida, Brownian Motion, Springer-Verlag, Berlin, 1980.

30. K. Ichihara and H. Kunita, A classification of the second order degenerate elliptic operators and its probabilistic characterization, Z. Wahrscheinlichkeitstheorie, Verw. Geb; 30, 1974, 235-254.

31. N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam, 1981.

32. T. Kato, Perturbation Theory for Linear Operators, Springer- Verlag, Berlin, 1966.

33. W. Kliemann, Recurrence and Invariant Measures for Degenerate Diffusions, Ann. Prob; 15, 1987, 690-707.

34. J.J. Kohn, Pseudo-differential Operators and Hypoellipticity, Proc. Symp. Pure Math; 23, A.M.S., Providence, R.F., 1973, 67-70.

35. S. Kotani, Limit Theorems of Hypoelliptic Diffusion Processes, in Probability Theory and Mathematical Statistics, Lectures Notes in Mathematics, ed. K. Ito and J.V. Prokhorov, Springer-Verlag, Berlin, 1983.

36. H. Kunita, Supports of Diffusion Processes and Controlability Problems, Proc. Inter. Symp. Stochastic Differential Equations, Kyoto, 1976, Ed. K. Ito, J. Wiley, New York, 1978, 163-185.

37. H. Kunita, Stochastic Differential Equations and Stochastic Flow of Diffeomorphism, in École d'Été de Probabilité, Saint-Flour XII, edited P.L. Hennequin, Lecture Notes in Mathematics 1097, Springer-Verlag, Berlin, 1982, 144-305.

38. F. Ledrappiér, Quelques Proprietes des Exposants Caracteristiques, in École d' Été de Probabilité, Saint- Flour XII, edited P.L. Hennequin, Lecture Notes in Mathematics 1097, Springer-Verlag, Berlin, 1982, 305-396.

39. E. Le Page, Théoreme limite pour les produits des matrices aleatories, in Prob. Measures on Groups Lecture Notes in Mathematics, 928, edited H. Heyer, Springer-Verlag, Berlin, 1982, 258-303.

40. Y. Miyahara, Ultimate Boundedness of Systems Governed by Stochastic Differential Equations, Nagoya Math. J.; 47, 1972, 111-144.

41. Y. Miyahara, Invariant Measures of Ultimately Bounded Stochastic Processes, Nagoya Math. J. 49, 1973, 149-153.

42. H.K. Moffatt, Geophysical and Astrophysical Turbulence, in Advances in Turbulence, ed. G. Compte-Bellot and J. Mathieu, Springer-Verlag, Berlin, 1987, 228-244.

43. H.K. Moffatt, Dynamo Instability and Feedback in a Stochastically Driven System, in Statistical Mechanics and Turbulence, Lecture Notes in Physics, edited by M. Rosenblatt and C.V. Atta, Springer-Verlag, Berlin, 1975.

44. H.K. Moffatt, Some Developments in the Theory of Turbulence, J. Fluid Mech; 106, 1981, 27-47.

45. H.K. Moffatt, Magnetic Field Generation in Eletrically Conducting Fluids, CUP, Cambridge, 1978.

46. S.A. Molchanov, A.A. Ruzmaiken, D.D. Sokolov and Ya B. Zel'dovich, Kinematic Dynamo Problem in a Linear Velocity Field, J. Fluid Mech; 144, 1984, 1-11.

47. S.A. Molchanov, A.A. Ruzmaiken, D.D. Sokolov, A Dynamo Theorem, Geophys. Astrophys. Fluid Dynamics, 30, 1984, 242-259.

48. S.A. Molchanov, A.A. Ruzmaiken and D.D. Sokolov, Kinematic Dynamo in Random Fields, Sov. Phys. Usp; 28, 1985, 307-327.

49. E.N. Parker, Cosmical Magnetic Fields, Oxford, University Press, Oxford, 1979.

50. D.W. Stroock, On Certain Systems of Parabolic Equations, Comm. Pure Appl. Math; 23, 1970, 447-457.

51. D.W. Stroock, An Introduction to the Theory of Large Deviations, Springer-Verlag, Berlin, 1984.

52. S. Takanobu, Diagonal Estimates of Transition Probability Densities of Certain Degenerate Diffusion Processes, J. Functional Anal; 91, 1990, 221-236.

53. S. Takanobu, Diagonal Short Time Asymptotics of Heat Kernels for Certain Degenerate Second Order Differential Operator of Hörmander type, Publ. RIMS, Kyoto University, 24, 1988, 169- 203.

54. S.R.S. Varadhan, Asymptotic Probabilities and Differential Equations, Comm. Pure Appl. Math; 29, 1966, 261-286.

55. M.M. Vishik, I.C. Dynamo, Vychislitel,'naya Seismologiya (Computational Seismology), Vol. 19, 1984, 186-215.

Publicações do Instituto de Matemática da UFRGS
Cadernos de Matemática e Estatística

Série A: Trabalho de Pesquisa

1. Marcos Sebastiani - Transformation des Singularités - MAR/89.
2. Jaime Bruck Ripoll - On a Theoremof R. Langevin About Curvature and Complex Singularities - MAR/89.
3. Eduardo Cisneros, Miguel Ferrero e Maria Inés Gonzales - Prime Ideals of Skew Polynomial Rings and Skew Laurent Polynomial Rings - ABR/89.
4. Oclide José Dotto - $\epsilon$ - Dilations - JUN/89.
5. Jaime Bruck Ripoll - A Characterization of Helicoids - JUN/89.
6. Mark Thompson, V. B. Moscatelli - Asymptotic Distribution of Liusternik-Schnirelman Eigenvalues for Elliptic Nonlinear Operators -JUL/89.
7. Mark Thompson - The Formula of Weyl for Regions with a Self- Similar Fractal Boundary - JUL/89.
8. Jaime Bruck Ripoll - A Note on Compact Surfaces with Non Zero Constant Mean Curvature - OUT/89.
9. Jaime Bruck Ripoll - Compact $\epsilon$ - Convex Hypersurfaces - NOV/89.
10. Jandyra Maria G. Fachel - Coeficientes de Correlação Tipo -Contingência - JAN/90.
11. Jandyra Maria G. Fachel - The Probability of Ocurrence of Heywood Cases - JAN/90.
12. Jandyra Maria G. Fachel - Heywood Cases in Unrestricted Factor Analysis - JAN/90.
13. Julio Cesar R. Claeyssen, Tereza Tsukazan - Dynamical Solutions of Linear Matrix Differential Equations - JAN/90.
14. Maria T. Albanese - Behaviour of the Likelihood Function in Latent Trait Analysis of Binary Data - ABR/91.
15. Maria T. Albanese - Measurement of the Latent Variable in Latent Trait Analysis of Binary Data - ABR/91.
16. Maria T. Albanese - Adequacy of the Asymptotic Variance-Covariance Matrix using Bootstrap and Jackknife Techniques in Latent Trait Analysis of Binary Data - ABR/91.

17. Maria T. Albanese - Latent Variable Models for Binary Response - ABR/91.

18. Mark Thompson - Kinematic Dynamo in Random Flows - DEZ/90.

19. Jaime Bruck Ripoll, Marcos Sebastiani - The Generalized Map and Applications - AGO/91.

20. Jaime Bruck Ripoll, Suzana Fornari, Katia Frensel - Hypersurfaces with Constant Mean Curvature in the Complex Hyperbolic Space - AGO/91.

Universidade Federal do Rio Grande Sul
Reitor: Professor Tuiskon Dick

Instituto de Matemática
Diretor: Professor Aron Taitelbaum
Núcleo de Atividades Extra Curriculares
Coordenador: Professora Jandyra G. Fachel
Secretária: Rosaura Monteiro Pinheiro

Os Cadernos de Matemática e Estatística publicam as seguintes séries:
Série A: Trabalho de Pesquisa
Série B: Trabalho de Apoio Didático
Série C: Colóquio de Matemática SBM/UFRGS
Série D: Trabalho de Graduação
Série E: Dissertações de Mestrado
Série F: Trabalho de Divulgação
Série G: Textos para Discussão

Toda correspondência com solicitação de números publicados e demais informações deverá ser enviada para:

NAEC - Núcleo de Atividades Extra Curriculares
Instituto de Matemática - UFRGS
Av. Bento Gonçalves, 9500
91.500 - Agronomia - POA/RS
Telefone: 36.11.59 ou 36.17.85 Ramal: 252