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Questões sobre Produtos Cruzados Parciais

Questions on Partial Crossed Products

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Resumo

Neste trabalho estudamos condições necessárias e suficientes para que o produto cruzado parcial $R *_{\alpha}^{w} G$ seja um anel totalmente fracamente primo, e estudamos uma descrição do radical primo do produto cruzado parcial quando o anel base R é um anel totalmente fracamente primo.

Também estudamos condições necessárias e suficientes para a comutatividade e a simplicidade de $R *_{\alpha}^{w} G$. Quando $R = C(X)$ é a álgebra das funções contínuas definidas sobre um espaço topológico X com valores nos números complexos e $C(X) *_{\alpha} G$ é o skew anel de grupo parcial associado a uma ação parcial α de um grupo topológico G sobre $C(X)$, estudamos a simplicidade de $C(X) *_{\alpha} G$ usando propriedades topológicas de X e os resultados obtidos sobre a simplicidade de $R *_{\alpha}^{w} G$.

Abstract

In this work we study necessary and sufficient conditions for the partial crossed product $R *_\alpha^w G$ to be a fully weakly prime ring, and we give a description of the prime radical of the partial crossed product when the base ring R is a fully weakly prime ring.

Also, we study necessary and sufficient conditions for the commutativity and simplicity of $R *_\alpha^w G$. When $R = C(X)$ is algebra of continuous functions defined on a topological space X with values in the complex numbers and $C(X) *_\alpha G$ is the associated partial skew group ring of a partial action α of a topological group G on $C(X)$, we study the simplicity of $C(X) *_\alpha G$ using topological properties of X and the results about the simplicity of $R *_\alpha^w G$.

Contents

Introdução

Ações parciais de grupos foram introduzidas na teoria das álgebras de operadores como uma abordagem geral para estudar C^*-álgebras geradas por isometrias parciais (ver, em particular, [20] e [21]), e produtos cruzados, ver [15], estão no centro de uma rica interação entre sistemas dinâmicos e álgebras de operadores (ver, por exemplo, [28] e [33]). A noção geral de ação parcial (contínua) torcida de um grupo localmente compacto sobre uma C^*-álgebra e produtos cruzados correspondentes foram introduzidos em [20]. Os correspondentes algébricos para algumas das noções acima mencionadas foram introduzidos e estudados em [14], estimulando novas investigações, por exemplo, em [7], [17], [22] e suas referências.

Dada uma ação parcial é natural perguntar se esta é restrição de uma ação global. Tal ação global é denominada uma ação envolvente (ou uma globalização) da ação parcial, e seu estudo foi iniciado na tese de doutorado de F. Abadie [2] (ver também [1]) e independentemente por J. Kellendonk e M. Lawson em [26]. Em [14], entre outros resultados, Dokuchaev e Exel provam que existem ações parciais sem ação envolvente e apresentam um critério para a existência de uma ação envolvente. A existência de uma ação envolvente para uma ação parcial tem um papel importante quando queremos generalizar resultados já conhecidos de ações globais (ver por exemplo [17], [10], [12] e [22]). Em algum momento, ao longo do segundo capítulo, assumiremos que a ação parcial possui uma ação envolvente.

No que segue, apresentamos uma breve descrição dos capítulos que compõem esta tese.

No primeiro capítulo, apresentamos as principais definições e resultados necessários para o desenvolvimento dos demais capítulos.

No segundo capítulo, descrevemos o radical primo do produto cruzado parcial quando o anel base é um anel totalmente fracamente primo e estudamos condições necessárias e suficientes para que o produto cruzado parcial seja um anel totalmente fracamente primo. Além disso, como consequência de nossas técnicas, estudamos condições necessárias e suficientes para que o produto cruzado parcial seja um anel quase totalmente primo, e isto generaliza resultados apresentados em [25]. Este capítulo é parte do artigo intitulado "*Partial crossed products and fully weak prime rings*", que foi submetido à publicação.

No terceiro capítulo, descrevemos completamente o centro e estudamos a comutatividade do produto cruzado parcial. Também estudamos condições necessárias e suficientes para a simplicidade do produto cruzado parcial, e isto generaliza resultados apresentados em [31] e [32]. Além disso, quando $R = C(X)$, a álgebra das funções contínuas definida sobre um espaço topológico X com valores nos números complexos, consideramos uma ação parcial de um grupo topológico G em X e sua extensão para $C(X)$. Estudamos algumas propriedades topológicas da ação parcial de G sobre X que implicam algumas propriedades algébricas de $C(X)$. Também aplicamos os resultados sobre a simplicidade de $R *_{\alpha}^{w} G$ para estudar a simplicidade do produto cruzado parcial sobre $C(X)$. Concluímos este capítulo com alguns exemplos onde são aplicam os resultados e mostramos que algumas das nossas hipóteses para obter a simplicidade $C(X) *_{\alpha} G$ não são supérfluas. Este capítulo é parte do artigo intitulado "*Simplicity of partial crossed product*", que foi submetido à publicação em conjunto com Wagner Cortes e Alexandre T. Baraviera.

Introduction

Partial actions of groups have been introduced in the theory of operator algebras as a general approach to study C^*-algebras generated by partial isometries (see, in particular, [20] and [21]), and crossed products classically, as well-pointed out in [15], are the center of the rich interplay between dynamical systems and operator algebras (see, for instance, [28] and [33]). The general notion of (continuous) twisted partial action of a locally compact group on a C^*-algebra and the corresponding crossed product were introduced in [20]. Algebraic counterparts for some of the above mentioned notions were introduced and studied in [14], stimulating further investigations, for instance, in [17], [22], [7] and references therein.

Given a partial action it is natural to ask if this is a restriction of a global action. Such global action is called an enveloping action (or a globalization) of partial action, and its study was initiated in the PhD Thesis of F. Abadie [2] (see also [1]) and independently by J. Kellendonk and M. Lawson in [26]. In [14], among other results, Dokuchaev and Exel proved that there exist partial actions without an enveloping action and give a criteria for the existence of an enveloping action. The existence of an enveloping action for a partial action has an important role when we want to generalize well known results of global actions (see, for example, [17], [10], [12] and [22]). At some point, throughout the second chapter, we assume that the partial action has an enveloping action.

In what follows, we present a brief description of the chapters of this thesis.

In the first chapter we present the main definitions and results that are necessary for the development in the remaining chapters.

In the second chapter we describe the prime radical of partial crossed products when the base ring is a fully weakly prime ring, and we study necessary and sufficient conditions for the partial crossed product to be fully weakly prime. Moreover, as consequence of our techniques, we study necessary and sufficient conditions for the partial crossed product to be an almost fully prime ring, and this generalize results presented in [25]. This chapter is part of the article entitled "*Partial crossed products and fully weak prime rings*" that was submitted for publication.

In the third chapter we completely describe the center and we study the commutativity of the partial crossed product. We also study necessary and sufficient conditions for the simplicity of the partial crossed product, and this generalizes results presented in [31] and [32]. Moreover, when $R = C(X)$, the algebra of the continuous functions defined on a topological space X with values in the complex numbers, we consider a partial action of a topological group G on X and its extension to $C(X)$. We study some topological properties of the partial action of G on X that will imply some algebraic properties on $C(X)$. We also apply the results about the simplicity of $R *_{\alpha}^{w} G$ to study the simplicity of the partial crossed product over $C(X)$. We conclude this chapter with some examples where we apply the results and we show that some of our assumptions to obtain the simplicity of $C(X) *_{\alpha} G$ are not superfluous. This chapter is part of the article entitled "*Simplicity of partial crossed product*" that was submitted for publication and it was a joint article with Wagner Cortes and Alexandre T. Baraviera.

Chapter 1

Preliminaries

In this chapter we present the main definitions and properties that are important for the development of the subsequent chapters. The definitions and results presented here are well known, we will expose them in order to fix notation and for the reader's convenience.

1.1 Crossed product

Let T be a ring with identity, $Aut(T)$ the group of automorphisms of the ring T and G a group. We assume that G acts by automorphisms on T, i.e. there is a group homomorphism $\beta : G \to Aut(T)$ such that for each $g \in G$ we associate an automorphism β_g of T.

Suppose that there is an application $u : G \times G \to \mathcal{U}(T)$ (twisting) which for each pair $(g, h) \in G \times G$ associates the invertible element $u_{g,h}$ of T, where $\mathcal{U}(T)$ denotes the group of units of T. The (*global*) *crossed product* $T *_\beta^u G$ of G on T is the set of all finite sums $\sum_{g \in G} t_g \delta_g$, where δ_g's are symbols, with the usual addition and multiplication determined by rule

$$(s_g\delta_g)(t_h\delta_h) = s_g\beta_g(t_h)u_{g,h}\delta_{gh},$$

for all $s_g, t_h \in T$ and $g, h \in G$.

The associativity of $T *_\beta^u G$ is equivalent to the assertions, for all $g, h, l \in G$:

(i) $\beta_g \circ \beta_h(t) = u_{g,h}\beta_{gh}(t)u_{g,h}^{-1}$, for all $t \in T$;

(ii) $\beta_g(u_{h,l})u_{g,hl} = u_{g,h}u_{gh,l}$.

Let β be a twisted global action of a group G on T. An ideal I of T is said to be *β-invariant* if $\beta_g(I) = I$, for all $g \in G$. An ideal P of T is said to be *β-prime* if for any β-invariant ideals I and J of T with $IJ \subseteq P$ we have either $I \subseteq P$ or $J \subseteq P$. The ring T is said to be *β-prime* if the zero ideal is β-prime.

In what follows, we will see some results that will be needed during the text. The first is known as Incomparability Theorem and will be generalized in Corollary 2.1.13.

Lemma 1.1.1. ([35], Theorem 16.6(iii)) *Let $T *_\beta^u G$ be a crossed product with G a finite group. If P_1 and P_2 are prime ideals of $T *_\beta^u G$ such that $P_1 \cap T = P_2 \cap T$, then $P_1 = P_2$.*

Lemma 1.1.2. ([35], Theorem 16.2(i)) *Let $T *_\beta^u G$ be a crossed product with G a finite group and T a β-prime ring. Then a prime ideal P of $T *_\beta^u G$ is minimal if and only if $P \cap T = 0$.*

Lemma 1.1.3. ([35], Lemma 1.3) *Let $T *_\beta^u G$ be a crossed product and H a normal subgroup of G. Then $T *_\beta^u G = (T *_\beta^u H) * (G/H)$ where the latter is some crossed product of the group G/H over the ring $T *_\beta^u H$.*

1.2 Twisted partial actions

Let A be an associative non-necessarily unital ring, we remind that the ring of multipliers $\mathcal{M}(A)$ is the set

$$\mathcal{M}(A) = \{(\mathcal{R}, L) \in End(_AA) \times End(A_A) : (a\mathcal{R})b = a(Lb), \forall a, b \in A\}$$

with the following operations:

(i) $(\mathcal{R}, L) + (\mathcal{R}', L') = (\mathcal{R} + \mathcal{R}', L + L')$;

(ii) $(\mathcal{R}, L)(\mathcal{R}', L') = (\mathcal{R}' \circ \mathcal{R}, L \circ L')$.

Here we use the right hand side notation for homomorphisms of left A-modules, while for homomorphisms of right modules the usual notation shall be used. In particular, we write $a \mapsto a\mathcal{R}$ and $a \mapsto La$ for $\mathcal{R} : {}_AA \to {}_AA$, $L : A_A \to A_A$ with $a \in A$. For the multiplier $w = (\mathcal{R}, L) \in \mathcal{M}(A)$ and $a \in A$ we set $aw = a\mathcal{R}$ and $wa = La$. Thus one always has $(aw)b = a(wb)$, for all $a, b \in A$. The first (resp. second) components of the elements of $\mathcal{M}(A)$ are called right (resp. left) multipliers of A. It is convenient to point out that if A is a unital ring, then we have that $A \simeq \mathcal{M}(A)$, see ([14], Proposition 2.3). So, in this case, each invertible multiplier may be considered as an invertible element of A.

The following definition appears in ([16], Definition 2.1).

Definition 1.2.1. A *twisted partial action* of a group G on a ring R is a triple

$$\alpha = \big(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}\big),$$

where for each $g \in G$, D_g is a two-sided ideal in R, $\alpha_g : D_{g^{-1}} \to D_g$ is an isomorphism of rings and for each $(g, h) \in G \times G$, $w_{g,h}$ is an invertible element from $\mathcal{M}(D_g D_{gh})$, satisfying the following postulates, for all $g, h, t \in G$:

(i) $D_g^2 = D_g$ and $D_g D_h = D_h D_g$;

(ii) $D_e = R$ and α_e is the identity map of R;

(iii) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$;

(iv) $\alpha_g \circ \alpha_h(a) = w_{g,h}\alpha_{gh}(a)w_{g,h}^{-1}$, for all $a \in D_{h^{-1}} D_{h^{-1}g^{-1}}$;

(v) $w_{g,e} = w_{e,g} = 1$;

(vi) $\alpha_g(aw_{h,t})w_{g,ht} = \alpha_g(a)w_{g,h}w_{gh,t}$, for all $a \in D_{g^{-1}} D_h D_{ht}$.

Note that if $w_{g,h} = 1_g 1_{gh}$, $\forall g, h \in G$, then we have the partial action defined by Dokuchaev and Exel in ([14], Definition 1.1) and when $D_g = R$, $\forall g \in G$, we have that α is a twisted global action.

Remark 1.2.2. If each ideal D_g of R is generated by a central idempotent 1_g, then $D_g = 1_g R$ and, for all $g, h \in G$, we have that $D_g D_h = D_g \cap D_h$ is a unital ring with identity $1_g 1_h$. Consequently, for all $g, h \in G$, we have that $\mathcal{M}(D_g D_{gh}) \simeq D_g D_{gh}$ and so each invertible multiplier $w_{g,h}$ may be considered as an invertible element of $D_g D_{gh}$.

Let $\beta = \big(T, \{\beta_g\}_{g\in G}, \{u_{g,h}\}_{(g,h)\in G\times G}\big)$ be a twisted global action of a group G on a (non-necessarily unital) ring T and R an ideal of T generated by a central idempotent 1_R. We can restrict β for R as follows. Put $D_g = R \cap \beta_g(R) = R \cdot \beta_g(R)$ we have that each D_g has identity $1_R \beta_g(1_R)$. Then defining $\alpha_g = \beta_g|_{D_{g^{-1}}}$, $g \in G$, the items (i), (ii) and (iii) of Definition 1.2.1 are satisfied. Furthermore, defining $w_{g,h} = u_{g,h} 1_R \beta_g(1_R) \beta_{gh}(1_R)$, $g, h \in G$, we have that (iv), (v) e (vi) are also satisfied. So we indeed have obtained a twisted partial action of G on R.

The following definition appears in ([16], Definition 2.2).

Definition 1.2.3. A twisted global action $\big(T, \{\beta_g\}_{g\in G}, \{u_{g,h}\}_{(g,h)\in G\times G}\big)$ of a group G on an associative (non-necessarily unital) ring T is said to be an *enveloping action* (or a *globalization*) for a twisted partial action α of G on a ring R if there exists a monomorphism $\varphi : R \to T$ such that, for all g and h in G:

(i) $\varphi(R)$ is an ideal of T;

(ii) $T = \sum_{g\in G} \beta_g(\varphi(R))$;

(iii) $\varphi(D_g) = \varphi(R) \cap \beta_g(\varphi(R))$;

(iv) $\varphi \circ \alpha_g(a) = \beta_g \circ \varphi(a)$, for all $a \in D_{g^{-1}}$;

(v) $\varphi(a w_{g,h}) = \varphi(a) u_{g,h}$ and $\varphi(w_{g,h} a) = u_{g,h} \varphi(a)$, for all $a \in D_g D_{gh}$.

In ([16], Theorem 4.1), the authors studied necessary and sufficient conditions for a twisted partial action α of a group G on a ring R has an enveloping action. Moreover, they studied which rings satisfy such conditions.

Suppose that (R, α, w) has an enveloping action (T, β, u). In this case, we may assume that R is an ideal of T and we can rewrite the conditions of the Definition 1.2.3 as follows:

(i') R is an ideal of T;

(ii') $T = \sum_{g \in G} \beta_g(R)$;

(iii') $D_g = R \cap \beta_g(R)$, for all $g \in G$;

(iv') $\alpha_g(a) = \beta_g(a)$, for all $a \in D_{g^{-1}}$ and $g \in G$;

(v') $aw_{g,h} = au_{g,h}$ and $w_{g,h}a = u_{g,h}a$, for all $a \in D_g D_{gh}$ and $g, h \in G$.

We recall from ([18], p. 345) that a ring S is *left* (*right*) s-*unital* if for any $r \in S$ we have that $r \in Sr$ ($r \in rS$). A ring S is said to be s-*unital* if it is right and left s-unital. We clearly have that every unital ring is s-unital. Note that if each ideal D_g of R is generated by a central idempotent 1_g and if (R, α, w) has an enveloping action (T, β, u), then T is s-unital. In this case, for each $g \in G$ we have that $T1_g = R1_g = D_g = R \cap \beta_g(R) = T1_R \cap T\beta_g(1_R) = T1_R\beta_g(1_R)$ and it follows that $1_g = 1_R\beta_g(1_R)$.

Given a twisted partial action α of a group G on a ring R, we recall from ([15], Definition 2.2) that the *partial crossed product* $R *_\alpha^w G$ is the direct sum

$$\bigoplus_{g \in G} D_g\delta_g,$$

where δ_g's are symbols, with the usual addition and multiplication defined by

$$(a_g\delta_g)(b_h\delta_h) = \alpha_g(\alpha_g^{-1}(a_g)b_h)w_{g,h}\delta_{gh}.$$

By ([15], Theorem 2.4) we have that $R *_\alpha^w G$ is an associative ring whose identity is $1_R\delta_1$. Moreover, we have the injective morphism $\phi : R \to R *_\alpha^w G$, defined by $r \mapsto r\delta_1$ and we can consider $R *_\alpha^w G$ an extension of R.

1.3 *FPR, AFPR* and *FWPR*

Now, we review some definitions and results on rings with "many" prime ideals. The following definition appears in ([5], Definition 1.1) and ([25], p. 86).

Definition 1.3.1. *Let S be a ring.*

(i) S is said to be a *fully prime ring* (*FPR*) if every ideal of S is prime.

(ii) S is said to be an *almost fully prime ring* (*AFPR*) if every proper ideal of S is prime and S is not a prime ring.

The following result appears in ([5], Theorem 1.2).

Proposition 1.3.2. *A ring S is a FPR if and only if the set of all the ideals of S is linearly ordered by inclusion and all the ideals of S are idempotent.*

Example 1.3.3.

(i) Let V be a right vector space over a division ring D and $\mathrm{End}_D(V)$ the endomorphisms ring of V. By fact that $\mathrm{End}_D(V)$ is a von Neumann regular ring, if J is any ideal of $\mathrm{End}_D(V)$ and $x \in J$, then there exists $y \in \mathrm{End}_D(V)$ such that $x = xyx$. Thus $x = xyx = (xy)x \in J^2$ and so every ideal of $\mathrm{End}_D(V)$ is idempotent. Moreover, by ([38], Theorem III.14), the ideals of $\mathrm{End}_D(V)$ are of the form $I_c = \{f \in \mathrm{End}_D(V) : \dim f(V) < c\}$ where c is any infinite cardinal number such that $c \leqslant \dim(V)$. Note that, if $c < d \leqslant \dim(V)$ are infinite cardinal numbers, then $I_c \subseteq I_d$ and so the ideals of $\mathrm{End}_D(V)$ are linearly ordered by inclusion. Hence, by Proposition 1.3.2, $\mathrm{End}_D(V)$ is a *FPR*.

(ii) Let R be a *FPR* with exactly one proper ideal P. For $p_1, p_2 \in P$ and $r_1, r_2 \in R$, let $S = P \oplus R$ with the usual addiction and multiplication defined by

$(p_1, r_1)(p_2, r_2) = (p_1r_2 + r_1p_2, r_1r_2)$. Then S has exactly two proper ideals, namely: $Q_1 = \{(p, p') : p, p' \in P\}$ and $Q_2 = \{(p, 0) : p \in P\}$. We easily see that S is an *AFPR*.

At this point it is convenient to point out that any *FPR* only have one maximal ideal. The following two results were proved in ([41], Theorems 2.1 and 2.2).

Lemma 1.3.4. *Let S be a ring whose set of ideals is not linearly ordered by inclusion. Then S is an AFPR if and only if*

(i) *all ideal of S is idempotent and it has exactly two minimal ideals;*

(ii) *each minimal ideal of S is contained in all nonzero ideal of S that is not minimal ideal;*

(iii) *the set of ideals of S that are not minimal is linearly ordered by inclusion.*

Lemma 1.3.5. *Let S be a ring whose set of ideals is linearly ordered by inclusion. Then S is an AFPR if and only if S has only one minimal ideal and every ideal of S except the minimal one is idempotent.*

In what follows, we denote by $Nil_*(S)$ the *prime radical* of a ring S, i.e. the intersection of all prime ideals of S, and in an abuse of notation we denote the zero ideal simply by 0. The proof of the next result follows directly from the Lemmas 1.3.4 and 1.3.5.

Lemma 1.3.6. *The following statements hold:*

(i) *If S is an AFPR whose set of ideals is not linearly ordered by inclusion, then $Nil_*(S) = P_1 \cap P_2 = 0$, where P_1 and P_2 are the minimal ideals of S.*

(ii) *If S is an AFPR whose set of ideals is linearly ordered by inclusion, then $Nil_*(S) = P_0$, where P_0 is the minimal prime ideal of S that is nilpotent.*

The following definitions appear in ([24], p. 1078).

Definition 1.3.7. Let S be a ring.

(i) A proper ideal I of S is said to be *weakly prime ideal* if for any ideals J and K of R with $0 \neq JK \subseteq I$ we have either $J \subseteq I$ or $K \subseteq I$.

(ii) S is said to be a *fully weakly prime ring* (*FWPR*) if every proper ideal of S is weakly prime.

Example 1.3.8.

(i) Obviously, all proper prime ideal of a ring S is weakly prime. Now, we shall see the converse is not true. Let R be a ring and M an R-bimodule. Define $R \star M = \{(r, m) : r \in R \text{ and } m \in M\}$ with component-wise addition and multiplication defined by $(r, m)(s, n) = (rs, rn + ms)$. Then $R \star M$ is a ring whose ideals are precisely of the form $I \star N$, where I is an ideal of R and N is a submodule (a bimodule) of M containing IM and MI. Let R be a prime ring with exactly one proper ideal P (e.g., the ring of linear transformations of a vector space V over a field F where $\dim_F V = \aleph_0$). Let $S_1 = R \star P$ and $P_1 = 0 \star P$ the nonzero minimal ideal of S_1. Now, let $S_2 = S_1 \star P_1$. Then all ideal of S_2 is weakly prime and $Q_1 = P_1 \star P_1$ and $Q_2 = P_1 \star 0$ are nonzero nilpotent ideals of S_2. Hence, Q_2 is weakly prime, but is not prime, because $0 = (Q_1)^2 \subseteq Q_2$ and $Q_1 \nsubseteq Q_2$, see ([24], Example 5).

(ii) Let R be a ring such that $R^2 = 0$ and K a field. Then $S = K \oplus R \oplus R$, with component-wise addition and multiplication, is not a *FWPR*, since the ideal $I = K \oplus 0 \oplus R$ is not weakly prime, because $0 \neq (K \oplus R \oplus 0)^2 \subseteq I$ and $K \oplus R \oplus 0 \nsubseteq I$.

(iii) Let $S = Ke_1 \oplus Ke_2$, where e_1 and e_2 are orthogonal central idempotents and K is a field. Then S is a *FWPR*.

Let S be a ring. Recall that S is said to be a *right noetherian* if every non-empty set of ideals of S contains a maximal element. In what follows, we denote the sum of all ideals of S whose square is zero by $N(S)$ and the *Jacobson radical* of S, i.e. the intersection of all the maximal right ideals of S, by $J(S)$.

Lemma 1.3.9. ([24], Theorem 1) *Suppose that S is a FWPR and $S^2 = S$. Then $Nil_*(S) = N(S)$ and $(Nil_*(S))^2 = (N(S))^2 = 0$.*

Lemma 1.3.10. ([24], Corollary 2) *Suppose that S is a right noetherian FWPR with identity. Then $Nil_*(S) = N(S) = J(S)$ and $(J(S))^2 = 0$.*

Lemma 1.3.11. ([24], Proposition 1) *If an ideal I of a ring S is a weakly prime ideal that is not prime, then $I^2 = 0$.*

1.4 Topological notions

In this subsection, we review some definitions and results on topological spaces that will be used in the Chapter 3. We begin with the definition of partial actions of topological groups on topological spaces, see [1]. We refer to [6] and [37], for the basic concepts of Topology.

Definition 1.4.1. Let G be a topological group and X a topological space. A *partial action* α of G on X is a family of open subsets $\{X_t\}_{t\in G}$ of X and homeomorphisms $\alpha_t\colon X_{t^{-1}} \to X_t$ such that the following properties hold, for all $s, t \in G$:

(i) $X_e = X$ and $\alpha_e = id_X$;

(ii) $\alpha_t(X_{t^{-1}} \cap X_s) = X_t \cap X_{ts}$;

(iii) $\alpha_t(\alpha_s(x)) = \alpha_{ts}(x),$ for all $x \in \alpha_{s^{-1}}(X_s \cap X_{t^{-1}})$;

(iv) The set $\Gamma_\alpha = \left\{(t, x) \in G \times X : t \in G, x \in X_{t^{-1}}\right\}$ is open in $G \times X$ and the function $\varphi\colon \Gamma_\alpha \to X$ defined by $\varphi(t, x) = \alpha_t(x)$ is continuous.

We denote it by the triple (X, α, G) and it is called a *partial dynamical system*, see [21].

Next, we give a well known non-trivial example of partial dynamical system.

Example 1.4.2. (non complete flows) Consider a smooth vector field $\mathfrak{X} \colon X \to TX$ on a manifold X, and for any $p \in X$ let ϕ_p be the flow of X through p, i.e. the solution of the differential equation

$$\frac{d}{dt}\phi_p(t) = \mathfrak{X}\big(\phi_p(t)\big)$$

with initial condition $\phi_p(0) = p$, defined on its maximal interval (a_p, b_p). For any $t \in \mathbb{R}$, set $X_{-t} = \big\{p \in X : t \in (a_p, b_p)\big\}$, $\alpha_t \colon X_{-t} \to X_t$ such that $\alpha_t(p) = \phi_p(t)$, and $\alpha = \big(\{X_t\}_{t\in\mathbb{R}}, \{\alpha_t\}_{t\in\mathbb{R}}\big)$. Now $(X, \alpha, \mathbb{R})$ is a partial dynamical system (and, if the manifold is compact, it is in fact a global action, and so, a usual dynamical system), see ([1], Example 1.2).

From ([1], Theorem 1.1) we have that any partial action of a topological group G on a topological space X has an enveloping action (X^e, β, G). For convenience we briefly recall the construction of ([1], Theorem 1.1): First, define the action $\gamma \colon G \times G \times X \to G \times X$ as $\gamma_s(t, x) = (st, x)$ and introduce the equivalence relation on $G \times X$ defined as follows:

$$(t, x) \sim (s, y) \Leftrightarrow x \in X_{t^{-1}s} \text{ and } \alpha_{s^{-1}t}(x) = y.$$

Then we have the topological space $X^e = G \times X/\sim$ and denote the equivalence class of $(g, x) \in G \times X$, as usual, by $[g, x] \in X^e$. The global action β is just the restriction of the action γ to the equivalence classes. The quotient map $q \colon G \times X \to X^e$ is defined by $q(g, x) = [g, x]$; it is also possible to introduce the injective morphism $i \colon X \to X^e$ given by $i(x) = q(e, x)$, that is an injective continuous morphism. Moreover, for each $x \in X_{g^{-1}}$, we have that

$$\big(\alpha_g(x)\big) = q\big(e, \alpha_g(x)\big) = q(g, x) = q\big(\gamma_g(e, x)\big) = \beta_g\big(q(e, x)\big) = \beta_g\big(i(x)\big)$$

and X is open in X^e. The triple (X^e, β, G) is called the *enveloping action* of (X, α, G).

Let (X, α, G) be a partial dynamical system and consider the algebra of continuous functions defined on topological space X with values in the complex numbers

$$C(X) = \{f : X \to \mathbb{C} \text{ continuous}\},$$

with the usual addition of functions and multiplication defined by

$$(ff')(x) = f(x)f'(x), \text{ for any } f, f' \in C(X).$$

Following [21] we can extend the partial action α of G on X to the algebra $C(X)$ with ideals $C(X_t)$ and isomorphisms $\alpha_t : C(X_{t^{-1}}) \to C(X_t)$ defined by $\alpha_t(f)(x) = f(\alpha_{t^{-1}}(x))$ for each $t \in G$ and the following properties are easily verified:

(a) $C(X_e) = C(X)$ and $\alpha_e = id_{C(X)}$;

(b) $\alpha_t(C(X_{t^{-1}})) \cap C(X_s)) = C(X_t) \cap C(X_{ts})$;

(c) $\alpha_t(\alpha_s(f)) = \alpha_{ts}(f)$, for all $f \in \alpha_{s^{-1}}(C(X_s) \cap C(X_{t^{-1}}))$.

We denote this partial action by α again. Following [14] the partial skew group ring $C(X) *_\alpha G$ is the set of all finite formal sums $\sum_{g \in G} a_g \delta_g$, where $a_g \in C(X_g)$, with the usual addition and multiplication defined by rule

$$(a_g\delta_g)(a_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)a_h)\delta_{gh}.$$

Note that, $C(X) *_\alpha G$ is associative, because (X, α, G) has an enveloping action (X^e, β, G) and in this case $C(X) *_\alpha G$ is a subring of the skew group ring $C(X^e) * G$, see [14].

Chapter 2

Partial Crossed Products and Fully Weakly Prime Rings

In this chapter we describe the prime radical of partial crossed products when the base ring is a fully weakly prime ring. We describe necessary and sufficient conditions for the partial crossed product to be fully weakly prime. As consequence of our techniques, we study necessary and sufficient conditions for the partial crossed product to be an almost fully prime ring and this generalize some results of [25].

Throughout this chapter $\alpha = \big(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}\big)$ is a twisted partial action of a group G on a ring R such that all the ideals D_g are generated by central idempotents 1_g, unless otherwise stated.

2.1 General results

In this section we look at some general results about twisted partial action, which will be used to develop the third section of this chapter.

Definition 2.1.1. Let α be a twisted partial action of a group G on a ring R. An ideal I of R is said to be *α-invariant* if $\alpha_g(I \cap D_{g^{-1}}) \subseteq I \cap D_g$, for all $g \in G$.

Note that the definition above is equivalent to $\alpha_g(I \cap D_{g^{-1}}) = I \cap D_g$, for all $g \in G$. If I is an α-invariant ideal of R we define $I *_\alpha^w G$ as the set of all finite sums $\sum_{g \in G} a_g \delta_g$ such that $a_g \in I \cap D_g$, for all $g \in G$, with the usual addition and multiplication determined by rule

$$(a_g \delta_g)(b_h \delta_h) = a_g \alpha_g(b_h 1_{g^{-1}}) w_{g,h} \delta_{gh}.$$

Lemma 2.1.2. *Let α be a twisted partial action of a group G on R.*

(i) *If J is an ideal of $R *_\alpha^w G$, then $J \cap R$ is an α-invariant ideal of R such that $(J \cap R) *_\alpha^w G \subseteq J$.*

(ii) *If I is an α-invariant ideal of R, then $I *_\alpha^w G$ is an ideal of $R *_\alpha^w G$ such that $(I *_\alpha^w G) \cap R = I$.*

Proof. (i) Clearly $J \cap R$ is an ideal of R and $(J \cap R) *_\alpha^w G \subseteq J$. Moreover, $J \cap R$ is α-invariant, because if $x \in J \cap R \cap D_{g^{-1}}$, we have that

$$\alpha_g(x) = 1_g \alpha_g(x) = (1_g \alpha_g(x) w_{g,g^{-1}}) w_{g,g^{-1}}^{-1} = (1_g \delta_g)(x \delta_{g^{-1}}) w_{g,g^{-1}}^{-1} \in J.$$

(ii) Clearly $I *_\alpha^w G$ is a subring of $R *_\alpha^w G$ and $(I *_\alpha^w G) \cap R = I$. Moreover, $I *_\alpha^w G$ is an ideal of $R *_\alpha^w G$, because if $b_h \delta_h \in I *_\alpha^w G$ and $a_g \delta_g \in R *_\alpha^w G$, we have that $(b_h \delta_h)(a_g \delta_g) = b_h \alpha_h(a_g 1_{h^{-1}}) w_{h,g} \delta_{hg} \in I *_\alpha^w G$ and $(a_g \delta_g)(b_h \delta_h) = a_g \alpha_g(b_h 1_{g^{-1}}) w_{g,h} \delta_{gh} \in I *_\alpha^w G$, since I is an α-invariant ideal of R. □

Definition 2.1.3. Let α be a twisted partial action of a group G on R.

(i) An α-invariant ideal P of R is said to be *α-prime* if for any α-invariant ideals I and J of R with $IJ \subseteq P$ we have either $I \subseteq P$ or $J \subseteq P$.

(ii) The *α-prime radical* of R is the intersection of all α-prime ideals of R and we denote it by $Nil_\alpha(R)$.

Lemma 2.1.4. *Let α be a twisted partial action of a group G on R.*

(i) *If Q is a prime ideal of $R *_{\alpha}^{w} G$, then $Q \cap R$ is an α-prime ideal of R.*

(ii) *If P is an α-prime ideal of R, then there exists a prime ideal Q of $R *_{\alpha}^{w} G$ such that $Q \cap R = P$.*

Proof. (i) Let Q be a prime ideal of $R *_{\alpha}^{w} G$ and I, J α-invariant ideals of R such that $IJ \subseteq Q \cap R$. Then $(I *_{\alpha}^{w} G)(J *_{\alpha}^{w} G) \subseteq Q$. By the fact that Q is prime we have that either $I *_{\alpha}^{w} G \subseteq Q$ or $J *_{\alpha}^{w} G \subseteq Q$. Thus either $I \subseteq Q \cap R$ or $J \subseteq Q \cap R$. Hence, $Q \cap R$ is an α-prime ideal of R.

(ii) Let P be an α-prime ideal of R. Then by Lema 2.1.2(ii), we have that $(P *_{\alpha}^{w} G) \cap R = P$. By Zorn's Lemma there exists an ideal Q in $R *_{\alpha}^{w} G$, maximal with the property $Q \cap R = P$. Now, it is easy to see that Q is a prime ideal of $R *_{\alpha}^{w} G$ such that $Q \cap R = P$. $\square$

Throughout the rest of section we assume that the twisted partial action α of G on R has an enveloping action (T, β, u), unless otherwise stated.

Lemma 2.1.5. *Let α be a twisted partial action of a group G on R and (T, β, u) its enveloping action. If $M = \left\{ \sum\limits_{g \in G} a_g \delta_g : a_g \in R \right\}$ and $N = \left\{ \sum\limits_{g \in G} a_g \delta_g : a_g \in \beta_g(R) \right\}$ then the following conditions hold:*

(i) $M(T *_{\beta}^{u} G) \subseteq M$;

(ii) $(T *_{\beta}^{u} G)N \subseteq N$;

(iii) $(R *_{\alpha}^{w} G)M \subseteq M$;

(iv) $N(R *_{\alpha}^{w} G) \subseteq N$;

(v) $MN = R *_{\alpha}^{w} G$;

(vi) $NM = T *_{\beta}^{u} G$;

(vii) $M(R *_\alpha^w G) \subseteq R *_\alpha^w G$;

(viii) $(R *_\alpha^w G)N \subseteq R *_\alpha^w G$.

Proof. The proofs of the first six items are similar the proofs of the ([14], Propositions 5.1, 5.2 and 5.3).

(vii) Item (i) implies that $MMN \subseteq MN$. Since, by item (v), $MN = R *_\alpha^w G$ it follows that $M(R *_\alpha^w G) \subseteq R *_\alpha^w G$.

$(viii)$ Item (ii) implies that $MNN \subseteq MN$. Since, by item (v), $MN = R *_\alpha^w G$ it follows that $(R *_\alpha^w G)N \subseteq R *_\alpha^w G$. $\square$

Note that if R is s-unital and (T, β, u) is the enveloping action of (R, α, w), then by ([18], Remark 2.5) we have that T is s-unital. Using this fact we obtain the next lemma that appears in the proof of ([16], Theorem 3.1).

Lemma 2.1.6. *Let R be a s-unital ring and α a twisted partial action of a group G on R with enveloping action (T, β, u). Then $T *_\beta^u G$ is s-unital.*

The proof of the next lemma is standard.

Lemma 2.1.7. *Let P' be an ideal of $T *_\beta^u G$. Then $P' \cap (R *_\alpha^w G)$ is an ideal of $R *_\alpha^w G$.*

Lemma 2.1.8. *There exists a bijective correspondence, via contraction, between the set of ideals of $R *_\alpha^w G$ and the set of ideals of $T *_\beta^u G$.*

Proof. Let P be an ideal of $R *_\alpha^w G$. Clearly NPM is a subring of $T *_\beta^u G$. Since $M(T *_\beta^u G) \subseteq M$ and $(T *_\beta^u G)N \subseteq N$ it follows that $NPM(T *_\beta^u G) \subseteq NPM$ and $(T *_\beta^u G)NPM \subseteq NPM$. Thus, NPM is an ideal of $T *_\beta^u G$.

Since $P = 1_R P 1_R$, $1_R \in N$ and $1_R \in M$ then $P \subseteq NPM \cap (R *_\alpha^w G)$. Now, for each $x \in NPM \cap (R *_\alpha^w G)$ we have that $x = 1_R x 1_R \in 1_R NPM 1_R$. By the fact that $MN = R *_\alpha^w G$, we have that $x = 1_R x 1_R \in 1_R NPM 1_R \subseteq (R *_\alpha^w G)P(R *_\alpha^w G) \subseteq P$. Thus, $NPM \cap (R *_\alpha^w G) \subseteq P$ and it follows that $NPM \cap (R *_\alpha^w G) = P$.

Next, let P' be an ideal of $T *_\beta^u G$. Since $M(T *_\beta^u G) \subseteq M$ and $MN = R *_\alpha^w G$, we obtain that $MP'N \subseteq R *_\alpha^w G$, and we easily see that $MP'N$ is a subring of

$R *_\alpha^w G$. By the fact that $(R *_\alpha^w G)M \subseteq M$ and $N(R *_\alpha^w G) \subseteq N$ it follows that $(R *_\alpha^w G)MP'N \subseteq MP'N$ and $MP'N(R *_\alpha^w G) \subseteq MP'N$. Thus, $MP'N$ is an ideal of $R *_\alpha^w G$.

By the fact that $MP'N \subseteq R *_\alpha^w G$ and P' is an ideal of $T *_\beta^u G$, we have that $MP'N \subseteq P' \cap (R *_\alpha^w G)$. Now, for each $x \in P' \cap (R *_\alpha^w G)$, we have $x = 1_R x 1_R \in 1_R P' 1_R \subseteq MP'N$. Thus, $P' \cap (R *_\alpha^w G) \subseteq MP'N$ and its follows that $P' \cap (R *_\alpha^w G) = MP'N$. $\square$

The following result is a direct consequence of proof of lemma above.

Corollary 2.1.9. *Let P' be an ideal of $T *_\beta^u G$ and P an ideal of $R *_\alpha^w G$ such that $P = P' \cap (R *_\alpha^w G)$. Then $P = MP'N$ and $P' = NPM$.*

We recall that given two rings R and S, bimodules ${}_RU_S$ and ${}_SV_R$ and maps $\theta : U \otimes_S V \to R$ and $\psi : V \otimes_R U \to S$ the collection $(R, S, U, V, \theta, \psi)$ is said to be a Morita context if the array

$$\begin{pmatrix} R & U \\ V & S \end{pmatrix}$$

with the usual formal operations of 2×2 matrices, is a ring. As in the Lemma 2.1.5, consider $M = \left\{ \sum_{g \in G} a_g \delta_g : a_g \in R \right\}$ and $N = \left\{ \sum_{g \in G} a_g \delta_g : a_g \in \beta_g(R) \right\}$. Using the Lemma 2.1.5 and similar arguments of ([16], Theorem 3.1), we have the Morita context $\left(R *_\alpha^w G, T *_\beta^u G, M, N, \theta, \psi\right)$, where θ and ψ are the obvious maps.

The next result appear in ([39], Proposition 2.3.2). We will show that it can be obtained as a consequence of Lemma 2.1.8 and Corollary 2.1.9.

Lemma 2.1.10. *There exists a bijective correspondence, via contraction, between the set of prime ideals of $R *_\alpha^w G$ and the set of prime ideals of $T *_\beta^u G$.*

Proof. Let P' be a prime ideal of $T *_\beta^u G$ and assume that I and J are ideals of $R *_\alpha^w G$ such that $IJ \subseteq P' \cap (R *_\alpha^w G) = P$. Since $IJ \subseteq P$ it follows that

$I'J' = (NIM)(NJM) \subseteq NIJM \subseteq NPM = P'$ and so either $I' \subseteq P'$ or $J' \subseteq P'$. Thus either $I = MI'N \subseteq MP'N = P$ or $J = MJ'N \subseteq MP'N = P$. Hence, $P = P' \cap (R *_{\alpha}^{w} G)$ is a prime ideal of $R *_{\alpha}^{w} G$. It can be seen by analogous way that if P is a prime ideal of $R *_{\alpha}^{w} G$, then is prime the ideal P' of $T *_{\beta}^{u} G$ such that $P = P' \cap (R *_{\alpha}^{w} G)$. $\square$

In what follows we will see some consequences of last result and they will be useful throughout this chapter.

Corollary 2.1.11. *There exists a bijective correspondence, via contraction, between the set of prime ideals P' of $T *_{\beta}^{u} G$ such that $P' \cap T = 0$ and the set of prime ideals P of $R *_{\alpha}^{w} G$ such that $P \cap R = 0$.*

Proof. Let P' be a prime ideal of $T *_{\beta}^{u} G$ such that $P' \cap T = 0$. Then by Lemma 2.1.10, there exists a prime ideal P of $R *_{\alpha}^{w} G$ such that $P = P' \cap (R *_{\alpha}^{w} G)$. Thus $P \cap R = (P' \cap (R *_{\alpha}^{w} G)) \cap R = P' \cap ((R *_{\alpha}^{w} G) \cap R) = P' \cap R \subseteq P' \cap T = 0$.

Now, let P be a prime ideal of $R *_{\alpha}^{w} G$ such that $P \cap R = 0$. Then by Lemma 2.1.10, there exists a prime ideal P' of $T *_{\beta}^{u} G$ such that $P = P' \cap (R *_{\alpha}^{w} G)$. Thus, $P' \cap R = P' \cap ((R *_{\alpha}^{w} G) \cap R) = (P' \cap (R *_{\alpha}^{w} G)) \cap R = P \cap R = 0$ and it follows that $(P' \cap T)1_R = 0$. Since $P' \cap T$ is β-invariant we have that $(P' \cap T)\beta_g(1_R) = 0$, for all $g \in G$. By the fact that $T = \sum_{g \in G} \beta_g(R)$ we obtain that $(P' \cap T)T = 0$ and, since T is s-unital, it follows that $P' \cap T \subseteq (P' \cap T)T = 0$. $\square$

Let α be a twisted partial action of a group G on R and I an α-invariant ideal of R. We define $I^* = \{t \in T : \beta_g(t)1_R \in I, \forall g \in G\}$.

The proof of the following lemma is analogous to the proof of ([7], Lemma 2.3), and will be omitted.

Lemma 2.1.12. *Suppose that (R, α, w) has an enveloping action (T, β, u). If I is an α-invariant ideal of R, then I^* is a β-invariant ideal of T, with $I^* \cap R = I$. Moreover, for any β-invariant ideal J of T with $J \cap R = I$ we have $J \subseteq I^*$. In* *addition, if I is α-prime, then I^* is β-prime and conversely if J is a β-prime ideal of T, then there exists an α-prime ideal I of R such that $I^* = J$.*

Let I be an α-invariant ideal of R. Then we can extend the twisted partial action α of G on R to a twisted partial action $\overline{\alpha}$ of G on R/I as follows: for each $g \in G$, we define $\overline{\alpha}_g : (D_{g^{-1}} + I)/I \to (D_g + I)/I$ by $\overline{\alpha}_g(a + I) = \alpha_g(a) + I$ and, for each $(g, h) \in G \times G$, we extend each $w_{g,h}$ to R/I by $\overline{w_{g,h}} = w_{g,h} + I$.

The next corollary generalizes ([35], Lemma 16.6(iii)).

Corollary 2.1.13. *Suppose that G is a finite group. If P_1 and P_2 are prime ideals of $R *_{\alpha}^{w} G$ such that $P_1 \cap R = P_2 \cap R$, then $P_1 = P_2$.*

Proof. Let $Q = P_1 \cap R = P_2 \cap R$. By Lemma 2.1.12, we have that the set $Q^* = \{t \in T : \beta_g(t)1_R \in Q, \forall g \in G\}$ is a β-prime ideal of T such that $Q^* \cap R = Q$. By similar arguments of ([22], Proposition 2.10), $(R/Q, \overline{\alpha}, \overline{w})$ has an enveloping action $(T/Q^*, \overline{\beta}, \overline{u})$. Thus we may assume that $Q = P_1 \cap R = P_2 \cap R = 0$, that R is α-prime, and that T is β-prime. By Corollary 2.1.11, there exist prime ideals P_1' and P_2' of $T *_{\beta}^{u} G$ such that $P_1' \cap T = P_2' \cap T = 0$. Hence, by Lemma 1.1.1 we have that $P_1' = P_2'$ and it follows that $P_1 = P_2$. □

Corollary 2.1.14. *Suppose that R is α-prime and G is a finite group. A prime ideal P of $R *_{\alpha}^{w} G$ is minimal if and only if $P \cap R = 0$.*

Proof. Let P be a minimal prime ideal of $R *_{\alpha}^{w} G$. Then, by Lemma 2.1.10, there exists a prime ideal P' of $T *_{\beta}^{u} G$ such that $P' \cap (R *_{\alpha}^{w} G) = P$. We claim that P' is minimal. In fact, let Q' be a prime ideal of $T *_{\beta}^{u} G$ such that $0 \neq Q' \subseteq P'$. Thus, $0 \neq Q = Q' \cap (R *_{\alpha}^{w} G) \subseteq P$ and by assumption $Q = P$. Hence, $Q' = P'$. Since R is α-prime, by Lemma 2.1.12 we have that T is β-prime. By Lemma 1.1.2 we have that $P' \cap T = 0$ and by Corollary 2.1.11, we obtain that $P \cap R = 0$.

Conversely, let P be a prime ideal of $R *_{\alpha}^{w} G$ such that $P \cap R = 0$ and Q a prime ideal of $R *_{\alpha}^{w} G$ such that $0 \neq Q \subseteq P$. Then $Q \cap R \subseteq P \cap R = 0$. Thus,

$Q \cap R = P \cap R$ and by Corollary 2.1.13, we have that $Q = P$. So, P is a minimal prime ideal in $R *_{\alpha}^{w} G$. □

Lemma 2.1.15. *If I is a nonzero β-invariant ideal of T then $I \cap R \neq 0$.*

Proof. Suppose that $I \cap R = 0$. Then $I1_R = I \cap R = 0$. By the fact that I is β-invariant we have that $I\beta_g(1_R) = 0$, for all $g \in G$, and it follows that $IT = 0$. Since T is s-unital we have that $I \subseteq IT = 0$, which is a contradiction. □

We finish this section with some results that have independent interest.

By ([27], Theorem 3.1) there is a bijective correspondence between the ideals of S and the ideals of the ring of matrices $\mathbb{M}_n(S)$. In the next result we use this fact without further mention.

Proposition 2.1.16. *S is a FWPR if and only if $\mathbb{M}_n(S)$ is a FWPR.*

Proof. Suppose that S is a *FWPR*. Let J be a proper ideal of $\mathbb{M}_n(S)$ and assume that A and B are ideals of $\mathbb{M}_n(S)$ such that $0 \neq AB \subseteq J$. Then there exist ideals I, K and L of S such that $J = \mathbb{M}_n(I)$, $A = \mathbb{M}_n(K)$, $B = \mathbb{M}_n(L)$ and $0 \neq KL \subseteq I$. By the fact that S is a *FWPR* we have that either $K \subseteq I$ or $L \subseteq I$. Thus either $A \subseteq J$ or $B \subseteq J$. So, $\mathbb{M}_n(S)$ is a *FWPR*.

Conversely, suppose that $\mathbb{M}_n(S)$ is a *FWPR*. Let K be a proper ideal of $\mathbb{M}_n(S)$ and assume that I and J are ideals of S such that $0 \neq IJ \subseteq K$. Then $0 \neq \mathbb{M}_n(I)\mathbb{M}_n(J) \subseteq \mathbb{M}_n(K)$. By assumption on $\mathbb{M}_n(S)$, either $\mathbb{M}_n(J) \subseteq \mathbb{M}_n(K)$ or $\mathbb{M}_n(I) \subseteq \mathbb{M}_n(K)$. Thus either $I \subseteq K$ or $J \subseteq K$. So, S is a *FWPR*. □

By similar reasoning of the proposition above we have the following result, which completes ([5], Theorem 2.1).

Proposition 2.1.17. *The following statements hold:*

(i) *S is a FPR if and only if $\mathbb{M}_n(S)$ is a FPR.*

(ii) *S is an AFPR if and only if $\mathbb{M}_n(S)$ is an AFPR.*

2.2 Prime radicals of partial crossed products

In this section, we describe the prime radical of partial crossed products when the base ring is a fully weakly prime ring.

Now, we need the following definition.

Definition 2.2.1. Let α be a twisted partial action of a group G on R. We say that a proper α-invariant ideal Q of R is *weakly α-prime* if for any α-invariant ideals A and B of R with $0 \neq AB \subseteq Q$ we have either $A \subseteq Q$ or $B \subseteq Q$.

The proof of the following lemma is analogous to the proof of ([24], Proposition 1) and we give it here for the reader's convenience.

Lemma 2.2.2. *Let P be a weakly α-prime ideal of R that is not α-prime. Then $P^2 = 0$.*

Proof. By assumption there exist α-invariant ideals I and J of R such that $I \nsubseteq P$, $J \nsubseteq P$ and $0 = IJ \subseteq P$. If $P^2 \neq 0$, then $0 \neq P^2 \subseteq (I + P)(J + P) \subseteq P$, which implies that either $I \subseteq P$ or $J \subseteq P$, this is a contradiction. $\square$

Given a nonzero element $a = \sum_{g\in G} a_g\delta_g$ of $R *_\alpha^w G$, the *support* of a is defined by $\text{supp}(a) = \{g \in G : a_g \neq 0\}$. The following result generalize ([39], Proposition 2.3.4).

Lemma 2.2.3. *If R is semiprime, then $R *_\alpha^w G$ is semiprime.*

Proof. Let $a = \sum_{g\in G} a_g\delta_g \in R *_\alpha^w G$ such that $a(R *_\alpha^w G)a = 0$. Suppose that $a \neq 0$, then there exists $s \in \text{supp}(a)$ and note that

$$1_{s^{-1}}\delta_{s^{-1}}a(R *_\alpha^w G)1_{s^{-1}}\delta_{s^{-1}}a \subseteq 1_{s^{-1}}\delta_{s^{-1}}a(R *_\alpha^w G)a = 0.$$

Hence, $1_{s^{-1}}\delta_{s^{-1}}aR1_{s^{-1}}\delta_{s^{-1}}a = 0$ and it follows that

$$\alpha_{s^{-1}}(a_s)w_{s^{-1},s}R\alpha_{s^{-1}}(a_s)w_{s^{-1},s} = 0.$$

Consequently, $\alpha_{s^{-1}}(a_s)w_{s^{-1},s} = 0$, since R is semiprime. Hence $a_s = 0$, a contradiction, because $s \in \text{supp}(a)$. So, $R *_\alpha^w G$ is semiprime. $\square$

Since $Nil_*(R)$ is a α-invariant ideal of R, the twisted partial action α of G on R induce a twisted partial action of G on $R/Nil_*(R)$. We denote this partial action by α again.

Proposition 2.2.4. *For any twisted partial action α of a group G on R we have $Nil_\alpha(R) *_\alpha^w G \subseteq Nil_*(R *_\alpha^w G) \subseteq Nil_*(R) *_\alpha^w G$.*

Proof. For any prime ideal P of $R *_\alpha^w G$, by Lemma 2.1.4(i), $Q = P \cap R$ is an α-prime ideal of R. Since $Nil_\alpha(R) \subseteq Q$, then by Lemma 2.1.2($i$), we have that $Nil_\alpha(R) *_\alpha^w G \subseteq Q *_\alpha^w G = (P \cap R) *_\alpha^w G \subseteq P$. Hence, $Nil_\alpha(R) *_\alpha^w G \subseteq Nil_*(R *_\alpha^w G)$.

Moreover, it is well known that $Nil_*(R)$ is a semiprime ideal of R and thus $R/Nil_*(R)$ is a semiprime ring. By Lemma 2.2.3, we have that $(R/Nil_*(R)) *_\alpha^w G$ is semiprime. Since $(R/Nil_*(R)) *_\alpha^w G \cong (R *_\alpha^w G)/(Nil_*(R) *_\alpha^w G)$ it follows that $Nil_*(R *_\alpha^w G) \subseteq Nil_*(R) *_\alpha^w G$. □

Proposition 2.2.5. *For any twisted partial action α of a group G on R we have $Nil_*(R *_\alpha^w G) \cap R = Nil_\alpha(R)$.*

Proof. By Lemma 2.1.4(ii), for any α-prime ideal Q of R there exists a prime ideal P of $R *_\alpha^w G$ such that $P \cap R = Q$ and so $Nil_*(R *_\alpha^w G) \cap R \subseteq P \cap R = Q$. Hence, $Nil_*(R *_\alpha^w G) \cap R \subseteq Nil_\alpha(R)$.

By Proposition 2.2.4, we have that $Nil_\alpha(R) *_\alpha^w G \subseteq Nil_*(R *_\alpha^w G)$ and so $(Nil_\alpha(R) *_\alpha^w G) \cap R \subseteq Nil_*(R *_\alpha^w G) \cap R$. So, by Lemma 2.1.2($ii$) we have that $Nil_\alpha(R) = (Nil_\alpha(R) *_\alpha^w G) \cap R \subseteq Nil_*(R *_\alpha^w G) \cap R$. □

Lemma 2.2.6. *If R is a FWPR, then $Nil_*(R) = Nil_\alpha(R)$.*

Proof. By Proposition 2.2.4, we have that $Nil_\alpha(R) *_\alpha^w G \subseteq Nil_*(R) *_\alpha^w G$. Thus, $Nil_\alpha(R) = Nil_\alpha(R) *_\alpha^w G \cap R \subseteq Nil_*(R) *_\alpha^w G \cap R = Nil_*(R)$. By Lemma 1.3.9 we have that $Nil_*(R)$ is nilpotent. Then, since $Nil_*(R)$ is α-invariant and $Nil_\alpha(R)$ is α-semiprime, it follows that $Nil_*(R) \subseteq Nil_\alpha(R)$. So, $Nil_*(R) = Nil_\alpha(R)$. □

From now on for any ring S, we denote by $Nil(S)$ the sum of all nilpotent ideals of S and by $J(S)$ the Jacobson radical of S. Next, we give a description of the prime radical of partial crossed when the base ring is a *FWPR*.

Theorem 2.2.7. *If R is a FWPR, then*

$$Nil_*(R *_\alpha^w G) = Nil(R) *_\alpha^w G = Nil_\alpha(R) *_\alpha^w G = Nil_*(R) *_\alpha^w G.$$

Proof. By Lemmas 1.3.9 and 2.2.6 we have that $Nil_*(R) = Nil(R) = Nil_\alpha(R)$. Since, by Proposition 2.2.4, $Nil_\alpha(R) *_\alpha^w G \subseteq Nil_*(R *_\alpha^w G) \subseteq Nil_*(R) *_\alpha^w G$ it follows that $Nil_*(R *_\alpha^w G) = Nil(R) *_\alpha^w G = Nil_\alpha(R) *_\alpha^w G = Nil_*(R) *_\alpha^w G$. □

Using Corollary 1.3.10 and the theorem above we have the following result.

Corollary 2.2.8. *If R is a noetherian FWPR, then $Nil_*(R *_\alpha^w G) = J(R) *_\alpha^w G$.*

Corollary 2.2.9. *Suppose that $R *_\alpha^w G$ is a FWPR and R is a noetherian FWPR. Then $Nil_*(R *_\alpha^w G) = J(R *_\alpha^w G) = J(R) *_\alpha^w G$.*

Proof. By analogous reasoning of ([7], Corollary 3.4), $R *_\alpha^w G$ is noetherian. Now, using Lemma 1.3.10 and Theorem 2.2.7 we have the result. □

Lemma 2.2.10. *If $R *_\alpha^w G$ is a FWPR, then $(Nil_\alpha(R))^2 = 0$.*

Proof. By Proposition 2.2.5, we have that $Nil_*(R *_\alpha^w G) \cap R = Nil_\alpha(R)$, and since $R *_\alpha^w G$ is a *FWPR*, by Lemma 1.3.9, we have that $(Nil_*(R *_\alpha^w G))^2 = 0$. So, $(Nil_\alpha(R))^2 = (Nil_*(R *_\alpha^w G) \cap R)^2 \subseteq (Nil_*(R *_\alpha^w G))^2 = 0$. □

Let α be a twisted partial action of an infinite cyclic group G on R and (T, β, u) its enveloping action. By similar arguments of ([9], Lemma 1.13), we can show that if L is an α-prime ideal of R, then $L *_\alpha^w G$ is a prime ideal of $R *_\alpha^w G$. Now, using the Proposition 2.2.4 we have the following result.

Proposition 2.2.11. *Let α be a twisted partial action of $\mathbb{Z}$ on R. Then*

$$Nil_*(R *_\alpha^w \mathbb{Z}) = Nil_\alpha(R) *_\alpha^w \mathbb{Z}.$$

2.3 Partial crossed products, *FWPR* and *AFPR*

In this section we study necessary and sufficient conditions for the partial crossed products to be a fully weakly prime ring. As a consequence of our techniques we obtain the results for the partial crossed products to be a *FPR* and we obtain necessary and sufficient conditions for the partial crossed products to be an *AFPR*. Moreover, we give some examples to show that our results are not an easy generalization of the global case.

Throughout this section we assume that the twisted partial action α of G on R has an enveloping action (T, β, u), unless otherwise stated.

In ([25], Definition 1) a ring T is said to be a β-*FPR* if every β-invariant ideal of T is β-prime and in ([25], p. 86) a ring T is said to be a β-*AFPR* if every proper β-invariant ideal of T is β-prime and T is not β-prime. Now, we need the following definitions.

Definition 2.3.1. Let α be a twisted partial action of a group G on a ring R.

(i) We say that R is an α-*FPR* if every α-invariant ideal of R is α-prime.

(ii) We say that R is an α-*AFPR* if every proper α-invariant ideal of R is α-prime and R is not a α-prime ring.

(iii) We say that R is an α-*FWPR* if every proper α-invariant ideal of R is weakly α-prime.

Definition 2.3.2. Let β be a twisted global action of a group G on a ring T. We say that T is a β-*FWPR* if every proper β-invariant ideal of T is weakly β-prime.

The proof of the following result is similar of Proposition 1.3.2, and will be omitted.

Proposition 2.3.3. *A ring S is an α-FPR if and only if the set of all the α-invariant ideals of S is linearly ordered by inclusion and all α-invariant ideal of S is idempotent.*

We recall that given an α-invariant ideal I of R, we have that

$$I^* = \{t \in T : \beta_g(t)1_R \in I, \forall g \in G\}$$

is a β-invariant ideal of T such that $I^* \cap R = I$, see Lemma 2.1.12. In the next result we use this fact without further mention.

Lemma 2.3.4. *R is an α-FWPR if and only if T is a β-FWPR.*

Proof. Let P be a proper β-invariant ideal of T and assume that A and B are nonzero β-invariant ideals of T such that $0 \neq AB \subseteq P$. Thus, by similar arguments of the Lemma 2.1.15, we have $0 \neq (A \cap R)(B \cap R) \subseteq P \cap R$, with $0 \neq A \cap R = A1_R$ and $0 \neq B \cap R = B1_R$. Hence, by assumption we have that either $A1_R \subseteq P \cap R \subseteq P$ or $B1_R \subseteq P \cap R \subseteq P$. Since A, B and P are β-invariant ideals of T, it follows that $A\beta_g(1_R) \subseteq P$ or $B\beta_g(1_R) \subseteq P$, for all $g \in G$. So, we have that either $AT \subseteq P$ or $BT \subseteq P$ and since T is s-unital, it follows that $A \subseteq AT \subseteq P$ or $B \subseteq BT \subseteq P$. Thus P is weakly β-prime and we have that T is a β-*FWPR*.

Conversely, let Q be a proper α-invariant ideal of R and assume that I and J are α-invariant ideals of R such that $0 \neq IJ \subseteq Q$. Since $I^* \cap R = I$ and $J^* \cap R = J$ we have that $0 \neq I^*J^* \subseteq Q^*$. By assumption we have that either $I^* \subseteq Q^*$ or $J^* \subseteq Q^*$ and it follows that either $I \subseteq Q$ or $J \subseteq Q$. Thus Q is weakly α-prime and we have that R is an α-*FWPR*. □

The proof of the following result is analogous to the proof of lemma above.

Proposition 2.3.5. *(i) R is an α-FPR if and only T is a β-FPR.*

(ii) R is an α-AFPR if and only if T is a β-AFPR.

Suppose that G is an infinite cyclic group generated by σ. In this case, note that $T *_{\beta}^{u} G$ is the twisted skew Laurent polynomial ring $T\langle x;\sigma,u\rangle$ whose set of elements consists of finite sums $\sum_{i=n}^{m} a_i x^i$, where $m,n \in \mathbb{Z}$, with the usual addition of polynomials and multiplication determined by rule

$$(a_i x^i)(a_j x^j) = a_i \sigma^i(a_j) u_{\sigma^i,\sigma^j} x^{i+j}.$$

For each $i,j \in \mathbb{Z}$ we denote u_{σ^i,σ^j} simply by $u_{i,j}$. As a subring of $T\langle x;\sigma,u\rangle$ we have $T[x;\sigma,u]$, the twisted skew polynomial ring whose elements are the polynomials $\sum_{i=0}^{n} a_i x^i$ with the usual addition and multiplication defined as before and we denote by $lc(f) = a_n$ the leading coefficient of $f = \sum_{i=0}^{n} a_i x^i$. We define T_m as the set of $f \in T[x;\sigma,u]$ such that $\tau(f) \leq m$, where $\tau(f)$ denote the degree of the polynomial f. Now, let J be a nonzero ideal of $T\langle x;\sigma,u\rangle$. We define $J \cap T_m$ as the set $f \in J \cap T\langle x;\sigma,u\rangle$ such that $\tau(f) \leq m$. Moreover, for an element $\sum_{i=n}^{m} a_i x^i \in T\langle x;\sigma,u\rangle$ we define

$$\sigma^j\big(\sum_{i=m}^{n} a_i x^i\big) = \sum_{i=m}^{n} \sigma^j(a_i) x^i = x^j\big(\sum_{i=m}^{n} a_i x^i\big)x^{-j}$$

and an ideal I of $T\langle x;\sigma,u\rangle$ is said to be T-disjoint if $I \cap T = 0$. Now, using these facts and with minor adaptations from ([36], Lemma 2.11) we have the following result.

Lemma 2.3.6. *Let I be a nonzero T-disjoint ideal of $T\langle x;\sigma,u\rangle$ and $f \in I$ a nonzero polynomial of minimal degree n such that $lc(f) = a$. Suppose that $m \geqslant n$ and $g \in I \cap T_m$. If $a_j \in T$ and i_j is a non-negative integer for each $j \in \{1,2,...,m-n\}$, then there exists $h \in T_{m-n}$ such that*

$$h a_0 \sigma^{i_0}(f) = g \prod_{j=0}^{m-n} \sigma^{-n}\big(a_{m-n-j}\sigma^{i_{m-n-j}}(a)\big),$$

for all $a_0 \in T$ and $i_0 \in \mathbb{Z}$.

Now we are ready to prove the next proposition that partially generalizes ([8], Lemma 2.7).

Proposition 2.3.7. *Suppose that T is σ-prime and P is a nonzero T-disjoint ideal of $T\langle x;\sigma,u\rangle$. If P is a prime ideal, then P is maximal in the set of T-disjoint ideals.*

Proof. Let I be a T-disjoint ideal of $T\langle x;\sigma,u\rangle$ such that $P \subseteq I$. Let $f \in I$ be a polynomial of minimal degree n in I such that $lc(f) = a$ and $g \in P$ a nonzero polynomial of minimal degree m in P such that $lc(g) = b$. Suppose that $m > n$. For each $g \in P \cap T_m \subseteq I \cap T_m$, by Lemma 2.3.6, there exists $h \in T_{m-n}$ such that

$$ha_0\sigma^{i_0}(f) = g\prod_{j=0}^{m-n}\sigma^{-n}\big(a_{m-n-j}\sigma^{i_{m-n-j}}(a)\big),$$

for all $a_0 \in T$ and $i_0 \in \mathbb{Z}$. Since $g \in P$ we obtain that $hT\sigma^{i_0}(f) \subseteq P$, for all $i_0 \in \mathbb{Z}$. Then, for each $tx^k \in T\langle x;\sigma,u\rangle$, we have that $ht\sigma^k(f)\sigma^k(c)x^k \in P$, where $c \in T$ is such that $fc = c$. By the fact that

$$htx^kfx^{-k}\sigma^k(c)x^k = htx^kfu_{-k,k}$$

we have that $htx^kfu_{-k,k}T \subseteq P$. In the proof of ([16], Theorem 4.1) we have that $u_{-k,k}T = Tu_{-k,k} = T$ and it follows that $htx^kfT \subseteq P$, which implies that $htx^kf \in P$. Consequently, $hT\langle x;\sigma,u\rangle f \subseteq P$ and since P is prime we have that either $h \in P$ or $f \in P$. Thus, either $f = 0$ or $h = 0$, which contradicts the fact that h and f are nonzero polynomials and it follows that $m = n$.

Next, let $f \in I$ and $g \in P$ such that $\tau(f) = n+1$ and $\tau(g) = n$. Then for $l = axtx^igcx^{-i} - f\sigma^{-(n+1)}\big(\sigma(t\sigma^i(b)u_{i,n}u_{n+i,-i})u_{1,n}\big)$, where $lc(g) = b$, $lc(f) = a$ and $t \in T$, we easily have that $\tau(l) = n$. Hence, $l \in I \cap T_n = P \cap T_n$ and it follows that $fT\sigma^{i-n}(b) \subseteq P$, for all $i \in \mathbb{Z}$. By similar arguments as before we obtain that $f \in P$ and consequently $I \cap T_{n+1} = P \cap T_{n+1}$. Now, proceeding by induction we have that $I \cap T_m = P \cap T_m$, for all $m \geqslant 0$. So, $I = P$. □

We recall that, given a twisted partial action α of G on a ring R, an ideal J of $R *_\alpha^w G$ is said to be *R-disjoint* if $J \cap R = 0$. The next lemma partially generalizes ([10], Corollary 2.12).

Lemma 2.3.8. *Suppose that G is an infinite cyclic group. If P_1 and P_2 are prime ideals of $R *_\alpha^w G$ such that $P_1 \cap R = P_2 \cap R$, then $P_1 = P_2$.*

Proof. Since $L = P_1 \cap R = P_2 \cap R$ is an α-invariant ideal of R, we consider the partial crossed product $Z = \big(R/L\big) *_\alpha^w G$. Note that the images of P_1 and P_2 in Z are R/L-disjoint prime ideals. Thus, we may assume that $P_1 \cap R = P_2 \cap R = 0$. Since P_i is prime, for $i = 1, 2$, it follows that R is α-prime and we hav that T is β-prime. Again by the fact that $P_i \cap R = 0$, we have that the prime ideal P_i' of $T *_\beta^u G$, such that $P_i = P_i' \cap R *_\alpha^w G$, satisfies $P_i' \cap T = 0$. Then, by Proposition 2.3.7, we have that P_i' is maximal in the set of T-disjoint ideals of $T *_\beta^u G$ and, since $P_i = P_i' \cap R *_\alpha^w G$, is not difficult see that P_i is maximal in the set of R-disjoint ideals of $R *_\alpha^w G$. Suppose, without loss of generality, that $P_1 \nsubseteq P_2$. Thus $P_1 \subsetneq P_1 + P_2$ and since, by Proposition 2.3.7, P_1 is maximal in the set of R-disjoint ideals, it follows that $(P_1 + P_2) \cap R \neq 0$. Hence, there exists an nonzero element $r = f + g \in P_1 + P_2$ which implies that $0 \neq a_0 + b_0 \in P_1 + P_2$, where $a_0 \in P_1$ and $b_0 \in P_2$. By the fact that $r \neq 0$ we have that either $a_0 \neq 0$ or $b_0 \neq 0$ and we obtain that either $P_1 \cap R \neq 0$ or $P_2 \cap R \neq 0$, which is a contradiction. Therefore $P_1 = P_2$. $\square$

In the next lemma we study partial crossed products by infinite cyclic groups that are *FWPR*.

Lemma 2.3.9. *Suppose that G is an infinite cyclic group. If $R *_\alpha^w G$ is a FWPR, then R is an α-FWPR and there exists a bijective correspondence between the set $\mathcal{L}_1$ of ideals of $R *_\alpha^w G$ that contains the prime radical of $R *_\alpha^w G$ and the set $\mathcal{L}_2$ of α-invariant ideals of R that contains the α-prime radical of R.*

Proof. Let A be a proper α-invariant ideal of R and assume that I and J are α-invariant ideals of R such that $0 \neq IJ \subseteq A$. By Lemma 2.1.2(ii), we have that $0 \neq (I *_\alpha^w G)(J *_\alpha^w G) \subseteq A *_\alpha^w G$ and by assumption we have that either $I *_\alpha^w G \subseteq A *_\alpha^w G$ or $J *_\alpha^w G \subseteq A *_\alpha^w G$. Consequently, either $I \subseteq A$ or $J \subseteq A$.

Note that, by Lemma 2.2.10 the intersection of all α-prime ideals of R is a nilpotent ideal. Now, we define $\Psi : \mathcal{L}_1 \to \mathcal{L}_2$ by $\Psi(L) = L \cap R$. If $L \in \mathcal{L}_1$ and L contains properly $Nil_*(R *_\alpha^w G)$, then L is a prime ideal, otherwise, by Lemma 1.3.11, we would have $L^2 = 0$, and hence $L = Nil_*(R *_\alpha^w G)$. Since L is a prime ideal, by Lemma 2.1.4(i) we have that $L \cap R$ is an α-prime ideal of R that contain the α-prime radical. By Lemma 2.3.8, we see easily that Ψ is injective. We show that Ψ is surjective. In fact, let K be an α-invariant ideal of R that contains properly $Nil_\alpha(R)$. If K was nilpotent, then we would have that $K \subseteq Nil_\alpha(R)$, this contradicts the assumption on K. Hence, $K^2 = K$ and, by Lemma 2.2.2, K is α-prime. Using the same techniques of ([9], Lemma 1.13), we obtain that $K *_\alpha^w G$ is a prime ideal of $R *_\alpha^w G$ that the contains the prime radical of $R *_\alpha^w G$. ☐

From now on we denote the set of ideals of R by $\mathcal{L}(R)$, the set of α-invariant ideals of R by $\alpha - \mathcal{L}(R)$ and the set of non-minimal ideals of R by $\overline{\mathcal{L}(R)}$. The next lemma is a partial converse of the lemma above.

Lemma 2.3.10. *Suppose that G is an infinite cyclic group. If R is an α-FWPR and the map $\varphi : \mathcal{L}(R *_\alpha^w G) \to \alpha - \mathcal{L}(R)$, defined by $\varphi(J) = J \cap R$, is bijective, then $R *_\alpha^w G$ is a FWPR.*

Proof. Let P be a proper ideal of $R *_\alpha^w G$ and assume that A and B are ideals of $R *_\alpha^w G$ such that $0 \neq AB \subseteq P$. Since φ is bijective, it follows that $0 \neq (A \cap R)(B \cap R) \subseteq (P \cap R)$ and by the fact that R is an α-*FWPR* we have that either $A \cap R \subseteq P \cap R$ or $B \cap R \subseteq P \cap R$. Again by the bijectivity of φ we have that either $A \subseteq P$ or $B \subseteq P$. Thus P is weakly prime and so $R *_\alpha^w G$ is a *FWPR*. ☐

Lemma 2.3.11. *$R *_\alpha^w G$ is a FWPR if and only if $T *_\beta^u G$ is a FWPR.*

Proof. Suppose that $R *_\alpha^w G$ is a *FWPR*. Let P' be a proper ideal of $T *_\beta^u G$ and assume that I' and J' are ideals of $T *_\beta^u G$ such that $0 \neq I'J' \subseteq P'$. Then by

Lemma 2.1.8 there exist ideals I, J and P of $R *_\alpha^w G$ such that $I = I' \cap (R *_\alpha^w G)$, $J = J' \cap (R *_\alpha^w G)$ and $P = P' \cap (R *_\alpha^w G)$ with $0 \neq IJ \subseteq P$. Since P is weakly prime, then either $I \subseteq P$ or $J \subseteq P$. Hence, either $I' \subseteq P'$ or $J' \subseteq P'$. Thus P' is weakly prime and so $T *_\beta^u G$ is a *FWPR*.

Using the Lemma 2.1.8 we show the converse with a similar reasoning. □

Using the Lemma 2.1.10, analogously to the lemma above, we obtain the following result.

Proposition 2.3.12. (i) *$R *_\alpha^w G$ is a FPR if and only if $T *_\beta^u G$ is a FPR.*

(ii) *$R *_\alpha^w G$ is an AFPR if and only if $T *_\beta^u G$ is an AFPR.*

Following the same arguments of Lemma 2.3.9 and the Corollary 2.1.13 we obtain the following result.

Proposition 2.3.13. *Suppose that G is a finite group. If $R *_\alpha^w G$ is a FWPR, then R is an α-FWPR and there is a bijective correspondence between the set $\mathcal{L}_1$ of ideals of $R *_\alpha^w G$ that contains the prime radical of $R *_\alpha^w G$ and the set $\mathcal{L}_2$ of all the α-invariant ideals of R that contains the α-prime radical of R.*

The next proposition is a partial converse of the proposition above.

Proposition 2.3.14. *Suppose that G is finite. If R is an α-FWPR and the map $\phi : \mathcal{L}(R *_\alpha^w G) \to \alpha - \mathcal{L}(R)$, defined by $\varphi(J) = J \cap R$, is bijective, then $R *_\alpha^w G$ is a FWPR.*

Proof. By similar arguments of Lemma 2.3.10 we have the result. □

Definition 2.3.15. A group G is said to be *polycyclic-by-finite* if there exists a series $\{1\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$ such that G_i is normal in G_{i+1} and G_{i+1}/G_i is either an infinite cyclic group or a finite group for all $i \geqslant 1$ and G_1 is an infinite cyclic group.

Let α be a twisted partial action of a group G on a ring R with an enveloping action (T, β, u). For any subgroup G_i of G we consider the twisted partial action α_i as the restriction of partial action α to G_i, the twisted global action β_i as the restriction of global action β to G_i and $\beta_{i+1,i}$ the twisted global action induce of G_{i+1}/G_i.

Now we are ready to prove the first principal result of this section.

Theorem 2.3.16. *Suppose that G is a polycyclic-by-finite group, R is an α_1-FWPR and that exists a bijective correspondence between the set of α_1-invariant ideals of R and the set of β_1-invariant ideals of T by map $I \mapsto I^* = \{t \in T : \beta_g(t)1_R \in I, \forall g \in G_1\}$. If R is an α-FWPR and the map $\varphi : \mathcal{L}(R *_\alpha^w G) \to \alpha - \mathcal{L}(R)$, defined by $\varphi(P) = P \cap R$, is bijective, then $R *_\alpha^w G$ is a FWPR.*

Proof. First, note that $\Psi : \mathcal{L}(T *_\beta^u G) \to \beta - \mathcal{L}(T)$, defined by $\Psi(J) = J \cap T$, is bijective. In fact, clearly Ψ is surjective. Moreover, if I' and J' are ideals of $T *_\beta^u G$ such that $I' \cap R = J' \cap R$, by Lemma 2.1.8, there exist ideals I and J of $R *_\alpha^w G$ such that $I = I' \cap (R *_\alpha^w G)$ and $J = J' \cap (R *_\alpha^w G)$. So $I \cap R = I' \cap R = J' \cap R = J \cap R$. Since $I \cap R = J \cap R$ is α-invariant and φ is bijective, we have that $I = J$ and, by Lemma 2.1.8, it follows that $I' = J'$. Hence, Ψ is injective.

Now, considering β_1 the restriction of β to G_1, we easily obtain that $\Psi_1 : \mathcal{L}(T *_{\beta_1}^u G_1) \to \beta_1 - \mathcal{L}(T)$ defined by $\Psi_1(J) = J \cap T$ is bijective. By assumption and the same arguments of Lemma 2.3.4 we have that T is a β_1-*FWPR*. Thus, by Lemma 2.3.10 we have that $T *_{\beta_1}^u G_1$ is a *FWPR*. Using the bijectivity of Ψ and Ψ_1 we easily obtain that $\Psi_2 : \mathcal{L}\big((T *_{\beta_1}^u G_1) * (G_2/G_1)\big) \to \beta_{2,1} - \mathcal{L}(T *_{\beta_1}^u G_1)$ is bijective. By the fact that $T *_{\beta_1}^u G_1$ is a *FWPR* we have in particular that $T *_{\beta_1}^u G_1$ is a $\beta_{2,1}$-*FWPR*. Now, using either Lemma 2.3.10 or Proposition 2.3.14, if G_2/G_1 is either infinite cyclic or finite we have that $T *_{\beta_2}^u G_2$ is a *FWPR*. So, using induction and either Lemma 2.3.10 or Proposition 2.3.14 we obtain that $T *_\beta^u G$ is a *FWPR*. Hence, by Lemma 2.3.11 we have that $R *_\alpha^w G$ is a *FWPR*. $\square$

The next result generalizes ([25], Theorem 2) and the proof follows by the same arguments used in Theorem 2.3.16 and in Proposition 2.3.13.

Proposition 2.3.17. *Suppose that G is a finite group. Then $R *_\alpha^w G$ is a FPR if and only if R is an α-FPR and the map $\phi : \mathcal{L}(R *_\alpha^w G) \to \alpha - \mathcal{L}(R)$, defined by $\phi(I) = I \cap R$, is bijective.*

Lemma 2.3.18. *Let α be a twisted partial action of a finite group G on R.*

(i) *Suppose that R is an AFPR whose set of ideals is not linearly ordered by inclusion and let Q_1 and Q_2 be the minimal ideals of R. Then R is not α-prime if and only if Q_1 and Q_2 are α-invariant.*

(ii) *Suppose that $R *_\alpha^w G$ is an AFPR whose set of ideals is not linearly ordered by inclusion and let P_1 and P_2 be the minimal ideals of $R *_\alpha^w G$. Then $P_1 \cap R = 0$ if and only if $P_2 \cap R = 0$.*

Proof. (i) Suppose that R is not α-prime. Then, there exist nonzero α-invariant ideals A and B of R such that $AB = 0$. Since R is an *AFPR*, Q_1 is prime and by the fact that $AB = 0 \subseteq Q_1$, it follows that $0 \neq A \subseteq Q_1$ or $0 \neq B \subseteq Q_1$. Thus either $A = Q_1$ or $B = Q_1$, and it follows that Q_1 is α-invariant. By analogous reasoning we obtain that Q_2 is α-invariant.

Conversely, suppose that Q_1 and Q_2 are α-invariant. Since Q_1 and Q_2 are minimal ideals, we have that $Q_1Q_2 \subseteq Q_1 \cap Q_2 = 0$ and so R is not α-prime.

(ii) If $P_1 \cap R = 0$, since P_1 is a prime ideal of $R *_\alpha^w G$, by Lemma 2.1.4(i) we have that $0 = P_1 \cap R$ is an α-prime ideal of R. Thus, R is α-prime and since P_2 is minimal, by Corollary 2.1.14 we have that $P_2 \cap R = 0$.

By similar arguments we have the converse. □

Lemma 2.3.19. *Suppose that α is a twisted partial action of a group G on a ring R which is an AFPR. If the map $\phi : \mathcal{L}(R *_\alpha^w G) \to \mathcal{L}(R)$, defined by $P \mapsto P \cap R$, is bijective, then all proper ideals of $R *_\alpha^w G$ are prime.*

Proof. Using the same arguments of Lemma 2.3.10 we obtain the result. □

The next result partially generalizes ([25], Theorems 4 and 5) and we study sufficient conditions for the partial crossed product to be an *AFPR* when the base ring is an *AFPR*.

Theorem 2.3.20. *Suppose that α is a twisted partial action of a finite group G on R which is an AFPR. If one of the following conditions is satisfied*

(i) *The map $\phi_1 : \mathcal{L}(R *_\alpha^w G) \to \mathcal{L}(R)$, defined by $P \mapsto P \cap R$, is bijective;*

(ii) (a) *$R *_\alpha^w G$ has exactly two minimal ideals, P_1 and P_2, which are prime;*

(b) *the map $\phi_2 : \overline{\mathcal{L}(R *_\alpha^w G)} \to \overline{\mathcal{L}(R)}$, defined by $P \mapsto P \cap R$, is bijective.*

(iii) (a) *$R *_\alpha^w G$ has only one minimal ideal P_0 which is prime and nilpotent;*

(b) *the map $\phi_3 : \overline{\mathcal{L}(R *_\alpha^w G)} \to \overline{\mathcal{L}(R)}$, defined by $P \mapsto P \cap R$, is bijective.*

then $R *_\alpha^w G$ is an AFPR.

Proof. Suppose that (i) holds. We have two cases to be considered:

(*Case* 1 - $\mathcal{L}(R)$ is linearly ordered by inclusion)

By Lemma 1.3.5, R has a unique minimal nilpotent ideal Q_0 and we easily obtain that $P_0 = Q_0 *_\alpha^w G$ is a nilpotent minimal ideal of $R *_\alpha^w G$. Hence, $R *_\alpha^w G$ is not prime and, by similar techniques of Lemma 2.3.10, we obtain that all proper ideals of $R *_\alpha^w G$ are prime. So, $R *_\alpha^w G$ is an *AFPR*.

(*Case* 2 - $\mathcal{L}(R)$ is not linearly ordered by inclusion)

By Lemma 1.3.4, R has two minimal ideals, Q_1 and Q_2. Since ϕ is bijective, there exist nonzero ideals P_1 and P_2 of $R *_\alpha^w G$ such that $P_1 \cap R = Q_1$ and $P_2 \cap R = Q_2$. It is not difficult to show that P_1 and P_2 are minimal prime ideals of $R *_\alpha^w G$. Thus $P_1 P_2 = 0$ and it follows that $R *_\alpha^w G$ is not prime. Moreover, by Lemma 2.3.19 all proper ideals of $R *_\alpha^w G$ are prime. So, $R *_\alpha^w G$ is an *AFPR*.

Suppose that (ii) holds. Let P be a proper ideal of $R *_\alpha^w G$. If P is minimal then P is prime, by item (a). If P is not minimal, then $P \in \overline{\mathcal{L}(R *_\alpha^w G)}$ and by the fact that ϕ_2 is bijective, by similar arguments of Lemma 2.3.10, we obtain that P is prime. So any proper ideal of $R*_\alpha^w G$ is prime. Moreover, since P_1 and P_2 are minimal ideals of $R*_\alpha^w G$, we have $P_1P_2=0$ and it follows that $R*_\alpha^w G$ is not prime. Hence, $R*_\alpha^w G$ is an *AFPR*.

Suppose that (iii) holds. By analogous reasoning of item (ii) we obtain that any proper ideal of $R*_\alpha^w G$ is prime. Since, by item (a), the minimal ideal P_0 is nilpotent it follows that $R*_\alpha^w G$ is not prime. So, $R*_\alpha^w G$ is an *AFPR*. □

The next theorem generalizes ([25], Theorems 6 and 7).

Theorem 2.3.21. *Let α be a twisted partial action of a finite group G on R. Then $R*_\alpha^w G$ is an AFPR if and only if either*

(i) (a) *R is an α-AFPR;*

(b) *the map $\phi_1:\mathcal{L}(R*_\alpha^w G)\to\alpha-\mathcal{L}(R)$, defined by $P\mapsto P\cap R$, is bijective;*

or (ii) (a) *R is an α-FPR;*

(b) *the minimal ideals of $R*_\alpha^w G$ are prime;*

(c) *the map $\phi_2:\overline{\mathcal{L}(R*_\alpha^w G)}\to\alpha-\mathcal{L}(R)$, defined by $P\mapsto P\cap R$, is bijective.*

Proof. Suppose that $R*_\alpha^w G$ is an *AFPR*. We have two cases to be considered:

(*Case* 1 - $\mathcal{L}(R*_\alpha^w G)$ is not linearly ordered by inclusion)

By Lemma 1.3.4(i), $R*_\alpha^w G$ has two minimal ideals, P_1 and P_2, which are prime because $R*_\alpha^w G$ is an *AFPR*. Now, we have the following subcases:

(*Subcase* 1.1 - $P_1\cap R\neq 0$)

By analogous reasoning of Proposition 2.3.17 we have that all nonzero α-invariant ideals of R are α-prime. Since $P_1\cap R\neq 0$, by Lemma 2.3.18(ii), we have that $P_2\cap R\neq 0$. Thus $(P_1\cap R)(P_2\cap R)\subseteq P_1P_2=0$ and therefore R is not α-prime. Hence, R is an α-*AFPR*.

Using the Corollary 2.1.13 and the Lemma 2.1.2(ii), we easily obtain that ϕ_1 is bijective.

(*Subcase* 1.2 - $P_1 \cap R = 0$)

By the same arguments of Subcase 1.1 we have that all nonzero α-invariant ideals of R are α-prime. Since, by Lemma 2.1.4(i), $P_1 \cap R = 0$ is α-prime it follows that R is α-prime and so R is an α-*FPR*.

Note that for each $P \in \overline{\mathcal{L}(R *_\alpha^w G)} \backslash \{0\}$ we have that $\phi_2(P) = P \cap R \neq 0$, otherwise P would be minimal by Corollary 2.1.14 since R is α-prime. By the fact that $Nil_\alpha(R) = 0$ and $Nil_*(R *_\alpha^w G) = P_1 \cap P_2 = 0$, by similar reasoning of Proposition 2.3.13 we have that ϕ_2 is bijective.

(*Case* 2 - $\mathcal{L}(R *_\alpha^w G)$ is linearly ordered by inclusion)

By Lemma 1.3.5, $R *_\alpha^w G$ has a unique minimal nilpotent ideal P_0 which is prime because $R *_\alpha^w G$ is an *AFPR*. Now, we have the following subcases:

(*Subcase* 2.1 - $P_0 \cap R \neq 0$)

By analogous arguments of Proposition 2.3.17 we have that all nonzero α-invariant ideals of R are α-prime. By the fact that $(P_0)^2 = 0$ we have that $(P_0 \cap R)(P_0 \cap R) \subseteq (P_0)^2 = 0$, with $P_0 \cap R \neq 0$. Hence, R is not α-prime and it follows that R is an α-*AFPR*. Moreover, analogously to Subcase 1.1, we obtain that ϕ_1 is bijective.

(*Subcase* 2.2 - $P_0 \cap R = 0$)

By similar arguments of Subcase 1.2, we obtain that R is an α-*FPR* and ϕ_2 is bijective.

Conversely, suppose that (i) holds. Since ϕ_1 is bijective, by analogous reasoning of Proposition 2.3.17, we show that all proper ideal of $R *_\alpha^w G$ is prime. Since R is an α-*AFPR*, there exists nonzero α-invariant ideals A and B of R such that $AB = 0$. Hence, $(A *_\alpha^w G)(B *_\alpha^w G) = 0$ and we have that $R *_\alpha^w G$ is not prime. So, $R *_\alpha^w G$ is an *AFPR*.

Suppose that (ii) holds. Let P be a proper ideal of $R *_\alpha^w G$. If P is minimal then, by item (b), P is prime. If P is not minimal, let I and J be ideals of $R *_\alpha^w G$ such that $IJ \subseteq P$. Then $(I \cap R)(J \cap R) \subseteq P \cap R$. Since R is an α-*FPR* we have that $P \cap R$ is α-prime and it follows that either $I \cap R \subseteq P \cap R$ or $J \cap R \subseteq P \cap R$. By the fact that ϕ_2 is bijective, we have that $I = (I \cap R) *_\alpha^w G$, $J = (J \cap R) *_\alpha^w G$ and $P = (P \cap R) *_\alpha^w G$. Consequently either $I = (I \cap R) *_\alpha^w G \subseteq (P \cap R) *_\alpha^w G = P$ or $J = (J \cap R) *_\alpha^w G \subseteq (P \cap R) *_\alpha^w G = P$. Hence, all proper ideals of $R *_\alpha^w G$ are prime. Now, if $\mathcal{L}(R *_\alpha^w G)$ is not linearly ordered by inclusion, then there exists nonzero ideals I and J of $R *_\alpha^w G$ such that $I \nsubseteq J$ and $J \nsubseteq I$. Note that $I \cap J = 0$, otherwise $I \cap J$ would be prime and we would obtain that $I \subseteq I \cap J \subseteq J$ or $J \subseteq I \cap J \subseteq I$, which is a contradiction. Consequently, $IJ \subseteq I \cap J = 0$ and we have that $R *_\alpha^w G$ is not a prime ring. If $\mathcal{L}(R *_\alpha^w G)$ is linearly ordered by inclusion, since $R *_\alpha^w G$ has a minimal ideal P_0 and $P_0 \cap R$ is an α-invariant ideal of R, there exists an ideal $P \in \overline{\mathcal{L}(R *_\alpha^w G)}$ such that $P \cap R = P_0 \cap R$. Note that $P = 0$, otherwise we would have that P is prime and by Corollary 2.1.13 that $P = P_0$. Thus $P_0 \cap R = 0$. Note that $R *_\alpha^w G$ is not a prime ring, because if 0 was prime, by Corollary 2.1.13 we would have that $P_0 = 0$, which is a contradiction. So, $R *_\alpha^w G$ is an *AFPR*. $\square$

It is natural to ask if R is either *FWPR* or *AFPR* or *FPR* or the set of ideals is linearly ordered by inclusion, then T would be either *FWPR* or *AFPR* or *FPR* or the set of ideals of T is linearly ordered by inclusion. The examples below show that this is not the case and show that our results are not an easy generalization of the global case.

Example 2.3.22.

(i) Let K be a field, $\{e_i : i \in \mathbb{Z}\}$ a set of orthogonal central idempotents and $T = \oplus_{i \in \mathbb{Z}} K e_i$. We defined a global action of the infinite cyclic group G generated by σ on T by $\sigma(e_i) = e_{i+1}$, for all $i \in \mathbb{Z}$. If $R = Ke_0$, then clearly we have a partial action of the group G on R. Note that R is a *FPR*, but T is not a *FPR*. Moreover, all ideals of R are linearly ordered by inclusion, but the set of ideals of T is not linearly ordered by inclusion. In turn, if $R = Ke_0 \oplus Ke_1$, then R is an *AFPR*, but T is not an *AFPR*.

(ii) Let K be a field, $\{e_1, e_2, e_3, e_4\}$ a set of orthogonal central idempotents and $T = \oplus_{i=1}^{4} Ke_i$. We define a global action of the finite cyclic group of order 4 generated by σ on T by $\sigma(e_1) = e_2$, $\sigma(e_2) = e_3$, $\sigma(e_3) = e_4$ and $\sigma(e_4) = e_1$. If $R = Ke_1$, then clearly we have a partial action of G on R. Note that R is a *FPR*, but T is not a *FPR*. Moreover, all ideals of R are linearly ordered by inclusion, but the set of ideals of T is not linearly ordered by inclusion.

(iii) Let T and σ as in the item (ii). If $R = Ke_1 \oplus Ke_2$, then we clearly have a partial action of G on R. Note that R is an *AFPR*, but T is not an *AFPR*. Moreover, note that R is a *FWPR*, but T is not a *FWPR* because let $I = Ke_1 \oplus Ke_3$ and $J = Ke_1 \oplus Ke_2$. Then $0 \neq IJ \subseteq Ke_1 \oplus Ke_4$ but $I \nsubseteq Ke_1 \oplus Ke_4$ and $J \nsubseteq Ke_1 \oplus Ke_4$.

Remark. Let β be a twisted global action of a group G on a ring T. If the set of all the ideals of T are linearly ordered by inclusion, then all the ideals of T are β-invariant. If it is possible to generalize this fact to twisted partial actions, then it is possible to prove the converse of Theorem 2.3.20.

Chapter 3

Simplicity of Partial Crossed Products

In this chapter, we study necessary and sufficient conditions for the commutativity and simplicity of $R *_\alpha^w G$. Furthermore, considering $R = C(X)$ the algebra of continuous functions defined on a topological space X with values in the complex numbers and $C(X) *_\alpha G$ the partial skew group ring, where α is a partial action of a topological group G on $C(X)$, we study some topological properties of G on X to obtain some results on the algebra $C(X)$. Also, we study the simplicity of $C(X) *_\alpha G$ using topological properties of X and the results about the simplicity of partial crossed product obtained for $R *_\alpha^w G$. Moreover, we give some examples to apply our results about the simplicity and to show that our assumptions are necessary to obtain the simplicity of $C(X) *_\alpha G$.

3.1 Commutativity and simplicity of partial crossed products

Let R be a ring and X a non-empty subset of R. The *centralizer* of X in R is the set $C_R(X) = \{r \in R : rx = xr, \forall x \in X\}$. It is easy to see that $C_R(X)$ is a subring of R. Note that if $X = R$, then the centralizer $C_R(X)$ is the center of R and it is denoted by $Z(R)$.

From now on, we assume that R is a ring with identity 1_R and

$$\alpha = \big(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}\big)$$

is a twisted partial action of G on R such that all the ideals D_g, $g \in G$, are generated by central idempotents 1_g. Note that this is not sufficient for a twisted partial action of a group G on a ring R to have an enveloping action, see ([16], Theorem 4.1).

The next result was proved in ([34], Lemma 2.1).

Lemma 3.1.1. *Let α be a twisted partial action of a group G on R. Then*

$$C_{R*_\alpha^w G}(R) = \Big\{\sum_{g\in G} a_g\delta_g \in R *_\alpha^w G : a_g\alpha_g(r1_{g^{-1}}) = ra_g, \forall r \in R \text{ and } \forall g \in G\Big\}.$$

Let R be a commutative ring. We denote the annihilator of an element $a \in R$ by $\mathrm{ann}(a)$. When R is commutative we have the following consequence.

Corollary 3.1.2. *Let α be a twisted partial action of a group G on a commutative ring R. Then*

$$C_{R*_\alpha^w G}(R) = \Big\{\sum_{g\in G} a_g\delta_g \in R*_\alpha^w G : \alpha_g(r1_{g^{-1}}) - r1_g \in \mathrm{ann}(a_g), \forall r \in R \text{ and } \forall g \in G\Big\}.$$

Proof. By assumption and by Lemma 3.1.1, for all $r \in R$ and $g \in G$, we have

$$\sum_{g\in G} a_g\delta_g \in C_{R*_\alpha^w G}(R) \Leftrightarrow a_g\alpha_g(r1_{g^{-1}}) = ra_g \Leftrightarrow a_g\alpha_g(r1_{g^{-1}}) = a_g r1_g$$
$$\Leftrightarrow a_g(\alpha_g(r1_{g^{-1}}) - r1_g) = 0 \Leftrightarrow \alpha_g(r1_{g^{-1}}) - r1_g \in \mathrm{ann}(a_g). \qquad \square$$

We say that R is *maximal commutative* in $R *_\alpha^w G$ if $R = C_{R*_\alpha^w G}(R)$. Note that if R is commutative, then $R \subseteq C_{R*_\alpha^w G}(R)$. Using Corollary 3.1.2 we obtain the following result.

Corollary 3.1.3. *Let α be a twisted partial action of a group G on a commutative ring R. Then R is maximal commutative in $R *_\alpha^w G$ if and only if for all $g \in G\backslash\{e\}$ and $a_g \in D_g\backslash\{0\}$, there exists $r \in R$ such that $\alpha_g(r1_{g^{-1}}) - r1_g \notin \operatorname{ann}(a_g)$.*

Using Corollary 3.1.3 we have the following.

Corollary 3.1.4. *Let α be a twisted partial action of a group G on a commutative ring R and suppose that for each $g \in G\backslash\{e\}$ there exists $r \in R$ such that $\alpha_g(r1_{g^{-1}}) - r1_g$ is not a zero divisor in D_g. Then R is maximal commutative in $R *_\alpha^w G$.*

Proposition 3.1.5. *Let α be a twisted partial action of a group G on R which is maximal commutative in $R *_\alpha^w G$. Then for each $g \in G\backslash\{e\}$ such that $D_g \neq 0$ we have $\alpha_g \neq id_{D_g}$.*

Proof. Suppose that there exists $h \in G\backslash\{e\}$ such that $D_h \neq \{0\}$ with $\alpha_h = id_{D_h}$. Thus, $D_h = D_{h^{-1}}$ and we have that $1_h = 1_{h^{-1}}$. Let $a_h\delta_h \neq 0$. Then for each $r \in R$,

$$(a_h\delta_h)(r\delta_e) = a_h\alpha_h(r1_{h^{-1}})w_{h,e}\delta_{he} = a_hr1_{h^{-1}}1_h\delta_h = a_hr\delta_h = (r\delta_e)(a_h\delta_h).$$

Hence, $a_h\delta_h \in C_{R*_\alpha^w G}(R)$ which contradicts the fact that R is maximal commutative in $R *_\alpha^w G$. $\square$

The following example shows that the assumption in Proposition 3.1.5 is not superfluous.

Example 3.1.6. Let R be a commutative ring and G any group. We define the following partial action: $D_e = R$, $D_g = 0$, for all $g \in G\backslash\{e\}$, $\alpha_e = id_R$ and $\alpha_g \equiv 0$, for all $g \in G\backslash\{e\}$. We easily obtain that R is maximal commutative in $R *_\alpha^w G$ and $\alpha_g = id_{D_g}$, for all $g \in G\backslash\{e\}$.

Definition 3.1.7. Let $\alpha = \big(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{g,h\in G}\big)$ be a twisted partial action of G on R. We say that w is *symmetric* if $w_{g,h} = w_{h,g}$, for all $g, h \in G$.

Corollary 3.1.8. *Let α be a twisted partial action of a group G on R. If R is commutative, G is abelian and w is symmetric, then $C_{R*_\alpha^w G}(R)$ is commutative.*

Proof. Let $x, y \in C_{R*_\alpha^w G}(R)$ such that $x = \sum\limits_{g\in G} a_g\delta_g$ and $y = \sum\limits_{h\in G} b_h\delta_h$. By Lemma 3.1.1, we have that $a_g\alpha_g(b_h1_{g^{-1}}) = b_h a_g$ and $b_h\alpha_h(a_g1_{h^{-1}}) = a_g b_h$, for all $g, h \in G$. By the fact that R is commutative, we have that $a_g\alpha_g(b_h1_{g^{-1}}) = b_h\alpha_h(a_g1_{h^{-1}})$, for all $g, h \in G$. Since G is abelian and w is symmetric, we have that

$$\begin{aligned}\Big(\sum_{g\in G} a_g\delta_g\Big)\Big(\sum_{h\in G} b_h\delta_h\Big) &= \sum_{g,h\in G} a_g\alpha_g(b_h1_{g^{-1}})w_{g,h}\delta_{gh}\\ &= \sum_{g,h\in G} b_h\alpha_h(a_g1_{h^{-1}})w_{h,g}\delta_{hg}\\ &= \Big(\sum_{h\in G} b_h\delta_h\Big)\Big(\sum_{g\in G} a_g\delta_g\Big).\end{aligned}$$

So, $C_{R*_\alpha^w G}(R)$ is commutative. $\square$

We recall that, given a nonzero element $a = \sum_{g\in G} a_g\delta_g \in R*_\alpha^w G$, the *support* of a is defined by $\text{supp}(a) = \{g \in G : a_g \neq 0\}$. Moreover, we denote $|\text{supp}(a)|$ as the cardinality of the support of the element a.

Lemma 3.1.9. *Let α be a twisted partial action of a group G on R. If R is commutative, then $I \cap C_{R*_\alpha^w G}(R) \neq 0$, for all nonzero ideal I of $R *_\alpha^w G$.*

Proof. For each $g \in G$, we define $T_g : R *_\alpha^w G \to R *_\alpha^w G$ by

$$T_g\Big(\sum_{h\in G} a_h\delta_h\Big) = \Big(\sum_{h\in G} a_h\delta_h\Big)(1_g\delta_g).$$

It is easy to verify that T_g is an homomorphism of left $R *_\alpha^w G$-modules such that $T_g(I) \subseteq I$, for each ideal I of $R*_\alpha^w G$ and for each $g \in G$. Note that for each $0 \neq a = \sum_{h\in G} a_h\delta_h$, with $a_e = 0$, there exists $p \in \text{supp}(a)$ such that

$$c=\sum_{l\in G}c_l\delta_l := T_{p^{-1}}\Big(\sum_{h\in G}a_h\delta_h\Big)=\sum_{h\in G}a_h\alpha_h(1_{p^{-1}}1_{h^{-1}})w_{h,p^{-1}}\delta_{hp^{-1}}$$

satisfies $c_e=a_pw_{p,p^{-1}}\neq 0$ and $1\leq|\text{supp}(c)|\leq|\text{supp}(a)|$.

For each $r\in R$ we define $K_r: R\ast_\alpha^w G\to R\ast_\alpha^w G$ by

$$K_r\Big(\sum_{h\in G}a_h\delta_h\Big)=\big(r\delta_e\big)\Big(\sum_{h\in G}a_h\delta_h\Big)-\Big(\sum_{h\in G}a_h\delta_h\Big)\big(r\delta_e\big).$$

It is easy to see that K_r is an homomorphism of additive abelian groups such that $K_r(I)\subseteq I$, for each ideal I of $R\ast_\alpha^w G$ and for each $r\in R$. Since R is commutative and $r1_e-\alpha_e(r1_{e^{-1}})=0$, we have

$$\begin{aligned}
K_r\Big(\sum_{h\in G}a_h\delta_h\Big) &= \big(r\delta_e\big)\Big(\sum_{h\in G}a_h\delta_h\Big)-\Big(\sum_{h\in G}a_h\delta_h\Big)\big(r\delta_e\big)\\
&= \Big(\sum_{h\in G}r\alpha_e(a_h1_{e^{-1}})w_{e,h}\delta_{eh}\Big)-\Big(\sum_{h\in G}a_h\alpha_h(r1_{h^{-1}})w_{h,e}\delta_{he}\Big)\\
&= \Big(\sum_{h\in G}ra_h1_h\delta_h\Big)-\Big(\sum_{h\in G}a_h\alpha_h(r1_{h^{-1}})\delta_h\Big)\\
&= \Big(\sum_{h\in G}a_hr1_h\delta_h\Big)-\Big(\sum_{h\in G}a_h\alpha_h(r1_{h^{-1}})\delta_h\Big)\\
&= \sum_{h\in G}a_h\big(r1_h-\alpha_h(r1_{h^{-1}})\big)\delta_h\\
&= \sum_{h\in G\setminus\{e\}}a_h\big(r1_h-\alpha_h(r1_{h^{-1}})\big)\delta_h.
\end{aligned}$$

Consequently, for each $r\in R$, the map K_r always annihilates the coefficient of δ_e and it follows that $\big|\text{supp}\big(K_r(\sum_{h\in G}a_h\delta_h)\big)\big|<\big|\text{supp}(\sum_{h\in G}a_h\delta_h)\big|$, for each $0\neq\sum_{h\in G}a_h\delta_h$ with $a_e\neq 0$.

By assumption on R and Corollary 3.1.2 we have $C_{R\ast_\alpha^w G}(R)=\bigcap_{r\in R}\ker(K_r)$. For each element $\sum_{h\in G}a_h\delta_h\in R\ast_\alpha^w G\setminus C_{R\ast_\alpha^w G}(R)$, we choose $r\in R$ such that $\sum_{h\in G}a_h\delta_h\notin\ker(K_r)$. Thus, for each $z=\sum_{g\in G}z_g\delta_g\in R\ast_\alpha^w G\setminus C_{R\ast_\alpha^w G}(R)$ with $z_e\neq 0$, we choose $r\in R$ such that $1\leq|K_r(z)|<|\text{supp}(z)|$.

Finally, we are able to show that $I\cap C_{R\ast_\alpha^w G}(R)\neq 0$, for each nonzero ideal I of $R\ast_\alpha^w G$. In fact, let I be a nonzero ideal of $R\ast_\alpha^w G$ and $0\neq z=\sum_{h\in G}a_h\delta_h\in I$. If $z\in C_{R\ast_\alpha^w G}(R)$ the proof is complete. Now, suppose that $z\notin C_{R\ast_\alpha^w G}(R)$. Then, applying T_g's and K_r's in a suitable way we obtain the nonzero element $b\delta_e \in I$ such that $0 \neq b\delta_e \in I \cap C_{R*^w_\alpha G}(R)$, since $T_g(I) \subseteq I$ and $K_r(I) \subseteq I$. $\square$

Using Lemma 3.1.9 we immediately obtain the following result.

Corollary 3.1.10. *Let α be a twisted partial action of a group G on R. If R is maximal commutative in $R *^w_\alpha G$, then $I \cap R \neq 0$, for all nonzero ideal I of $R *^w_\alpha G$.*

We recall that a ring S with a twisted partial action γ of G is said to be γ-simple if the unique γ-invariant ideals of S are the trivial ideals.

Corollary 3.1.11. *Let α be a twisted partial action of a group G on R. If R is α-simple and maximal commutative in $R *^w_\alpha G$, then $R *^w_\alpha G$ is simple.*

Proof. Let I be a nonzero ideal of $R *^w_\alpha G$. Then $I \cap R$ is an α-invariant ideal of R. By assumption and by Corollary 3.1.10, we have that $I \cap R \neq 0$. Since R is α-simple, then $I \cap R = R$. So, $R *^w_\alpha G$ is simple. $\square$

The proof of the following lemma is standard.

Lemma 3.1.12. *Let α be a twisted partial action of a group G on R. If $R *^w_\alpha G$ is simple, then R is α-simple.*

Using Corollary 3.1.11 and Lemma 3.1.12, we obtain the first principal result of this section, which generalizes ([31], Theorem 6.13).

Theorem 3.1.13. *Let α be a twisted partial action of a group G on R. Suppose that R is maximal commutative in $R *^w_\alpha G$. Then $R *^w_\alpha G$ is simple if and only if R is α-simple.*

Lemma 3.1.14. *Let α be a twisted partial action of a group G on R. The center of $R *_{\alpha}^{w} G$ is*

$$Z(R *_{\alpha}^{w} G) = \left\{ \sum_{g \in G} r_g \delta_g \quad : \quad r_{ts^{-1}} w_{ts^{-1},s} = \alpha_s(r_{s^{-1}t} 1_{s^{-1}}) w_{s,s^{-1}t}, \right.$$
$$\left. r_s \alpha_s(a 1_{s^{-1}}) = a r_s, \forall a \in R \text{ and } \forall s,t \in G \right\}.$$

Proof. Let $\sum_{g \in G} r_g \delta_g \in Z(R *_{\alpha}^{w} G)$. For any $a \in R$, we have

$$\begin{aligned}
\left(\sum_{g \in G} r_g \delta_g\right)(a\delta_e) &= \sum_{g \in G} r_g \alpha_g(a 1_{g^{-1}}) w_{g,e} \delta_{ge} \\
&= \sum_{g \in G} r_g \alpha_g(a 1_{g^{-1}}) 1_g \delta_g \\
&= \sum_{g \in G} r_g \alpha_g(a 1_{g^{-1}}) \delta_g
\end{aligned}$$

and

$$\begin{aligned}
(a\delta_e)\left(\sum_{g \in G} r_g \delta_g\right) &= \sum_{g \in G} a \alpha_e(r_g 1_{e^{-1}}) w_{e,g} \delta_{eg} \\
&= \sum_{g \in G} a r_g 1_g \delta_g \\
&= \sum_{g \in G} a r_g \delta_g.
\end{aligned}$$

Then, replacing g by s, we have that $r_s \alpha_s(a 1_{s^{-1}}) = a r_s$, for all $a \in R$ and $s \in G$.

Moreover, for all $s \in G$, we have

$$\begin{aligned}
\left(\sum_{g \in G} r_g \delta_g\right)(1_s \delta_s) &= \sum_{g \in G} r_g \alpha_g(1_s 1_{g^{-1}}) w_{g,s} \delta_{gs} \\
&= \sum_{g \in G} r_g 1_g 1_{gs} w_{g,s} \delta_{gs} \\
&= \sum_{g \in G} r_g 1_{gs} w_{g,s} \delta_{gs} \\
&= \sum_{t \in G} r_{ts^{-1}} 1_t w_{ts^{-1},s} \delta_t \\
&= \sum_{t \in G} r_{ts^{-1}} w_{ts^{-1},s} \delta_t
\end{aligned}$$

and

$$\begin{aligned}(1_s\delta_s)\left(\sum_{g\in G} r_g\delta_g\right) &= \sum_{g\in G}\alpha_s(r_g1_{s^{-1}})w_{s,g}\delta_{sg}\\ &= \sum_{t\in G}\alpha_s(r_{s^{-1}t}1_{s^{-1}})w_{s,s^{-1}t}\delta_t.\end{aligned}$$

Hence, $r_{ts^{-1}}w_{ts^{-1},s} = \alpha_s(r_{s^{-1}t}1_{s^{-1}})w_{s,s^{-1}t}$, $\forall s,t\in G$, and we have

$$Z(R*_\alpha^w G) \subseteq \left\{ \sum_{g\in G} r_g\delta_g \ : \ r_{ts^{-1}}w_{ts^{-1},s} = \alpha_s(r_{s^{-1}t}1_{s^{-1}})w_{s,s^{-1}t}, \right.$$
$$\left. r_s\alpha_s(a1_{s^{-1}}) = ar_s, \forall a\in R \text{ and } \forall s,t\in G \right\}.$$

On the other hand, let $\sum_{g\in G} r_g\delta_g \in R*_\alpha^w G$ such that $r_s\alpha_s(a1_{s^{-1}}) = ar_s$ and $r_{ts^{-1}}w_{ts^{-1},s} = \alpha_s(r_{s^{-1}t}1_{s^{-1}})w_{s,s^{-1}t}$, for all $a\in R$ and $s,t\in G$. Then, for any $\sum_{s\in G} a_s\delta_s \in R*_\alpha^w G$ we have that

$$\begin{aligned}\left(\sum_{g\in G} r_g\delta_g\right)\left(\sum_{s\in G} a_s\delta_s\right) &= \sum_{s,g\in G} r_g\alpha_g(a_s1_{g^{-1}})w_{g,s}\delta_{gs}\\ &= \sum_{s,g\in G} a_sr_gw_{g,s}\delta_{gs}\\ &= \sum_{t,s\in G} a_sr_{ts^{-1}}w_{ts^{-1},s}\delta_t\\ &= \sum_{t,s\in G} a_s\alpha_s(r_{s^{-1}t}1_{s^{-1}})w_{s,s^{-1}t}\delta_t\\ &= \sum_{g,s\in G} a_s\alpha_s(r_g1_{s^{-1}})w_{s,g}\delta_{sg}\\ &= \left(\sum_{s\in G} a_s\delta_s\right)\left(\sum_{g\in G} r_g\delta_g\right).\end{aligned}$$

So, $\sum_{g\in G} r_g\delta_g$ commutes with any element of $R*_\alpha^w G$. $\square$

In the next three corollaries we obtain a description of the center of partial crossed product when we assume some other assumptions either on R or on the twisted partial action α.

Corollary 3.1.15. *Let α be a twisted partial action of a group G on R. If $\alpha_g = id_{D_g}$, $\forall g \in G$, then*

$$Z(R *_\alpha^w G) = \left\{ \sum_{g\in G} r_g\delta_g : r_s \in Z(R), r_{ts^{-1}}w_{ts^{-1},s} = r_{s^{-1}t}w_{s,s^{-1}t}, \forall s,t \in G \right\}.$$

Proof. Let $\sum_{g\in G} r_g\delta_g$ be an element of $Z(R *_\alpha^w G)$. By assumption, we have that $\alpha_g(r_g) = r_g$, for all $r_g \in D_g$, and $1_{g^{-1}} = 1_g$, for all $g \in G$. Thus,

$$\alpha_s(r_{s^{-1}t}1_{s^{-1}})w_{s,s^{-1}t} = r_{s^{-1}t}1_s w_{s,s^{-1}t}, \text{ for all } s,t \in G, \tag{3.1}$$

$$\alpha_s(a1_{s^{-1}}) = a1_s, \forall a \in R \text{ and } \text{ for all } s \in G. \tag{3.2}$$

Since $w_{s,s^{-1}t} \in D_sD_{ss^{-1}t} = D_sD_t \subseteq D_s$, we have $1_sw_{s,s^{-1}t} = w_{s,s^{-1}t}$. Using Lemma 3.1.14 and equality (3.1), we obtain that

$$r_{ts^{-1}}w_{ts^{-1},s} = r_{s^{-1}t}w_{s,s^{-1}t}, \text{ for all } s,t \in G.$$

By the fact that $r_sa \in D_s, \forall a \in R$, we have $r_sa1_s = r_sa$. Hence, by Lemma 3.1.14 and equality (3.2), we obtain that $r_sa = ar_s$, for all $a \in R$, i.e. $r_s \in Z(R)$. □

Corollary 3.1.16. *Let α be a twisted partial action of a group G on R and suppose that G is abelian and w is symmetric. Then*

$$Z(R *_\alpha^w G) = \left\{ \sum_{g\in G} r_g\delta_g \;:\; \alpha_s(r_{s^{-1}t}1_{s^{-1}}) = r_{s^{-1}t}1_t, r_s\alpha_s(a1_{s^{-1}}) = ar_s, \forall a \in R \text{ and } \forall s,t \in G \right\}.$$

Proof. By assumption we have that $w_{ts^{-1},s} = w_{s^{-1}t,s} = w_{s,s^{-1}t}$, for all $s,t \in G$. Since $w_{s,s^{-1}t} \in D_sD_{ss^{-1}t} = D_sD_t \subseteq D_t$, we have $w_{s,s^{-1}t} = 1_tw_{s,s^{-1}t}$. Thus, by Lemma 3.1.14, it follows that $\alpha_s(r_{s^{-1}t}1_{s^{-1}})w_{s,s^{-1}t} = r_{s^{-1}t}1_tw_{s,s^{-1}t}$, for all $s,t \in G$. Hence, $\alpha_s(r_{s^{-1}t}1_{s^{-1}}) = r_{s^{-1}t}1_t$, for all $s,t \in G$. □

Corollary 3.1.17. *Let α be a twisted partial action of a group G on R and suppose that G is abelian and w is symmetric. If one of the following conditions is satisfied*

(i) *R is commutative;*

(ii) $\alpha_g = id_{D_g}$, for all $g \in G$,

then

$$Z(R *_\alpha^w G) = \left\{ \sum_{g \in G} r_g \delta_g \quad : \quad \alpha_s(r_{s^{-1}t} 1_{s^{-1}}) = r_{s^{-1}t} 1_t, \big(\alpha_s(a 1_{s^{-1}}) - a\big) \in \text{ann}(r_s), \right.$$
$$\left. \forall a \in R \text{ and } \forall s, t \in G \right\}.$$

Proof. Note that, since G is abelian and w is symmetric, by Corollary 3.1.16, we obtain the first equality.

Suppose that (i) holds. By Lemma 3.1.14 and assumption we have that $r_s \alpha_s(a 1_{s^{-1}}) = a r_s = r_s a$. Thus we obtain that $\big(\alpha_s(a 1_{s^{-1}}) - a\big) \in \text{ann}(r_s)$, for all $a \in R$ and $s \in G$.

Suppose that (ii) holds. Since $\alpha_g = id_{D_g}$, for all $g \in G$, by Corollary 3.1.15 we have that $r_s \in Z(R)$, $\forall s \in G$. Thus $a r_s = r_s a$, for all $a \in R$, and by similar argument as before, we obtain that $\big(\alpha_s(a 1_{s^{-1}}) - a\big) \in \text{ann}(r_s)$, for all $a \in R$ and $s \in G$. $\square$

We need the following result to show when $R *_\alpha^w G$ is commutative.

Lemma 3.1.18. *Let α be a twisted partial action of a group G on R. If $R *_\alpha^w G$ is commutative, then $w_{g,g^{-1}} = w_{g^{-1},g}$, for all $g \in G$.*

Proof. By the fact that $w_{g,g^{-1}} \in D_g D_{gg^{-1}} = D_g D_e = D_g$ and $w_{g^{-1},g} \in D_{g^{-1}} D_{g^{-1}g} = D_{g^{-1}} D_e = D_{g^{-1}}$, for all $g \in G$, we have that $1_g w_{g,g^{-1}} = w_{g,g^{-1}}$ and $1_{g^{-1}} w_{g^{-1},g} = w_{g^{-1},g}$. Since $(1_g \delta_g)(1_{g^{-1}} \delta_{g^{-1}}) = (1_{g^{-1}} \delta_{g^{-1}})(1_g \delta_g)$, for all $g \in G$, it follows that

$$\begin{aligned}
w_{g,g^{-1}} \delta_e &= 1_g w_{g,g^{-1}} \delta_e = \alpha_g(1_{g^{-1}}) w_{g,g^{-1}} \delta_{gg^{-1}} \\
&= (1_g \delta_g)(1_{g^{-1}} \delta_{g^{-1}}) = (1_{g^{-1}} \delta_{g^{-1}})(1_g \delta_g) \\
&= \alpha_{g^{-1}}(1_g) w_{g^{-1},g} \delta_{g^{-1}g} = 1_{g^{-1}} w_{g^{-1},g} \delta_e \\
&= w_{g^{-1},g} \delta_e.
\end{aligned}$$

So, $w_{g,g^{-1}} = w_{g^{-1},g}$, for all $g \in G$. $\square$

The next result provides necessary and sufficient conditions for the commutativity of the partial crossed product which generalizes ([31], Corollary 4) and generalizes partially ([11], Proposition 2).

Theorem 3.1.19. *Let α be a twisted partial action of G on R. The partial crossed product $R *_{\alpha}^{w} G$ is commutative if and only if R is commutative, G is abelian, w is symmetric and $\alpha_g = id_{D_g}$, for all $g \in G$.*

Proof. Suppose that $R *_{\alpha}^{w} G$ is commutative. Then, in particular, R is commutative.

We show that $D_g = D_{g^{-1}}$, for all $g \in G$. In fact, since $w_{g,g^{-1}}$ is invertible in $D_g D_{gg^{-1}} = D_g D_e = D_g$, there exists $w_{g,g^{-1}}^{-1} \in D_g$ such that $w_{g,g^{-1}} \cdot w_{g,g^{-1}}^{-1} = 1_g$. By Lemma 3.1.18, $w_{g,g^{-1}} = w_{g^{-1},g}$ and since $w_{g^{-1},g} \in D_{g^{-1}}$ it follows that $1_g \in D_{g^{-1}}$. Thus, $D_g \subseteq D_{g^{-1}}$ and analogously we obtain the other inclusion. Consequently, $1_g = 1_{g^{-1}}$, for all $g \in G$. By the fact that $1_g\delta_g \in Z(R *_{\alpha}^{w} G)$ and using Lemma 3.1.14, we have that $1_g\alpha_g(a1_{g^{-1}}) = a1_g$, for all $a \in R$. So, for any $a \in D_g = D_{g^{-1}}$, we have $\alpha_g(a) = a$. Hence, $\alpha_g = id_{D_g}$, for all $g \in G$.

Since $(1_g\delta_g)(1_h\delta_h) = (1_h\delta_h)(1_g\delta_g)$, for all $g, h \in G$, it follows that

$$\begin{aligned} w_{g,h}\delta_{gh} &= 1_h 1_{g^{-1}} w_{g,h}\delta_{gh} = \alpha_g(1_h 1_{g^{-1}}) w_{g,h}\delta_{gh} \\ &= (1_g\delta_g)(1_h\delta_h) = (1_h\delta_h)(1_g\delta_g) \\ &= \alpha_h(1_g 1_{h^{-1}}) w_{h,g}\delta_{hg} = 1_g 1_{h^{-1}} w_{h,g}\delta_{hg} = w_{h,g}\delta_{hg}. \end{aligned}$$

From equality above, we obtain that $gh = hg$, for all $g, h \in G$, and also that $w_{g,h} = w_{h,g}$, for all $g, h \in G$. Therefore, G is abelian and w is symmetric.

Conversely, suppose that R is commutative, $\alpha_g = id_{D_g}$, for all $g \in G$, G is abelian and w is symmetric. Let $\sum_{g\in G} a_g\delta_g$ and $\sum_{h\in G} b_h\delta_h$ be elements of $R *_{\alpha}^{w} G$. Then, we have

$$\begin{aligned}
\Big(\sum_{g\in G} a_g\delta_g\Big)\Big(\sum_{h\in G} b_h\delta_h\Big) &= \sum_{g,h\in G} a_g\alpha_g(b_h1_{g^{-1}})w_{g,h}\delta_{gh} = \sum_{g,h\in G} a_gb_h1_{g^{-1}}w_{g,h}\delta_{gh} \\
&= \sum_{g,h\in G} a_gb_h1_gw_{g,h}\delta_{gh} = \sum_{g,h\in G} a_gb_hw_{g,h}\delta_{gh} \\
&= \sum_{g,h\in G} a_gb_hw_{h,g}\delta_{gh} = \sum_{g,h\in G} a_gb_hw_{h,g}\delta_{hg} \\
&= \sum_{g,h\in G} b_ha_g1_hw_{h,g}\delta_{hg} = \sum_{g,h\in G} b_ha_g1_{h^{-1}}w_{h,g}\delta_{hg} \\
&= \sum_{g,h\in G} b_h\alpha_h(a_g1_{h^{-1}})w_{h,g}\delta_{hg} = \Big(\sum_{h\in G} b_h\delta_h\Big)\Big(\sum_{g\in G} a_g\delta_g\Big)
\end{aligned}$$

and it follows that $R *_{\alpha}^{w} G$ is commutative. $\square$

Lemma 3.1.20. *Let α be a twisted partial action of a group G on R. Then for every nonzero ideal I of $R *_{\alpha}^{w} G$ we have that $I \cap C_{R*_{\alpha}^{w}G}\big(Z(R)\big) \neq 0$.*

Proof. It is enough to show that if $I \cap C_{R*_{\alpha}^{w}G}\big(Z(R)\big) = 0$, then $I = 0$. Let $x = \sum_{h\in G} a_h\delta_h \in I$. If $x \in C_{R*_{\alpha}^{w}G}\big(Z(R)\big)$, then $x = 0$ by assumption. Thus, we assume that there exists $z \in I \backslash C_{R*_{\alpha}^{w}G}(Z(R))$ and we choose $x \in I\backslash C_{R*_{\alpha}^{w}G}\big(Z(R)\big)$ among the elements of $I\backslash C_{R*_{\alpha}^{w}G}\big(Z(R)\big)$ such that $|\mathrm{supp}(x)|$ is minimal. Note that, for any $p \in \mathrm{supp}(x)$, $x' = x1_{p^{-1}}\delta_{p^{-1}} \in I\backslash C_{R*_{\alpha}^{w}G}\big(Z(R)\big)$. In fact, note that $(a_p\delta_p)(1_{p^{-1}}\delta_{p^{-1}}) = a_p\alpha_p(1_{p^{-1}})w_{p,p^{-1}}\delta_{p,p^{-1}} = a_pw_{p,p^{-1}}\delta_e$. If $x' \in C_{R*_{\alpha}^{w}G}\big(Z(R)\big)$, by the fact that $I \cap C_{R*_{\alpha}^{w}G}\big(Z(R)\big) = 0$, we would have $x' = 0$. Thus, $a_pw_{p,p^{-1}} = 0$ and we have that $a_p = 0$, which contradicts the fact that $p \in \mathrm{supp}(x)$. Since $(a_p\delta_p)(1_{p^{-1}}\delta_{p^{-1}}) = a_pw_{p,p^{-1}}\delta_e$, with $a_pw_{p,p^{-1}} \neq 0$, we have that $e \in \mathrm{supp}(x')$. Moreover, since $x' \neq 0$, it follows that $|\mathrm{supp}(x')| = |\mathrm{supp}(x)|$. Hence, we may assume that $e \in \mathrm{supp}(x)$.

For each $r \in Z(R)$, let $x'' = rx - xr$. Since $r \in Z(R)$ and $r1_e - \alpha_e(r1_{e^{-1}}) = 0$, we have that

$$
\begin{aligned}
x'' &= (r\delta_e)\Big(\sum_{h\in G} a_h\delta_h\Big) - \Big(\sum_{h\in G} a_h\delta_h\Big)(r\delta_e) \\
&= \Big(\sum_{h\in G} r\alpha_e(a_h 1_{e^{-1}})w_{e,h}\delta_{eh}\Big) - \Big(\sum_{h\in G} a_h\alpha_h(r1_{h^{-1}})w_{h,e}\delta_{he}\Big) \\
&= \Big(\sum_{h\in G} ra_h 1_h\delta_h\Big) - \Big(\sum_{h\in G} a_h\alpha_h(r1_{h^{-1}})\delta_h\Big) \\
&= \Big(\sum_{h\in G} a_h r1_h\delta_h\Big) - \Big(\sum_{h\in G} a_h\alpha_h(r1_{h^{-1}})\delta_h\Big) \\
&= \sum_{h\in G} a_h\big(r1_h - \alpha_h(r1_{h^{-1}})\big)\delta_h \\
&= \sum_{h\in G\setminus\{e\}} a_h\big(r1_h - \alpha_h(r1_{h^{-1}})\big)\delta_h.
\end{aligned}
$$

Consequently, $e \notin \mathrm{supp}(x'')$ and it follows that $|\mathrm{supp}(x'')| < |\mathrm{supp}(x)|$. Since $x'' \in I$, by the minimality of $|\mathrm{supp}(x)|$, we obtain that $rx = xr$, for all $r \in Z(R)$. So, $x \in C_{R*_\alpha^w G}(Z(R))$ which contradicts the choose of x. Therefore, $I = 0$.

The proof is complete. □

We are in conditions to prove the second principal result of this section.

Theorem 3.1.21. *Suppose that $C_{R*_\alpha^w G}\big(Z(R)\big)$ is a simple ring. Then $R *_\alpha^w G$ is simple if and only if R is α-simple.*

Proof. If $R *_\alpha^w G$ is simple, by Lemma 3.1.12, R is α-simple.

Conversely, suppose that R is α-simple and let I be a nonzero ideal of $R *_\alpha^w G$. Note that, if $I \cap R = 0$, by Lemma 3.1.20, we have $I \cap C_{R*_\alpha^w G}\big(Z(R)\big) \neq 0$ and, since $C_{R*_\alpha^w G}\big(Z(R)\big)$ is simple, we obtain that $1_R \in I$, a contradiction because we are considering $I \cap R = 0$. Hence $I \cap R \neq 0$ and the result follows. □

In the next lemma, we denote δ_g by g and we consider $e = g_1$.

Lemma 3.1.22. *If R is α-simple and G is abelian, then $I \cap C_{R*_\alpha^w G}(R) \neq 0$, for all nonzero ideal I of $R *_\alpha^w G$.*

Proof. Let $0 \neq x = \sum_{i=1}^{n} a_i g_i \in I$ such that $|\text{supp}(x)|$ is minimal. By similar arguments of Lemma 3.1.20, we may assume that $e = g_1 \in \text{supp}(x)$. Since R is α-simple and $a_1 \neq 0$, the set $\left\{\alpha_g\big(a_1 1_{g^{-1}}\big) : g \in G\right\}$ generates R as an ideal. Hence $1 = \sum\limits_j \sum\limits_k r_{kj}\alpha_{g_j}\big(a_1 1_{g_j^{-1}}\big)s_{kj}$, for some $r_{kj}, s_{kj} \in R$ and $g_j \in G$. Let

$$y = \sum_j \sum_k r_{kj} 1_{g_j} g_j x 1_{g_j^{-1}} g_j^{-1} w^{-1}_{g_j, g_j^{-1}} s_{kj}.$$

Then we have that

$$\begin{aligned}
y &= \sum_j \sum_k r_{kj}\alpha_{g_j}\big(a_1 1_{g_j^{-1}}\big)s_{kj} + \\
&\quad + \sum_j \sum_k r_{kj}\left(\sum_{i=2}^{n} \alpha_{g_j}\big(a_i 1_{g_j^{-1}}\big) w_{g_j, g_i} w_{g_j g_i, g_j^{-1}} \alpha_{g_i}\left(w^{-1}_{g_j, g_j^{-1}} 1_{g_i^{-1}}\right) g_i\right) s_{kj} \\
&= 1 + \sum_j \sum_k r_{kj}\left(\sum_{i=2}^{n} \alpha_{g_j}\big(a_i 1_{g_j^{-1}}\big) w_{g_j, g_i} w_{g_j g_i, g_j^{-1}} \alpha_{g_i}\left(w^{-1}_{g_j, g_j^{-1}} 1_{g_i^{-1}}\right) g_i\right) s_{kj} \\
&= 1 + \sum_{i=2}^{n}\left(\sum_j \sum_k r_{kj}\alpha_{g_j}\big(a_i 1_{g_j^{-1}}\big) w_{g_j, g_i} w_{g_j g_i, g_{j^{-1}}} \alpha_{g_i}\left(w^{-1}_{g_j, g_j^{-1}} s_{kj} 1_{g_i^{-1}}\right)\right) g_i.
\end{aligned}$$

Thus, $y \in I$ is such that $|\text{supp}(y)|$ is minimal and we may assume that

$$x = 1 + \sum_{i=2}^{n} a_i g_i.$$

For each $r \in R$, the element $x' = rx - xr$ satisfies $|\text{supp}(x')| < |\text{supp}(x)|$. By the fact that $|\text{supp}(x)|$ is minimal and $x' \in I$, we have that $rx = xr$, for all $r \in R$, and so $x \in C_{R*_\alpha^w G}(R)$. Hence, $I \cap C_{R*_\alpha^w G}(R) \neq 0$. $\square$

Theorem 3.1.23. *Suppose that G is abelian and $C_{R*_\alpha^w G}(R)$ is simple. Then $R *_\alpha^w G$ is simple if and only if R is α-simple.*

Proof. If $R *_\alpha^w G$ is simple then, by Lemma 3.1.12, R is α-simple.

Conversely, suppose that R is α-simple and let I be a nonzero ideal of $R *_\alpha^w G$. Note that by Lemma 3.1.20, we have that $I \cap C_{R*_\alpha^w G}\big(Z(R)\big) \neq 0$. Since $C_{R*_\alpha^w G}\big(Z(R)\big)$ is simple, then we have that $1_R \in I$. So, $I = R *_\alpha^w G$. $\square$

Lemma 3.1.24. *If R is α-simple and G is abelian, then $I \cap Z(R *_{\alpha}^{w} G) \neq 0$, for all nonzero ideal I of $R *_{\alpha}^{w} G$.*

Proof. Let I be a nonzero ideal of $R *_{\alpha}^{w} G$ and x a nonzero element of I such that $|\mathrm{supp}(x)|$ is minimal. By proof of Lemma 3.1.22, we have $x = 1\delta_e + \sum_{g \neq e} a_g \delta_g$ and $xr = rx$, for all $r \in R$. By the fact that $(1_g\delta_g)(1\delta_e) = (1\delta_e)(1_g\delta_g)$, it follows that $\big|\mathrm{supp}\big((1_g\delta_g)x - x(1_g\delta_g)\big)\big| < |\mathrm{supp}(x)|$ and, since $(1_g\delta_g)x - x(1_g\delta_g) \in I$, we obtain that $(1_g\delta_g)x = x(1_g\delta_g)$, for all $g \in G$. Hence, $x \in Z(R *_{\alpha}^{w} G)$ and consequently $x \in I \cap Z(R *_{\alpha}^{w} G)$. □

Now, we are ready to prove the last result of this section which generalizes partially ([32], Theorem 1.2).

Theorem 3.1.25. *Suppose that G is an abelian group. Then $R *_{\alpha}^{w} G$ is simple if and only if $Z(R *_{\alpha}^{w} G)$ is a field and R is α-simple.*

Proof. Suppose that $R *_{\alpha}^{w} G$ is simple. Thus, for each $0 \neq x \in Z(R *_{\alpha}^{w} G)$, we have that $(R *_{\alpha}^{w} G)x = x(R *_{\alpha}^{w} G) = R *_{\alpha}^{w} G$ and it follows that there exists $x^{-1} \in R *_{\alpha}^{w} G$ such that $xx^{-1} = x^{-1}x = 1$. Note that for any $a \in R *_{\alpha}^{w} G$ we obtain that $x(x^{-1}a) = (xx^{-1})a = a(xx^{-1}) = (ax)x^{-1} = (xa)x^{-1} = x(ax^{-1})$, i.e. $x(x^{-1}a) = x(ax^{-1})$. Hence, $x^{-1}a = ax^{-1}$, for any $a \in R *_{\alpha}^{w} G$, and we have that $x^{-1} \in Z(R *_{\alpha}^{w} G)$. Thus x is invertible in $Z(R *_{\alpha}^{w} G)$ and so $Z(R *_{\alpha}^{w} G)$ is a field. Moreover, by Lemma 3.1.12, R is α-simple.

Conversely, suppose that $Z(R *_{\alpha}^{w} G)$ is a field and R is α-simple. Let I be a nonzero ideal of $R *_{\alpha}^{w} G$. By Lemma 3.1.24 we have that $I \cap Z(R *_{\alpha}^{w} G) \neq 0$. So, $I = R *_{\alpha}^{w} G$ and it follows that $R *_{\alpha}^{w} G$ is simple. □

Now, we finish with the following example where we apply the results of this section to conclude that the partial crossed product is not simple. Moreover, it shows that the assumptions on Theorem 3.1.25 are not superfluous.

Example 3.1.26. Let $T = Ke_1 \oplus Ke_2 \oplus Ke_3$, where K is a field and $\{e_1, e_2, e_3\}$ are central orthogonal idempotents, and $R = Ke_1$. We define the action of $\mathbb{Z}$

on T as follows: $\sigma(e_1) = e_2$, $\sigma(e_2) = e_3$ and $\sigma(e_3) = e_1$. We have the following induced partial action of $\mathbb{Z}$ on R: $D_j = R$, for $j \equiv 0(mod\, 3)$, and $D_k = 0$, for $k \equiv 1(mod\, 3)$ and $k \equiv 2(mod\, 3)$, with isomorphisms $\alpha_j = id_R$, for $j \equiv 0(mod\, 3)$, and $\alpha_k = 0$, for $k \equiv 1(mod\, 3)$ and $k \equiv 2(mod\, 3)$. By Theorem 3.1.19, $R *_\alpha G$ is commutative. Since $R *_\alpha G = Z(R *_\alpha G)$ is not field and, by Theorem 3.1.25, $R *_\alpha G$ is not simple.

3.2 Applications

In this section, we study some topological properties and applications of the last section to the C^*-algebra of type $C(X)$, where X is a topological space.

3.2.1 Some properties of partial dynamical systems

Given a partial dynamical system (X, α, G) the *partial orbit* of a point $x \in X$ is the set $O^\alpha(x) = \big\{\alpha_t(x) : x \in X_{t^{-1}}, t \in G\big\}$.

A partial dynamical system is said to be *transitive* if there exists some $x_0 \in X$ such that $O^\alpha(x_0)$ is dense in X, i.e. $\overline{O^\alpha(x_0)} = X$. If for every $x \in X$, $O^\alpha(x)$ is dense in X, we say that the partial dynamical system is *minimal*.

We remind that a topological space X is *compact* if, for every collection $\{U_i\}_{i \in I}$ of open sets in X whose union is X, there exists a finite sub-collection $\{U_{i_j}\}_{j=1}^n$ whose union is also X.

In the next result we assume that X is a compact metric space and we can state a condition that implies transitivity.

Theorem 3.2.1. *Let (X, α, G) be a partial dynamical system such that X is a compact metric space. Then the following conditions are equivalent:*

(i) *(X, α, G) is transitive.*

(ii) *Given any two non-empty open sets U and V in X, there exists some $g \in G$ such that $\alpha_g\big(U \cap X_{g^{-1}}\big) \cap V \neq \emptyset$.*

Proof. Since $(i) \Rightarrow (ii)$ is clear from the definition, then we concentrate on the proof of the other implication. We want to show that for any real number $\omega > 0$, there exists an orbit that is $\omega-$dense, i.e. such that any point of X is at a distance smaller than ω from the orbit. For this purpose, take a covering of X by open balls of radius ω. Thus, by compacity, we extract a finite subcovering $B_1, \ldots, B_N$ such that $X \subset B_1 \cup \cdots \cup B_N$. Hence, by assumption, there exists some $t_1 \in G$ such that $\alpha_{t_1}\big(B_1 \cap X_{t_1^{-1}}\big) \cap B_2 \neq \emptyset$ and this implies that $\alpha_{t_1^{-1}}\big(B_2 \cap X_{t_1}\big) \cap B_1 =: B_{12} \neq \emptyset$, which is an open set. Note that by assumption, this set has some image intersecting the open set B_3, because of this we have some $t_2 \in G$ such that $\alpha_{t_2}\big(B_{12} \cap X_{t_2^{-1}}\big) \cap B_3 \neq \emptyset$. Now we have the open set $\alpha_{t_2^{-1}}\big(B_3 \cap X_{t_2}\big) \cap B_{12} =: B_{123} \neq \emptyset$. Repeating this procedure N times we get an open set $B_{12\cdots N} \neq \emptyset$ such that any point in it has images in B_1, B_2, ..., B_N, being $\omega-$dense, as desired.

Since we have ω-dense orbits for any $\omega > 0$ then we have in fact dense orbits for this partial dynamical system. $\square$

Now, we have the following result.

Proposition 3.2.2. *Let (X, α, G) be a partial dynamical system, with X compact, (X^e, β, G) its enveloping action and $x_0 \in X$. The following statements hold:*

(i) *If $\overline{O^\alpha(x_0)} = X$ then $\overline{\beta_g\big([e, O^\alpha(x_0)]\big)} = \beta_g\big([e, X]\big)$, for all $g \in G$.*

(ii) *If (X, α, G) is transitive then (X^e, β, G) is transitive.*

Proof. (i) We clearly have $\beta_g\big([e, O^\alpha(x_0)]\big) \subseteq \beta_g\big([e, X]\big)$. We claim that $\beta_g\big([e, X]\big)$ is closed. In fact, we have

$$\beta_g\big([e, X]\big) = [g, X] = \bigcup_{x \in X} [g, x].$$

If $[g, x_n] \to [g, x]$ then we have $x_n \to x$ and since X is closed, we have that $x \in X$. Hence, $[g, x] \in [g, X] = \beta_g\big([e, X]\big)$ and so $\overline{\beta_g\big([e, O^\alpha(x_0)]\big)} = \beta_g\big([e, X]\big)$.

(ii) If (X, α, G) is transitive then, given $x \in X$, there exists $x_0 \in X$ and a sequence $\{g_n\}_{n\in\mathbb{N}} \subset G$ such that $\alpha_{g_n}(x_0) \to x$. Now, take a point $[g, x] \in X^e$. Then we have

$$\beta_{gg_n}\big([e, x_0]\big) = \beta_g\big([g_n, x_0]\big) = \beta_g\big([e, \alpha_{g_n}(x_0)]\big) = \big[g, \alpha_{g_n}(x_0)\big] \to [g, x].$$

So, (X^e, β, G) is also transitive. $\square$

Remark 3.2.3. As a particular case of the proposition above we have that if $\overline{O^\alpha(x)} = X$ then $\overline{O^\beta(x)} := \overline{\big\{\beta_g(x) : g \in G\big\}} = X^e$.

The following definitions appear in [21]

Definition 3.2.4. *Let (X, α, G) be a partial dynamical system.*

(i) We say that a set $Y \subseteq X$ is α-*invariant* if $\alpha_g\big(Y \cap X_{g^{-1}}\big) = Y \cap X_g$, for all $g \in G$.

(ii) An ideal I of $C(X)$ is said to be α-*invariant* if $\alpha_g\big(I \cap C(X_{g^{-1}})\big) = I \cap C(X_g)$, for all $g \in G$. Moreover, $C(X)$ is α-simple if the unique α-invariant ideals are 0 and $C(X)$.

Proposition 3.2.5. *Let (X, α, G) be a minimal partial dynamical system. Then, the unique α-invariant open subsets of X are $\emptyset$ and X.*

Proof. Suppose that X contains a proper α-invariant open subset U. Then there exists some $x_0 \in X \backslash U$. Since $\overline{O^\alpha(x_0)} = X$ and U is open, U contains some point of $O^\alpha(x_0)$. Once that U is a α-invariant set, it must contain all the orbit, a contradiction because we are considering $x_0 \in X \backslash U$. $\square$

Remark 3.2.6. It is easy to see that for any α-invariant ideal I of $C(X)$ there exists an open α-invariant subset $Y \subseteq X$ such that $I = C(Y)$.

Lemma 3.2.7. *Let (X, α, G) be a partial dynamical system. The following conditions are equivalent:*

(i) $C(X)$ *is* α*-simple.*

(ii) X *does not have proper* α*-invariant closed subsets.*

(iii) (X, α, G) *is minimal.*

Proof. $(i) \Rightarrow (ii)$

Suppose that X contains a proper α-invariant closed subset S. We easily see that $X \backslash S$ is a proper α-invariant open subset of X. Hence, $C(X \backslash S)$ is a proper α-invariant ideal of $C(X)$, which is a contradiction.

$(ii) \Rightarrow (iii)$

Let x be arbitrary element of X. Then we clearly have that $O^{\alpha}(x)$ is an α-invariant subset of X. It is not difficult to show that $\overline{O^{\alpha}(x)}$ is a closed α-invariant subset of X. By assumption, we have that $\overline{O^{\alpha}(x)} = X$. So (X, α, G) is minimal.

$(iii) \Rightarrow (i)$

Let I be an α-invariant ideal of $C(X)$. By Remark 3.2.6, $I = C(U)$ for some α-invariant open subset $U \subseteq X$. By Proposition 3.2.5, we have that either $U = \emptyset$ or $U = X$ and so $I = 0$ or $I = C(X)$. $\square$

3.2.2 Simplicity of $C(X) *_{\alpha} G$

Throughout this subsection (X, α, G) is a partial dynamical system, $C(X)$ is the algebra of continuous functions defined on topological space X with values in the complex numbers, α will denote the extended partial action of G on X to $C(X)$, and $C(X) *_{\alpha} G$ will be the partial skew group ring. Moreover, we denote the centralizer of $C(X)$ in $C(X) *_{\alpha} G$ by $\mathcal{A}$ and we will call it the *commutant* of $C(X)$ in $C(X) *_{\alpha} G$, i.e. $\mathcal{A} = \{a \in C(X)_{\alpha} * G : af = fa, \text{ for all } f \in C(X)\}$.

Definition 3.2.8. For any $g \in G\backslash\{e\}$, we set:

(i) $Per^g_{C(X)}(X) = \big\{x \in X_g : f(x) = f(\alpha_{g^{-1}}(x)),\ \text{for all } f \in C(X)\big\}$;

(ii) $Sep^g_{C(X)}(X) = \big\{x \in X_g : f(x) \neq f(\alpha_{g^{-1}}(x)),\ \text{for some } f \in C(X)\big\}$.

A topological space X is said to be *Hausdorff* if for any distinct points $a, b \in X$, there exist open subsets A and B contained in X such that $A \cap B = \emptyset$, with $a \in A$ and $b \in B$.

Let us define, for each $f \in C(X)$, the set $Supp(f) = \big\{x \in X : f(x) \neq 0\big\}$.

Definition 3.2.9. A partial dynamical system (X, α, G) is said to be *topologically free* if for each $g \in G\backslash\{e\}$, the set $\theta_g = \big\{a \in X_{g^{-1}} : \alpha_g(a) = a\big\}$ has empty interior.

The following remark has independent interest.

Remark 3.2.10. Let $Gr(\alpha) = \big\{(t, x, y) \in G \times X \times X : x \in X_{t^{-1}}, \alpha_t(x) = y\big\}$ and suppose that $Gr(\alpha)$ is closed. Then by ([1], Proposition 1.2) the enveloping space X^e is a Hausdorff space and we easily have that X is Hausdorff. We claim that for each $g \in G\backslash\{e\}$, θ_g is a closed set. In fact, for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset \theta_g$ such that $x_n \to x$ we have, by the continuity of $i : X \to X^e$ and β_g, that $\beta_g\big(i(x_n)\big) \to \beta_g\big(i(x)\big)$. By the fact that $\beta_g\big(i(x_n)\big) = i\big(\alpha_g(x_n)\big) = i(x_n)$, we obtain that $i(x_n) \to \beta_g\big(i(x)\big)$ and $i(x_n) \to i(x)$. Thus, $\beta_g\big(i(x)\big) = i(x)$. Since $i\big(\alpha_g(x)\big) = [e, x] = [g, x] = \beta_g\big(i(x)\big) = i(x)$, it follows that $\alpha_g(x) = x$.

We have the following result, where item (*ii*) generalizes ([40], Theorem 3.3).

Theorem 3.2.11. *Let (X, α, G) be a partial dynamical system. The following statements hold:*

(i) *The commutant of $C(X)$ in $C(X) *_\alpha G$ is*

$$\mathcal{A} = \Bigg\{\sum_{g\in G} a_g\delta_g \in C(X) *_\alpha G : aa_g = \alpha_g(\alpha_{g^{-1}}(a_g)a), \forall a \in C(X)\Bigg\}.$$

Moreover, $C(X) \subseteq \mathcal{A}$.

(ii) *The subalgebra* $\mathcal{A}$ *of* $C(X) *_\alpha G$ *is*

$$\mathcal{A} = \Big\{ \sum_{g\in G} a_g\delta_g \in C(X) *_\alpha G : a_g\big|_{Sep^g_{C(X)}(X)} \equiv 0 \Big\}.$$

(iii) *If* X *is a Hausdorff space, then*

$$\mathcal{A} = \Big\{ \sum_{g\in G} a_g\delta_g \in C(X) *_\alpha G : Supp(a_g) \subseteq \theta_g \Big\}.$$

(iv) *Suppose that* G *is abelian. Then* $\mathcal{A}$ *is the maximal commutative subalgebra of* $C(X) *_\alpha G$ *that contains* $C(X)$.

Proof. (i) The proof follows from Lemma 3.1.1.

(ii) Let $\sum_{g\in G} a_g\delta_g \in \mathcal{A}$. Then by item (i),

$$\alpha_g\big(\alpha_{g^{-1}}(a_g)f\big) = fa_g,$$

for any $f \in C(X)$. Note that for each $b \in Sep^g_{C(X)}(X)$ there exists $h \in C(X)$ such that $h(b) \neq h\big(\alpha_{g^{-1}}(b)\big)$. Thus,

$$\alpha_g\big(\alpha_{g^{-1}}(a_g)h\big)(b) = (ha_g)(b) \Leftrightarrow \big(\alpha_{g^{-1}}(a_g)h\big)\big(\alpha_{g^{-1}}(b)\big) = h(b)a_g(b) \Leftrightarrow$$
$$\alpha_{g^{-1}}(a_g)\big(\alpha_{g^{-1}}(b)\big)h\big(\alpha_{g^{-1}}(b)\big) = h(b)a_g(b) \Leftrightarrow a_g(b)h\big(\alpha_{g^{-1}}(b)\big) = a_g(b)h(b) \Leftrightarrow$$
$$a_g(b)\big(h(\alpha_{g^{-1}}(b)) - h(b)\big) = 0.$$

Hence, $a_g(b) = 0$. So, $a_g\big|_{Sep^g_{C(X)}(X)} \equiv 0$.

(iii) Let $\sum_{g\in G} a_g\delta_g$ be an element of $\mathcal{A}$. Then by item (i), for any $f \in C(X)$ and $x \in X_g$, we have that

$$\alpha_g\big(\alpha_{g^{-1}}(a_g)f\big)(x) = (fa_g)(x) \Leftrightarrow a_g(x)\big(f(\alpha_{g^{-1}}(x)) - f(x)\big) = 0.$$

Thus, for each $x \in X_g$ such that $a_g(x) \neq 0$, we have that $f\big(\alpha_{g^{-1}}(x)\big) = f(x)$. Since X is Hausdorff, we obtain $\alpha_{g^{-1}}(x) = x$, which implies that $x \in \theta_{g^{-1}}$. By the fact that $\theta_{g^{-1}} = \theta_g$, we have that $x \in \theta_g$ and so $Supp(a_g) \subseteq \theta_g$.

(iv) This proof is similar to the proof of ([40], Proposition 2.1) and we give it here for reader's convenience. We start by observing that every commutative subalgebra of $C(X) *_\alpha G$ is contained in $\mathcal{A}$. So it remains to show that $\mathcal{A}$ is commutative, which is a consequence of Corollary 3.1.8. $\square$

The next definition appears in ([40]).

Definition 3.2.12. Let B be a topological space and $\emptyset \neq A \subseteq B$. Then A is said to be a *domain of uniqueness* for B if for any continuous function $f : B \to \mathbb{C}$ we have that $f|_A = 0 \Rightarrow f = 0$.

Now we are in position to state the next result, where item (ii) generalizes ([31], Lemma 8.2).

Theorem 3.2.13. *Let (X, α, G) be a partial dynamical system. The following statements hold:*

- (i) *$C(X) = \mathcal{A}$ if and only if for any $g \in G\backslash\{e\}$, $Sep^g_{C(X)}(X)$ is a domain of uniqueness for X_g.*
- (ii) *Suppose that X is Hausdorff. Then $C(X) = \mathcal{A}$ if and only if (X, α, G) is topologically free.*

Proof. (i) Suppose that $C(X) = \mathcal{A}$. Let $a_g \in X_g$, $g \neq e$, such that $a_g\big|_{Sep^g_{C(X)}(X)} = 0$. Then, by Theorem 3.2.11(ii), $a_g\delta_g \in \mathcal{A}$. Hence, by assumption, $a_g = 0$ and so $Sep^g_{C(X)}(X)$ is a domain of uniqueness for X_g, with $g \neq e$.

Conversely, suppose that for each $g \in G\backslash\{e\}$, $Sep^g_{C(X)}(X)$ is a domain of uniqueness for X_g. Let $\sum_{h\in G} a_h\delta_h \in \mathcal{A}$. Then, by Theorem 3.2.11(ii), we have that $a_h\big|_{Sep^h_{C(X)}(X)} = 0$, for each $h \in G\backslash\{e\}$. Hence, $a_h = 0$, for all $h \in G\backslash\{e\}$. So, $\mathcal{A} \subseteq C(X)$ and we have that $C(X) = \mathcal{A}$.

(ii) Suppose that (X, α, G) is not topologically free. So, for some $g \in G\backslash\{e\}$, there exists a non-empty open subset V contained in θ_g, and it follows that

$C(V)$ is a nonzero subalgebra of $C(X)$. Hence, there exists $0 \neq f \in C(V)$. Note that $Supp(f) \subseteq \theta_g$. Thus, by Theorem 3.2.11(iii), $f\delta_g \in A$, which contradicts the assumption.

Conversely, suppose that $C(X) \neq \mathcal{A}$. Then, there exists $\sum_{j=1}^{n} a_{g_j}\delta_{g_j} \in \mathcal{A}$ with $g_i \neq e$ and $a_{g_i} \neq 0$, for some $1 \leq i \leq n$. Let $x \in X_{g_i}$ such that $a_{g_i}(x) \neq 0$. We claim that $Supp(a_{g_i})$ contains an open set. In fact, suppose that for any open set $V \subseteq X_{g_i}$ we have that $V \nsubseteq Supp(a_{g_i})$. Hence, there exists $x_1 \in X_{g_i}$ such that $a_{g_i}(x_1) = 0$. Since X is Hausdorff, there exists open subsets A_1 and A_2 of X_g such that $A_1 \cap A_2 = \emptyset$ with $x \in A_1$ and $x_1 \in A_2$. By the assumption on a_{g_i} there exists $x_2 \in A_1$ such that $a_{g_i}(x_2) = 0$. Proceeding by this way we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X_g$ such that $x_n \to x$ and $a_{g_i}(x_n) = 0$. Since a_{g_i} is continuous, then $a_{g_i}(x_n) \to a_{g_i}(x)$, which implies $a_{g_i}(x) = 0$, this is a contradiction. Thus, there exists an open set $A \subseteq Supp(a_{g_i}) \subseteq \theta_{g_i}$, which contradicts the fact that (X, α, G) is topologically free. So, $C(X) = \mathcal{A}$. $\square$

Definition 3.2.14. Let (X, α, G) be a partial dynamical system. We say that a point $z \in X$ is *periodic* if there exists $g \in G\backslash\{e\}$ such that $z \in X_{g^{-1}}$ and $\alpha_g(z) = z$.

Lemma 3.2.15. *Suppose that X is infinite and the cardinality of the partial orbits of periodic points is finite. If (X, α, G) is minimal, then (X, α, G) is topologically free.*

Proof. Suppose that there exists $g \in G\backslash\{e\}$ such that $\theta_g \neq \emptyset$. Then any point $x \in \theta_g$ has a finite partial orbit. Since (X, α, G) is minimal, we have $\overline{O^{\alpha}(x)} = X$, which contradicts the fact that X is infinite. $\square$

Now we are ready to prove the main result of this section that generalizes ([31], Theorem 8.6).

Theorem 3.2.16. *Let (X, α, G) be partial dynamical system such that X is an infinite Hausdorff space and the cardinality of the partial orbits of periodic points of X is finite. The following conditions are equivalent:*

(i) (X, α, G) *is a minimal dynamical system.*

(ii) $C(X)$ *is maximal commutative in* $C(X) *_\alpha G$ *and* $C(X)$ *is* α*-simple.*

(iii) $C(X) *_\alpha G$ *is simple.*

Proof. $(i) \Rightarrow (ii)$

By the Lemma 3.2.15 we have that (X, α, G) is topologically free and by Theorem 3.2.13(ii) we have that $C(X)$ is maximal commutative. Since (X, α, G) is minimal, by Lemma 3.2.7 we get that $C(X)$ is α-simple.

$(ii) \Rightarrow (iii)$

The proof follows from Theorem 3.1.13.

$(iii) \Rightarrow (i)$

For each $x \in X$, we have that $\overline{O^\alpha(x)}$ is an α-invariant closed subset of X. Thus $X \backslash \overline{O^\alpha(x)}$ is an α-invariant open subset of X. Suppose that $\overline{O^\alpha(x)} \subsetneq X$. Then $C\big(X \backslash \overline{O^\alpha(x)}\big)$ is a proper α-invariant ideal of $C(X)$, which contradicts the Lemma 3.1.12. So, (X, α, G) is minimal. $\square$

Remark 3.2.17. It is convenient to point out if $G = \mathbb{Z}^n$, $n \geqslant 1$, in Lemma 3.2.15 we obtain that the cardinality of the partial orbits is finite. Thus, in this case, we do need to assume the assumption that cardinality of the partial orbits of periodic points of X is finite in Theorem 3.2.16.

Next, we give an example to show that the assumption of the finiteness of the cardinality of the partial periodic points in Theorem 3.2.16 is not superfluous.

Example 3.2.18. Let $X = (\mathbb{Z}_6)^{\mathbb{N}}$ be the topological space with the discrete topology and the additive topological group $H = (\mathbb{Z}_6)^{\mathbb{N}} \times (\mathbb{Z}_6)^{\mathbb{N}}$ with product topology. We define the global (hence, partial) action $\beta : H \times X \to X$ by

$$\beta_{((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}})}\big((z_i)_{i\in\mathbb{N}}\big) = (x_i)_{i\in\mathbb{N}} + (y_i)_{i\in\mathbb{N}} + (z_i)_{i\in\mathbb{N}} = (x_i + y_i + z_i)_{i\in\mathbb{N}}.$$

We clearly have that this action is well defined. Note that the element $(w_i)_{i\in\mathbb{N}}$ with $w_i = \overline{3}$, for all $i \in \mathbb{N}$, is periodic, because for $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ such that $x_i = y_i = \overline{3}$, for all $i \in \mathbb{N}$, we have that

$$\beta_{((x_i)_{i\in\mathbb{N}},(y_i)_{i\in\mathbb{N}})}\big((w_i)_{i\in\mathbb{N}}\big) = (x_i)_{i\in\mathbb{N}} + (y_i)_{i\in\mathbb{N}} + (w_i)_{i\in\mathbb{N}} = (x_i + y_i + w_i)_{i\in\mathbb{N}} = (w_i)_{i\in\mathbb{N}}.$$

It is not difficult to see that the cardinality of the orbit $(w_i)_{i\in\mathbb{N}}$ is infinite, X is Hausdorff and the unique β-invariant open sets are the trivial open sets, that is, $\emptyset$ and X. Hence, X is minimal, but X is not topologically free because the set θ_g, $g \in G$, has non-empty interior since the topology is discrete. So, by Theorem 3.2.13, $C(X) \subsetneq \mathcal{A}$. Therefore, the equivalent conditions on Theorem 3.2.16 does not hold in this case.

3.3 Examples

In this section, we present some examples which we apply some the principal results of this article. All the examples of this section are build on metric spaces, and so they are also Hausdorff spaces.

Example 3.3.1. (the horseshoe) The horseshoe is a well known model in dynamical systems theory; it appears naturally in systems presenting homoclinic points and is the paradigm of the hyperbolic dynamical systems, see, for example, [39]. The dynamics is a diffeomorphism F defined on the sphere S^2. Typically one is interested on the restriction of this dynamics to the subset $\mathcal{Q} \subset S^2$ that is homeomorphic to the unitary square $Q = [0,1] \times [0,1]$. Since this set is closed, we just relax the condition of X_t being open sets on the definition of a partial dynamical system to include also closed sets in what follows.

In order to keep the presentation clear, we only describe the dynamics induced by F over the closed square Q and we call it f, assuming that this last one is affine at some part of its domain. The diffeomorphism f maps bijectively the horizontal strips $[0,1] \times [0,1/3]$ and $[0,1] \times [2/3,1]$, respectively, to the vertical strips $[0,1/3] \times [0,1]$ and $[2/3,1] \times [0,1]$, the horizontal strips being the domain where f is affine.

We can now see the horseshoe as a partial action of $\mathbb{Z}$ defined as follows: take

$$X_n := Q \cap f^n(Q) \text{ and } \alpha_n(x) := f^n(x) \text{ for } n \in \mathbb{Z}, x \in X_{-n}.$$

Then $\big((X_n)_{n\in\mathbb{Z}}, (\alpha_n)_{n\in\mathbb{Z}}, \mathbb{Z}\big)$ is a partial dynamical system on the square $Q = X_0$.

Since α_n is always affine on its domain, it is not hard to see that each α_n has at most a finite number of fixed points for any $n \neq 0$; hence, for each $n \neq 0$ the set of the fixed points of α has empty interior.

It is also possible to define a limit set $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(Q)$ that is homeomorphic to $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$. Over Σ_2 we can define an homeomorphism known as shift, defined as follows:

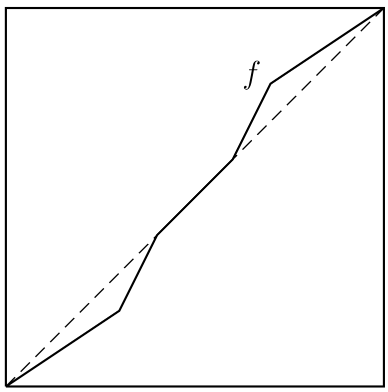
$$x \in \Sigma_2, x = (x_i)_{i\in\mathbb{Z}}, \text{ then } \big(\sigma(x)\big)_i = x_{i+1}.$$

This map is conjugate to f restricted to Λ, i.e. there exists an homeomorphism $h\colon \Lambda \to \Sigma_2$ such that $hf = \sigma h$. Hence, the restriction $(\Lambda, \alpha|_\Lambda, \mathbb{Z})$ is in fact a global action. By means of the conjugation we get that fixed points of $\alpha|_\Lambda$ correspond to the fixed points of the shift over Σ_2, showing that they are finite for any α_n, $n \in \mathbb{Z}$.

Note that the dynamics of the horseshoe is topologically free (since the sets of fixed points have empty interior) but it is not transitive: just take the open balls $B_r\big((1/2, 0.2)\big)$ and $B_r\big((1/2, 0.8)\big)$, for some positive $r < 1/10$; calling one by U and the other by V its is easy to see that they violate the criterium established in Theorem 3.2.1.

Since $(X_0, \alpha, \mathbb{Z})$ it is not transitive, it is not minimal and, by Lemma 3.2.7, $C(X_0)$ is not α-simple. Hence, by Theorem 3.2.16 we have that $C(X_0) *_\alpha \mathbb{Z}$ is not simple.

Example 3.3.2. We can use two dynamics f and g defined on the closed interval $[0, 1]$ and such that $f \circ g = g \circ f$, defined as follows: f has an interval of fixed points, fix 0 and 1, 0 is an attractor on the first interval and 1 is an attractor on the third interval (see the picture); for g just take the identity. Now we can consider a global (hence, partial) action of $G = \mathbb{Z}^2$ on $[0, 1]$ where $\alpha_{(m,n)}(x) = f^m \circ g^n(x)$. For $t = (0, 1)$, the interior of the θ_t is not empty and so the system is not topologically free. And the dynamics in fact is the dynamics of f, that is not transitive, since any open subset of the middle interval can not contain points belonging to a dense orbit. In fact, it does not satisfy the criterium for transitivity of Theorem 3.2.1, since given two open and disjoint subsets of the medium interval, U and V, there exists no $g \in G$ such that $\alpha_g(U) \cap V \neq \emptyset$



So, by Theorem 3.2.13 and Theorem 3.2.16 the algebra $C(X) *_\alpha \mathbb{Z}^2$ is not simple.

Example 3.3.3. Consider the map $R_\omega\colon [0,1] \to [0,1]$ defined by

$$R_\omega(x) = (x+\omega) mod(1).$$

It is well known that R_ω has dense orbits if and only if ω is irrational. We use now $f = R_\omega$, ω irrational, g the identity, and the set is $[0,1]$ with 0 identified with 1. Now we can consider, as in Example 3.3.2 above, the global action of $G = \mathbb{Z}^2$ defined in the same way. Restricting this global action to the set $(1/3, 2/3)$ we get a partial action, that is not topologically free, has dense orbits, being transitive. So, by Theorems 3.2.13 and 3.2.16 we have that the algebra $C(X) *_\alpha \mathbb{Z}^2$ is not simple.

We finish with the following example that gives an easy application of Theorems 3.2.13 and 3.2.16 to show the simplicity.

Example 3.3.4. Let $G = \mathbb{R}^*$ be the multiplicative group and $X = \mathbb{R}$ with usual topology. We consider the global (hence, partial) action $\alpha : G \times X \to X$ by $\alpha(x, y) = xy$. It is easy to see that the unique periodic element is $z = 0$ and we clearly have that X is topologically free. Moreover, the unique α-invariant open subsets are the trivial ones. So, by Theorems 3.2.13 and 3.2.16 we have that $C(\mathbb{R}) *_\alpha \mathbb{R}^*$ is simple.

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Some frequently used notations

AFPR	almost fully prime ring	11
FPR	fully prime ring	11
FWPR	fully weakly prime ring	13
$\mathbb{C}$	field of complex numbers	16
$\mathbb{R}$	field of real numbers	15
$\mathbb{Z}$	ring of integers	28
$J(S)$	Jacobson radical of S	14
$\mathbb{M}_n(S)$	ring of $n \times n$ matrices with entries from S	24
$N(S)$	sum of all ideals of S whose square is zero	14
$Nil(S)$	sum of all nilpotent ideals of S	27
$Nil_\alpha(S)$	α-prime radical of S	18
$Nil_*(S)$	prime radical of S	12
$\mathcal{A}$		60
$Per^g_{C(X)}(X)$		60
$Sep^g_{C(X)}(X)$		60
$Supp(f)$		60
supp(a)		25
T_m		30
α-*AFPR*		28
α-*FPR*		28
α-*FWPR*		28
β-*AFPR*		28
β-*FPR*		28
β-*FWPR*		28
θ_g		60

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