ON A THEOREM OF R. LANGEVIN ABOUT CURVATURE AND COMPLEX SINGULARITIES

Jaime B. Ripoll

- Trabalho de Pesquisa -
Série A2/MAR/89

ON A THEOREM OF R. LANGEVIN ABOUT CURVATURE AND COMPLEX SINGULARITIES

Jaime B. Ripoll

A classical result due to R. Langevin asserts that for a given polinomial $p:(C^n,o) \longrightarrow (C,o)$ with an isolated singularity at the origin, the following formula holds:

$$\lim_{\epsilon\to o}\lim_{t\to o}\int_{M_t\cap B_\epsilon} K\,\omega_t = c_{n-1}(\mu^n+\mu^{n-1}) \qquad (*)$$

where B_ϵ is a ball centered at the origin with radius ϵ, ω_t is the volume form of $M_t \equiv p^{-1}(t)$, $t \in C$, $t \sim o$, induced by the usual hermitian inner product of C^n, K is the Lipschitz-Killing curvature of M_t, μ^n is the Milnor number of M_o at o, μ^{n-1} the Milnor number at o of $M_o \cap H$ where H is a generic complex hyperplane through the origin of C^n and $c_{n-1} = (1/2)\mathrm{vol}(S^{2n-1})$.

Prof. Langevin commented that the above formula should be true in any complex manifold with a hermitian metric. We obtain here the following result:

Theorem 1. Let N be a 3-dimensional (over C) complex manifold with an hermitian metric. Let $f:N \longrightarrow C$ be an analytic map with an isolated singularity at p_0 with $f(p_0) = o$. Then, the following formula holds:

$$\lim_{\epsilon\to o}\lim_{t\to o}\int_{M_t\cap B_\epsilon} K\,\omega_t = c_2(\mu^3+\mu^2) \qquad (**)$$

where B_ϵ is a geodesic ball centered at p_0 with radius ϵ, ω_t is the volume form of $M_t \equiv f^{-1}(t)$ determined by the metric induced from N, K is the Lipschitz-Killing curvature of M_t, μ^3 the Milnor number of M_o at p_o and μ^2 the Milnor number of $M_o \cap P$ where P is a complex generic hypersurface through p_o of N.

To prove this result we introduce the concept of translation in a complex hermitian manifold N and we define the polar curve associated to an isolated singularity of a hypersurface of N. We also introduce the Gauss map of a complex hypersurface of N associated to a given translation, and then we apply similar techniques used for proving the above result in $\mathbb{C}^n$. Using this Gauss map we can define, as in $\mathbb{C}^n$, by taking the determinant of its derivative, another curvature which we call the *translation curvature* of the hypersurface. We prove then that formula (*) holds for the translation curvature (Corollary 2.2).

It follows from our results a generalization of a theorem of Linda Ness about the curvature of algebraic curves (see [N], Theorem 4.1 of this paper and Theorem of [L]).

I want to thank Marcos Sebastiani for his aid on the realization of this work.

1. Translations.

Let N be a complex n-dimensional manifold with an hermitian metric $\langle\ ,\ \rangle_c$ and let us consider the complex orthonormal frame bundle $O_c(N)$ of N, that is:

$O_c(N) = \{(p,\beta) \mid p \in N,\ \beta \text{ is an orthonormal basis over } \mathbb{C} \text{ of } T_p(N)\}$
and let $\pi: O_c(N) \longrightarrow N$ be the projection $(p,\beta) \longrightarrow p$.

A complex translation in an open set $U \subset N$ is obtained by taking a section of $O_c(N)$ over U, that is, an analityc map $T_c: U \longrightarrow O_c(N)$ such that $\pi \circ T_c = Id_U$. Then, given $p \in U$ and $X \in T_p(N)$, we can define the translation $X^\sim$ of X on U by setting:

$$X^\sim(q) = \langle X, X_1(p)\rangle_c X_1(q) + \ldots + \langle X, X_n(p)\rangle_c X_n(q) \qquad q \in U$$

where $T_c(q)$ is the orthonormal basis $\{X_1(q), \ldots, X_n(q)\}$ of $T_q(N)$. Similarly, we can define the translation $H^\sim$ of any complex subspace $H \subset T_p(N)$.

Let us choose $p_o \in N$ and assume that a translation is defined in a neighbourhood U of p_o. Denote by CP^{n-1} the complex projective space of complex lines of $T_{p_o}(N)$ and let M be a complex hypersurface of N contained in U. Then, the *Gauss map* $\gamma_c: M \longrightarrow CP^{n-1}$ of M is defined by:

$$\gamma_c(p) = H \Longleftrightarrow H^\sim(p) = (T_p(M))^\perp \qquad (a)$$

where $H^\sim$ is the translation of $H \subset T_{p_o}(M)$ on U.

Let $f: N \longrightarrow C$ be an analytic map with an isolated singularity at $p_o \in N$ such that $f(p_o) = o$. Given $H \in CP^{n-1}$, it determines a polar curve Γ_H by the condition:

$$p \in \Gamma_H \Longleftrightarrow T_p(f^{-1}(t)) = (H^\sim(p))^\perp \qquad p \in U$$

where $t \in C$ is such that $f(p) = t$.

We have the following result:

Theorem 1.1 Let N be a n-dimensional complex manifold with a hermitian metric. Let $f: N \longrightarrow C$ be an analytic map with an isolated singularity at p_o with $f(p_o) = o$. Then, the following formula holds:

$$\lim_{\epsilon \to o} \lim_{t \to o} \int_{M_t \cap B_\epsilon} \gamma_c^*(\lambda) = c_{n-1}(\mu^n + \mu^{n-1}) \qquad (***)$$

where $\gamma_c: M_t \longrightarrow CP^{n-1}$ is the Gauss map of M_t and λ the usual volume form of CP^{n-1}.

Proof.

We will prove that the limit above equals to $c_{n-1} I(\Gamma_H, M_o)_{p_o}$ = intersection index between Γ_H and M_o at p_o. Theorem 1.1 follows then from a result of Teissier (see Theorem 2 of [L]).

As in the Lemma of [L], one has:

$$\int_{M_t \cap B_\epsilon} \gamma_o^*(\lambda) = \int_{CP^{n-1}} \tau(M_t \cap B_\epsilon, H)\lambda(H^\perp)$$

where

$$\tau(M_t \cap B_\epsilon, H) = \sum_{p \in B_\epsilon} I(M_t, \Gamma_H)_p$$

so that in order to compute the limite for $t \to o$ and $\epsilon \to o$ as in [L] we have just to assure that the function $t \to \tau(M_t \cap B_\epsilon, H)$ is bounded.

Clearly, $\tau(M_t \cap B_\epsilon, H)$ is finite for $t \neq o$ and, since t goes to o, we have just to see that $\lim_{t \to o} \tau(M_t \cap B_\epsilon, H)$ is finite. To do this, let us consider the Nash Transformation $N_f \subset B_\epsilon \times CP^{n-1}$ of f restrict to B_ϵ. Let $\pi: N_f \longrightarrow B_\epsilon$ and $\gamma: N_f \to CP^{n-1}$ be the projections $(x,H) \to x$ and $(x,H) \to H^\perp$, respectively. Therefore, it is easy to see that

$$\tau(M_t \cap B_\epsilon, H) = \mathrm{card}(\pi^{-1}(M_t) \cap \gamma^{-1}(H^\perp)) \qquad t \neq o$$

so that

$$\lim_{t \to o} \tau(M_t \cap B_\epsilon, H) = \mathrm{card}(\pi^{-1}(M_o) \cap \gamma^{-1}(H^\perp)).$$

But

$$\pi(\pi^{-1}(M_o) \cap \gamma^{-1}(H^\perp)) = \{p \in M_o \cap B_\epsilon \mid p \neq o \text{ and } T_p(M_o) = H\} \cup \{o\}$$

is analytic and compact and hence finite.

It follows that $\pi^{-1}(M_o) \cap \gamma^{-1}(H^\perp)$ is finite since

$$\pi: \pi^{-1}(M_o) \cap \gamma^{-1}(H^\perp) \longrightarrow \pi(\pi^{-1}(M_o) \cap \gamma^{-1}(H^\perp))$$

is bijective. □

We will prove that the integrals in (**) and (***) coincide for $n = 3$. For, we have to relate the differentiall form $\gamma_c{}^*(\sigma)$ with geometric invariants of M.

2. *The real Gauss map associated to a translation.*

In this section, N will be a n-dimensional (over R) Riemannian manifold with a Riemannian metric $\langle\ ,\ \rangle$. Let U be an open set of N with a translation determined by a section of $O_r(N)$ over U, where $O_r(N)$ is the real orthonormal frame bundle of N.

Let M be a m-dimensional Riemannian manifold isometrically embedded in N. Let us assume that $M \subset U$. Let $p \in M$ and let $\eta \in T_p(N)$ be an unitary normal vector to $T_p(M)$.

We recall that the 2^{nd} fundamental form at p defined by η is given by :

$$A_p(\eta)(X,Y) = \langle \nabla_X \bar{Y}, \eta \rangle \qquad X,Y \in T_p(M)$$

where $\bar{Y}$ is a vector field in a neighbourhhod of p tangent to M such that $\bar{Y}(p) = Y$, and ∇ is the Riemannian connetion of N determined by $\langle\ ,\ \rangle$. Let $K_p(\eta) = \det A_p(\eta)$. Therefore, if the normal vector bundle N(M) of M is orientable,

$$K_p = \frac{1}{c_{k-1}} \int_{SN_p(M)} K_p(\eta)\, d\theta(\eta)$$

is the Lipschitz-Killing curvature of the embedding $M \subset N$, where SN(M) is the bundle of spheres correspondent to N(M), θ the volume form of $SN_p(M)$ and c_{k-1} the volume of S^{k-1}, $k = n - m$.

We introduce now the 2^{nd} fundamental form $\hat{A}$ associated to the given translation by setting:

$$\hat{A}_p(\eta)(X,Y) = \langle \nabla_X Y^{\wedge}, \eta \rangle \qquad X, Y \in T_p(M)$$

where $Y^{\sim}$ is the translation of Y on U.

Set

$$K^{\sim}_p(\eta) = \det(A_p(\eta) - \hat{A}_p(\eta))$$

and

$$K^{\sim}_p = \int_{SN_p(M)} K^{\sim}_p(\eta)\theta(\eta).$$

We will call $K^{\sim}$ the translation curvature of M.

The map:

$$\gamma_r:SN(M) \longrightarrow S^{n-1},\ \gamma_r((p,\eta)) = (\eta^{\sim})(p_o) \qquad \text{(b)}$$

is the "(real) Gauss map" of M associated to the given translation.

The result that follows relates the Gauss map associated to a translation and the translation curvature. It generalizes a well known theorem in Euclidean spaces relating the usual Gauss map in $\mathbb{R}^n$ and the Lipschitz-Killing curvature.

2.1 Theorem. Let M and N be orientable and m odd. Let σ be the volume form of S^{n-1} and ω the one of M. Then, $K^{\sim}_p\omega(p)$, $p \in M$, is the integral along the fiber of $SN(M) \longrightarrow M$ of $\gamma_r{}^*(\sigma)$ divided by c_{k-1}.

In [S] Sebastiani proves this result for the case that N is a Lie group with a left invariant metric and the translation is the usual left translation defined with the group operation (see the Theorem of §3 of [S] and its Corollary). Since its proof is the same as in our case, we omit the proof of Theorem 2.1.

Now, let us come back to the complex case. Let N be a complex n-dimensional (over C) manifold with an hermitian metric. Assume that a (complex) translation is given in an open set U of N, this translation being determined by a section $T_c:U \longrightarrow O_c(N)$.

If $\langle\ ,\ \rangle_c$ denotes the hermitian metric of N, $\langle\ ,\ \rangle_r \equiv \mathrm{re}(\langle\ ,\ \rangle_c)$

defines a Riemannian metric on N, and T_c induces a section T_r of $O_r(N)$ over U by setting $T_r(p) = \{X_1(p), iX_1(p), \ldots, X_n(p), iX_n(p)\}$, where $T_c(p) = \{X_1(p), \ldots, X_n(p)\}$, $p \in U$.

We can combine Theorems 1.1 and 2.1 to obtain the following corollary:

2.2 Corollary. With the same notations and hipothesis above, the following formula holds:

$$\lim_{\epsilon \to 0} \lim_{t \to 0} \int_{M_t \cap B_\epsilon} K^{\sim}_p \omega(p) = c_{n-1}(\mu^n + \mu^{n-1})$$

where $K^{\sim}$ is the translation curvature of M_t.

Proof. Let $\gamma_r : SN(M_t) \to S^{2n-1}$ and $\gamma_c : M_t \to CP^{n-1}$ be the real Gauss map and the complex Gauss map of M_t associated to the given translation, defined by (b) and (a), respectively. It is not difficult to prove, therefore, that the following diagramme is commutative:

$$\begin{array}{ccc} SN(M_t) & \xrightarrow{\gamma_r} & S^{2n-1} \\ P\downarrow & & \downarrow \pi \\ M_t & \xrightarrow{\gamma_c} & CP^{n-1} \end{array}$$

where π is the projection of the Hopf fiber. Hence, we obtain:

$$\int_{M_t \cap B_\epsilon} K^{\sim}_p \omega(p) = \frac{1}{c_2} \int_{SN(M_t \cap B_\epsilon)} \gamma_r^*(\sigma) = \frac{1}{c_2} \int_{SN(M_t \cap B_\epsilon)} (\gamma_c \circ P)^*(\lambda) = \int_{M_t \cap B_\epsilon} \gamma_c^*(\lambda)$$

which proves the corollary. □

3. *Proof of Theorem 1.*

heorem 1 follows easily from Corollary 2.2. In fact: since the second member of the formula is distinct from zero, we have from Corollary 2.2 that $\lim_{p\to p_o} K^\sim_p = \infty$, where $K^\sim_p$ is the translation curvature of M_t at $p \in M_t$. Furthermore, since

$$\lim_{p\to p_o} \inf \{\det \mathring{A}_p(\eta) \mid \|\eta\| = 1\} = \inf \{\det \mathring{A}_{p_o}(\eta) \mid \|\eta\| = 1\} < \infty$$

it follows from the definition of $K^\sim$ and that $n = 3$ that $\lim_{p\to p_o}(K^\sim_p - K_p) = o$, and this proves the Theorem.□

4. Curvature of complex hypersurfaces.

In [L] Langevin reobtains, by using formula (*), a Theorem due to Linda Ness about the curvature of algebraic curves of CP^2 converging to algebraic curve with an isolated singularity (see Theorem §III of [L]). Using formula of Corollary 2.2 above, we prove here the following generalization, which also gives a simpler proof of Theorem §III of [L]:

4.1 Theorem. Let N be a complex n-dimensional manifold with an hermitian metric. Let M be a complex hypersurface of N with an isolated singularity at p. Let M_t be a family of complex hypersurfaces of N converging to M when t goes to infinity. Then

$$\lim_{t\to\infty} \inf_{M_t} K_I = -\infty$$

where K_I denotes the intrinsic sectional curvatures of M_t.

Proof. Let $p_t \in M_t$ be such that $\lim_{t\to\infty} p_t = p$. Then, it follows from formula of Corollary 2.2 and the definition of $K^\sim$ that

$$\lim_{t\to\infty} \lambda_{p_t} = \infty$$

where $\lambda_t = \max\{ \lambda \mid \lambda$ is an eigenvalue of $A_{p_t}(\eta)$ for some $\eta \in T_{p_t}(M_t)^\perp$, $\|\eta\| = 1\}$. If X_t is an eigenvector associated to λ_t, then $-\lambda_t$ is an eigenvalue with eigenvector iX_t.

Denote by $K(P_t)$ and $\bar{K}(P_t)$ the sectional curvatures of M_t and N, respectively, at p_t, determined by the plane P_t generated by X_t and iX_t. Let $P \subset T_p(N)$ be such that $\lim_{t\to\infty} P_t = P$. Then, from the Gauss Equation of an isometric immersion, one has:

$$K(P_t) = \bar{K}(P_t) - \lambda_t^2$$

hence

$$\lim_{t\to\infty} K(P_t) = \bar{K}(P) - \lim_{t\to\infty} \lambda_t^2 = -\infty. \square$$

REFERENCES

[L] Langevin, R., *Courbure et singularités complexes*, Comment. Math. Helvetici 54(1979) 6-16

[S] Sebastiani, M., *La deuxième forme fondamentale dans les groupes de Lie*, pre-print

Jaime B. Ripoll
Universidade Federal do R. G. do Sul
Instituto de Matematica
Av. Bento Goncalves 9500
91500 - Porto Alegre - RS
BRAZIL

Série A: Trabalho de Pesquisa.

1. Marcos Sebastiani - <u>Transformation des Singularités</u> - MAR/89
2. Jaime B. Ripoll - <u>On a Theorem of R. Langevin About Curvature and Complex Singularities</u> - MAR/89