A CHARACTERIZATION OF HELICOIDS

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- Trabalho de Pesquisa-

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Some results proving the uniqueness of the helicoid have been recently obtained. For exemple, W. Meeks and H. Rosenberg prove that the plane and the helicoid are the only properly embedded simply connected minimal surfaces in R^3 with infinite symmetry group [1]. Also, a result due to E. Toubiana proves that an embedded minimal two punctured sphere in R^3/T, T a translation, is a helicoid, provided the total curvature is finite [3].

We prove in this paper the uniqueness of the helicoids as a solution of Plateau's Problem with respect to two different boundary conditions. To state the results, we introduce some notations.

Let $\gamma_1, \gamma_2 : R \longrightarrow R^3$ be the helix $\gamma_1(t) = (a_1 \sin t, a_1 \cos t, bt)$, $\gamma_2(t) = (a_2 \sin t, a_2 \cos t, bt)$, $a_1 > a_2 \geq 0$. Set $\epsilon = a_1 - a_2$, $\Gamma_\epsilon = \gamma_1(R) \cup \gamma_2(R)$ and $\bar{\Gamma}_\epsilon = \gamma_1([p,q]) \cup \gamma_2([p,q]) \cup L_1 \cup L_2$, where $p < q$, L_1 is the line segment connecting $\gamma_1(p)$ to $\gamma_2(p)$ and L_2 is the line segment connecting $\gamma_1(q)$ to $\gamma_2(q)$.

We will say that a surface is cilindrically bounded if it is contained in some cylinder of R^3.

We prove the following results:

THEOREM 1. Let S be a complete connected properly immersed minimal surface in R^3, with boundary Γ_ϵ, invariant by an infinite group of isometries of R^3 acting freely in R^3. Suppose that S is cilindrically bounded. Then there exists $\epsilon_o > 0$ such that if $\epsilon < \epsilon_o$ then S is an helicoid.

THEOREM 2. For the same ϵ_o of Theorem 1, if $\epsilon < \epsilon_o$ and if S is a compact connected minimal surface with boundary $\bar{\Gamma}_\epsilon$ then S is an helicoid.

Theorem 1 is false if we take helix γ_1 and γ_2 with $a_1a_2<o$. Counter-examples are given by the helicoidal minimal surfaces, as commented after the proof of Theorem 1.

In order to prove these results, we use a one-parameter family of minimal helicoidal surfaces together with the Tangency Principle (see Lemma 1 of [2]).

PRELIMINARIES. By a helicoidal surface of R^3 we mean a surface which is invariant by a helicoidal one-parameter subgroup of isometries of R^3. Recall that, up to conjugation, a helicoidal subgroup $\{\phi_t\}$ of isometries of R^3 acts in R^3 by $\phi_t(x,y,z) = (x\cos t+y\sin t, -x\sin t+y\cos t, z+bt)$. The real number b is called the pitch of $\{\phi_t\}$. Therefore, it is easy to see that, up to a congruence, any complete helicoidal surface is generated, under the action of $\{\phi_t\}$, by a curve in the plane z = o. Let γ be any such a curve, and let S be the helicoidal surface generated by γ. Suppose that γ is given in polar coordinates by $r = r(t)$ and $\theta = \theta(t)$, t being its arc length. Then, straightforward computations show that the following system of ordinary differential equations must be satisfied in order that γ generates a helicoidal minimal surface with pitch b:

$$(*)\quad \begin{cases} \dot{r}^2 + r^2\dot{\theta}^2 = 1 \\ (1 + b^2r^2)[r(\dot{\theta}\ddot{r} - \dot{r}\ddot{\theta}) - r^2\dot{\theta}^3 - 2\dot{r}^2\dot{\theta}] + b^2r^2\dot{\theta} = 0. \end{cases}$$

REMARKS. It follows from theorem of existence and uniqueness of the solutions of a second order ordinary differential equation with respect to the initial conditions that, given (r_o, θ_o) and (r_o', θ_o') such that $r_o'^2 + r_o\theta_o'^2 = 1$, there exists one and only one curve $r = r(t)$ and $\theta = \theta(t)$ parametrized by arc length satisfying (*) such that $r(o) = r_o$, $\theta(o) = \theta_o$ and $\dot{r}(o) = r_o'$, $\dot{\theta}(o) = \theta_o'$.

The lines θ = constant, when parametrized by arc length, satisfy (*). The surfaces generated by such lines are the well known helicoids. It follows from the above remark that given any curve $r = r(t)$ $\theta = \theta(t)$ parametrized by arc length satisfying (*), we have $\dot{\theta}(t) \neq o$ everywhere unless θ = constant and the curve generates a helicoid.

Proof of the Theorem 1. Observe that $\gamma_1(R)$ and $\gamma_2(R)$ are the orbits of the points (o, a_1, o) and (o, a_2, o) in the plane $z = o$ under the action of $\{\theta_t\}$.

Let $\{c_s\}$, $s \leq o$, be the one parameter family of curves parametrized by arc length satisfying (*), determined in cartesian coordinates by the initial conditions:

$$(**) \qquad \begin{aligned} c_s(o) &= (o, s, o) \\ \dot{c}_s(o) &= (1, o, o). \end{aligned}$$

Observe that c_o is the x-axis.

Let us denote by $r = r_s(t)$ and $\theta = \theta_s(t)$ the polar coordinates of the curve c_s. From (**), we see that $\dot{r}_s(o) = o$. By another hand, from (*), for any solution $r = r(t)$ of (*) we obtain:

$$\ddot{r} = \frac{1}{r^3(1+b^2r^2)}$$

at any critical point of $r = r(t)$. This shows that $r = r_s(t)$ is an increasing function in the interval $[o, +\infty)$, decreasing in $(-\infty, o]$ and that $r_s(o) = -s$ is a global minimal value for it.

Given $s < 0$, define t_o as the non negative minimal value of the parameter t such that $\gamma_s(t_0)$ or $\gamma_s(-t_0)$ intersects the y-axis, and set $y_s = \min\{d(\gamma_s(t_0),0), d(\gamma_s(-t_0),0)\}$. We put $y_s = \infty$ if t_0 does not exist. Set $\epsilon_0 = \inf\{y_s \mid s<0\}$, $0 \leq \epsilon_0 \leq \infty$.

We have $\epsilon_0 > 0$. In fact: since $r_s(t) \geq -s$ for any $s < 0$, we have that $\lim_{s\to-\infty} y_s = \infty$. By another hand, when s approaches to 0, c_s approaches the x-axis, and this implies also that $\lim_{s\to 0} y_s = \infty$.

Let $\{S_s\}$ be the one parameter family of helicoidal minimal surfaces generated by the curves c_s.

Let S be a complete connected properly immersed minimal surface with boundary $\Gamma_\epsilon = \gamma_1(R) \cup \gamma_2(R)$, which $\epsilon < \epsilon_0$, contained in a cylinder C, and let us assume also that S is invariant by an infinite group G of isometries of R^3 acting freely in it. Since G leaves Γ_ϵ invariant, C/G is compact. Therefore, since S is properly immersed, S/G is compact that is, $S = G(S')$, S' compact.

Since $r_s(t) \geq -s$ for any t and since $\{\phi_t\}$ leaves invariant any cylinder centered on the z-axis, it follows that $S_s \subset R^3 \setminus C_s$ for some s, where C_s is a cylinder centered on the z-axis with radius $-s$. Therefore, there exists $s_0 \leq 0$ such that $S_s \cap S = \emptyset$ if $s < s_0$.

Let $s_1 = \sup\{ s < 0 \mid S_s \cap S = \emptyset\}$. Suppose that $s_1 > 0$. Then, from the choice of ϵ_0 and since $\epsilon < \epsilon_0$, $\partial S_{s_1} \cap \partial S = \emptyset$. Since $S = G(S')$, S' compact, S and S_{s_1} must be tangent in an interior point, and this implies that $S = S_{s_1}$, which is an absurd.

Therefore $s_1 = 0$, and this implies that S does not intersect the helicoid generated by the x-axis in the plane $z = 0$. We can then rotate this helicoid around the z-axis until it touches S. Then, by the Tangency Principle, S will be a piece of a helicoid. □

The proof above can also be used to prove Theorem 2.

We observe that the surfaces S_s are simply connected, properly immersed, minimal and with infinite symmetry group. Therefore, from the result of Meeks and Rosenberg mentioned in the introduction, they can not be embedded, showing the necessity of the embeddednes condition in their result.

The fact that the surfaces S_s are not embedded implies that $s_0 < \infty$. It follows also from this that Theorem 1 is false if $a_1a_2 < 0$.

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