THE FORMULA OF WEYL FOR REGIONS WITH A SELF-SIMILAR FRACTAL BOUNDARY

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-Trabalho de Pesquisa-

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THE FORMULA OF WEYL FOR REGIONS WITH A SELF-SIMILAR FRACTAL BOUNDARY

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Introduction

In the paper [1] M.V. Berry speculates that for general sets $\Omega \subset R^n$ with boundary $\partial\Omega$, having general nonintegral Hausdorff measure D and d respectively it should be possible to show that the Laplacian operator defined in Ω has an asymptotic distribution of frequencies $N(\lambda)$ given by $N(\lambda) = \mu_D(\Omega)\lambda^{D/2} + O(\mu_d(\partial\Omega))\lambda^{d/2}$ as $\lambda \to \infty$. Here $\mu_s(\Omega)$ is the s-Hausdorff measure of the set Ω. In fact, it is not all clear how to formulate this problem when Ω is not an open set of R^n and even in this case it seems doubtful that the result is true as formulated. Here, we show that for regions with a boundary displaying greater geometric regularity, as is the case, for example, for fractal boundaries generated by similarity transformations the result holds in the form

$$N(\lambda) = \mathrm{vol}(\Omega)\lambda^{n/2} + O(\lambda^{\frac{n-1}{2}+\frac{1}{2(n-s+1)}}) \quad \text{as} \quad \lambda \to \infty,$$

where s is the (Hausdorff) similarity dimension of $\partial\Omega$ and $n-1 < s < n$.

In such a case, from our point of view all that is involved is that the lattice of cubes of side-length ρ gives a relatively efficient covering of the boundary in the sense that only $O(\rho^{-s})$ disjoint cubes are needed in the covering as $\rho \to 0$. This follows from a result given in [4] for self-similar fractal sets satisfying the open condition. It follows that we may apply the localisation technique of Courant-Weyl (extended to more general operators by Métivier in [5] and Moscatelli and Thompson in [7]). We follow the definitions and formulation of self-similar fractals as first given by Hutchinson in [3] and in resumed form by Falconer in [2] (see also [4]).

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1. Preliminaries

1.1 Let Ω be an open region in R^n with fractal boundary $\partial\Omega$ which is self-similar in the sense of Hutchinson [3] and generated in the following way. Let $\mathcal{S} = (S_1, \ldots, S_m)$ be a set of similitudes on R^n with constants $(r_1, r_2, \ldots, r_m)$. That is to say $|S_j(x) - S_j(y)| = r_j|x-y|$, $x, y \in R^n$, $j = 1, \ldots, m$. It is assumed that the open set condition is satisfied in the sense that there exists a non-empty open set O such that

$$(1) \quad \bigcup_{i=1}^{m} S_i O \subset O$$

$$(2) \quad S_i O \cap S_j O = \phi \quad \text{if} \quad i \neq j \quad .$$

Setting $\overline{r} = \max_{j=1,\ldots,m} r_j$ we assume that $\overline{r} < 1/2$ to avoid degeneracies (see [3]). Then it was shown by Hutchinson that there exists a unique compact set K such that

$$K = \bigcup_{j=1}^{m} S_j K \quad .$$

We assume that Ω has as its boundary $\partial\Omega$ such a K. In case $0 \cap K \neq \phi$ we say that $\mathcal{S}$ satisfies the strong open condition. The similarity dimension D of $\partial\Omega$ is then defined by $\sum_{i=1}^{m} r_i^D = 1$. Recall that the k-dimensional Hausdorff measure $H^k(E)$ of a set $E \subset R^n$ is defined as follows: For $k \geq 0$, let $H_\delta^k(E) = \inf\{ \sum_{m=1}^{\infty} \frac{\Gamma(1/2)^k}{\Gamma(k/2+1)} 2^{-k} (\operatorname{diam} E_i)^k :$ $E \subset \bigcup_{i=1}^{\infty} E_i ; \operatorname{diam} E_i \leq \delta\}$ and $H^k(E) = \lim_{\delta \to 0} H^k(E) = \sup_{\delta > 0} H_\delta^k(E)$.

For self-similar fractals it is known that the similarity dimension and the Hausdorff dimension are equal. We observe that in two dimensions the Koch snowflake may be generated in this way (see [4]); $D = \frac{\log 4}{\log 3}$. Call a finite subset F of K ε-separated if $\operatorname{dist}(x,x') \geq \varepsilon$ for all $x,x' \in F$ such that $x \neq x'$. Let $N(\varepsilon,K)$ be the maximal cardinality of an ε-separated subset of K; this is called the packing function. The packing dimension D_p is defined by $D_p = \lim_{\varepsilon \downarrow 0} \frac{N(\varepsilon,K)}{\log \varepsilon^{-1}}$ provided that the limit exists. One recalls that the packing dimension (often called capacity) was introduced by Pontrjagin and Schnirelman in 1932. There is of course a considerable volume of mathematical folklore as to be equivalence of various notions of fractal dimension. In the present case one has the result of Lalley in [4] (Corollary to Theorem 1).

Lemma 1 (Lalley)

For a fractal self-similar boundary $\partial\Omega$ satisfying the strong open condition $D = D_p$ and hence $s = D_p$.

1.2 Let V and H be two separable Hilbert space with V embedded continuously in H and let $\mathcal{A}(u,v)$ be symmetric continuous coercive bilinear form on V . Then we call $(V,H,\mathcal{A})$ a variational triplet. We know that if V is dense in H then there exists an unbounded self-adjoint operator A such that $(Au,v) = \mathcal{A}(u,v)$, $u \in D(A)$, $v \in V$. If the injection of V is compact then the resolvent of A in H is compact and the spectrum of A consists of a set of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ accumulating at infinity, which we can clearly order as to multiplicity $\lambda_1 \leq \lambda_2 \leq \dots$. Let $E_\lambda(V,H,\mathcal{A})$ be the set of all closed subspaces E in V such that the form $\mathcal{A} - \lambda$ is strongly coercive on E . Then, we define the distribution function of the variational triplet $N(\lambda,V,H,\mathcal{A})$ by

$$N(\lambda,V,H,\mathcal{A}) = \inf_{E \in E_\lambda(V,H,\mathcal{A})} \operatorname{codim}_V E .$$

It is well known that

$$N(\lambda,V,H,\mathcal{A}) = \#\{\lambda_j \leq \lambda\} .$$

If $\mathcal{A}_1(u,v)$ is a further continuous symmetric bilinear form on V and $\mathcal{A}_1(u,u) \leq \mathcal{A}(u,u)$, $u \in V$, we have

P1 $$N(\lambda,V,H,\mathcal{A}_1) \geq N(\lambda,V,H,\mathcal{A}) .$$

It is clear that

P2 $\qquad N(\lambda,V,H,\mathcal{A}+t) = N(\lambda-t,V,H,\mathcal{A})\ , \quad t\in R\ ,$

and

P3 $\qquad N(\lambda,V,H,\alpha\mathcal{A}) = N(\alpha^{-1}\lambda,V,\mathcal{A})\ , \quad \alpha>0\ .$

For these and many other properties, perhaps the most concise (and general) reference is the paper [5].
As well we recall the notion of n-diameters of Kolmogorov (see [7] section 2) for further information and references.
If A is a bounded subset of a normed space E the n-diameter $d_n(A,E)$ of A in E is defined by

$$d_n(A,E) = \inf_{G\in G_n(E)} \sup_{x\in A} \inf_{y\in G} \|x-y\|_E\ ,$$

where $G_n(E)$ denotes the set of subspaces of E of dimension less than or equal to n . Further, if V is a Hilbert space continuously embedded in $L^2(\Omega)$, $\Omega\subset R^n$ and ω is an open subset of Ω we denote by SV the unit ball of V and by $SV|\omega$, the set of restrictions to ω, of elements of SV . Then we define $N(\lambda,V,L^2(\omega))$ by

$$N(\lambda,V,L^2(\omega)) = \sum_{\lambda d_n^2(SV|_\omega,L^2(\omega))\geq 1} 1 .$$

In case $\omega\equiv\Omega$ this of course coincides asymptotically with the distribution function $N(\lambda^{1/2},\iota_V)$ of the injection of V into $L^2(\Omega)$ using the notion of the widths of a compact operator since $\lambda_j = d_{j-1}(\iota_V)$, $j=1,2,\ldots$ (see [7] section 1.16.1).

Let us observe that the problem posed by Berry may be formulated in more general terms with respect to the variational triplet where V is taken to be $H_0^1(\Omega)$, the usual Sobolev space, H to be $L^2(\Omega)$ and $\mathcal{A}(u,v)$ is defined by $\mathcal{A}(u,v) = \int_\Omega \sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} a_{\alpha\beta}(x) D^\alpha u D^\beta v\, dx$ satisfying the following hypotheses C:

C1 (1) $a_{\alpha\beta}(x) = \overline{a_{\beta\alpha}(x)}$ and the $a_{\alpha\beta}(x)$ are uniformly Hölder continuous of order σ on $\overline{\Omega}$ (without loss of generality we may assume this to be the case on $\Omega_I \cup \Omega_\partial$, see section 2)

(2) the bilinear form is continuous and formally symmetric on $H_0^1(\Omega)$

(3) there exists a positive constant c_0 such that $c_0 \|u\|_{1,2}^2 \leq \mathcal{A}(u)$.

From C(3) we may induce a natural norm $|\ |_V$ on V from $\mathcal{A}(u)$ and obviously the injection ι_V of V in $L^2(\Omega)$ is compact from Rellich's lemma. Note also that the injection ι_j of $H^1(Q_j)$ into $L^2(Q_j)$ is compact.

2. The formula of Weyl

We follow the classical localisation procedure of Courant-Weyl. Thus we cover Ω by a grid of cubes Q_i, $i \in F$ of side-length $\rho/2\pi$. For each cube Q_i we associate an augmented cube $\hat{Q}_i$ of side-length $5\rho/2\pi$

with the same centre as Q_i and with faces parallel to those of Q_i. Let Q_i, $i \in G$ be the set of cubes in the cover $\mathcal{F} = \{Q_i\}_{i \in F}$ such that $Q_i \cap \partial\Omega \neq \phi$. Without loss of genrality, we may assume $\hat{Q}_i \subseteq \bigcup_{k \in F} Q_k$, $i = 1,\ldots,G$.

Lemma 2

There exists a cover of Ω by cubes $\{Q_i\}_{i \in F' \cup F''}$ F' and F'' disjoint such that

(i) $Q_i \subset \Omega$, $i \in F'$;

(ii) $\partial\Omega \subset \Omega_\partial = \bigcup_{i \in F''} Q_i$ and $\# F'' \leq 5^n O(\rho^{-s})$;

(iii) $\lambda(\Omega) - \lambda(\Omega_I) = O(\rho^{n-s})$, as $\rho \to 0$;

where we have set $\Omega_I = \bigcup_{i \in F'} Q_i$ and λ denotes Lebesgue measure in R^n.

Proof

Choose points $q_i \in Q_i \cap \partial\Omega$, $i \in G$.
By definition not more than $N(\rho/2\pi, \partial\Omega)$ of the q_i may be $\rho/2\pi$-separated. Let H be the set of all i such that the q_i are $\rho/2\pi$-separated. By definition and lemma 1 $\# H \leq N(\rho/2\pi, \partial\Omega) \leq \text{const } \rho^{-s}$. Also any $p \in \partial\Omega$ belongs to some Q_j, $j \in G$ and if $j \notin H$ then p is within a distance $\rho/2\pi$ of some point $q_i \in H$ and, hence, contained in the aumented cube $\hat{Q}_i$. It follows that
$\partial\Omega \subset \bigcup_{i \in H} \hat{Q}_i$.
Each $\hat{Q}_i$, $i \in H$ may be decomposed into the union of 5^n cubes of side-length $\rho/2\pi$, possibly with overlapping.

It follows that $\partial\Omega \subset \bigcup_{i \in F''} Q_i$ with some index set F'' such that $F'' \leq 5^n O(\rho^{-s})$, the latter bound following from the estimate on $\# H$. Let $\{Q_j\}_{j \in F'}$ be the set of cubes of cover F contained in Ω such that $j \notin F''$. The we have

$$\Omega_I = \bigcup_{i \in F'} Q_i \subset \Omega \subset \bigcup_{i \in F'} Q_i \cup \bigcup_{i \in F''} Q_i = \Omega_I \cup \Omega_\partial .$$

Note that

$$0 \leq \lambda(\Omega \quad \Omega_I) \leq \lambda(\Omega_\partial) \leq O(5^n \rho^{-s})\rho^n = O(\rho^{n-s}) \quad \text{as} \quad \rho \to 0 .$$

Let Q_θ be the cube of side-length $\rho/2\pi$ and centre $\theta\rho/4\pi$, $\theta \in Z^n$. Let us introduce the approximating bilinear forms $\mathcal{A}_\theta(u,v)$ (with the associated operator A_θ) defined on $H^1(Q_\theta)$ via

$$\mathcal{A}_\theta(u,v) = \sum_{|\alpha|=|\beta|=1} \int_{Q_\theta} a_{\alpha\beta}(\theta\rho/4 \quad) D^\alpha u D^\beta v dx$$

and define $\tilde{\mathcal{A}}(u) = \sum_\theta \tilde{\mathcal{A}}_\theta(u)$ on $H_o^1(\Omega)$. Then by C1(1) and interpolation inequalities one has the elementary bound (see [5] and [6]):

$$|\mathcal{A}(u) - \tilde{\mathcal{A}}(u)| \leq (O(\rho^s)+\delta)\mathcal{A}(u) + O(\rho^{-1}+\delta^{-1}))|u|^2, \; u \in H_o^1(\Omega), (6)$$

(6) implies that $\tilde{\mathcal{A}}(u)$ is coercive on $H_o^1(\Omega)$.
Let $V_o = \oplus_{i \in F'} H_o^1(Q_i)$. Define Z_λ by

$$Z_\lambda = \{u \in H_o^1(\Omega) : (\bar{\mathcal{A}} - \lambda)(u,v) = 0 \quad \text{for all} \quad v \in V_o\} .$$

We also set $Z_\lambda(Q_\theta) = \{u \in H^1(Q_\theta) : A_\theta u = \lambda u\}$.

Let $a(x,\xi)$ be the principal symbol of A and set

$$\mu(\Omega) = (2\pi)^{-n} \int_\Omega dx \int_{a(x,\xi)<1} d\xi \quad ; \quad \mu_\theta = (2\pi)^{-n} \int_{a(\theta\rho/4\pi,\xi)<1} d\xi$$

and $\tilde{\mu}(\Omega) = \sum_\theta \mu_\theta$ for cubes Q_θ ccurring in the set $\{Q_j\}_{j \in F'}$.

We recall the following results of Métivier:

Proposition 2.7 of [5] implies that

$$N(\lambda, H_o^1(\Omega), L^2(\Omega), \tilde{\mathcal{A}}) \le N(\lambda, V_o, L^2(\Omega), \tilde{\mathcal{A}}) + N(\lambda, Z_\lambda, L^2(\Omega), \tilde{\mathcal{A}}), \quad (7)$$

Proposition 4.1 of [5] states that

$$N(\lambda, Z_\lambda(Q_\theta), L^2(Q_\theta), \|\ \|^2_{1,2}) \le c(1+\rho^{n-1}(\lambda+\mu)^{(n-1)/2}), \lambda, \mu \ge 0 \quad (8)$$

and finally that $|N(\lambda, H_o^1(Q_\theta), L^2(Q_\theta), \mathcal{A}_\theta) - \mu_\theta \lambda^{n/2}|$

$$\le c(1+\rho^{n-1}\lambda^{(n-1)/2}) \qquad (9).$$

We need certain modifications of technical Lemmas proved in [5] namely

<u>Lemma 3</u>

Under the hypotheses of Lemma 1 we have

$$N(\lambda, H_o^1(\Omega), L^2(\Omega_\partial)) \le \text{const } \rho^{n-s}\lambda^{n/2}$$

<u>Proof</u>

One has the inequalities

$$N(\lambda, H_o^1(\Omega), L^2(\Omega_\partial) \le N(\lambda, H^1(\Omega), L^2(\Omega_\partial)) \le N(\lambda, H^1(\Omega_\partial), L^2(\Omega_\partial)$$

$$\le \text{const} \sum_{j \in F''} N(\lambda^{1/2}, \iota_j) \quad \text{, by} \qquad \text{definition}$$

and the additive property for the distribution functions of the injections of direct sums. It follows from the qualitative theory of Sobolev spaces [see [7] 4.10.2] and Lemma 2(ii) that

$$N(\lambda, H_o^1(\Omega), L^2(\Omega_\partial)) \leq \text{const } \rho^{n-s}\lambda^{n/2} .$$

Lemma 4

Under the hypotheses of Lemma 1 we have

$$N(\lambda, Z_\lambda, \bar{\mathcal{A}}) \leq \text{const } \rho^{n-s}\lambda^{n/2}$$

if $\lambda \geq \text{const}(\rho^{-1} + \delta^{-1})$.

Proof

From (6) it follows that

$$\mathcal{A}(u) \leq 2(\bar{\mathcal{A}}(u) + (O(\rho^{-1}) + O(\delta^{-1}))|u|^2)$$

and hence from (5) that

$$\|u\|^2_{1,2} \leq c_o^{-1}(\bar{\mathcal{A}}(u) + (O(\rho^{-1}) + O(\delta^{-1}))|u|^2) \qquad (10)$$

Hence, from P1 and (10) we have

$$N(\lambda, Z_\lambda, \bar{\mathcal{A}}) \leq N(\zeta, Z_\lambda, \|\ \|^2_{1,2}) \qquad (11)$$

where we have set $\zeta = \dfrac{2\lambda + O(\rho^{-1}) + O(\delta^{-1})}{c_o}$.

We set $\bar{\omega} = \Omega \cap \Omega_\partial$.

Then, we have

$$N(\zeta, Z_\lambda, \|\ \|^2_{1,2}) \leq N(\zeta, Z_\lambda, L^2(\bar{\omega}))$$

$$+ \sum_{j \in F'} N(\zeta, Z_\lambda(Q_j), L^2(Q_j), \|\ \|^2_{H^{1,2}(Q_j)})$$

as in [5],

$$\leq N(\zeta, H_o^1(\Omega), L^2(\Omega_\partial)$$

$$+ \sum_{j\in F'} N(\lambda, Z_\lambda(Q_j), L^2(Q_j)), \| \ \|^2_{H^{1,2}(Q_j)}$$

$$\leq \text{const } (\rho^{n-s}\lambda^{n/2} + \rho^{-1}\lambda^{(n-1)/2}) \qquad (12)$$

by Lemma 3, (8) and Lemma 2(iii) if $\lambda \geq \text{const}(\rho^{-1} + \delta^{-1})$. The result follows from (11) and (12).

<u>Theorem</u>

Under the hypotheses of Lemma 1 and conditions C the following estimate holds:

$$|N(\lambda, H_o^1(\Omega), \mathcal{A}) - \mu(\Omega)\lambda^{n/2}| \leq \text{const}(\rho^{n-s}\lambda^{n/2} + \rho^{-1}\lambda^{(n-1)/2} + \rho^{\sigma}\lambda^{n/2} + \rho^{-n})$$

for $\lambda \geq \text{const } \rho^{-1}$ and λ sufficiently large.

<u>Proof</u>

We set $\lambda_- = \lambda(1 - O(\rho^\sigma + \delta)) - O(\rho^{-1} + \delta^{-1})$

$$\lambda_+ = (1 + O(\rho^\sigma + \delta))(\lambda + O(\rho^{-1} + \delta^{-1})) \ .$$

Then one sees from (6) and P1 that

$$N(\lambda_-, V_o, L^2(\Omega), \tilde{\mathcal{A}}) \leq N(\lambda, H_o^1(\Omega), \mathcal{A}) \leq N(\lambda_+, H_o^1(\Omega), \tilde{\mathcal{A}}) \ , \qquad (13)$$

while from (7) we have

$$N(\lambda_+, H_o^1(\Omega), \tilde{\mathcal{A}}) \leq N(\lambda_+, V_o, L^2(\Omega), \tilde{\mathcal{A}}) + N(\lambda, Z_\lambda; L^2(\Omega), \tilde{\mathcal{A}}) \ , \quad (14)$$

It follows that choosing $\delta = \lambda^{-1/2}$, if $\lambda \geq \text{const } \rho^{-1}$, we have, using the additive property of distribution functions of orthogonal variational triplets (see Proposition 2.8 of [5]), (9), Lemma 4, (13) and (14) that

$$\bar{\mu}(\Omega)\lambda_-^{n/2} + O(\lambda_-^{(n-1)/2}\rho^{-1}) + O(\rho^{-n}) \leq N(\lambda, H_o^1(\Omega), \mathcal{A})$$

$$\leq \bar{\mu}(\Omega)\lambda_+^{n/2} + O(\lambda_+^{(n-1)/2}\rho^{-1}) + O(\rho^{n-s})\lambda^{n/2} + O(\rho^{-n}) \ .$$

However, $|\mu(\Omega) - \bar{\mu}(\Omega)| \leq \text{const}\ \rho^s$, hence, the result follows.

Corollary 1

$$N(\lambda, H_o^1(\Omega), \| \ \|_{1,2}^2) = \mu(\Omega)\lambda^{n/2} + O(\lambda^{n-1/2+1/2(n-1+s)})$$

as $\lambda \to \infty$.

Proof

In this case we choose the form $\mathcal{A}(u) = \| \ \|_{1,2}^2$. As the form has constant coefficients all of the approximating terms fall out in the calculations and no term corresponding to $\rho^\sigma \lambda^{n/2}$ occurs. Now choose $\rho = \lambda^{-1/2(n-1+s)}$ to obtain the result.

Corollary 2

If $0 < n-s < \sigma$

$$N(\lambda, H_o^1(\Omega), \mathcal{A}) = \mu(\mu)\lambda^{n/2} + O(\lambda^{n-1/2 + 1/2(n-1+s)})$$

Proof

In this case $\rho^\sigma \lambda^{n/2} = o(\rho^{n-s}\lambda^{n/2})$ and then choose $\rho = \lambda^{-1/2(n-1+s}$ to obtain the result.

Observation

All the preceding analysis may be carried out if the conclusion of Lemma 1 is true. The analysis may be extended to higher order coercive elliptic operators (in divergence form) on $H_0^m(\Omega)$.

References

1. M.V. Berry, Some geometrical aspects of ware motion: wave front dislocations, diffraction, catastrophes, diffractals, in Geometry of the Laplace operator, Proc. Symp. Pure Math; ed. R. Osserman, A. Weinstein, 13-28, A.M.S.; Providence, Rhode Island, 1980.

2. K.J. Falconer, The geometry of fractal sets, Cambridge University Press, Cambridge, 1985.

3. J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math.J.; 30, 1981, 713-747.

4. P. Lalley, The packing and covering functions of some self-similar fractals, Indiana Univ. Math.J.; 37, 1988, 699-709.

5. G. Métivier, Valeurs propres de problèmes aux limites elliptiques irréguliers, Bull. Soc. Math. France, Memoire, 51-52, 1977, 125-219.

6. V.B. Moscatelli, M. Thompson, Asymptotic eigenvalue distributions for pairs of elliptic bilinear forms, Richerche di Matematica (Napoli), 32, 1983, 41-61.

7. H. Triebel, Interpolation theory, functions spaces, differential operators, North-Holland, New York, 1978.

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