Universidade Federal do Rio Grande do Sul

Escola de Administração

Programa de Pós-Graduação em Administração

Leonardo Riegel Sant'Anna

# Essays on Index Tracking and Portfolio Optimization

Brazil

September 2017

Sant'Anna, Leonardo Riegel
Essays on Index Tracking and Portfolio Optimization/ Leonardo Riegel Sant'Anna. – Brazil, September 2017-
124 p. : il. (algumas color.) ; 30 cm.

Supervisor: Tiago Pascoal Filomena, PhD

Tese (Doutorado) – Universidade Federal do Rio Grande do Sul
Escola de Administração
Programa de Pós-Graduação em Administração, September 2017.

1. Index tracking. 2. Otimização de carteiras de investimento. 3. Algoritmo genético. 4. Gráficos de controle. 5. Cointegração e Correlação. I. Tiago Pascoal Filomena. II. Essays on Index Tracking and Portfolio Optimization.

Leonardo Riegel Sant'Anna

# Essays on Index Tracking and Portfolio Optimization

Tese apresentada como requisito parcial para a obtenção do título de Doutor em Administração pelo Programa de Pós-graduação em Administração da Universidade Federal do Rio Grande do Sul.

**Tiago Pascoal Filomena, PhD**
Thesis Supervisor

**Cristiano Machado Costa, PhD**
Committee Member

**Denis Borenstein, PhD**
Committee Member

**Flávio Sanson Fogliatto, PhD**
Committee Member

**João Frois Caldeira, PhD**
Committee Member

Brazil

September 2017

# Acknowledgements

Gostaria de iniciar estas linhas agradecendo obviamente aos meus pais, Suzana e Rubinho, que são os principais responsáveis por eu saber dar valor ao esforço, ao trabalho com dedicação e vontade e, principalmente, com honestidade e humildade. Além disso, obrigado também aos meus irmãos Rodrigo e Felipe e, mais recentemente, ao meu sobrinho Matheus, por fazer parte desta família extremamente especial que me torna uma pessoa muito abençoada.

Muito obrigado também a todos os demais familiares, Avós e Avôs paternos e maternos, Tias, Tios e Primos. Trata-se de uma família enorme em quantidade e em qualidade, e sou muito grato por tê-los perto de mim.

Passando ao ambiente acadêmico, agradeço à Universidade Federal do Rio Grande do Sul pelo ensino de ótima qualidade, e agradeço imensamente ao meu Orientador de Mestrado e Doutorado, Professor Tiago Pascoal Filomena. Sua orientação e amizade foram extremamente fundamentais desde o primeiro dia deste longo processo de Pós-Graduação, e os resultados que obtive absolutamente não seriam os mesmos sem sua liderança e aconselhamentos.

Agradeço aos professores do Programa de Pós-Graduação da Escola de Administração, em especial aos Professores das áreas de Modelagem Quantitativa e Pesquisa Operacional, onde conclui o Mestrado, e aos Professores da área de Finanças, onde estou em vias de finalizar o Doutorado. Além do Professor Tiago, obrigado também aos Professores Denis Borenstein e João Frois Caldeira, pela fundamental participação e orientação nos artigos em que trabalhei até o momento. Sou muito grato por ter sempre recebido atenção de todos, sem exceção.

Por fim, agradeço aos colegas e amigos que fiz durante o período de Pós-Graduação, que certamente estarão em minha memória para o resto da vida. Obrigado aos amigos Daniel Vancin e Pablo C. Guedes pelos estudos e artigos que fizemos (e faremos) juntos, obrigado ao Danilo Navarro, Filipe Gabriel Rosa, Henrique Ramos e Mariana Davi, pelos estudos durante as aulas de Finanças e pelos ótimos momentos fora das salas de aula, e aos amigos Alan D. de Oliveira, Camilo Bornia-Poulsen, Carla Marcolin, Guilherme Bucco, Henrique C. Martins, Maura Ferreira e Tito Grillo, pela excelente companhia durante todo o período de Pós-Graduação.

Muito obrigado!

*"Who is John Galt?"*
Ayn Rand, "Atlas Shrugged".

# Abstract

This dissertation focus on portfolio optimization models designed for index tracking investment strategy. The final content is composed by three articles. The first article is entitled "Index Tracking with Controlled Number of Assets Using a Hybrid Heuristic Combining Genetic Algorithm and Non-linear Programming", which has been accepted for publication in the Annals of Operations Research. The second article is "Index Tracking and Enhanced Indexing using Cointegration and Correlation with Endogenous Portfolio Selection", and it has been accepted for publication in The Quarterly Review of Economics and Finance. Lastly, the third article is "Investigating the Use of Statistical Process Control Charts for Index Tracking Portfolios", and it has already been submitted and is currently under review.

In the first paper, we discuss the index tracking strategy using mathematical programming. First, we use a non-linear programming formulation for the index tracking problem, considering a limited number of assets. Since the problem is difficult to be solved in reasonable time by commercial mathematical packages, we apply a hybrid solution approach, combining mathematical programming and genetic algorithm. We show the efficiency of the proposed approach comparing the results with optimal solutions, with previous developed methods, and from real-world market indexes. The computational experiments focus on Ibovespa (the most important Brazilian market index), but we also present results for consolidated markets such as S&P 100 (USA), FTSE 100 (UK) and DAX (Germany). The proposed framework shows its ability to obtain very good results (gaps from the optimal solution smaller than 5% in 8 minutes of CPU time) even for a highly volatile index from a developing country.

In the second paper, the attention is turned to the analysis of two alternative methods to solve the index tracking optimization problem. This article investigates the out-of-sample performance of cointegration and correlation methods for index tracking (IT) and enhanced indexing (EIT) strategies applied to Brazilian and U.S. market data. Our goal is to compare both methods as we strongly explore cointegration in relation to previous studies: we make the portfolio selection endogenous to the problem for this approach. The tests are performed using data from 2004 to 2014 with samples of 57 assets for Brazilian data and 96 assets for U.S. data; portfolios are built using combinations of at most 10 of these assets. Despite the extensive tests carried out, the overall result shows similar performance for both methods. For IT in the Brazilian market, there was a trade-off between better tracking error and higher turnover for cointegration (with the opposite for correlation), this pattern was not clear in the U.S. market. The outcome for the EIT also does not clearly favor cointegration or correlation.

Finally, the third article is dedicated to the discussion regarding the use of statistical process control charts to regulate index tracking portfolios. In this article, our goal is to introduce a statistical process control charts approach (SPC) to monitor the rebalancing process of index tracking (IT) portfolios. SPC methods derive from statistics and engineering as tools to control production process. We use exponentially weighted moving average (EWMA) control charts to monitor IT portfolios based on two combined charts: portfolios' tracking error performance and portfolios' volatility. As a result, we endogenously control the rebalancing process of the portfolios based on their performance and on their risk conditions over time. Computational tests are performed to evaluate the developed approach in comparison with the traditional fixed period strategy, using data from Brazilian and U.S. market from 2005 to 2014. Cointegration and correlation methods are applied to form the portfolios. The results show that SPC approach can be a viable alternative to portfolio rebalancing.

**Keywords**: index tracking, portfolio optimization, genetic algorithm, correlation, cointegration, control charts.

# Resumo

Esta tese tem foco no tema de otimização de carteiras de investimento modeladas para estratégia de investimento de index tracking. O conteúdo final é composto por três artigos. O primeiro artigo é intitulado "Index Tracking with Controlled Number of Assets Using a Hybrid Heuristic Combining Genetic Algorithm and Non-linear Programming", e foi aceito para publicação na revista Annals of Operations Research. O segundo artigo é "Index Tracking and Enhanced Indexing using Cointegration and Correlation with Endogenous Portfolio Selection", e foi aceito para publicação na revista Quarterly Review of Economics and Finance. Por fim, o terceiro artivo é "Investigating the Use of Statistical Process Control Charts for Index Tracking Portfolios", o qual já foi submetido e está atualmente em processo de revisão.

No primeiro artigo, discutimos a estratégia de investimento de index tracking usando programação matemática. Primeiro, usamos uma formulação de programação não linear para o problema de index tracking, considerando um número limitado de ações. Devido à dificuldade de solução do problema em um intervalo de tempo razoável por pacotes matemáticos comerciais, aplicamos uma abordagem de solução híbrida, combinando programação matemática e algoritmo genético. Com a aplicação de testes, demonstramos a eficiência da abordagem proposta comparando os resultados com soluções ótimas, com métodos previamente desenvolvidos, e com dados reais de índices de mercado. Os experimentos computacionais focam no Ibovespa (o mais popular índice do mercado brasileiro), e também apresentamos resultados para mercados consolidados tais quais S&P 100 (Estados Unidos), FTSE 100 (Reino Unido) and DAX (Alemanha). A estrutura proposta apresenta sua abilidade para obter ótimos resultados (resultados com gap em relação às soluções ótimas menores que 5% em 8 minutos de tempo de processamento) até mesmo para índices de mercado com alta volatilidade em um mercado em desenvolvimento.

No segundo artigo, a atenção é voltada para a análise de dois métodos alternativos entre si para solução do problema de otimização de index tracking. Esse artigo investiga o desempenho "fora da amostra" dos métodos de correlação e cointegração para as estratégias de index tracking (IT) e enhanced indexing (EIT) aplicadas aos dados de mercado Brasileiro e Norte-americano. Nosso objetivo é comparar ambos os métodos na medida em que exploramos fortemente a cointegração em relação a estudos prévios: nós transformamos a seleção do portfólio endógena ao problema de otimização nessa abordagem. Os testes foram executados utilizando dados de 2004 a 2014 com amostras de 57 ações para dados brasileiros, e 96 ações para dados dos Estados Unidos; carteiras foram construídas usando combinações de no máximo 10 ações. Apesar da realização de testes extensivos, os resultados gerais demonstraram desempenho similar para ambos os métodos. Para IT no mercado brasileiro, foi verificado um trade-off entre melhor erro de tracking e maior turnover com cointegração (com resultados opostos para correlação), sendo que este mesmo padrão não foi encontrado para dados norte-americanos. Os resultados para EIT também não apresentação claro favorecimento para cointegração ou correlação.

Por fim, o terceiro artigo é dedicado à discussão a respeito do uso de processo estatístico de gráficos de controle para regulação de carteiras de index tracking. Nesse artigo, nosso objetivo é introduzir uma abordagem baseada em gráficos de controle (SPC) para monitorar o processo de rebalanceamento de carteiras de index tracking. O método de SPC é derivado da Estatística e da Engenharia, como ferramenta para controle de processos de produção. Para cumprir os objetivos, aplicamos gráficos de controle EWMA (do inglês, *exponentially weighted moving average*) para monitorar carteiras de IT baseadas no uso combinado de dois gráficos de controle: desempenho de carteiras em termos de erro de tracking e em termos de volatilidade. Assim, visamos tornar endógeno o controle do processo de rebalanceamento das carteiras baseado em seu desempenho e em suas condições de risco ao longo do tempo. Testes computacionais foram realizados para avaliar a abordagem desenvolvida em comparação com a estratégia tradicional de rebalanceamento (que consiste no uso de janelas fixas de tempo para atualização das carteiras), usando dados dos mercados brasileiro e norte-americano de 2005 a 2014. Os métodos de cointegração e correlação foram aplicados para otimização das carteiras. Os resultados demonstraram que a abordagem com SPC pode ser uma alternativa viável para o processo de rebalanceamento de carteiras.

**Palavras-chave**: index tracking, otimização de carteiras de investimento, algoritmo genético, correlação, cointegração, gráficos de controle.

# Contents

# Introduction

The goal of this dissertation is to discuss the index tracking investment strategy through the presentation of three articles. Index tracking (IT) is a strategy that seeks to mimic the performance of a market index. For instance, it is widely used to construct ETF assets (exchange-traded-funds). DeFusco, Ivanov e Karels (2011) have discussed the relevance of ETF assets in the American market, meanwhile Borges, Jr. e Yoshinaga (2012) have showed the recent boom in the use of ETFs in Brazil.

Since the goal of IT portfolios is to replicate an index performance, IT optimization models usually have the objective to minimize the variance of the tracking error (TE). The TE consists in the difference between portfolio and index returns over time. Also, IT models frequently deal with binary constraints in order to control the number of assets in the portfolio. Therefore, portfolios can have reduced size in order to diminish transaction and management costs. Coleman, Li e Henniger (2006) have already shown that the TE minimization problem is NP-hard when a binary constraint on the total number of assets is used. As a result, a heuristic method should be used to solve the optimization model in reasonable time so that it fits commercial practices, specially when large databases are used.

In the first part of this dissertation, we present an study that focuses on this relevant issue for IT models: to present an heuristic approach that solves the TE minimization problem with a binary constraint to form reduced portfolios. The optimization model is a non-linear programming formulation for the IT problem, and the model is solved through the use of a hybrid solution approach that combines mathematical programming and genetic algorithm. The hybrid approach combines the use of a genetic algorithm and the optimization solver IBM® ILOG CPLEX. Also, an algorithm is presented to produce initial responses as a lead to start the algorithm and improve the results.

The solution approach is widely stressed in order to verify the quality of the solutions. The tests involve smaller databases, such as the data instance relative to the Germany market and the DAX index (30 assets), and the Brazilian market and the Ibovespa index (67 assets), as well as larger data instances for the British market and the FTSE 100 index (96 assets) and the American market and the S&P 100 index (97 assets). Also, two original tests are performed "crossing" markets: the attempt to form portfolios using Brazilian assets to track two foreign index (FTSE 100 and S&P 100).

Overall, the results show the quality of the obtained results in terms of gap values as well as in terms of tracking error performance over time. The solution approach was able to generate 10-asset portfolios with gap average below 5% and CPU time around 8 minutes. Also, the TE quality of the portfolios can be noticed based on the lower values obtained for all four markets analyzed and also for the tests crossing foreign markets.

In the second part, we present an article that seeks to explore two methods commonly used to solve the IT problem. The literature on index tracking has a variety of approaches to solve

IT models, such as optimization models (KONNO; WIJAYANAYAKE, 2001), simulation and optimization (CONSIGLIO; ZENIOS, 2001), heuristics (BEASLEY; MEADE; CHANG, 2003; OH; KIM; MIN, 2005; MARINGER; OYEWUMI, 2007; JEURISSEN; BERG, 2008; GUASTAROBA; SPERANZA, 2012; SCOZZARI et al., 2013), cointegration (ALEXANDER; DIMITRIU, 2002; ALEXANDER; DIMITRIU, 2005a; DUNIS; HO, 2005; CALDEIRA; PORTUGAL, 2010) and quadratic programming combined with fundamental analysis for asset selection (JANSEN; DIJK, 2002; COLEMAN; LI; HENNIGER, 2006).

Among those methods, two of them have already been compared in previous studies (ALEXANDER; DIMITRIU, 2005a; GROBYS, 2010): cointegration and correlation. However, overall, such comparisons do not explore extensively specially the cointegration method. Meanwhile, some studies present more robust techniques to solve IT optimization with cointegration (DUNIS; HO, 2005; CALDEIRA; PORTUGAL, 2010). So, in our study, the goal is to perform a comparison between cointegration and correlation while we perform the cointegration tests based on the robust techniques presented in previous literature. The cointegrated portfolios are solved with an approach that makes the portfolio selection endogenous to the optimization process. Also, we follow suggestions from Alexander e Dimitriu (2005a) to use a larger database (up to 57 assets, from the Brazilian market) combined with more restrict portfolios (relative to the sample size; portfolios have at most 10 assets). Therefore, we expect to achieve more consistent results about the comparison.

Overall, the results are similar for both the methods; specifically, we were able to notice the tendency of cointegration to generate portfolios with better tracking error performance, along with larger turnover values. As a result, on the one hand, cointegration delivers better tracking error, but also implies larger transaction costs (represented by turnover values). On the other hand, correlation results in portfolios with larger TE over time, but also in reduced turnover (i.e. smaller transaction costs). Thus, we can highlight a trade-off between TE performance and costs. Also, as a secondary goal, we seek to present tests for enhanced indexing (EIT) strategy in the Brazilian stock market. Such strategy aims to adapt the IT concept in order to generate portfolios with excess returns consistently over time. The results demonstrate the capacity of EIT strategy to add additional value to the portfolio, relative to the market index.

Finally, the third part of this dissertation is an article in which we seek to present an alternative approach to control the rebalancing process for IT portfolios. When forming an index fund, the first choice would be to select the index to be tracked. Next, it would be to pick up the assets and their respective weights in the portfolio. Finally, it would be to define the portfolio rebalancing strategy. The second step has already been dealt by several IT optimization models using different approaches such as optimization (KONNO; WIJAYANAYAKE, 2001; MEZALI; BEASLEY, 2013), simulation and optimization (CONSIGLIO; ZENIOS, 2001), heuristics (BEASLEY; MEADE; CHANG, 2003; GUASTAROBA; SPERANZA, 2012; SCOZZARI et al., 2013), cointegration (ALEXANDER; DIMITRIU, 2005a), and quadratic programming (COLEMAN; LI; HENNIGER, 2006). The aim of these models is to minimize the tracking error (TE), i.e. the difference between portfolio and index returns. Notice that stock market indexes, for instance, usually have hundreds of assets and a full replication of the index would strongly increase transaction costs. Thus, in order to reduce the number of assets and decrease transaction and management costs, IT models usually control the maximum size of each portfolio. However, the third step (the portfolio rebalancing strategy) has received little attention from the literature. So, as an attempt to shed new light on the rebalancing strategy issue, this paper focus its attention on the introduction of a new approach to regulate IT portfolios over time, based on the portfolio performance in terms of tracking error as well as in terms of the portfolio and market volatilities.

Regardless of the diversity of the applied optimization models and techniques for IT portfolios, one similar characteristic stands out: the use of fixed time windows to portfolio rebalancing. For instance, Alexander e Dimitriu (2005a) used four specific intervals: 2-week, monthly, quarterly, and semi-annual rebalancing, while Canakgoz e Beasley (2008) and Krink, Mittnik e Paterlini (2009) considered monthly rebalancing. An alternative approach to the use of the traditional fixed rebalancing intervals is the application of control charts to monitor portfolio updates over time. The idea behind the use of control charts is the attempt to maintain portfolios inside preset limits; once the portfolio is out of the boundaries, the optimization model is reassessed and the portfolio is re-estimated. Such method should make the portfolio rebalancing strategy dynamic over time, favoring two issues as follows: (i) to permit the manager to quickly update the portfolio, if sudden market movements make the portfolio deviate from its objective, and (ii) to allow the portfolio to remain unchanged if its performance is in accordance to the expected, thus avoiding unnecessary rebalancing activities and diminishing transaction and management costs.

Statistical process control (SPC) was originated in statistics and engineering and has as main characteristic the application of statistical methods to control industrial processes. One of the tools derived from SPC is the use of control charts (WHEELER; CHAMBERS, 1992; MONTGOMERY, 1996) – graphs employed to regulate processes based on upper and lower boundaries. The idea supporting control charts consists in (i) to define a statistical value correspondent to the process to be regulated, and (ii) to set upper and lower control limits (UCL, LCL) as boundaries for the statistical value time series. In this article, our goal is to employ an approach based on EWMA control charts to regulate IT portfolios. Thereby, we expect to obtain portfolios controlled by control charts with performance similar to portfolios based on fixed rebalancing intervals. The use of fixed rebalancing time windows relies solely on the measure of tracking error (TE). In contrast, we design the control chart approach to update portfolios based on two measures: TE values, and the index volatility. Thus, the portfolio rebalancing process will become dynamic, as it will adapt to different market conditions over time. For instance, during periods in which market volatility is larger than usual, we assume IT portfolios could accept larger TE values, as long as the volatility of the portfolio is inside predefined control limits (which implies that the risk level of the portfolio is in line with the market risk).

Overall, our results present SPC portfolios with performance similar to portfolios with fixed windows when we observe cumulative returns, tracking error and volatility over time for each portfolio. On the one hand, SPC portfolios using correlation method performed well in both the Brazilian and the American markets. On the other hand, the SPC portfolios using cointegration behaved well only in the Brazilian market, whereas the cointegration approach presented larger tracking errors not only for SPC portfolios but also for portfolios with fixed windows. Thus, the outcomes showed that SPC portfolios (using either correlation or cointegration) could be a viable rebalancing strategy for IT portfolios especially in a more volatile environment such as the Brazilian market.

This dissertation is divided as follows. Part I presents the first article entitled “Index Tracking with Controlled Number of Assets Using a Hybrid Heuristic Combining Genetic Algorithm and Non-linear Programming”, that has been accepted for publication in the Annals of Operations Research. Part II presents the article “Index Tracking and Enhanced Indexing using Cointegration and Correlation with Endogenous Portfolio Selection”, that has been accepted for publication in The Quarterly Review of Economics and Finance. Finally, Part III presents the article “Investigating the Use of Statistical Process Control Charts for Index Tracking Portfolios”, that has already been submitted and is currently under review.

# Part I

# Index Tracking with Controlled Number of Assets Using a Hybrid Heuristic Combining Genetic Algorithm and Non-linear Programming

# Abstract

In this paper, we discuss the index tracking strategy using mathematical programming. First, we use a non-linear programming formulation for the index tracking problem, considering a limited number of assets. Since the problem is difficult to be solved in reasonable time by commercial mathematical packages, we apply a hybrid solution approach, combining mathematical programming and genetic algorithm. We show the efficiency of the proposed approach comparing the results with optimal solutions, with previous developed methods, and from real-world market indexes. The computational experiments focus on Ibovespa (the most important Brazilian market index), but we also present results for consolidated markets such as S&P 100 (USA), FTSE 100 (UK) and DAX (Germany). The proposed framework shows its ability to obtain very good results (gaps from the optimal solution smaller than 5% in 8 minutes of CPU time) even for a highly volatile index from a developing country.

**Keywords:** index tracking, portfolio optimization, genetic algorithm.

**Note:** this article has been accepted for publication in the Annals of Operations Research.

# 1 Introduction

Index tracking (index fund) is a passive investment strategy that seeks to replicate the performance of a market index. This strategy can be used, for example, to form an ETF (Exchange-Traded Fund – a security that represents an index fund, and which, in recent years, has become popular in Brazil) or to reproduce a market indicator such as inflation (rather than following a stock index). The decision of an investor in choosing a passive investment fund is based on the efficient markets hypothesis proposed by Fama (1970) which was also discussed, for example, by Frino e Gallagher (2001) and Fama e French (2010). When forming an index fund, the first choice would be to produce an exact replica of the index. However, the disadvantage of this strategy is that it leads to portfolios with large quantities of stocks, thereby generating a larger number of transactions and higher costs (BARRO; CANESTRELLI, 2009; CANAKGOZ; BEASLEY, 2008). Thus, to reduce transaction and management costs, index tracking funds commonly limit the amount of assets in the portfolio as discussed by Maringer e Oyewumi (2007), Murray e Shek (2012) and Scozzari et al. (2013) – just to cite a few.

In the contemporary literature, various methods were adopted in attempting to solve the index tracking (IT) problem, such as optimization (KONNO; WIJAYANAYAKE, 2001), simulation and optimization (CONSIGLIO; ZENIOS, 2001), heuristics (BEASLEY; MEADE; CHANG, 2003; OH; KIM; MIN, 2005; MARINGER; OYEWUMI, 2007; JEURISSEN; BERG, 2008; GUASTAROBA; SPERANZA, 2012; SCOZZARI et al., 2013), cointegration (ALEXANDER; DIMITRIU, 2005a; DUNIS; HO, 2005) and quadratic programming combined with fundamental analysis for asset selection (JANSEN; DIJK, 2002; COLEMAN; LI; HENNIGER, 2006). Although these models present interesting ideas, they were developed and tested oriented to mature markets, such as in the US, Japan, and in some countries of Europe, where the volatility is relatively lower compared to developing markets. This paper introduces an attempt to understand the performance of these index tracking methods in a developing country, specifically, the Brazilian stock market.

In general, index funds limit the number of assets in the portfolio, increasing the concentration risks. Adding higher concentration risks to markets that are not mature might demand new features of the index tracking model. For instance, in the year 2013, a company called OGX weighted more than 4% of the most important Brazilian stock index[1], the Ibovespa. This same company filed for Chapter 11 in the year 2014. This simple example shows the risks of concentration in a developing country. Furthermore, during periods of increased market volatility, it would be appropriate to conduct several tests to support decision-making in relation to rebalancing the portfolio, towards establishing whether it is better for the new portfolio to have fewer, or slightly more stocks based on tracking forecasts. Again, this demands a rapid analysis to update the index tracking portfolio. Thus, a method that can process information in a short period of time gives flexibility in a high volatile environment in which rebalancing might be more frequent.

[1] Newspaper: Valor Economico, August 2013: <http://www.valor.com.br/financas/3235752/ogx-e-acao-que-mais-ganha-peso-na-nova-carteira-do-ibovespa>

Moreover, faster computational methods have an increasing importance in the era of higher frequency trading (GENCAY et al., 2001; ALDRIDGE, 2009). There is no reason to doubt the use of index tracking for higher frequency trading in the near future. In summary, although there is a vast literature on the application of mathematical programming to the index tracking problem, the requirements and difficulties involved in developing country markets were neglected by the current methods and techniques.

We apply a mathematical model aiming at minimizing the mean squared difference between return of a portfolio and an index (tracking error minimization) with a limited number of assets in the portfolio. As the objective is to solve this problem in a short processing time and for markets with different levels of volatility, we developed a hybrid approach, combining mathematical programming and genetic algorithm (GA) methods. Since the choice of the initial generation directly affects the performance and the convergence process of the GA, fundamental aspects when dealing with highly volatile markets, we designed a simple, but effective initial solution generator for both speeding up and enhance the solution process of our hybrid method.

We demonstrate the effectiveness and efficiency of the developed approach by adopting the Brazilian Ibovespa index as central reference, with a sample of 67 assets for the period January/2009 to July/2012. Our central goal is to form portfolios containing up to 5 and 10 assets with a CPU time of less than 10 minutes and solutions with a maximum gap from the optimal solution of around 10% – in line with the results presented in Beasley, Meade e Chang (2003). Thus, this study is an extension of Sant'Anna, Filomena e Borenstein (2014), in which it was demonstrated the impossibility of forming portfolios containing 20 assets (for this exactly same data sample) with optimization times shorter than 1 hour and gaps smaller than 10%. We also apply the method to the S&P 100 (USA), FTSE 100 (UK) and DAX (Germany) to verify the method performance when applied to less volatile markets. Tests were performed with out-of-sample intervals of 20, 60, 120 and 240 business days (basically monthly, quarterly, semi-annual and annual rebalancing). Finally, to provide evidence of the flexibility of the method, we discuss an exercise in which the S&P 100 and FTSE 100 are tracked using just assets traded in the Brazilian market. Based on the results obtained (gaps from the optimal solution under 5% in 8 minutes of CPU time) even for highly volatile indexes, we can conclude that our developed method suits both markets with lower volatility (which is the case of the S&P 100, FTSE 100 and DAX) and with higher volatility (which happens specially in stock markets of emerging countries, such as the Brazilian), being a very competitive method to solve the index tracking problem.

The contribution of this paper is twofold. On the one hand, we contribute to the solution of the index tracking problem, offering a hybrid method that solves this problem with efficiency and efficacy, without requiring prior knowledge from the index being tracked. On the other hand, the empirical testing also presents innovations. The method was tested and applied in an uncommon market and tracking exercises. Overall, we developed a framework that is able to efficiently generate portfolios that closely follow the behavior of market indexes with different volatility patterns.

This paper is organized as follows. Section 2 presents a brief literature review on index tracking, emphasizing the contributions of our developed method to the current literature. In Section 3, we introduce the mathematical formulation of the developed model. Section 4 describes with details the developed solution approach. Section 5 presents the experiments carried out and the computational analysis of the hybrid solution approach. Finally, Section 6 concludes the paper and points out further research on the topic.

# 2 Literature Review

Index tracking optimization has attracted the attention of several researchers in the area of operations research and computational finance. The initial research works were mainly directed to formal mathematical formulations of the problem. Konno e Wijayanayake (2001) developed a non-linear programming model for the problem. Consiglio e Zenios (2001) were the first authors to develop a hybrid model, employing simulation and optimization, to follow a composite index of the international bond market. Gaivoronski, Krylov e Wijst (2005) discussed different approaches for index tracking with transaction costs constraints and control of the amount of assets in the portfolios. These authors demonstrated that the tracking error is strongly influenced by the amount of assets used and the length of the out-of-sample intervals. Alexander e Dimitriu (2005a) and Dunis e Ho (2005) applied a statistical cointegration index tracking method, arguing that it tends to form more stable portfolios over time, with quarterly rebalancing periods providing the best results. Jansen e Dijk (2002) and Coleman, Li e Henniger (2006) applied initially fundamental analysis methods to select the assets that will compose each portfolio (i.e. portfolio composition is exogenous to the optimization), then, a quadratic programming modeling is performed.

The importance of using heuristic approaches for portfolio optimization problems with larger samples and cardinality constraints can be noticed in a number of heuristics' studies. Derigs e Nickel (2004) used a heuristic method defined as 2-phase Simulated Annealing approach for the index tracking problem. Maringer e Oyewumi (2007), Krink, Mittnik e Paterlini (2009), and Scozzari et al. (2013) focused on a heuristic approach, the so-called Differential Evolution. These authors employed quadratic integer optimization models with constraint on the amount of assets in the portfolios. Also, Krink, Mittnik e Paterlini (2009) and Scozzari et al. (2013) centered their efforts on dealing with specific trading constraints for European markets (UCITS constraints). Gilli e Schumann (2012) described the use of several heuristics with emphasis on Threshold Accepting.

Angelelli, Mansini e Speranza (2012) and Guastaroba e Speranza (2012) combined linear integer programming and a heuristic framework, the so-called Kernel Search, for index tracking. These articles used databases with a total of eight indexes, some of them were employed by Beasley, Meade e Chang (2003), and samples comprising up to 2,151 assets to compose portfolios with a maximum of 90 assets.

In a very relevant contribution to the index tracking problem, Beasley, Meade e Chang (2003) developed a general formulation for the problem and used GA to solve it. An index tracking formulation is presented using constraints on the portfolio maximum number of assets, transaction costs, and rebalancing control. Their solution method is based on "population heuristic" which takes into account the complexity of the problem and the need for rapid responses. Tests were carried out using five benchmarks, namely Hang Seng Index (Hong Kong), DAX (Germany), FTSE (UK), S&P 100 (USA), and Nikkei (Japan), all from developed countries. Since then, evolutionary heuristics and GA have been applied by several authors (OH; KIM; MIN, 2005; MARINGER; OYEWUMI, 2007; JEURISSEN; BERG, 2008; RUIZ-TORRUBIANO; SUÁREZ, 2009).

Oh, Kim e Min (2005) developed a two-phase index tracking algorithm based on GA. The algorithm's first step is to select, based on the firms' indicators, the assets that will be part of the chosen portfolio. Then, the selected assets are used to minimize the objective function. A limitation of this study is that it requires extra information such as trading volume, market capitalization and weight of the industries in the portfolio, not only assets and index returns. Jeurissen e Berg (2008) and Ruiz-Torrubiano e Suárez (2009) developed hybrid methods, combining non-linear programming and GA. Ruiz-Torrubiano e Suárez (2009) assessed the performance of the hybrid method on the benchmark problems described in Beasley, Meade e Chang (2003). Although good results were obtained for real world problems, the method presented some convergence problems when the number of assets in the universe was high.

In terms of contributions, we can start pointing out our developed method additions to the literature. We designed a hybrid solution method, combining genetic algorithm and non-linear programming modeling, in the same direction as proposed by Jeurissen e Berg (2008) and Ruiz-Torrubiano e Suárez (2009). However, our study presents a different way of measuring the index tracking in comparison with the former research, without the need of estimating the covariance matrix of the portfolio, a quite complex task (FILOMENA; LEJEUNE, 2012). Our study is also, regarding some aspects, more ambitious than the model studied by Ruiz-Torrubiano e Suárez (2009), because it considers a much wider experimental setting, including comparisons with optimal solutions, which is not the case in the aforementioned paper. An extra contribution to the literature is the inclusion of Kernel Search ideas to our method. The initial candidate solution is obtained by a special designed procedure, as discussed by Angelelli, Mansini e Speranza (2012) and (GUASTAROBA; SPERANZA, 2012), in opposite to the randomly generated method (BEASLEY; MEADE; CHANG, 2003; RUIZ-TORRUBIANO; SUÁREZ, 2009), and the fundamental variables based method (OH; KIM; MIN, 2005) applied in the literature on index tracking. Furthermore, we contribute by finding near optimal solutions requiring very short CPU times. As we have discussed, this might be important not only in a high-frequency trading environment (GENCAY et al., 2001; ALDRIDGE, 2009) but also in developing markets in which volatility may present high variations during a short period of time. The quality of the heuristic solutions are compared to optimal solutions obtained from CPLEX, a well-known commercial mathematical package.

Another set of contributions is related to the empirical tests. We targeted our efforts on an index tracking environment barely explored by the operations research community: the Brazilian market. To show that our method is not market specific, we also experimented our model on the S&P 100, FTSE 100 and DAX. Given that our method does not need the tracking assets to be in the benchmark portfolio, we examined the tracking of the S&P 100 and FTSE 100 with just Brazilian stocks and the US dollar. This exercise is quite relevant, since it is common for a country to forbid the buying of assets from other countries, due to strict legislation. This tracking exercise across different markets was discussed but not presented by Beasley, Meade e Chang (2003). Furthermore, acknowledging the differences not only on the models, but also on the data, we provided a *rough comparison* with the results presented by Beasley, Meade e Chang (2003) and Guastaroba e Speranza (2012) for the S&P 100, FTSE 100 and DAX. This should not be taken as a strict comparison, but rather a validation process, in which results from previous studies are used to give an idea of our model tracking error performance. The data used in our experiments is available on-line for future comparisons.[1]

[1] <https://www.dropbox.com/s/ws1t6gorp4cvwkx/idx_tracking.zip?dl=0>

# 3 Model

In this section, we present the optimization model. Before describing the model, we need to present the following notation:

$r_{it}$ = daily return of asset $i$ at time $t$;
$R_t$ = daily return of the index at time $t$;
$t$ = time (each trading day);
$T$ = total of trading in-sample days;
$N$ = set of assets in the sample;
$K$ = threshold amount of assets in the portfolio;
$\vartheta$ = minimum error between the portfolio and index at each $t$;
$\theta$ = maximum error between the portfolio and index at each $t$.

Equations (3.5) and (3.6) show, respectively, how the values of $R_t$ and $r_{it}$ are computed based on the market database (in general, comprised of the index and assets' average daily returns), as follows:

$$R_t = \frac{average\, index_t}{average\, index_{t-1}} - 1 \tag{3.1}$$

$$r_{i,t} = \frac{average\, price_{i,t}}{average\, price_{i,t-1}} - 1 \tag{3.2}$$

Based on this set of parameters, we developed a model based on the objective function described in Gaivoronski, Krylov e Wijst (2005), a well-known reference for the index tracking problem. The central goal is to minimize the average tracking error (TE) of the portfolio over all considered in-sample periods, defined in the objective function as the variance of the difference between the returns from the portfolio relative to the index. There are two decision variables. Let $x_i$ be a continuous variable, indicating the weight of asset $i \in N$ in the portfolio (that is, $x$ represents the composition of the portfolio). Let $z_i$ be a binary decision variable, with $z_i = 1$ if the asset $i$ is included in portfolio, and $z_i = 0$ otherwise. The index tracking problem can be formulated as a non-linear mixed-integer programming model as follows:

Model NLMIP$(N, K)$:

$$\min \frac{1}{T}\sum_{t=1}^{T}\Big[\sum_{i\in N} x_i r_{it} - R_t\Big]^2 \tag{3.3}$$

s.t.

$$\sum_{i\in N} x_i r_{it} - R_t \geq \vartheta \qquad \forall t \in T \tag{3.4}$$

$$\sum_{i\in N} x_i r_{it} - R_t \leq \theta \qquad \forall t \in T \tag{3.5}$$

$$\sum_{i\in N} x_i = 1 \tag{3.6}$$

$$\sum_{i\in N} z_i \leq K \tag{3.7}$$

$$x_i \leq z_i \qquad \forall i \in N \tag{3.8}$$

$$x_i \geq 0 \qquad \forall i \in N \tag{3.9}$$

$$z_i \in \{0,1\} \qquad \forall i \in N \tag{3.10}$$

The objective function (3.3) seeks to minimize the difference between the variance of the portfolio's and the index's returns. Constraints (3.4) and (2.5) restrict the difference in return between the portfolio and the index to a minimum value and a maximum value at each time/moment $t$ (parameters $\vartheta$ and $\theta$); the purpose of both constraints is to avoid tracking error peaks over the in-sample period. Constraint (2.6) states that 100% of the wealth available should be allocated. Together, constraints (3.1) and (3.2) define the maximum number of assets in the portfolio (parameter $K$). Constraints (3.3) and (3.4) define the range of the decision variables.

Coleman, Li e Henniger (2006) showed that the tracking error minimization problem, with a restriction on the total number of assets, is NP-hard. Since for high volatility markets this problem should not be solved quickly, we developed an heuristic method to solve the problem.

# 4 Heuristic Method

As the proposed non-linear mixed-integer program is not likely to be solvable in a reasonable time for instances of real-world size, we propose a heuristic solution method. The heuristic decomposes the problem into two intertwined subproblems as follows: (i) To define a subset $Z = \{z \in N | \, |Z| = K\}$ of assets that will compose the portfolio; and (ii) To define the weight of each asset $x_i, i \in Z$ to form a valid portfolio. The first problem is solved using a GA. The second one is solved using a non-linear programming model with only continuous variables, embedded into the GA. The next sections described with details the developed heuristic.

## 4.1 The Genetic Algorithm

The genetic algorithm in our heuristic approach has the objective of selecting $K$ assets of the set $N$ of total assets. We use a natural, compact and efficient encoding. Each chromosome corresponds to a binary vector $I$ with $N$ positions. Each gene position $I[i] = 1$ represents that the asset $i$ is in the portfolio, e.g., asset $i \in Z$, otherwise $I[i] = 0$. Given the encoding technique, the genetic algorithm typically consists of an initial population, genetic operations and fitness evaluation. These elements are presented below.

### 4.1.1 Initial Population

The choice of the initial generation directly affects the performance and the convergence process of the GA. Algorithm 1 presents the specially designed routine to provide an initial population for the GA, considering the specificities (objectives and constraints) of the index tracking problem.

**Algorithm 1** Routine to generate an initial population

**Input:** A set of $N$ assets, the number of assets $K$ in the portfolio, counter $L$, population size $P$
**Output:** Initial population of size $P$, where each individual is a portfolio with $K$ assets

**Step 1:** Solve model NLP$(N, K)$ (3.3)–(2.6) and (3.3) and store its solution vector $\mathbf{x}$

**Step 2:** Select the $K + L$ greater values of $x_i$, and store the corresponding assets $i$ in set $B$

**Step 3:** Define the set of portfolios $F$ resulting from the combination of $K$ assets in set $B$

**Step 4:** For all portfolios $s \in F$ solve model NLP$(\{s\}, K)$ and store its solution

**Step 5:** Select the $P$ portfolios with the smallest $TE$, i.e. smallest result of the objective function, from the $\begin{pmatrix} K+L \\ K \end{pmatrix}$ possible combinations.

This routine is quite fast, since model NLP is a relaxed version of Model NLMIP, without the constraints related to the amount of assets – i.e. without constraints (3.1), (3.2) and (3.4) – therefore containing only continuous variables. The routine is dependent on a new parameter,

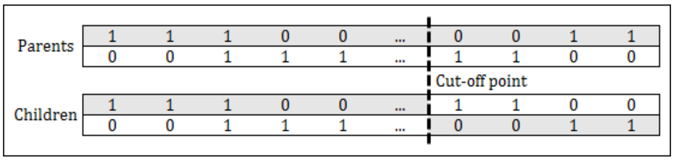



Figure 1 – Crossover illustration

counter $L$, that specifies the number of additional assets to be considered in a portfolio of size $K$ to introduce diversity in the initial population, avoiding an initial population that would lead the GA to a local optimum, and resulting in the algorithm stagnation. The value of this parameter is determined based on experimentation. High values would improve the initial population, but it will require extra CPU time. Initial experiments suggest values in the interval $[2, 4]$, depending on the number of assets in the set.

### 4.1.2 Genetic Operators

Two genetic operators were applied in our heuristic, namely crossover and mutation. These operators are responsible for introducing diversity in the space of solutions, creating a new generation at each iteration of the GA.

The main genetic operator in our implementation is crossover. In this operator, two individuals (parents) are reproduced based on the parameter crossover rate, that defines the probability that these individuals can be selected to mate. Offspring are produced following the 1-point order crossover procedure (JEURISSEN; BERG, 2008). For this, one cut-off point is randomly set, defining a boundary for a series of copying operations. The crossover operator creates offspring that preserve the order and position of symbols in a subsequence of one parent, delimited by the cut-off point, while preserving the remaining symbols from the second parent. Figure 1 illustrates how this operator works. From two individuals, 1 and 2, individual 3 is formed with an initial piece of individual 1 and an end piece of individual 2 (dashed line), and individual 4 is formed with the initial part of individual 2 and the end part of individual 1.

The resulting individuals generated by the crossover operator can have a number of assets $\sum_i I[i] \neq K$, that make this specific individual unacceptable. A feasibility routine was developed to avoid such situation. In case of an incorrect total amount, some positions are randomly changed. For instance, in portfolios consisting of 5 assets: if the individual has six positions equal to 1, one of them will be randomly selected and changed to 0, so that the portfolio has 5 assets. Analogue operations are carried out for individuals with smaller number of assets. The feasibility routine also takes care of repetitive individuals, keeping the population as diverse as possible.

The other GA operator implemented was mutation. Since all individuals suffering mutations are validated, the procedure is quite simple. Each individual has a set of positions altered from 0 to 1, and another set altered from 1 to 0, keeping the individual with $\sum_i I[i] = K$. The changed positions are randomly set. Single and double mutations were implemented. In double mutations, four genes are changed, not just two. Parameter $Num\_Mut$ defines the option for single ($Num\_Mut = 1$) or double mutation ($Num\_Mut = 2$).

### 4.1.3 Fitness Evaluation

Given a population of individuals, we need to determine a fitness value for each individual, defining its potential to remains in the population into the generation. The tracking error $TE = \frac{1}{T}\sum_{t=1}^{T}[\sum_{i \in N} x_i r_{it} - R_t]^2$ (Equation 3.3) would be the natural fitness value of each solution. However, each individual has only the information of which assets may compose a portfolio, not the required weights ($x_i$), necessary to compute $TE$. To define the assets' weights, for each portfolio, model NLP (3.3)–(2.6) and (3.3) is used. Next section describes how this model is connected with the GA.

## 4.2 Algorithm Summary

Algorithm 2 presents an overview of the hybrid heuristic for the index tracking problem.

**Algorithm 2** Hybrid heuristic overview

```
Initial Solution:
Generate starting population using Algorithm 1
j ← 1
Genetic Operators:
repeat
    Crossover
    Validation
    Mutation
Non-Linear Programming:
    for all individual k ∈ Population do
        Set Z(k) = {i ∈ N | I[i] = 1}
        Solve model NLP(Z(k))
        Set fitness(k) ← (1/T) Σ_{t=1}^{T} [ Σ_{i∈Z(k)} x_i r_it − R_t ]^2
    Elitist Strategy: Select the best P fitted individuals.
    j ← j + 1
until (j > Num_Iteration)
```

In order to completely define a portfolio, the heuristic combines GA and non-linear mathematical programming. As stated before, the GA is responsible for finding the assets in the future portfolio, applying genetic operator, while the non-linear programming defines the weight of each asset (defined by the GA) in the portfolio. For each individual $k$ at the current population (initial individuals plus the new individual generated through crossover and mutation), the fitness $k$ is determined by solving a relaxed version of model NLMIP$(K, N)$, in which all binary variables are considered implicitly as parameters rather than decision variables. This relaxed version solves the problem NLP (3.3)–(2.6) and (3.3) over set $Z(k)$, that contains only the assets of individual $k$, obtaining the assets' weights for each portfolio ($x_i, i \in Z$) and its correspondent fitness value (the result of the objective function). Although model NLP$(Z)$ is non-linear, the current commercial mathematical packages are able to solve this model using very low CPU times due to the reduced number of continuous decision variables and the absence of an integer constraint. As we are using a minimization problem, the smaller the fitness value, the better the individual. It should be noticed that, even though we set $K$ assets to compose each individual, the optimal portfolio might have less than $K$ assets, since the optimization requires $x_i \geq 0$.

After a single iteration of the algorithm finishes, the population in the next generation suffers the elite strategy in order to keep the fixed number of the population. This strategy selects the $P$ individuals with the smallest fitness.

# 5 Computational Experiments

The tests were performed using an Intel® Core$^{\text{TM}}$ i7-3770 @ 3.40GHz computer with 8GB RAM. The solution approach was coded with C++ programming language using IBM ILOG CPLEX® version 12.6 as the commercial solver. As noted above, this paper proposes to extend the results obtained in Sant'Anna, Filomena e Borenstein (2014), in which portfolios to track the Ibovespa index were formed with 40, 30 and 20 assets from the same sample of 67 assets that composed the Ibovespa and that was used in this paper. The results obtained on that occasion demonstrated not only the quality of the optimization in terms of tracking but also the possibility of using it in a commercial environment by comparing the results with a Brazilian ETF market asset (BOVA11). The main idea in this study was to form smaller portfolios, generating benefits in terms of operating costs without diminishing the quality of solutions.

Subsection 5.1 describes the experimental settings. In subsection 5.2, we perform a validation process, through rough comparison, due to different data set, with results reported in the literature. In subsection 5.3, we present the main results obtained with the heuristic for the Ibovespa, verifying the quality of the solutions in terms of their gap responses. Finally, in subsection 5.4, we perform additional tests aiming to emulate foreign indexes using only Brazilian assets, in such a way to explore the flexibility of our method.

## 5.1 Database and the Description of the Tests

For the Ibovespa, the sample consisted of 67 assets out of 69 that were quoted on the Ibovespa index from May to August, 2012. For the S&P 100, the sample consisted of 97 assets. For the DAX index, we used 30 assets, and for the FTSE 100, we used 96 assets. Historical prices for Ibovespa and S&P 100 were obtained using Economatica® database; historical prices for DAX and FTSE 100 were obtained from Bloomberg database. Each set of assets contains 871 daily stock price observations, adjusted for dividends, stock splits, etc. – therefore, we have 870 observations of daily returns. For the Ibovespa, price range goes from Jan/2009 to July/2012; for the remainder three indexes, price range is from Jan/2009 to June/2012.

For the tests, we adopted $T = 150$ (in-sample period), which corresponds to the use of historical prices of 150 trading days for the optimization (range of seven to eight months), to form a portfolio that will be further employed to estimate the behavior of the market index in the following days. This is in accordance with Gaivoronski, Krylov e Wijst (2005) who recommended longer in-sample intervals in order to form more stable portfolios. Each formed portfolio had its return projected on the $n$ subsequent trading days. In these projections, $n$ is equal to 20, 60, 120 and 240 (i.e. monthly, quarterly, semi-annual and annual rebalancing). Thus, to form the first portfolio with $n = 60$, for instance, we used the data in the interval $1 \leq t \leq 150$, and the portfolio was projected on the interval $151 \leq t \leq 210$. The second portfolio will be formed with data from $61 \leq t \leq 210$, and the portfolio was projected in the interval $211 \leq t \leq 270$, and so on. Overall, 36 portfolios were formed for $n = 20$; 12 for $n = 60$; 6 for $n = 120$; and 3 for $n = 240$.

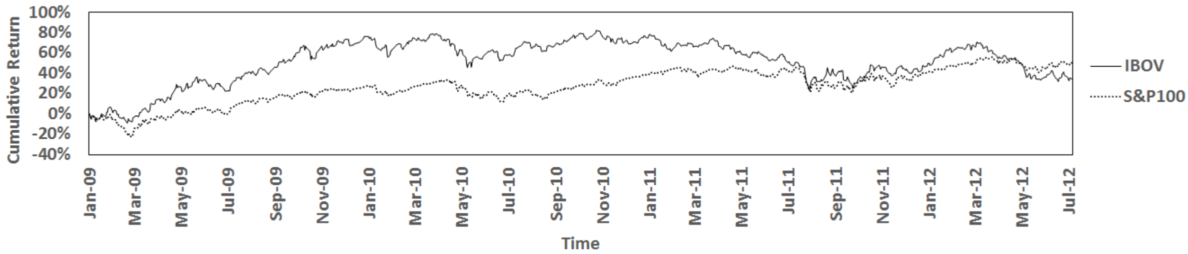



Figure 2 – Cumulated Returns of S&P 100 and Ibovespa, from January 2009 to July 2012

Considering $T = 150$, the projection of the portfolios in rolling horizon started in Aug/2009 for all tested indexes, while the full projections finished in July/2012 for Ibovespa and in June/2012 for S&P 100, FTSE 100 and DAX.

As an illustration, Fig. 2 presents the cumulative returns for the Ibovespa and the S&P 100 from Jan/2009 to July/2012. It should be noticed the difference in the volatility between Ibovespa and S&P 100, which emphasizes our goal of using the proposed solution method to solve the index tracking problem in several markets, with different volatility patterns.

The objective of the heuristic was to form portfolios containing 5 and 10 assets for the Ibovespa, and portfolios with 10 assets for the S&P 100, FTSE 100 and DAX indexes. Based on Beasley, Meade e Chang (2003) and Oh, Kim e Min (2005), the runtime of the algorithm was defined at around 5 minutes (specifically between 4.5 minutes and 5.5 minutes), while we expect to get answers with an average gap of below 10% – this is the runtime for Initial_Test and Tests 1 and 2 described in Table 1. Subsequently, the time was increased to around 8 minutes (specifically between 7.5 minutes and 8.5 minutes) – runtime for Test 3 also described in Table 1 – in order to obtain solutions with a gap of less than 5% (which is a standard gap used in other portfolio optimization studies, such as Filomena e Lejeune (2014)).

Table 1 describes the four different tests that were performed (Initial_Test and Tests 1, 2 and 3) and their respective parameters. Initial_Test is the base scenario with a computational time of around 5 minutes (runtime strictly between 4.5 and 5.5 minutes, as mentioned above). In Tests 1 and 2, parameters $P$ and $Num\_Iteration$ were jointly changed and the computational time remained within 5 minutes. For Test 3, parameter $P$ was maintained within the same value as in Initial_Test and the parameter $Num\_Iteration$ was increased, therefore the computational time was increased and set to around 8 minutes (strictly between 7.5 and 8.5 minutes). The Mutation Rate and $Num\_Mut$ parameters were randomly set in Initial_Test and then changed, being increased in Test 2 as an attempt to form portfolios with greater diversity among individuals.

For the optimization model, $\vartheta$ and $\theta$ were defined to be equal in magnitude, only changing their sign. The values were maintained at $\vartheta = -0.01$ and $\theta = 0.01$ (i.e., 1% in module). This was the smaller interval we could find. For smaller intervals between these parameters, no solution was found for some portfolios. For the tests across different markets that involved Ibovespa, S&P 100 and FTSE 100 (which will be further explained), $\vartheta$ and $\theta$ were altered to 5% in module, since the model could not be solved with smaller values. We have adopted crossover rate of 100% in all experiments, so ensuring the generation of a larger amount of new individuals at each iteration, seeking to avoid local minimum points. For all tests carried out, counter $L$ (required for generating the initial population) was set to 2. The use of higher values did not have a significant impact in the results for the market indexes involved in the experiments.

Table 1 – Description of the parameters adopted in the tests

| Portfolio Size | | 5-Assets | 10-Assets |
|---|---|---|---|
| Initial_Test | P | 20 | 20 |
| | Crossover Rate | 1 | 1 |
| | Mutation Rate | 0.85 | 0.85 |
| | Num_Mut | 1 | 1 |
| | Num_Iteration | 30 | 25 |
| Test 1 | P | 15 | 15 |
| | Crossover Rate | 1 | 1 |
| | Mutation Rate | 0.85 | 0.85 |
| | Num_Mut | 1 | 1 |
| | Num_Iteration | 35 | 30 |
| Test 2 | P | 10 | 10 |
| | Crossover Rate | 1 | 1 |
| | Mutation Rate | 0.90 | 0.90 |
| | Num_Mut | 2 | 2 |
| | Num_Iteration | 50 | 45 |
| Test 3 | P | 20 | 20 |
| | Crossover Rate | 1 | 1 |
| | Mutation Rate | 0.80 | 0.80 |
| | Num_Mut | 1 | 1 |
| | Num_Iteration | 50 | 45 |

## 5.2 Validation

The performance of the developed heuristic was evaluated comparing the results with previous methods described in the literature. In order to evaluate these performances, we use the same tracking error definition from Beasley, Meade e Chang (2003) and Guastaroba e Speranza (2012) as follows:

$$TE^B = \frac{1}{T}\Big[\sum_{t=1}^{T} |r_t^p - R_t|^2\Big]^{1/2} \tag{5.1}$$

where $r_t^p = \sum_{i \in N} x_i r_{it}$ is the return of portfolio $p$ at period $t$.

Since we formed six portfolios for semiannual rebalancing interval, and three portfolios for annual rebalancing, the tracking errors for our tests, presented in Tables 2, 3, 5 and 6, correspond to the average $TE^B$ of the six (or three) portfolios.

Initially, we ran the developed heuristic towards obtaining the projection of the selected portfolios considering semiannual and annual rebalancing using the S&P 100, FTSE 100 and DAX indexes. As an illustration, Figure 3 compares the performance of the selected portfolios using the S&P 100 index and annual rebalancing interval. The tracking error was, in general, good, with the model emulating quite closely the market index during the whole out-of-sample period. We can only notice small tracking errors in two periods, in which the difference between

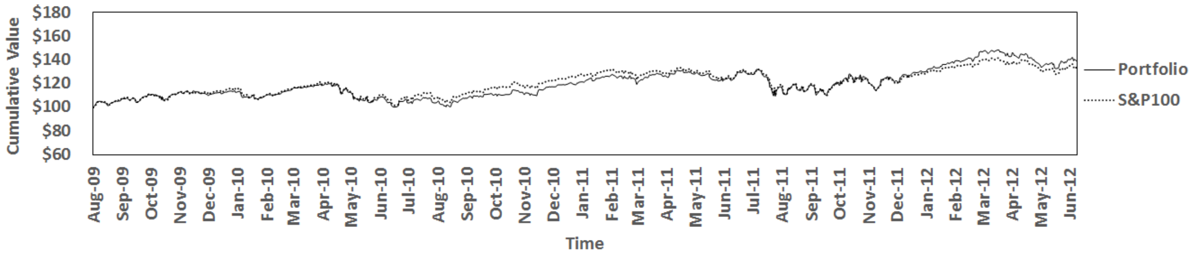



Figure 3 – S&P 100 – Test 3: forecast for annual rebalancing

the cumulative returns from the portfolio and the index reached -6.5 percentage points (Oct, 10th 2010) and 5.3 percentage points (March, 14th 2012).

Next, we performed a predictive validation (BORENSTEIN, 1998) with previous reported results in the literature. We selected the research works of Beasley, Meade e Chang (2003) and Guastaroba e Speranza (2012) as references. They used the S&P 100, FTSE 100 and DAX indexes, forming portfolios with 10 assets. While in Beasley, Meade e Chang (2003) and Guastaroba e Speranza (2012), the DAX sample was composed by 85 stocks, our sample contained only 30 assets, due to structural changes in the index over time. For these validation experiments, we employed the parametrization defined in Test 3 (see Table 1), situation in which the heuristic obtained the best results in previous tests (as described in subsection 5.3). It is important to highlight that, since we do not employ the same databases from the aforementioned studies, this is just a rough comparison presented as a validation of our algorithm. Table 2 succinctly presents the experimental settings and average $TE^B$ values computed by the three research works using DAX, FTSE 100 and S&P 100 indexes.

From the obtained results in Table 2, we can notice that the $TE^B$ values slightly decrease as the rebalancing intervals become larger for DAX, FTSE 100 and S&P 100. Comparing our results with Beasley, Meade e Chang (2003) and Guastaroba e Speranza (2012), we obtained smaller $TE^B$ values for all analyzed indexes. Using semiannual rebalancing, we reduced the average $TE^B$ for the DAX, FTSE 100 and S&P 100 indexes by factors of 9.40, 4.48, and 4.35, respectively. Using annual rebalancing, the reduction factors were 12.57, 5.88, and 6.14, respectively. Particularly for the DAX index, these were expected results, since our sample size for this index was smaller as presented in Table 2. Based on the obtained results, our heuristic was considered a valid and competitive method for the index tracking problem, considering market indexes from developed countries.

## 5.3 Case Study 1: Ibovespa Market Index

Table 3 shows the results of the application of Initial_Test parametrization (see Table 1) in the Ibovespa market index, in terms of the $TE^B$ and the monthly turnover of the portfolios. The average error decreases as the rebalancing interval is extended, a counterintuitive result. Nevertheless, average monthly turnover decreases with larger rebalancing intervals, as expected.

Figure 4 shows the forecast for the Ibovespa tracking with the portfolios with 5 and

Table 2 – Rough comparison with Beasley, Meade e Chang (2003) and Guastaroba e Speranza (2012)

| Index | N* | C** | C/N | $TE^B$ | Rebalancing |
|---|---|---|---|---|---|
| Beasley, Meade e Chang (2003) | | | | | |
| DAX | 85 | 10 | 0.12 | 0.2049% | - |
| FTSE 100 | 89 | 10 | 0.11 | 0.0958% | - |
| S&P 100 | 98 | 10 | 0.10 | 0.1032% | - |
| Guastaroba e Speranza (2012) | | | | | |
| DAX | 85 | 10 | 0.12 | 0.2048% | - |
| FTSE 100 | 89 | 10 | 0.11 | 0.0958% | - |
| S&P 100 | 98 | 10 | 0.10 | 0.1032% | - |
| Developed method | | | | | |
| DAX | 30 | 10 | 0.33 | 0.0218% | Semiannual |
| | 30 | 10 | 0.33 | 0.0163% | Annual |
| FTSE 100 | 96 | 10 | 0.10 | 0.0214% | Semiannual |
| | 96 | 10 | 0.10 | 0.0163% | Annual |
| S&P 100 | 97 | 10 | 0.10 | 0.0237% | Semiannual |
| | 97 | 10 | 0.10 | 0.0168% | Annual |

N* = Sample size; C** = Portfolio size.

Table 3 – Ibovespa - Results of the Initial_Test (portfolios with 5 and 10 assets)

| | 10-Asset Portfolios | | | | 5-Asset Portfolios | | | |
|---|---|---|---|---|---|---|---|---|
| Out-of-sample interval | 20 | 60 | 120 | 240 | 20 | 60 | 120 | 240 |
| Tracking Error ($TE^B$) | | | | | | | | |
| Average | 0.055% | 0.032% | 0.024% | 0.017% | 0.078% | 0.049% | 0.034% | 0.023% |
| Minimum | 0.037% | 0.025% | 0.017% | 0.015% | 0.046% | 0.036% | 0.025% | 0.022% |
| Maximum | 0.088% | 0.045% | 0.037% | 0.021% | 0.131% | 0.071% | 0.052% | 0.027% |
| Std Deviation | 0.014% | 0.006% | 0.007% | 0.003% | 0.023% | 0.010% | 0.009% | 0.003% |
| Monthly Turnover | | | | | | | | |
| Average | 36.142% | 13.126% | 7.728% | 4.141% | 34.185% | 15.598% | 8.862% | 4.650% |
| Minimum | 7.982% | 6.181% | 4.298% | 3.578% | 1.241% | 3.878% | 6.040% | 3.384% |
| Maximum | 81.263% | 20.067% | 9.566% | 4.783% | 78.719% | 27.661% | 13.146% | 5.706% |
| Std Deviation | 16.006% | 5.587% | 1.912% | 0.607% | 20.699% | 7.859% | 2.489% | 1.175% |

10 assets (results for the Initial_Test parametrization) for an out-of-the-sample interval of 120 business days (semi-annual rebalancing). We can observe that the portfolios' curves are relatively close to the index's curve, with the portfolio with 10 assets being the closest (in accordance with what is seen in Table 3) for the 120-period interval: average tracking error and standard deviation are lower for portfolios with 10 rather than 5 assets.

In order to analyze the quality of the solutions of the heuristic for Ibovespa, we opted to check the gap of the solutions obtained, considering optimal solutions obtained by CPLEX, a well-known commercial mathematical package. For this purpose, we adopted a 20-period rebalancing interval, in which we formed a total of 36 portfolios with 5 assets and 36 ones with 10 assets. To check the gap, we randomly selected one third of the 36 portfolios obtained (for each type of portfolio). Gaps from the optimal solutions were calculated using gap=$(OF_H/OF_O) - 1$,

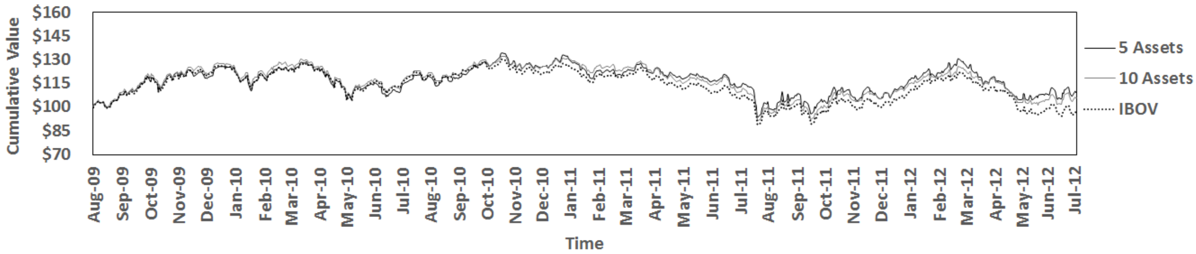



Figure 4 – Ibovespa - Initial_Test: forecast for semi-annual rebalancing

where $OF_H$ is the heuristic solution, and $OF_O$ is the optimal solution offered by CPLEX (GUASTAROBA; SPERANZA, 2012).

Table 4 shows the average gap for each of the parametrizations described in Table 1. As already mentioned, runtimes for Initial_Test and Tests 1 and 2 are around 5 minutes, while the runtime for Test 3 is around 8 minutes. In order to obtain the optimal solutions for each of the 24 portfolios randomly selected (12 portfolios with 5 assets, and another 12 with 10 assets), CPLEX required on average 7 hours of CPU time for each portfolio, reaching more than 10 hours in some cases. Surprisingly, better results were found with a smaller number of assets in the portfolio, using our method. This fact might be consequence of the Ibovespa volatility or the in-sample chosen periods. Since GA is a stochastic experimental method, the solution found is highly dependent on experiments and the set of random variables generated in our heuristic approach. Therefore, further experimentation may be required towards a conclusive explanation for this counter-intuitive behavior. Nevertheless, it is possible to point out some interesting aspects captured by the conducted experiments as follows.

In the case of 10 assets, the gaps were larger in the Initial_Test; however, the average value remained at 7.68%, with a minimum gap of only 3%. The gap was above 10% for only 2 of the 12 tested portfolios (maximum gap 17.58%). In the case of portfolios with 5 assets, we obtained an average gap of only 2.44%. Furthermore, the algorithm provided optimal solutions in 7 of the 12 test cases. Although the maximum gap found was 15.1%, there was only one portfolio with a gap bigger than 10%.

For portfolios with 5 assets, using settings defined in Tests 1 and 2, changes in the parameters did not significantly influenced the solutions provided by the algorithm. The average, maximum and standard deviation values of the gaps increased in Tests 1 and 2, which tends to imply larger tracking errors. Thus, we observed an advantage in the Initial_Test results, which suggests that the use of a larger population tends to produce better results. Test 3 outperformed all other parametrization settings. This result was expected, since a higher CPU time was offered to this situation. The average gap decreased to only 1.05%, with a maximum gap below 5%, and the optimal solution was obtained in 8 out of the 12 test cases.

For 10-asset portfolios, in relation to the Initial_Test parametrization, the average gap varied in contrasting directions in Tests 1 and 2. In Test 1, the average gap increased; while in Test 2, the average gap decreased. However, there was an increase in the standard deviation of the errors. Thus, we concluded that these altered parameters did not significantly influence the heuristic results. Test 3 also obtained the best results for 10-asset portfolios. The average gap was only 3.77%, with a maximum gap below 10%; that is, all the 12 generated portfolios had a gap below 10%, and the heuristic provided optimal solution for 1 of the 12 test cases. We observed again that a small increase in processing time is able to have a significant impact on the solutions.

Table 4 – Gap values for the 5 and 10- asset portfolios (with monthly rebalancing)

| Average Gap for 10-Asset Portfolios | | | | |
|---|---|---|---|---|
| Tests | Initial_Test | Test 1 | Test 2 | Test 3 |
| Average | 7.680% | 7.950% | 6.610% | 3.780% |
| Minimum | 3.050% | 0.000% | 0.000% | 0.000% |
| Maximum | 17.580% | 19.710% | 17.020% | 7.860% |
| Std Deviation | 4.050% | 5.630% | 6.450% | 2.210% |
| Average Gap for 5-Asset Portfolios | | | | |
| Tests | Initial_Test | Test 1 | Test 2 | Test 3 |
| Average | 2.440% | 4.020% | 4.710% | 1.050% |
| Minimum | 0.000% | 0.000% | 0.000% | 0.000% |
| Maximum | 15.130% | 23.020% | 15.660% | 4.730% |
| Std Deviation | 4.440% | 6.720% | 6.340% | 1.690% |

Table 5 presents the results applying the developed heuristic for the Ibovespa index, considering analogue experimental conditions described in Section 5.2 and semiannual and annual rebalancing intervals. The main difference is in the higher relation $\frac{C}{N}$, since Ibovespa has a smaller number of assets in its portfolio relative to FTSE 100 and S&P 100. Analyzing the results for our heuristic method, we can see a rather higher tracking error for the Ibovespa index than for the S&P 100. These are expected results given that the former presents a higher volatility than the latter.

However, considering that the gaps of the solutions of the algorithm indicate that these solutions are at least close to the optimal responses, we can conclude that it is rarely possible to conduct tracking of the Ibovespa index specially using portfolios of only 5 assets (at least considering longer rebalancing intervals). Intuitively, tracking an index with only 5 assets generates higher concentration risks. Any abnormal behavior in one of this 5 assets should create great impact in the portfolio. Also, we can see that the error for the Ibovespa index increases considerably from 10 to 5 assets. For 10 assets, we can compare the Ibovespa results with those obtained for markets with lower volatility, such as the S&P, and the FTSE 100 (see Table 2). This is a significant result, considering the volatility of the Ibovespa index.

## 5.4 Case Study 2: Emulating Other Indexes Using Stocks in the Brazilian Market

Finally, we formalized two tests for index tracking across different markets. To do so, we opted to use Brazilian assets (that compose the Ibovespa sample) to follow FTSE 100 and S&P 100 indexes. From an economic viewpoint, the possibility of following a foreign index (from a mature market) using Brazilian assets (an example of emerging and more unstable market)

Table 5 – Ibovespa reults with semiannual and annual rebalancing

| N* | C** | C/N | $TE^B$ | Rebalancing |
|---|---|---|---|---|
| Ibovespa Index – 10-Asset Portfolios | | | | |
| 67 | 10 | 0.15 | 0.0240% | Semiannual |
| 67 | 10 | 0.15 | 0.0172% | Annual |
| Ibovespa Index – 5-Asset Portfolios | | | | |
| 67 | 05 | 0.07 | 0.0343% | Semiannual |
| 67 | 05 | 0.07 | 0.0235% | Annual |

N* = Sample size; C** = Portfolio size.

should be considered important not only as a possibility to emulate assets not locally available but also as a hedge operation. Through the use of a portfolio capable of tracking a mature market, the investor is able to diversify the risks of the market in which he/she is inserted.

Computational tests were performed using the 67 assets that compose the Brazilian sample plus the US Dollar currency exchange quotation. The US Dollar price series was extracted from Economatica (the same database in which we obtained stock prices series for the assets that compose the Ibovespa). The inclusion of the US Dollar in our sample is relevant since we are now attempting to track foreign indexes. We could have used another currency, like the Sterling Pound, specially to help with the FTSE tracking. However, differently from the US Dollar, there is no structured and liquid market for the Sterling Pound in Brazil. Thus, the US Dollar was selected not only by being the currency in which the S&P 100 is quoted, but also as a way of connecting Brazil and its risks (country and currency risks) to the rest of the world.

In order to perform this experiment, we need to introduce the following small changes in model NLMIP(N,K), to allow not only long, but also short positions in the portfolio: (i) we eliminated constraints (3.3), towards accepting short positions in our portfolio; and (ii) we must consider the absolute value of $x_i$ in constraint (3.2), replacing this constraint by $|x_i| \leq z_i, \forall i \in N$. The latter alteration was a relevant practical adjustment, allowing our model to achieve better tracking results specially in periods in which Ibovespa and FTSE 100/S&P 100 are moving in opposite directions. The new formulation of model NLMIP(N,K) was then used in Algorithms 1 and 2.

Table 6 – Results for the tests across different markets: using Ibovespa to track FTSE 100 and S&P 100 (semiannual and annual rebalancing)

| N* | C** | C/N | $TE^B$ | Rebalancing |
|---|---|---|---|---|
| Ibovespa-S&P 100 | | | | |
| 68 | 10 | 0.15 | 0.0898% | Semiannual |
| 68 | 10 | 0.15 | 0.0650% | Annual |
| Ibovespa-FTSE 100 | | | | |
| 68 | 10 | 0.15 | 0.0834% | Semiannual |
| 68 | 10 | 0.15 | 0.0592% | Annual |

N* = Sample size; C** = Portfolio size.

Table 6 presents the results for the tests performed using assets in Ibovespa to track S&P 100 (“Ibovespa-S&P 100”) and FTSE 100 (“Ibovespa-FTSE 100”), considering semiannual and annual rebalancing. The results for both indexes were very similar, with a slight better average $TE^B$ for the FTSE 100. As in previous sections, longer rebalancing intervals resulted in smaller TE values.

Analyzing the results for both indexes, larger average $TE^B$ values for both semiannual and annual rebalancing periods were obtained in comparison with the results presented in Table 2. This was an expected result, given the high volatility discrepancy between Ibovespa and S&P 100/FTSE 100 indexes (See Figure 2 for an illustration). In addition, it seems reasonable to accept that a practical ETF composed by Brazilian stocks aiming to track a foreign index should necessarily accept larger tracking error values over time. Therefore, we consider the results obtained in both tests quite promising.

# 6 Conclusions

In this study, we applied an index tracking model with controlled number of assets to several market indexes, with different volatility patterns. Considering the tight constraint on the amount of assets to constitute the portfolios, this problem has already been proved to be an NP-hard problem. As a consequence, we presented a hybrid heuristic approach to solve the problem, combining genetic algorithm and non-linear mathematical programming method.

Initially, we validated our method, using results from benchmark methods described in Beasley, Meade e Chang (2003) and Guastaroba e Speranza (2012) with respect to S&P 100, FTSE 100, and DAX indexes. The obtained results demonstrated that our method is competitive with these aforementioned studies, giving the different computational settings of the performed experiments. Next, we applied the heuristic approach for the Ibovespa. The central objectives of forming portfolios containing 5 and 10 assets with solutions gaps below 10%, and processing times of about 5 minutes were achieved. Also, by increasing the time to about 8 minutes, it was possible to obtain responses with an average gap below 5% for both 5 and 10-asset portfolios, thus providing strong evidence to suggest that the heuristic provides good quality solutions. With the heuristic, it was possible to form portfolios combining a small number of assets, short-length processing times, and average gap values below 10%, comparing with optimal solutions obtained by CPLEX. Also, with a slightly longer processing time, about 8 minutes, the average gap obtained was below 5%.

In practice, the set of financial analysts would run the heuristic several times in one day, with different parametrizations and data scenarios, towards setting up the position of the new portfolio, especially in situations involving abrupt market movements. We consider that eight minutes is still an appropriate time for real-world environments. Obviously, processing times above 8 minutes should lead to higher quality answers in terms of gap. However, new tests (with higher computational time) were not performed since eight minutes were enough to obtain solutions with average gap below 5%, as initially expected.

Furthermore, we performed tests in an attempt to employ index tracking across different markets. The obtained results were quite promising, emphasizing the flexibility of our method. The developed solution approach was able to make it possible to emulate FTSE 100 and S&P 100 indexes using Brazilian stocks.

Summarizing the experimental results, we can consider the results of the heuristic quite satisfactory. We got near-optimal responses with a computational time of about 5 minutes for problems that would require several processing hours to generate optimal solutions. The results of the experiments also confirmed that the heuristic has fulfilled the proposed objectives both for high and low volatile market indexes. Actually, the developed method is quite competitive with previous related methods for mature market indexes, considering simultaneously efficiency and efficacy, with the advantage of being tested and validated for less stable, and therefore, more volatile market indexes. As a consequence, we believe that our method can be applied with success to other developing markets in Latin America, Africa, and Asia.

Future research proceeds to expand the range of the application of the method to other market indexes, including multi-market ones, and to improve the effectiveness of the heuristic, introducing more sophisticated techniques in the GA, such as the use of specifically designed penalty functions to cope with the index tracking problem constraints, increasing the efficiency of the search algorithm used for optimization.

# Part II

# Index Tracking and Enhanced Indexing using Cointegration and Correlation with Endogenous Portfolio Selection

# Abstract

This article investigates the out-of-sample performance of cointegration and correlation methods for index tracking (IT) and enhanced indexing (EIT) strategies applied to Brazilian and U.S. market data. Our goal is to compare both methods as we strongly explore cointegration in relation to previous studies: we make the portfolio selection endogenous to the problem for this approach. The tests are performed using data from 2004 to 2014 with samples of 57 assets for Brazilian data and 96 assets for U.S. data; portfolios are built using combinations of at most 10 of these assets. Despite the extensive tests carried out, the overall result shows similar performance for both methods. For IT in the Brazilian market, there was a trade-off between better tracking error and higher turnover for cointegration (with the opposite for correlation), this pattern was not clear in the U.S. market. The outcome for the EIT also does not clearly favor cointegration or correlation.

**Keywords:** index tracking, enhanced index tracking, cointegration, correlation.

**Note:** this article has been accepted for publication in The Quarterly Review of Economics and Finance.

# 1 Introduction

As opposed to traditional active funds that seek to beat the market by using many varieties of stock picking strategies, index tracking (IT) is a passive fund (portfolio) in which the concern is to reproduce (track) the risk-return profile of some specified benchmark. Because indexed portfolios have lower fees and better diversification than most actively managed investments, the passive strategy has been broadly considered by more than just a few portfolio managers. In the United States, the first index mutual fund was launched in 1976 by Vanguard Group Inc and has recently become a popular investment vehicle in different markets and economies (FRINO; GALLAGHER, 2001; CREMERS et al., 2016). As showed by Appel, Gormley e Keim (2016), "passive investors have grown significantly in recent years; the share of equity mutual fund assets held in passively managed funds tripled over the 1998–2014 period to 33.5%, and the share of total U.S. market capitalization held by passively managed funds quadrupled to more than 8%". The theoretical foundation for index tracking comes from the theory of market efficiency (ALEXANDER; DIMITRIU, 2005b). Currently, the literature documents that active funds do not consistently outperform benchmark indexes and suggests that passive funds represent an appropriate alternative, especially if significant transaction costs are involved (MALKIEL, 1995; FRINO; GALLAGHER, 2001). Moreover, passive funds can be used for investors as a tool to reduce their exposure to nonsystemic risks (CHEN; HUANG, 2010).

The simplest way to track an index is to hold all of its assets in the same relative quantities: the so-called *full replication*. However, full replication has a number of disadvantages. In particular, certain assets with small weights proportionally incur high transaction costs, and frequent revisions of the index result in persistent transactions. Portfolio rebalancing can be especially problematic in times of high volatility; fluctuating prices can cause many assets to be added and deleted from the index. Because one of the main advantages of passive management is reduced transaction costs—especially for those tracking large indexes—it may be desirable to hold fewer assets than the number of assets contained on the index. Nonetheless, this approach results in tracking portfolios that do not match the performance of the target index as closely as full replication. Another relevant approach is represented by synthetic replication toward equity derivatives (i.e., future contracts), which usually have singularly lower transaction costs. However, using rolling contracts to dynamically track the underlying index is rather expensive and risky. To avoid these disadvantages, or at least to mitigate their effects, partial replication can be applied by selecting a smaller subset of assets. Even then, an optimal procedure should be designed to address asset selection and allocation.

In the same direction, the problem known as enhanced index tracking (EIT; or enhanced indexation) aims to reproduce the performance of a stock market index and generate excess returns (adding alpha) while minimizing tracking errors. Indeed, enhanced index tracking is a strategy derived from IT (GRINOLD, 1989; MILLER; MECKEL, 1999) and represents a dual-objective problem that seeks the optimal decisions for outperforming its benchmark index without incurring much additional risk. Such strategies come in two basic forms: derivative based and stock based. The intention of derivative-based enhanced index tracking is to provide exposure to the desired equity market through a derivative and to the enhanced return through an asset other than an equity investment (LI; BAO, 2014). Because the drawbacks incurred by active management (such as higher risk and transaction costs) could not reduce or eliminate the advantages of index tracking, passively managed enhanced index tracking could bring fund managers competitive advantages after deducting the costs to track and reward customers.

Enhanced index funds execute essentially index-oriented investment strategies. However, the enhanced index manager is permitted to engage in limited (risk-controlled) active strategies designed to beat the benchmark index with minimum likelihood of underperforming it in a given period. Comparisons of both the strategies and the performance of index and enhanced index funds provide insights into the potential benefits available to investors who exploit passive funds by executing less rigid strategies.

Because the aim of index funds is to replicate an index's performance, the goal of index tracking optimization models is to minimize the difference between the return of a benchmark index and the return of an index fund (*or tracking error*). Different approaches to addressing this problem have been proposed in the literature, such as Focardi e Fabozzi (2004), who adopted a clustering-based methodology to determine an optimal tracking portfolio. They use Euclidean distances between stock price series as a basis for hierarchical clustering. Once clusters of stocks have been formed, they select one (or more) stocks from each cluster to include in the tracking portfolio, which could also be applied to enhanced indexation. Alexander e Dimitriu (2004) developed a model to select an optimal tracking portfolio using principal component analysis in the stock returns. Furthermore, Corielli e Marcellino (2006) considered multiple factors by employing a simple selection heuristic to produce a portfolio that avoids long-term factors that could lead to tracking errors. Wu et al. (2007) employed a goal programming approach by defining desired values for both excess returns and tracking errors. Canakgoz e Beasley (2008) and Li, Sun e Bao (2011) employed multi-objective optimization with the goals of maximizing index outperformance and minimizing tracking errors.

In contrast to traditional portfolio optimization models employing stock returns on the basis of correlation (ROLL, 1992), there are also advocates of index tracking on the basis of cointegration. This method's objective is to find long-term relationships between the subset of stocks constituting the portfolio and the index – for example, Alexander (1999), Alexander e Dimitriu (2005a), Alexander e Dimitriu (2005b), Acosta-González, Armas-Herrera e Fernández-Rodríguez (2015). As pointed out in the seminal paper of Alexander e Dimitriu (2005a), the usual correlation-based approach produces relatively unstable tracking portfolios. Alexander (1999) and Dunis e Ho (2005) proposed the use of cointegration to capture stable long-run equilibrium between the tracking fund and the benchmark index. Cointegration is an appropriate technique used to model long-run asset price dynamics. Its key characteristics, i.e., mean reverting tracking error, enhanced weight stability, and better use of the information comprised in stock prices, allow a flexible design of various funded and self-financing trading strategies, from index and enhanced index tracking to long-short market neutral and alpha transfer techniques.

As argued by Alexander e Dimitriu (2005a), the advantage of using cointegration for index tracking is, on the one hand, a stationary price difference between the benchmark and the portfolio by construction; therefore, the tracking portfolio is tied to the benchmark index in the long run. On the other hand, stock weights, based on cointegration and a long history of prices, have more stability producing the benefits of less frequent portfolio rebalancing with respect to using correlation to carry out indexation. These benefits rest on making full use of the information in stock prices before their detrending, which permits a long-run relationship between equity prices and the market index.

Because both cointegration and correlation approaches stand out for the index tracking problem, Alexander e Dimitriu (2005a) presented a study focused on comparing the two methods; the authors tested IT strategy using cointegration and correlation with the Dow Jones Industrial Average index from 1990 to 2003 (sample of 30 assets). Portfolios were built with 20, 25, and 30 assets. The results were similar for both methods; however, the authors argued that tests with more constrained portfolios (relative to the sample size) could highlight differences between the methods. Furthermore, a naive approach is used to select the assets that compose the portfolios, i.e., portfolio selection is exogenous to the problem, which limits the results obtained because it does not explore portfolios with alternative combinations of assets. A similar comparison was conducted by Grobys (2010); nevertheless, such a study also applies exogenous portfolio selection instead of including selection into the optimization process.

A number of recent studies, such as Dunis e Ho (2005) and Thomaidis (2013), specifically use cointegration, whereas some studies develop particular techniques related to correlation – for example, Gaivoronski, Krylov e Wijst (2005), Maringer e Oyewumi (2007), Barro e Canestrelli (2009), Scozzari et al. (2013). Finally, some studies on EIT exist in the contemporary literature – for instance, Alexander e Dimitriu (2002), Alexander e Dimitriu (2005a), Wu et al. (2007), Thomaidis (2013). Specifically, Alexander e Dimitriu (2005a) used cointegration and correlation for the IT strategy and for long-short portfolios (a strategy similar to EIT). However, as already mentioned, a drawback of this study is the naive portfolio selection. Wu et al. (2007) combined correlation and goal programming to solve the EIT problem. Thomaidis (2013) addressed EIT strategy using cointegration and a cardinality constraint to control for portfolio size. This author introduced a routine to make the portfolio selection endogenous to the problem. However, no study compared correlation and cointegration methods with EIT strategy.

In this study, we attempt to compare correlation and cointegration using IT and EIT strategies. This comparison is similar to that carried out by Alexander e Dimitriu (2005a) for IT and long-short strategies. Nonetheless, first, we expand the cointegration technique by building cointegrated portfolios using simulations, similar to Thomaidis (2013), instead of performing exogenous portfolio selection. Such a practice should lead to more consistent use of a cointegration approach. Second, we selected databases with a long interval (a total of 11 years, from 2004 to 2014) as reinforcement of our comparison by covering different market conditions. Third, we adopted two markets: Brazil with the Ibovespa index, and the United States, with the S&P 100 index. Finally, we follow the suggestion in Alexander e Dimitriu (2005a) and form reduced portfolios in relation to the sample size—portfolios limited to only 10 assets out of a sample comprised of up to 57 assets for the Ibovespa and 96 assets for the S&P 100. The combination of these features should provide even more robustness to the results, making our study different from previous articles.

Despite the extensive tests carried out, the overall results showed similar performance for both methods used. In the case of the Brazilian market, a trade-off was formed between tracking error performance and costs. Cointegration outcomes were associated with lower tracking errors and larger turnover values, whereas correlation results presented portfolios with slightly worse performance but lower volatility and reduced costs. Such a trade-off was unclear in the U.S. market, which had mixed results for the TE with some advantage for correlation in terms of transaction costs (represented by turnover values). Thereby, our conclusions follow Alexander e Dimitriu (2005a), particularly for the Brazilian environment, in which the results did not show an advantage for any method. The same pattern was found for EIT strategy; the cointegration approach resulted in good responses for both markets when we used 1-year in-sample data for portfolio selection, whereas the correlation also had good quality responses using 2-year in-sample data. As a result, regardless of the extension of our tests, we conclude that both methods can deliver results with similar overall performance.

This article is organized as follows. Section 2 presents a theoretical review of cointegration and correlation. Section 3 discusses the methodology behind the tests. Section 4 presents the results, and Section 5 provides the conclusions.

# 2 Theoretical Review

The theoretical review is separated into two parts. Subsection 2.1 describes the methodological aspects of cointegration, whereas Subsection 2.2 discusses correlation.

## 2.1 The Cointegration Approach

Originally, cointegration was introduced by Granger (1981a) and further developed by Engle e Granger (1987). Its main concept lies in the fact that a linear combination of two or more non-stationary time series might be stationary. Thus, if such a stationary linear combination exists, then the non-stationary time series is said to be cointegrated. Cointegration is the property in which two or more time series share a common stochastic trend. The fundamental observation that justifies the application of the cointegration concept in the analysis of stock prices is that a set of non-stationary stock prices can present a common stochastic trend in levels – see Stock e Watson (1988).

The application of cointegration to asset allocation was pioneered by Lucas (1997) and Alexander (1999). On the one hand, Lucas (1997) discussed a continuous-time framework for risk-averse management, showing that strategic asset allocation can generate encouragingly good results considering a long-run equilibrium. On the other hand, Alexander (1999) proposed a cointegration framework designed for optimal pairs of trading identification. The presented characteristics, i.e., mean reverting tracking error, enhanced weight stability, and better use of the information comprised in stock prices, allow a flexible design of various levered and self-financing trading strategies, such as index tracking, enhanced index tracking, and long-short market neutral.

Since then, extensive studies have used the concept of cointegration applied in financial econometrics. Two central studies on cointegration and portfolio selection are Alexander e Dimitriu (2005a), Alexander e Dimitriu (2005b), who constructed cointegration-based portfolios to examine whether classic and enhanced index tracking provide potential sources of "alpha". These studies mentioned that cointegration portfolios could improve traditional tracking-error variance models. Additionally, Dunis e Ho (2005) proposed using cointegration to capture stable long-run equilibria between an index tracking fund and the benchmark index[1]. The theoretical guidelines for cointegration are as follows:

[1] Other examples of studies related to cointegration are as follows: Ahlgren e Antell (2002) applied cointegration to detect comovements among foreign markets; Benzoni, Collin-Dufresne e Goldstein (2007) studied portfolio choice under cointegrated stock and labor markets; Pan (2007) used cointegration to analyze shocks in stock returns caused by firms' corporate events; Bansal e Kiku (2011) used a CVAR (Conditional Value at Risk—CVAR) setup to study optimal long-term asset allocation; Chiu e Wong (2011) considered the mean-variance criterion and investigated the optimal dynamic trading strategy of cointegrated assets with a general correlation structure; and Acosta-González, Armas-Herrera e Fernández-Rodríguez (2015) focused on the role of stock-picking.

**Definition 1**. *A nonstationary stochastic process $X_t$ is said to be integrated to order 1, or $X_t \sim I(1)$, if the first difference of the time series forms a stationary series denoted by $I(0)$.*

Many stock price series are $I(1)$. Therefore, let $X_{1t}, X_{2t}, \ldots, X_{Kt}$ be a sequence of $I(1)$ time series. If there are nonzero real numbers $\beta_1, \beta_2, \ldots, \beta_K$ such that:

$$\beta_1 X_{1t} + \beta_2 X_{2t} + \ldots + \beta_K X_{Kt} \tag{2.1}$$

becomes an $I(0)$ series, then $X_{1t}, X_{2t}, \ldots, X_{Kt}$ are said to be cointegrated. In other words, we call $X_t \sim I(1)$ cointegrated with the $\beta \neq 0$ cointegrating vector if $\beta' X \sim I(0)$, i.e., it can be made stationary. Cointegrated price series present a stationary long-run stable equilibrium relationship with the associated property of mean reversion. Thus, the linear combination of cointegrated price series is stationary and always reverts to the mean of the stationary series.

The regression in Equation 2.1 is suitable for finding cointegrating coefficients for portfolio construction. The most popular cointegration test has been developed by Engle e Granger (1987) and Engle e Yoo (1987), who proposed a two-step estimation method. The first step consists of estimating a long-run equilibrium relationship. The second step consists of estimating a dynamic error-correction relationship using lagged residuals.

Following Alexander e Dimitriu (2005a), we assume a natural logarithmic formulation of the cointegration-based index tracking. In the first step, we consider a linear model as:

$$\log\left(I_t\right) = \beta_0 + \sum_{i=1}^{K} \beta_i \log\left(p_{i,t}\right) + \epsilon_t, \tag{2.2}$$

where $I_t$ and $p_{i,t}$ represent, respectively, the index and the $i^{\text{th}}$ constituent asset price. By normalizing the cointegration coefficients $\beta_i$ (for $i = 1, 2, \ldots, K$) to sum up to one, we determine the proportional weights for each stock.

The second step, in Engle and Granger's two-step procedure, is to test for the unit root in the residual process of the cointegrating regression previously described. By exploiting the definition of $I(1)$ series and given that both log of index and log of prices are $I(1)$ stochastic processes, then if $\beta$ represents a cointegration relationship, $\epsilon \sim I(0)$, the residuals are supposed to be stationary processes. Thus, the loss function is defined as:

$$L(\epsilon_t) = \frac{\widehat{\rho} - 1}{se\left(\widehat{\rho}\right)} \tag{2.3}$$

where $\rho$ is defined as a regression coefficient in the following auxiliary residual regression:

$$\epsilon_t = \alpha + \rho\epsilon_{t-1} + \sum_{i=1}^{d} \gamma \Delta\epsilon_{t-1} + \upsilon_t, \tag{2.4}$$

where $se\left(\widehat{\rho}\right)$ and $d$ are, respectively, the relative standard error and the lag-order. In general, the constant term $\alpha$ can be left out to improve the efficiency of the estimate. The null hypothesis is $H_0 : \rho = 1$, i.e., there is a unit root. Under the null of no cointegration, the estimated residual is $I(1)$ because $I_t$ and $p_{i,t}$ are $I(1)$, and all parameters are zero in the long run, which is the well-known Augmented Dickey Fuller test (ADF). Because lower ADF test statistics result in higher stationarity of residuals (meaning stronger equilibrium relationships between the index and the subset of selected stocks), the aim is to select the vector $\beta$ such that $L(\epsilon_t)$ is minimized.

## 2.2 Optimization using Correlation Method

Index tracking using correlation usually consists of minimizing the variance of the difference between portfolio and index returns (ROLL, 1992). The tracking error (TE) is defined as:

$$TE_t = \sum_{i=1}^{N} w_i r_{it} - R_t. \tag{2.5}$$

where $w_i$ is the weight of asset $i$ in the portfolio, $r_{it}$ is the return of asset $i$ in $t$, and $R_t$ is the index return in $t$. The optimization problem consists of minimizing the variance of TE, $\sigma^2_{TE}$:

$$\sigma^2_{TE} = \frac{1}{T} \sum_{t=1}^{T} \Big[ \sum_{i=1}^{N} w_i r_{it} - R_t \Big]^2. \tag{2.6}$$

Constraints could be imposed on this minimization, such as to define that the sum of all assets' weights in the portfolio should equal one or to define boundaries for minimum and maximum weights for each asset (OH; KIM; MIN, 2005; KRINK; MITTNIK; PATERLINI, 2009; GUASTAROBA; SPERANZA, 2012). In this study, we impose the constraints described in Subsection 3.2.

Alexander e Dimitriu (2005a) pointed out the difference between Equations 2.2 and 2.6: in cointegration, we minimize the spread between portfolio and index prices; in correlation, we minimize the spread between index and portfolio returns. Theoretically, the spread between portfolio and index prices (cointegration) should have mean reversion and low volatility during out-of-sample intervals because the spread should return to zero in the in-sample period on the basis of the definition of ordinary least squares (OLS). In contrast, the correlation approach does not necessarily present a spread with mean reversion because of the use of returns instead of prices.

# 3 Methodology

In this Section, we first describe the conditions for both cointegration (Subsection 3.1) and correlation (Subsection 3.2). Subsequently, in Subsection 3.3, we introduce the approach for EIT tests.

## 3.1 Cointegration Tests

For cointegration, the first step consists of applying the ADF test on the price series of the index and each asset (we use the natural logarithm of daily prices). The test must be performed at the level—when the series most likely will be $I(1)$—and first difference—at which the series most likely will be $I(0)$. If the ADF test reveals that an asset has price series with order different from 1 at a level, then this asset should be excluded from the sample.

Subsequently, for each in-sample interval, the process consists of estimating Equation 2.2 using OLS, with index price series as a dependent variable and asset price series as an independent variable. To estimate each regression, we randomly select 10 assets to compose the portfolio. With those results, we run the ADF test on the residuals (Equation 2.4). If the ADF test finds stationary residuals, we conclude that the index and assets price series are cointegrated and the portfolio obtained can be accepted as a candidate for that in-sample interval. However, if the residuals are non-stationary, the portfolio should be discarded.

As mentioned, asset selection is random for each regression. In our tests, 100,000 regressions were estimated in sequence for each portfolio that referred to the Ibovespa index and 35,000 regressions for the portfolios that referred to the S&P 100. Notice that, although we adopt 57 assets to track the Ibovespa, we use 96 assets to track the S&P 100. Thus, the S&P 100 consumes significantly higher computational memory, which constrains the maximum number of randomly generated regressions to 35,000. Each regression uses a unique combination of 10 assets. For each regression, we estimate Equations 2.2 and 2.4 in sequence; if the cointegration conditions are met, the resulting portfolio is kept as a candidate. Among the 100,000 (or 35,000) regressions tested, all of the candidate portfolios are selected (i.e., all combinations of 10 assets that meet the cointegration requirements described in Subsection 2.1). The chosen one is the portfolio whose regression has the smallest sum of the squared residuals. Given such an approach, we make the portfolio selection endogenous to the problem, which differs from the current literature that does not extensively explore the cointegration method.

## 3.2 Correlation Tests

For correlation tests, the portfolios were optimized by minimizing Equation 2.6. The natural logarithm of assets and index daily returns are calculated. Minimization is performed with the complete database and without needing to generate different combinations of assets. Thus, we do not randomly select 10 assets in advance as we do in cointegration tests. The following constraints are also part of the minimization in the same direction as Sant'Anna et al. (2016):

$$\sum_{i=1}^{N} w_i = 1 \tag{3.1}$$

$$w_i \leq z_i \qquad \forall i = 1, 2, ...N \tag{3.2}$$

$$\sum_{i=1}^{N} z_i \leq C \tag{3.3}$$

$$w_i \geq v_i(-1) \qquad \forall i = 1, 2, ...N \tag{3.4}$$

$$\sum_{i=1}^{N} v_i \leq K \tag{3.5}$$

$$C + K \leq J \tag{3.6}$$

$$z_i \in \{0, 1\} \qquad \forall i = 1, 2, ...N \tag{3.7}$$

$$v_i \in \{0, 1\} \qquad \forall i = 1, 2, ...N \tag{3.8}$$

Constraint 3.1 indicates that 100% of the wealth is invested in the portfolio. Constraints 3.2 to 3.6 are imposed to limit each portfolio to 10 assets. $C$ assets will have long positions in the portfolio, and $K$ assets will have short positions. Constraint 3.6 limits the portfolio size to $C + K \leq J$ because we design portfolios with 10 assets, then $J = 10$. $z_i$ and $v_i$ are binary variables with value 1 if asset $i$ is in the portfolio, and 0 otherwise. As long as we use constraints 3.1 to 3.8 in the correlation tests, cointegration and correlation portfolios are formed with the same boundaries: the sum of the asset weights equals 1 and the long and short positions in each portfolio are accepted.

## 3.3 Enhanced Index Tracking (EIT) Tests

According to Alexander e Dimitriu (2005a), EIT consists of working with enhanced indexes constructed by adding annual excess returns to the original index (uniformly distributed over daily returns). We use enhanced indexes (indexes "plus") with annual excess returns of 5%, 10%, and 15%. Once we have the enhanced indexes, the remaining methodology is exactly the same from IT tests for both correlation and cointegration methods. The only difference is the substitution of the original index by each of the three indexes plus. For instance, in case of portfolios with an annual excess return of 5%, all of the tests are performed by substituting the original index by the artificial enhanced index, which accounts for the index plus 5% annual excess return—the so-called index "plus" 5%. As a result, we expect to obtain portfolios for which an annual excess return of 5% can be added in relation to the original index. For instance, Roll (1992) and Stucchi (2015) argued that indexes are not efficient; consequently, it is possible to form portfolios that seek to outperform an index, which is the goal of EIT strategy.

# 4 Results

In this section, we focus on the empirical results. All tests were implemented using an Intel® Core$^{TM}$ i7-3770 @ 3.40 GHz computer with 8 GB RAM. AMPL and IBM® ILOG CPLEX version 12.6 were used for correlation tests. For cointegration tests, we used the software MATLAB®. The computational runtime remained at shorter than five minutes for both cointegration portfolios (each portfolio resulting from performing 100,000 regressions in sequence in the case of the Ibovespa index, or 35,000 in the case of the S&P 100), as well as each correlation portfolio. We refer to the test described in Appendix A to justify the massive execution of regressions to select each cointegrated portfolio. In Appendix A, we show that the portfolio ranking generated by the cointegration method yields better TE.

For the correlation methodology, the literature in general analyses the gap response for each optimized portfolio (ANGELELLI; MANSINI; SPERANZA, 2012; GUASTAROBA; SPERANZA, 2012). The gap signals the distance that the obtained solution is from the true optimal solution. Commonly, a gap under 5% is satisfactory (FILOMENA; LEJEUNE, 2014). However, because we do not have the gap information for each cointegration response, our choice was to relax the gap discussion. In such a way, we have the same condition for both correlation and cointegration methods, which is the uncertainty related to the true optimal response. Next, we provide a description of the databases and the basic methodology of the tests (Subsection 4.1), followed by the results for both index tracking portfolios (Subsection 4.2) and enhanced indexing (Subsection 4.3).

## 4.1 Database Description and Basic Methodology

The collected databases have daily closing prices adjusted for dividends and splits for both Ibovespa and the S&P 100 from Jan/2004 to Dec/2014. The Brazilian database has the Ibovespa index and 57 assets obtained from Economatica (a financial database widely used in Brazil). The U.S. database has the S&P 100 index and 96 assets downloaded from Bloomberg. For both databases, the indexes were not reconstructed (simulated) only on the basis of the assets that compose each sample, as discussed in Beasley, Meade e Chang (2003). Although changes over time in the composition of the index can strongly affect comovements of index and asset prices, we opted not to rebuild the indexes given these structural changes over time, particularly the Ibovespa.

The in-sample intervals are equal to 1 and 2 years for each optimization. For index tracking portfolios with each of these two intervals, we analyze the out-of-sample results for 1, 2, 3, 6, and 12 months. Thus, for instance, considering a rolling horizon window, for a 1-year in-sample interval and 1-month out-of-sample interval, the first portfolio is selected with data from Jan/2005 to Dec/2005, and is projected for Jan/2006. For a 2-month out-of-sample, the first portfolio's performance is verified during Jan/2006 and Feb/2006, and so on. For a 2-year in-sample, the first portfolio is chosen with data from Jan/2004 to Dec/2005, following the same out-of-sample procedure with projections from Jan/2006 to Dec/2014. All portfolios are limited to 10 assets, and we accept long and short positions. For simplification purposes, we only use a 1-year out-of-sample interval for enhanced indexing portfolios. In summary, on the one hand, considering the IT tests, 108 portfolios were obtained for each combination of method, index and 1-month in-sample time interval; 54 portfolios were obtained if we consider 2-month in-sample interval; 36 portfolios for 3-month in-sample; 18 portfolios for 6-month in-sample, and 9 portfolios for 1-year in-sample interval. On the other hand, regarding the EIT tests, 9 portfolios were selected for each combination of in-sample time interval, method, and index.

## 4.2 Results for Index Tracking

For convenience, we define cointegration portfolios using 1-year in-sample data as C1y.1m (1-month out-of-sample interval), C1y.2m (2 months), C1y.3m (3 months), C1y.6m (6 months), and C1y.1y (1 year). For 2-year in-sample data, the portfolios are C2y.1m, C2y.2m, C2y.3m, C2y.6m, and C2y.1y. Following the same logic, correlation portfolios are R1y.1m, R1y.2m, R1y.3m, R1y.6m, R1y.1y, R2y.1m, R2y.2m, R2y.3m, R2y.6m, and R2y.1y. Tables 7 and 8 present the annual tracking error values for each portfolio, respectively, with Ibovespa and S&P 100 indexes. The annual TE is defined as the difference between each portfolio's cumulative return and the index cumulative return in each year (Equation 2.5), considering each out-of-sample interval. Moreover, descriptive results for IT tests are presented in Tables 9 (Ibovespa) and 10 (S&P 100): Panel A (cointegration and correlation portfolios using 1-year in-sample) and Panel B (portfolios using 2-year in-sample).

First, on the basis of the Brazilian data described in Table 7, we notice good quality portfolio performance in terms of TE—particularly for cointegration portfolios using the 1-year in-sample. For the 1-year in-sample, portfolios have overall performance very close to the index for shorter rebalancing intervals (C1y.1m and C1y.2m) and for some longer intervals (C1y.6m) as it can be inferred from the average annual TE (Table 7) and cumulative return values (Table 9).

Compared with cointegration portfolios, in most cases, correlation portfolios have larger average tracking errors (see Table 7). Only portfolios R1y.3m and R2y.1y have average annual TE under 2%.

However, the same conclusions cannot be made for the S&P 100. On the basis of Table 8, we notice that, in general, the average tracking error for correlation is better than the average tracking error for cointegration. Comparing the cumulative return from Table 10, the results from cointegration and correlation are mixed. For instance, the cumulative return from R1y.1m (correlation) is closer to the index when compared with C1y.1m (cointegration). The opposite is true when R1y.1y (correlation) is compared with C1y.1y (cointegration).

In terms of tracking errors, cointegration is favored for the Brazilian data, whereas the results are mixed for the U.S. data. This mixed outcome for tracking errors is in the same direction as the results presented by Alexander e Dimitriu (2005a). Thus, despite the advantage presented by cointegration for the Brazilian market, it is not possible to make an argument for the best method in terms of tracking errors. A summary of these results is observed in Figure 5, which plots the results for the Ibovespa and the S&P 100 considering the 2-year in-sample with a 3-month out-of-sample portfolio. On the one hand, Figure 5a exhibits for the Ibovespa a better cumulative tracking error obtained with cointegration. On the other hand, Figure 5b presents for the S&P 100 a better cumulative tracking error attained with correlation.

Two other metrics often used to describe tracking quality is the volatility and tracking error volatility of the synthetic portfolio. Unlike as suggested in the literature (STOCK; WATSON, 1988; ALEXANDER; DIMITRIU, 2005a), when comparing portfolios for each method with equal in-sample and out-of-sample intervals, cointegration portfolios present daily and annual volatility slightly larger than correlation portfolios not only for the Ibovespa (Table 9) but also for the S&P 100 (Table 10). The differences are small but very consistent, with larger differences observed for the S&P 100. In terms of TE volatility, our results are in line with Alexander e Dimitriu (2005a), who also obtained smaller TE volatility using correlation. In eight out of 10 combinations, the volatility of the tracking error favored correlations in relation to cointegration for the Ibovespa (see Std Dev of Daily TE on Table 9). For the S&P 100, correlation was significantly better than cointegration in all combinations (see Std Dev of Daily TE on Table 10).

Furthermore, correlation portfolios have monthly turnovers that are significantly smaller than cointegration for all cases in both the Ibovespa and the S&P 100 on the basis of Tables 9 and 10. Thus, correlation resulted in portfolios with slightly lower volatility and higher stability (smaller monthly turnover), meaning lower transaction costs. This result is in line with Alexander e Dimitriu (2005a), who also obtained smaller turnover values for correlation portfolios.

In conclusion, we point out the higher costs associated with keeping closer track of the index in the case of the Brazilian market. For the Ibovespa, a trade-off exists between tracking error and costs. According to our results for the Brazilian market, to demand smaller TE values (that were obtained using cointegration) implies to accept higher costs (reflected by turnover values), whereas accepting larger TE values (obtained using correlation) leads to smaller costs over time. Such a trade-off is not clear for the U.S. market. The outcome for tracking errors for the U.S. market was mixed and did not favor cointegration or correlation. However, correlation presented better turnover performance than cointegration. Thus, on the one hand, for the Brazilian market, the overall results did not favor any method, which follows the conclusions from Alexander e Dimitriu (2005a). On the other hand, for the American market, the results were not conclusive for TE with advantages for correlation on the turnover.

Table 7 – Annual Tracking Error for Indexing Portfolios using Cointegration and Correlation with **Ibovespa index**[1].

| Portfolios | Cointegration | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | C1y.1m | C1y.2m | C1y.3m | C1y.6m | C1y.1y | C2y.1m | C2y.2m | C2y.3m | C2y.6m | C2y.1y |
| TE 2006 | 1.335% | 3.559% | 0.813% | 1.985% | 2.752% | 5.365% | 7.679% | 9.031% | 8.663% | 3.074% |
| TE 2007 | 13.083% | 7.547% | 8.479% | -6.159% | 0.613% | 4.517% | 2.320% | 0.819% | 3.847% | 6.074% |
| TE 2008 | -1.618% | 1.318% | -0.045% | 8.062% | 15.064% | 7.642% | 10.685% | 11.808% | 13.727% | 8.811% |
| TE 2009 | 1.697% | 3.223% | -0.559% | 1.737% | 0.483% | -4.001% | -7.548% | -6.265% | -5.246% | -9.699% |
| TE 2010 | -2.612% | -1.831% | 0.962% | 1.888% | -1.490% | 4.576% | 2.479% | 0.157% | 0.968% | 3.896% |
| TE 2011 | -0.127% | -1.221% | -0.741% | -5.283% | -0.466% | -0.142% | 4.353% | 1.386% | -2.819% | -0.385% |
| TE 2012 | -0.788% | -4.466% | -1.967% | -3.803% | 1.347% | -0.056% | -2.574% | -8.956% | -5.565% | -0.349% |
| TE 2013 | -0.283% | 1.648% | 3.418% | 3.863% | 4.917% | 7.358% | 9.576% | 18.379% | 19.539% | 5.665% |
| TE 2014 | -4.499% | -12.736% | 0.426% | 0.874% | -10.836% | -9.834% | -3.808% | -11.375% | -6.800% | -12.700% |
| Average | 0.688% | -0.329% | 1.199% | 0.351% | 1.376% | 1.714% | 2.574% | 1.665% | 2.924% | 0.487% |

| Portfolios | Correlation | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | R1y.1m | R1y.2m | R1y.3m | R1y.6m | R1y.1y | R2y.1m | R2y.2m | R2y.3m | R2y.6m | R2y.1y |
| TE 2006 | 4.994% | 7.396% | 3.948% | 2.500% | 2.269% | 7.671% | 6.486% | 7.567% | 4.421% | 5.960% |
| TE 2007 | 9.147% | 11.569% | 11.105% | 12.883% | 8.991% | 6.000% | 6.923% | 6.915% | 6.100% | 4.253% |
| TE 2008 | 17.944% | 15.000% | 11.177% | 14.295% | 12.311% | 13.645% | 9.769% | 13.882% | 14.367% | 9.026% |
| TE 2009 | -4.195% | -7.506% | -4.860% | -2.843% | -4.637% | -1.466% | -1.055% | -0.220% | -3.691% | -3.726% |
| TE 2010 | 0.319% | -0.542% | -2.757% | -2.158% | -0.952% | -0.348% | -0.381% | 0.825% | -0.178% | 0.192% |
| TE 2011 | 5.626% | 5.468% | 1.708% | 2.058% | 4.638% | 3.910% | 3.631% | 5.748% | 6.594% | 6.313% |
| TE 2012 | 2.106% | 7.314% | 0.745% | 7.551% | 2.082% | 3.416% | 5.090% | 1.838% | 6.330% | 0.079% |
| TE 2013 | 9.314% | 9.424% | 1.317% | 1.493% | 2.199% | 4.174% | 2.825% | 5.596% | 5.605% | 6.164% |
| TE 2014 | -8.613% | -5.399% | -6.026% | 10.097% | -2.685% | 9.658% | -7.953% | -10.633% | -12.887% | -11.953% |
| Average | 4.071% | 4.747% | 1.817% | 5.097% | 2.691% | 5.184% | 2.815% | 3.502% | 2.962% | 1.812% |

[1] Annual Tracking Error refers to the difference between the cumulative return of each portfolio and the cumulative return of the index in each year. Average Tracking Error refers to the average value of the nine annual errors previously computed.

Table 8 – Annual Tracking Error for Indexing Portfolios using Cointegration and Correlation with **S&P 100 index**[1].

| Portfolios | Cointegration | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | C1y.1m | C1y.2m | C1y.3m | C1y.6m | C1y.1y | C2y.1m | C2y.2m | C2y.3m | C2y.6m | C2y.1y |
| TE 2006 | 15.223% | 2.596% | 10.489% | 9.184% | 1.867% | 3.173% | 5.752% | 2.911% | 2.409% | 0.245% |
| TE 2007 | 2.788% | 3.124% | 3.028% | -8.406% | -1.427% | 7.572% | 4.074% | 6.970% | 3.027% | 9.242% |
| TE 2008 | 9.756% | 11.242% | 17.620% | -6.562% | 6.237% | -1.928% | 6.063% | 6.390% | 2.571% | 14.972% |
| TE 2009 | 4.827% | 19.453% | 14.577% | 7.788% | 7.095% | 5.746% | 8.930% | 9.281% | 11.212% | 19.976% |
| TE 2010 | -1.674% | 1.407% | 5.796% | -0.744% | -4.570% | 4.622% | 6.603% | 5.395% | 8.652% | 3.976% |
| TE 2011 | -1.382% | 1.289% | 0.291% | 5.319% | 2.833% | -11.615% | 6.470% | 4.410% | 4.869% | 8.340% |
| TE 2012 | 4.107% | 5.603% | 2.119% | 3.296% | 2.798% | 4.223% | 6.566% | 3.008% | 0.389% | 1.908% |
| TE 2013 | 4.566% | -0.489% | -0.007% | 4.758% | 5.562% | -3.667% | 0.785% | 0.085% | 4.922% | 4.877% |
| TE 2014 | -7.891% | -7.679% | -2.429% | 0.199% | 3.195% | 9.169% | 3.996% | 3.963% | 5.873% | 5.752% |
| Average | 3.369% | 4.061% | 5.721% | 1.648% | 2.621% | 1.922% | 5.471% | 4.713% | 4.880% | 7.699% |

| Portfolios | Correlation | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | R1y.1m | R1y.2m | R1y.3m | R1y.6m | R1y.1y | R2y.1m | R2y.2m | R2y.3m | R2y.6m | R2y.1y |
| TE 2006 | 4.120% | 2.788% | 3.602% | 0.987% | 1.919% | 5.371% | 1.851% | 1.671% | 3.954% | 2.189% |
| TE 2007 | 9.029% | 7.283% | 8.521% | 6.707% | 4.707% | 5.769% | 3.930% | 5.877% | 5.405% | 3.798% |
| TE 2008 | 1.999% | 0.907% | 10.591% | 2.342% | -2.518% | 6.685% | 8.183% | 0.417% | 4.530% | 3.786% |
| TE 2009 | -1.883% | 2.910% | 1.280% | 3.962% | 3.637% | -3.849% | -1.778% | 3.043% | 2.128% | 0.363% |
| TE 2010 | 0.634% | 3.633% | -0.601% | 0.451% | 2.055% | -2.034% | -0.904% | -1.249% | -0.314% | -1.515% |
| TE 2011 | 6.479% | 6.700% | 9.695% | 6.210% | 8.888% | 2.237% | 3.178% | 2.840% | 4.664% | 6.567% |
| TE 2012 | 2.085% | 3.812% | 5.688% | 4.039% | 2.139% | 5.209% | 4.472% | 4.770% | 1.696% | 1.075% |
| TE 2013 | -1.631% | -1.218% | -1.224% | -0.990% | 2.150% | 0.006% | 0.808% | 0.721% | 1.757% | 0.320% |
| TE 2014 | 3.487% | 7.711% | 0.951% | 3.116% | 3.772% | -3.856% | -4.301% | -4.126% | 0.690% | 1.093% |
| Average | 2.702% | 3.836% | 4.278% | 2.981% | 2.972% | 1.727% | 1.715% | 1.551% | 2.723% | 1.964% |

[1] Annual Tracking Error refers to the difference between the cumulative return of each portfolio and the cumulative return of the index in each year. Average Tracking Error refers to the average value of the nine annual errors previously computed.

Table 9 – Results for Index Tracking Portfolios using Cointegration and Correlation with **Ibovespa index**[1].

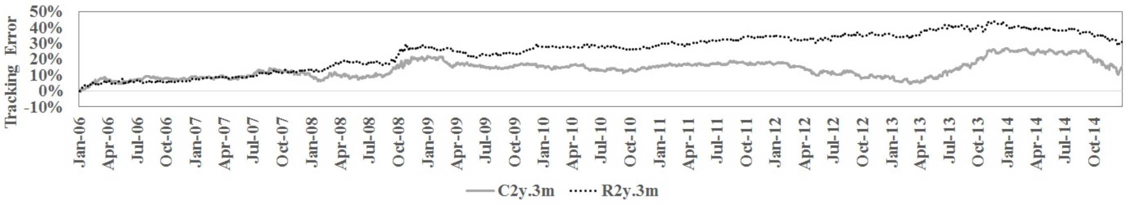
| *Panel A: Calibration period: 1 year* | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Portfolios | IBOVESPA | C1y.1m | C1y.2m | C1y.3m | C1y.6m | C1y.1y | R1y.1m | R1y.2m | R1y.3m | R1y.6m | R1y.1y |
| Daily Average Return | 0.02% | 0.02% | 0.02% | 0.02% | 0.02% | 0.02% | 0.03% | 0.04% | 0.03% | 0.04% | 0.03% |
| Cumulative Return | 40.20% | 44.75% | 35.60% | 49.34% | 41.72% | 50.94% | 76.96% | 83.05% | 56.68% | 86.20% | 64.54% |
| Daily Volatility | 1.84% | 1.91% | 1.91% | 1.92% | 1.88% | 1.88% | 1.86% | 1.85% | 1.85% | 1.83% | 1.86% |
| Annual Volatility | 29.14% | 30.30% | 30.32% | 30.47% | 29.87% | 29.85% | 29.55% | 29.31% | 29.31% | 29.09% | 29.59% |
| Daily Average TE | – | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.02% | 0.02% | 0.01% | 0.02% | 0.01% |
| Std Dev of Daily TE | – | 0.44% | 0.45% | 0.44% | 0.47% | 0.49% | 0.35% | 0.35% | 0.37% | 0.54% | 0.45% |
| Asymmetry | -0.010 | -0.027 | 0.037 | -0.021 | -0.002 | 0.029 | 0.111 | 0.058 | 0.114 | 0.450 | 0.105 |
| Kurtosis | 5.958 | 5.404 | 4.802 | 5.477 | 4.482 | 4.749 | 5.278 | 4.888 | 4.982 | 7.197 | 4.906 |
| Correlation | – | 0.973 | 0.972 | 0.974 | 0.968 | 0.965 | 0.982 | 0.982 | 0.980 | 0.957 | 0.970 |
| Beta | – | 1.012 | 1.012 | 1.018 | 0.993 | 0.989 | 0.996 | 0.988 | 0.986 | 0.955 | 0.985 |
| Monthly Average Turnover | – | 64.51% | 34.13% | 22.94% | 12.25% | 5.91% | 20.49% | 14.34% | 10.88% | 6.62% | 4.16% |
| *Panel B: Calibration period: 2 years* | | | | | | | | | | | |
| Portfolios | IBOVESPA | C2y.1m | C2y.2m | C2y.3m | C2y.6m | C2y.1y | R2y.1m | R2y.2m | R2y.3m | R2y.6m | R2y.1y |
| Daily Average Return | 0.02% | 0.03% | 0.03% | 0.02% | 0.03% | 0.02% | 0.04% | 0.03% | 0.03% | 0.03% | 0.02% |
| Cumulative Return | 40.20% | 55.80% | 63.54% | 55.36% | 66.69% | 44.76% | 85.83% | 64.51% | 70.69% | 65.83% | 55.48% |
| Daily Volatility | 1.84% | 1.92% | 1.90% | 1.93% | 1.95% | 1.90% | 1.90% | 1.84% | 1.84% | 1.84% | 1.84% |
| Annual Volatility | 29.14% | 30.42% | 30.12% | 30.64% | 30.92% | 30.21% | 30.13% | 29.19% | 29.13% | 29.24% | 29.22% |
| Daily Average TE | – | 0.01% | 0.01% | 0.01% | 0.01% | 0.00% | 0.02% | 0.01% | 0.01% | 0.01% | 0.01% |
| Std Dev of Daily TE | – | 0.45% | 0.46% | 0.47% | 0.49% | 0.45% | 0.61% | 0.37% | 0.38% | 0.40% | 0.41% |
| Asymmetry | -0.010 | -0.016 | 0.056 | -0.018 | 0.020 | 0.042 | 0.736 | -0.049 | -0.081 | 0.013 | 0.017 |
| Kurtosis | 5.958 | 4.711 | 4.217 | 4.439 | 4.445 | 4.036 | 13.404 | 4.993 | 4.594 | 4.982 | 4.867 |
| Correlation | – | 0.972 | 0.970 | 0.970 | 0.969 | 0.971 | 0.947 | 0.980 | 0.979 | 0.977 | 0.976 |
| Beta | – | 1.015 | 1.003 | 1.020 | 1.028 | 1.007 | 0.980 | 0.982 | 0.979 | 0.980 | 0.979 |
| Monthly Average Turnover | – | 59.55% | 31.80% | 22.23% | 10.96% | 5.35% | 14.31% | 9.64% | 7.32% | 5.49% | 2.95% |

[1] Daily Average Return accounts for the average of the daily returns during the entire out-of-sample interval, from 2006 to 2014. Cumulative Return accounts for the cumulative returns from 2006 to 2014. Daily Volatility accounts for the standard deviation ($\sigma$) of the daily returns from 2006 to 2014, whereas annual volatility refers to $\sigma \times \sqrt{252}$. Daily Average TE and Std Dev of Daily TE account for the average and standard deviation of the daily tracking error from 2006 to 2014. Asymmetry, Kurtosis, Correlation, and Beta values were computed on the basis of the daily returns from 2006 to 2014 for each portfolio and index.

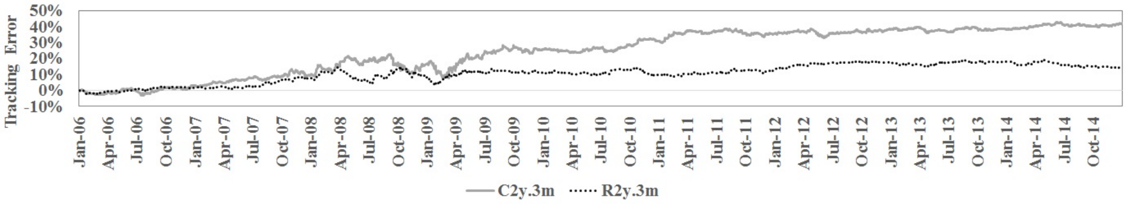
Table 10 – Results for Index Tracking Portfolios using Cointegration and Correlation with **S&P 100 index**[1].

| *Panel A: Calibration period: 1 year* | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Portfolios | S&P 100 | C1y.1m | C1y.2m | C1y.3m | C1y.6m | C1y.1y | R1y.1m | R1y.2m | R1y.3m | R1y.6m | R1y.1y |
| Daily Average Return | 0.02% | 0.03% | 0.04% | 0.04% | 0.03% | 0.03% | 0.03% | 0.04% | 0.04% | 0.03% | 0.03% |
| Cumulative Return | 46.60% | 76.92% | 83.15% | 98.09% | 61.44% | 70.19% | 70.92% | 81.13% | 85.11% | 73.43% | 73.35% |
| Daily Volatility | 1.29% | 1.47% | 1.46% | 1.44% | 1.43% | 1.43% | 1.35% | 1.34% | 1.35% | 1.34% | 1.35% |
| Annual Volatility | 20.53% | 23.39% | 23.17% | 22.89% | 22.72% | 22.65% | 21.41% | 21.23% | 21.37% | 21.22% | 21.50% |
| Daily Average TE | – | 0.01% | 0.02% | 0.02% | 0.01% | 0.01% | 0.01% | 0.02% | 0.02% | 0.01% | 0.01% |
| Std Dev of Daily TE | – | 0.43% | 0.43% | 0.41% | 0.42% | 0.38% | 0.29% | 0.29% | 0.28% | 0.29% | 0.30% |
| Asymmetry | -0.281 | -0.188 | -0.241 | 0.003 | -0.550 | -0.244 | -0.064 | -0.322 | 0.023 | -0.170 | -0.324 |
| Kurtosis | 10.645 | 11.393 | 10.875 | 10.587 | 10.079 | 10.583 | 13.451 | 10.329 | 13.569 | 11.271 | 11.492 |
| Correlation | – | 0.959 | 0.958 | 0.960 | 0.958 | 0.966 | 0.977 | 0.977 | 0.978 | 0.977 | 0.975 |
| Beta | – | 1.092 | 1.082 | 1.070 | 1.060 | 1.065 | 1.019 | 1.010 | 1.018 | 1.010 | 1.021 |
| Monthly Average Turnover | – | 79.65% | 39.60% | 27.66% | 15.03% | 7.78% | 40.73% | 23.69% | 17.74% | 10.26% | 5.98% |
| *Panel B: Calibration period: 2 years* | | | | | | | | | | | |
| Portfolios | S&P 100 | C2y.1m | C2y.2m | C2y.3m | C2y.6m | C2y.1y | R2y.1m | R2y.2m | R2y.3m | R2y.6m | R2y.1y |
| Daily Average Return | 0.02% | 0.03% | 0.04% | 0.04% | 0.04% | 0.05% | 0.03% | 0.03% | 0.03% | 0.03% | 0.03% |
| Cumulative Return | 46.60% | 63.90% | 95.84% | 89.02% | 90.53% | 115.89% | 62.14% | 62.04% | 60.57% | 71.11% | 64.28% |
| Daily Volatility | 1.29% | 1.53% | 1.54% | 1.51% | 1.52% | 1.52% | 1.33% | 1.32% | 1.33% | 1.35% | 1.35% |
| Annual Volatility | 20.53% | 24.25% | 24.52% | 24.03% | 24.20% | 24.17% | 21.06% | 20.96% | 21.14% | 21.35% | 21.36% |
| Daily Average TE | – | 0.01% | 0.02% | 0.02% | 0.02% | 0.03% | 0.01% | 0.01% | 0.01% | 0.01% | 0.01% |
| Std Dev of Daily TE | – | 0.47% | 0.50% | 0.47% | 0.50% | 0.51% | 0.28% | 0.28% | 0.29% | 0.29% | 0.31% |
| Asymmetry | -0.281 | -0.115 | -0.062 | -0.043 | 0.019 | -0.159 | -0.209 | -0.091 | -0.171 | -0.288 | -0.405 |
| Kurtosis | 10.645 | 11.050 | 10.570 | 11.199 | 12.730 | 8.800 | 10.147 | 10.400 | 10.258 | 10.904 | 10.564 |
| Correlation | – | 0.957 | 0.952 | 0.955 | 0.951 | 0.948 | 0.977 | 0.978 | 0.976 | 0.977 | 0.973 |
| Beta | – | 1.130 | 1.137 | 1.118 | 1.121 | 1.116 | 1.002 | 0.998 | 1.005 | 1.016 | 1.012 |
| Monthly Average Turnover | – | 76.05% | 38.17% | 27.92% | 13.86% | 7.45% | 26.23% | 16.52% | 11.82% | 7.20% | 5.06% |

[1] Daily Average Return accounts for the average of daily returns during the entire out-of-sample interval from 2006 to 2014. Cumulative Return accounts for the cumulative returns from 2006 to 2014. Daily Volatility accounts for the standard deviation ($\sigma$) of daily returns from 2006 to 2014, whereas Annual Volatility refers to $\sigma \times \sqrt{252}$. Daily Average TE and Std Dev of Daily TE account for the average and standard deviations of the daily tracking errors from 2006 to 2014. Asymmetry, Kurtosis, Correlation, and Beta values were computed on the basis of daily returns from 2006 to 2014 for each portfolio and index.




(a) Tracking error time series for portfolios C2y.3m and R2y.3m using **Ibovespa**.




(b) Tracking error time series for portfolios C2y.3m and R2y.3m using **S&P 100**.

Figure 5 – Tracking error time series for portfolios C2y.3m and R2y.3m using the Ibovespa index and the S&P 100 index.

## 4.3 Results for Enhanced Index Tracking

As previously mentioned, for the enhanced indexing strategy (EIT), we performed the tests using only one year as the out-of-sample interval (in-sample periods remain at one and two years). The indexes "plus" were constructed by adding 5%, 10%, and 15% annual excess returns to each index. For convenience, we defined the portfolios as C1y.1y+5%, C1y.1y+10%, C1y.1y+15%, C2y.1y+5%, C2y.1y+10%, C2y.1y+15%, R1y.1y+5%, R1y.1y+10%, R1y.1y+15%, R2y.1y+5%, R2y.1y+10%, and R2y.1y+15%.

Tables 11 (Ibovespa) and 12 (S&P 100) present the results for the EIT portfolios. First, we again observe good responses in the cointegration portfolios for the 1-year in-sample in the Brazilian market, similar to the index tracking portfolios. In this case, we observe annual average returns increasing constantly as we improve enhancements to the index. Nonetheless, although overall performance improves through the enhancing strategy, annual volatility remains stable; for instance, the annual volatility for portfolio C1y.1y is 29.85%, whereas the annual volatility for portfolio C1y.1y+15% is 29.19%. The increasing Information Ratio[1] from 0.018 for portfolio C1y.1y+5% to 0.049 for portfolio C1y.1y+15% shows that value is being added to the portfolio over time and volatility remains stable.

For the American market (Table 12), cointegrated portfolios with a 1-year in-sample also deliver consistent results, such as for the Brazilian market. We notice larger cumulative returns for the enhanced portfolios in relation to the portfolio C1y.1y, as well as stable annual volatility (except for portfolio C1y.1y+10%). Monthly turnover values also remain stable, particularly in the U.S. markets. In conclusion, cointegration portfolios with a 1-year in-sample clearly exhibited good performance for enhanced index tracking.

[1] According to Wu et al. (2007), an important measure for EIT portfolios, is the Information Ratio (IR)—TE daily average of a portfolio divided by its TE volatility. Therefore, we computed the IR (annual average) for each portfolio and the IR for all of 2006–2014. The results are presented in Tables 11 and 12. A positive IR shows that the portfolio incorporates excess returns over time. Consequently, the increase in each portfolio's volatility should be compensated for by its excess return.

The same conclusion cannot be reached for correlation portfolios with a 1-year in-sample. Using Ibovespa, portfolios R1y.1y+5%, R1y.1y+10%, and R1y.1y+15% presented cumulative returns inferior to portfolio R1y.1y, as well as inferior IR. Using the S&P 100, although portfolios R1y.1y+5%, R1y.1y+10%, and R1y.1y+15% have better performance than R1y.1y, we see that the cumulative return decreases as we increase the index enhancement from 5% to 15%. Such results show that, relative to the correlation approach, cointegration tends to be a more resilient approach for enhanced portfolios that seek to add value consistently over time, at least if we consider using the 1-year in-sample.

For cointegration portfolios with a 2-year in-sample in the Brazilian market, the portfolios also present increasing cumulative returns; however, different from the 1-year in-sample, annual volatility is also larger as we increase the index enhancement from 5% to 15%. Consequently, we do not note larger IR values; for instance, portfolios C2y.1y+5% and C2y.1y+15% have similar IR (annual average): 0.015 and 0.016. In conclusion, although returns are increasing for enhanced portfolios using cointegration and a 2-year in-sample, portfolio volatility is also increasing, which is different from the results for cointegration and a 1-year in-sample. In the U.S. market, cointegration with a 2-year in-sample did not result in good responses. Cumulative returns diminish as we increase the index from 5% to 15%, and IR strongly decreases. Therefore, we notice again good responses for cointegration in the Brazilian market for the 2-year in-sample data interval. The results are inconsistent for the U.S. market.

The conclusion changes if we consider correlation portfolios with a 2-year in-sample in relation to cointegration portfolios. For the Brazilian market, portfolios R2y.1y+5%, R2y.1y+10%, and R2y.1y+15% have results that are similar to those of C1y.1y+5%, C1y.1y+10%, and C1y.1y+15%: the cumulative return robustly increases as we change from index plus 5% to 15%, whereas volatility actually decreases. As the IR (2006–2014) goes from 0.021 to 0.044, we observe that the responses obtained are of good quality because the enhanced portfolios are incorporating value without a corresponding increase in portfolio volatility over time.

The conclusions are in a similar direction for correlation portfolios with a 2-year in-sample and U.S. data. Enhanced portfolios have larger returns than the portfolio R2y.1y, whereas volatility substantially decreases; in fact, portfolio R2y.1y+15% has annual volatility that is smaller than that of portfolio R2y.1y. IR rates confirm the good quality of the correlated portfolios using a 2-year in-sample, as was the case in the Brazilian market. Additionally, given Brazilian and U.S. data, correlation portfolios using a 2-year in-sample have reduced monthly average turnovers in relation to cointegrated portfolios with a 2-year in-sample. This finding confirms that correlation best fits the EIT problem if we consider 2-year intervals for the optimization data.

Overall, conclusions cannot be drawn in favor of any of the methods with respect to the EIT problem. The cointegration approach resulted in good responses using a 1-year in-sample to optimize the portfolios, whereas correlation resulted in better outcomes using a 2-year in-sample. Different from the index tracking results, which allowed us to slightly distinguish the approaches in terms of tracking error and volatility, the responses for enhanced index tracking clearly did not benefit from any method.

Table 11 – Results for Enhanced Indexing Portfolios using Cointegration and Correlation with **Ibovespa index**[1].

| Portfolios | IBOVESPA | Cointegration | | | |
|---|---|---|---|---|---|
| | | C1y.1y | C1y.1y+5% | C1y.1y+10% | C1y.1y+15% |
| Annual Average Return | 4.47% | 5.66% | 6.80% | 9.48% | 10.63% |
| Cumulative Return | 40.20% | 50.94% | 61.19% | 85.34% | 95.71% |
| Annual Volatility | 29.14% | 29.85% | 29.82% | 30.43% | 29.19% |
| Sharpe Ratio | – | 0.484 | 0.832 | 1.580 | 1.910 |
| Monthly Average Turnover | – | 5.91% | 6.67% | 7.16% | 6.84% |
| IR (Annual Average) | – | 0.007 | 0.021 | 0.036 | 0.047 |
| IR (2006-2014) | – | 0.010 | 0.018 | 0.037 | 0.049 |
| | | C2y.1y | C2y.1y+5% | C2y.1y+10% | C2y.1y+15% |
| Annual Average Return | 4.47% | 4.97% | 7.41% | 7.52% | 8.57% |
| Cumulative Return | 40.20% | 44.76% | 66.68% | 67.67% | 77.14% |
| Annual Volatility | 29.14% | 30.21% | 30.16% | 31.34% | 31.64% |
| Sharpe Ratio | – | 0.228 | 1.049 | 1.065 | 1.101 |
| Monthly Average Turnover | – | 5.35% | 7.15% | 9.00% | 10.63% |
| IR (Annual Average) | – | 0.002 | 0.015 | 0.026 | 0.016 |
| IR (2006-2014) | – | 0.005 | 0.021 | 0.018 | 0.020 |

| Portfolios | IBOVESPA | Correlation | | | |
|---|---|---|---|---|---|
| | | R1y.1y | R1y.1y+5% | R1y.1y+10% | R1y.1y+15% |
| Annual Average Return | 4.47% | 7.17% | 6.82% | 6.64% | 6.97% |
| Cumulative Return | 40.20% | 64.54% | 61.41% | 59.72% | 62.75% |
| Annual Volatility | 29.14% | 29.59% | 29.18% | 28.94% | 29.52% |
| Sharpe Ratio | – | 1.043 | 0.921 | 0.824 | 1.004 |
| Monthly Average Turnover | – | 4.16% | 5.01% | 6.11% | 6.66% |
| IR (Annual Average) | – | 0.027 | 0.019 | 0.013 | 0.014 |
| IR (2006-2014) | – | 0.024 | 0.023 | 0.019 | 0.019 |
| | | R2y.1y | R2y.1y+5% | R2y.1y+10% | R2y.1y+15% |
| Annual Average Return | 4.47% | 6.16% | 7.17% | 8.01% | 9.46% |
| Cumulative Return | 40.20% | 55.48% | 64.56% | 72.08% | 85.13% |
| Annual Volatility | 29.14% | 29.22% | 29.54% | 28.90% | 28.82% |
| Sharpe Ratio | – | 0.690 | 1.126 | 1.334 | 1.596 |
| Monthly Average Turnover | – | 2.95% | 3.34% | 3.99% | 4.52% |
| IR (Annual Average) | – | 0.017 | 0.030 | 0.033 | 0.043 |
| IR (2006-2014) | – | 0.017 | 0.021 | 0.033 | 0.044 |

[1] Annual Average Return refers to the average of the Annual Returns (cumulative return in each year from 2006 to 2014). Cumulative Return refers to the cumulative returns from 2006 to 2014. Annual Volatility refers to $\sigma \times \sqrt{252}$ ($\sigma$ refers to the standard deviation of daily returns from 2006 to 2014). The Sharpe Ratio of each portfolio refers to the difference between the portfolio's cumulative return and the index's cumulative return from 2006 to 2014, divided by the standard deviation of the portfolio's daily returns from 2006 to 2014. The Annual Information Ratio for each portfolio refers to the average of the daily tracking errors in each year divided by the standard deviation of the daily tracking errors in each year. IR (Annual Average) refers to the average of the Annual IR values. IR (2006–2014) refers to the IR computed using the daily tracking errors from 2006 to 2014.

Table 12 – Results for Enhanced Indexing Portfolios using Cointegration and Correlation with **S&P 100 index**[1].

| Portfolios | S&P 100 | Cointegration | | | |
|---|---|---|---|---|---|
| | | C1y.1y | C1y.1y+5% | C1y.1y+10% | C1y.1y+15% |
| Annual Average Return | 5.18% | 7.80% | 8.85% | 8.57% | 9.20% |
| Cumulative Return | 46.60% | 70.19% | 79.61% | 77.14% | 82.82% |
| Annual Volatility | 20.53% | 22.65% | 22.44% | 24.25% | 22.78% |
| Sharpe Ratio | – | 1.047 | 1.522 | 1.350 | 1.393 |
| Monthly Average Turnover | – | 7.78% | 7.17% | 7.90% | 7.99% |
| IR (Annual Average) | – | 0.028 | 0.028 | 0.029 | 0.038 |
| IR (2006-2014) | – | 0.027 | 0.036 | 0.026 | 0.035 |
| | S&P 100 | C2y.1y | C2y.1y+5% | C2y.1y+10% | C2y.1y+15% |
| Annual Average Return | 5.18% | 12.88% | 10.02% | 9.33% | 8.66% |
| Cumulative Return | 46.60% | 115.89% | 90.18% | 83.95% | 77.97% |
| Annual Volatility | 20.53% | 24.17% | 24.16% | 23.06% | 26.25% |
| Sharpe Ratio | – | 1.944 | 1.576 | 1.341 | 1.088 |
| Monthly Average Turnover | – | 7.45% | 7.91% | 10.45% | 14.96% |
| IR (Annual Average) | – | 0.064 | 0.055 | 0.040 | 0.019 |
| IR (2006-2014) | – | 0.060 | 0.037 | 0.029 | 0.016 |

| Portfolios | S&P 100 | Correlation | | | |
|---|---|---|---|---|---|
| | | R1y.1y | R1y.1y+5% | R1y.1y+10% | R1y.1y+15% |
| Annual Average Return | 5.18% | 8.15% | 9.34% | 9.03% | 8.61% |
| Cumulative Return | 46.60% | 73.35% | 84.08% | 81.24% | 77.49% |
| Annual Volatility | 20.53% | 21.50% | 20.96% | 21.28% | 21.69% |
| Sharpe Ratio | – | 0.751 | 1.490 | 1.372 | 1.180 |
| Monthly Average Turnover | – | 5.98% | 6.34% | 7.89% | 10.01% |
| IR (Annual Average) | – | 0.050 | 0.055 | 0.040 | 0.031 |
| IR (2006-2014) | – | 0.039 | 0.054 | 0.036 | 0.023 |
| | S&P 100 | R2y.1y | R2y.1y+5% | R2y.1y+10% | R2y.1y+15% |
| Annual Average Return | 5.18% | 7.14% | 8.90% | 12.09% | 10.25% |
| Cumulative Return | 46.60% | 64.28% | 80.11% | 108.83% | 92.24% |
| Annual Volatility | 20.53% | 21.36% | 20.73% | 20.86% | 20.12% |
| Sharpe Ratio | – | 0.791 | 1.425 | 2.099 | 1.835 |
| Monthly Average Turnover | – | 5.06% | 4.86% | 6.02% | 6.29% |
| IR (Annual Average) | – | 0.027 | 0.048 | 0.077 | 0.047 |
| IR (2006-2014) | – | 0.025 | 0.047 | 0.079 | 0.051 |

[1] Annual Average Return refers to the average of Annual Returns (cumulative return in each year from 2006 to 2014). Cumulative Return refers to the cumulative returns from 2006 to 2014. Annual Volatility refers to $\sigma \times \sqrt{252}$ ($\sigma$ refers to the standard deviation of daily returns from 2006 to 2014). The Sharpe Ratio of each portfolio refers to the difference between the portfolio's cumulative return and the index's cumulative return from 2006 to 2014, divided by the standard deviation of a portfolio's daily returns from 2006 to 2014. The Annual Information Ratio for each portfolio refers to the average of the daily tracking errors in each year divided by the standard deviation of the daily tracking errors in each year; IR (Annual Average) refers to the average of Annual IR values; IR (2006–2014) refers to the IR computed using daily tracking errors from 2006 to 2014.

# 5 Conclusions

Considerable evidence shows that superior returns to investment performance are elusive and that, on average, professional investment managers fail to outperform passive benchmarks. In turn, methods of optimally tracking a benchmark, especially when full replication of the benchmark is not desired or practical, have received attention from both academics and practitioners. The portfolio optimization problem for index tracking (IT) and enhanced indexing (EIT) has been solved using a number of approaches in the recent literature, and two are highlighted: cointegration and correlation. This study focused on comparing both approaches, in line with previous comparisons made by Alexander e Dimitriu (2005a). To enforce the comparison, we expanded the cointegration technique by generating portfolios using simulations, a technique that made the portfolio selection endogenous to the optimization process. We also used a long data sample interval (2004–2014) and databases from two markets (Brazil—Ibovespa index, and U.S.—S&P 100 index). Furthermore, we formed reduced portfolios (limited to 10 assets) in relation to the sample sizes. Such choices were made to give robustness to the findings.

The tests for IT revealed no significant difference between the results of both methods. For the Brazilian market, cointegration responses had good quality in terms of tracking error performance; however, in opposition to what would be expected from the theoretical literature, correlation portfolios presented lower volatility than cointegration portfolios. Furthermore, correlation portfolios generally had lower monthly turnover values, which represents reduced transaction costs. Thereby, a trade-off between better performance (cointegration) and diminished costs (correlation) was noticed. Regarding the U.S. market, such a trade-off between performance and costs was not clear, and both approaches had portfolios with similar performance in terms of TE. However, correlation presented advantages in turnover and volatility. Overall, the mixed results are in line with previous findings from Alexander e Dimitriu (2005a).

The same conclusions can be drawn for EIT tests. Cointegration approach results stood out for in-sample data with 1-year intervals, whereas correlation outcomes presented good quality for in-sample data with 2-year intervals. In conclusion, regardless of extending the cointegration testing methodology, we could not differentiate the performance of portfolios formed using each of the two methods.

Future research could be designed with specific changes in portfolio constraints, such as barring short positions from portfolios. Moreover, other tests could be performed using different markets and particular periods (strong bull or bear markets) as an experiment to delimit the eventual qualities of the methods in specific market conditions.

# Part III

# Investigating the Use of Statistical Process Control Charts for Index Tracking Portfolios

# Abstract

In this article, our goal is to introduce a statistical process control charts approach (SPC) to monitor the rebalancing process of index tracking (IT) portfolios. SPC methods derive from statistics and engineering as tools to control production process. We use exponentially weighted moving average (EWMA) control charts to monitor IT portfolios based on two combined charts: portfolios' tracking error performance and portfolios' volatility. As a result, we endogenously control the rebalancing process of the portfolios based on their performance and on their risk conditions over time. Computational tests are performed to evaluate the developed approach in comparison with the traditional fixed period strategy, using data from Brazilian and U.S. market from 2005 to 2014. Cointegration and correlation methods are applied to form the portfolios. The results show that SPC approach can be a viable alternative to portfolio rebalancing.

**Keywords:** EWMA control chart, index track, cointegration, correlation.

**Note:** this article has already been submitted and is currently under review.

# 1 Introduction

Index tracking (IT) is a passive investment strategy that seeks to mimic the performance of a market index. This strategy can be used to form an ETF (Exchange-Traded Fund – a security that represents an index fund) or to reproduce a market indicator such as inflation (rather than following a market index). The decision of investing in such a passive investment fund is based on the efficient market hypothesis proposed by Fama (1970), who claims that investment funds tend to underperform the market in the long run, which implies that investors would be better off following the market.

When forming an index fund, the first choice would be to select the index to be tracked. Next, it would be to pick up the assets and their respective weights in the portfolio. Finally, it would be to define the portfolio rebalancing strategy. The second step has already been dealt by several IT optimization models using different approaches such as optimization (KONNO; WIJAYANAYAKE, 2001; MEZALI; BEASLEY, 2013), simulation and optimization (CONSIGLIO; ZENIOS, 2001), heuristics (BEASLEY; MEADE; CHANG, 2003; GUASTAROBA; SPERANZA, 2012; SCOZZARI et al., 2013), cointegration (ALEXANDER; DIMITRIU, 2005a), and quadratic programming (COLEMAN; LI; HENNIGER, 2006). The aim of these models is to minimize the tracking error (TE), i.e. the difference between portfolio and index returns. Notice that stock market indexes, for instance, usually have hundreds of assets and a full replication of the index would strongly increase transaction costs. Thus, in order to reduce the number of assets and decrease transaction and management costs, IT models usually control the maximum size of each portfolio. However, the third step (the portfolio rebalancing strategy) has received little attention from the literature. So, as an attempt to shed new light on the rebalancing strategy issue, this paper focus its attention on the introduction of a new approach to regulate IT portfolios over time, based on the portfolio performance in terms of tracking error as well as in terms of the portfolio and market volatilities.

Regardless of the diversity of the applied optimization models and techniques for IT portfolios, one similar characteristic stands out: the use of fixed time windows to portfolio rebalancing. For instance, Alexander e Dimitriu (2005a) used four specific intervals: 2-week, monthly, quarterly, and semi-annual rebalancing, while Canakgoz e Beasley (2008) and Krink, Mittnik e Paterlini (2009) considered monthly rebalancing. An alternative approach to the use of the traditional fixed rebalancing intervals is the application of control charts to monitor portfolio updates over time. The idea behind the use of control charts is the attempt to maintain portfolios inside preset limits; once the portfolio is out of the boundaries, the optimization model is reassessed and the portfolio is re-estimated. Such method should make the portfolio rebalancing strategy dynamic over time, favoring two issues as follows: (i) to permit the manager to quickly update the portfolio, if sudden market movements make the portfolio deviate from its objective, and (ii) to allow the portfolio to remain unchanged if its performance is in accordance to the expected, thus avoiding unnecessary rebalancing activities and diminishing transaction and management costs.

Statistical process control (SPC) was originated in statistics and engineering and has as main characteristic the application of statistical methods to control industrial processes. One of the tools derived from SPC is the use of control charts (WHEELER; CHAMBERS, 1992; MONTGOMERY, 1996) – graphs employed to regulate processes based on upper and lower boundaries. The idea supporting control charts consists in (i) to define a statistical value correspondent to the process to be regulated, and (ii) to set upper and lower control limits (UCL, LCL) as boundaries for the statistical value time series. After the process is started, the statistical value is cumulatively computed over time and, once it falls outside the control limits, the process must be stopped for corrections and then restarted. In the past decades, the use of this instrument has been expanded (LUCAS; CROSIER, 1982; WOODALL, 1986; MONTGOMERY, 1996; HAWKINS; OLWELL, 1998; KOEHLER; MARKS; O'CONNELL, 2001; ROGERSON, 2006; HUANG, 2014) and many sorts of control charts have been proposed for different environments. For economy's sake we refer to the aforementioned references for details on control charts types.

More recently, some studies have used control charts in portfolio optimization problems (GOLOSNOY; SCHMID, 2007; GOLOSNOY, 2007; GOLOSNOY; RAGULIN; SCHMID, 2011). The most used control chart techniques (GOLOSNOY; RAGULIN; SCHMID, 2011) are the exponentially weighted moving average (EWMA) (LUCAS; SACCUCCI, 1990) and the cumulated sum (CUMSUM) (LUCAS; CROSIER, 1982). Nevertheless, such studies were designed for the use of control charts with minimum variance portfolios, not for IT strategy. Consequently, the SPC methods in those articles were based on the covariance matrix of stock returns and can not be directly adapted to IT, since the covariance matrix is not a central information for IT optimization. As a result, a new adaptation of SPC becomes necessary to use control charts with IT portfolios. The literature on control charts has already shown similar performance between CUSUM and EWMA methods for general implementations (LUCAS; CROSIER, 1982; GOLOSNOY; RAGULIN; SCHMID, 2011). Since there is a grater simplicity in the use of EWMA charts, we chose to explore this type of control chart in our study. Further, as pointed out by Golosnoy, Ragulin e Schmid (2011), EWMA does not require a specification of the out-of-control state of the process for its implementation, while CUSUM charts require to know the expected shifts for optimality.

In this article, our goal is to employ an approach based on EWMA control charts to regulate IT portfolios. Thereby, we expect to obtain portfolios controlled by control charts with performance similar to portfolios based on fixed rebalancing intervals. The use of fixed rebalancing time windows relies solely on the measure of tracking error (TE). In contrast, we design the control chart approach to update portfolios based on two measures: TE values, and the index volatility. Thus, the portfolio rebalancing process will become dynamic, as it will adapt to different market conditions over time. For instance, during periods in which market volatility is larger than usual, we assume IT portfolios could accept larger TE values, as long as the volatility of the portfolio is inside predefined control limits (which implies that the risk level of the portfolio is in line with the market risk). Although the natural purpose of IT portfolios is to emulate an index, some trading operations to rebalance the portfolio could be avoided if the portfolio is not submitted to increasing risk (relative to the market). Consequently, the use of control charts might favor lower transaction costs associated to the portfolio allocation for some market conditions. As a result, we conjecture SPC portfolios (portfolios regulated by control charts) may present more frequent updates (relative to portfolios with fixed rebalancing windows) during intervals when the stock market has larger volatility, in contrast with fewer portfolio updates when the market is facing increasing stability.

To evaluate the developed approach in comparison with the traditional fixed time-window rebalancing strategy, we performed extensive computational testing through the use of different market conditions, sample sizes, and applying two optimization methods to form portfolios, seeking to assure robustness to our results. The experiments were carried out using data from two markets, Brazil (Ibovespa index and 67 stocks) and USA (S&P 100 index and 96 stocks), that considerably differ in terms of market volatility over time. We used long data sample intervals (from 2005 to 2014), in order to cover diverse market conditions, with strong crisis intervals, periods with larger stability, and so on. Further, we applied two well-known IT protfolio optimization methods, namely cointegration (ALEXANDER, 1999; ALEXANDER; DIMITRIU, 2005a) and correlation (BEASLEY; MEADE; CHANG, 2003; GAIVORONSKI; KRYLOV; WIJST, 2005). We also experimented with small size portfolios (limited to 15 assets), as well as larger portfolios (with at most 30 assets).

Overall, our results present SPC portfolios with performance similar to portfolios with fixed windows when we observe cumulative returns, tracking error and volatility over time for each portfolio. On the one hand, SPC portfolios using correlation method performed well in both the Brazilian and the American markets. On the other hand, the SPC portfolios using cointegration behaved well only in the Brazilian market, whereas the cointegration approach presented larger tracking errors not only for SPC portfolios but also for portfolios with fixed windows. Thus, the outcomes showed that SPC portfolios (using either correlation or cointegration) could be a viable rebalancing strategy for IT portfolios especially in a more volatile environment such as the Brazilian market.

To support this conclusion, we performed tests for difference in means and difference in variances among SPC portfolios and portfolios with fixed windows, and also between each portfolio and the market index. The analysis of portfolios daily returns and portfolio volatility allowed us to find only a few statistically significant differences between SPC portfolios and the SP 100 index, and also between SPC portfolios and fixed portfolios, in the U.S. market during 2009 and 2010. Those results led us to infer that control charts for IT portfolios using correlation can be a viable tool for IT portfolio rebalancing strategy, given some market conditions (especially with larger instability) and the objectives of the fund (to accept larger TE as compensation for fewer portfolio updates and lower transaction costs). Particularly, control charts offered good results, considering both modeling approaches, when applied to a more volatile environment, such as the Brazilian market.

This article is organized as follows. Section 2 presents the theoretical background, both in terms of using EWMA control charts and the application of cointegration and correlation in IT optimization. Section 3 discusses the developed SPC based approach to IT portfolio rebalancing. Section 4 presents the two implemented models towards IT optimization. The computational experiments and comparison with fixed time windows strategy are described in detail in Section 5. Finally, we present the conclusions in Section 6.

# 2 Background

## 2.1 EWMA Control Charts

Lucas e Crosier (1982) describe the EWMA control charts as follows. Let $x_t$ be a time series composed by an i.i.d. (independent and identically distributed) variable with variance $\sigma_x^2$. Then, the chart is based on the statistical value $z_t$ as follows:

$$z_t = \lambda x_t + (1-\lambda) z_{t-1}$$

where $\lambda \in \{0,1\}$ is the exponential smoothing constant, and the variance of $z_t$ is $\sigma_{z_t}^2 = \sigma_x^2[1-(1-\lambda)^{2t}]\left(\frac{\lambda}{2-\lambda}\right)$, that converges to its asymptotic value when $t \to \infty$:

$$\lim_{t\to\infty} \sigma_{z_t}^2 = \sigma_x^2 \Big(\frac{\lambda}{2-\lambda}\Big)$$

Furthermore, to regulate the statistical value $z_t$, we define upper and lower control limits (UCL, LCL) based on the asymptotic volatility $\sigma_{z_t}$ according to Equations 2.1 and 2.2:

$$\text{UCL} = (Target) + L\sigma_z = (Target) + L\sigma_x \Big(\frac{\lambda}{2-\lambda}\Big)^{1/2}, \tag{2.1}$$

$$\text{LCL} = (Target) - L\sigma_x \Big(\frac{\lambda}{2-\lambda}\Big)^{1/2}. \tag{2.2}$$

In line with EWMA basic concepts (LUCAS; SACCUCCI, 1990), $\sigma_x^2$ is defined based on past observations and is kept fixed. Commonly, $L$ is set to create a confidence interval around 99%. $Target$ is the expected value of series $x_t$, and $\lambda$ has a weighting effect, diminishing the weight of the initial observations as the process advances on time. For instance, $\lambda = 1$ characterizes a "memoryless" process.

## 2.2 Index Tracking Optimization with Cointegration

The concept of cointegration originated by Granger (1981b) is based on the existence of an equilibrium relation between two or more non-stationary economic series of which a linear combination could either be stationary or have a lower degree of integration than the original series. Formally, the collection $(X, Y, Z)$ is cointegrated if (HAMILTON, 1994): (i) all of the series are integrated of order 1 (or $I(1)$ series); and (ii) a linear combination of this collection of $I(1)$ series is integrated of order zero (or $I(0)$ series). The importance of cointegration in the modeling of non-stationary economic series becomes clear in the Granger representation theorem (ENGLE; GRANGER, 1987), in which integrated variables sharing an equilibrium relation turn out to be either stationary or have a lower degree of integration than the original series.

The concept of cointegration has presented important applications in economics and finance (CHIU; WONG, 2013). Although there are numerous applications of cointegration in the literature, its use in asset allocation has only recently been put forward. Alexander (1999) presented the use of cointegration for portfolio optimization and index tracking. The potential application of cointegration in asset allocation was discussed by Alexander, Giblin e Weddington (2002). Also, Alexander e Dimitriu (2005a) empirically showed that when tracking an index based on the minimization of tracking-error variance becomes more difficult, cointegration relationship can enhance performance – similar results were found by Dunis e Ho (2005). Additionally, Benzoni, Collin-Dufresne e Goldstein (2007) studied portfolio choice under cointegrated stock market dividends and labor income. A mean-variance portfolio allocation problem under the presence of cointegrated assets with a general correlation structure was investigated by Chiu e Wong (2011). Liu e Timmermann (2013) developed a theoretical model of "convergence" trading that uses cointegration to detect and exploit temporary mispricings. Cointegration has also been used to test interdependence between many different stock markets (HUANG; YANG; HU, 2000; VORONKOVA, 2004; GUPTA; GUIDI, 2012).

In order to infer the out-of-sample portfolio weights, we estimate multiple constrained regressions of the cointegration equation given a pre-specified in-sample calibration period. Under the restriction of no short sales, all stock weights must be positive. This is achieved by applying a non-negative least squares (NNLS) estimation that ensures non-negativity on the regression coefficients. The cointegration equation takes the following form:

$$P_t = \beta_{0,t} + \sum_{i=1}^{N} \beta_{i,t} S_{i,t} + \epsilon_t \tag{2.3}$$

where $P_t$ is the index log-price ("target series") in time $t$, $S_{it}$ is the log-price of asset $i$ in period $t$ ("cointegrating series"), and $\epsilon_t$ is a zero-mean "tracking error". By normalizing the cointegration coefficients $\beta_i$ (for $i = 1, 2, \ldots, N$) to sum up to one, we determine the proportional weights for each stock. We impose the restriction to avoid short positions to limit liquidity issues and higher transaction costs.

The second step consists in applying the unit root test in the residuals $\widehat{\varepsilon}_t$ from Equation 2.3. In order to test the null hypothesis of no cointegration ($\mathcal{H}_0 : \gamma = 0$), we apply the Augmented Dickey-Fuller (ADF) test on the residuals of the cointegration equation (2.4). Let $q$ be the lag order of the AR process, $\widehat{\varepsilon}_t$ the estimated error term from the cointegration regression and $\Delta\widehat{\varepsilon}_t$ the change between two error terms, the Dickey-Fuller regression takes the following form:

$$\Delta\hat{\epsilon}_t = \gamma\hat{\epsilon}_{t-1} + \sum_{i=1}^{p} \phi_i \Delta\hat{\epsilon}_{t-i} + u_t. \tag{2.4}$$

We consider the critical values suggested by MacKinnon (1992), MacKinnon (2010) at 1% level of significance. We also point out that the ADF statistics are only insignificantly affected by the non-negativity constraint that are imposed on the cointegration coefficients.

## 2.3 Index Tracking Optimization with Correlation

Let $R_t$ be the index return in period $t$, $r_{i,t}$ be the return of asset $i$ in period $t$, and $w_i$ be the weight of asset $i$ in the portfolio. The tracking error (TE) on period $t$ can be defined as follows (ROLL, 1992):

$$TE_t = \sum_{i \in N} w_i r_{i,t} - R_t. \tag{2.5}$$

In addition, we have that the index tracking using correlation approach consists of minimizing the TE variance $\sigma^2_{TE}$, computed as follows:

$$\sigma^2_{TE} = \frac{1}{T} \sum_{t=1}^{T} \Big( \sum_{i \in N} w_i r_{i,t} - R_t \Big)^2. \tag{2.6}$$

Constraints could be imposed on this minimization to represent investors' preferences to select each portfolio, called real features. Such features may include transaction costs, transaction lots, cardinality constraints, investment threshold constraints, and decision dependency constraints (MANSINI; OGRYCZAK; SPERANZA, 2014). In this study, the optimization model will be detailed in Section 4.2.

# 3 Analysis of IT Portfolios Using EWMA Control Charts

To adapt EWMA control charts to the analysis of index tracking (IT) portfolios, this study proposes two graphical analysis: (i) control on portfolios' daily returns and (ii) control on portfolios' daily volatility. Our goal is to have a control tool that adjusts its conditions to market volatility and does not rely solely on the tracking error measure (TE). Sections 3.1 and 3.2 describe the design of SPC1 (control chart on daily returns) and SPC2 (control chart on daily volatility), respectively.

## 3.1 Control chart on daily returns (SPC1)

To define the first control chart (SPC1), we take the portfolio daily returns as the time series to be controlled. Further, we need to define the reference volatility $\sigma_x$ to compute the upper and lower control limits (UCL, LCL), according to Equations 2.1 and 2.2. In general, $\sigma_x$ is computed based on the time series to be controlled (LUCAS; SACCUCCI, 1990), the portfolio daily returns in our case. Nevertheless, we chose to compute the reference volatility $\sigma$ using the index daily returns and the exponentially weighted moving average volatility approach; thereby, we are computing the market volatility, not the IT portfolio volatility. This model gives more weight to recent observations and can increase the predictive ability by capturing the tendency of periods with more persistent instability – also known as volatility clustering or the "GARCH effect". The model also incorporates the majority of the stylized features of historical return series such as latency of volatility as well as the long-term mean reverting. As a result, the exponentially weighted volatility helps to balance out the over- and under-estimation of volatility. The exponentially weighted volatility can be defined recursively as follows:

$$\sigma_t = (1-\alpha)\epsilon_{t-1}^2 + \alpha\sigma_{t-1}$$

where $\alpha \in \{0, 1\}$. The implementation requires an initial value for $\sigma_1$, which could be set to the average volatility over the first $m < T$ days or also be set to the full sample volatility. The single parameter $\alpha$ is usually set to 0.94 for daily data and 0.97 for monthly data based on recommendations from Bollen (2015).

To start the chart and compute $z_{t^*}$, we need to set up parameters $\sigma$ and $z_{t^*-1}$, where $t^*$ is the first day out-of-sample in our empirical tests and the control chart is computed cumulatively as of $t^*$ (we refer the reader to Section 5.1 for a description of our database and experimental settings in more detail). Concerning $\sigma$, we compute the volatility of index daily returns from $t = 1$ to $t = t^* - 1$. The starting value $z_{t^*-1}$ is often taken to be the target value (LUCAS; CROSIER, 1982). We set $z_{t^*-1}$ equal to the index daily return immediately previous to $t^*$, i.e. $R_{t^*-1}$. In addition, the $Target$ value for each $t$ is the index return $R_t$. Finally, we adopt $\lambda = 0.5$ and $L = 1.96$ (i.e. confidence interval around 95%, since we have a two-tailed distribution).

As a result, we use the index daily returns to compute both UCL and LCL; in the meantime, the portfolio daily returns are the series to be controlled. Consequently, we are comparing the magnitude of the variations of the index and each portfolio.

In summary, we construct both UCL and LCL using the volatility of a data series (index daily returns), while the statistical value to be controlled is constructed using another data series (portfolio daily returns). This is not a common practice in the control chart literature. The regular choice would be to use the same data series to compute the control limits and the statistical value. This emphasizes our main purpose of keeping portfolios daily returns “regulated” by the index. We aim to maintain portfolios and index returns as close as possible, thus minimizing the TE. Nevertheless, during periods in which the index presents larger volatility, the control chart should accept larger TE values, as long as the portfolio volatility remains under control (which will be determined by the second control chart SPC2 described in the next section). Under those circumstances, the EWMA charts will be continually adapting itself to the market volatility and the environment conditions.

## 3.2 Control chart on daily volatility (SPC2)

In the second control chart SPC2, we define the tracking error (TE) volatility of each portfolio as the time series to be controlled. The TE volatility is calculated using the exponential smoothing approach. In order to compute the first value $x_{t^*}$, we evaluate the volatility of the TE in-sample from $t^* - 119$ to $t^*$ (which represents the historical volatility during the past six months approximately), i.e. the time series $x_t$ is computed for $t = t^*, t^* + 1, t^* + 2, \dots, T$. Further, we need to set $\sigma$ towards computing the control limits UCL and LCL. Based on Equation 2.6, the TE volatility is defined as follows:

$$\sigma_{TE} = \Big[\frac{1}{T}\sum_{t=1}^{T}\Big(\sum_{i\in N} w_i r_{it} - R_t\Big)^2\Big]^{1/2} \tag{3.1}$$

where $\sigma = [\text{Var(idx)} + \text{Var(portfolio)} - 2 \times \text{Cov(idx,portfolio)}]^{1/2}$. In this formula, the variances of the index and the portfolio daily returns, as well as the covariance, are computed cumulatively starting in $t^*$ and updated daily as the portfolio and index values are recalculated for each $t$, $t = t^*, t^* + 1, t^* + 2, \dots, T$.

Finally, in Equations 2.1 and 2.2, we define the $Target$ as zero, since we want portfolios with lower TE volatility over time (considering that TE volatility equal to zero implies index and portfolio with equal daily returns). As we set $Target = 0$, we use the initial value $z_{t^*-1} = 0$. We maintain $\lambda = 0.5$ and set $L = 1.64$, so that we continue to deal with a 95% confidence interval, as we are now considering an one-tailed distribution (only upper control limit).

Having set charts SPC1 and SPC2, the statistical value for each chart is computed for $z_t$, $t = t^*, t^* + 1, t^* + 2, \dots, T$, i.e. each business day out of sample. For each $t$, once the statistical value overcomes the control limits in both charts, the portfolio is recomputed using data from $t - \beta$ to $t$, where $\beta$ is a parameter to be defined based on the historical data and the optimization method employed. The portfolio is updated in $t + 1$.

Considering both SPC1 and SPC2 combined, we set up the control measures over the portfolios in terms of their performance as well as their volatility. The performance is usually the main driver to point out the need of updating a IT portfolio. However, a few points out of the control limits regarding the portfolio performance may be only outlier points. In this sense, the use of TE volatility should be understood as a tool to highlight the detachment of portfolio returns from the index (because TE volatility will indicate the rising portfolio volatility, in relation to the index volatility), reinforcing the need to update the portfolio.

# 4 Index Tracking Portfolio Optimization

## 4.1 Tests using Cointegration

For cointegration tests, the overall process follows the guidelines from Alexander e Dimitriu (2005a). Given a set of $N$ assets and an in-sample data interval, the construction of one cointegrated portfolio consists in defining $n \in N$ assets, in order to estimate Equations 2.3 and 2.4. Once the cointegration requirements explained in Section 2.2 are met, the estimated portfolio is a valid candidate for the in-sample interval analyzed.

Next, among all candidate portfolios, we have to select one to be used. To do so, we use a random approach, thereby allowing us to explore more extensively the cointegration method. For each in-sample interval, we run $M$ estimations in sequence, each estimation consists of applying Equations 2.3 and 2.4, using the index and a subset of $n$ assets randomly selected, where $n$ is the size of each portfolio. For each different combination of $n$ assets, if the cointegration requirements are fulfilled, then the portfolio is kept as a candidate. Among all candidate portfolios in each in-sample interval, we chose the one whose regression in Equation 2.3 has the smallest sum of the squared residuals. In our tests, portfolios are limited to 15 and 30 assets ($n = 15$ or $n = 30$). We also set $M = 25,000$ for portfolios with $n = 30$, and $M = 50,000$ for portfolios with $n = 15$. The use of a smaller number of simulations with 30-asset portfolios is due to the larger consumption of computational memory, thus, diminishing the number of regressions that we were able to perform. Sant'Anna, Filomena e Caldeira (2017) describe in details the approach used.

Regarding the use of ordinary least squares (OLS) to estimate Equation 2.3, a relevant detail should receive attention. The use of OLS would allow each portfolio to have long and short positions. Nonetheless, the use of short positions may be a tricky situation in portfolio management, since it is subjected to the possibility of renting each stock, thus raising some concern regarding transaction costs as well as of liquidity restrictions. Moreover, short positions are also associated to larger variability in rental costs. As a result, we estimate Equation 2.3 with non-negative least squares (NNLS) to accept only long positions.

Another relevant topic about cointegration is related to assets whose price series do not have order 1, i.e. are not $I(1)$. As mentioned in Section 2.2, we usually find index and assets price series integrated to order 1. Furthermore, in our tests, each asset has its price series tested in each in-sample interval to confirm its order. For instance, the first portfolio is optimized with data from $t = 1$ to $t = 480$, so each asset's price series is analyzed for this interval. If the *i-th* asset is not $I(1)$ in this in-sample interval, then this asset is excluded from the optimization using data from $t = 1$ to $t = 480$. In case we have less assets $I(1)$ than the size of the portfolio (example: 25 assets $I(1)$ for 30-asset portfolios), we simply form the portfolio with the assets respecting the required condition, i.e. all stocks $I(1)$.

In addition to testing each stock, it is also possible that the index presents an order different from 1 for some in-sample interval. In such situations, our choice is to maintain the previously obtained portfolio. To justify such choice, we should consider that the fundamental theory supporting cointegration for index tracking is the use of a number of price series that are integrated to the same order, which should be different from zero. So, an asset with order $I(0)$ can naturally be excluded from the sample, because assets are independent variables in Equation 2.3. However, the index can not be excluded because it is the dependent variable, different from each stock. As a result, during moments in which the index is stationary, there is only one option to perform the regression: to use only assets that also are $I(0)$ in the same in-sample interval. Nevertheless, to run a regression using index and assets $I(0)$ is equivalent to work under the assumption that index and portfolios would maintain a stationary behavior in the long term, and such assumption contradicts the efficient market hypothesis. For this reason, although running a regression with index and assets $I(0)$ is acceptable from an econometric viewpoint, it is not a reasonable choice from a financial perspective. So, in situations when the index is $I(0)$, we opted to maintain the previous portfolio.

## 4.2 Tests using Correlation

The tests using correlation approach to optimize IT portfolios were performed using the mixed integer non-linear programming model previously adopted in Sant'Anna et al. (2016). The weight of asset $i$ is defined by the decision variable $w_i$, meanwhile the binary decision variable $z_i$ equals 1 if asset $i$ is included in the portfolio, and 0 otherwise. The model is stated as follows:

$$\min \frac{1}{T}\sum_{t=1}^{T}\Big(\sum_{i\in N} w_i r_{it} - R_t\Big)^2 \tag{4.1}$$

s.t.

$$\sum_{i\in N} w_i r_{it} - R_t \geq \vartheta \qquad \forall t \in T \tag{4.2}$$

$$\sum_{i\in N} w_i r_{it} - R_t \leq \theta \qquad \forall t \in T \tag{4.3}$$

$$\sum_{i\in N} w_i = 1 \tag{4.4}$$

$$w_i \leq z_i \qquad \forall i \in N \tag{4.5}$$

$$\sum_{i\in N} z_i \leq K \tag{4.6}$$

$$w_i \geq 0 \qquad \forall i \in N \tag{4.7}$$

$$z_i \in \{0,1\} \qquad \forall i \in N \tag{4.8}$$

The objective function 4.1 minimizes the TE variance. Constraints 4.2 and 4.3 restrict the difference between daily returns of the index and portfolios, as we seek to avoid extreme values in tracking error during the in-sample interval; the minimum and maximum values for such difference are respectively the parameters $\vartheta$ and $\theta$, which are set during the tests to receive values as minimal as possible. Constraint 4.4 assures that 100% of the wealth is invested in the portfolio, and constraints 4.5 and 4.6, when combined, restrict each portfolio to $K$ assets, while constraint 4.7 allows the portfolios to have only long positions. Finally, constraint 4.8 refers to the binary decision variable $z_i$, that is equal to 1 if the asset $i$ is included in the portfolio, and 0 otherwise.

The database for correlation has assets and index daily returns in natural logarithm. The minimization is carried out with the complete database, without the need to select only non-stationary assets, as it is done with cointegration.

# 5 Empirical Tests

All experiments were performed using a CPU Intel® Core$^{TM}$ i7-3770 @ 3.40GHz computer with 8GB RAM. The correlation tests were computed using IBM® ILOG CPLEX 12.6.3; in the meantime, MATLAB® was chosen to run the cointegration tests. The CPU time to form each portfolio, for both cointegration and correlation methods, has been limited to around 5 minutes per portfolio, which is a reasonable time for real-world conditions.

It is relevant to notice that the literature related to portfolio optimization using correlation also considers the gap response as a central issue. The gap signals how far the obtained responses are from the optimal responses. However, since we do not have the gap information for the solutions obtained in cointegration tests, we relax this discussion and disregard the gap information for correlation responses either.

To account for transaction costs during the portfolio re-balancing process, we assume that transaction costs can be discounted from the daily return of asset $i$ in day $t$ as follows (HAN, 2006; CORTE; SARNO; TSIAKAS, 2009; DO; FAFF, 2012):

$$r_{i,t} = \ln\Big(\frac{P_{i,t}}{P_{i,t-1}}\Big) + \ln\Big(\frac{1-C}{1+C}\Big) \tag{5.1}$$

where $C$ represents the transaction costs. We used 0.2% for transaction costs, in line with Alexander e Dimitriu (2002) and Dunis e Ho (2005). This percentage refers mainly to brokerage fees. Rental costs are not a concern in our study, since we do not accept short positions.

## 5.1 Experimental settings

The data instance concerning the Brazilian stock market is composed by the Ibovespa index and 67 stocks. Historical data was downloaded from software Economatica, a financial database widely used in Brazil. The data instance related to the American market has the S&P 100 index and a sample of 96 assets, and their historical prices were downloaded from Bloomberg database. Both data instances have daily close prices adjusted for dividends and splits; the database is available at: <https://www.dropbox.com/s/uzq5uboxylyc1lp/spc-article-database.zip?dl=0>. For all tests, we set portfolios with two sizes: portfolios limited to 15 stocks and limited to 30 stocks.

For each market, the data sample has daily historical returns for 2,400 data points (each data point is one business day). The database for the Brazilian market has daily prices from Jan 21, 2005 to Oct 10, 2014. The database for the American market goes from Feb 03, 2005 to Aug 18, 2014. Both databases were arranged so that the first data point out of sample (the first business day out of sample) is the first business day in 2007 (which corresponds to Jan 02, 2007, in the case of the Brazilian market, and Jan 03, 2007 in the case of the American market).

The tests using cointegration approach were performed using in-sample interval with 480 data points (around 2 years), analogously to Alexander e Dimitriu (2002), while the tests using correlation were performed using in-sample interval with 120 data points (around 6 months), similar to Filomena e Lejeune (2014). Therefore, the first cointegrated portfolio is formed with data from $t = 1$ to $t = 480$ and the first correlated portfolio is formed with data from $t = 361$ to $t = 480$. As a result, out of sample intervals start in $t = 481$ for both markets.

Plus, to compare the results obtained for the portfolios regulated by control charts, we also construct portfolios with fixed rebalancing time windows. In this case, we optimize portfolios with 20, 60 and 120 business days, i.e. monthly, quarterly and semi-annual rebalancing.

## 5.2 Results: descriptive analysis

We start the description of the results with a qualitative discussion to get some initial insides, then, on next section we present the detailed statistical inference. Descriptive results are presented in Tables 13, 14, 15, 16 and 17, and Figure 6. For convenience, we defined cointegrated portfolios using at most 15 assets and fixed rebalancing interval with 20 data points as CT15-20; in the same way, we defined the portfolios using 60 and 120 data points for rebalancing intervals as CT15-60 and CT15-120, respectively. Cointegrated portfolios using at most 30 assets were named CT30-20, CT30-60 and CT30-120, while portfolios using correlation approach were CR15-20, CR15-60, CR15-120, CR30-20, CR30-60, and CR30-120. Finally, portfolios based on control charts were named CT15-SPC, CT30-SPC, CR15-SPC and CR30-SPC.

The portfolios were compared using several performance measures, as follows: (i) Annual average returns refer to the average of cumulative returns for each year from 2007 to 2014. (ii) Cumulative returns refer to the returns cumulatively compounded from 2007 to 2014 (from the first day out of sample to the last one). (iii) Volatility daily returns refer to the standard deviation ($\sigma$) of daily returns time series from 2007 to 2014. (iv) Annual volatility refers to $\sigma \times \sqrt{252}$. (v) Daily TE average refers to the average of daily tracking error time series from 2007 to 2014 (where daily tracking error refers to the portfolio daily return minus the index daily return – Equation 2.5). (vi) Daily TE volatility refers to the standard deviation of daily TEs from 2007 to 2014.

Initially, we observe in Table 13 (results for the Brazilian market) that SPC portfolios limited to 15 assets (CT15-SPC and CR15-SPC) have a slightly superior performance in relation to portfolios using 60 and 120 days as update intervals (a portfolio's superior performance implies cumulative return of the portfolio closest to the index return). Further, portfolio CT15-SPC has an apparent superior performance than portfolio CT15-20. In contrast, portfolio CR15-SPC has a slightly inferior performance than portfolio CR15-20. Nonetheless, from Table 15, we have that CR15-SPC is submitted to 69 updates from 2007 to 2014, while CR15-20 has 96 rebalancing moments, which implies a clear advantage of CR15-SPC in terms of transaction costs – such advantage is reflected in the monthly turnover values: 23.25% for CR15-SPC against 34.37% for CR15-20. Moreover, CT15-SPC has similar monthly turnover to CT15-60, even though the former also has apparently better performance than the later.

Further, as we keep observing the results for the Brazilian market, we can also highlight the outcomes in terms of annual volatility. In Table 13, in the case of portfolios limited to 15 assets, SPC portfolios have a slightly larger (or equal) annual volatility than 15-asset portfolios with fixed rebalancing windows. Although it implies SPC portfolios are presenting a larger risk, we should consider that the control charts were designed to update each portfolio if its volatility increases along with the enlargement of the TE. Thereby, even though SPC portfolios have a larger risk, we must consider that such increasing risk is mitigated by the fact that the volatility is monitored on a daily basis.

If we observe the results for 30-asset portfolios with Ibovespa, it is possible to conclude that portfolios using SPC approach had a good overall performance. CT30-SPC has a slightly superior performance than the three portfolios with fixed rebalancing intervals, and it is submitted to 72 updates from 2007 to 2014 (see Table 15). Even though CT30-SPC has larger annual volatility than CT30-20, CT30-60 and CT30-120, this is not a drawback since SPC portfolios are also regulated by their volatility, as previously mentioned. In case of the correlation portfolios, CR30-SPC has a slightly poorer performance than CR30-20; nonetheless, the former is updated only 63 times from 2007 to 2014 and it has significant lower monthly turnover than the later, i.e. the loss in TE performance is relatively compensated by some gain in transaction costs – which creates a trade-off condition that is certainly relevant for the investor to consider.

As we turn attention to the results for the S&P 100 and U.S. data (Table 14), the SPC portfolios using either correlation or cointegration do not present quality similar to the tests with Brazilian data. Overall, CR15-SPC and CR30-SPC have superior performance than portfolios with 20 and 120 as fixed rebalancing windows, in terms of both annual average return and cumulative return. However, the performance is poorer than portfolios updated each 60 business days. The same pattern is found for portfolio CT15-SPC, while portfolio CT30-SPC has performance inferior than all portfolios with fixed updates when we observe either cumulative returns or volatility. Such findings suggest that the use of control charts might be a better choice especially for markets with historically larger volatility, which is the case of Brazil.

Table 13 – Descriptive results for the **Brazilian stock market**[1].

| Results | Ibovespa | CT15-20 | CT15-60 | CT15-120 | CT15-SPC | CT30-20 | CT30-60 | CT30-120 | CT30-SPC |
|---|---|---|---|---|---|---|---|---|---|
| Annual average return | 3.18% | 3.74% | 5.66% | 3.81% | 3.33% | 2.59% | 5.49% | 6.04% | 3.63% |
| Cumulative return | 28.43% | 34.10% | 55.34% | 34.87% | 29.96% | 22.74% | 53.32% | 59.92% | 32.98% |
| Volatility of daily returns | 1.86% | 1.88% | 1.82% | 1.86% | 1.90% | 1.82% | 1.79% | 1.78% | 1.86% |
| Annual volatility | 29.50% | 29.81% | 28.93% | 29.58% | 30.22% | 28.82% | 28.36% | 28.30% | 29.50% |
| Daily TE average | - | 0.00% | 0.01% | 0.00% | 0.00% | 0.00% | 0.01% | 0.01% | 0.00% |
| Daily TE volatility | - | 0.39% | 0.40% | 0.41% | 0.40% | 0.33% | 0.35% | 0.35% | 0.35% |
| Monthly turnover | - | 54.00% | 20.72% | 10.55% | 22.86% | 30.74% | 13.63% | 8.20% | 13.76% |

| Results | Ibovespa | CR15-20 | CR15-60 | CR15-120 | CR15-SPC | CR30-20 | CR30-60 | CR30-120 | CR30-SPC |
|---|---|---|---|---|---|---|---|---|---|
| Annual average return | 3.18% | 2.55% | 4.61% | 7.26% | 4.01% | 3.08% | 5.72% | 6.89% | 4.06% |
| Cumulative return | 28.43% | 22.27% | 43.40% | 75.25% | 36.93% | 27.47% | 56.10% | 70.40% | 37.45% |
| Volatility of daily returns | 1.86% | 1.87% | 1.85% | 1.83% | 1.87% | 1.85% | 1.83% | 1.81% | 1.85% |
| Annual volatility | 29.50% | 29.67% | 29.40% | 29.10% | 29.67% | 29.32% | 29.05% | 28.73% | 29.37% |
| Daily TE average | - | 0.00% | 0.01% | 0.02% | 0.00% | 0.00% | 0.01% | 0.01% | 0.00% |
| Daily TE volatility | - | 0.30% | 0.31% | 0.34% | 0.31% | 0.24% | 0.25% | 0.27% | 0.24% |
| Monthly turnover | - | 34.37% | 14.85% | 9.50% | 23.25% | 23.44% | 11.67% | 7.12% | 13.73% |

[1] Annual average returns refer to the average of cumulative returns for each year from 2007 to 2014. Cumulative returns refer to the returns cumulatively compounded from 2007 to 2014 (from the first day out of sample to the last one). Volatility daily returns refer to the standard deviation ($\sigma$) of daily returns time series from 2007 to 2014. Annual volatility refers to $\sigma \times \sqrt{252}$. Daily TE average refers to the average of daily tracking error time series from 2007 to 2014 (where daily tracking error refers to the portfolio daily return minus the index daily return). Daily TE volatility refers to the standard deviation of daily tracking errors from 2007 to 2014.

Table 14 – Descriptive results for the **American stock market**[1].

| Results | S&P 100 | CT15-20 | CT15-60 | CT15-120 | CT15-SPC | CT30-20 | CT30-60 | CT30-120 | CT30-SPC |
|---|---|---|---|---|---|---|---|---|---|
| Annual average return | 3.59% | 3.95% | 7.38% | 4.46% | 2.29% | 7.73% | 6.21% | 6.02% | 9.77% |
| Cumulative return | 32.59% | 36.31% | 76.70% | 41.81% | 19.88% | 81.40% | 61.94% | 59.60% | 110.73% |
| Volatility of daily returns | 1.38% | 1.55% | 1.52% | 1.71% | 1.69% | 1.51% | 1.46% | 1.49% | 1.54% |
| Annual volatility | 21.87% | 24.53% | 24.16% | 27.08% | 26.82% | 23.97% | 23.24% | 23.65% | 24.51% |
| Daily TE average | - | 0.00% | 0.02% | 0.01% | 0.00% | 0.02% | 0.01% | 0.01% | 0.03% |
| Daily TE volatility | - | 0.50% | 0.53% | 0.62% | 0.60% | 0.33% | 0.28% | 0.30% | 0.35% |
| Monthly turnover | - | 74.45% | 26.41% | 14.32% | 37.48% | 52.64% | 19.55% | 11.39% | 21.73% |

| Results | S&P 100 | CR15-20 | CR15-60 | CR15-120 | CR15-SPC | CR30-20 | CR30-60 | CR30-120 | CR30-SPC |
|---|---|---|---|---|---|---|---|---|---|
| Annual average return | 3.59% | 3.07% | 4.82% | 3.64% | 4.49% | 4.23% | 6.24% | 5.37% | 5.78% |
| Cumulative return | 32.59% | 27.40% | 45.76% | 33.13% | 42.09% | 39.30% | 62.28% | 51.97% | 56.72% |
| Volatility of daily returns | 1.38% | 1.43% | 1.41% | 1.44% | 1.39% | 1.41% | 1.39% | 1.40% | 1.40% |
| Annual volatility | 21.87% | 22.63% | 22.44% | 22.93% | 22.14% | 22.37% | 22.14% | 22.24% | 22.29% |
| Daily TE average | - | 0.00% | 0.01% | 0.00% | 0.00% | 0.00% | 0.01% | 0.01% | 0.01% |
| Daily TE volatility | - | 0.27% | 0.26% | 0.27% | 0.25% | 0.17% | 0.17% | 0.16% | 0.18% |
| Monthly turnover | - | 57.12% | 21.46% | 11.56% | 27.00% | 43.44% | 16.72% | 9.15% | 18.01% |

[1] Annual average returns refer to the average of cumulative returns for each year from 2007 to 2014. Cumulative returns refer to the returns cumulatively compounded from 2007 to 2014 (from the first day out of sample to the last one). Volatility daily returns refer to the standard deviation ($\sigma$) of daily returns time series from 2007 to 2014. Annual volatility refers to $\sigma \times \sqrt{252}$. Daily TE average refers to the average of daily tracking error time series from 2007 to 2014 (where daily tracking error refers to the portfolio daily return minus the index daily return). Daily TE volatility refers to the standard deviation of daily tracking errors from 2007 to 2014.

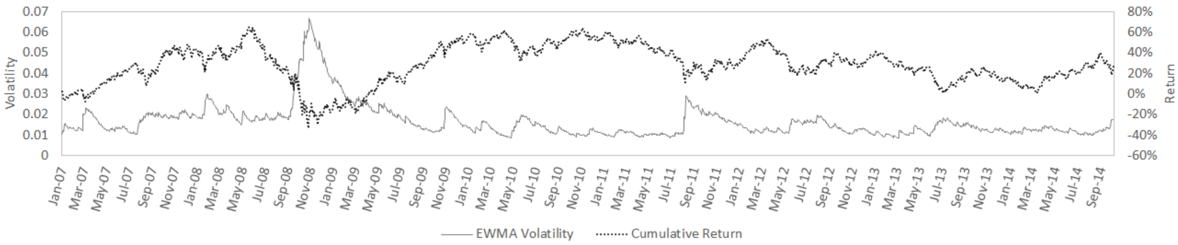
Concerning the number of updates (Table 15), it is clear that the financial crisis of 2007-2008 had a direct impact on the rebalancing process of SPC portfolios for both markets. The high volatility in this period significantly increased the number of updates, since the monitored values were frequently outside the control interval. However, from 2009 onwards, the SPC portfolios were rarely rebalanced for the American market. For the Brazilian market, rebalancing returned especially in the period 2013-2014. Although the number of updates was smaller in comparison with 2007-2008, the values were higher than in the previous years, mainly for portfolios formed using cointegration. It is important to observe that 2013 and 2014 were very bad years for Ibovespa, with devaluation of 16.1% and 7%, respectively – such results can be visually noticed in Figure 6a.

Table 15 – Number of updates per portfolio per year.

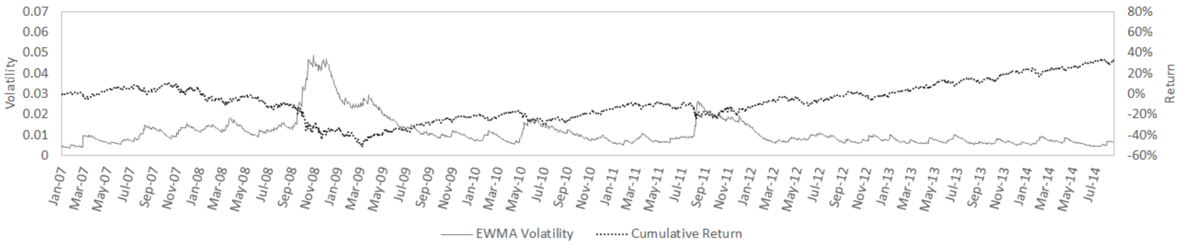
| Index | Ibovespa | | | |
|---|---|---|---|---|
| Portfolios | CT15-SPC | CT30-SPC | CR15-SPC | CR30-SPC |
| 2007 | 10 | 10 | 8 | 10 |
| 2008 | 21 | 20 | 19 | 20 |
| 2009 | 0 | 6 | 2 | 2 |
| 2010 | 0 | 0 | 5 | 2 |
| 2011 | 5 | 5 | 3 | 3 |
| 2012 | 4 | 7 | 8 | 7 |
| 2013 | 16 | 14 | 17 | 15 |
| 2014 | 18 | 10 | 7 | 4 |
| Total | 74 | 72 | 69 | 63 |
| Index | S&P 100 | | | |
| Portfolios | CT15-SPC | CT30-SPC | CR15-SPC | CR30-SPC |
| 2007 | 26 | 13 | 21 | 16 |
| 2008 | 25 | 22 | 20 | 23 |
| 2009 | 0 | 4 | 6 | 5 |
| 2010 | 0 | 0 | 2 | 0 |
| 2011 | 0 | 0 | 0 | 1 |
| 2012 | 0 | 0 | 0 | 0 |
| 2013 | 0 | 0 | 2 | 1 |
| 2014 | 0 | 0 | 3 | 2 |
| Total | 51 | 39 | 54 | 48 |

Based on the obtained results, we can initially infer that control charts for IT portfolios fit more accurately markets that are subjected to larger volatility, which is the case of Brazil. In contrast, the use of fixed rebalancing intervals is enough for developed markets, that tend to be consistently stable in the long run. This reasoning may be complemented by the data in Table 15, which shows the number of updates per year per SPC portfolio. For both indexes, it can be noticed that all eight portfolios have a greater number of updates especially in 2007 and 2008 for the S&P 100; and 2007, 2008 and 2013 for the Ibovespa. Figures 6a and 6b help to observe peaks of volatility in those years, when shocks suddenly affect cumulative returns. In calmer periods, the portfolios using SPC obtained very good performance for both markets, with a very reduced number of updates. For instance, portfolio CT15-SPC was not updated for the whole period 2009-2014.

Especially for the Ibovespa, we can notice the quality of all four SPC portfolios (CT15-SPC, CT30-SPC, CR15-SPC and CR30-SPC) during intervals that present more volatility such as 2008 and 2010. On the one hand, we notice apparent reduced annual TEs in 2008 (Table 16) as




(a) **Ibovespa** - EWMA Volatility vs. Cumulative Return




(b) **S&P 100** - EWMA Volatility vs. Cumulative Return

Figure 6 – Ibovespa and S&P 100: EWMA volatility vs. Cumulative Return from 2007 to 2014.

a consequence of more frequent portfolio updates, specially for portfolios CT30-SPC (annual TE in 2008: 0.06%; portfolio was updated 20 times in this year) and CR15-SPC (annual TE in 2008: -0.34%; portfolio was updated 19 times in this year). On the other hand, despite the rising risk in years such as 2009 and 2011, SPC portfolios were able to maintain reduced annual TE values (Table 16: reduced annual TE in 2009 for SPC portfolios, except for CT15-SPC) combined with a smaller number of updates (for instance, only 3 rebalancing moments for portfolios CR15-SPC and CR30-SPC).

Table 16 – Annual tracking error for the **Brazilian stock market**[1].

| Annual TE | CT15-20 | CT15-60 | CT15-120 | CT15-SPC | CT30-20 | CT30-60 | CT30-120 | CT30-SPC |
|---|---|---|---|---|---|---|---|---|
| 2007 | -5.20% | 6.23% | 5.23% | 3.83% | -0.28% | 5.99% | 5.56% | 3.62% |
| 2008 | 1.62% | 0.67% | -7.45% | -4.65% | 0.08% | 2.78% | 2.50% | 0.06% |
| 2009 | 4.35% | 8.25% | 0.72% | 9.71% | -3.09% | -0.37% | -5.20% | -3.19% |
| 2010 | 4.41% | 9.49% | 3.34% | 2.58% | 0.97% | 6.59% | 5.11% | 1.46% |
| 2011 | -0.91% | 0.80% | 1.31% | -1.93% | -3.40% | 2.23% | 1.06% | 0.89% |
| 2012 | 1.27% | 2.76% | 0.56% | 6.99% | 2.92% | -1.02% | 1.40% | 1.44% |
| 2013 | 11.93% | 9.92% | 16.20% | 11.45% | 10.53% | 9.56% | 14.21% | 11.44% |
| 2014 | -15.06% | -14.84% | -8.95% | -17.83% | -14.39% | -10.10% | -6.67% | -14.37% |
| Average | 0.30% | 2.91% | 1.37% | 1.27% | -0.83% | 1.96% | 2.25% | 0.17% |

| Annual TE | CR15-20 | CR15-60 | CR15-120 | CR15-SPC | CR30-20 | CR30-60 | CR30-120 | CR30-SPC |
|---|---|---|---|---|---|---|---|---|
| 2007 | 6.00% | 2.52% | 9.36% | 6.58% | 3.72% | 6.91% | 6.69% | 5.38% |
| 2008 | -1.19% | 3.29% | 15.00% | -0.34% | 0.53% | 1.11% | 6.97% | -2.36% |
| 2009 | -1.90% | 3.51% | 4.64% | 0.92% | -5.84% | 3.79% | 3.26% | 4.31% |
| 2010 | -0.99% | 0.04% | 1.05% | -0.72% | -2.78% | -0.87% | 0.52% | -0.99% |
| 2011 | 0.43% | 5.82% | 2.37% | 0.94% | 1.17% | 3.93% | 3.77% | 2.00% |
| 2012 | 1.24% | 1.96% | 1.56% | 6.17% | 6.58% | 8.59% | 5.43% | 9.08% |
| 2013 | 8.19% | 7.50% | 5.35% | 8.66% | 6.91% | 8.75% | 11.27% | 7.50% |
| 2014 | -16.26% | -15.71% | -12.47% | -14.27% | -13.91% | -11.03% | -12.24% | -13.70% |
| Average | -0.56% | 1.12% | 3.36% | 0.99% | -0.45% | 2.65% | 3.21% | 1.40% |

[1] Annual TE value for each year refers to the cumulative tracking error calculated for each year using Equation (2.5). The Average annual TE refers to the average of annual tracking errors from 2007 to 2014.

Table 17 – Annual tracking error for the **American stock market**[1].

| Annual TE | CT15-20 | CT15-60 | CT15-120 | CT15-SPC | CT30-20 | CT30-60 | CT30-120 | CT30-SPC |
|---|---|---|---|---|---|---|---|---|
| 2007 | 2.51% | 7.07% | -2.62% | -4.02% | -0.52% | 0.43% | 1.42% | 3.51% |
| 2008 | -3.12% | 11.18% | -1.68% | -6.82% | 11.86% | 3.46% | 1.83% | 7.43% |
| 2009 | 6.52% | 12.88% | 8.04% | 11.64% | 18.22% | 12.92% | 13.62% | 24.74% |
| 2010 | 2.94% | -5.84% | -1.43% | -3.89% | 0.83% | 3.25% | 1.01% | 1.86% |
| 2011 | -4.44% | -6.15% | 7.13% | -9.29% | 0.78% | 0.25% | -2.77% | -1.42% |
| 2012 | 6.67% | 2.44% | 1.57% | 2.93% | -2.98% | -2.59% | 0.83% | 4.98% |
| 2013 | -6.52% | 7.03% | 1.85% | 11.38% | 1.51% | 2.19% | 1.22% | 8.17% |
| 2014 | 1.64% | -0.27% | -3.03% | -1.94% | 0.33% | 1.48% | 3.99% | 2.36% |
| Average | 0.77% | 3.54% | 1.23% | 0.00% | 3.75% | 2.67% | 2.64% | 6.45% |
| Annual TE | CR15-20 | CR15-60 | CR15-120 | CR15-SPC | CR30-20 | CR30-60 | CR30-120 | CR30-SPC |
| 2007 | 3.55% | 7.19% | 0.64% | 0.04% | 0.52% | 3.53% | 3.38% | 0.47% |
| 2008 | 5.37% | 3.02% | 1.13% | 1.29% | 2.05% | 2.94% | 3.84% | 1.82% |
| 2009 | -5.07% | -2.81% | -2.30% | -3.13% | 5.84% | 7.18% | 4.19% | 6.51% |
| 2010 | -3.44% | -0.02% | 0.47% | 1.65% | -0.47% | 2.66% | 0.67% | 1.78% |
| 2011 | -1.95% | 2.92% | 5.56% | 8.08% | -1.09% | 0.88% | 1.25% | 1.07% |
| 2012 | 2.71% | 0.61% | -0.76% | 1.08% | -1.39% | 1.70% | 0.80% | 3.85% |
| 2013 | -7.30% | -2.52% | -4.20% | -6.55% | 0.32% | 3.22% | -0.68% | 0.66% |
| 2014 | -2.65% | -0.96% | -1.89% | 2.85% | -1.12% | -0.83% | -1.01% | 1.69% |
| Average | -1.10% | 0.93% | -0.17% | 0.66% | 0.58% | 2.66% | 1.56% | 2.23% |

[1] Annual TE value for each year refers to the cumulative tracking error calculated for each year using Equation (2.5). The Average annual TE refers to the average of annual tracking errors from 2007 to 2014.

Finally, interesting results were found in the case of SPC portfolios for the S&P 100. As presented in Table 15, CT15-SPC and CT30-SPC are only updated essentially in 2007 and 2008 (CT30-SPC is also updated four times in 2009). Such results highlight an idea of the proposed methodology related to control charts: to update a portfolio when both its TE and its volatility (in relation the index volatility) increase jointly. For instance, from Table 17, we have higher TE values in 2009, 2011 and 2013 for the portfolio CT15-SPC, and in 2009 and 2011 for CT30-SPC; however, as it can be noticed in Figure 6b, the index volatility in those years also has peaks (specially in 2009 and 2011), which leads to the enlargement of the boundaries in the second control chart (SPC2). As a result, the portfolio is held regardless of the higher TE, because we understand that the risk associated with the portfolio is similar to the market risk (represented by the index).

Overall, the descriptive statistical findings for SPC portfolios, in terms of performance (cumulative returns), volatility and transaction costs (represented by turnover values) led to the evaluation of the control chart approach as a viable IT portfolio rebalancing strategy especially for the Brazilian market. In general, this strategy had similar statistical results in a comparison with fixed time rebalancing policies. With data from Brazilian market, SPC portfolios had an apparent good performance for both optimization methods tested as well as for both types of portfolios (15-asset and 30-asset portfolios). However, considering the U.S. data, SPC portfolios using cointegration had apparent poorer performance than portfolios with fixed rebalancing intervals, while SPC portfolios using a correlation based optimization model had probably an equivalent performance compared to fixed rebalancing strategies. On next section, the statistical inference results are discussed towards a more conclusive comparison.

## 5.3 Statistical analysis: difference in means and difference in variances

Descriptive results analyzed in the previous section showed some differences among the portfolios in terms of performance and TE. Nonetheless, such differences were very small in most of the cases. In this sense, the use of some statistical analysis could shed new light on the results, as we seek to highlight differences or similarities among the compared strategies (control charts and fixed rebalancing intervals). Thus, this section describes statistical tests performed to analyze differences in means and differences in variances between portfolios' daily returns and portfolios' daily tracking errors. Both tests for difference in means and in variances were carried out using bootstrapping technique, following the discussion that arises from the market efficiency hypothesis and the potential dependence among assets daily returns (DEMIGUEL; NOGALES; UPPAL, 2014).

To run the tests for difference in means of two time series, we used the following theoretical structure. Given two time series $y_{1,t}$ and $y_{2,t}$, $t = 1, 2, \ldots, T$, we apply a t-test to verify the null hypothesis $H_0 : \mu_{y_1} = \mu_{y_2}$. In the case of the use of bootstrapping, given two time series $y_{1,t}$ and $y_{2,t}$, we must randomly select $V$ values in each of the two time series, $V \subset T$, thus forming the two subsets $y^s_{1,v}$ and $y^s_{2,v}$, $v = 1, 2, \ldots, V$; then, we test the null hypothesis $H_0 : \mu_{y^s_1} = \mu_{y^s_2}$. The random sampling must be performed $S$ times and, for each sampling, we constructed a statistical value $z_s = \mu_{y^s_{1,v}} - \mu_{y^s_{2,v}}$. As a result, we formed a set $z_s$, $s = 1, 2, \ldots, S$, which is used to compute the lower and upper limits (CI- and CI+) of the confidence interval with a $1-\alpha$ confidence. The null hypothesis to check the difference in means is not rejected if 0 (zero) falls inside [CI-,CI+].

Regarding the test for difference in variances of two time series, the procedure was similar, with only a few changes. A F-test was applied to verify the null hypothesis $H_0 : \sigma_{y_1} = \sigma_{y_2}$. The random sampling was analogous to the previous test. The statistical value $z_s$ was defined as $z_s = \sigma_{y^s_{1,v}} / \sigma_{y^s_{2,v}}$. After setting lower and upper confidence intervals for $z_s$, the null hypothesis to verify the difference in variances is not rejected if 1 falls inside [CI-,CI+].

In the case of tests for difference in means, we used the daily returns of index and portfolios out of sample for each year from 2007 to 2014. We tested the index daily returns against each portfolio's daily returns per year. Also, we tested the daily returns of SPC portfolios against the daily returns of the corresponding portfolios with fixed rebalancing intervals (for example: CT15-SPC vs. CT15-20, CT15-SPC vs. CT15-60, CT15-SPC vs. CT15-120, CT30-SPC vs. CT30-20, and so on). After using daily returns, we tested for difference in means of the daily TE of the portfolios (again, for each year from 2007 to 2014). The tests considered the analysis of each SPC portfolio against the corresponding portfolios with fixed rebalancing windows, as it was done with daily returns. Concerning the tests for difference in variances, we also used the time series of daily returns and carried out the tests as it was done with the tests for difference in means with daily returns, for each year from 2007 to 2014 (i.e. we tested each SPC portfolio against the index, and also each SPC portfolio against the corresponding portfolios with fixed updates).

For all tests, we defined $V = 50$, $S = 1,000$ and $\alpha = 0.05$ (confidence interval equal to 95%). Table 25 presents the results for the tests of difference in means of daily returns using Ibovespa index and cointegration portfolios limited to 15 assets. Since the results from all tests for difference in means and variances resulted in 40 Tables, we present the complete results in Appendices B (tests for difference in means of daily returns), C (tests for difference in means of daily TEs) and D (tests for difference in variances of daily returns).

The results in Appendix B show the statistical similarity of means (i.e. *Fail to reject* $H_0$) for daily returns in all tested cases. Therefore, we can infer that all portfolios formed during our optimization tests have similarity in their daily returns, based on the comparison between daily returns out of sample. As a result, it is reasonable to argue on the analogous performance of the SPC and fixed-time rebalancing strategies in the long run, with only slightly differences between the performance of the portfolios during each year.

Also, we have found similarity between means of daily TE values in all cases (see Appendix C) for both the Ibovespa and S&P 100 indexes. In relation to the hypothesis test for difference in variances in daily returns, it can be confirmed the similarity between index and portfolios in all cases using Ibovespa index. Thus, we can reasonably expect that the Ibovespa index and all formed portfolios, using both rebalancing strategies (control charts and fixed updates), present similar volatility over time in their daily returns. However, this inference could not be traced for the tests related with the S&P 100 index. Table 19 presents the tested cases in which the hypotheses $H_0$ was rejected. Specially in the case of portfolio CT15-SPC, we can notice in Appendix D, Table 53 (d), that this portfolio did not have similarity in terms of variance with

Table 18 – Test results for difference in means of daily returns out of sample, in the case of Ibovespa index and cointegration portfolios limited to 15 assets.

(a)

| IBOV vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0074 | 0.0074 | Fail to reject H0 |
| 2008 | -0.0089 | 0.0099 | Fail to reject H0 |
| 2009 | -0.0104 | 0.0110 | Fail to reject H0 |
| 2010 | -0.0057 | 0.0048 | Fail to reject H0 |
| 2011 | -0.0047 | 0.0037 | Fail to reject H0 |
| 2012 | -0.0043 | 0.0050 | Fail to reject H0 |
| 2013 | -0.0046 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0045 | 0.0064 | Fail to reject H0 |

(b)

| IBOV vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0074 | 0.0070 | Fail to reject H0 |
| 2008 | -0.0107 | 0.0081 | Fail to reject H0 |
| 2009 | -0.0096 | 0.0111 | Fail to reject H0 |
| 2010 | -0.0052 | 0.0045 | Fail to reject H0 |
| 2011 | -0.0040 | 0.0044 | Fail to reject H0 |
| 2012 | -0.0042 | 0.0050 | Fail to reject H0 |
| 2013 | -0.0042 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0047 | 0.0060 | Fail to reject H0 |

(c)

| IBOV vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0076 | 0.0067 | Fail to reject H0 |
| 2008 | -0.0102 | 0.0082 | Fail to reject H0 |
| 2009 | -0.0107 | 0.0115 | Fail to reject H0 |
| 2010 | -0.0056 | 0.0048 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0044 | Fail to reject H0 |
| 2012 | -0.0039 | 0.0049 | Fail to reject H0 |
| 2013 | -0.0045 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0038 | 0.0062 | Fail to reject H0 |

(d)

| IBOV vs. CT15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0075 | 0.0070 | Fail to reject H0 |
| 2008 | -0.0092 | 0.0095 | Fail to reject H0 |
| 2009 | -0.0112 | 0.0101 | Fail to reject H0 |
| 2010 | -0.0053 | 0.0049 | Fail to reject H0 |
| 2011 | -0.0045 | 0.0046 | Fail to reject H0 |
| 2012 | -0.0047 | 0.0048 | Fail to reject H0 |
| 2013 | -0.0054 | 0.0031 | Fail to reject H0 |
| 2014 | -0.0042 | 0.0058 | Fail to reject H0 |

the index for most of the years ($H_0$ is rejected in 2009, 2010, 2012 and 2013). Such results help to explain the larger TE values presented in Table 17 for this portfolio, specially in 2009 and 2013.

Table 19 – Cases that $H_0$ is rejected - Difference in variances.

| Case | Year | Table in Appendix C |
|---|---|---|
| S&P 100 vs. CT15-20 | 2009 | Table 33 |
| S&P 100 vs. CT15-60 | 2009 | Table 33 |
| S&P 100 vs. CT15-120 | 2009 | Table 33 |
| S&P 100 vs. CT15-SPC | 2009, 2010, 2012, 2013 | Table 33 |
| S&P 100 vs. CT30-20 | 2009 | Table 34 |
| S&P 100 vs. CT30-SPC | 2009 | Table 34 |
| CT30-SPC vs. CT30-20 | 2010 | Table 38 |

Thus, the results from the statistical inference are in the same direction with the descriptive statistics. SPC portfolios with correlation performed as well as the fixed rebalancing strategies considering both Brazilian and American markets. In contrast, despite performing well for the Brazilian market, SPC with cointegration presented some differences in the volatility for the American market when compared to other strategies and the S&P 100 index.

# 6 Conclusions

In this article, we implemented an alternative approach to regulate index tracking (IT) portfolios. Considering some studies about control chart methodology for the regulation of production processes in industry (LUCAS; CROSIER, 1982; WOODALL, 1986; MONTGOMERY, 1996; HAWKINS; OLWELL, 1998; ROGERSON, 2006), we adapted such approach for the control of IT portfolios. Through the use of control charts, we attempted to introduce a new rebalancing option to the common practice in past literature, which is the used of fixed rebalancing time windows to update portfolios regardless of the market conditions in terms of volatility, or regardless of the tracking error performance (TE). By using control charts, our goal was to make the rebalancing decision endogenous to the optimization problem, avoiding unnecessary updates during periods in which the market volatility is low and the TE is inside boundaries, and quickly updating the portfolios when tracking error and volatility are out of control.

Tests were performed using two indexes: Ibovespa (Brazilian market) and S&P 100 (American market), for portfolios with at most 15 and 30 assets, and data samples from 2005 to 2014. Portfolios were also computed with three fixed rebalancing intervals (monthly, quarterly and semiannual rebalancing) to compare their results with the portfolios using control charts. Further, the tests considered two optimization methods (cointegration and correlation), as we sought to stress the proposed control chart approach.

Overall, the SPC portfolios (regulated by control charts) and the fixed rebalancing interval strategies were, similar in terms of their returns, TEs, and volatilities. They differed regarding volatility mostly during the years of 2009 and 2010. On the one hand, the SPC portfolios for correlation performed well in both Brazilian and U.S. markets. But, on the contrary, the SPC portfolios with cointegration presented consistent performance only for the Brazilian market. The cointegration method presented some problems especially in the tests using U.S. data and the S&P 100, not only for the SPC portfolios but also for the fixed rebalancing strategies. Thus, the results showed that control charts could be a viable rebalancing strategy for IT portfolios, especially using correlation. Moreover, our analysis led to more consistent outcomes when using the SPC approach in a market with larger volatility (which is the Brazilian case), contrasting with the findings with U.S. data, where portfolios with fixed rebalancing intervals seemed to fit the IT portfolio optimization due to the solid market stability in the long term.

Future studies could explore another markets in order to extend control chart experiments. Furthermore, tests for specific moments, such as strong bull or bear market periods, could be run to analyze if the control chart approach best fits IT portfolios in specific moments. The use of cointegration with SPC for IT should be also revisited, enlarging the time frame of this study.

# Bibliography

ACOSTA-GONZáLEZ, E.; ARMAS-HERRERA, R.; FERNáNDEZ-RODRíGUEZ, F. On the index tracking and the statistical arbitrage choosing the stocks by means of cointegration: the role of stock picking. *Quantitative Finance*, v. 15, n. 6, p. 1075–1091, 2015. Cited 2 times on pages 44 and 47.

AHLGREN, N.; ANTELL, J. Testing for cointegration between international stock prices. *Applied Financial Economics*, Taylor & Francis, v. 12, n. 12, p. 851–861, 2002. Cited on page 47.

ALDRIDGE, I. *High-frequency Trading: a practical guide to algorithmic strategies and trading systems*. [S.l.]: John Wiley and Sons, 2009. v. 459. Cited 2 times on pages 19 and 22.

ALEXANDER, C. Optimal hedging using cointegration. *Philosophical Transactions of the Royal Society Series A*, The Royal Society, v. 357, n. 1758, p. 2039–2058, 1999. Cited 4 times on pages 44, 47, 68, and 71.

ALEXANDER, C.; DIMITRIU, A. The cointegration alpha: enhanced index tracking and long-short equity market neutral strategies. *ISMA Discussion papers in Finance*, v. 8, 2002. Cited 3 times on pages 13, 45, and 79.

ALEXANDER, C.; DIMITRIU, A. Sources of Over-Performance in Equity Markets. *Journal of Portfolio Management*, v. 30, n. 4, p. 170–185, 2004. Cited on page 44.

ALEXANDER, C.; DIMITRIU, A. Indexing and statistical arbitrage: tracking error or cointegration? *The Journal of Portfolio Management*, Institutional Investor Journals, v. 31, n. 2, p. 50–63, 2005. Cited 18 times on pages 13, 14, 18, 21, 44, 45, 46, 47, 48, 49, 51, 53, 54, 63, 66, 68, 71, and 76.

ALEXANDER, C.; DIMITRIU, A. Indexing, cointegration and equity market regimes. *International Journal of Finance & Economics*, Wiley Online Library, v. 10, n. 3, p. 213–231, 2005. Cited 3 times on pages 43, 44, and 47.

ALEXANDER, C.; GIBLIN, I.; WEDDINGTON, W. Cointegration and asset allocation: A new active hedge fund strategy. *Research in International Business and Finance*, v. 16, p. 65–90, 2002. Cited on page 71.

ANGELELLI, E.; MANSINI, R.; SPERANZA, M. G. Kernel search: a new heuristic framework for portfolio selection. *Computational Optimization and Applications*, Springer, v. 51, n. 1, p. 345–361, 2012. Cited 3 times on pages 21, 22, and 52.

APPEL, I. R.; GORMLEY, T. A.; KEIM, D. B. Passive investors, not passive owners. *Journal of Financial Economics*, v. 121, n. 1, p. 111 – 141, 2016. Cited on page 43.

BANSAL, R.; KIKU, D. Cointegration and long-run asset allocation. *Journal of Business & Economic Statistics*, v. 29, n. 1, p. 161–173, 2011. Cited on page 47.

BARRO, D.; CANESTRELLI, E. Tracking error: a multistage portfolio model. *Annals of Operations Research*, Springer, v. 165, n. 1, p. 47–66, 2009. Cited 2 times on pages 18 and 45.

BEASLEY, J. E.; MEADE, N.; CHANG, T.-J. An evolutionary heuristic for the index tracking problem. *European Journal of Operational Research*, Elsevier, v. 148, n. 3, p. 621–643, 2003. Cited 14 times on pages 13, 18, 19, 21, 22, 23, 31, 32, 33, 34, 39, 52, 66, and 68.

BENZONI, L.; COLLIN-DUFRESNE, P.; GOLDSTEIN, R. S. Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *Journal of Finance*, v. 62, n. 5, p. 2123–2167, 2007. Cited 2 times on pages 47 and 71.

BOLLEN, B. What should the value of lambda be in the exponentially weighted moving average volatility model? *Applied Economics*, Taylor & Francis, v. 47, n. 8, p. 853–860, 2015. Cited on page 73.

BORENSTEIN, D. Towards a practical method to validate decision support systems. *Decision Support Systems*, Elsevier, v. 23, n. 3, p. 227–239, 1998. Cited on page 33.

BORGES, E. C.; JR., W. E.; YOSHINAGA, C. E. Exchange traded funds versus fundos indexados no brasil. *Revista de Finanças Aplicadas*, v. 1, p. 1–15, 2012. Cited on page 12.

CALDEIRA, J. F.; PORTUGAL, M. S. Long-short market neutral and index tracking strategies based on cointegrated portfolios. *Brazilian Review of Finance (in portuguese)*, Brazilian Society of Finance, v. 8, n. 4, p. 469–504, 2010. Cited on page 13.

CANAKGOZ, N. A.; BEASLEY, J. E. Mixed-integer programming approaches for index tracking and enhanced indexation. *European Journal of Operational Research*, Elsevier, v. 196, p. 384–399, 2008. Cited 4 times on pages 14, 18, 44, and 66.

CHEN, J.-H.; HUANG, C.-Y. An analysis of the spillover effects of exchange-traded funds. *Applied Economics*, Taylor & Francis, v. 42, n. 9, p. 1155–1168, 2010. Cited on page 43.

CHIU, M. C.; WONG, H. Y. Mean-variance portfolio selection of cointegrated assets. *Journal of Economic Dynamics and Control*, v. 35, n. 8, p. 1369–1385, August 2011. Cited 2 times on pages 47 and 71.

CHIU, M. C.; WONG, H. Y. Optimal investment for an insurer with cointegrated assets: Crra utility. *Insurance: Mathematics and Economics*, v. 52, n. 1, p. 52–64, 2013. Cited on page 70.

COLEMAN, T. F.; LI, Y.; HENNIGER, J. Minimizing tracking error while restricting the number of assets. *Journal of Risk*, v. 8, n. 4, p. 33–56, 2006. Cited 6 times on pages 12, 13, 18, 21, 25, and 66.

CONSIGLIO, A.; ZENIOS, S. A. Integrated simulation and optimization models for tracking international fixed income indices. *Mathematical Programming*, v. 89, p. 311–339, 2001. Cited 4 times on pages 13, 18, 21, and 66.

CORIELLI, F.; MARCELLINO, M. Factor based index tracking. *Journal of Banking & Finance*, v. 30, n. 8, p. 2215–2233, August 2006. Cited on page 44.

CORTE, P. D.; SARNO, L.; TSIAKAS, I. An economic evaluation of empirical exchange rate models. *Review of Financial Studies*, v. 22, n. 9, p. 3491–3530, September 2009. Cited on page 79.

CREMERS, M.; FERREIRA, M. A.; MATOS, P.; STARKS, L. Indexing and active fund management: International evidence. *Journal of Financial Economics*, v. 120, n. 3, p. 539–560, 2016. Cited on page 43.

DEFUSCO, R. A.; IVANOV, S. I.; KARELS, G. V. The exchange traded funds' pricing deviation: analysis and forecasts. *Journal of Economics and Finance*, Springer, v. 35, n. 2, p. 181–197, 2011. Cited on page 12.

DEMIGUEL, V.; NOGALES, F. J.; UPPAL, R. Stock return serial dependence and out-of-sample portfolio performance. *Review of Financial Studies*, v. 27, n. 4, p. 1031–1073, 2014. Cited on page 88.

DERIGS, U.; NICKEL, N.-H. On a local-search heuristic for a class of tracking error minimization problems in portfolio management. *Annals of Operations Research*, Springer, v. 131, n. 1-4, p. 45–77, 2004. Cited on page 21.

DO, B.; FAFF, R. Are pairs trading profits robust to trading costs? *Journal of Financial Research*, v. 35, n. 2, p. 261–287, 06 2012. Cited on page 79.

DUNIS, C. L.; HO, R. Cointegration portfolios of european equities for index tracking and market neutral strategies. *Journal of Asset Management*, Palgrave Macmillan, v. 6, n. 1, p. 33–52, 2005. Cited 8 times on pages 13, 18, 21, 44, 45, 47, 71, and 79.

ENGLE, R.; YOO, B. S. Forecasting and testing in co-integrated systems. *Journal of Econometrics*, v. 35, p. 143–159, 1987. Cited on page 48.

ENGLE, R. F.; GRANGER, C. Cointegration and error correction: representation, estimation and testing. *Econometrica*, JSTOR, v. 55, p. 251–276, 1987. Cited 3 times on pages 47, 48, and 70.

FAMA, E. Efficient capital markets: a review of theory and empirical work. *The Journal of Finance*, Wiley Online Library, v. 25, p. 383–417, 1970. Cited 2 times on pages 18 and 66.

FAMA, E.; FRENCH, K. Luck versus skill in the cross-section of mutual fund returns. *The Journal of Finance*, Wiley Online Library, v. 65, n. 5, p. 1915–1947, 2010. Cited on page 18.

FILOMENA, T. P.; LEJEUNE, M. A. Stochastic portfolio optimization with proportional transaction costs: convex reformulations and computational experiments. *Operations Research Letters*, Elsevier, v. 40, p. 212–217, 2012. Cited on page 22.

FILOMENA, T. P.; LEJEUNE, M. A. Warm-start heuristic for stochastic portfolio optimization with fixed and proportional transaction costs. *Journal of Optimization Theory and Applications*, Springer, v. 161, p. 308–329, 2014. Cited 3 times on pages 31, 52, and 80.

FOCARDI, S.; FABOZZI, F. A methodology for index tracking based on time-series clustering. *Quantitative Finance*, v. 4, n. 4, p. 417–425, 2004. Cited on page 44.

FRINO, A.; GALLAGHER, D. R. Tracking s&p 500 index funds. *The Journal of Portfolio Management*, Institutional Investor Journals, v. 28, p. 44–55, 2001. Cited 2 times on pages 18 and 43.

GAIVORONSKI, A. A.; KRYLOV, S.; WIJST, N. van der. Optimal portfolio selection and dynamic benchmark tracking. *European Journal of Operational Research*, Elsevier, v. 163, p. 115–131, 2005. Cited 5 times on pages 21, 24, 30, 45, and 68.

GENCAY, R.; DACOROGNA, M.; MULLER, U. A.; PICTET, O.; OLSEN, R. *An Introduction to High-frequency Finance*. [S.l.]: Academic press, 2001. Cited 2 times on pages 19 and 22.

GILLI, M.; SCHUMANN, E. Heuristic optimisation in financial modelling. *Annals of Operations Research*, Springer, v. 193, n. 1, p. 129–158, 2012. Cited on page 21.

GOLOSNOY, V. Sequential monitoring of minimum variance portfolio. *AStA Advances in Statistical Analysis*, Springer, v. 91, n. 1, p. 39–55, 2007. Cited on page 67.

GOLOSNOY, V.; RAGULIN, S.; SCHMID, W. Cusum control charts for monitoring optimal portfolio weights. *Computational Statistics & Data Analysis*, Elsevier, v. 55, n. 11, p. 2991–3009, 2011. Cited on page 67.

GOLOSNOY, V.; SCHMID, W. Ewma control charts for monitoring optimal portfolio weights. *Sequential Analysis*, Taylor & Francis, v. 26, n. 2, p. 195–224, 2007. Cited on page 67.

GRANGER, C. W. J. Some properties of time series data and their use in econometric model specification. *Journal of Econometrics*, v. 16, n. 1, p. 121–130, May 1981. Cited on page 47.

GRANGER, C. W. J. Some properties of time series data and their use in econometric model specification. *Journal of Econometrics*, v. 16, n. 1, p. 121–130, May 1981. Cited on page 70.

GRINOLD, R. C. The fundamental law of active management. *The Journal of Portfolio Management*, Institutional Investor Journals, v. 15, n. 3, p. 30–37, 1989. Cited on page 43.

GROBYS, K. Correlation versus cointegration: do cointegration based index-tracking portfolios perform better? evidence from the swedish stock-market. *German Journal for Young Researchers*, German Journal for Young Researchers, v. 2, p. 72–78, 2010. Cited 2 times on pages 13 and 45.

GUASTAROBA, G.; SPERANZA, M. G. Kernel search: an application to the index tracking problem. *European Journal of Operational Research*, Elsevier, v. 217, p. 54–68, 2012. Cited 13 times on pages 13, 18, 21, 22, 23, 32, 33, 34, 35, 39, 49, 52, and 66.

GUPTA, R.; GUIDI, F. Cointegration relationship and time varying co-movements among Indian and Asian developed stock markets. *International Review of Financial Analysis*, Elsevier, v. 21, p. 10–22, 2012. Cited on page 71.

HAMILTON, J. D. *Time Series Analysis*. Princeton, New Jersey, USA: Princeton University Press, 1994. Cited on page 70.

HAN, Y. Asset allocation with a high dimensional latent factor stochastic volatility model. *Review of Financial Studies*, Soc Financial Studies, v. 19, n. 1, p. 237–271, 2006. Cited on page 79.

HAWKINS, D. M.; OLWELL, D. H. *Cumulative Sum Charts and Charting for Quality Improvement*. [S.l.]: Springer Science & Business Media, 1998. Cited 2 times on pages 67 and 91.

HUANG, B.-N.; YANG, C.-W.; HU, J. W.-S. Causality and cointegration of stock markets among the United States, Japan and the South China growth triangle. *International Review of Financial Analysis*, Elsevier, v. 9, n. 3, p. 281–297, 2000. Cited on page 71.

HUANG, C. C. Max control chart with adaptive sample sizes for jointly monitoring process mean and standard deviation. *Journal of the Operational Research Society*, Springer, v. 65, n. 12, p. 1788–1799, 2014. Cited on page 67.

JANSEN, R.; DIJK, R. van. Optimal benchmark tracking with small portfolios. *The Journal of Portfolio Management*, Institutional Investor Journals, v. 28, n. 2, p. 33–39, 2002. Cited 3 times on pages 13, 18, and 21.

JEURISSEN, R.; BERG, J. van den. Optimized index tracking using a hybrid genetic algorithm. In: IEEE. *Evolutionary Computation, 2008. CEC 2008.(IEEE World Congress on Computational Intelligence). IEEE Congress on*. [S.l.], 2008. p. 2327–2334. Cited 4 times on pages 13, 18, 22, and 27.

KOEHLER, A.; MARKS, N.; O'CONNELL, R. Ewma control charts for autoregressive processes. *Journal of the Operational Research Society*, JSTOR, v. 52, n. 6, p. 699–707, 2001. Cited on page 67.

KONNO, H.; WIJAYANAYAKE, A. Minimal cost index tracking under nonlinear transactions costs and minimal transactions unit constraints. *International Journal of Theoretical and Applied Finance*, v. 4, p. 939–958, 2001. Cited 4 times on pages 13, 18, 21, and 66.

KRINK, T.; MITTNIK, S.; PATERLINI, S. Differential evolution and combinatorial search for constrained index-tracking. *Annals of Operations Research*, Springer, v. 172, n. 1, p. 153–176, 2009. Cited 4 times on pages 14, 21, 49, and 66.

LI, Q.; BAO, L. Enhanced index tracking with multiple time-scale analysis. *Economic Modelling*, v. 39, n. C, p. 282–292, 2014. Cited on page 44.

LI, Q.; SUN, L.; BAO, L. Enhanced index tracking based on multi-objective immune algorithm. *Expert Systems with Applications*, v. 38, n. 5, p. 6101 – 6106, 2011. ISSN 0957-4174. Cited on page 44.

LIU, J.; TIMMERMANN, A. Optimal convergence trade strategies. *Review of Financial Studies*, v. 26, n. 4, p. 1048–1086, 2013. Cited on page 71.

LUCAS, A. *Strategic and tactical asset allocation and the effect of long-run equilibrium relations*. [S.l.], 1997. Cited on page 47.

LUCAS, J. M.; CROSIER, R. B. Fast initial response for cusum quality-control schemes: give your cusum a head start. *Technometrics*, Taylor & Francis, v. 24, n. 3, p. 199–205, 1982. Cited 4 times on pages 67, 70, 73, and 91.

LUCAS, J. M.; SACCUCCI, M. S. Exponentially weighted moving average control schemes: properties and enhancements. *Technometrics*, Taylor & Francis, v. 32, n. 1, p. 1–12, 1990. Cited 3 times on pages 67, 70, and 73.

MACKINNON, J. G. Critical values for cointegration tests. In: ENGLE, R.; GRANGER, C. (Ed.). *Long-Run Economic Relationships: Readings in Cointegration*. Oxford: Oxford University Press, 1992. cap. 13, p. 274–290. Cited on page 71.

MACKINNON, J. G. *Critical Values for Cointegration Tests*. [S.l.], 2010. Cited on page 71.

MALKIEL, B. G. Returns from Investing in Equity Mutual Funds 1971 to 1991. *Journal of Finance*, v. 50, n. 2, p. 549–72, June 1995. Cited on page 43.

MANSINI, R.; OGRYCZAK, W.; SPERANZA, M. G. Twenty years of linear programming based portfolio optimization. *European Journal of Operational Research*, Elsevier, v. 234, n. 2, p. 518–535, 2014. Cited on page 72.

MARINGER, D.; OYEWUMI, O. Index tracking with constrained portfolios. *Intelligent Systems in Accounting, Finance and Management*, Wiley Online Library, v. 15, p. 57–71, 2007. Cited 5 times on pages 13, 18, 21, 22, and 45.

MEZALI, H.; BEASLEY, J. E. Quantile regression for index tracking and enhanced indexation. *Journal of the Operational Research Society*, Springer, v. 64, n. 11, p. 1676–1692, 2013. Cited 2 times on pages 13 and 66.

MILLER, T.; MECKEL, T. S. Beating index funds with derivatives. *The Journal of Portfolio Management*, Institutional Investor Journals, v. 25, n. 5, p. 75–87, 1999. Cited on page 43.

MONTGOMERY, D. *Introduction to Statistical Quality Control*. [S.l.]: New York: John Wiley, 1996. Cited 3 times on pages 14, 67, and 91.

MURRAY, W.; SHEK, H. A local relaxation method for the cardinality constrained portfolio optimization problem. *Computational Optimization and Applications*, Springer, v. 53, n. 3, p. 681–709, 2012. Cited on page 18.

OH, K. J.; KIM, T. Y.; MIN, S. Using genetic algorithm to support portfolio optimization for index fund management. *Expert Systems with Applications*, Elsevier, v. 28, p. 371–379, 2005. Cited 5 times on pages 13, 18, 22, 31, and 49.

PAN, M.-S. Permanent and transitory components of earnings, dividends, and stock prices. *The Quarterly Review of Economics and Finance*, Elsevier, v. 47, n. 4, p. 535–549, 2007. Cited on page 47.

ROGERSON, P. A. Formulas for the design of cusum quality control charts. *Communications in Statistics - Theory and Methods*, Taylor & Francis, v. 35, n. 2, p. 373–383, 2006. Cited 2 times on pages 67 and 91.

ROLL, R. A mean/variance analysis of tracking error. *The Journal of Portfolio Management*, Institutional Investor Journals, v. 18, n. 4, p. 13–22, 1992. Cited 4 times on pages 44, 49, 51, and 72.

RUIZ-TORRUBIANO, R.; SUÁREZ, A. A hybrid optimization approach to index tracking. *Annals of Operations Research*, Springer, v. 166, n. 1, p. 57–71, 2009. Cited on page 22.

SANT'ANNA, L. R.; FILOMENA, T. P.; BORENSTEIN, D. Index tracking with control on the number of assets. *Brazilian Review of Finance (in portuguese)*, Brazilian Society of Finance, v. 12, n. 1, p. 89–119, 2014. Cited 2 times on pages 19 and 30.

SANT'ANNA, L. R.; FILOMENA, T. P.; CALDEIRA, J. Index tracking and enhanced indexing using cointegration and correlation with endogenous portfolio selection. *The Quarterly Review of Economics and Finance*, v. 65, p. 146–157, 2017. Cited on page 76.

SANT'ANNA, L. R.; FILOMENA, T. P.; GUEDES, P. C.; BORENSTEIN, D. Index tacking with controlled number of assets using a hybrid heuristic combining genetic algorithm and non-linear programming. *Annals of Operations Research (forthcoming)*, 2016. Cited 2 times on pages 51 and 77.

SCOZZARI, A.; TARDELLA, F.; PATERLINI, S.; KRINK, T. Exact and heuristic approaches for the index tracking problem with UCITS constraints. *Annals of Operations Research*, Springer, v. 205, p. 235–250, 2013. Cited 5 times on pages 13, 18, 21, 45, and 66.

STOCK, J. H.; WATSON, M. W. Testing for common trends. *Journal of the American Statistical Association*, Taylor & Francis, v. 83, n. 404, p. 1097–1107, 1988. Cited 2 times on pages 47 and 54.

STUCCHI, P. A unified approach to portfolio selection in a tracking error framework with additional constraints on risk. *The Quarterly Review of Economics and Finance*, Elsevier, v. 56, p. 165–174, 2015. Cited on page 51.

THOMAIDIS, N. S. On the application of cointegration analysis in enhanced indexing. *Applied Economics Letters*, Taylor & Francis, v. 20, n. 4, p. 391–396, 2013. Cited on page 45.

VORONKOVA, S. Equity market integration in central european emerging markets: A cointegration analysis with shifting regimes. *International Review of Financial Analysis*, Elsevier, v. 13, n. 5, p. 633–647, 2004. Cited on page 71.

WHEELER, D. J.; CHAMBERS, D. S. *Understanding statistical process control*. Knoxville, TN, US: SPC Press, 1992. Cited 2 times on pages 14 and 67.

WOODALL, W. H. Weaknesses of the economic design of control charts. *Technometrics*, Taylor & Francis, v. 28, n. 4, p. 408–409, 1986. Cited 2 times on pages 67 and 91.

WU, L.-C.; CHOU, S.-C.; YANG, C.-C.; ONG, C.-S. Enhanced index investing based on goal programming. *The Journal of Portfolio Management*, Institutional Investor Journals, v. 33, n. 3, p. 49–56, 2007. Cited 3 times on pages 44, 45, and 59.

# Appendix

# APPENDIX A – Justification for Using Numerous Random Regressions with Cointegration

As mentioned in Section 4.1, the index tracking tests with cointegration consist of obtaining 108 final portfolios for each index (Ibovespa and S&P 100) and for each in-sample interval (1 or 2 years) if we consider monthly rebalancing. The process of obtaining each of the 108 final cointegration portfolios involves performing 100,000 regressions in sequence using the Ibovespa index, or 35,000 using the S&P 100 index. Each regression is a random combination of 10 assets, representing an acceptable candidate portfolio as long as the regression meets the requirements described in Section 2.1.

To verify the relevance of performing thousands of regressions to define each final portfolio, we carried out the following experiment. For each sequence of 35,000 regressions using U.S. data and the S&P 100 index, in addition to selecting the best portfolio (with a cointegration regression with the lowest value for the sum of the squared residuals), we also selected the best 100 and the worst 100 candidate portfolios. Then, we computed the cumulative tracking error in-sample for each of these 200 portfolios and took the absolute values of the cumulative tracking errors. Thus, we verified the average of this absolute TE of the 100 best and 100 worst portfolios. The results for the average TE of the 100 best and the 100 worst (with a 1-year in-sample) are presented in Table 20. In all of the 108 cases, the average TE of the 100 best is lower than the average TE of the 100 worst. Overall, the average TE of the 100 best is 4.04%, whereas the average TE of the 100 worst is 10.44%.

The same experiment was performed for the tests using the S&P 100 and the 2-year in-sample. In only two out of 108 cases, the average TE of the 100 worst portfolios was lower than the average TE of the 100 best. Altogether, for the 2-year in-sample, the 100 best portfolios had an average TE of 7.68%, whereas the average TE of the 100 worst was 20.80%. We omit the table with outcomes for the 2-year in-sample given that it presents results similar to those in Table 20. In summary, such results clarify the importance of running thousands of regressions to select each cointegration portfolio.

Table 20 – Average Tracking Error of the 100 Best/Worst Estimated Regressions per Portfolio using S&P 100 Index and 1-Year In-sample.

| Portfolio | 100 Best | 100 Worst | Portfolio | 100 Best | 100 Worst | Portfolio | 100 Best | 100 Worst | Portfolio | 100 Best | 100 Worst |
|---|---|---|---|---|---|---|---|---|---|---|---|
| P1 | 1.58% | 15.69% | P28 | 3.03% | 9.29% | P55 | 2.51% | 7.65% | P82 | 7.09% | 14.89% |
| P2 | 2.29% | 12.88% | P29 | 2.54% | 10.27% | P56 | 2.98% | 4.91% | P83 | 4.44% | 11.73% |
| P3 | 1.99% | 9.27% | P30 | 2.35% | 9.09% | P57 | 1.58% | 4.49% | P84 | 4.64% | 6.70% |
| P4 | 2.67% | 11.07% | P31 | 1.85% | 9.39% | P58 | 2.37% | 5.47% | P85 | 3.59% | 6.85% |
| P5 | 3.35% | 13.24% | P32 | 2.00% | 24.06% | P59 | 3.16% | 6.73% | P86 | 2.69% | 11.17% |
| P6 | 2.68% | 10.86% | P33 | 1.74% | 47.87% | P60 | 1.97% | 5.61% | P87 | 2.97% | 9.20% |
| P7 | 2.33% | 18.02% | P34 | 3.03% | 36.99% | P61 | 2.75% | 8.35% | P88 | 2.40% | 10.85% |
| P8 | 2.41% | 12.36% | P35 | 2.62% | 12.39% | P62 | 3.96% | 5.93% | P89 | 3.60% | 11.40% |
| P9 | 2.08% | 7.81% | P36 | 2.53% | 10.22% | P63 | 4.60% | 7.22% | P90 | 5.67% | 11.25% |
| P10 | 4.02% | 10.96% | P37 | 2.57% | 9.94% | P64 | 3.57% | 6.07% | P91 | 4.04% | 9.78% |
| P11 | 5.08% | 19.47% | P38 | 2.51% | 12.55% | P65 | 3.76% | 8.49% | P92 | 4.32% | 9.40% |
| P12 | 3.57% | 17.44% | P39 | 2.90% | 22.97% | P66 | 6.38% | 13.21% | P93 | 2.86% | 10.75% |
| P13 | 5.75% | 20.31% | P40 | 2.89% | 9.25% | P67 | 8.24% | 13.16% | P94 | 3.37% | 13.93% |
| P14 | 5.64% | 14.41% | P41 | 2.29% | 8.35% | P68 | 5.41% | 11.30% | P95 | 4.82% | 11.04% |
| P15 | 4.07% | 6.98% | P42 | 2.11% | 8.00% | P69 | 3.09% | 8.45% | P96 | 6.55% | 14.06% |
| P16 | 5.11% | 8.00% | P43 | 2.16% | 7.44% | P70 | 2.03% | 5.00% | P97 | 7.44% | 10.83% |
| P17 | 6.15% | 10.15% | P44 | 2.16% | 8.11% | P71 | 2.05% | 4.33% | P98 | 5.57% | 8.79% |
| P18 | 6.82% | 10.32% | P45 | 1.51% | 5.12% | P72 | 1.56% | 4.56% | P99 | 4.86% | 7.26% |
| P19 | 5.87% | 10.55% | P46 | 2.64% | 4.41% | P73 | 2.25% | 3.10% | P100 | 5.01% | 7.29% |
| P20 | 5.41% | 9.75% | P47 | 4.80% | 8.09% | P74 | 1.77% | 5.41% | P101 | 4.15% | 6.52% |
| P21 | 5.19% | 8.93% | P48 | 8.62% | 12.90% | P75 | 2.12% | 6.64% | P102 | 4.32% | 5.87% |
| P22 | 4.74% | 9.80% | P49 | 7.92% | 9.39% | P76 | 2.90% | 3.34% | P103 | 5.24% | 7.57% |
| P23 | 5.53% | 14.16% | P50 | 10.21% | 12.76% | P77 | 2.60% | 4.77% | P104 | 3.83% | 4.00% |
| P24 | 3.18% | 10.67% | P51 | 21.89% | 29.74% | P78 | 2.76% | 4.10% | P105 | 4.33% | 5.56% |
| P25 | 4.49% | 10.20% | P52 | 11.36% | 19.99% | P79 | 2.91% | 3.86% | P106 | 4.66% | 5.72% |
| P26 | 2.28% | 7.52% | P53 | 7.38% | 7.73% | P80 | 3.74% | 8.18% | P107 | 3.57% | 4.45% |
| P27 | 4.10% | 18.90% | P54 | 2.38% | 9.99% | P81 | 5.36% | 7.89% | P108 | 3.84% | 6.09% |

# APPENDIX B – Test Results for Difference Between Means of Daily Returns

Results for the hypothesis test for the difference between two means using time series of **daily returns** out-of-sample of index and portfolios.

Table 21 – Test results for difference between means of daily returns out-of-sample, for Ibovespa index and cointegrated portfolios with at most 15 assets.

(a)

| IBOV vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0074 | 0.0074 | Fail to reject H0 |
| 2008 | -0.0089 | 0.0099 | Fail to reject H0 |
| 2009 | -0.0104 | 0.0110 | Fail to reject H0 |
| 2010 | -0.0057 | 0.0048 | Fail to reject H0 |
| 2011 | -0.0047 | 0.0037 | Fail to reject H0 |
| 2012 | -0.0043 | 0.0050 | Fail to reject H0 |
| 2013 | -0.0046 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0045 | 0.0064 | Fail to reject H0 |

(b)

| IBOV vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0074 | 0.0070 | Fail to reject H0 |
| 2008 | -0.0107 | 0.0081 | Fail to reject H0 |
| 2009 | -0.0096 | 0.0111 | Fail to reject H0 |
| 2010 | -0.0052 | 0.0045 | Fail to reject H0 |
| 2011 | -0.0040 | 0.0044 | Fail to reject H0 |
| 2012 | -0.0042 | 0.0050 | Fail to reject H0 |
| 2013 | -0.0042 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0047 | 0.0060 | Fail to reject H0 |

(c)

| IBOV vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0076 | 0.0067 | Fail to reject H0 |
| 2008 | -0.0102 | 0.0082 | Fail to reject H0 |
| 2009 | -0.0107 | 0.0115 | Fail to reject H0 |
| 2010 | -0.0056 | 0.0048 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0044 | Fail to reject H0 |
| 2012 | -0.0039 | 0.0049 | Fail to reject H0 |
| 2013 | -0.0045 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0038 | 0.0062 | Fail to reject H0 |

(d)

| IBOV vs. CT15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0075 | 0.0070 | Fail to reject H0 |
| 2008 | -0.0092 | 0.0095 | Fail to reject H0 |
| 2009 | -0.0112 | 0.0101 | Fail to reject H0 |
| 2010 | -0.0053 | 0.0049 | Fail to reject H0 |
| 2011 | -0.0045 | 0.0046 | Fail to reject H0 |
| 2012 | -0.0047 | 0.0048 | Fail to reject H0 |
| 2013 | -0.0054 | 0.0031 | Fail to reject H0 |
| 2014 | -0.0042 | 0.0058 | Fail to reject H0 |

Table 22 – Test results for difference between means of daily returns out-of-sample, for Ibovespa index and cointegrated portfolios with at most 30 assets.

(a)

| IBOV vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0073 | 0.0070 | Fail to reject H0 |
| 2008 | -0.0095 | 0.0094 | Fail to reject H0 |
| 2009 | -0.0102 | 0.0111 | Fail to reject H0 |
| 2010 | -0.0046 | 0.0052 | Fail to reject H0 |
| 2011 | -0.0044 | 0.0043 | Fail to reject H0 |
| 2012 | -0.0045 | 0.0048 | Fail to reject H0 |
| 2013 | -0.0057 | 0.0032 | Fail to reject H0 |
| 2014 | -0.0040 | 0.0062 | Fail to reject H0 |

(b)

| IBOV vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0073 | 0.0069 | Fail to reject H0 |
| 2008 | -0.0101 | 0.0085 | Fail to reject H0 |
| 2009 | -0.0103 | 0.0104 | Fail to reject H0 |
| 2010 | -0.0047 | 0.0045 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0038 | Fail to reject H0 |
| 2012 | -0.0040 | 0.0052 | Fail to reject H0 |
| 2013 | -0.0045 | 0.0035 | Fail to reject H0 |
| 2014 | -0.0044 | 0.0056 | Fail to reject H0 |

(c)

| IBOV vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0069 | 0.0074 | Fail to reject H0 |
| 2008 | -0.0091 | 0.0083 | Fail to reject H0 |
| 2009 | -0.0105 | 0.0119 | Fail to reject H0 |
| 2010 | -0.0053 | 0.0044 | Fail to reject H0 |
| 2011 | -0.0047 | 0.0041 | Fail to reject H0 |
| 2012 | -0.0043 | 0.0051 | Fail to reject H0 |
| 2013 | -0.0048 | 0.0031 | Fail to reject H0 |
| 2014 | -0.0042 | 0.0053 | Fail to reject H0 |

(d)

| IBOV vs. CT30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0073 | 0.0072 | Fail to reject H0 |
| 2008 | -0.0096 | 0.0088 | Fail to reject H0 |
| 2009 | -0.0100 | 0.0110 | Fail to reject H0 |
| 2010 | -0.0057 | 0.0045 | Fail to reject H0 |
| 2011 | -0.0049 | 0.0040 | Fail to reject H0 |
| 2012 | -0.0043 | 0.0052 | Fail to reject H0 |
| 2013 | -0.0055 | 0.0033 | Fail to reject H0 |
| 2014 | -0.0046 | 0.0058 | Fail to reject H0 |

Table 23 – Test results for difference between means of daily returns out-of-sample, for Ibovespa index and correlated portfolios with at most 15 assets.

(a)

| IBOV vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0074 | 0.0067 | Fail to reject H0 |
| 2008 | -0.0085 | 0.0095 | Fail to reject H0 |
| 2009 | -0.0118 | 0.0103 | Fail to reject H0 |
| 2010 | -0.0051 | 0.0052 | Fail to reject H0 |
| 2011 | -0.0045 | 0.0041 | Fail to reject H0 |
| 2012 | -0.0045 | 0.0049 | Fail to reject H0 |
| 2013 | -0.0038 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0037 | 0.0071 | Fail to reject H0 |

(b)

| IBOV vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0076 | 0.0067 | Fail to reject H0 |
| 2008 | -0.0084 | 0.0099 | Fail to reject H0 |
| 2009 | -0.0118 | 0.0091 | Fail to reject H0 |
| 2010 | -0.0056 | 0.0050 | Fail to reject H0 |
| 2011 | -0.0045 | 0.0039 | Fail to reject H0 |
| 2012 | -0.0044 | 0.0053 | Fail to reject H0 |
| 2013 | -0.0043 | 0.0048 | Fail to reject H0 |
| 2014 | -0.0039 | 0.0067 | Fail to reject H0 |

(c)

| IBOV vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0075 | 0.0068 | Fail to reject H0 |
| 2008 | -0.0096 | 0.0093 | Fail to reject H0 |
| 2009 | -0.0110 | 0.0094 | Fail to reject H0 |
| 2010 | -0.0053 | 0.0050 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0043 | Fail to reject H0 |
| 2012 | -0.0045 | 0.0048 | Fail to reject H0 |
| 2013 | -0.0044 | 0.0038 | Fail to reject H0 |
| 2014 | -0.0041 | 0.0064 | Fail to reject H0 |

(d)

| IBOV vs. CR15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0076 | 0.0068 | Fail to reject H0 |
| 2008 | -0.0085 | 0.0108 | Fail to reject H0 |
| 2009 | -0.0108 | 0.0102 | Fail to reject H0 |
| 2010 | -0.0049 | 0.0050 | Fail to reject H0 |
| 2011 | -0.0047 | 0.0040 | Fail to reject H0 |
| 2012 | -0.0043 | 0.0051 | Fail to reject H0 |
| 2013 | -0.0041 | 0.0048 | Fail to reject H0 |
| 2014 | -0.0042 | 0.0062 | Fail to reject H0 |

Table 24 – Test results for difference between means of daily returns out-of-sample, for Ibovespa index and correlated portfolios with at most 30 assets.

(a)

| IBOV vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0080 | 0.0066 | Fail to reject H0 |
| 2008 | -0.0088 | 0.0097 | Fail to reject H0 |
| 2009 | -0.0109 | 0.0102 | Fail to reject H0 |
| 2010 | -0.0052 | 0.0055 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0043 | Fail to reject H0 |
| 2012 | -0.0043 | 0.0047 | Fail to reject H0 |
| 2013 | -0.0042 | 0.0040 | Fail to reject H0 |
| 2014 | -0.0035 | 0.0067 | Fail to reject H0 |

(b)

| IBOV vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0078 | 0.0074 | Fail to reject H0 |
| 2008 | -0.0095 | 0.0102 | Fail to reject H0 |
| 2009 | -0.0110 | 0.0107 | Fail to reject H0 |
| 2010 | -0.0053 | 0.0053 | Fail to reject H0 |
| 2011 | -0.0048 | 0.0039 | Fail to reject H0 |
| 2012 | -0.0046 | 0.0044 | Fail to reject H0 |
| 2013 | -0.0042 | 0.0041 | Fail to reject H0 |
| 2014 | -0.0044 | 0.0061 | Fail to reject H0 |

(c)

| IBOV vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0076 | 0.0073 | Fail to reject H0 |
| 2008 | -0.0088 | 0.0087 | Fail to reject H0 |
| 2009 | -0.0111 | 0.0109 | Fail to reject H0 |
| 2010 | -0.0052 | 0.0049 | Fail to reject H0 |
| 2011 | -0.0045 | 0.0040 | Fail to reject H0 |
| 2012 | -0.0044 | 0.0048 | Fail to reject H0 |
| 2013 | -0.0040 | 0.0038 | Fail to reject H0 |
| 2014 | -0.0042 | 0.0062 | Fail to reject H0 |

(d)

| IBOV vs. CR30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0077 | 0.0066 | Fail to reject H0 |
| 2008 | -0.0092 | 0.0098 | Fail to reject H0 |
| 2009 | -0.0101 | 0.0100 | Fail to reject H0 |
| 2010 | -0.0047 | 0.0052 | Fail to reject H0 |
| 2011 | -0.0050 | 0.0043 | Fail to reject H0 |
| 2012 | -0.0049 | 0.0047 | Fail to reject H0 |
| 2013 | -0.0044 | 0.0041 | Fail to reject H0 |
| 2014 | -0.0037 | 0.0065 | Fail to reject H0 |

Table 25 – Test results for difference between means of daily returns out-of-sample among portfolios, using Ibovespa index and cointegrated portfolios with at most 15 assets.

(a)

| CT15-SPC vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0068 | 0.0073 | Fail to reject H0 |
| 2008 | -0.0091 | 0.0094 | Fail to reject H0 |
| 2009 | -0.0113 | 0.0105 | Fail to reject H0 |
| 2010 | -0.0049 | 0.0056 | Fail to reject H0 |
| 2011 | -0.0040 | 0.0047 | Fail to reject H0 |
| 2012 | -0.0054 | 0.0042 | Fail to reject H0 |
| 2013 | -0.0053 | 0.0038 | Fail to reject H0 |
| 2014 | -0.0057 | 0.0060 | Fail to reject H0 |

(b)

| CT15-SPC vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0067 | 0.0071 | Fail to reject H0 |
| 2008 | -0.0087 | 0.0095 | Fail to reject H0 |
| 2009 | -0.0122 | 0.0101 | Fail to reject H0 |
| 2010 | -0.0050 | 0.0052 | Fail to reject H0 |
| 2011 | -0.0040 | 0.0048 | Fail to reject H0 |
| 2012 | -0.0048 | 0.0045 | Fail to reject H0 |
| 2013 | -0.0052 | 0.0037 | Fail to reject H0 |
| 2014 | -0.0053 | 0.0057 | Fail to reject H0 |

(c)

| CT15-SPC vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0075 | 0.0078 | Fail to reject H0 |
| 2008 | -0.0086 | 0.0100 | Fail to reject H0 |
| 2009 | -0.0117 | 0.0106 | Fail to reject H0 |
| 2010 | -0.0047 | 0.0057 | Fail to reject H0 |
| 2011 | -0.0039 | 0.0045 | Fail to reject H0 |
| 2012 | -0.0053 | 0.0047 | Fail to reject H0 |
| 2013 | -0.0054 | 0.0037 | Fail to reject H0 |
| 2014 | -0.0056 | 0.0052 | Fail to reject H0 |

Table 26 – Test results for difference between means of daily returns out-of-sample among portfolios, using Ibovespa index and cointegrated portfolios with at most 30 assets.

(a)

| CT30-SPC vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0074 | 0.0075 | Fail to reject H0 |
| 2008 | -0.0097 | 0.0088 | Fail to reject H0 |
| 2009 | -0.0103 | 0.0115 | Fail to reject H0 |
| 2010 | -0.0044 | 0.0052 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0046 | Fail to reject H0 |
| 2012 | -0.0045 | 0.0049 | Fail to reject H0 |
| 2013 | -0.0049 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0050 | 0.0057 | Fail to reject H0 |

(b)

| CT30-SPC vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0075 | 0.0071 | Fail to reject H0 |
| 2008 | -0.0098 | 0.0084 | Fail to reject H0 |
| 2009 | -0.0105 | 0.0098 | Fail to reject H0 |
| 2010 | -0.0043 | 0.0048 | Fail to reject H0 |
| 2011 | -0.0042 | 0.0046 | Fail to reject H0 |
| 2012 | -0.0042 | 0.0049 | Fail to reject H0 |
| 2013 | -0.0042 | 0.0045 | Fail to reject H0 |
| 2014 | -0.0049 | 0.0056 | Fail to reject H0 |

(c)

| CT30-SPC vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0073 | 0.0068 | Fail to reject H0 |
| 2008 | -0.0099 | 0.0080 | Fail to reject H0 |
| 2009 | -0.0099 | 0.0106 | Fail to reject H0 |
| 2010 | -0.0049 | 0.0045 | Fail to reject H0 |
| 2011 | -0.0042 | 0.0046 | Fail to reject H0 |
| 2012 | -0.0042 | 0.0050 | Fail to reject H0 |
| 2013 | -0.0046 | 0.0045 | Fail to reject H0 |
| 2014 | -0.0057 | 0.0050 | Fail to reject H0 |

Table 27 – Test results for difference between means of daily returns out-of-sample among portfolios, using Ibovespa index and correlated portfolios with at most 15 assets.

(a)

| CR15-SPC vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0077 | 0.0067 | Fail to reject H0 |
| 2008 | -0.0090 | 0.0095 | Fail to reject H0 |
| 2009 | -0.0114 | 0.0103 | Fail to reject H0 |
| 2010 | -0.0055 | 0.0049 | Fail to reject H0 |
| 2011 | -0.0041 | 0.0045 | Fail to reject H0 |
| 2012 | -0.0048 | 0.0052 | Fail to reject H0 |
| 2013 | -0.0047 | 0.0040 | Fail to reject H0 |
| 2014 | -0.0051 | 0.0056 | Fail to reject H0 |

(b)

| CR15-SPC vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0079 | 0.0069 | Fail to reject H0 |
| 2008 | -0.0105 | 0.0084 | Fail to reject H0 |
| 2009 | -0.0116 | 0.0097 | Fail to reject H0 |
| 2010 | -0.0056 | 0.0047 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0044 | Fail to reject H0 |
| 2012 | -0.0049 | 0.0054 | Fail to reject H0 |
| 2013 | -0.0048 | 0.0042 | Fail to reject H0 |
| 2014 | -0.0055 | 0.0053 | Fail to reject H0 |

(c)

| CR15-SPC vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0070 | 0.0077 | Fail to reject H0 |
| 2008 | -0.0103 | 0.0077 | Fail to reject H0 |
| 2009 | -0.0103 | 0.0097 | Fail to reject H0 |
| 2010 | -0.0054 | 0.0050 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0047 | Fail to reject H0 |
| 2012 | -0.0049 | 0.0047 | Fail to reject H0 |
| 2013 | -0.0050 | 0.0039 | Fail to reject H0 |
| 2014 | -0.0055 | 0.0054 | Fail to reject H0 |

Table 28 – Test results for difference between means of daily returns out-of-sample among portfolios, using Ibovespa index and correlated portfolios with at most 30 assets.

(a)

| CR30-SPC vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0073 | 0.0071 | Fail to reject H0 |
| 2008 | -0.0090 | 0.0094 | Fail to reject H0 |
| 2009 | -0.0107 | 0.0104 | Fail to reject H0 |
| 2010 | -0.0051 | 0.0050 | Fail to reject H0 |
| 2011 | -0.0047 | 0.0046 | Fail to reject H0 |
| 2012 | -0.0048 | 0.0045 | Fail to reject H0 |
| 2013 | -0.0041 | 0.0041 | Fail to reject H0 |
| 2014 | -0.0056 | 0.0059 | Fail to reject H0 |

(b)

| CR30-SPC vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0074 | 0.0080 | Fail to reject H0 |
| 2008 | -0.0103 | 0.0091 | Fail to reject H0 |
| 2009 | -0.0100 | 0.0106 | Fail to reject H0 |
| 2010 | -0.0060 | 0.0048 | Fail to reject H0 |
| 2011 | -0.0047 | 0.0044 | Fail to reject H0 |
| 2012 | -0.0043 | 0.0048 | Fail to reject H0 |
| 2013 | -0.0037 | 0.0039 | Fail to reject H0 |
| 2014 | -0.0062 | 0.0052 | Fail to reject H0 |

(c)

| CR30-SPC vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0075 | 0.0070 | Fail to reject H0 |
| 2008 | -0.0099 | 0.0085 | Fail to reject H0 |
| 2009 | -0.0103 | 0.0122 | Fail to reject H0 |
| 2010 | -0.0052 | 0.0048 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0040 | Fail to reject H0 |
| 2012 | -0.0044 | 0.0041 | Fail to reject H0 |
| 2013 | -0.0039 | 0.0036 | Fail to reject H0 |
| 2014 | -0.0058 | 0.0051 | Fail to reject H0 |

Table 29 – Test results for difference between means of daily returns out-of-sample, for S&P 100 index and cointegrated portfolios with at most 15 assets.

(a)

| S&P 100 vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0034 | 0.0024 | Fail to reject H0 |
| 2008 | -0.0059 | 0.0048 | Fail to reject H0 |
| 2009 | -0.0112 | 0.0095 | Fail to reject H0 |
| 2010 | -0.0036 | 0.0044 | Fail to reject H0 |
| 2011 | -0.0028 | 0.0034 | Fail to reject H0 |
| 2012 | -0.0024 | 0.0023 | Fail to reject H0 |
| 2013 | -0.0026 | 0.0029 | Fail to reject H0 |
| 2014 | -0.0034 | 0.0020 | Fail to reject H0 |

(b)

| S&P 100 vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0036 | 0.0024 | Fail to reject H0 |
| 2008 | -0.0047 | 0.0062 | Fail to reject H0 |
| 2009 | -0.0130 | 0.0120 | Fail to reject H0 |
| 2010 | -0.0035 | 0.0049 | Fail to reject H0 |
| 2011 | -0.0029 | 0.0035 | Fail to reject H0 |
| 2012 | -0.0027 | 0.0024 | Fail to reject H0 |
| 2013 | -0.0032 | 0.0018 | Fail to reject H0 |
| 2014 | -0.0032 | 0.0027 | Fail to reject H0 |

(c)

| S&P 100 vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0036 | 0.0025 | Fail to reject H0 |
| 2008 | -0.0049 | 0.0058 | Fail to reject H0 |
| 2009 | -0.0139 | 0.0111 | Fail to reject H0 |
| 2010 | -0.0043 | 0.0038 | Fail to reject H0 |
| 2011 | -0.0035 | 0.0022 | Fail to reject H0 |
| 2012 | -0.0027 | 0.0027 | Fail to reject H0 |
| 2013 | -0.0031 | 0.0022 | Fail to reject H0 |
| 2014 | -0.0025 | 0.0033 | Fail to reject H0 |

(d)

| S&P 100 vs. CT15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0030 | 0.0029 | Fail to reject H0 |
| 2008 | -0.0052 | 0.0058 | Fail to reject H0 |
| 2009 | -0.0135 | 0.0118 | Fail to reject H0 |
| 2010 | -0.0049 | 0.0033 | Fail to reject H0 |
| 2011 | -0.0025 | 0.0045 | Fail to reject H0 |
| 2012 | -0.0026 | 0.0027 | Fail to reject H0 |
| 2013 | -0.0037 | 0.0023 | Fail to reject H0 |
| 2014 | -0.0034 | 0.0030 | Fail to reject H0 |

Table 30 – Test results for difference between means of daily returns out-of-sample, for S&P 100 index and cointegrated portfolios with at most 30 assets.

(a)

| S&P 100 vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0031 | 0.0025 | Fail to reject H0 |
| 2008 | -0.0060 | 0.0047 | Fail to reject H0 |
| 2009 | -0.0101 | 0.0118 | Fail to reject H0 |
| 2010 | -0.0036 | 0.0036 | Fail to reject H0 |
| 2011 | -0.0028 | 0.0031 | Fail to reject H0 |
| 2012 | -0.0025 | 0.0026 | Fail to reject H0 |
| 2013 | -0.0033 | 0.0021 | Fail to reject H0 |
| 2014 | -0.0030 | 0.0030 | Fail to reject H0 |

(b)

| S&P 100 vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0031 | 0.0031 | Fail to reject H0 |
| 2008 | -0.0059 | 0.0049 | Fail to reject H0 |
| 2009 | -0.0097 | 0.0111 | Fail to reject H0 |
| 2010 | -0.0040 | 0.0038 | Fail to reject H0 |
| 2011 | -0.0033 | 0.0027 | Fail to reject H0 |
| 2012 | -0.0021 | 0.0028 | Fail to reject H0 |
| 2013 | -0.0026 | 0.0026 | Fail to reject H0 |
| 2014 | -0.0028 | 0.0030 | Fail to reject H0 |

(c)

| S&P 100 vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0031 | 0.0028 | Fail to reject H0 |
| 2008 | -0.0061 | 0.0055 | Fail to reject H0 |
| 2009 | -0.0099 | 0.0098 | Fail to reject H0 |
| 2010 | -0.0039 | 0.0034 | Fail to reject H0 |
| 2011 | -0.0032 | 0.0027 | Fail to reject H0 |
| 2012 | -0.0028 | 0.0020 | Fail to reject H0 |
| 2013 | -0.0027 | 0.0027 | Fail to reject H0 |
| 2014 | -0.0034 | 0.0025 | Fail to reject H0 |

(d)

| S&P 100 vs. CT30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0031 | 0.0030 | Fail to reject H0 |
| 2008 | -0.0073 | 0.0043 | Fail to reject H0 |
| 2009 | -0.0114 | 0.0098 | Fail to reject H0 |
| 2010 | -0.0038 | 0.0041 | Fail to reject H0 |
| 2011 | -0.0032 | 0.0033 | Fail to reject H0 |
| 2012 | -0.0027 | 0.0025 | Fail to reject H0 |
| 2013 | -0.0030 | 0.0023 | Fail to reject H0 |
| 2014 | -0.0035 | 0.0026 | Fail to reject H0 |

Table 31 – Test results for difference between means of daily returns out-of-sample, for S&P 100 index and correlated portfolios with at most 15 assets.

(a)

| S&P 100 vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0030 | 0.0032 | Fail to reject H0 |
| 2008 | -0.0068 | 0.0052 | Fail to reject H0 |
| 2009 | -0.0099 | 0.0089 | Fail to reject H0 |
| 2010 | -0.0035 | 0.0035 | Fail to reject H0 |
| 2011 | -0.0032 | 0.0030 | Fail to reject H0 |
| 2012 | -0.0028 | 0.0019 | Fail to reject H0 |
| 2013 | -0.0026 | 0.0027 | Fail to reject H0 |
| 2014 | -0.0028 | 0.0031 | Fail to reject H0 |

(b)

| S&P 100 vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0031 | 0.0028 | Fail to reject H0 |
| 2008 | -0.0058 | 0.0053 | Fail to reject H0 |
| 2009 | -0.0091 | 0.0094 | Fail to reject H0 |
| 2010 | -0.0041 | 0.0033 | Fail to reject H0 |
| 2011 | -0.0034 | 0.0027 | Fail to reject H0 |
| 2012 | -0.0030 | 0.0018 | Fail to reject H0 |
| 2013 | -0.0025 | 0.0030 | Fail to reject H0 |
| 2014 | -0.0026 | 0.0032 | Fail to reject H0 |

(c)

| S&P 100 vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0034 | 0.0029 | Fail to reject H0 |
| 2008 | -0.0060 | 0.0051 | Fail to reject H0 |
| 2009 | -0.0106 | 0.0105 | Fail to reject H0 |
| 2010 | -0.0038 | 0.0036 | Fail to reject H0 |
| 2011 | -0.0033 | 0.0029 | Fail to reject H0 |
| 2012 | -0.0024 | 0.0021 | Fail to reject H0 |
| 2013 | -0.0024 | 0.0026 | Fail to reject H0 |
| 2014 | -0.0029 | 0.0031 | Fail to reject H0 |

(d)

| S&P 100 vs. CR15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0032 | 0.0029 | Fail to reject H0 |
| 2008 | -0.0063 | 0.0048 | Fail to reject H0 |
| 2009 | -0.0093 | 0.0105 | Fail to reject H0 |
| 2010 | -0.0037 | 0.0037 | Fail to reject H0 |
| 2011 | -0.0034 | 0.0029 | Fail to reject H0 |
| 2012 | -0.0022 | 0.0024 | Fail to reject H0 |
| 2013 | -0.0026 | 0.0024 | Fail to reject H0 |
| 2014 | -0.0028 | 0.0030 | Fail to reject H0 |

Table 32 – Test results for difference between means of daily returns out-of-sample, for S&P 100 index and correlated portfolios with at most 30 assets.

(a)

| S&P 100 vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0030 | 0.0030 | Fail to reject H0 |
| 2008 | -0.0062 | 0.0052 | Fail to reject H0 |
| 2009 | -0.0090 | 0.0105 | Fail to reject H0 |
| 2010 | -0.0041 | 0.0031 | Fail to reject H0 |
| 2011 | -0.0029 | 0.0031 | Fail to reject H0 |
| 2012 | -0.0024 | 0.0023 | Fail to reject H0 |
| 2013 | -0.0027 | 0.0027 | Fail to reject H0 |
| 2014 | -0.0030 | 0.0030 | Fail to reject H0 |

(b)

| S&P 100 vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0031 | 0.0027 | Fail to reject H0 |
| 2008 | -0.0057 | 0.0053 | Fail to reject H0 |
| 2009 | -0.0104 | 0.0094 | Fail to reject H0 |
| 2010 | -0.0038 | 0.0036 | Fail to reject H0 |
| 2011 | -0.0029 | 0.0031 | Fail to reject H0 |
| 2012 | -0.0028 | 0.0022 | Fail to reject H0 |
| 2013 | -0.0029 | 0.0025 | Fail to reject H0 |
| 2014 | -0.0027 | 0.0031 | Fail to reject H0 |

(c)

| S&P 100 vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0032 | 0.0028 | Fail to reject H0 |
| 2008 | -0.0054 | 0.0052 | Fail to reject H0 |
| 2009 | -0.0094 | 0.0093 | Fail to reject H0 |
| 2010 | -0.0034 | 0.0036 | Fail to reject H0 |
| 2011 | -0.0026 | 0.0032 | Fail to reject H0 |
| 2012 | -0.0026 | 0.0020 | Fail to reject H0 |
| 2013 | -0.0027 | 0.0027 | Fail to reject H0 |
| 2014 | -0.0028 | 0.0031 | Fail to reject H0 |

(d)

| S&P 100 vs. CR30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0028 | 0.0030 | Fail to reject H0 |
| 2008 | -0.0063 | 0.0049 | Fail to reject H0 |
| 2009 | -0.0086 | 0.0105 | Fail to reject H0 |
| 2010 | -0.0033 | 0.0036 | Fail to reject H0 |
| 2011 | -0.0027 | 0.0029 | Fail to reject H0 |
| 2012 | -0.0027 | 0.0020 | Fail to reject H0 |
| 2013 | -0.0029 | 0.0025 | Fail to reject H0 |
| 2014 | -0.0031 | 0.0025 | Fail to reject H0 |

Table 33 – Test results for difference between means of daily returns out-of-sample among portfolios, using S&P 100 index and cointegrated portfolios with at most 15 assets.

(a)

| CT15-SPC vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0030 | 0.0030 | Fail to reject H0 |
| 2008 | -0.0063 | 0.0051 | Fail to reject H0 |
| 2009 | -0.0135 | 0.0140 | Fail to reject H0 |
| 2010 | -0.0039 | 0.0060 | Fail to reject H0 |
| 2011 | -0.0038 | 0.0033 | Fail to reject H0 |
| 2012 | -0.0032 | 0.0027 | Fail to reject H0 |
| 2013 | -0.0024 | 0.0039 | Fail to reject H0 |
| 2014 | -0.0039 | 0.0026 | Fail to reject H0 |

(b)

| CT15-SPC vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0035 | 0.0025 | Fail to reject H0 |
| 2008 | -0.0047 | 0.0055 | Fail to reject H0 |
| 2009 | -0.0138 | 0.0157 | Fail to reject H0 |
| 2010 | -0.0034 | 0.0058 | Fail to reject H0 |
| 2011 | -0.0037 | 0.0033 | Fail to reject H0 |
| 2012 | -0.0033 | 0.0027 | Fail to reject H0 |
| 2013 | -0.0030 | 0.0027 | Fail to reject H0 |
| 2014 | -0.0035 | 0.0035 | Fail to reject H0 |

(c)

| CT15-SPC vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0033 | 0.0027 | Fail to reject H0 |
| 2008 | -0.0050 | 0.0050 | Fail to reject H0 |
| 2009 | -0.0139 | 0.0165 | Fail to reject H0 |
| 2010 | -0.0042 | 0.0053 | Fail to reject H0 |
| 2011 | -0.0043 | 0.0021 | Fail to reject H0 |
| 2012 | -0.0033 | 0.0033 | Fail to reject H0 |
| 2013 | -0.0028 | 0.0035 | Fail to reject H0 |
| 2014 | -0.0028 | 0.0038 | Fail to reject H0 |

Table 34 – Test results for difference between means of daily returns out-of-sample among portfolios, using S&P 100 index and cointegrated portfolios with at most 30 assets.

(a)

| CT30-SPC vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0032 | 0.0030 | Fail to reject H0 |
| 2008 | -0.0054 | 0.0067 | Fail to reject H0 |
| 2009 | -0.0111 | 0.0139 | Fail to reject H0 |
| 2010 | -0.0038 | 0.0036 | Fail to reject H0 |
| 2011 | -0.0031 | 0.0033 | Fail to reject H0 |
| 2012 | -0.0024 | 0.0031 | Fail to reject H0 |
| 2013 | -0.0030 | 0.0029 | Fail to reject H0 |
| 2014 | -0.0023 | 0.0034 | Fail to reject H0 |

(b)

| CT30-SPC vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0031 | 0.0032 | Fail to reject H0 |
| 2008 | -0.0052 | 0.0068 | Fail to reject H0 |
| 2009 | -0.0104 | 0.0128 | Fail to reject H0 |
| 2010 | -0.0040 | 0.0039 | Fail to reject H0 |
| 2011 | -0.0030 | 0.0029 | Fail to reject H0 |
| 2012 | -0.0022 | 0.0030 | Fail to reject H0 |
| 2013 | -0.0027 | 0.0031 | Fail to reject H0 |
| 2014 | -0.0025 | 0.0033 | Fail to reject H0 |

(c)

| CT30-SPC vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0030 | 0.0031 | Fail to reject H0 |
| 2008 | -0.0050 | 0.0066 | Fail to reject H0 |
| 2009 | -0.0108 | 0.0122 | Fail to reject H0 |
| 2010 | -0.0042 | 0.0039 | Fail to reject H0 |
| 2011 | -0.0032 | 0.0029 | Fail to reject H0 |
| 2012 | -0.0031 | 0.0022 | Fail to reject H0 |
| 2013 | -0.0026 | 0.0032 | Fail to reject H0 |
| 2014 | -0.0029 | 0.0031 | Fail to reject H0 |

Table 35 – Test results for difference between means of daily returns out-of-sample among portfolios, using S&P 100 index and correlated portfolios with at most 15 assets.

(a)

| CR15-SPC vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0029 | 0.0032 | Fail to reject H0 |
| 2008 | -0.0060 | 0.0052 | Fail to reject H0 |
| 2009 | -0.0106 | 0.0084 | Fail to reject H0 |
| 2010 | -0.0036 | 0.0032 | Fail to reject H0 |
| 2011 | -0.0027 | 0.0036 | Fail to reject H0 |
| 2012 | -0.0032 | 0.0017 | Fail to reject H0 |
| 2013 | -0.0028 | 0.0026 | Fail to reject H0 |
| 2014 | -0.0030 | 0.0030 | Fail to reject H0 |

(b)

| CR15-SPC vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0032 | 0.0030 | Fail to reject H0 |
| 2008 | -0.0059 | 0.0058 | Fail to reject H0 |
| 2009 | -0.0098 | 0.0099 | Fail to reject H0 |
| 2010 | -0.0037 | 0.0030 | Fail to reject H0 |
| 2011 | -0.0032 | 0.0027 | Fail to reject H0 |
| 2012 | -0.0030 | 0.0018 | Fail to reject H0 |
| 2013 | -0.0025 | 0.0029 | Fail to reject H0 |
| 2014 | -0.0027 | 0.0032 | Fail to reject H0 |

(c)

| CR15-SPC vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0030 | 0.0031 | Fail to reject H0 |
| 2008 | -0.0058 | 0.0054 | Fail to reject H0 |
| 2009 | -0.0099 | 0.0097 | Fail to reject H0 |
| 2010 | -0.0038 | 0.0032 | Fail to reject H0 |
| 2011 | -0.0030 | 0.0032 | Fail to reject H0 |
| 2012 | -0.0029 | 0.0020 | Fail to reject H0 |
| 2013 | -0.0023 | 0.0029 | Fail to reject H0 |
| 2014 | -0.0030 | 0.0028 | Fail to reject H0 |

Table 36 – Test results for difference between means of daily returns out-of-sample among portfolios, using S&P 100 index and correlated portfolios with at most 30 assets.

(a)

| CR30-SPC vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0033 | 0.0029 | Fail to reject H0 |
| 2008 | -0.0065 | 0.0058 | Fail to reject H0 |
| 2009 | -0.0111 | 0.0099 | Fail to reject H0 |
| 2010 | -0.0042 | 0.0033 | Fail to reject H0 |
| 2011 | -0.0032 | 0.0027 | Fail to reject H0 |
| 2012 | -0.0020 | 0.0026 | Fail to reject H0 |
| 2013 | -0.0026 | 0.0026 | Fail to reject H0 |
| 2014 | -0.0030 | 0.0029 | Fail to reject H0 |

(b)

| CR30-SPC vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0030 | 0.0027 | Fail to reject H0 |
| 2008 | -0.0056 | 0.0060 | Fail to reject H0 |
| 2009 | -0.0108 | 0.0091 | Fail to reject H0 |
| 2010 | -0.0038 | 0.0031 | Fail to reject H0 |
| 2011 | -0.0028 | 0.0028 | Fail to reject H0 |
| 2012 | -0.0022 | 0.0025 | Fail to reject H0 |
| 2013 | -0.0029 | 0.0024 | Fail to reject H0 |
| 2014 | -0.0025 | 0.0031 | Fail to reject H0 |

(c)

| CR30-SPC vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0033 | 0.0027 | Fail to reject H0 |
| 2008 | -0.0049 | 0.0059 | Fail to reject H0 |
| 2009 | -0.0106 | 0.0094 | Fail to reject H0 |
| 2010 | -0.0035 | 0.0034 | Fail to reject H0 |
| 2011 | -0.0030 | 0.0029 | Fail to reject H0 |
| 2012 | -0.0022 | 0.0025 | Fail to reject H0 |
| 2013 | -0.0026 | 0.0026 | Fail to reject H0 |
| 2014 | -0.0025 | 0.0033 | Fail to reject H0 |

# APPENDIX C – Test Results for Difference Between Means of Daily Tracking Error Values

Results for the hypothesis test for the difference between two means using time series of **daily tracking error values** out-of-sample.

Table 37 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using Ibovespa index and cointegrated portfolios with at most 15 assets.

(a)

| Ibovespa: CT15-SPC vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0009 | 0.0009 | Fail to reject H0 |
| 2008 | -0.0017 | 0.0014 | Fail to reject H0 |
| 2009 | -0.0006 | 0.0020 | Fail to reject H0 |
| 2010 | -0.0013 | 0.0010 | Fail to reject H0 |
| 2011 | -0.0018 | 0.0007 | Fail to reject H0 |
| 2012 | -0.0007 | 0.0018 | Fail to reject H0 |
| 2013 | -0.0011 | 0.0025 | Fail to reject H0 |
| 2014 | -0.0017 | 0.0015 | Fail to reject H0 |

(b)

| Ibovespa: CT15-SPC vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0011 | 0.0008 | Fail to reject H0 |
| 2008 | -0.0023 | 0.0008 | Fail to reject H0 |
| 2009 | -0.0002 | 0.0024 | Fail to reject H0 |
| 2010 | -0.0013 | 0.0008 | Fail to reject H0 |
| 2011 | -0.0015 | 0.0009 | Fail to reject H0 |
| 2012 | -0.0009 | 0.0015 | Fail to reject H0 |
| 2013 | -0.0005 | 0.0027 | Fail to reject H0 |
| 2014 | -0.0016 | 0.0013 | Fail to reject H0 |

(c)

| Ibovespa: CT15-SPC vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0011 | 0.0007 | Fail to reject H0 |
| 2008 | -0.0023 | 0.0008 | Fail to reject H0 |
| 2009 | -0.0002 | 0.0024 | Fail to reject H0 |
| 2010 | -0.0013 | 0.0008 | Fail to reject H0 |
| 2011 | -0.0013 | 0.0010 | Fail to reject H0 |
| 2012 | -0.0006 | 0.0016 | Fail to reject H0 |
| 2013 | -0.0007 | 0.0025 | Fail to reject H0 |
| 2014 | -0.0014 | 0.0016 | Fail to reject H0 |

Table 38 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using Ibovespa index and cointegrated portfolios with at most 30 assets.

(a)

| Ibovespa: CT30-SPC vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0009 | 0.0009 | Fail to reject H0 |
| 2008 | -0.0014 | 0.0010 | Fail to reject H0 |
| 2009 | -0.0010 | 0.0014 | Fail to reject H0 |
| 2010 | -0.0005 | 0.0013 | Fail to reject H0 |
| 2011 | -0.0009 | 0.0014 | Fail to reject H0 |
| 2012 | -0.0010 | 0.0011 | Fail to reject H0 |
| 2013 | -0.0017 | 0.0011 | Fail to reject H0 |
| 2014 | -0.0009 | 0.0015 | Fail to reject H0 |

(b)

| Ibovespa: CT30-SPC vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0011 | 0.0006 | Fail to reject H0 |
| 2008 | -0.0019 | 0.0006 | Fail to reject H0 |
| 2009 | -0.0012 | 0.0013 | Fail to reject H0 |
| 2010 | -0.0009 | 0.0011 | Fail to reject H0 |
| 2011 | -0.0010 | 0.0013 | Fail to reject H0 |
| 2012 | -0.0007 | 0.0013 | Fail to reject H0 |
| 2013 | -0.0014 | 0.0015 | Fail to reject H0 |
| 2014 | -0.0011 | 0.0012 | Fail to reject H0 |

(c)

| Ibovespa: CT30-SPC vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0011 | 0.0007 | Fail to reject H0 |
| 2008 | -0.0019 | 0.0006 | Fail to reject H0 |
| 2009 | -0.0011 | 0.0013 | Fail to reject H0 |
| 2010 | -0.0010 | 0.0011 | Fail to reject H0 |
| 2011 | -0.0007 | 0.0011 | Fail to reject H0 |
| 2012 | -0.0008 | 0.0012 | Fail to reject H0 |
| 2013 | -0.0014 | 0.0015 | Fail to reject H0 |
| 2014 | -0.0011 | 0.0010 | Fail to reject H0 |

Table 39 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using Ibovespa index and correlated portfolios with at most 15 assets.

(a)

| Ibovespa: CR15-SPC vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0011 | 0.0006 | Fail to reject H0 |
| 2008 | -0.0014 | 0.0009 | Fail to reject H0 |
| 2009 | -0.0011 | 0.0012 | Fail to reject H0 |
| 2010 | -0.0012 | 0.0005 | Fail to reject H0 |
| 2011 | -0.0006 | 0.0009 | Fail to reject H0 |
| 2012 | -0.0011 | 0.0011 | Fail to reject H0 |
| 2013 | -0.0017 | 0.0014 | Fail to reject H0 |
| 2014 | -0.0014 | 0.0020 | Fail to reject H0 |

(b)

| Ibovespa: CR15-SPC vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0008 | 0.0008 | Fail to reject H0 |
| 2008 | -0.0021 | 0.0005 | Fail to reject H0 |
| 2009 | -0.0014 | 0.0009 | Fail to reject H0 |
| 2010 | -0.0013 | 0.0004 | Fail to reject H0 |
| 2011 | -0.0007 | 0.0009 | Fail to reject H0 |
| 2012 | -0.0010 | 0.0012 | Fail to reject H0 |
| 2013 | -0.0019 | 0.0013 | Fail to reject H0 |
| 2014 | -0.0013 | 0.0017 | Fail to reject H0 |

(c)

| Ibovespa: CR15-SPC vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0008 | 0.0009 | Fail to reject H0 |
| 2008 | -0.0021 | 0.0006 | Fail to reject H0 |
| 2009 | -0.0014 | 0.0008 | Fail to reject H0 |
| 2010 | -0.0012 | 0.0004 | Fail to reject H0 |
| 2011 | -0.0006 | 0.0009 | Fail to reject H0 |
| 2012 | -0.0012 | 0.0010 | Fail to reject H0 |
| 2013 | -0.0020 | 0.0012 | Fail to reject H0 |
| 2014 | -0.0013 | 0.0017 | Fail to reject H0 |

Table 40 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using Ibovespa index and correlated portfolios with at most 30 assets.

(a)

| Ibovespa: CR30-SPC vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0007 | 0.0006 | Fail to reject H0 |
| 2008 | -0.0011 | 0.0008 | Fail to reject H0 |
| 2009 | -0.0007 | 0.0011 | Fail to reject H0 |
| 2010 | -0.0009 | 0.0003 | Fail to reject H0 |
| 2011 | -0.0006 | 0.0008 | Fail to reject H0 |
| 2012 | -0.0008 | 0.0008 | Fail to reject H0 |
| 2013 | -0.0012 | 0.0013 | Fail to reject H0 |
| 2014 | -0.0016 | 0.0013 | Fail to reject H0 |

(b)

| Ibovespa: CR30-SPC vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0008 | 0.0005 | Fail to reject H0 |
| 2008 | -0.0016 | 0.0007 | Fail to reject H0 |
| 2009 | -0.0009 | 0.0009 | Fail to reject H0 |
| 2010 | -0.0010 | 0.0002 | Fail to reject H0 |
| 2011 | -0.0007 | 0.0006 | Fail to reject H0 |
| 2012 | -0.0007 | 0.0009 | Fail to reject H0 |
| 2013 | -0.0012 | 0.0011 | Fail to reject H0 |
| 2014 | -0.0017 | 0.0011 | Fail to reject H0 |

(c)

| Ibovespa: CR30-SPC vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0008 | 0.0005 | Fail to reject H0 |
| 2008 | -0.0015 | 0.0007 | Fail to reject H0 |
| 2009 | -0.0008 | 0.0010 | Fail to reject H0 |
| 2010 | -0.0008 | 0.0003 | Fail to reject H0 |
| 2011 | -0.0008 | 0.0005 | Fail to reject H0 |
| 2012 | -0.0007 | 0.0009 | Fail to reject H0 |
| 2013 | -0.0014 | 0.0012 | Fail to reject H0 |
| 2014 | -0.0017 | 0.0012 | Fail to reject H0 |

Table 41 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using S&P 100 index and cointegrated portfolios with at most 15 assets.

(a)

| S&P 100: CT15-SPC vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0013 | 0.0010 | Fail to reject H0 |
| 2008 | -0.0025 | 0.0016 | Fail to reject H0 |
| 2009 | -0.0047 | 0.0061 | Fail to reject H0 |
| 2010 | -0.0007 | 0.0026 | Fail to reject H0 |
| 2011 | -0.0017 | 0.0013 | Fail to reject H0 |
| 2012 | -0.0014 | 0.0012 | Fail to reject H0 |
| 2013 | -0.0004 | 0.0022 | Fail to reject H0 |
| 2014 | -0.0018 | 0.0007 | Fail to reject H0 |

(b)

| S&P 100: CT15-SPC vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0014 | 0.0009 | Fail to reject H0 |
| 2008 | -0.0017 | 0.0027 | Fail to reject H0 |
| 2009 | -0.0052 | 0.0078 | Fail to reject H0 |
| 2010 | -0.0004 | 0.0027 | Fail to reject H0 |
| 2011 | -0.0017 | 0.0013 | Fail to reject H0 |
| 2012 | -0.0014 | 0.0011 | Fail to reject H0 |
| 2013 | -0.0013 | 0.0012 | Fail to reject H0 |
| 2014 | -0.0013 | 0.0012 | Fail to reject H0 |

(c)

| S&P 100: CT15-SPC vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0015 | 0.0007 | Fail to reject H0 |
| 2008 | -0.0020 | 0.0024 | Fail to reject H0 |
| 2009 | -0.0062 | 0.0073 | Fail to reject H0 |
| 2010 | -0.0012 | 0.0021 | Fail to reject H0 |
| 2011 | -0.0028 | 0.0003 | Fail to reject H0 |
| 2012 | -0.0013 | 0.0014 | Fail to reject H0 |
| 2013 | -0.0011 | 0.0015 | Fail to reject H0 |
| 2014 | -0.0008 | 0.0017 | Fail to reject H0 |

Table 42 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using S&P 100 index and cointegrated portfolios with at most 30 assets.

(a)

| S&P 100: CT30-SPC vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0006 | 0.0008 | Fail to reject H0 |
| 2008 | -0.0009 | 0.0020 | Fail to reject H0 |
| 2009 | -0.0025 | 0.0049 | Fail to reject H0 |
| 2010 | -0.0007 | 0.0009 | Fail to reject H0 |
| 2011 | -0.0008 | 0.0011 | Fail to reject H0 |
| 2012 | -0.0009 | 0.0010 | Fail to reject H0 |
| 2013 | -0.0011 | 0.0007 | Fail to reject H0 |
| 2014 | -0.0003 | 0.0013 | Fail to reject H0 |

(b)

| S&P 100: CT30-SPC vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0007 | 0.0008 | Fail to reject H0 |
| 2008 | -0.0004 | 0.0020 | Fail to reject H0 |
| 2009 | -0.0020 | 0.0044 | Fail to reject H0 |
| 2010 | -0.0009 | 0.0007 | Fail to reject H0 |
| 2011 | -0.0011 | 0.0008 | Fail to reject H0 |
| 2012 | -0.0005 | 0.0014 | Fail to reject H0 |
| 2013 | -0.0006 | 0.0012 | Fail to reject H0 |
| 2014 | -0.0005 | 0.0013 | Fail to reject H0 |

(c)

| S&P 100: CT30-SPC vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0007 | 0.0008 | Fail to reject H0 |
| 2008 | -0.0003 | 0.0021 | Fail to reject H0 |
| 2009 | -0.0022 | 0.0036 | Fail to reject H0 |
| 2010 | -0.0009 | 0.0006 | Fail to reject H0 |
| 2011 | -0.0011 | 0.0007 | Fail to reject H0 |
| 2012 | -0.0014 | 0.0007 | Fail to reject H0 |
| 2013 | -0.0005 | 0.0012 | Fail to reject H0 |
| 2014 | -0.0008 | 0.0009 | Fail to reject H0 |

Table 43 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using S&P 100 index and correlated portfolios with at most 15 assets.

(a)

| S&P 100: CR15-SPC vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0005 | 0.0007 | Fail to reject H0 |
| 2008 | -0.0019 | 0.0007 | Fail to reject H0 |
| 2009 | -0.0026 | 0.0010 | Fail to reject H0 |
| 2010 | -0.0010 | 0.0008 | Fail to reject H0 |
| 2011 | -0.0005 | 0.0011 | Fail to reject H0 |
| 2012 | -0.0015 | 0.0001 | Fail to reject H0 |
| 2013 | -0.0007 | 0.0007 | Fail to reject H0 |
| 2014 | -0.0007 | 0.0007 | Fail to reject H0 |

(b)

| S&P 100: CR15-SPC vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0006 | 0.0006 | Fail to reject H0 |
| 2008 | -0.0011 | 0.0012 | Fail to reject H0 |
| 2009 | -0.0022 | 0.0013 | Fail to reject H0 |
| 2010 | -0.0014 | 0.0004 | Fail to reject H0 |
| 2011 | -0.0010 | 0.0006 | Fail to reject H0 |
| 2012 | -0.0013 | 0.0002 | Fail to reject H0 |
| 2013 | -0.0004 | 0.0008 | Fail to reject H0 |
| 2014 | -0.0008 | 0.0009 | Fail to reject H0 |

(c)

| S&P 100: CR15-SPC vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0006 | 0.0006 | Fail to reject H0 |
| 2008 | -0.0012 | 0.0011 | Fail to reject H0 |
| 2009 | -0.0019 | 0.0015 | Fail to reject H0 |
| 2010 | -0.0013 | 0.0004 | Fail to reject H0 |
| 2011 | -0.0008 | 0.0010 | Fail to reject H0 |
| 2012 | -0.0012 | 0.0005 | Fail to reject H0 |
| 2013 | -0.0004 | 0.0009 | Fail to reject H0 |
| 2014 | -0.0008 | 0.0006 | Fail to reject H0 |

Table 44 – Test results for difference between means of daily tracking error out-of-sample among portfolios, using S&P 100 index and correlated portfolios with at most 30 assets.

(a)

| S&P 100: CR30-SPC vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0006 | 0.0003 | Fail to reject H0 |
| 2008 | -0.0009 | 0.0008 | Fail to reject H0 |
| 2009 | -0.0019 | 0.0009 | Fail to reject H0 |
| 2010 | -0.0011 | 0.0001 | Fail to reject H0 |
| 2011 | -0.0008 | 0.0007 | Fail to reject H0 |
| 2012 | -0.0002 | 0.0008 | Fail to reject H0 |
| 2013 | -0.0004 | 0.0006 | Fail to reject H0 |
| 2014 | -0.0003 | 0.0006 | Fail to reject H0 |

(b)

| S&P 100: CR30-SPC vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0007 | 0.0001 | Fail to reject H0 |
| 2008 | -0.0005 | 0.0010 | Fail to reject H0 |
| 2009 | -0.0021 | 0.0006 | Fail to reject H0 |
| 2010 | -0.0008 | 0.0003 | Fail to reject H0 |
| 2011 | -0.0007 | 0.0008 | Fail to reject H0 |
| 2012 | -0.0004 | 0.0006 | Fail to reject H0 |
| 2013 | -0.0004 | 0.0004 | Fail to reject H0 |
| 2014 | -0.0001 | 0.0008 | Fail to reject H0 |

(c)

| S&P 100: CR30-SPC vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | -0.0007 | 0.0002 | Fail to reject H0 |
| 2008 | -0.0005 | 0.0011 | Fail to reject H0 |
| 2009 | -0.0016 | 0.0009 | Fail to reject H0 |
| 2010 | -0.0005 | 0.0006 | Fail to reject H0 |
| 2011 | -0.0006 | 0.0008 | Fail to reject H0 |
| 2012 | -0.0005 | 0.0006 | Fail to reject H0 |
| 2013 | -0.0003 | 0.0006 | Fail to reject H0 |
| 2014 | 0.0000 | 0.0009 | Fail to reject H0 |

# APPENDIX D – Test Results for Difference Between Variances of Daily Returns

Results for the hypothesis test for the difference between variances of two time series using **daily returns** out-of-sample of index and portfolios.

Table 45 – Test results for difference between variances of daily returns out-of-sample, for Ibovespa index and cointegrated portfolios with at most 15 assets.

(a)

| IBOV vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7284 | 1.3406 | Fail to reject H0 |
| 2008 | 0.6414 | 1.4978 | Fail to reject H0 |
| 2009 | 0.7131 | 1.3714 | Fail to reject H0 |
| 2010 | 0.8070 | 1.4752 | Fail to reject H0 |
| 2011 | 0.6911 | 1.5082 | Fail to reject H0 |
| 2012 | 0.6922 | 1.1875 | Fail to reject H0 |
| 2013 | 0.7672 | 1.3161 | Fail to reject H0 |
| 2014 | 0.6054 | 1.1813 | Fail to reject H0 |

(b)

| IBOV vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7081 | 1.3023 | Fail to reject H0 |
| 2008 | 0.7578 | 1.7004 | Fail to reject H0 |
| 2009 | 0.7175 | 1.3322 | Fail to reject H0 |
| 2010 | 0.7729 | 1.3931 | Fail to reject H0 |
| 2011 | 0.7120 | 1.6092 | Fail to reject H0 |
| 2012 | 0.7732 | 1.3092 | Fail to reject H0 |
| 2013 | 0.7476 | 1.2792 | Fail to reject H0 |
| 2014 | 0.6905 | 1.2968 | Fail to reject H0 |

(c)

| IBOV vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6811 | 1.2584 | Fail to reject H0 |
| 2008 | 0.6540 | 1.4780 | Fail to reject H0 |
| 2009 | 0.6990 | 1.3660 | Fail to reject H0 |
| 2010 | 0.8157 | 1.4621 | Fail to reject H0 |
| 2011 | 0.7728 | 1.7457 | Fail to reject H0 |
| 2012 | 0.7921 | 1.3452 | Fail to reject H0 |
| 2013 | 0.7551 | 1.2873 | Fail to reject H0 |
| 2014 | 0.6108 | 1.1771 | Fail to reject H0 |

(d)

| IBOV vs. CT15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7142 | 1.3049 | Fail to reject H0 |
| 2008 | 0.6407 | 1.4414 | Fail to reject H0 |
| 2009 | 0.7276 | 1.3709 | Fail to reject H0 |
| 2010 | 0.7532 | 1.3564 | Fail to reject H0 |
| 2011 | 0.6903 | 1.4822 | Fail to reject H0 |
| 2012 | 0.6688 | 1.1719 | Fail to reject H0 |
| 2013 | 0.7160 | 1.2246 | Fail to reject H0 |
| 2014 | 0.6016 | 1.1173 | Fail to reject H0 |

Table 46 – Test results for difference between variances of daily returns out-of-sample, for Ibovespa index and cointegrated portfolios with at most 30 assets.

(a)

| IBOV vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7345 | 1.3471 | Fail to reject H0 |
| 2008 | 0.7218 | 1.5551 | Fail to reject H0 |
| 2009 | 0.7649 | 1.4969 | Fail to reject H0 |
| 2010 | 0.8403 | 1.5457 | Fail to reject H0 |
| 2011 | 0.7622 | 1.6120 | Fail to reject H0 |
| 2012 | 0.7255 | 1.2342 | Fail to reject H0 |
| 2013 | 0.8216 | 1.4209 | Fail to reject H0 |
| 2014 | 0.6887 | 1.3088 | Fail to reject H0 |

(b)

| IBOV vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7187 | 1.3274 | Fail to reject H0 |
| 2008 | 0.7355 | 1.7327 | Fail to reject H0 |
| 2009 | 0.7511 | 1.4457 | Fail to reject H0 |
| 2010 | 0.8483 | 1.5481 | Fail to reject H0 |
| 2011 | 0.7485 | 1.7054 | Fail to reject H0 |
| 2012 | 0.7949 | 1.3702 | Fail to reject H0 |
| 2013 | 0.8494 | 1.4480 | Fail to reject H0 |
| 2014 | 0.7103 | 1.3445 | Fail to reject H0 |

(c)

| IBOV vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6977 | 1.3102 | Fail to reject H0 |
| 2008 | 0.7658 | 1.7681 | Fail to reject H0 |
| 2009 | 0.7900 | 1.4611 | Fail to reject H0 |
| 2010 | 0.8697 | 1.5847 | Fail to reject H0 |
| 2011 | 0.7852 | 1.7329 | Fail to reject H0 |
| 2012 | 0.8049 | 1.3546 | Fail to reject H0 |
| 2013 | 0.8365 | 1.4083 | Fail to reject H0 |
| 2014 | 0.6809 | 1.3105 | Fail to reject H0 |

(d)

| IBOV vs. CT30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7194 | 1.3048 | Fail to reject H0 |
| 2008 | 0.6721 | 1.5190 | Fail to reject H0 |
| 2009 | 0.7664 | 1.4689 | Fail to reject H0 |
| 2010 | 0.8227 | 1.4981 | Fail to reject H0 |
| 2011 | 0.7364 | 1.4869 | Fail to reject H0 |
| 2012 | 0.7116 | 1.2421 | Fail to reject H0 |
| 2013 | 0.7218 | 1.2204 | Fail to reject H0 |
| 2014 | 0.6589 | 1.2890 | Fail to reject H0 |

Table 47 – Test results for difference between variances of daily returns out-of-sample, for Ibovespa index and correlated portfolios with at most 15 assets.

(a)

| IBOV vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7035 | 1.2897 | Fail to reject H0 |
| 2008 | 0.6806 | 1.4792 | Fail to reject H0 |
| 2009 | 0.7204 | 1.3632 | Fail to reject H0 |
| 2010 | 0.7254 | 1.3046 | Fail to reject H0 |
| 2011 | 0.6642 | 1.5350 | Fail to reject H0 |
| 2012 | 0.7264 | 1.2384 | Fail to reject H0 |
| 2013 | 0.8016 | 1.3663 | Fail to reject H0 |
| 2014 | 0.6572 | 1.2434 | Fail to reject H0 |

(b)

| IBOV vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7179 | 1.3075 | Fail to reject H0 |
| 2008 | 0.6859 | 1.5715 | Fail to reject H0 |
| 2009 | 0.7326 | 1.3778 | Fail to reject H0 |
| 2010 | 0.6999 | 1.2752 | Fail to reject H0 |
| 2011 | 0.7077 | 1.5530 | Fail to reject H0 |
| 2012 | 0.7211 | 1.2646 | Fail to reject H0 |
| 2013 | 0.8233 | 1.3826 | Fail to reject H0 |
| 2014 | 0.6454 | 1.2230 | Fail to reject H0 |

(c)

| IBOV vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7025 | 1.2609 | Fail to reject H0 |
| 2008 | 0.7538 | 1.6418 | Fail to reject H0 |
| 2009 | 0.7406 | 1.4027 | Fail to reject H0 |
| 2010 | 0.7331 | 1.3244 | Fail to reject H0 |
| 2011 | 0.6601 | 1.5466 | Fail to reject H0 |
| 2012 | 0.7231 | 1.2780 | Fail to reject H0 |
| 2013 | 0.8114 | 1.4046 | Fail to reject H0 |
| 2014 | 0.5812 | 1.1095 | Fail to reject H0 |

(d)

| IBOV vs. CR15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7042 | 1.3077 | Fail to reject H0 |
| 2008 | 0.6696 | 1.4973 | Fail to reject H0 |
| 2009 | 0.7205 | 1.3786 | Fail to reject H0 |
| 2010 | 0.7408 | 1.3057 | Fail to reject H0 |
| 2011 | 0.6780 | 1.4866 | Fail to reject H0 |
| 2012 | 0.7095 | 1.2612 | Fail to reject H0 |
| 2013 | 0.8200 | 1.3557 | Fail to reject H0 |
| 2014 | 0.6970 | 1.2909 | Fail to reject H0 |

Table 48 – Test results for difference between variances of daily returns out-of-sample, for Ibovespa index and correlated portfolios with at most 30 assets.

(a)

| IBOV vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7300 | 1.3096 | Fail to reject H0 |
| 2008 | 0.6666 | 1.5278 | Fail to reject H0 |
| 2009 | 0.7466 | 1.3838 | Fail to reject H0 |
| 2010 | 0.7328 | 1.3253 | Fail to reject H0 |
| 2011 | 0.6832 | 1.4858 | Fail to reject H0 |
| 2012 | 0.7455 | 1.2969 | Fail to reject H0 |
| 2013 | 0.8237 | 1.4291 | Fail to reject H0 |
| 2014 | 0.6764 | 1.2912 | Fail to reject H0 |

(b)

| IBOV vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7383 | 1.3290 | Fail to reject H0 |
| 2008 | 0.6997 | 1.6397 | Fail to reject H0 |
| 2009 | 0.7233 | 1.4048 | Fail to reject H0 |
| 2010 | 0.7419 | 1.3264 | Fail to reject H0 |
| 2011 | 0.6873 | 1.5517 | Fail to reject H0 |
| 2012 | 0.7525 | 1.3188 | Fail to reject H0 |
| 2013 | 0.8346 | 1.4253 | Fail to reject H0 |
| 2014 | 0.6701 | 1.2631 | Fail to reject H0 |

(c)

| IBOV vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7140 | 1.2778 | Fail to reject H0 |
| 2008 | 0.7629 | 1.7029 | Fail to reject H0 |
| 2009 | 0.7305 | 1.4629 | Fail to reject H0 |
| 2010 | 0.7628 | 1.3593 | Fail to reject H0 |
| 2011 | 0.6989 | 1.5873 | Fail to reject H0 |
| 2012 | 0.7969 | 1.3553 | Fail to reject H0 |
| 2013 | 0.8180 | 1.4111 | Fail to reject H0 |
| 2014 | 0.6054 | 1.1835 | Fail to reject H0 |

(d)

| IBOV vs. CR30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7320 | 1.2907 | Fail to reject H0 |
| 2008 | 0.6785 | 1.4719 | Fail to reject H0 |
| 2009 | 0.7594 | 1.4203 | Fail to reject H0 |
| 2010 | 0.7657 | 1.3559 | Fail to reject H0 |
| 2011 | 0.6884 | 1.5560 | Fail to reject H0 |
| 2012 | 0.7306 | 1.3088 | Fail to reject H0 |
| 2013 | 0.8307 | 1.4311 | Fail to reject H0 |
| 2014 | 0.6476 | 1.2483 | Fail to reject H0 |

Table 49 – Test results for difference between variances of daily returns out-of-sample among portfolios, using Ibovespa index and cointegrated portfolios with at most 15 assets.

(a)

| CT15-SPC vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7040 | 1.3374 | Fail to reject H0 |
| 2008 | 0.6271 | 1.4323 | Fail to reject H0 |
| 2009 | 0.7320 | 1.3605 | Fail to reject H0 |
| 2010 | 0.6827 | 1.2441 | Fail to reject H0 |
| 2011 | 0.6710 | 1.4751 | Fail to reject H0 |
| 2012 | 0.7398 | 1.3107 | Fail to reject H0 |
| 2013 | 0.7460 | 1.2326 | Fail to reject H0 |
| 2014 | 0.7288 | 1.3979 | Fail to reject H0 |

(b)

| CT15-SPC vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7607 | 1.3656 | Fail to reject H0 |
| 2008 | 0.5622 | 1.2846 | Fail to reject H0 |
| 2009 | 0.7254 | 1.4191 | Fail to reject H0 |
| 2010 | 0.7263 | 1.3710 | Fail to reject H0 |
| 2011 | 0.6379 | 1.3580 | Fail to reject H0 |
| 2012 | 0.6724 | 1.1557 | Fail to reject H0 |
| 2013 | 0.7243 | 1.2589 | Fail to reject H0 |
| 2014 | 0.6368 | 1.1712 | Fail to reject H0 |

(c)

| CT15-SPC vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7756 | 1.4087 | Fail to reject H0 |
| 2008 | 0.6405 | 1.4644 | Fail to reject H0 |
| 2009 | 0.7477 | 1.3849 | Fail to reject H0 |
| 2010 | 0.6836 | 1.2299 | Fail to reject H0 |
| 2011 | 0.5848 | 1.3160 | Fail to reject H0 |
| 2012 | 0.6344 | 1.0939 | Fail to reject H0 |
| 2013 | 0.7303 | 1.2238 | Fail to reject H0 |
| 2014 | 0.7105 | 1.3697 | Fail to reject H0 |

Table 50 – Test results for difference between variances of daily returns out-of-sample among portfolios, using Ibovespa index and cointegrated portfolios with at most 30 assets.

(a)

| CT30-SPC vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7744 | 1.3692 | Fail to reject H0 |
| 2008 | 0.7053 | 1.5917 | Fail to reject H0 |
| 2009 | 0.7279 | 1.3739 | Fail to reject H0 |
| 2010 | 0.7807 | 1.4463 | Fail to reject H0 |
| 2011 | 0.7088 | 1.4745 | Fail to reject H0 |
| 2012 | 0.7610 | 1.3295 | Fail to reject H0 |
| 2013 | 0.8459 | 1.5168 | Fail to reject H0 |
| 2014 | 0.7480 | 1.4015 | Fail to reject H0 |

(b)

| CT30-SPC vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7406 | 1.3759 | Fail to reject H0 |
| 2008 | 0.7538 | 1.7245 | Fail to reject H0 |
| 2009 | 0.7172 | 1.3468 | Fail to reject H0 |
| 2010 | 0.7750 | 1.4206 | Fail to reject H0 |
| 2011 | 0.7216 | 1.5491 | Fail to reject H0 |
| 2012 | 0.8112 | 1.4371 | Fail to reject H0 |
| 2013 | 0.8789 | 1.4954 | Fail to reject H0 |
| 2014 | 0.7892 | 1.4857 | Fail to reject H0 |

(c)

| CT30-SPC vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7329 | 1.3646 | Fail to reject H0 |
| 2008 | 0.7603 | 1.7358 | Fail to reject H0 |
| 2009 | 0.7174 | 1.3712 | Fail to reject H0 |
| 2010 | 0.8069 | 1.4492 | Fail to reject H0 |
| 2011 | 0.7942 | 1.6252 | Fail to reject H0 |
| 2012 | 0.8311 | 1.5020 | Fail to reject H0 |
| 2013 | 0.8810 | 1.4804 | Fail to reject H0 |
| 2014 | 0.7378 | 1.3850 | Fail to reject H0 |

Table 51 – Test results for difference between variances of daily returns out-of-sample among portfolios, using Ibovespa index and correlated portfolios with at most 15 assets.

(a)

| CR15-SPC vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7440 | 1.3469 | Fail to reject H0 |
| 2008 | 0.6710 | 1.4781 | Fail to reject H0 |
| 2009 | 0.7464 | 1.3983 | Fail to reject H0 |
| 2010 | 0.7443 | 1.3352 | Fail to reject H0 |
| 2011 | 0.6805 | 1.5091 | Fail to reject H0 |
| 2012 | 0.7320 | 1.3196 | Fail to reject H0 |
| 2013 | 0.7801 | 1.3309 | Fail to reject H0 |
| 2014 | 0.6905 | 1.2769 | Fail to reject H0 |

(b)

| CR15-SPC vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7528 | 1.3645 | Fail to reject H0 |
| 2008 | 0.6975 | 1.5767 | Fail to reject H0 |
| 2009 | 0.7381 | 1.3885 | Fail to reject H0 |
| 2010 | 0.7543 | 1.3221 | Fail to reject H0 |
| 2011 | 0.6948 | 1.5962 | Fail to reject H0 |
| 2012 | 0.7672 | 1.3175 | Fail to reject H0 |
| 2013 | 0.7920 | 1.3482 | Fail to reject H0 |
| 2014 | 0.6626 | 1.2547 | Fail to reject H0 |

(c)

| CR15-SPC vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7222 | 1.2811 | Fail to reject H0 |
| 2008 | 0.7362 | 1.7073 | Fail to reject H0 |
| 2009 | 0.7319 | 1.4231 | Fail to reject H0 |
| 2010 | 0.7498 | 1.3611 | Fail to reject H0 |
| 2011 | 0.6795 | 1.5655 | Fail to reject H0 |
| 2012 | 0.7710 | 1.3102 | Fail to reject H0 |
| 2013 | 0.7818 | 1.3277 | Fail to reject H0 |
| 2014 | 0.6141 | 1.1441 | Fail to reject H0 |

Table 52 – Test results for difference between variances of daily returns out-of-sample among portfolios, using Ibovespa index and correlated portfolios with at most 30 assets.

(a)

| CR30-SPC vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7422 | 1.3460 | Fail to reject H0 |
| 2008 | 0.6911 | 1.5509 | Fail to reject H0 |
| 2009 | 0.7215 | 1.3892 | Fail to reject H0 |
| 2010 | 0.7037 | 1.2826 | Fail to reject H0 |
| 2011 | 0.6489 | 1.4287 | Fail to reject H0 |
| 2012 | 0.7519 | 1.3495 | Fail to reject H0 |
| 2013 | 0.7715 | 1.3127 | Fail to reject H0 |
| 2014 | 0.7397 | 1.4239 | Fail to reject H0 |

(b)

| CR30-SPC vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7439 | 1.3735 | Fail to reject H0 |
| 2008 | 0.6972 | 1.5933 | Fail to reject H0 |
| 2009 | 0.7020 | 1.3765 | Fail to reject H0 |
| 2010 | 0.7148 | 1.3362 | Fail to reject H0 |
| 2011 | 0.6633 | 1.4884 | Fail to reject H0 |
| 2012 | 0.7836 | 1.3714 | Fail to reject H0 |
| 2013 | 0.7700 | 1.2847 | Fail to reject H0 |
| 2014 | 0.6972 | 1.3791 | Fail to reject H0 |

(c)

| CR30-SPC vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7422 | 1.3501 | Fail to reject H0 |
| 2008 | 0.7238 | 1.6791 | Fail to reject H0 |
| 2009 | 0.7157 | 1.3943 | Fail to reject H0 |
| 2010 | 0.7441 | 1.3486 | Fail to reject H0 |
| 2011 | 0.6935 | 1.4929 | Fail to reject H0 |
| 2012 | 0.8118 | 1.3930 | Fail to reject H0 |
| 2013 | 0.7695 | 1.2893 | Fail to reject H0 |
| 2014 | 0.6704 | 1.3002 | Fail to reject H0 |

Table 53 – Test results for difference between variances of daily returns out-of-sample, using S&P 100 index and cointegrated portfolios with at most 15 assets.

(a)

| S&P 100 vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7289 | 1.3719 | Fail to reject H0 |
| 2008 | 0.5397 | 1.2822 | Fail to reject H0 |
| 2009 | 0.4375 | 0.9552 | Reject H0 |
| 2010 | 0.6130 | 1.2207 | Fail to reject H0 |
| 2011 | 0.5501 | 1.1486 | Fail to reject H0 |
| 2012 | 0.6998 | 1.2402 | Fail to reject H0 |
| 2013 | 0.6175 | 1.1862 | Fail to reject H0 |
| 2014 | 0.6629 | 1.3633 | Fail to reject H0 |

(b)

| S&P 100 vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6631 | 1.2750 | Fail to reject H0 |
| 2008 | 0.6340 | 1.4866 | Fail to reject H0 |
| 2009 | 0.3611 | 0.7772 | Reject H0 |
| 2010 | 0.5898 | 1.1986 | Fail to reject H0 |
| 2011 | 0.6435 | 1.3206 | Fail to reject H0 |
| 2012 | 0.6966 | 1.2152 | Fail to reject H0 |
| 2013 | 0.6007 | 1.1844 | Fail to reject H0 |
| 2014 | 0.5817 | 1.2494 | Fail to reject H0 |

(c)

| S&P 100 vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6078 | 1.1907 | Fail to reject H0 |
| 2008 | 0.3774 | 1.0895 | Fail to reject H0 |
| 2009 | 0.2780 | 0.6380 | Reject H0 |
| 2010 | 0.5972 | 1.1843 | Fail to reject H0 |
| 2011 | 0.6436 | 1.3334 | Fail to reject H0 |
| 2012 | 0.6908 | 1.2314 | Fail to reject H0 |
| 2013 | 0.6164 | 1.1394 | Fail to reject H0 |
| 2014 | 0.6051 | 1.2854 | Fail to reject H0 |

(d)

| S&P 100 vs. CT15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7022 | 1.3587 | Fail to reject H0 |
| 2008 | 0.5117 | 1.2021 | Fail to reject H0 |
| 2009 | 0.3001 | 0.6479 | Reject H0 |
| 2010 | 0.5121 | 0.9866 | Reject H0 |
| 2011 | 0.4480 | 1.0017 | Fail to reject H0 |
| 2012 | 0.5493 | 0.9938 | Reject H0 |
| 2013 | 0.4513 | 0.8668 | Reject H0 |
| 2014 | 0.5490 | 1.1861 | Fail to reject H0 |

Table 54 – Test results for difference between variances of daily returns out-of-sample, using S&P 100 index and cointegrated portfolios with at most 30 assets.

(a)

| S&P 100 vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7019 | 1.3664 | Fail to reject H0 |
| 2008 | 0.5145 | 1.2610 | Fail to reject H0 |
| 2009 | 0.4608 | 0.9749 | Reject H0 |
| 2010 | 0.7981 | 1.5910 | Fail to reject H0 |
| 2011 | 0.6346 | 1.4056 | Fail to reject H0 |
| 2012 | 0.6469 | 1.2005 | Fail to reject H0 |
| 2013 | 0.6541 | 1.2615 | Fail to reject H0 |
| 2014 | 0.7454 | 1.6433 | Fail to reject H0 |

(b)

| S&P 100 vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7157 | 1.3749 | Fail to reject H0 |
| 2008 | 0.5607 | 1.2919 | Fail to reject H0 |
| 2009 | 0.5301 | 1.1014 | Fail to reject H0 |
| 2010 | 0.7624 | 1.4772 | Fail to reject H0 |
| 2011 | 0.6698 | 1.4295 | Fail to reject H0 |
| 2012 | 0.6841 | 1.2693 | Fail to reject H0 |
| 2013 | 0.6758 | 1.2764 | Fail to reject H0 |
| 2014 | 0.7616 | 1.7680 | Fail to reject H0 |

(c)

| S&P 100 vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.8291 | 1.5610 | Fail to reject H0 |
| 2008 | 0.5244 | 1.2109 | Fail to reject H0 |
| 2009 | 0.5429 | 1.1232 | Fail to reject H0 |
| 2010 | 0.7615 | 1.4748 | Fail to reject H0 |
| 2011 | 0.6580 | 1.4327 | Fail to reject H0 |
| 2012 | 0.6936 | 1.2650 | Fail to reject H0 |
| 2013 | 0.6656 | 1.2723 | Fail to reject H0 |
| 2014 | 0.6946 | 1.5940 | Fail to reject H0 |

(d)

| S&P 100 vs. CT30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6855 | 1.3156 | Fail to reject H0 |
| 2008 | 0.5123 | 1.2283 | Fail to reject H0 |
| 2009 | 0.4417 | 0.9447 | Reject H0 |
| 2010 | 0.5839 | 1.1883 | Fail to reject H0 |
| 2011 | 0.6072 | 1.2420 | Fail to reject H0 |
| 2012 | 0.6672 | 1.1953 | Fail to reject H0 |
| 2013 | 0.5557 | 1.0630 | Fail to reject H0 |
| 2014 | 0.6009 | 1.3665 | Fail to reject H0 |

Table 55 – Test results for difference between variances of daily returns out-of-sample, using S&P 100 index and correlated portfolios with at most 15 assets.

(a)

| S&P 100 vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6600 | 1.3017 | Fail to reject H0 |
| 2008 | 0.5886 | 1.3671 | Fail to reject H0 |
| 2009 | 0.6366 | 1.3932 | Fail to reject H0 |
| 2010 | 0.6521 | 1.3302 | Fail to reject H0 |
| 2011 | 0.6885 | 1.4452 | Fail to reject H0 |
| 2012 | 0.6948 | 1.3169 | Fail to reject H0 |
| 2013 | 0.6673 | 1.2572 | Fail to reject H0 |
| 2014 | 0.6492 | 1.4392 | Fail to reject H0 |

(b)

| S&P 100 vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6643 | 1.2988 | Fail to reject H0 |
| 2008 | 0.6364 | 1.4419 | Fail to reject H0 |
| 2009 | 0.5944 | 1.2483 | Fail to reject H0 |
| 2010 | 0.6589 | 1.3984 | Fail to reject H0 |
| 2011 | 0.6836 | 1.4208 | Fail to reject H0 |
| 2012 | 0.7016 | 1.3069 | Fail to reject H0 |
| 2013 | 0.6968 | 1.3000 | Fail to reject H0 |
| 2014 | 0.6505 | 1.4915 | Fail to reject H0 |

(c)

| S&P 100 vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6798 | 1.2843 | Fail to reject H0 |
| 2008 | 0.5499 | 1.2625 | Fail to reject H0 |
| 2009 | 0.6347 | 1.3436 | Fail to reject H0 |
| 2010 | 0.6797 | 1.3711 | Fail to reject H0 |
| 2011 | 0.7250 | 1.4980 | Fail to reject H0 |
| 2012 | 0.6975 | 1.3170 | Fail to reject H0 |
| 2013 | 0.7316 | 1.4135 | Fail to reject H0 |
| 2014 | 0.6887 | 1.5339 | Fail to reject H0 |

(d)

| S&P 100 vs. CR15-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6995 | 1.3273 | Fail to reject H0 |
| 2008 | 0.6214 | 1.4557 | Fail to reject H0 |
| 2009 | 0.6699 | 1.3539 | Fail to reject H0 |
| 2010 | 0.7491 | 1.5042 | Fail to reject H0 |
| 2011 | 0.7548 | 1.5113 | Fail to reject H0 |
| 2012 | 0.7248 | 1.3955 | Fail to reject H0 |
| 2013 | 0.6975 | 1.3771 | Fail to reject H0 |
| 2014 | 0.6107 | 1.3832 | Fail to reject H0 |

Table 56 – Test results for difference between variances of daily returns out-of-sample, using S&P 100 index and correlated portfolios with at most 30 assets.

(a)

| S&P 100 vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7199 | 1.4236 | Fail to reject H0 |
| 2008 | 0.6284 | 1.4961 | Fail to reject H0 |
| 2009 | 0.6123 | 1.3206 | Fail to reject H0 |
| 2010 | 0.6673 | 1.3242 | Fail to reject H0 |
| 2011 | 0.6903 | 1.4301 | Fail to reject H0 |
| 2012 | 0.7175 | 1.3622 | Fail to reject H0 |
| 2013 | 0.6837 | 1.3397 | Fail to reject H0 |
| 2014 | 0.6522 | 1.4338 | Fail to reject H0 |

(b)

| S&P 100 vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7265 | 1.3915 | Fail to reject H0 |
| 2008 | 0.6596 | 1.5022 | Fail to reject H0 |
| 2009 | 0.6087 | 1.2943 | Fail to reject H0 |
| 2010 | 0.6958 | 1.3750 | Fail to reject H0 |
| 2011 | 0.6588 | 1.4112 | Fail to reject H0 |
| 2012 | 0.7096 | 1.3031 | Fail to reject H0 |
| 2013 | 0.7110 | 1.3748 | Fail to reject H0 |
| 2014 | 0.6970 | 1.5258 | Fail to reject H0 |

(c)

| S&P 100 vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6965 | 1.3734 | Fail to reject H0 |
| 2008 | 0.6237 | 1.3828 | Fail to reject H0 |
| 2009 | 0.6636 | 1.3728 | Fail to reject H0 |
| 2010 | 0.7101 | 1.3911 | Fail to reject H0 |
| 2011 | 0.6602 | 1.4415 | Fail to reject H0 |
| 2012 | 0.7476 | 1.3835 | Fail to reject H0 |
| 2013 | 0.7147 | 1.3956 | Fail to reject H0 |
| 2014 | 0.6900 | 1.5333 | Fail to reject H0 |

(d)

| S&P 100 vs. CR30-SPC | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6995 | 1.3773 | Fail to reject H0 |
| 2008 | 0.6069 | 1.4702 | Fail to reject H0 |
| 2009 | 0.5936 | 1.3222 | Fail to reject H0 |
| 2010 | 0.7249 | 1.4822 | Fail to reject H0 |
| 2011 | 0.6885 | 1.4960 | Fail to reject H0 |
| 2012 | 0.7157 | 1.3329 | Fail to reject H0 |
| 2013 | 0.7304 | 1.4142 | Fail to reject H0 |
| 2014 | 0.7110 | 1.5060 | Fail to reject H0 |

Table 57 – Test results for difference between variances of daily returns out-of-sample among portfolios, using S&P 100 index and cointegrated portfolios with at most 15 assets.

(a)

| CT15-SPC vs. CT15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7337 | 1.3896 | Fail to reject H0 |
| 2008 | 0.6681 | 1.6429 | Fail to reject H0 |
| 2009 | 0.9700 | 2.1961 | Fail to reject H0 |
| 2010 | 0.8616 | 1.6417 | Fail to reject H0 |
| 2011 | 0.8278 | 1.7739 | Fail to reject H0 |
| 2012 | 0.9294 | 1.6510 | Fail to reject H0 |
| 2013 | 0.9810 | 1.9009 | Fail to reject H0 |
| 2014 | 0.8249 | 1.7421 | Fail to reject H0 |

(b)

| CT15-SPC vs. CT15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6682 | 1.2737 | Fail to reject H0 |
| 2008 | 0.7778 | 2.0600 | Fail to reject H0 |
| 2009 | 0.8033 | 1.8047 | Fail to reject H0 |
| 2010 | 0.8614 | 1.6509 | Fail to reject H0 |
| 2011 | 0.9525 | 1.9871 | Fail to reject H0 |
| 2012 | 0.9511 | 1.6656 | Fail to reject H0 |
| 2013 | 0.9747 | 1.8092 | Fail to reject H0 |
| 2014 | 0.7261 | 1.5292 | Fail to reject H0 |

(c)

| CT15-SPC vs. CT15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6093 | 1.2234 | Fail to reject H0 |
| 2008 | 0.4829 | 1.3825 | Fail to reject H0 |
| 2009 | 0.6580 | 1.5358 | Fail to reject H0 |
| 2010 | 0.8656 | 1.6447 | Fail to reject H0 |
| 2011 | 0.9251 | 2.0421 | Fail to reject H0 |
| 2012 | 0.9347 | 1.6544 | Fail to reject H0 |
| 2013 | 0.9677 | 1.7723 | Fail to reject H0 |
| 2014 | 0.7643 | 1.6580 | Fail to reject H0 |

Table 58 – Test results for difference between variances of daily returns out-of-sample among portfolios, using S&P 100 index and cointegrated portfolios with at most 30 assets.

(a)

| CT30-SPC vs. CT30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7183 | 1.3709 | Fail to reject H0 |
| 2008 | 0.6357 | 1.5462 | Fail to reject H0 |
| 2009 | 0.6899 | 1.5028 | Fail to reject H0 |
| 2010 | 1.0052 | 1.8895 | Reject H0 |
| 2011 | 0.7706 | 1.5873 | Fail to reject H0 |
| 2012 | 0.7476 | 1.3358 | Fail to reject H0 |
| 2013 | 0.8584 | 1.6271 | Fail to reject H0 |
| 2014 | 0.7778 | 1.8025 | Fail to reject H0 |

(b)

| CT30-SPC vs. CT30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7722 | 1.4382 | Fail to reject H0 |
| 2008 | 0.6761 | 1.6018 | Fail to reject H0 |
| 2009 | 0.8006 | 1.6787 | Fail to reject H0 |
| 2010 | 0.9480 | 1.8505 | Fail to reject H0 |
| 2011 | 0.7988 | 1.6685 | Fail to reject H0 |
| 2012 | 0.8094 | 1.4198 | Fail to reject H0 |
| 2013 | 0.8415 | 1.6517 | Fail to reject H0 |
| 2014 | 0.8954 | 1.8779 | Fail to reject H0 |

(c)

| CT30-SPC vs. CT30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.8739 | 1.6015 | Fail to reject H0 |
| 2008 | 0.6404 | 1.4371 | Fail to reject H0 |
| 2009 | 0.8251 | 1.7156 | Fail to reject H0 |
| 2010 | 0.9328 | 1.7843 | Fail to reject H0 |
| 2011 | 0.7798 | 1.6071 | Fail to reject H0 |
| 2012 | 0.8001 | 1.3826 | Fail to reject H0 |
| 2013 | 0.8594 | 1.7162 | Fail to reject H0 |
| 2014 | 0.7620 | 1.7917 | Fail to reject H0 |

Table 59 – Test results for difference between variances of daily returns out-of-sample among portfolios, using S&P 100 index and correlated portfolios with at most 15 assets.

(a)

| CR15-SPC vs. CR15-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7028 | 1.3314 | Fail to reject H0 |
| 2008 | 0.6465 | 1.4963 | Fail to reject H0 |
| 2009 | 0.6564 | 1.4301 | Fail to reject H0 |
| 2010 | 0.6146 | 1.2566 | Fail to reject H0 |
| 2011 | 0.6412 | 1.3195 | Fail to reject H0 |
| 2012 | 0.6823 | 1.2810 | Fail to reject H0 |
| 2013 | 0.6582 | 1.2555 | Fail to reject H0 |
| 2014 | 0.6983 | 1.5498 | Fail to reject H0 |

(b)

| CR15-SPC vs. CR15-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7171 | 1.3454 | Fail to reject H0 |
| 2008 | 0.6653 | 1.5722 | Fail to reject H0 |
| 2009 | 0.6298 | 1.3245 | Fail to reject H0 |
| 2010 | 0.6295 | 1.2779 | Fail to reject H0 |
| 2011 | 0.6734 | 1.3634 | Fail to reject H0 |
| 2012 | 0.7079 | 1.3200 | Fail to reject H0 |
| 2013 | 0.6829 | 1.3250 | Fail to reject H0 |
| 2014 | 0.7114 | 1.5995 | Fail to reject H0 |

(c)

| CR15-SPC vs. CR15-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6921 | 1.3309 | Fail to reject H0 |
| 2008 | 0.5991 | 1.3940 | Fail to reject H0 |
| 2009 | 0.6613 | 1.4200 | Fail to reject H0 |
| 2010 | 0.6287 | 1.2922 | Fail to reject H0 |
| 2011 | 0.6960 | 1.4190 | Fail to reject H0 |
| 2012 | 0.6881 | 1.2812 | Fail to reject H0 |
| 2013 | 0.7087 | 1.4137 | Fail to reject H0 |
| 2014 | 0.7594 | 1.6733 | Fail to reject H0 |

Table 60 – Test results for difference between variances of daily returns out-of-sample among portfolios, using S&P 100 index and correlated portfolios with at most 30 assets.

(a)

| CR30-SPC vs. CR30-20 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7036 | 1.3553 | Fail to reject H0 |
| 2008 | 0.6705 | 1.5009 | Fail to reject H0 |
| 2009 | 0.6899 | 1.4883 | Fail to reject H0 |
| 2010 | 0.6557 | 1.3148 | Fail to reject H0 |
| 2011 | 0.6426 | 1.3887 | Fail to reject H0 |
| 2012 | 0.7382 | 1.3831 | Fail to reject H0 |
| 2013 | 0.6865 | 1.3599 | Fail to reject H0 |
| 2014 | 0.6326 | 1.3557 | Fail to reject H0 |

(b)

| CR30-SPC vs. CR30-60 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.7086 | 1.3936 | Fail to reject H0 |
| 2008 | 0.7167 | 1.6202 | Fail to reject H0 |
| 2009 | 0.6651 | 1.4004 | Fail to reject H0 |
| 2010 | 0.6842 | 1.3283 | Fail to reject H0 |
| 2011 | 0.6355 | 1.3763 | Fail to reject H0 |
| 2012 | 0.7096 | 1.3390 | Fail to reject H0 |
| 2013 | 0.7188 | 1.4058 | Fail to reject H0 |
| 2014 | 0.6685 | 1.4336 | Fail to reject H0 |

(c)

| CR30-SPC vs. CR30-120 | | | |
|---|---|---|---|
| Year | CI- | CI+ | |
| 2007 | 0.6981 | 1.3576 | Fail to reject H0 |
| 2008 | 0.6527 | 1.5125 | Fail to reject H0 |
| 2009 | 0.7449 | 1.6162 | Fail to reject H0 |
| 2010 | 0.6717 | 1.3968 | Fail to reject H0 |
| 2011 | 0.6900 | 1.3990 | Fail to reject H0 |
| 2012 | 0.7604 | 1.3818 | Fail to reject H0 |
| 2013 | 0.7051 | 1.3757 | Fail to reject H0 |
| 2014 | 0.6682 | 1.4550 | Fail to reject H0 |