Universidade Federal do Rio Grande do Sul
Instituto de Matemática
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O Primeiro Autovalor do Laplaciano em Variedades Riemannianas

The First Eigenvalue of the Laplace Operator in Riemannian Manifolds

Tese de Doutorado

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Since this is a personal part of this work, it will be written in portuguese.

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Resumo

Propriedades do primeiro autovalor e da primeira autofunção do operador laplaciano em variedades riemannianas são estudadas.

Para variedades em que se pode estimar o laplaciano de funções distância, estimativas explícitas para o primeiro autovalor do laplaciano em domínios duplamente conexos são obtidas. Então observamos que hipóteses sobre as curvaturas da variedade e do bordo do domínio permitem estimar o laplaciano da distância.

Além disso, autofunções em domínios não compactos do espaço hiperbólico $\mathbb{H}^n$ são estudadas. Mostramos que domínios contidos em horobolas não admitem autofunções limitadas associadas ao autovalor $\lambda(\mathbb{H}^n)$, mas se o fecho assintótico do domínio contém um aberto de $\partial_\infty \mathbb{H}^n$, então ele admite uma autofunção positiva que se anula em $\partial\Omega \cup \partial_\infty\Omega$. A existência e o perfil de autofunções de autovalor $\lambda(\mathbb{H}^n)$ em $\mathbb{H}^n$, em $\mathbb{H}^n \backslash B_r(o)$, em horobolas, em hiperbolas e no complementar de horobolas são analisados. Para alguns desses domínios apresentamos uma expressão explícita para a autofunção que depende apenas da distância à fronteira.

Finalmente, técnicas de simetrização de Schwarz são adaptadas para variedades permitindo-nos obter estimativas para normas de autofunções. Primeiro um argumento de comparação demonstra que variedades mais simétricas maximizam certas normas. Obtemos também uma estimativa diretamente da função isoperimétrica da variedade.

Palavras-chave: Estimativas para autovalores/autofunções do operador laplaciano; Autofunções no espaço hiperbólico.

Abstract

Some properties of the first eigenvalue λ and the first eigenfunction of the Laplace operator in a Riemannian manifold are studied.

Assuming a bound for the Laplacian of the distance function, explicit estimates for the first eigenvalue of a doubly connected domain are presented. Then some assumptions on the curvatures of the manifold and its boundary are made in order to have an estimate for the Laplacian of the distance function.

Furthermore eigenfunctions of non compact domains in the hyperbolic space $\mathbb{H}^n$ are studied. We prove that a domain contained in a horoball does not admit a bounded eigenfunction of eigenvalue $\lambda(\mathbb{H}^n)$, but if the closure of the domain contains an open set of $\partial_\infty \mathbb{H}^n$, then it admits a positive eigenfunction that vanishes on $\partial\Omega \cup \partial_\infty \Omega$. The existence and the profile of eigenfunctions of eigenvalue $\lambda(\mathbb{H}^n)$ in $\mathbb{H}^n$, in $\mathbb{H}^n \backslash B_r(o)$, in horoballs, hiperballs and in the complement of a horoball are analysed. For some of these domains we present an explicit expression for the eigenfunction that depends only on the distance to the boundary.

Finally Schwarz symmetrization techniques are adapted for manifolds implying in estimates for the norm of the eigenfunctions. First a comparison argument proves that highly symmetric manifolds maximize some norm and then an estimated obtained directly from the isoperimetric function of the manifold is presented.

Key-words: Estimates for eigenvalues/eigenfunctions of the Laplacian Operator; Eigenfunctions in the hyperbolic space.

Contents

Introduction

Different aspects of the first eigenvalue of the Laplace operator in domains contained in Riemannian manifolds will be analyzed. The first eigenvalue of the Laplacian operator in a subset Ω of a manifold is the smallest real value λ for which there is a function $u \in C^2(\Omega)$, with $-\Delta u = \lambda u$ in Ω, that vanishes on $\partial\Omega$.

The text starts reviewing some basic facts on geometry. Then, in the second chapter estimates for the first eigenvalue of doubly connected domains in Riemannian manifolds are presented. These results were published at [20]. Explicit lower estimates for the first eigenvalue of the Laplacian operator are obtained, without any assumption on the mean convexity of the boundary of the domain, assuming either an upper bound of the sectional curvature, a lower bound of the Ricci curvature, or in highly symmetric manifolds where the Laplacian of the distance function to a fixed point depends only on the distance. Asymptotic properties are also analyzed. In many cases our estimates improve the classical and more recent ones.

The third chapter concentrates only in the hyperbolic space. In this part the focus is on the eigenfunction associated to the eigenvalue of the whole space. The existence and the behavior of positive solutions to some Dirichlet eigenvalue problems for unbounded domains of the hyperbolic space $\mathbb{H}^n$ are studied. If the domain is contained in a horoball, we prove that the problem has no bounded solution. However, if the domain contains a hyperball, then there is a solution that converges to 0 at infinity and can be extended continuously to the asymptotic boundary. In particular, this result holds for hyperballs. In this case, we also present a second kind of solution that only exists if there is some relation between the curvature of the hypersphere and n. This part of the research was done with the advice of Professor Dr. Leonardo Bonorino and is in preprint phase.

In the last chapter, we present results from a research that is still in progress. The results from Chapter 4 relate the isoperimetric profile, the eigenvalue and the eigenfunction of Riemannian manifolds. This work was also done with contributions of Professor Dr. Leonardo Bonorino.

Although this text divides in three disjoint parts, it concentrates in the problem of the first Dirichlet eigenvalue of the Laplacian operator in Riemannian manifolds, which, as it is well known, is associated to physical aspects of the domain, motivating much of its study.

Chapter 1

Preliminaries

There are some preliminaries that are common to all chapters. The definitions and results presented here are well known, but we expose them in order to fix notation and to simplify the reading.

1.1 The Second Fundamental Form

This section follows [11]. Let M be a complete n-dimensional Riemannian manifold with Riemannian connection ∇. Let N be a hypersurface of M and η a unit normal vector at a point p of N. The second fundamental form $S_\eta(p)$ of N with respect to η at p is

$$S_\eta(p)(v) = -\nabla_v \widetilde{\eta}, \tag{1.1}$$

where $\widetilde{\eta}$ is any extension of η normal to N in a neighboorhood of p. The mean curvature of N (with respect to η) at p is the trace of $S_\eta(p)$ divided by the dimension of N and is denoted by H_p.

If V is a vector field along a curve α, we denote its covariant derivative $\nabla_{\alpha'} V$ by V'.

Let us define some very useful functions, the functions S_K and C_K for $K \in \mathbb{R}$:

$$S_K(t) = \begin{cases} \dfrac{\sinh(\sqrt{-K}t)}{\sqrt{-K}} \text{ if } K < 0; \\ t \text{ if } K = 0; \\ \dfrac{\sin(\sqrt{K}t)}{\sqrt{K}} \text{ if } K > 0 \end{cases}$$

and

$$C_K(t) = S'_K(t).$$

We observe that if $\mathbb{Q}^n(K)$ is a simply connected space of constant curvature K, then its metric is given by $dt^2 + S_K(t)^2 d\theta^2$.

1.2 The Hyperbolic Space $\mathbb{H}^n$

When $K = -1$, $\mathbb{Q}^n(-1)$ is the hyperbolic space denoted by $\mathbb{H}^n$. Chapter 3 concentrates in the study of eigenfunctions of the Laplace operator defined in this space and in some special subsets of it. The aim of this section is to present $\mathbb{H}^n$, which we also do following [11].

There are two models of the hyperbolic space that are interesting for this work. The first is the semi-space model, which has a nice expression for the metric and therefore makes some computations easier. It consists in taking

$$H^n = \{p = (p_1, ..., p_n) \in \mathbb{R}^n \mid p_n > 0\}$$

with the metric

$$k_{ij}(p) = \frac{\delta_{ij}}{p_n^2}.$$

Two Riemannian metrics g_1, g_2 in a manifold M are said conformal if there is a function $\mu : M \to \mathbb{R}$, such that

$$g_1(p)(u,v) = \langle u, v\rangle_{p,1} = \mu(p) g_2(p)(u,v) = \mu(p)\langle u, v\rangle_{p,2}.$$

The metric k of H^n is conformal to the euclidean metric of $\mathbb{R}^n$ with $\mu(p) = 1/p_n^2$.

For the sake of completeness we demonstrate here that the manifold defined above has indeed sectional curvature -1. Then we present its umbilical hypersurfaces and look at them in the ball model of $\mathbb{H}^n$.

Proposition 1.2.1. *If g_1 and g_2 are two conformal metrics in the differentiable manifold M, $g_2 = \mu g_1$, Riemannian connection $\overline{\nabla}$ in $M_2 = (M, g_2)$ is given by*

$$\overline{\nabla}_X Y = \nabla_X Y + S(X,Y),$$

where ∇ is the Riemannian connection of $M_1 = (M, g_1)$ and

$$S(X,Y) = \frac{1}{2\mu}\{X(\mu)Y + Y(\mu)X - \langle X, Y\rangle_1 \operatorname{grad}\mu\}$$

with the gradient computed in the metric g_1.

The proof of this proposition requires only the verification of the properties of the Levi-Civita connection, which are just computations that will be omitted.

Applying the above result, we obtain an expression for the Riemannian connection of H^n. If $\{e_1, ..., e_n\}$ is the canonical basis of $T_p\mathbb{R}^n$ for all $p \in \mathbb{R}^n$, which is a orthogonal basis of T_pH^n for all $p \in H^n$, then

$$\overline{\nabla}_{e_i}e_j = \nabla_{e_i}e_j + S(e_i, e_j) \text{ and}$$

$$\begin{aligned}
\overline{\nabla}_{e_i}e_j &= 0 + 0 = 0 \text{ if } i \neq j, i, j \in \{1, ..., n-1\} \\
\overline{\nabla}_{e_i}e_j &= 0 - \frac{\operatorname{grad}\mu}{2\mu} = \frac{e_n}{p_n} \text{ if } i = j \in \{1, ..., n-1\} \\
\overline{\nabla}_{e_n}e_j &= 0 + \frac{e_n(\mu)}{2\mu} = \frac{-e_j}{p_n} \text{ if } j \neq n \\
\overline{\nabla}_{e_j}e_n &= [e_j, e_n] + \overline{\nabla}_{e_n}e_j = \frac{-e_j}{p_n} \text{ if } j \neq n \\
\overline{\nabla}_{e_n}e_n &= 0 + 2\frac{e_n(\mu)}{2\mu} - \frac{\operatorname{grad}\mu}{2\mu} = -\frac{e_n}{p_n}.
\end{aligned}$$

From these formulas, one concludes that

$$R(e_i, e_j)e_i = \nabla_{e_j}\nabla_{e_i}e_i - \nabla_{e_i}\nabla_{e_j}e_i + \nabla_{[e_i,e_j]}e_i = \frac{-e_j}{p_n^2}$$

and if σ is a plane in T_pH^n generated by $\{e_i, e_j\}$, then

$$K(\sigma) = \frac{\langle R(e_i, e_j)e_i, e_j\rangle}{\|e_i\|^2\|e_j\|^2} = -1 \text{ since } \|e_k\| = p_n^{-2}\ \forall k.$$

This implies that all planes in T_pH^n have sectional curvature -1, concluding that H^n is a model for $\mathbb{H}^n$.

Definition 1.2.2. *An immersion $x : N^{n-1} \to M^n$ is called umbilical if for all $p \in x(N)$,*

$$\langle S_\eta(p)(v), w\rangle = \lambda(p)\langle v, w\rangle\ \forall v, w \in T_px(N),$$

for all η unitary vector field orthonormal to $x(N)$.

One can demonstrate that if M has constant sectional curvature, then λ has to be constant. Hence the umbilical hypersurfaces of H^n have constant mean curvature. In order to find them, we need two more propositions.

Proposition 1.2.3. *If (M, g_1) and (M, g_2) are conformal, then $x : N^{n-1} \to M$ is umbilical for the metric g_1 if and only if it is umbilical for g_2. Besides, if λ is the coefficient of x for g_1 and $g_2 = \mu g_1$, then*

$$\overline{\lambda} = \frac{2\lambda\mu - \eta(\mu)}{2\mu\sqrt{\mu}} \tag{1.2}$$

is the coefficient for g_2 when the normal vector field is $\frac{\eta}{\sqrt{\mu}}$.

Proof. We have to demonstrate that

$$\langle -\overline{\nabla}_X \frac{\eta}{\sqrt{\mu}}, Y\rangle_2 = \overline{\lambda}\langle X, Y\rangle_2.$$

First notice that

$$\langle -\overline{\nabla}_X \frac{\eta}{\sqrt{\mu}}, Y\rangle_2 = \frac{\langle -\overline{\nabla}_X \eta, Y\rangle_2}{\sqrt{\mu}} - X(\mu^{-1/2})\langle \eta, Y\rangle_2 = \frac{\langle -\overline{\nabla}_X \eta, Y\rangle_2}{\sqrt{\mu}}.$$

From Proposition 1.2.1,

$$\langle -\overline{\nabla}_X \eta, Y\rangle_2 = \langle -\nabla_X \eta, Y\rangle_2 - \langle S(X,\eta), Y\rangle_2.$$

Since x is umbilical for g_1,

$$\langle -\nabla_X \eta, Y\rangle_2 = \mu\langle -\nabla_X \eta, Y\rangle = \mu\lambda\langle X, Y\rangle = \lambda\langle X, Y\rangle_2$$

and from the definition of S,

$$\begin{aligned}\langle S(X,\eta), Y\rangle_2 &= \tfrac{1}{2\mu}\{X(\mu)\langle \eta, Y\rangle_2 + \eta(\mu)\langle X, Y\rangle_2 - \langle X, \eta\rangle\langle \operatorname{grad}\mu, Y\rangle_2\} \\ &= \tfrac{\eta(\mu)}{2\mu}\langle X, Y\rangle_2.\end{aligned}$$

Hence

$$\langle -\overline{\nabla}_X \frac{\eta}{\sqrt{\mu}}, Y\rangle_2 = \frac{\langle -\overline{\nabla}_X \eta, Y\rangle_2}{\sqrt{\mu}} = \frac{\lambda\langle X, Y\rangle_2 - \frac{\eta(\mu)}{2\mu}\langle X, Y\rangle_2}{\sqrt{\mu}},$$

concluding the proof. □

Now it became sufficient knowing the umbilical hypersurfaces of $\mathbb{R}^n$ in order to know the umbilical hypersurfaces of $\mathbb{H}^n$.

Proposition 1.2.4. *If $x : N^{n-1} \to \mathbb{R}^n$ is umbilical, then $x(N)$ is contained in a hyperplane or in a $n-$sphere.*

Proof. Since $\mathbb{R}^n$ has constant sectional curvature 0, λ is constant.

If $\lambda = 0$, $S_\eta(p)(v) = 0 \ \forall v \in T_p x(N)$, $\forall p \in x(N)$. This implies that the vector field η is constant, meaning that $x(N)$ is contained in a hyperplane of $\mathbb{R}^n$.

If $\lambda \neq 0$, we claim that

$$y : N \to \mathbb{R}^n, \ \ y(p) = x(p) + \frac{\eta(p)}{\lambda}$$

is constant. If T and Y are vector fields of N and one thinks of y as a vector field, then

$$\langle \nabla_T y, Y \rangle = \langle \nabla_T x, Y \rangle + \frac{1}{\lambda}\langle \nabla_T \eta, Y \rangle = \langle T, Y \rangle - \frac{\lambda}{\lambda}\langle T, Y \rangle = 0.$$

Then y is constant. If $y(N) = \{x_0\}$, then $|x(p) - x_0| = 1/\lambda$, implying that $x(N)$ is contained in the sphere of radius $1/\lambda$ centered at x_0. □

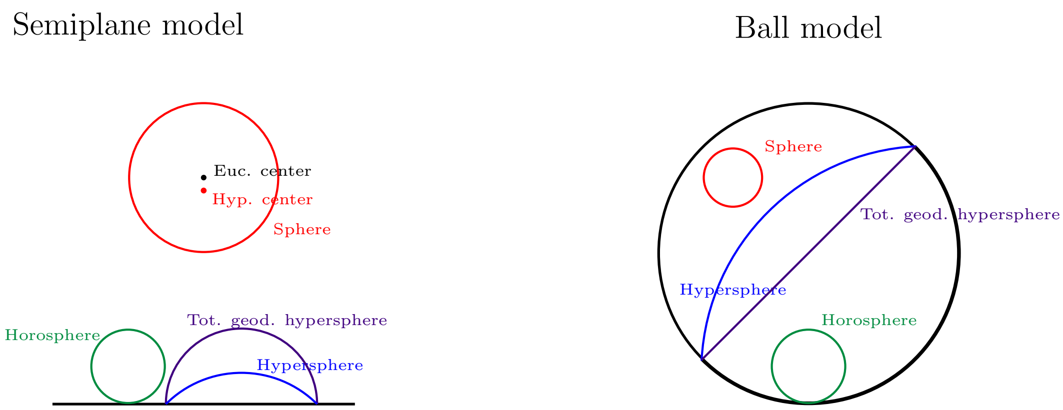
As a consequence of the last two propositions the next theorem is demonstrated.

Theorem 1.2.5. *The umbilical hypersurfaces of $\mathbb{H}^n$ in the semispace model H^n are the intersections of H^n with euclidean $n-1$-spheres or $n-1$-planes.*

The umbilical hypersurfaces of $\mathbb{H}^n$ divide into three classes: The spheres, which are the Euclidean spheres entirely contained in H^n, the horospheres, represented by the spheres that are tangent to ∂H^n and by the hyperplanes parallel to ∂H^n and finally the hyperspheres, the intersections with H^n of spheres of $\mathbb{R}^n$ that are not entirely contained in $\{x_n \geq 0\}$ and all hyperplanes that are not parallel to $\{x_n = 0\}$.

In order to compute the mean curvature of these hypersurfaces it is sufficient apply formula (1.2) from Proposition 1.2.3. This implies that if Σ is a sphere of $\mathbb{R}^n$ of radius r and with center at high p_n, the mean curvature of $\Sigma \cap H^n$ in H^n is given by p_n/r, if the normal vector points inwards. If Γ is a hyperplane with normal vector η whose component in the e_n direction is α_n, then the mean curvature of Γ is α_n.

On the other hand, it will be useful to know the mean curvature of these hypersurfaces as a function of the hyperbolic distance. Knowing that the geodesics in H^n are represented by arcs of circles centered at ∂H^n and by vertical lines, one proves the following facts: The spheres contained in H^n are in fact geodesic spheres of $\mathbb{H}^n$ and if d is the hyperbolic distance to the center of the sphere, the mean curvature of the sphere is $\coth d$. The horospheres represented by spheres tangent to ∂H^n at the same point are all level sets of the distance function to any of these horospheres and, as observed above, they all have mean curvature $p_n/r = 1$, if the normal vector points to ∂H^n. To consider the mean curvature of the hyperspheres related to the hyperbolic distance one has to observe that the hyperspheres that cross ∂H^n at the same set are also a family of equidistant surfaces. If one fixes H_0 as the intersection of a sphere centered at ∂H^n and H^n, which is totally geodesic (the mean curvature is $0/r = 0$), and H is a hypersphere equidistant to H_0, then the mean curvature of H is $\tanh d$, where d is the distance between them and the normal vector η points to ∂H^n. In Chapter 3 we will analyse the profile of eigenfunctions defined in some very symmetric subsets of $\mathbb{H}^n$, which are bounded by umbilical hypersurfaces.



In order to see the symmetries of $\mathbb{H}^n$ and its asymptotic boundary, the ball model becomes more natural. Consider

$$B^n = \{p \in \mathbb{R}^n \mid |p| < 1\},$$

where $|\cdot|$ denotes the Euclidean norm. Introducing in B^n, the metric

$$h_{ij}(p) = \frac{4\delta_{ij}}{(1 - |p|^2)^2},$$

B^n is a model of $\mathbb{H}^n$. In order to see this, it is sufficient to present an isometry from B^n to H^n. Take

$$f : B^n \to H^n, \qquad f(p) = 2\frac{p - p_0}{|p - p_0|^2} - (0, ..., 0, 1).$$

It is easy to prove that f is a bijection and that is derivative satisfies

$$|df_p(u)| = 2\frac{|u|}{|p - p_0|^2},$$

which implies that it is an isometry.

In this model, the geodesic spheres are also represented by euclidean spheres. The horospheres are the intersection of spheres that are tangent to ∂B^n and B^n. Intuitively they are the spheres centered at infinity. The totally geodesic hyperspheres H_0 are represented by the Euclidean spheres of $\mathbb{R}^n$ that meet ∂B^n orthogonally and the hyperspheres equidistant to H_0 are are euclidean spheres that meet ∂B^n at the same set as H_0.

We shortly present the asymptotic boundary of $\mathbb{H}^n$ and its topology. For details, see [16].

A Hadamard manifold is a complete simply connected Riemannian manifold of negative sectional curvature. The fact that it has negative curvature means that it is, in some sense, larger than the Euclidean space, since the geodesics in M spread more than in $\mathbb{R}^n$. Thus the asymptotic boundary of M is also larger and the question whether and how a function extends continuously to the boundary becomes more interesting. In order to think about this question in Chapter 3, we define the asymptotic boundary.

For the rest of this section M is assumed to be a Hadamard manifold.

Definition 1.2.6. *Two unit speed geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are asymptotic if there is $C > 0$, such that $d_M(\gamma_1(t), \gamma_2(t)) < C \; \forall \; t \in [0, \infty)$.*

Being asymptotic is an equivalence relation in the set of geodesic rays in M. The asymptotic boundary of M, denoted by $\partial_\infty M$ is the set of all equivalence classes of geodesic rays γ. The equivalence class of γ is denoted by $\gamma(\infty)$. It can be demonstrated that given $p \in M$, $\partial_\infty M$ is in a one-to-one correspondence with

$$S_p = \{v \in T_pM \mid \|v\| = 1\},$$

meaning that $\partial_\infty M$ can be viewed as the set of all oriented directions from the point p.

The closure of M is then defined by $\overline{M} = \partial_\infty M \cup M$. In order to endow $\overline{M}$ with a topology, the so-called cone topology, we define for $p \neq q$, γ_{pq} the unique normalized geodesic ray such that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(t) = q$ for some $t > 0$. If $p \in M$ and $q \in \partial_\infty M$ we denote by γ_{pq} the geodesic ray such that the $\gamma_{pq}(0) = p$ and $\gamma_{pq}(\infty) = q$. One can prove that γ_{pq} exists and is unique.

Now fix a point $p \in M$. Given $u, v \in T_pM$ denote by $\sphericalangle_p(u, v)$ the angle between u and v. If $x, y \in \overline{M}$, define $\sphericalangle_p(x, y) = \sphericalangle_p(\gamma'_{px}(0), \gamma'_{py}(0))$.

Definition 1.2.7. *Given $v \in T_pM$, $\|v\| = 1$, $\delta > 0$ and $r > 0$, we define the cone with vertex p, axis v and opening angle δ by*

$$C(v, \delta) = \{x \in \overline{M} | \sphericalangle_p(v, \gamma'_{px}(0)) < \delta\}$$

and the truncated cone of radius r is

$$T(v, \delta, r) = C(v, \delta) \backslash \overline{\{x \in M \big| d(x, p) < r\}}.$$

The set of all truncated cones at p and all the open geodesic balls centered at p is a local basis of a topology in $\overline{M}$, the cone topology. Under this topology $\overline{M}$ is homeomorphic to a n dimensional ball and $\partial_\infty M$ to a $n-1$ dimensional sphere. Finally, if $S \subset M$ set $\partial_\infty S = \overline{S} \cap \partial_\infty M$, where $\overline{S}$ is the closure of S in the cone topology.

In the particular case of $\mathbb{H}^n$ represented in the ball model, given a unit speed geodesic ray $\gamma : [0, \infty) \to B^n$, $\lim_{t\to\infty} \gamma(t)$ is a well defined point in ∂B^n, considering in ∂B^n the usual topology induced by the inclusion $i : B^n \to \mathbb{R}^n$. The correspondence

$$\gamma(\infty) \longmapsto \lim_{t\to\infty} \gamma(t)$$

is a homeomorphism between $\partial_\infty B^n$ and ∂B^n. Since the map above can be viewed as a continuous extension of the identification of $\mathbb{H}^n$ and B^n to the closure of $\mathbb{H}^n$, $\overline{B^n}$ is homeomorphic to $\overline{\mathbb{H}^n}$ with the cone topology.

1.3 The Laplace Operator

This work concentrates on the first Dirichlet eigenvalue of the Laplace operator in different domains in Riemannian manifolds.

Definition 1.3.1. *Given a function $u : M \to \mathbb{R}$, $u \in C^2(M)$, the Laplacian of u is $\Delta u = \operatorname{div}(\operatorname{grad} u)$.*

In local coordinates the Laplace Operator is given by

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{j,k\in\{1,\ldots,n\}} \partial_j (g^{jk}\sqrt{g}\partial_k f))$$

where $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ are the metric coefficients, $g = \det(g)_{ij}$ and g^{ij} are the elements of the inverse matrix of $(g)_{ij}$. For a proof of this expression, see [12].

Therefore the Laplacian is an elliptic operator and uniformly elliptic in compact sets. Consequently, results such as the strong maximum principle for subharmonic functions also hold in Riemannian manifolds.

Definition 1.3.2. *Given a bounded domain (open connected set) Ω in a Riemannian manifold M, its first eigenvalue is defined as the smallest $\lambda \in \mathbb{R}$, for which the problem*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

has a non trivial solution. It is denoted by $\lambda(\Omega)$.

We demonstrate a comparison lemma about eigenfunctions.

Lemma 1.3.3. *(Comparison Principle I) Let $u, v : \bar{\Omega} \to \mathbb{R}$, Ω a bounded domain, be such that $-\Delta u = \lambda u$ and $-\Delta v = \lambda v$, for $\lambda < \lambda(\Omega)$. If $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.*

Proof. Define $\Omega' = \{x \in \Omega | v(x) > u(x)\}$. Then $v - u$ is a positive eigenfunction associated to λ in $\Omega' \subseteq \Omega$. But $\lambda(\Omega') \geq \lambda(\Omega) > \lambda$, a contradiction. □

Definition 1.3.4. *The first eigenvalue of a non compact manifold M is defined by*

$$\lambda(M) = \inf\{\lambda_1(\mathcal{O}) \mid \mathcal{O} \textit{ is a bounded domain of } M\},$$

where $\lambda_1(\mathcal{O})$ is the first eigenvalue of the Laplacian on $\mathcal{O}$.

The Laplacian of the distance function

In this work we often consider functions that present some symmetry, usually functions that depend only on the distance to a point or a submanifold.

Assume that $g : M \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$ are C^2 functions, then

$$\Delta (u \circ g)(x) = u''(g(x)) |\operatorname{grad} g(x)|^2 + u'(g(x)) \Delta g(x).$$

If $N \hookrightarrow M$ is a hypersurface of M and d is the distance to N, then

$$\Delta (u \circ d)(x) = u''(d(x)) - u'(d(x))(n-1) H_d(x),$$

where $H_d(x)$ is the mean curvature of the parallel hypersuface of distance $d(x)$, $H_{d(x)} = \{y \in M \mid d(y) = d(x)\}$, oriented with normal vector $\operatorname{grad} d$.

It is also useful to observe that from section 1.2, if the manifold is the hyperbolic space, we have some expressions for the laplacian of functions that depend only on distances. If $g(x) = r(x)$ is the distance between x and some fixed point in $\mathbb{H}^n$, then

$$\Delta (u \circ r)(x) = u''(r) + (n-1)\coth(r) u'(r). \tag{1.4}$$

If $g(x) = d(x)$ is the distance to a horosphere H that bounds a horoball B, then

$$\Delta (u \circ d)(x) = u''(d) - (n-1) u'(d) \text{ if } x \in B \tag{1.5}$$

and

$$\Delta (u \circ d)(x) = u''(d) + (n-1) u'(d) \text{ if } x \notin B. \tag{1.6}$$

Finally, if $g(x) = d(x)$ is the distance to a totally geodesic hypersphere H, then

$$\Delta (u \circ d)(x) = u''(d) + (n-1)\tanh(d) u'(d). \tag{1.7}$$

1.4 Co-area formula, isoperimetric functions and isoperimetric inequalities

Let M be a complete Riemannian manifold. The co-area formula (see [31], page 81 or [25]) states that given a mesuarable $n-$dimensional subset $U \subset M$, a function $f \in H^1(U)$ and an integrable function $g : U \to \mathbb{R}$,

$$\int_U g\,dx = \int_{-\infty}^{+\infty} \int_{\{f=t\}} g \frac{1}{|\operatorname{grad} f|} dH^{n-1}(\{f=t\}) dt. \tag{1.8}$$

Here $H^1(U)$ denotes the Sobolev space $W^{1,2}(U)$. Actually this formula holds in a larger set of functions, but we will apply it only for H^1 functions.

In [6], isoperimetric functions on M are defined.

Definition 1.4.1. *Consider M a complete Riemannian manifold. An isoperimetric function on M is a function $H : [0, \mathrm{vol}(M)] \to \mathbb{R}$ that satisfies*

$$H(\mathrm{vol}(\Omega)) \leq \mathrm{vol}(\partial\Omega) \; \forall \; \Omega \subset\subset M. \tag{1.9}$$

If $\mathrm{vol}(M) = \infty$, *H is defined in $\mathbb{R}_+$.*

Remark 1.4.2. *We call isoperimetric profile the greatest isoperimetric real function,*

$$h(v) = \inf\{\mathrm{vol}(\partial\Omega) \bigm| \mathrm{vol}(\Omega) = v\} \quad v \in [0, \infty).$$

Chapter 2

Lower estimates for the first eigenvalue on doubly connected domains

In this chapter we obtain lower estimates for the first eigenvalue $\lambda = \lambda(\Omega)$ of the Dirichlet problem for the Laplacian operator in relatively compact smooth domains Ω of a Riemannian manifold. Our results are specially adapted for doubly connected domains, that is, the boundary of the domain has two connected components. This seems to be of interest since the classical results as Faber-Krahn (Theorem 2 of Chapter IV, [12]), Cheng [14] are more effective for domains homeomorphic to balls. Nevertheless we also have results for such domains, which in the particular case of M being a spherically symmetric Riemannian manifold coincide with the result of Barroso and Bessa at [3]. We also observe that most of well known and more recent results require mean convexity of the boundary of the domain, (see [29], [22] and [23]) condition which is more difficult to be satisfied for domains with boundary connectivity.

Other results on first eigenvalue lower estimates are related to isoperimetric constants or to functions/vector fields that can be constructed on the domain (see [13], [2], [5], [7]). These estimates can be of hard computation, since usually an infimum or a supremum must be taken. Once one has the necessary information about the domain, our estimates are explicit. This is usually the case when the domain is (or is contained in) a geodesic annulus $A_{r,r+R}(x_0) := B_{r+R}(x_0)\backslash\overline{B_r(x_0)}$, $B_r(x_0)$ being a geodesic ball with radius r. Moreover, analysing the asymptotic behaviour of these annuli estimates in complete non compact manifolds and using well known properties of Busemann functions we obtain estimates for the first eigenvalue of domains contained in horoannuli (see Definition 2.1.5). Explicit lower estimates of the first eigenvalue of annuli and horoannuli can be obtained assuming either an upper bound of the sectional curvature of M (Section 2.3), a lower bound of the Ricci curvature of M (Section 2.4) or in highly symmetric manifolds where the Laplacian of the distance function to a fixed point depends only on the distance (Section 2.5). We also obtain results that relate first eigenvalue properties with curvature and volume (Subsections 2.3.1 and 2.4.1).

Finally, we would like to observe that what essentially allows our results to be effectively applied to domains with boundary connectivity is that the assumption on the convexity of the domains is not needed at all. Our proofs differ completely from more recent papers as Yang, Ling, Lu, which are based on gradient estimates of the first eigenfunction, technique developed by Li on the 70's [24]. The main idea of our proofs is to use the distance function to one of the connected components of the boundary of the domain to the construction of explicit barriers from which the estimates are obtained. With this technique it becomes clearer the influence of the extrinsic curvature of the boundary of Ω in M on a lower bound for $\lambda(\Omega)$. Section 2.5.1 illustrates better this fact. We want also to mention that to any recent work that the authors have notice, or to any classical one, there are examples of domains in spatial forms where our results give better estimates (see Section 2.5).

2.1 General results

In order to clarify the notation we define two constants associated to a function f.

Definition 2.1.1. *Given a C^1 function $f : [r, r+R] \to \mathbb{R}_+$, define g_f^+ as the solution of*

$$\begin{cases} (g' f^{n-1})' = -f^{n-1} \text{ in } (r, r+R) \\ g'(r) = 0 \\ g(r+R) = 0. \end{cases}$$

and $C^+(f) = g_f^+(r)$ the maximum value of g_f^+.

Define also g_f^-, *the solution of*

$$\begin{cases} (g'f^{n-1})' = -f^{n-1} \text{ in } (r, r+R) \\ g'(r+R) = 0 \\ g(r) = 0. \end{cases}$$

and $C^-(f) = g_f^-(r+R)$ *the maximum value of* g_f^-.

The constants associated to f have an explicit expression which can be obtained integrating the ODEs that define them.

Lemma 2.1.2. *Given a* C^1 *function* $f : [r, r+R] \to \mathbb{R}_+$,

$$C^+(f) = \int_r^{r+R} \frac{\int_r^s f(\tau)^{n-1} d\tau}{f(s)^{n-1}} ds \text{ and } C^-(f) = \int_r^{r+R} \frac{\int_s^{r+R} f(\tau)^{n-1} d\tau}{f(s)^{n-1}} ds.$$

Theorem 2.1.3. *Let* M^n *be a complete Riemannian manifold,* $n \geq 1$, *and* N^k *a submanifold of* M, $0 \leq k < n$. *Set*

$$d(x) := d(x, N) = \inf \{d(x, y) \mid y \in N\}, \ x \in M,$$

where d *is the Riemannian distance in* M. *Let* Ω *be a relatively compact open subset of* M *such that* $d|_{\Omega \setminus N}$ *is smooth. Denote by* $\lambda = \lambda(\Omega)$ *the first positive eigenvalue of* Ω.

A) Assuming that $\Omega \cap N = \emptyset$ *and setting*

$$0 \leq r := \inf \{d(x) \mid x \in \Omega\} < \sup \{d(x) \mid x \in \Omega\} =: r + R$$

we have:

A1) If $f : [r, r+R] \to \mathbb{R}_+$ *is a* C^1 *function satisfying*

$$\frac{\Delta d(x)}{n-1} \geq \frac{f'(d(x))}{f(d(x))}, \ x \in \Omega \text{ then } \lambda \geq \frac{1}{C^+(f)}. \tag{2.1}$$

A2) If $f : [r, r+R] \to \mathbb{R}_+$ *is a* C^1 *function satisfying*

$$\frac{\Delta d(x)}{n-1} \leq \frac{f'(d(x))}{f(d(x))}, \ x \in \Omega, \text{ then } \lambda \geq \frac{1}{C^-(f)}. \tag{2.2}$$

B) Assuming that $N \subset \overline{\Omega}$ *and setting*

$$R = \sup \{d(x) \mid x \in \Omega\},$$

if $f : [0, R] \to \mathbb{R}_+$ *is a* C^1 *function such that*

$$\frac{\Delta d(x)}{n-1} \geq \frac{f'(d(x))}{f(d(x))}, \ x \in \Omega \setminus N, \text{ then } \lambda \geq \frac{1}{C^+(f)}.$$

Recall that a geodesic ray γ in a complete non compact manifold M is a geodesic $\gamma : [0,\infty) \to M$ such that the length of an arc of γ connecting any two points is the distance of these points. It is easy to prove that if M is complete and non compact, then for any $p \in M$ there exists a geodesic ray starting from p. Since $|d(x,\gamma(t)) - t| \le d(x,\gamma(0))$, the map $t \mapsto d(x,\gamma(t)) - t$ is uniformly bounded on compact sets and it is non increasing. Hence the limit as $t \to \infty$ exists and the convergence is uniform on compact sets (see [27]).

Definition 2.1.4. *If γ is a geodesic ray on M, the Busemann function $b : M \to \mathbb{R}$ associated to γ is*

$$b(x) = \lim_{t\to\infty} \left(d(x,\gamma(t)) - t\right). \tag{2.3}$$

Definition 2.1.5. *A horoball is a ball centered at infinity, $B(\infty) = \{x \in M \mid b(x) < C\}$, a horosphere is the boundary of a horoball and a horoannulus of width R, $A_R(\infty) = b^{-1}\left((-R/2, R/2)\right)$, is an annulus centered at infinity.*

Given a C^0 function $f : [a,\infty) \to \mathbb{R}$, $a > 0$, and $R > 0$, set

$$\Gamma_R^+(f) = \limsup_{t\to\infty} \left\{C^+\left(f|_{[t-R/2,t+R/2]}\right)\right\}^{-1}$$
$$\Gamma_R^-(f) = \limsup_{t\to\infty} \left\{C^-\left(f|_{[t-R/2,t+R/2]}\right)\right\}^{-1}.$$

Corollary 2.1.6. *Let M be a complete non compact Riemannian manifold. Consider a geodesic ray $\gamma : [0,\infty) \to M$ and let b the Busemann function associated to γ. Assume that Ω is a bounded domain contained in $A_R(\infty)$ and that there is a $t_0 \in \mathbb{R}$ such that the distance function $d_{\gamma(t)}$ to $\gamma(t)$ is smooth in Ω for all $t > t_0$.*

1) If $f : [a,\infty) \to \mathbb{R}_+$, $a < t_0 - R/2$, is a C^1 function satisfying

$$\frac{\Delta d_{\gamma(t)}(x)}{n-1} \ge \frac{f'(d_{\gamma(t)}(x))}{f(d_{\gamma(t)}(x))},\ x \in \Omega,\ t > t_0,$$

then

$$\lambda \ge \Gamma_R^+(f).$$

2) If $f : [a,\infty) \to \mathbb{R}_+$, $a < t_0 - R/2$, is a C^1 function satisfying

$$\frac{\Delta d_{\gamma(t)}(x)}{n-1} \le \frac{f'(d_{\gamma(t)}(x))}{f(d_{\gamma(t)}(x))},\ x \in \Omega,\ t > t_0,$$

then

$$\lambda \ge \Gamma_R^-(f).$$

2.1.1 Finding f explicitly in Theorem 2.1.3 and a geometric interpretation of estimates (2.1) and (2.2).

We observe that $\Delta d(x)/(n-1)$ is minus the mean curvature $H_d(x)$ at x of the parallel hypersurface

$$N_{d(x)} = \{y \in M \mid d(x) = d(y)\},$$

oriented with normal vector grad$d$ (see (1.1)). A way of finding explicitly a function f satisfying the hypothesis of Theorem 2.1.3 is setting

$$h_1 : [r, r+R] \to \mathbb{R},\ h_1(t) = \inf\{-H_t(x) \mid x \in N_t\}$$
$$h_2 : [r, r+R] \to \mathbb{R},\ h_2(t) = \sup\{-H_t(x) \mid x \in N_t\}$$

and taking f_i as a solution of the ODE $h_i(d) = f_i'(d)/f_i(d)$, $i = 1, 2$.

As a consequence of (2.1) and (2.2) we obtain

$$\lambda \geq \max\left\{\{C^+(f_1)\}^{-1}, \{C^-(f_2)\}^{-1}\right\}. \tag{2.4}$$

We note that

$$C^+(f_1) = \int_r^{r+R} \frac{\int_r^t f_1(s)^{n-1} ds}{f_1(t)^{n-1}} dt = \int_r^{r+R} \frac{V_1(r,t)}{S_1(t)} dt \tag{2.5}$$

$$C^-(f_2) = \int_r^{r+R} \frac{\int_s^{r+R} f_2(\tau)^{n-1} d\tau}{f_2(s)^{n-1}} ds = \int_r^{r+R} \frac{V_2(t, r+R)}{S_2(t)} dt, \tag{2.6}$$

where $S_i(t)$ is the area of the geodesic sphere of radius t and $V_i(a,b)$ is the volume of the annulus of inner radius a and outer radius b in the spherically symmetric manifold $\mathbb{R} \times \mathbb{S}^{n-1}$ with the metric $dt^2 + f_i^2(t) ds^2$. If the mean curvature of the parallel hypersurfaces depends only on the distance then $h_1 = h_2 = -H$ and $f := f_1 = f_2$. In the cases where the function f (area of the geodesic spheres) is an increasing function of the radius, estimate (2.5) is better than (2.6). But if f is neither increasing nor decreasing it is hard to say which expression will give a better estimate, occurring cases where (2.6) is better than (2.5) (an example is given at the end of Section 2.5.2). Theorem 2.1.3 is specially interesting when the mean curvature of the parallel hypersurfaces can be explicitly computed, as is the case of geodesic spheres and annuli in rank 1 symmetric spaces (see Section 2.5).

2.2 Proof of the general results

Theorem 2.1.3 is a consequence of the following general inequality which is used to obtain C^0 estimates of the solutions of Poisson's equation (see Theorem 3.7 of [17]). To see that it is a consequence of the theorem below, take u the eigenfunction associated to the first eigenvalue of Ω and apply it.

Theorem 2.2.1. *Consider M, N, Ω and the functions f and d as in Theorem 2.1.3, part A1. Then*

$$\sup_{\Omega}|u| \leq \sup_{\partial\Omega}|u| + C^{+}(f)\sup_{\Omega}|\Delta u|\,, \tag{2.7}$$

for all $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$.

Proof. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be given. We may assume that $\sup_\Omega |\Delta u| < \infty$ otherwise (2.7) is trivially satisfied. Define $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ by

$$v(x) = \sup_{\partial\Omega}|u| + g(d(x))\sup_{\Omega}|\Delta u|$$

where $g = g_f^+ \in C^2\left((r, r+R)\right) \cap C^0([r, r+R])$ defined in Definition 2.1.1. Now we use the Maximum Principle to demonstrate that $u \leq v$ in Ω. First, if $y \in \partial\Omega$,

$$v(y) - u(y) = \sup_{\partial\Omega}|u| + g(d(y))\sup_{\Omega}|\Delta u| - u(y) \geq 0$$

because $g_f^+ \geq 0$.

Besides

$$\Delta(v-u) = \Delta(g \circ d)\sup_{\Omega}\Delta u - \Delta u \leq 0,$$

if, and only if, $\Delta(g \circ d) \leq -1$. But from the hypothesis on Δd and from the fact that g_f^+ is decreasing,

$$\Delta(g \circ d) = g''(d) + g'(d)\Delta d \leq g''(d) + (n-1)g'(d)\frac{f'}{f}.$$

Hence $\Delta(g \circ d) \leq -1$, if

$$g''(d) + (n-1)g'(d)\frac{f'}{f} \leq -1,$$

which is equivalent, by multiplying by f^{n-1}, to the ODE satisfied by g. Hence, by the Maximum Principle, $u \leq v$ and, taking $C^+(f)$ as the supremum of g_f^+, the proof of the theorem case A1 is concluded. $\square$

The proof of part A2 of Theorem 2.1.3 is the same of part A1 except that g is chosen to be g_f^-. In part B we observe that using $g = g_f^+$ to define v, although $v - u$ is not smooth on Ω, it is still superharmonic if $\Delta(v-u) \geq 0$ in $\Omega \backslash N$ since $(v-u)_N = \sup_\Omega (v-u)$. Consequently, the Maximum Principle implies the same inequality.

2.2.1 Proof of Corollary 2.1.6

The proof of Corollary 2.1.6 consists in considering annuli centered at $\gamma(t)$ and letting t go to infinity. Precisely: First note that for any $x \in \Omega$ we have

$$\lim_{t\to\infty} \left(d_{\gamma(t)}(x) - t \right) \in [-R/2, R/2]$$

and the convergence is uniform in compact sets. Thus, given $\epsilon > 0$, there is t_0 such that $\Omega \subset A_{t-R/2-\epsilon, t+R/2+\epsilon}(\gamma(t))$ for all $t \geq t_0$. Hence,

$$\lambda \geq \left\{ C^+(f|_{[t-R/2-\epsilon, t+R/2+\epsilon]}) \right\}^{-1}$$

and the result follows by letting t go to infinity.

Remark 2.2.2. *In the first case of Corollary 2.1.6, where Δd is estimated from below, if f has the property that f'/f is decreasing, then the function*

$$r \mapsto \left\{ C^+(f|_{[r,r+R]}) \right\}^{-1}$$

is decreasing. In this case, if the estimate for the Laplacian is valid for the distance to any point $y \in M$, $\lambda\left(A_{r,r+R}\right)$ can be estimated by the expression given in the first case of Corollary 2.1.6 for any $r > 0$. The property of f'/f being decreasing geometrically means that the mean curvature of the geodesic spheres in the rotationally symmetric space related to f (which is $-f'/f$) increases with the sphere radius. This happens in the Euclidean and hyperbolic spaces, fact that will be used in the next sections.

2.3 Estimates assuming an upper bound for the sectional curvature of M

In this section, by considering an upper bound for the sectional curvature of M, we obtain estimates from below of the mean curvature of the parallel hypersurfaces to the one of the connected component of $\partial\Omega$ which allows the application of parts A1 and B of Theorem 2.1.3. We begin by defining a kind of domain for which our results apply very well.

Definition 2.3.1. *Let M^n be a Riemannian manifold. We say that a domain Ω of M is a one side normal neighboorhood with width R of a hypersurface N of M if there is an unitary normal vector field η to N such that*

$$(p,t) \mapsto \exp_p(t\eta(p)),\ (p,t) \in N \times [0,R]$$

is a diffeomorphism onto Ω, where exp *is the exponential map of M.*

Note that if N is a geodesic sphere of radius r and η points to the connected component of $M\backslash N$ which does not contain the center x_0 of N then $\Omega = A_{r,r+R}(x_0)$.

Theorem 2.3.2. *Let Ω be a one side normal neighboorhood of N of width R and define*

$$\Lambda(N) = \sup_{p\in N} \sup_{v\in T_pN,\ |v|=1} \langle S_\eta(p)(v), v\rangle.$$

Assume that $f : [0,R] \to (0,\infty)$ is a function in $C^2([0,R])$ satisfying

$$\Lambda(N) \leq -\frac{f'(0)}{f(0)}. \tag{2.8}$$

Let $\gamma : [0,R] \to M$ be an arc-length geodesic such that $\gamma(0) \in N$ and $\gamma'(0) = \eta(\gamma(0))$, where η is the unit normal vector of N pointing to Ω. Assume also that given any $t \in (0,R]$ and any non zero $v \in \{\gamma'(t)\}^\perp$ it holds

$$K_M(\gamma'(t), v) \leq -\frac{f''(t)}{f(t)},$$

where $K_M(\gamma'(t), v)$ is the sectional curvature of N on the plane determined by γ' and v. Then for any $t \in [0,R]$

$$-H_t(\gamma(t)) = \frac{\Delta d(\gamma(t))}{n-1} \geq \frac{f'(t)}{f(t)},$$

where d is the distance to N.

From Theorem 2.1.3, part A1 and Theorem 2.3.2 we obtain:

Corollary 2.3.3. *Let Ω be a one side normal neighboorhood of N. Assume the hypothesis of theorem above. Then*

$$\lambda(\Omega) \geq \left\{C^{+}(f)\right\}^{-1}.$$

Before proving Theorem 2.3.2, we analyse both theorem and corollary for one side normal neighboorhoods of submanifolds N contained in manifolds whose sectional curvature is bounded by a constant K according to the sign of K. Let $\Omega \subset M$ be a one side normal neighboorhood of a submanifold N.

1. Suppose that $K_M \leq K < 0$ and $\Lambda(N) < -\sqrt{-K}$, then taking

   $$f(t) = \frac{\sinh(\sqrt{-K}(r_0+t))}{\sqrt{-K}} \text{ with } r_0 = \frac{1}{\sqrt{-K}}\text{arccoth}\left(\frac{-\Lambda}{\sqrt{-K}}\right)$$

   it follows from the theorem that

   $$-H_t(\gamma(t)) \geq \sqrt{-K}\coth\left(\sqrt{-K}(r_0+t)\right)$$

   and from the corollary that the first eigenvalue of Ω satisfies

   $$\lambda(\Omega) \geq \left\{C^{+}\left(\frac{\sinh^{n-1}(\sqrt{-K}t)}{\sqrt{-K}}\Big|_{r_0, r_0+R}\right)\right\}^{-1} \tag{2.9}$$

2. Assume now that $K_M \leq 0$ and $\Lambda(N) < 0$, then taking $f(t) = r_0 + t$ wuth $r_0 = 1/(-\Lambda)$, by the theorem

   $$-H_t(\gamma(t)) \geq \frac{1}{r_0+t}$$

   and, by the corollary

   $$\lambda(\Omega) \geq C^{+}\left(t\Big|_{[r_0,r_0+R]}\right)^{-1} =$$

   $$\begin{cases} \left\{\dfrac{(r_0+R)^2 - r_0^2}{2n} + \dfrac{r_0^n}{n(n-2)}\left(\dfrac{1}{(R+r_0)^{n-2}} - \dfrac{1}{r_0^{n-2}}\right)\right\}^{-1} & \text{if } n > 2 \\ \left\{\dfrac{(r_0+R)^2 - r_0^2}{4} + \dfrac{r_0^2}{2}\ln\left(\dfrac{r_0}{r_0+R}\right)\right\}^{-1} & \text{if } n = 2. \end{cases} \tag{2.10}$$

3. If $K_M \leq K$, $K > 0$ and $\Lambda(N) < 0$, taking

$$f(t) = \frac{\sin(\sqrt{K}(r_0 + t))}{\sqrt{K}}, \ t \in [0, \pi/\sqrt{K}] \text{ and } r_0 = \frac{1}{\sqrt{K}}\text{arccot}\left(\frac{-\Lambda}{\sqrt{K}}\right)$$

we obtain

$$-H_t(\gamma(t)) \geq \sqrt{K}\cot\left(\sqrt{K}(r_0 + t)\right)$$

and the eigenvalue estimate is

$$\lambda(\Omega) \geq \left\{C^+\left(\frac{\sin(\sqrt{K}t)}{\sqrt{K}}\Big|_{[r_0, r_0+R]}\right)\right\}^{-1}. \tag{2.11}$$

For proving Theorem 2.3.2 we use a well known Jacobi field comparison result (see [8]). To state it we first introduce some terminology. Let N be a submanifold of M and let γ be a geodesic segment orthogonal to N at $\gamma(0) \in N$. Recall that a Jacobi field J is an N-Jacobi field along the geodesic γ if it is orthogonal to γ, $J(0) \in T_{\gamma(0)}N$ and $S_{\gamma'(0)}J(0) + J'(0)$ is orthogonal to $T_{\gamma(0)}N$. The index form at γ is a bilinear form on the space $\mathcal{L}$ of all broken C^∞ vector fields V along γ, orthogonal to γ with $V(0) \in T_{\gamma(0)}N$ defined by

$$L_b(V, W) = \int_0^b \{\langle V', W'\rangle - \langle R(\gamma', V)\gamma', W\rangle\}(u)du - \langle S_{\gamma'(0)}V(0), W(0)\rangle \tag{2.12}$$

$V, W \in \mathcal{L}$. If there is an N-Jacobi field V along γ, $V \not\equiv 0$, $V(0) = 0$ and $V(b) = 0$, then $\gamma(b)$ is called a focal point of N.

Theorem 2.3.4. *(the basic inequality) Assume that N has no focal points on $\gamma(0, b]$. Then, given $V \in \mathcal{L}$ there is a unique N-Jacobi field J such that $J(b) = V(b)$. Besides, it holds $L_b(V, V) \geq L_b(J, J)$ with the equality occuring if and only if $V = J$.*

Proof of Theorem 2.3.2.
Choose $q \in \Omega$ such that $d(q) = b$, where d is the distance to N, and let $\gamma : [0, b] \to M$ be the minimizing arc length geodesic such that $\gamma(0) \in N$, $\gamma'(0) = \eta(\gamma(0))$ and $\gamma(b) = q$. Then $\text{grad}d(q) = \gamma'(b)$ and

$$\Delta d = \sum_{i=1}^{n}\langle\nabla_{E_i}\text{grad}d, E_i\rangle = \sum_{i=1}^{n-1}\langle\nabla_{E_i}\text{grad}d, E_i\rangle,$$

where $\{E_1, ..., E_{n-1}, E_n = \text{grad}d\}$ is an orthonormal basis of T_qM. Given $F \in T_qM \cap \gamma'(b)^\perp$ unitary, we have

$$\langle \nabla_F \text{grad}d, F\rangle = \langle Y(b), Y'(b)\rangle = L_b(Y, Y),$$

for Y a N-Jacobi field along γ such that $Y(b) = F$. Hence, we have to estimate $L_b(Y, Y)$ from below for any N-Jacobi field Y along γ such that and $Y(b) \in T_qM \cap \gamma'(b)^\perp$ and $|Y(b)| = 1$. It is easy to see that

$$L_b(Y, Y) \geq \frac{f'(0)}{f(0)}|Y(0)|^2 + \int_0^b |Y'(t)|^2 - |Y(t)|^2\frac{f''(t)}{f(t)}dt. \tag{2.13}$$

In order to estimate the right hand of (2.13) we compare it with the index form of a $\widetilde{N}-$Jacobi field of a rotationally symmetric manifold $\widetilde{M} = [0, R] \times \mathbb{S}^{n-1}$ with the Riemannian metric $ds^2 = dt^2 + f(t)^2 d\theta^2$, where $\widetilde{N} = \{0\} \times \mathbb{S}^{n-1}$. Let $\widetilde{\gamma} : [0, b] \to \widetilde{M}$ a geodesic orthogonal to $\widetilde{N}$ parametrized by arc length such that $\widetilde{\gamma}(0) \in \widetilde{N}$ and $\widetilde{\gamma}'(0)$ is orthogonal to $T_{\widetilde{\gamma}(0)}\widetilde{N}$. A $\widetilde{N}$-Jacobi field $\widetilde{Y}$ along $\widetilde{\gamma}$ such that $|\widetilde{Y}(b)| = 1$ has the expression

$$\widetilde{Y}(t) = \frac{f(t)}{f(b)}V(t)$$

where V is a parallel unitary vector field along $\widetilde{\gamma}$ such that $V(0) \in T_{\gamma(0)}\widetilde{N}$. Let $\{e_1(t), ..., e_n(t)\}$ be orthonormal parallel vector fields along γ such that

$$e_1(t) = \gamma'(t),\ e_n(b) = Y(b)$$

and $\{\widetilde{e}_1(t), ..., \widetilde{e}_n(t)\}$ orthonormal parallel vector fields along $\widetilde{\gamma}$ such that

$$\widetilde{e}_1(t) = \widetilde{\gamma}'(t),\ \widetilde{e}_n(b) = \widetilde{Y}(b).$$

If V is a vector field along γ, there are unique functions $g_1, ..., g_n : [0, b] \to \mathbb{R}$, such that

$$V(t) = \sum_{i=1}^n g_i(t)e_i(t).$$

Defining

$$\Phi V(t) = \sum_{i=1}^n g_i(t)\widetilde{e}_i(t)$$

and applying Φ to (2.13), we conclude that:

$$\begin{aligned} L_b(Y,Y) &\geq \frac{f'(0)}{f(0)}|\Phi Y(0)|^2 + \int_0^b |\Phi Y'(t)|^2 - |\Phi Y(t)|^2 \frac{f''(t)}{f(t)} dt \\ &= L_b(\Phi Y, \Phi Y) \geq L_b(\widetilde{Y}, \widetilde{Y}) = \frac{f'(b)}{f(b)}. \end{aligned}$$

To conclude the proof just note that

$$\Delta d(\gamma(b)) = \sum_{i=1}^{n-1} L_b(Y_{E_i}, Y_{E_i}) \geq (n-1)\frac{f'(b)}{f(b)},$$

for Y_{E_i} a N-Jacobi field along γ such that $Y_{E_i}(b) = E_i$.

2.3.1 Applications to non compact manifolds

For the next results of this section we assume that M is non compact (besides complete).

Corollary 2.3.5. *Let M be a Riemannian manifold with sectional curvature $K_M \leq -k^2$ and Ω a bounded domain in M contained in an horoannulus of width R. Then*

$$\lambda(\Omega) \geq \frac{k^2(n-1)^2}{Rk(n-1) - 1 + e^{-(n-1)kR}}.$$

Proof. The corollary is an application of Corollary 2.1.6 for the function $f(x) = \sinh(kx)$. In fact, with this choice of f, we have

$$\begin{aligned} \lambda(\Omega) &\geq \Gamma_R^+(f) \qquad (2.14) \\ &= \limsup_{t\to\infty} \left\{ \int_{t-R/2}^{t+R/2} \frac{\int_{t-R/2}^{s} \sinh^{n-1}(k\tau)d\tau}{\sinh^{n-1}(ks)} ds \right\}^{-1} \\ &= \frac{k^2(n-1)^2}{Rk(n-1) - 1 + e^{-(n-1)kR}}. \end{aligned}$$

□

Definition 2.3.6. *The horowidth $w(\Omega)$ of a bounded domain Ω in a complete non compact Riemannian manifold M is defined by*

$$w(\Omega) = \inf\{R > 0 \mid \Omega \subset A_R(\infty), \text{ for any Busemann function on } M\}.$$

The definition is well-posed since for any $p \in \Omega$,

$$\Omega \subset b^{-1}\left(-\text{diam}(\Omega), \text{diam}(\Omega)\right) = A_{2\text{diam}(\Omega)}(\infty),$$

for any Busemann function b associated to a geodesic ray from p.

Corollary 2.3.7. *If Ω_m is any sequence of bounded domains in a Riemannian manifold M with sectional curvature $K_M \leq -k^2$ such that $\lambda\left(\Omega_m\right)$ converges to zero when m goes to infinity, then the horowidth $w\left(\Omega_m\right)$ converges to infinity.*

Proof. For the sake of contradiction, assume that $w(\Omega_m)$ does not converge to infinity. Then $\{w\left(\Omega_m\right)\}$ has a bounded subsequence and therefore we can assume $w(\Omega_m) \leq L$. Each Ω_m is contained in some horoannulus of width $R_m < L + 1$. By Corollary 2.3.5 above,

$$\begin{aligned}\lambda(\Omega_m) &\geq \frac{k^2(n-1)^2}{R_m k(n-1) - 1 + e^{-(n-1)kR_m}} \\ &\geq \frac{k^2(n-1)^2}{(L+1)k(n-1) - 1 + e^{-(n-1)k(L+1)}} > 0,\end{aligned}$$

a contradiction. □

Corollary 2.3.8. *If the sectional curvature of M is less than or equal to zero, then for each $c > 0$ given, there is a domain Ω with arbitrarily large volume such that $\lambda(\Omega) \geq c$.*

Proof. Considering that the expressions in (2.10) decrease with r_0, letting $r_0 \to \infty$, we conclude that $\lambda(A_{r,r+R}) \geq 2/R^2$ for all $r > 0$. Take $R = \frac{\sqrt{c}}{\sqrt{2}}$ and r large enough for which $\text{Vol}(A^{\mathbb{R}^n}_{r,r+R})$ is greater than a given positive value V. Since

$$\text{Vol}(A^M_{r,r+R}) \geq \text{Vol}(A^{\mathbb{R}^n}_{r,r+R}),$$

choosing $\Omega = A^M_{r,r+R}$ we have $\text{Vol}(\Omega) \geq V$ and $\lambda(\Omega) \geq \frac{2}{R^2} = c$. □

2.4 Estimates assuming a lower bound for the Ricci curvature of M

In this section, by considering a lower bound for the Ricci curvature of M, we obtain estimates from above of the mean curvature of the parallel hypersurfaces to one of the connected component of $\partial\Omega$ which allows the application of part A2 of Theorem 2.1.3. In the next theorem we use Definition 2.3.1.

Theorem 2.4.1. *Consider a one side normal neighboorhood Ω of N with width R. Let η be the normal unit vector of N pointing to Ω. Assume that there is a function $f : [0, R] \to (0, +\infty)$ in $C^2([0, R])$ satisfying:*

i) The mean curvature of N with respect to η satisfies

$$H_N \geq -\frac{f'(0)}{f(0)}.$$

ii) The radial Ricci curvature of M associated to N satisfies

$$\mathrm{Ric}_M(\gamma'(t), \gamma'(t)) \geq -(n-1)\frac{f''(t)}{f(t)},$$

where γ is a arc-length parametrized geodesic such that $\gamma(0) \in N$, $\gamma'(0) = \eta(\gamma(0))$.

Then

$$-H_t(\gamma(t)) = \frac{\Delta d_M(\gamma(t))}{(n-1)} \leq \frac{f'(t)}{f(t)},$$

where d is the distance to N.

This theorem is a generalization to submanifolds of the following mean curvature comparison theorem:

Theorem 2.4.2. *Let M be a Riemannian manifold with Ricci curvature satisfying*

$$\mathrm{Ric}_M \geq \mathrm{Ric}_{\mathbb{Q}^n(K)} = (n-1)K$$

and let d denote the distance function to $p \in M$. Then

$$-H_d \leq -H_d(\mathbb{Q}^n(K)) = (n-1)\frac{C_K(d)}{S_K(d)},$$

where $d_{\mathbb{Q}^n(K)}$ is the distance function in $d_{\mathbb{Q}^n(K)}$ and H_d is the mean curvature of the geodesic sphere of radius d oriented outwards.

The proofs of both theorems are alike. Both compare the Laplacian of the distance function in M with the Laplacian of the distance in a rotationally symmetric manifold. We now demonstrate Theorem 2.4.1.

Proof. Applying formula

$$\frac{1}{2}\Delta\left(|\mathrm{grad}\,\phi|^2\right) = |\mathrm{Hess}\phi|^2 + \langle \mathrm{grad}\,\phi, \mathrm{grad}\,(\Delta\phi)\rangle + \mathrm{Ric}\,(\mathrm{grad}\,\phi, \mathrm{grad}\,\phi)\,,$$

$\phi \in C^3(M)$, which is demonstrated in Chapter I of [27], to the distance function, we obtain

$$\begin{aligned} 0 &= |\mathrm{Hess}d|^2 + \langle \mathrm{grad}d, \mathrm{grad}(\Delta d)\rangle + \mathrm{Ric}(\mathrm{grad}d, \mathrm{grad}d) \\ &\geq \frac{1}{n-1}(\Delta d)^2 + \frac{\partial}{\partial r}(\Delta d) - (n-1)\frac{f''(d)}{f(d)}, \end{aligned} \tag{2.15}$$

where $\partial/\partial r$ means the differentiation in the direction of grad$d$. For $q \in \Omega$, $d(q) = b$, let $p \in N$ such that $d(q,p) = b$. Consider an arc length parametrized, minimizing geodesic $\gamma : [0,b] \to \Omega$ connecting p to q. Define the function

$$\Psi : (0,b] \to \mathbb{R},\ \Psi(t) = \Delta d(\gamma(t)).$$

So, by (2.15),

$$0 \geq \frac{1}{n-1}(\Psi)^2(t) + \Psi'(t) - (n-1)\frac{f''(t)}{f(t)}\ \forall\, t \in (0,b].$$

Besides,

$$\lim_{t\to 0}\Psi(t) = -H_N\left(\gamma(0)\right),$$

the mean curvature of N with respect to the normal vector $\eta(\gamma(0)) = \gamma'(0)$. Now consider $\widetilde{M}$ the rotationally symmetric manifold $[0,R]\times\mathbb{S}^{n-1}$ with metric $dt^2 + f(t)^2 d\theta^2$ and define $\widetilde{N} = \{0\} \times \mathbb{S}^{n-1}$. Consider $\widetilde{\gamma} : [0,b] \to \widetilde{M}$ be a minimizing geodesic orthogonal to $\widetilde{N}$ connecting $\widetilde{p} \in \widetilde{N}$ to $\widetilde{q}$. Then we define

$$\Psi_f(t) = \Delta(d_{\widetilde{M}})$$

and observe that for Ψ_f we have equality in (2.15), so that

$$0 = \frac{1}{n-1}(\Psi_f)^2 + \Psi_f' - (n-1)\frac{f''(t)}{f(t)}\ \forall\, t \in (0,b].$$

As in M,

$$\lim_{t\to 0}\Psi_f(t) = -H\left(\widetilde{\gamma}(0)\right) = (n-1)\frac{f'(0)}{f(0)}.$$

Hence there are two ODE with comparable initial conditions.

$$(1)\begin{cases} (n-1)\dfrac{f''}{f} \geq \dfrac{1}{n-1}(\Psi)^2 + \Psi' \;\; \forall\, t \in (0,b] \\ \lim\limits_{t\to 0} \Psi(t) = -H\left(\gamma(0)\right) \leq -H\left(\widetilde{\gamma}(0)\right), \end{cases}$$

$$(2)\begin{cases} (n-1)\dfrac{f''}{f} = \dfrac{1}{n-1}(\Psi_f)^2 + \Psi_f' \;\; \forall\, t \in (0,b] \\ \lim\limits_{t\to 0} \Psi_f(t) = -H\left(\widetilde{\gamma}(0)\right) \end{cases}$$

Comparing the ODE's and their initial conditions, one concludes that

$$\Psi(t) \leq \Psi_f(t)\; \forall t \in (0,b].$$

Hence, the proof of the theorem is complete. □

Corollary 2.4.3. *Consider a one side normal neighboorhood Ω of N with width R. Let η be the normal unit vector of N pointing to Ω. Suppose that the mean curvature of N with respect to η is less than or equal to the mean curvature of a geodesic sphere of radius r in $\mathbb{Q}^n(K)$ oriented outwards. Assume also that the radial Ricci curvature associated to N satisfies*

$$\mathrm{Ric}_M(\gamma', \gamma') \geq \mathrm{Ric}_{\mathbb{Q}^n(K)} = (n-1)K,$$

where γ is a arc-length parametrized geodesic satisfying $\gamma(0) \in N$, $\gamma'(0) = \eta(\gamma(0))$. Then

$$\lambda \geq \left\{ C^-(S_K\big|_{[r,r+R]}) \right\}^{-1}. \tag{2.16}$$

2.4.1 Applications to non compact manifolds

Up to the end of this section, we assume $\mathrm{Ric}_M \geq (n-1)K$ with $K = -k^2 < 0$. By Theorem 2.4.2,

$$\Delta d_M \leq (n-1)\frac{k\cosh(kd)}{\sinh(kd)},$$

which gives the eigenvalue estimate

$$\lambda(A_{r,r+R}) \geq l(r,R) := \left\{ C^- \left(\frac{\sinh(k\tau)}{k}\big|_{[r,r+R]} \right) \right\}^{-1}$$

for an annulus of inner radius r and outer radius $r+R$.

The next result and its proof are analogous of Corollary 2.3.5, applying now the second case of Corollary 2.1.6.

Corollary 2.4.4. *If Ω is a bounded domain in M contained in an horo-annulus of width R then*

$$\lambda(\Omega) \geq \frac{k^2(n-1)^2}{e^{k(n-1)R} - 1 - k(n-1)R}.$$

The next two results, relating the volume of the domain with its first eigenvalues, are consequences of Corollary 2.1.6.

Corollary 2.4.5. *Assume that* $\mathrm{Vol}(B_r) \geq Cr^\alpha$, *for* $r >> 0$, *where* $C > 0$ *and* $\alpha > 1$ *are constants. Then, given* $L, V > 0$, *there are domains* $\Omega \subset M$, *such that* $\mathrm{Vol}\,(\Omega) \geq V$ *and* $\lambda(\Omega) \geq L$.

Proof. Given $L > 0$, take R_0 such that

$$\frac{k^2(n-1)^2}{e^{k(n-1)R_0} - 1 - k(n-1)R_0} > L + 1.$$

Since

$$\lim_{r\to\infty} l(r, R_0) = \frac{k^2(n-1)^2}{e^{k(n-1)R_0} - 1 - k(n-1)R_0},$$

there is a large enough r_0 for which $l(r_0, R_0) > L$. As $l(r, R)$ is an increasing function of r,

$$\lambda(A_{r,r+R_0}) \geq l(r, R_0) \geq l(r_0, R_0),\ \forall r \geq r_0.$$

Since $\mathrm{Vol}(A_{r,r+R_0}) \geq Dr^{\alpha-1}R_0$ for a positive constant D, the corollary is demonstrated by taking $r > r_0$ large enough. □

The next result has a similar proof to the one of Corollary 2.3.7, now applying Corollary 2.4.4 above.

Corollary 2.4.6. *If Ω_m is any sequence of bounded domains in M such that $\lambda\,(\Omega_m)$ converges to zero when m goes to infinity, then the horowidth $w\,(\Omega_m)$ of Ω_m converges to infinity.*

2.5 Domains in rank 1 symmetric spaces

In this section we obtain some explicit estimates for geodesic balls and annuli in rank 1 symmetric spaces. We begin with the simplest case of spatial forms.

2.5.1 Spatial forms

Usually eigenvalues estimates can be more easily obtained in domains with highly symmetric boundary (see, for example [3]). However, with Theorem 2.1.3, one has effective estimates for domains with asymmetric boundary. We give an example in the Euclidean space. Let $\Omega \subset \mathbb{R}^3$ be a region bounded by an ellipsoid with minor axis 2a and major axis 2c and an outer parallel surface of distance R. From Theorem 2.1.3 we obtain

$$\lambda\left(\Omega\right) \geq \frac{6(c^2 + Ra)}{R^2(3c^2 + Ra)}.$$

As a way of giving some more justification on our above comment, we note that enclosing Ω in a circular annulus A and comparing the eigenvalue of Ω with the one of A we obtain

$$\lambda\left(\Omega\right) \geq \lambda\left(A\right) = \frac{\pi^2}{\left(R + c - a\right)^2},$$

which is worse than the previous one if $c - a$ is big.

Consider an horoannulus A in $\mathbb{H}^n(k)$, $k > 0$ such that the sectional curvature of the space is $-k^2$. Let A be bounded by two concentric horospheres, which are at a distance R. If $\Omega \subset A$ is a connected, open and bounded subset, then

$$\lambda(\Omega) \geq \frac{(n-1)^2k^2}{(n-1)kR + e^{-(n-1)kR} - 1}. \tag{2.17}$$

To see this, let r go to infinity in the expression of λ given by Theorem 2.1.3 with $f(r) = \sinh(kr)$:

$$\lambda(r,R) \geq \left\{ \int_0^R \frac{\int_0^t \sinh^{n-1}(k(r+s))ds.}{\sinh^{n-1}(k(r+t))} dt. \right\}^{-1}.$$

Since the estimate for an annulus of inner radius r is worse than (2.17), estimate (2.17) works for any annulus of $\mathbb{H}^n(k)$. Besides, the estimates obtained for annulus in $\mathbb{H}^n(k)$ are valid for annulus in manifolds of sectional curvature bounded from above by $-k^2$.

Applying Theorem 2.1.3 - part B for the particular case of a ball of radius R in $\mathbb{H}^2$, we estimate that

$$\lambda \geq \frac{1}{-R - 2\ln 2 + 2\ln(e^R + 1)},$$

which is not the best estimate neither for small values of R, nor for big ones. If the radius is less than 3, Ling [23]

$$\lambda \geq \frac{\pi^2}{R^2} - \frac{1}{2},$$

provides a better estimate and if it is greater than 5.4, Yau's result [30]

$$\lambda \geq \frac{\coth^2(R)}{4}$$

is better. But for the intermediate values of r, the authors didn't find any result that gives better estimates.

The last example in a spatial form is an annulus in the unit sphere $\mathbb{S}^2$. We must assume that $r + R < \pi$. Applying Theorem 2.1.3,

$$\lambda \geq \frac{1}{\frac{\cos(r)}{2}\left[\ln\left(\frac{1-\cos(r+R)}{1+\cos(r+R)}\right) - \ln\left(\frac{1-\cos(r)}{1+\cos(r)}\right)\right] - \ln\left(\frac{\sin(r+R)}{\sin(r)}\right)}.$$

This is an explicit estimate for the first eigenvalue of a doubly connected domain in $\mathbb{S}^2$ which is not mean convex in both connected components of its boundary.

2.5.2 Compact rank 1 symmetric spaces other than spatial forms

Since the mean curvature of geodesic spheres in rank 1 symmetric spaces can be explicitly computed in terms of their radius (see [9]), we may use our results (Theorem 2.1.3) to obtain explicit estimates of the first eigenvalue of their geodesic balls and annuli. We consider here only the case of compact spaces but clearly similar estimates can be obtained for balls and annuli in non compact rank one symmetric spaces.

Proposition 2.5.1. *Denote by* $\mathbb{CP}^n$ *the complex projective space of constant holomorphic sectional curvature* 4, $\mathbb{HP}^n$ *the quaternionic projective space of constant quaternionic sectional curvature* 4 *and* $\mathbb{C}a\mathbb{P}^2$ *the Cayley plane of constant Cayley sectional curvature* 4.

1. *If* B_R *is a geodesic ball with radius* R *of* $\mathbb{CP}^n$ *then*

$$\lambda(B_R) \geq \frac{-2n}{\ln(\cos R)}.$$

2. *If $A_{r,r+R}$ is an annulus in $\mathbb{CP}^n$, with inner radius r and outer radius $r+R$, then*

$$\lambda(A_{r,r+R}) \geq 2n \max\left\{\left\{\ln\left(\frac{\cos r}{\cos(r+R)}\right) - \sin^{2n} r \int_r^{r+R} \frac{dt}{\sin^{2n-1} t \cos t}\right\}^{-1},\right.$$
$$\left.\left\{-\ln\left(\frac{\cos r}{\cos(r+R)}\right) + \sin^{2n}(r+R) \int_r^{r+R} \frac{dt}{\sin^{2n-1} t \cos t}\right\}^{-1}\right\},$$

3. *If B_R is a geodesic ball with radius R of $\mathbb{HP}^n$ then*

$$\lambda(B_R) \geq \left\{C^+\left(\sin^{4n-1}(s)\cos^3(s),\ s \in [0,R]\right)\right\}^{-1}.$$

4. *If $A_{r,r+R}$ is an annulus in $\mathbb{HP}^n$, with inner radius r and outer radius $r+R$, then*

$$\lambda(A_{r,r+R}) \geq \max\left\{\left\{C^+\left(\sin^{4n-1}(s)\cos^3(s), s \in [r,r+R]\right)\right\}^{-1},\right.$$
$$\left.\left\{C^-\left(\sin^{4n-1}(s)\cos^3(s),\ s \in [r,r+R]\right)\right\}^{-1}\right\}.$$

5. *If B_R is a geodesic ball with radius R of $\mathbb{C}a\mathbb{P}^2$ then*

$$\lambda(B_R) \geq \left\{C^+\left(\sin^{15}(s)\cos^7(s),\ s \in [0,R]\right)\right\}^{-1}.$$

6. *If $A_{r,r+R}$ is an annulus in $\mathbb{C}a\mathbb{P}^2$, with inner radius r and outer radius $r+R$, then*

$$\lambda(A_{r,r+R}) \geq \max\left\{\left\{C^+\left(\sin^{15}(s)\cos^7(s),\ s \in [r,r+R]\right)\right\}^{-1},\right.$$
$$\left.\left\{C^-\left(\sin^{15}(s)\cos^7(s),\ s \in [r,r+R]\right)\right\}^{-1}\right\}.$$

Geometric Interpretation - part II

As an example of what has been commented in Subsection 2.1.1, in the complex projective space $\mathbb{CP}^2$ there are annuli for which the part A1 of Theorem 2.1.3 gives a better estimate and for which part A2 is better. Applying Equation

$$\frac{\Delta d(x)}{3} = \frac{f'(d(x))}{f(d(x))}$$

for the d the distance to a point, the solution is

$$f(t) = \sin t \cos^{\frac{1}{3}} t,$$

which is not a increasing function in the hole interval $[0, \frac{\pi}{2}]$. Taking $r = \frac{\pi}{4}$ and $R = \frac{\pi}{8}$ the first part (A1) gives $\lambda \geq 13.79$ while the second (A2) gives $\lambda \geq 12.12$, approximately. On the other hand, for $r = \frac{\pi}{3}$ and $R = \frac{\pi}{12}$ part A1 gives $\lambda \geq 26.12$ while part A2 gives $\lambda \geq 32.36$, approximately.

In [9] the mean curvature of the hypersurfaces parallel to totally geodesic submanifolds in rank 1 symmetric spaces is also calculated. This provides (Theorem 2.1.3) eigenvalue estimates for normal neighboorhoods of these submanifolds.

Chapter 3

The existence and the profile of eigenfunctions associated to the first eigenvalue of $\mathbb{H}^n$ in subsets of $\mathbb{H}^n$

In this chapter we study eigenfunctions of non compact sets of the hyperbolic space.

Recall that if Ω is a compact set and $\lambda(\Omega)$ is its first eigenvalue, then there is a unique, up to scalar multiplication, positive function $u : \Omega \to \mathbb{R}$, satisfying $-\Delta u = \lambda(\Omega)u$ in Ω and $u = 0$ on $\partial\Omega$. In Definition 1.3.4, the first eigenvalue of a non compact set was presented. From that definition, it is not clear whether there is a first eigenfunction associated to $\lambda(\Omega)$, fact that inspired us to inquire some questions. To start we will consider a first eigenfunction of a non compact set a solution of

$$\begin{cases} -\Delta u = \lambda(\Omega)u \text{ in } \Omega \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

not requiring that u is bounded nor that it converges to zero at infinity. Then we ask: Is there a first eigenfunction associated to the first eigenvalue of a non compact domain? Or more, is there a positive, bounded first eigenfunction associated to the first eigenvalue of a non compact domain? If there is, is it unique and how does it behave at infinity?

For example, the first eigenvalue of $\mathbb{R}^n$ is zero. Consequently, its first eigenfunction is a harmonic function. Liouville's Theorem states that any bounded harmonic function in $\mathbb{R}^n$ is constant. This makes looking for positive bounded eigenfunctions in $\mathbb{R}^n$ useless. We decided to see how this works in the hyperbolic space, which is the simplest non compact space, after the Euclidean. The hyperbolic space and some very symmetric subsets of it were presented in Section 1.2. For this sets we added a question to our list. Are there eigenfunctions of non compact domains that present the symmetry of the domain? Another point of interest is to determine whether an eigenfunction of M can be extended continuously to the asymptotic boundary of M (defined in Section 1.2), as the zero function.

The first eigenvalue of $\mathbb{H}^n$ or of any subset of it that contains arbitrary large balls was presented by McKean in [26], $\lambda_n = \frac{(n-1)^2}{4}$.

This chapter divides into two parts, the first focuses in finding symmetric eigenfunctions, i. e., functions that depend only on the distance to a point, if we are considering the whole space, or that depend only on the distance to the boundary, for the other cases. We prove the existence of bounded eigenfunctions associated to the whole space $\mathbb{H}^n$, to the exterior of balls, to the exterior of horoballs and to hyperballs. But we do not always have the continuous extension to the asymptotic boundary or the positivity of the eigenfunction. For odd dimensions, we also present an explicit expression for the solutions after a change of variables. The method applied to obtain these results is to transform the problem in an ODE problem, where the variable is the distance function and then study the ODE. The first part ends by proving that if a domain is contained in a horoball, it does not admit a bounded non trivial eigenfunction.

A natural question is what domains in $\mathbb{H}^n$ admit eigenfunctions associated to λ_n. We conclude that the answer is related to the question "how large is the asymptotic boundary of this domain?". From the first part of this chapter, we know that if the asymptotic boundary is so small that the domain is contained in a horoball, then it does not admit a bounded eigenfunction. On the other hand, in Section 3.2, we prove that if the domain Ω contains an open subset of $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ that intercepts $\partial_\infty \mathbb{H}^n$, it admits an eigenfunction that extends continuously to the zero function at the asymptotic boundary. Nevertheless, for $n \geq 4$, there are special hyperannuli (the region bounded by two parallel hyperspheres), which have only two points at infinity and that admit bounded eigenfunctions without continuous extension to the boundary at infinity.

3.1 Solutions with some symmetry

In this section, we investigate the existence of solutions for several domains of $\mathbb{H}^n$, namely the whole space and subsets whose boundary are umbilical hypersurfaces of $\mathbb{H}^n$.

3.1.1 Global Solutions

The problem

$$\begin{cases} \Delta u = -\lambda_n u \text{ in } \mathbb{H}^n \\ u \geq 0 \text{ is a bounded radially symmetric function around } o \in \mathbb{H}^n \end{cases}$$

has solutions, which are presented as integral formulas in [18]. There, the authors also exhibit unbounded functions defined in $\mathbb{H}^n \backslash \{o\}$. We study them here, following a different approach, since they are useful to construct explicit solutions of the problem defined outside a ball.

Any radial eigenfunction satisfies (1.4) for $r > 0$, where $r(x) = dist(x, o)$. Hence, making $s = \cosh r$, we obtain

$$(s^2 - 1)u'' + nsu' + \lambda_n u = 0, s \in (1, \infty).$$

Then we have to study

$$T_n(v) = -\lambda_n v,$$

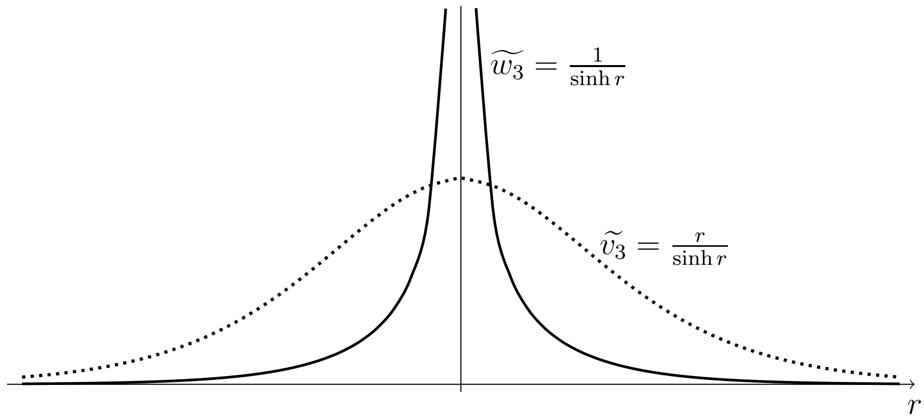
where $T_n(v) = (s^2 - 1)v'' + nsv'$. Our aim is to demonstrate the next theorem.

Theorem 3.1.1. *The problem $T_n(v) = -\lambda_n v$ in $(1, \infty)$ has two linearly independent solutions, v_n and w_n, that are positive decreasing functions and converge to zero as s goes to infinity. v_n corresponds to a global radial eigenfunction of $\mathbb{H}^n$ and w_n to a Green function of $\mathbb{H}^n$. Besides, for odd dimension, we have*

$$v_3(s) = \frac{\text{arccosh(s)}}{\sqrt{s^2 - 1}} = \frac{\ln\left(s + \sqrt{s^2 - 1}\right)}{\sqrt{s^2 - 1}}, \qquad w_3(s) = \frac{1}{\sqrt{s^2 - 1}}$$

$$v_{2m+3}(s) = (-1)^m \frac{d^m}{ds^m} v_3(s) \text{ and } w_{2m+3}(s) = (-1)^m \frac{d^m}{ds^m} w_3(s).$$

As a function of the distance to $o \in \mathbb{H}^3$, v_3 and w_3 have the following profiles:



In order to organize the proof of the theorem, we divide it in lemmas.

Lemma 3.1.2. *If v is a solution of $T_n(v) = -\lambda_n v$, then $u(r) = v(\cosh r)$ is a radial eigenfunction of the Laplacian in $\mathbb{H}^n \backslash \{o\}$. If v extends continuously to $[1, \infty)$, then u is a global eigenfunction.*

Proof. The first conclusion is a direct result of a change of variable. To prove the second one, it is sufficient to show that $u(r) = v(\cosh r)$ is a weak solution of $-\Delta u = \lambda_n u$ and, then, by the regularity theory of elliptic equation, it is a classical solution. For that, given a test function $\phi \in C_0^\infty(\mathbb{H}^n)$, apply the divergence theorem in some annulus $B_R(o)\backslash B_r(o)$ with the support of ϕ contained in $B_R(o)$, make $r \to 0$ and use the continuity of u at o. □

As a consequence of the above lemma, a solution v of $T_n(v) = -\lambda_n v$ has value zero for at most one $t > 1$. Otherwise a compact set (annulus) would have an eigenfunction of eigenvalue λ_n, a contradiction. Besides, if v extends continuously to one, then it cannot assume value zero.

Lemma 3.1.3. *If v is a solution of $T_n(v) = -\lambda_n v$, then v' is a solution of $T_{n+2}(v) = -\lambda_{n+2} v$.*

Differentiating the ODE, one proves the lemma.

From now on, we will look at the ODE $T_n(v) = -\lambda_n v$.

Lemma 3.1.4. *If v is a solution of $T_n(v) = -\lambda_n v$, that is bounded in $(1, 1+\epsilon)$ for a positive ϵ, then v' is also bounded in this interval. Hence, v is Lipschitz and the limit as $s \to 1$ of $v(s)$ exists.*

Proof. Assume that $|v| \leq M$ in $(1, 1+\epsilon)$. Since $T_n(v) = -\lambda_n v$,

$$\big((s^2-1)^{n/2}v'(s)\big)' = -\lambda_n(s^2-1)^{n/2-1}v(s). \tag{3.2}$$

For $1 < \rho < s < 1+\epsilon$,

$$\big((s^2-1)^{n/2}v'(s)\big) - \big((\rho^2-1)^{n/2}v'(\rho)\big) = -\lambda_n \int_\rho^s (t^2-1)^{n/2-1}v(t)dt. \tag{3.3}$$

Using that v is bounded, the right-hand side of (3.3) converges as $\rho \to 1$. Hence, there is the limit

$$\lim_{\rho \to 1^+} (\rho^2-1)^{n/2}v'(\rho) = L.$$

Observe that the bounds on v imply that

$$\big|\lambda_n \int_1^s (t^2-1)^{n/2-1}v(t)dt\big| \leq \frac{\lambda_n M}{n}(s^2-1)^{n/2}. \tag{3.4}$$

If $L = 0$, then

$$\begin{aligned} \big|(s^2-1)^{n/2}v'(s)\big| &= \big|-\lambda_n \int_1^s (t^2-1)^{n/2-1}v(t)dt\big| \\ &\leq \frac{\lambda_n M}{n}(s^2-1)^{n/2}. \end{aligned}$$

and

$$\big|v'(s)\big| \leq \frac{\lambda_n M}{n}$$

If $L \neq 0$, we get a contradiction. Indeed, making $\rho \to 1$ in (3.3) and using (3.4),

$$-\frac{\lambda_n M}{n}(s^2-1)^{n/2} \leq (s^2-1)^{n/2}v'(s) - L \leq \frac{\lambda_n M}{n}(s^2-1)^{n/2}.$$

Isolating $v'(s)$ and integrating in $(\rho_1, s_1) \subset (1, s)$,

$$\begin{aligned} -\frac{\lambda_n M}{n}(s_1-\rho_1) + L\int_{\rho_1}^{s_1}(s^2-1)^{-n/2}ds \quad &\leq v(s_1) - v(\rho_1) \\ \leq \frac{\lambda_n M}{n}(s_1-\rho_1) + L\int_{\rho_1}^{s_1}(s^2-1)^{-n/2}ds. \end{aligned}$$

If $L > 0$, the first inequality yields a contradiction ($\infty \leq 2M$) as $\rho_1 \to 1$. If $L < 0$, a contradiction ($2M \leq -\infty$) is obtained from the second inequality as $\rho_1 \to 1$. Hence $L = 0$ and

$$\left|v'(s)\right| \leq \frac{\lambda_n M}{n}$$

□

As a consequence, the set of bounded solutions of $T_n(v) = -\lambda_n v$ in $(1, \infty)$ has at most dimension 1. If there were v and w two bounded linearly independent solutions of the problem, $v - Cw$, for some constant $C \in \mathbb{R}$ would be an eigenfunction of a compact ball of eigenvalue λ_n, which cannot exist.

Lemma 3.1.5. *For $n \geq 2$, $\mathbb{H}^n$ has a global positive eigenfunction. It is a decreasing function of the distance to o and does not admit any critical point.*

Proof. From Lemmas 3.1.3 and 3.1.4, it is enough to prove the result for $n = 2$ and $n = 3$. If $n = 3$,

$$v_3(s) = \frac{\text{arccosh(s)}}{\sqrt{s^2 - 1}} = \frac{\ln\left(s + \sqrt{s^2 - 1}\right)}{\sqrt{s^2 - 1}},$$

is a bounded positive eigenfunction. It extends continuously to $s = 1$ and, using Lemma 3.1.2, it corresponds to a global eigenfunction.

In order to establish a solution v_2, we use the Frobenius method to solve ODE's. v_2 satisfies

$$(s^2 - 1)v_2'' + nsv' + (1/4)v = 0.$$

The method consists in assuming that

$$v_2(s) = \sum_{j=0}^{\infty} a_j (s-1)^{j+r} \text{ with } a_0 \neq 0.$$

Substituting it on the ODE, we conclude that $r = 0$ and obtain a first solution,

$$v_2(s) = \sum_{j=0}^{\infty} a_j (s-1)^j,$$

where

$$a_{j+1} = -\frac{(2j+1)^2}{8(j+1)^2} a_j.$$

This solution extends continuously to $[1, \infty)$ and a change of variable concludes the proof of the existence of a global eigenfunction.

Observe that if v is a solution of $T_n(v) = -\lambda_n v$, that extends continuously to $[1, \infty)$, then applying lemma 3.1.4 to v', we conclude that the limit $\lim_{s\to 1} v'(s)$ exists and by induction, it exists for all derivatives of v. Hence $\lim_{s\to 1} v'(s) = -a < 0$ by the ODE analyzed as $s \to 1$. Again looking at the ODE $T_n(v) = -\lambda_n v$, one concludes that any critical point of v is a maximum point, since $v > 0$. Consequently, there are no critical points and v is a non increasing function of s. Since the change of coordinates is increasing, the eigenfunction must be like v. □

Lemma 3.1.6. *The equation $T_n(v) = -\lambda_n v$ has an unbounded solution v with $\lim_{s\to 1^+} v(s) = +\infty$.*

Proof. Since the ODE is linear of second order, there are two linearly independent solutions. Let w_n be a solution linearly independent to v_n. w_n must be unbounded close to 1, because of Lemma 3.1.4. Since w_n is zero at at most one point, we can assume that there is $s_1 > 0$, such that $w_n > 0$ in $(1, s_1)$. If there is a critical point in $(1, s_1)$, it must be a maximum point. Hence w_n has at most one critical point and there is an interval $(1, s_2)$ in which w_n is monotone. A monotone unbounded positive function in $(1, s_1]$ has to satisfy

$$\lim_{s\to 1^+} w_n(s) = +\infty.$$

□

Lemma 3.1.7. *If v is a solution of $T_n(v) = -\lambda_n v$, then*

$$\lim_{s\to\infty} v(s) = 0$$

Proof. Since the derivative of v is also an eigenfunction, v has at most one critical point. Hence it is monotone for s sufficiently large. Consequently, if the limit is not zero, there are $L > 1,\ M > 0$ such that $s > L \ \Rightarrow\ |v(s)| > M$. We claim that this cannot happen. Manipulating expression (3.2) like we did in the proof of Lemma 3.1.4, we conclude that if $v(s) \le -M$ for all $s > L$, then

$$v'(s) \ge \frac{M\lambda_n}{ns} + \left(\frac{L^2-1}{s^2-1}\right)^{n/2} \left(v'(L) - \frac{M\lambda_n}{ns}\right) = \frac{M\lambda_n}{ns} + O(s^n)$$

and integrating in $[l, s]$,

$$v(s) \geq \frac{M\lambda_n}{n} \ln\left(\frac{s}{L}\right) + O(s^{n-1}),$$

yielding a contradiction when $s \to \infty$.

If $v(s) \geq M$ for all $s > L$, then

$$v'(s) \leq \frac{-M\lambda_n}{ns} + \left(\frac{L^2-1}{s^2-1}\right)^{n/2} \left(v'(L) + \frac{M\lambda_n}{ns}\right).$$

and a the contradiction is obtained analogously. □

At this point, all the claims from Theorem 3.1.1 about the global eigenfunction v_n have been demonstrated. We now focus on the unbounded solution of the eigenvalue problem.

Lemma 3.1.8. *The problem* $T_n(v) = -\lambda_n v$, *has a positive decreasing solution in* $(1, \infty)$ *with* $\lim_{s\to 1^+} v(s) = +\infty$.

This proof follows from an estimate from bellow for the bounded positive solution v_n presented in Lemma 3.1.5, which will be demonstrated separately in the next lemma.

Lemma 3.1.9. *If* v_n *is a bounded positive solution of* $T_n(v) = -\lambda_n v$, *then there is a constant* $C > 0$, *such that*

$$v_n(s) \geq C s^{-\frac{(n-1)}{2}} \ln s \;\forall\; s > 1. \tag{3.5}$$

Proof. We claim that $v_n'' > 0$ in $(1, \infty)$. Observe that from Lemma 3.1.4, v_n'' is bounded close to 1. Hence v_n'', that is a solution of $T_{n+4}(v) = -\lambda_n v$, converges to zero at infinity from Lemma 3.1.7. Therefore v_n'' is bounded in $(1, \infty)$ and from the remark after lemma 3.1.2 it does not change sign. Then, using that v_n is positive and goes to zero at infinity, we get $v_n'' > 0$. Thus

$$0 = T_n(v_n)(s) = (s^2-1)v_n''(s) + nsv_n'(s) + \lambda_n v_n(s) \leq s^2 v_n''(s) + nsv_n'(s) + \lambda_n v_n(s)$$

Let g_n be the function for which equality holds, i. e.,

$$0 = g_n''(s) + nsg_n'(s) + \lambda_n g_n(s).$$

Hence

$$(s^n v_n')' \geq -\lambda_n s^{n-2} v_n$$

and equality holds for g_n. We claim that if $v_n(1) = g_n(1)$ and $v_n > g_n$ in a neighboorhood of $s = 1$, then $v_n > g_n$ in $(1, \infty)$.

Take $u_n(s) = v_n(s) - g_n(s)$, which is zero at $s = 1$. For the sake of contradiction, assume that there is $s_1 > 1$ for which $u_n(s_1) = 0$ and $u_n > 0$ in $(1, s_1)$. Then

$$(s^n u_n')' \geq -\lambda_n s^{n-2} u_n$$

implies that

$$\frac{\int_1^{s_1} s^n (u_n')^2}{\int_1^{s_1} s^{n-2} u_n^2} \geq \lambda_n = \frac{(n-1)^2}{4}$$

But

$$\Lambda_n := \inf \left\{ \frac{\int_1^{s_1} s^n (u')^2}{\int_1^{s_1} s^{n-2} u^2} \mid u \in H_0^1\left([1, s_1]\right) \right\} > \lambda_n.$$

The function u that minimizes the above quotient satisfies the Cauchy-Euler ODE $s^2 u'' + nsu' + \Lambda_n u = 0$. Analysing the solutions for this ODE for different values of Λ_n, we conclude that it only can have a solution with two zero points in $[1, s_1]$ if $\Lambda_n > \lambda_n$.

We conclude that $u_n > 0$, demonstrating the claim.

From Lemma 3.1.4, the limit of the k^{th} derivative of v_n as $s \to 1$ exists and we will refer to it as $v_n^{(k)}(1)$. Since the eigenfunction is a decreasing function of the radius, the sign of $v_n^{(k)}(1)$ is positive for k even and negative for k odd. From the ODE $T_n(v_n) = -\lambda_n v_n$,

$$v_n'(1) = \frac{-\lambda_n v_n(1)}{n}.$$

Let us take g_n the solution of

$$\begin{cases} s^2 u'' + nsu' + \lambda_n u = 0 \\ u(1) = v_n(1) \\ u'(1) = v_n'(1) = \frac{-\lambda_n v_n(1)}{n}, \end{cases}$$

then $g_n''(1) = 0 < v_n''(1)$. Hence u_n is positive in a neighborhood of 1 and, by the claim, must be positive in $(1, \infty)$.

Solving the ODE, the function g_n is given by

$$g_n(s) = v_n(1)s^{-(n-1)/2}\left(\frac{(n-1)}{2n}\ln s + 1\right).$$

Finally,

$$v_n(s) \geq g_n(s) = v_n(1)s^{-(n-1)/2}\left(\frac{(n-1)}{2n}\ln s + 1\right) \geq Cs^{(n-1)/2}\ln s$$

for some positive constant C. □

We are now ready to prove Lemma 3.1.8.

Proof. Using D'Alembert's method we look for w, the second linearly independent solution of $T_n(v) = -\lambda_n v$, which is given by $w = f.v_n$, for some function f. Solving the ODE, we conclude that

$$f'(s) = \frac{C}{v_n^2(s)(s^2-1)^{n/2}}$$

and f can be taken as

$$f(s) = \int_S^s \frac{1}{v_n^2(t)(t^2-1)^{n/2}},$$

which is negative in $(1, S)$ and positive in (S, ∞). By the bound (3.5), the function f must be bounded in (S, ∞) and there is a positive constant K for which $f(s) \leq K$.

Hence a solution of the problem $T_n(v) = -\lambda_n v$ that is linearly independent from v_n is $w(s) = f(s)v_n(s)$. We have that $w(s) \leq Kv_n(s)$ $\forall s \geq 1$ and $w(s) < 0$ in $(1, S)$, therefore the function $w_n(s) = Kv_n(s) - w(s)$ is a positive unbounded solution of $T_n(v) = -\lambda_n v$. The argument presented in the proof of Lemma 3.1.6 implies that $\lim_{s\to 1^+} v(s) = +\infty$. □

To conclude the proof of Theorem 3.1.1, it only remains to observe that w_n, the positive unbounded eigenfunction, is a decreasing function. Since it is positive, the ODE implies that its critical points are all maximum points. But it goes to $+\infty$ as $s \to 1$. Then it does not admit any critical point and must be a decreasing function.

3.1.2 Solutions in the exterior of balls

Theorem 3.1.10. *The problem*

$$\begin{cases} \Delta u = -\lambda u \text{ in } \mathbb{H}^n \backslash B_R(0) \\ u = 0 \text{ in } \partial B_R(0) \\ u \geq 0 \text{ is a bounded function.} \end{cases}$$

has a radial solution for any $R \in (0, \infty)$. It is given by a linear combination of the two linearly independent solutions presented in the last section. Besides, this solution vanishes at the asymptotic boundary of $\mathbb{H}^n$.

Proof. Given $R \in (0, \infty)$, the solution u of the problem above must satisfy (1.4) with boundary condition $u(R) = 0$. Consequently it corresponds to a solution of $T_n(v) = -\lambda_n v$ in $(\cosh(R), \infty)$ with initial condition $v_n(\cosh R) = 0$, $v_n'(\cosh R) > 0$.

In Section 3.1.1, the ODE $T_n(v) = -\lambda_n v$ was studied and we demonstrated that it has two linearly independent positive decreasing solutions, both bounded in $[\cosh(R), \infty)$. A solution in $[\cosh(R), \infty)$ is just a linear combination of these two solutions, hence it converges to zero as s goes to infinity. ☐

Remark 3.1.11. *The radial function u viewed as a function of the distance to $\partial B_R(0)$ starts as an increasing function, attains its maximum value and then decreases converging to zero at infinity.*

The remark is justified by the fact that from (1.4) at any critical point of u, $u''(r) = -\lambda u(r) < 0$. Hence u has at most one critical point and behaves as described in the remark.

We also present two lemmas that compare radial eigenfunctions associated to different domains.

Lemma 3.1.12. *Consider the problem P_R :*

$$\begin{cases} -\Delta u = \lambda u \text{ in } \mathbb{H}^n \backslash B_R(o) \\ u = 0 \text{ on } \partial B_R(o) \\ u' = 1 \text{ on } \partial B_R(o) \end{cases}$$

Let u be the solution of P_r and v a solution of P_R with $R > r$. Then the function $\frac{v}{u}$ defined in $\mathbb{H}^n \backslash B_R(o)$ is a non decreasing function of the distance to o.

Proof. Observe that

$$\left(\frac{v}{u}\right)'(R) = \frac{v'(R)u(R) - u'(R)v(R)}{u^2(R)} \geq 0,$$

hence the function $\frac{v}{u}$ starts increasing. Assume for a contradiction that it is not an non decreasing function. Then it has a first local maximum at distance S from the origin. Define $C = \frac{v(S)}{u(S)}$ and consider the function $\widetilde{v}(x) = \frac{v(x)}{C}$. It is an eigenfunction that touches the function u from below. This contradicts the fact that the eigenfunctions have the same eigenvalue, because multiply $\widetilde{v}$ by a constant greater than 1 would give an eigenfunction of eigenvalue λ_n defined in a compact annulus, a contradiction. □

Lemma 3.1.13. *Let v be an eigenfuction of the whole hyperbolic space of eigenvalue λ_n, presented above. Let u be a solution of P_r from Lemma 3.1.12. Then the function $\frac{v}{u}$ defined in $\mathbb{H}^n \backslash B_r(o)$ is a non decreasing function of the distance to o.*

The proof of this lemma is the same as the one from Lemma 3.1.12.

3.1.3 Solutions in the exterior of horoballs

A horosphere H determines two non compact sets, its interior and its exterior. The equation satisfied by the radial eigenfunction associated to the exterior was presented in Section 1.3, expression (1.6).

Theorem 3.1.14. *Let B be a horoball in $\mathbb{H}^n$ and $\overline{B}$ its closure in $\overline{\mathbb{H}^n}$. The problem*

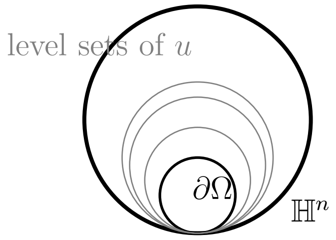
$$\begin{cases} \Delta u = -\lambda_n u \text{ in } \mathbb{H}^n \backslash B \\ u = 0 \text{ in } \partial B \\ u \geq 0 \text{ is a bounded radial function} \end{cases}$$

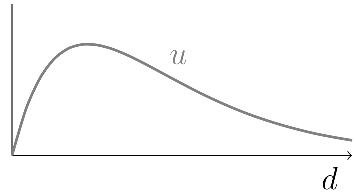
has a solution given by

$$u(d) = Cde^{-\sqrt{\lambda_n} d}, \text{ for any } C \in \mathbb{R},$$

where d is the distance to ∂B. It extends continuously to zero at $\partial_\infty \mathbb{H}^n \backslash \overline{B}$ and it cannot be extended to $\overline{B} \cap \partial_\infty \mathbb{H}^n$.

We present a picture of the profile of u.





Proof. The solution of the corresponding ODE is

$$u(d) = Cde^{-\sqrt{\lambda_n}d},$$

where d is the distance to the boundary horosphere H. To see that it extends continuously to $\partial_\infty \mathbb{H}^n \backslash \overline{B}$, take a point $p \in \partial_\infty \mathbb{H}^n \backslash \overline{B}$. Given $\epsilon > 0$, there is d_0 large enough such that $d > d_0$ implies $u(d) < \epsilon$. Consider the horosphere H_{d_0} parallel to H of distance d_0 from H and $\overline{B_{d_0}}$ the closed horoball bounded by H_{d_0}. The set $\overline{\mathbb{H}^n} \backslash \overline{B_{d_0}}$ contains an open set around p and where $u < \epsilon$. On the other hand, if $\{p\} = H \cap \partial_\infty \mathbb{H}^n$, any open set containing p intercepts all horospheres parallel to H, which are the level sets of u. Hence there is no a continuous extension of u to p. □

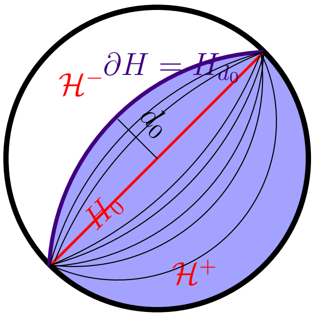
3.1.4 Solutions in the hyperballs

Now we show that for any hyperball H there exists a positive bounded eigenfunction associated to λ_n, that vanishes on the boundary and posses some symmetry, provided the mean curvature of ∂H, oriented in the opposite direction of the totally geodesic parallel to ∂H, is smaller than some constant that depends on n.

First, given a hyperball $H \subset \mathbb{H}^n$, consider H_0 the totally geodesic hypersphere parallel to ∂H. H_0 divides $\mathbb{H}^n$ in two connected components, $\mathcal{H}^+$ and $\mathcal{H}^-$. We choose $\mathcal{H}^+$ such that $\partial_\infty H = \partial_\infty \mathcal{H}^+$. Define the parameter $d(x)$ by

$$d(x) = \begin{cases} \text{dist}(x, H_0) \text{ for } x \in \mathcal{H}^+ \\ \\ -\text{dist}(x, H_0) \text{ for } x \notin \mathcal{H}^+ \end{cases} \tag{3.6}$$

The next figure represents a hyperball in $\mathbb{H}^2$, for which d_0, the distance with sign between ∂H and H_0 is negative. We observe that $d_0 \leq 0$, is equivalent to the region H being convex.



Then the problem

$$\begin{cases} -\Delta u = \lambda_n u \text{ in } H, \\ u \text{ depends only on the distance to } \partial H \\ u = 0 \text{ on } \partial H \end{cases} \tag{3.7}$$

corresponds to (see (1.7))

$$\begin{cases} u''(d) + (n-1)\tanh(d)\, u'(d) = -\lambda_n u(d) \quad \text{for} \quad d > d_0 \\ u(d_0) = 0. \end{cases} \tag{3.8}$$

Observe that the second condition in (3.7) means that the black hyperspheres represented in the last picture are level sets of the solution u.

Doing the change of variable $s = \sinh d$, we obtain

$$\begin{cases} (1+s^2)\dfrac{d^2u}{ds^2} + ns\dfrac{du}{ds} = -\dfrac{(n-1)^2}{4}u \quad \text{for} \quad s > s_0 \\ u(s_0) = 0, \end{cases} \tag{3.9}$$

where $s_0 = \sinh d_0$. Then, defining the operator

$$L_n(v) = (1+s^2)\frac{d^2v}{ds^2} + ns\frac{dv}{ds}$$

we have to study $L_n(u) = -\lambda_n u$. The first result to this problem is obtained differentiating this equation with respect to s, leading to the next lemma.

Lemma 3.1.15. *If u is a solution of $L_n(v) = -\lambda_n v$, then du/ds is a solution of $L_{n+2}(v) = -\lambda_{n+2} v$.*

This allows us to obtain eigenfunctions to higher dimensions from lower ones. For instance, from the solutions of $L_1 = -\lambda_1$, we can obtain the solutions of $L_3 = -\lambda_3$. For that, notice that if $n = 1$, equation (3.9) can be reduced to a first order ODE, whose the general solution is given by

$$w_1(s) = C_1 \ln |s + \sqrt{1+s^2}| + C_2 = C_1 \text{arcsinh}(s) + C_2.$$

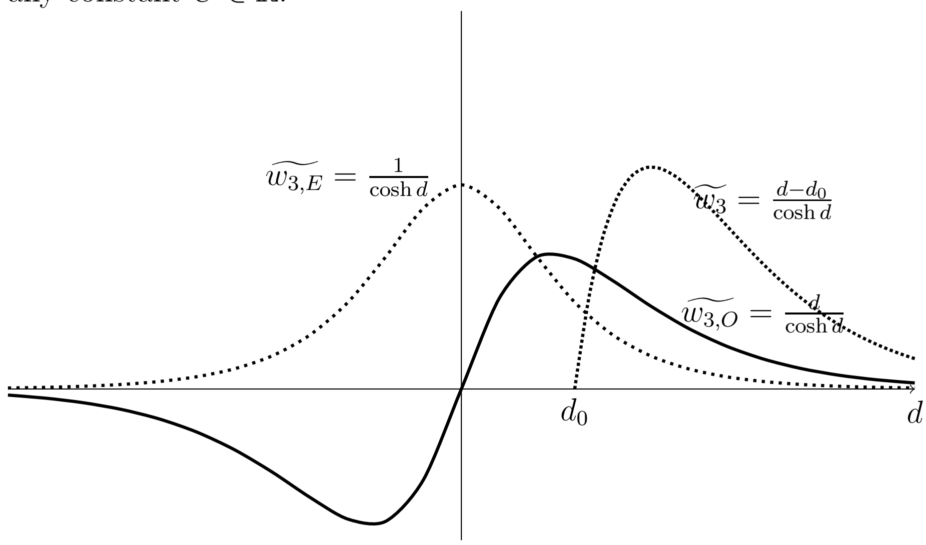
Since Lemma 3.1.15 guarantees that the derivative of w_1 is a family of solutions of (3.9) for $n = 3$, we can apply the reduction of order method to obtain the general solution in this dimension, that is given by

$$w_3(s) = C_1 \frac{1}{\sqrt{1+s^2}} + C_2 \frac{\text{arcsinh}(s)}{\sqrt{1+s^2}}.$$

Going back to the variable d, the distance with sign to H_0, we conclude that a solution in the hyperball with boundary $H_{d_0} = \{x \in \mathbb{H}^3 \mid d(x) = d_0\}$ for some fixed totally geodesic hypersphere H_0 in $\mathbb{H}^3$. is

$$\widetilde{w_3}(d) = C \frac{(d - d_0)}{\cosh d}$$

for any constant $C \in \mathbb{R}$.



Indeed, we can solve (3.9) for any odd dimension as we show in the next lemma.

Lemma 3.1.16. *The functions*

$$\frac{d^m}{dt^m}\left(\frac{1}{\sqrt{1+s^2}}\right) \quad \text{and} \quad \frac{d^m}{ds^m}\left(\frac{\text{arcsinh}(s)}{\sqrt{1+s^2}}\right)$$

are linearly independent solutions of $L_{2m+3}v = -\lambda_{2m+3}v$ *for* $m \in \{0, 1, \dots\}$.

Proof. Since

$$w_E(s) := \frac{1}{\sqrt{1+s^2}} \quad \text{and} \quad w_O(s) := \frac{\operatorname{arcsinh}(s)}{\sqrt{1+s^2}},$$

are solutions of $L_3 = -\lambda_3$, applying Lemma 3.1.15 n times, the derivatives of order n of these two functions are solutions of $L_{2n+3} = -\lambda_{2n+3}$. Moreover, w_E and w_O are even and odd functions respectively. Hence, if n is even, $d^n w_E/ds^n$ is even and $d^n w_O/ds^n$ is odd, and since they are not identically zero, these derivatives must be linearly independent. The same holds if n is odd. □

Remark 3.1.17. *At this point we wonder if there is some similar result for even dimension.*

First we need the next two results.

Lemma 3.1.18. *For any interval* $[a, b] \subset \mathbb{R}$, *it holds*

$$\inf \left\{ \frac{\int_a^b s^2 (v')^2 ds}{\int_a^b v^2 ds} \mid v \in H_0^1([a, b]) \backslash \{0\} \right\} \geq \frac{1}{4}.$$

The proof is given in the appendix.

Lemma 3.1.19. *If* $u \not\equiv 0$ *is a solution of* $L_2 = -\lambda_2$, *then* u *has at most one zero.*

Proof. Suppose that $a < b$ are consecutive zeros of $u(s)$ and that u is positive in (a, b). Observe that $L_2(u) = -\lambda_2 u$ is equivalent to

$$((1+s^2)u')' = -\frac{1}{4}u.$$

Multiplying this equation by u and integrating on $[a, b]$, we have

$$\frac{\displaystyle\int_a^b (1+s^2)(u')^2 ds}{\displaystyle\int_a^b u^2 ds} = \frac{1}{4}.$$

But this contradicts Lemma 3.1.18 and $\int_a^b (u')^2 ds > 0$. Hence, u cannot have more than one zero. $\square$

Now we can present a result that is similar to Lemma 3.1.16.

Lemma 3.1.20. *The eigenvalue problem $L_2 = -\lambda_2$ has two solutions defined on $\mathbb{R}$, v_E and v_O, that are even and odd functions respectively, such that v_E has no zero and v_O has only one zero. Furthermore*

$$\frac{d^m}{ds^m}(v_E) \quad \text{and} \quad \frac{d^m}{ds^m}(v_O)$$

are linearly independent solutions of $L_{2m+2}v = -\lambda_{2m+2}v$ for $m \in \{0, 1, \dots\}$.

Proof. Let v_E be the solution of $L_2 v = -\lambda_2 v$ that satisfies $v(0) = 1$ and $v'(0) = 0$. From the classical theory of ODE, v_E is defined globally. Moreover, since the coefficients of the zero and second order derivatives are even and the coefficient of the first order is odd, v_E is an even function. Hence, Lemma 3.1.19 implies that v_E has no zero. Using Lemma 3.1.15, we get that $d^n v_E/ds^n$ is solution of $L_{2n+2}v = -\lambda_{2n+2}v$.

Define v_O as being the solution of the same ODE such that $v(0) = 0$ and $v'(0) = 1$. Following the same argument, v_O satisfies the stated properties.

Observe that $d^n v_E/ds^n$ is not the zero function for any n, otherwise the $(n-1)$-derivative of v_E is constant, contradicting the fact that $L_{2n} = -\lambda_{2n}$ has no constant solution. The same holds for $d^n v_O/ds^n$. Hence, using the same argument as in Lemma 3.1.16, these derivatives are linearly independent. $\square$

Remark 3.1.21. *As a consequence of Lemma 3.1.16 and Lemma 3.1.20, any solution of $L_n = -\lambda_n$ is the $(n-3)/2$-derivative of some solution of $L_3 = -\lambda_3$ if n is odd, and is the $(n-2)/2$-derivative of some solution of $L_2 = -\lambda_2$ if n is even.*

Now we remind some basic result of ODE.

Lemma 3.1.22. *Let u be a non trivial global solution of $L_n = -\lambda_n$.*

(i) If s_0 is a critical point of u, then s_0 is a strict local maximum in case $u(s_0) > 0$ and a strict local minimum in case $u(s_0) < 0$.

(ii) If u has no zero in the interval I, then u has at most one critical point in I.

(iii) Between two consecutive zeros of u, there is only one critical point.

Proof. (i) This is a consequence of $(1+s_0^2)\dfrac{d^2u}{ds^2}(t_0) = -\lambda_n u(s_0)$.

(ii) Suppose, without loss of generality, that u is positive on I. Note that (i) implies that all critical points in I must be points of local maximum. If $s_1 < s_2$ are local maxima of u in I, then there exists a local minimum $s_0 \in (s_1, s_2) \subset I$. But this contradicts (i) and $u(s_0) > 0$.

(iii) Denoting two consecutive zeros by a and b, we just need to apply (ii) for $I = (a, b)$. □

Lemma 3.1.23. *Suppose that u is a non trivial solution of $L_n = -\lambda_n$ that has k zeros in $\mathbb{R}$, where $k \in \{0, 1, 2, \dots\}$. Then*

(i) $\lim_{s\to\pm\infty} u(s) = 0$

(ii) The derivative of u has $k+1$ zeros in $\mathbb{R}$.

Proof. (i) Suppose that $u(s)$ does not converge to zero as $s \to +\infty$. Denoting the largest zero of u by b, we can suppose that u is positive on (b, ∞). From (ii) of Lemma 3.1.22, either u increases in (b, ∞) (in case there is no critical point in this interval), or u increases up to the unique critical point, which is a local maximum according to (i) of that lemma, and decreases afterwards. In this second possibility u converges to some positive number, since we are assuming that it does not converge to zero. Anyway, in both cases, there exist $K > 0$ and $s_1 > \max\{b, 0\}$ such that $u(s) \geq K$ for $s \geq s_1$. Therefore

$$L_n(u)(s) \leq -\lambda_n K \quad \text{for} \quad s \geq s_1.$$

Multiplying this relation by $(1+s^2)^{n/2-1}$, we have

$$((1+s^2)^{n/2}u')' \leq -\lambda_n K(1+s^2)^{n/2-1} \leq -\lambda_n K s^{n-2}$$

for $s \geq s_1$. Integrating on $[s_1, s]$, we get

$$(1+s^2)^{n/2}u' \leq -\lambda_n K \frac{s^{n-1}}{n-1} + A,$$

where $A = (1+s_1^2)^{n/2}u'(s_1) + \lambda_n K s_1^{n-1}/(n-1)$. If $A < 0$, we can replace it by zero, and then we can suppose that $A \geq 0$. Naming $B = \lambda_n K/(n-1) > 0$, it follows that

$$u'(s) \leq -B\frac{s^{n-1}}{(1+s^2)^{n/2}} + A\frac{1}{(1+s^2)^{n/2}} < -\frac{B}{2^{n/2}} \cdot \frac{1}{s} + A\frac{1}{s^n}.$$

Hence, integrating on $[s_1, s]$,

$$u(s) \leq -\frac{B}{2^{n/2}} \ln s - A\frac{1}{(n-1)s^{n-1}} + C,$$

where C is some constant. Therefore, $u(s)$ is negative for s large, contradicting the positivity of u for $s > b$. In the same way $\lim_{s\to-\infty} u(s) = 0$.

(ii) Let $s_1 < \cdots < s_k$ be the zeros of u. In each interval (s_i, s_{i+1}) there is one critical point according to (iii) of Lemma 3.1.22. Note that the zeros of u cannot be critical points, otherwise, u is identically zero from the uniqueness of solution to initial value problems. Hence, in $[s_1, s_k]$ there are exactly $k-1$ critical points. In (s_k, ∞), there is a point of local maximum or local minimum since $u(s_k) = 0$ and $u(s) \to 0$ as $s \to +\infty$. This critical point is unique from (ii) of Lemma 3.1.22. By the same argument, there exists only one critical point in $(-\infty, s_1)$, completing the result. □

Lemma 3.1.24. *Let u be a solution of $L_n = -\lambda_n$, where $n \in \{2, 3, 4, \ldots\}$. Then, the number of roots of u is $\frac{n}{2}$ or $\frac{n-2}{2}$ if n is even and is $\frac{n-1}{2}$ or $\frac{n-3}{2}$ if n is odd. Moreover u converges to 0 as s goes to $\pm\infty$.*

Proof. Suppose that n is even. Using Remark 3.1.21, u is the $(n-2)/2$-derivative of some solution w of $L_2 = -\lambda_2$. According to Lemma 3.1.19, w has either no zero or one zero. Hence, from Lemma 3.1.23, u has $(n-2)/2$ or $n/2$ roots. If n is odd, the argument follows in the same way, since the

solutions of $L_3 = -\lambda_3$ also have at most one zero according to the expression $w_3 = C_1 w_E + C_2 w_O$.
Using that u has finite number of zeros, Lemma 3.1.23 implies that

$$\lim_{s \to \pm\infty} u(s) = 0.$$

□

This lemma implies that if u is a solution of $L_n = -\lambda_n$, then there exists a root s^* of u that is the largest one. Therefore, the restriction of u to $[s^*, \infty)$ is a solution to the initial value problem (3.9) with $s_0 = s^*$, that does not change sign. Hence the set

$$A = \{s_0 \mid \text{the problem (3.9) has a positive solution}\}$$

is non empty. To obtain the main claim of this section, we need to show that A is an interval.

Lemma 3.1.25. *If $n = 2$ or $n = 3$, then $A = \mathbb{R}$. If $n > 3$, there exists $S_n \in \mathbb{R}$ such that $A = [S_n, \infty)$.*

Proof. Consider first the cases $n = 2$ or $n = 3$. For any $s_0 \in \mathbb{R}$, if u is a solution of (3.9), then $u > 0$ or $u < 0$ in (s_0, ∞) since u can have at most one root from Lemma 3.1.24. Hence $s_0 \in A$.

Suppose now that $n > 3$ and $s_1 \in A$. Given $s_2 > s_1$, we have to show that $s_2 \in A$. For that, let u_1 be a solution of (3.9) with $s_0 = s_1$, that is positive on (s_1, ∞). From the classical results for ODE, there exists a global solution u_2 of (3.9), for $s_0 = s_2$ with $u_2'(0) = 1$. Then u_2 is non trivial and s_2 is a simple root. Suppose that u_2 has a root $s_3 > s_2$. We can assume that $u_2 > 0$ in (s_2, s_3) and consider

$$\alpha = \max_{[s_2, s_3]} \frac{u_2(s)}{u_1(s)}.$$

Observe that the maximum is positive and is attained for some $s^* \in (s_2, s_3)$, since $u_2(s_2) = u_2(s_3) = 0$. Hence $\alpha u_1(s^*) = u_2(s^*)$ and $\alpha u_1(s) \geq u_2(s)$ for $s \in [s_2, s_3]$. Thus $\alpha u_1'(s^*) = u_2'(s^*)$ and, from the uniqueness result, $\alpha u_1 = u_2$. However this contradicts $u_1(s_2) > 0$, and, therefore, u_2 has no roots larger than s_2. Then $s_2 \in A$, proving that A is an interval with $+\infty$ endpoint.

Moreover, if $n = 2m+2$ is even and $u = \frac{d^m}{ds^m}v_O$ or if $n = 2m+3$ is odd and $u = \frac{d^m}{ds^m}w_O$, then u has at least two roots from Lemmas 3.1.16, 3.1.20 and 3.1.23. Let s^* be the smallest root of u. Any eigenfunction that vanishes at $s = s^*$ is multiple of u and, therefore, has another root larger than s^*. Hence, problem (3.9) does not have positive solution and A is bounded by below by s^*. Let $S_n = \inf A > -\infty$. To complete the proof, we just need to show that $S_n \in A$. For that, take a decreasing sequence s_k that converges to S_n and a sequence of eigenfuctions such that $u_k(s_k) = 0$ and $u_k'(s_k) = 1$. Hence u_k is positive for $s > s_k$ and the sequence converges to some non trivial solution u of (3.9) for $s_0 = S_n$. Then $u \geq 0$ in (S_n, ∞). Furthermore, if $u(\tilde{s}) = 0$ for some $\tilde{s} > S_n$, then $u'(\tilde{s}) = 0$ and $u \equiv 0$, contradicting that is non trivial. Then $S_n \in A$. ☐

Now we present a characterization of S_n.

Lemma 3.1.26. *If $n = 2m+2$ is even, S_n is the largest root of $\frac{d^m}{ds^m}v_E$, and if $n = 2m+3$, S_n is the largest root of $\frac{d^m}{ds^m}w_E$. In particular, $S_4 = S_5 = 0$. Furthermore, $S_4 < S_6 < S_8 < \ldots$ and $S_5 < S_7 < S_9 < \ldots$.*

Proof. For any function $u : \mathbb{R} \to \mathbb{R}$, denote the largest root of u by $S(u)$. *Assertion 1:* if u and v are solutions of $L_n = -\lambda_n$ and $S(u) < S(v)$, then $S(u') < S(v')$. Suppose this is not true. Hence $S(u') \geq S(v')$. Note that $S(u) < S(u')$, since $u = 0$ at $S(u)$ and $\lim_{s\to+\infty} u(s) = 0$ imply that u has a critical point in $(S(u), \infty)$. (Indeed, $S(u')$ is the unique critical point of u in this interval from (ii) of Lemma 3.1.22.) Therefore $(S(u), S(u')] \supset [S(v), S(v')]$ and, assuming that u and v are positives in $(S(u), \infty)$ and $(S(v), \infty)$ respectively, we have $u > 0$ in $[S(v), S(v')]$. Hence,

$$\alpha := \max_{[S(v),S(v')]} \frac{v(s)}{u(s)}$$

is finite, positive and attained at some $s^* \in (S(v), S(v')]$. If $s^* \in (S(v), S(v'))$, then, using that $\alpha u - v \geq 0$ in $[S(v), S(v')]$ and $(\alpha u - v)(s^*) = 0$, we conclude that $(\alpha u - v)'(s^*) = 0$. Therefore, from the uniqueness of solution, $\alpha u = v$, contradicting $\alpha u(S(v)) > 0 = v(S(v))$. Hence $s^* = S(v')$. However, since $\alpha u \geq v$ and αu is increasing in $[S(v), S(v')]$ (u is increasing in $[S(u), S(u')]$), we have that $0 \leq \alpha u'(s^*) \leq v'(s^*) = 0$. Then, it follows again that $\alpha u = v$, which is an absurd. This completes assertion 1.

Observe that this argument also holds it $S(u) = -\infty$, that is the case if $u = v_E$ or $u = w_E$. Hence, for $n = 4$ (or $n = 5$), if v is a solution of $L_n = -\lambda_n$

with $S(v) \in \mathbb{R}$, then $S(v') > S(v'_E) = 0$ (or $S(v') > S(w'_E) = 0$). Therefore, $S_4 = S_5 = 0$, that is the largest zero of the first derivative of v_E and w_E. Thus the statement is true for $m = 1$, and, using induction and assertion 1, it holds for any $m \in \mathbb{N}$.

Finally, as we pointed out in assertion 1, $S(u) < S(u')$. Then, $S_{2m+2} = S(\frac{d^m}{ds^m} v_E) < S(\frac{d^{m+1}}{ds^{m+1}} v_E) = S_{2m+4}$. In a similar way, $S_5 < S_7 < \ldots$. □

Theorem 3.1.27. *Consider H a hyperball in $\mathbb{H}^n$. The problem*

$$\begin{cases} -\Delta u &= \lambda_n u \quad \text{in } H \\ u &= 0 \quad \text{on } \partial H \end{cases}$$

has a positive solution u that depends only on the variable d, if and only if minus the mean curvature of ∂H is larger than or equal to $S_n/\sqrt{S_n^2+1}$, where S_n is given by the previous lemmas. In particular, if $n = 2$ or $n = 3$, the problem has this kind of solution for any hyperball.

Proof. We have already shown that if $u = u(d)$ is such solution, then $u(s)$ must satisfy (3.9), where $s_0 = \sinh d_0$ and $\operatorname{arc\,tanh}(K) = d_0$. According to Lemma 3.1.25, problem (3.9) has a positive solution if and only if $s_0 \geq S_n$. This is equivalent to $\sinh d_0 \geq S_n$. Hence, using that $\operatorname{arc\,tanh}(K) = d_0$, it follows that K must be larger or equal than $S_n/\sqrt{S_n^2+1}$. □

Remark 3.1.28. *We have a similar result to the exterior of a hyperball. The solutions in this region that depends on d can be studied considering the problem (3.9) for $s < s_0$. Making the change of variable $z = -s$, we get the same equation, and therefore, we conclude a similar result as in Theorem 3.1.27 provided minus the mean curvature of ∂H is smaller or equal than $-S_n/\sqrt{S_n^2+1}$.*

If $n > 3$, according to Lemma 3.1.24 there exists a solution u of $L_n = -\lambda_n$ with at least two zeros. Suppose that s_1 and s_2 are consecutive zeros of u, that is positive in (s_1, s_2). Then $u(d)$ is a bounded positive eigenfunction associated to λ_n in the region $\{\operatorname{arcsinh} s_1 < d(x) < \operatorname{arcsinh} s_2\}$, that vanishes on the boundary. This proves the existence of eigenfunction to this hyperannulus bounded by the hyperspheres $\{d = \operatorname{arcsinh} s_1\}$ and $\{d = \operatorname{arcsinh} s_2\}$, provided $n > 3$.

3.1.5 Non existence of solutions in horoballs or subsets

The ODE that corresponds to the interior of a horoball was presented in Section 1.3 and has solution

$$v(r) = Kre^{\sqrt{\lambda_n}r}, K \in \mathbb{R}$$

which is not bounded. We will refer the function v as the usual eigenfunction of the horoball. It will be useful in the next proof.

Theorem 3.1.29. *Let B be a horoball in $\mathbb{H}^n$. The problem*

$$\begin{cases} \Delta u = -\lambda_n u \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega \\ u \text{ is a bounded function.} \end{cases}$$

does not have a non trivial solution for any $\Omega \subset B$.

Proof. Let us assume that there is a solution u_0 in Ω. Take v a positive eigenfunction associated to B, presented above.

Given $C > 0$, we claim that $u \leq Cv$.

Let d be the distance to ∂B. Since u_0 is a bounded function, and v increases with d, there is $d_0 > 0$, such that $Cv(x) \geq u_0(x)+1 \;\forall\, x$ with $d(x) \geq d_0$

In order to make thing easier, we set the notation

$$B_d = \begin{cases} \{x \in B| \mid d(x) = d\} \text{ if } d \geq 0 \\ \{x \notin B \mid d(x) = -d\} \text{ if } d < 0 \end{cases}$$

and $A_{a,b}$ is the horoannulus bounded by B_a and B_b.

Define $\Omega_1 = \{x \in \Omega | (u_0 - Cv)(x) \geq 0\} \subset A_{0,d_0} = \{x \in B | 0 \leq d(x) \leq d_0\}$. If Ω_1 is empty, the claim is shown. Otherwise, define

$$\widetilde{u_1} = (u_0 - Cv) \text{ and } u_1 = \frac{\widetilde{u_1}}{\max \widetilde{u_1}}.$$

Consider v_1 the usual eigenfunction associated to the interior of the horoball with boundary B_{-1} such that $v_1\big|_{B_{d_0}} = 1$.

Now construct a function w, which will allow us to repeat the process in a smaller region.

Given $p \in B_{d_0+1}$, take w_p the eigenfunction rotationally symmetric associated to $\mathbb{H}^n \backslash B_1(p)$ whose maximum value is 1. Let r_2 be the distance from $\partial B_1(p)$ to the sphere where w_p attains its maximum value. From Remark 3.1.11, we know that w_p is an increasing function of the distance to p at the points where the distance is smaller than r_2. Concentrate in the set

$$A_{1,r_2+1}(p) \cap A_{0,d_0} = \{x \in A_{0,d_0} | 1 \leq d(x,p) \leq r_2 + 1\}.$$

Notice that

$$\partial\left(A_{1,r_2+1}(p) \cap A_{0,d_0}\right) \subseteq B_{d_0} \cup \partial B_{r_2+1}(p).$$

On $\partial\left(A_{1,r_2+1}(p) \cap A_{0,d_0}\right) \cap B_{d_0}$, $u_1 \leq 0$ and $0 \leq w$.

On $\partial\left(A_{1,r_2+1}(p) \cap A_{0,d_0}\right) \cap \partial B_{r_2+1}(p)$, $w = 1$ and $u_1 \leq 1$. Hence, by Lemma 1.3.3, $w \geq u_1$ in $A_{1,r_2+1}(p) \cap A_{0,d_0}$. We define the function w in A_{0,d_0} in the following way: For each $x \in A_{0,d_0}$, there is a unique corresponding p such that the geodesic starting at the center of the horoball that passes though p contains x.

We define $w(x) = w_p(x)$. In this way, w is defined in the whole A_{0,d_0}, is an increasing function of the distance to B_{d_0} and $u_1 \leq w$ in $A_{d_0-r_2,d_0}$.

Finally, we define $r_1 < r_2$ as the distance between B_{d_0} and $B_{d_0-r_1} = \{x \in A_{d_0-r_2,d_0} | w(x) = v_1(x)\}$. Since w is an increasing function of the distance to B_{d_0}, is zero on B_{d_0} and 1 at distance r_2 and v_1 decreases with the distance to B_{d_0} in $A_{d_0-r_2,d_0}$ and has value 1 at distance 0, there must be $r_1 < r_2$. Moreover $w \leq v_1$ in $A_{d_0-r_1,d_0}$.

We conclude that $u_1 \leq v_1$ in $A_{d_0-r_1,d_0}$ and r_1 does not depend on u_1. If $r_1 \geq d_0$, the contradiction is found, because $v_1 < 1$ in $A_{d_0-r_1,d_0} \backslash B_{d_0}$ and u_1 has maximum 1 in $\Omega_1 \subset A_{d_0-r_1,d_0} \backslash B_{d_0}$.

If $r_1 < d_0$, we repeat the process in a way that the width decreases r_1 again. Take $\Omega_2 = \{x \in \Omega_1 | u_1(x) \geq v_1(x)\}$ and u_2 the eigenfunction of Ω_2 that is a positive multiple of $u_1 - v_1$ and has maximum 1. Consider v_2 the usual eigenfunction of the horoball that has boundary B_{-1-r_1} and has value 1 on $B_{-d_0-r_1}$. Now construct w_2 for $B_{d_0-r_1}$ in the same way that we constructed w for B_{d_0}. w_2 is then a kind of translation of w to a parallel horoball. That is why the distance r_2 between the maximum of w_2 and $B_{d_0-r_1}$ is the same r_2 from w. Since v_2 is also a kind of translation of v_1, r_1 is also the same as the r_1 above. Applying the argument of the first step, $u_2 \leq v_2$ in $A_{d_0-2r_1,d_0-r_1}$.

We repeat this process up to $B_0 \subset A_{d_0-kr_1,d_0-(k-1)r_1}$, where the contradiction happens by the fact that $v_k < 1$ in $A_{d_0-kr_1,d_0-(k-1)r_1}$ and $u_k \leq v_k$ has maximum 1 in $\Omega_k \subset A_{d_0-(k-1)r_1,d_0-kr_1}$. Hence u_0 cannot exist. □

3.2 Existence of solutions

In this section we study the existence of non negative bounded eigenfunctions that admit a continuous extension to $\partial_\infty \mathbb{H}^n$. We show that domains that contain an open set of $\partial \mathbb{H}^n$ admit such an eigenfunction.

Theorem 3.2.1. *Let Ω be an open set in $\overline{\mathbb{H}^n}$ that contains an open subset of $\partial_\infty \mathbb{H}^n$. Then Ω admits a positive bounded eigenfunction of eigenvalue λ_n that vanishes at $\partial_\infty \Omega$.*

Proof. Since Ω contains a truncated cone (defined in Section 1.2) it contains a totally geodesic hyperball H.

Fix p_1 and p_2 two points that are equidistant from ∂H, $p_1 \in H$ and $p_2 \notin H$. Consider u_i a global eigenfunction centered at p_i with $u_i(p_i) = 1$. Define

$$u_0 = \begin{cases} u_1 - u_2 \text{ in } H \\ 0 \text{ in } H^C. \end{cases}$$

Since $u_1 = u_2$ on ∂H, u_0 is a continuous function and it follows from u_i being a decreasing function of the distance to p_i that $u_0 \geq 0$.

Take x_0 a point in $\partial\Omega$ and consider the problem P_R

$$\begin{cases} -\Delta u = \lambda_n u \text{ in } \Omega_R = \Omega \cap B_R(x_0) \\ u = u_0 \text{ on } \partial\Omega_R \end{cases}$$

for R great enough such that $H \cap \Omega_R \neq \emptyset$

Let u_N be a solution of P_N, $N \in \mathbb{N}$. Since $u_N \geq u_0$ on $\partial\Omega_N$, by the comparison principle (Lemma 1.3.3), $u_N \geq u_0$ in Ω_N. On the other hand $u_N \leq u_1$ on $\partial\Omega_N$, hence also in Ω_N. We conclude

$$0 \leq u_0 \leq u_N \leq u_1 \text{ in } \Omega_N. \tag{3.10}$$

Taking $u = \lim_{N\to\infty} u_N$ in Ω, it follows from (3.10) that the sequence u_N is equibounded and, from standard PDE theory, it has a subsequence converging uniformly on compact subsets of $\overline{\Omega}$ to a solution u of

$$\begin{cases} -\Delta u = \lambda_n u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \\ u \geq 0 \text{ is bounded.} \end{cases}$$

Also from (3.10) u extends continuously to the asymptotic closure of the domain and is zero at $\partial_\infty\Omega$.

□

3.3 Appendix

Proof of Lemma 3.1.18: Naming the quotient that appears in the left-hand side of the statement by $Q_{a,b}(v)$, we have to prove that $Q_{a,b}(v) \geq 1/4$ for any non vanishing function $v \in H_0^1([a,b])$.

First consider an interval $[-c,c]$, that contains $[a,b]$, and the corresponding quotient $Q := Q_{-c,c}$, that is obtained from $Q_{a,b}$ by replacing a and b by $-c$ and c. Since any function $v \in H_0^1([a,b])$ can be extended to $[-c,c]$ as being zero outside $[a,b]$, the minimum of Q on $H_0^1([-c,c])$ is smaller or equal than the minimum of $Q_{a,b}$. Hence, it is enough to prove that $Q(v) \geq 1/4$ for any $v \in H_0^1([-c,c])$. For observe that

$$Q(v) = \frac{\displaystyle\int_{-c}^{c} t^2 (v')^2 dt}{\displaystyle\int_{-c}^{c} v^2 dt} = \lim_{k\to\infty} \frac{\displaystyle\int_{-c}^{c} g_k(t)(v')^2 dt}{\displaystyle\int_{-c}^{c} v^2 dt},$$

where

$$g_k(t) = \begin{cases} 1/k^2 & \text{if} \quad |t| < 1/k \\ t^2 & \text{if} \quad |t| \geq 1/k \end{cases}$$

Therefore, if we show that $\inf Q_k(v) \geq 1/4$ for any k, where $Q_k(v)$ is the quotient that appear in the right-hand side, the lemma is proved. Using classical techniques for eigenvalue problems, we can minimize Q_k for any k, since g_k is bounded by below by some positive constant. Moreover any minimum v_k of Q_k in $H_0^1([-c,c])$ satisfies

$$-(g_k(t)v_k'(t))' = \alpha_k v_k, \tag{3.11}$$

where $\alpha_k = \min Q_k(v) = Q_k(v_k)$. We prove now that $\alpha_k \geq 1/4$ for any k, completing the proof. Proceeding by contradiction, suppose that $\alpha := \alpha_k < 1/4$ for some k. From the fact that α_k is the first eigenvalue of the operator that appears in (3.11) and g_k is an even function, it follows that v_k is even. We can also compute the classical solutions of (3.11), since it has constants coefficients for $|t| < 1/k$ and it is a Cauchy-Euler equations for $|t| \geq 1/k$. For instance, one possibility for v_k, restricted to $[-c,0]$, is

$$v_k(t) = \begin{cases} \dfrac{|t|^{m_2}}{c^{m_2}} - \dfrac{|t|^{m_1}}{c^{m_1}} & \text{for} \quad -c \leq t \leq -1/k \\ A\cos(k\sqrt{\alpha}\,t) + B\sin(k\sqrt{\alpha}\,t) & \text{for} \quad -1/k < t \leq 0 \end{cases}$$

where

$$m_1 = \frac{-1+\sqrt{1-4\alpha}}{2}, \qquad m_2 = \frac{-1-\sqrt{1-4\alpha}}{2},$$

$$\begin{aligned} A = \; & -\left(\frac{1}{kc}\right)^{m_1}\left[\cos(\sqrt{\alpha}) - \frac{m_1}{\sqrt{\alpha}}\sin(\sqrt{\alpha})\right] \\ & +\left(\frac{1}{kc}\right)^{m_2}\left[\cos(\sqrt{\alpha}) - \frac{m_2}{\sqrt{\alpha}}\sin(\sqrt{\alpha})\right] \end{aligned}$$

and

$$\begin{aligned} B = \; & \left(\frac{1}{kc}\right)^{m_1}\left[\sin(\sqrt{\alpha}) + \frac{m_1}{\sqrt{\alpha}}\cos(\sqrt{\alpha})\right] \\ & -\left(\frac{1}{kc}\right)^{m_2}\left[\sin(\sqrt{\alpha}) + \frac{m_2}{\sqrt{\alpha}}\cos(\sqrt{\alpha})\right]. \end{aligned}$$

Hence, according to this expression, $v_k'(0) = k\sqrt{\alpha}\,B \neq 0$, since k, α and B are positive. To prove that $B > 0$, observe that

$$\sin(\sqrt{\alpha}) + \frac{m_2}{\sqrt{\alpha}}\cos(\sqrt{\alpha}) = \frac{\cos(\sqrt{\alpha})}{\sqrt{\alpha}}\left(\sqrt{\alpha}\tan(\sqrt{\alpha}) + m_2\right) < 0,$$

since $m_2 < -1/2$ and $\sqrt{\alpha}\tan(\sqrt{\alpha}) < 1/2$ for $\alpha < 1/4$. Using this and that $(1/kc)^{m_1} \leq (1/kc)^{m_2}$ (since $kc \geq 1$ and $m_1 > m_2$), we get

$$\left(\frac{1}{kc}\right)^{m_1}\left[\sin(\sqrt{\alpha}) + \frac{m_2}{\sqrt{\alpha}}\cos(\sqrt{\alpha})\right] > \left(\frac{1}{kc}\right)^{m_2}\left[\sin(\sqrt{\alpha}) + \frac{m_2}{\sqrt{\alpha}}\cos(\sqrt{\alpha})\right].$$

Thus

$$\begin{aligned} B &> \left(\frac{1}{kc}\right)^{m_1}\left[\sin(\sqrt{\alpha}) + \frac{m_1}{\sqrt{\alpha}}\cos(\sqrt{\alpha})\right] - \left(\frac{1}{kc}\right)^{m_1}\left[\sin(\sqrt{\alpha}) + \frac{m_2}{\sqrt{\alpha}}\cos(\sqrt{\alpha})\right] \\ &= \left(\frac{1}{kc}\right)^{m_1}\left[\frac{m_1 - m_2}{\sqrt{\alpha}}\cos(\sqrt{\alpha})\right] > 0. \end{aligned}$$

On the other hand, $v_k'(0) = 0$, since v_k is a C^2 function from the regularity theory and is even. Then, we have a contradiction. □.

Chapter 4

Isoperimetric Functions applyed to Eigenfunctions Estimates

4.1 Introduction

Results related to bounding the quotient

$$\frac{\|u\|_\infty}{\|u\|_2} \tag{4.1}$$

of an eigenfunction u defined in a compact domain of a manifold M are obtained. The results presented here hold in $\mathbb{R}^n$ and are generalized for a complete Riemannian manifold M.

In the first part symmetrization techniques are adopted to demonstrate a version of Chiti's Theorem, which provides an upper bound for the quotient (4.1) for the first eigenfunction of Ω given by the same quotient of the symmetrized u. The procedure consists in defining a symmetrized function u^* and adapt the original proof to this procedure. The definition of the symmetrized u follows from [6].

In the second part a result relating an isoperimetric function of M, an eigenvalue of Ω, which doesn't have to be the first, and the quotient (4.1) is obtained.

4.2 Symmetrization techniques on manifolds and Applications

Chiti's comparison argument [15] is a specialized comparison result which establishes a crossing property of the symmetric decreasing rearrangement of the eigenfunction and the first eigenfunction of the geodesic ball that has the same first eigenvalue. Chiti's argument was adapted to the the hemisphere of $\mathbb{S}^n$ [1] and to $\mathbb{H}^n$ [4], where it is observed that rearrangements and symmetrization techniques can also be applied in these spaces. In a general manifold these techniques cannot work, since they require symmetry of the space. Hence, in order to symmetrize a function defined in a manifold, we have to leave the manifold and define the symmetrized function in a model space.

As a motivation and to explain where the ideas of the result come from, we take a brief look at the symmetrization techniques in the Euclidean space.

4.2.1 Symmetrization in $\mathbb{R}^n$

Fix $U \subseteq \mathbb{R}^n$ a domain of finite measure and a measurable function $u : U \to \mathbb{R}$. The distribution function of u is

$$\mu(t) = |\{x \in U; |u(x)| > t\}| = |U_t|.$$

μ is a decreasing measurable function, hence differentiable almost everywhere. By the co-area formula (1.8),

$$\mu(t) = \int_{U_t} 1dx = \int_t^\infty \int_{\{u=s\}} \frac{1}{|\operatorname{grad} u|} dH^{n-1}(\{u = s\})ds,$$

then, at the points of differentiability,

$$\mu'(t) = -\int_{\{u=s\}} \frac{1}{|\operatorname{grad} u|} dH^{n-1}\left(\{u = t\}\right).$$

The decreasing rearrangement of u into $[0, \infty]$ is denoted by $u^{\#}$ and defined as the smallest decreasing function from $[0, \infty]$ into $[0, \infty]$ such that $u^{\#}(\mu(t)) \geq t$ for all t. More concisely,

$$u^{\#}(s) = \inf\{t \geq 0; \mu(t) < s\}.$$

An important property of $u^{\#}$ is that it has the same distribution function from u.

The symmetrized U is $U^{\star}$, the ball centered at the origin of $\mathbb{R}^n$ with the same measure of U. Then the spherically symmetric rearrangement of u is

$$u^*(x) = u^{\#}(n\omega_n|x|^n), \text{ where } n\omega_n = \operatorname{vol}\left(\mathbb{S}^{n-1}\right) = \frac{\pi^{n/2}}{\Gamma(1+n/2)},$$

which is rotationally symmetric and also has μ as the distribution function.

Principle of Polya and Szegö states that

$$\int_U |\operatorname{grad} u|^2 \geq \int_U^* |\operatorname{grad} u^*|^2$$

and hence Rayleigh's quotient reduces when one symmetrises.

This construction, the isoperimetric inequality and some computations allow the proof of two beautiful facts:

- Faber-Krahn Theorem, which says that round balls minimize eigenvalues among sets with the same volume;

- Chiti's Theorem, which says that balls maximize the quotient

$$\frac{\|u\|_{L^\infty}}{\|u\|_{L^2}}$$

  among all domains of eigenvalue λ.

4.2.2 Symmetrization in a manifold M

We try to extend the above results to a manifold. The greatest difficulty is that a general manifold doesn't have a round ball, so the symmetrization will deal with two manifolds. Given a manifold M and a function on M, how do we obtain a "symmetrization" of this function? Inspired by a book of P. Bérard ([6], chapter IV) we choose one that preserves the isoperimetric function.

The model space M^*

Consider M a Riemannian manifold and H an isoperimetric function on M (Definition 1.4.1).

Our first idea was to construct a model space M^* which is rotationally symmetric and has isoperimetric profile H, but what turns out to be needed is a rotationally symmetric model in which the balls centered at the origin satisfy equality for the function H, but they don't have to be isoperimetric domains (realize equality in the isoperimetric inequality of M^*).

The model space is then $M^* = [0, \mathrm{vol}(M)) \times \mathbb{S}^{n-1}$ with metric $dt^2 + f(t)^2 d\theta^2$, $f(0) = 0$, $f'(0) = 1$. The function function f will be determined in the next lemma.

In M^*, the volume of a ball centered at the origin ($\mathcal{O} = 0 \times \mathbb{S}^{n-1}$) of radius r, denoted by $B_r(\mathcal{O})$, is given by

$$V(r) = n\omega_n \int_0^r f^{n-1}(s)ds, \text{ if } f(s) > 0 \text{ in } (0, r)$$

and the area ($(n-1)$-dimensional volume) of the sphere $\partial B_r(\mathcal{O})$, is

$$A(r) = V'(r) = n\omega_n f^{n-1}(r), \text{ if } f(s) > 0 \text{ in } (0, r). \tag{4.2}$$

Lemma 4.2.1. *If*

$$r = \int_0^{V(r)} \frac{ds}{H(s)} \;\forall\, r \in [0, R], \tag{4.3}$$

then

$$H(V(r)) = A(r) \;\forall\, r \in [0, R],$$

i. e., the balls $B_r(\mathcal{O})$ in M^* realize equality for the function H.

Proof.

$$r = \int_0^{V(r)} \frac{ds}{H(s)} \;\forall r \in [0, R] \Rightarrow 1 = \frac{V'(r)}{H(V(r))} \Rightarrow H(V(r)) = A(r).$$

□

Given a Riemannian manifold M with isoperimetric function H, such that $1/H$ is an integrable function in $[0, \mathrm{vol}(M))$, we define $V(r)$ as the solution of the integral equation

$$r = \int_0^{V(r)} \frac{ds}{H(s)} \;\forall\, r \in [0, \mathrm{vol}(M)]$$

with initial condition $V(0) = 0$. We will assume that the isoperimetric function H is nice enough to guarantee the existence of V and that M^* is a

Riemannian manifold, meaning that the function f defined by (4.2) has the needed properties for $dt^2 + f(t)^2 d\theta^2$ to be a Riemannian metric. If H has the necessary properties, we define the model space M^* as described above and are able to construct rearrangements.

Rearrangements

Given a domain $\Omega \subset M$ and a measurable function $u : \Omega \to \mathbb{R}$, we define $\Omega^* \subset M^*$ and a symmetrized $u^* : \Omega^* \to \mathbb{R}$, equimeasurable to u.

The symmetrized domain Ω^* is the ball $B_r(\mathcal{O}) = [0, r) \times \mathbb{S}^{n-1}$ whose volume is equal to the volume of Ω,

$$\text{vol}(\Omega) = n\omega_n \int_0^r f^{n-1}(s)ds.$$

Observe that

$$|\partial\Omega| \geq H(\text{vol}(\Omega)) = H(\text{vol}(\Omega^*)) = |\partial\Omega^*|. \tag{4.4}$$

Hence our new symmetrization process preserves the volume and reduces the perimeter (area of the boundary), like the process from the Euclidean space. We are now ready to define the function u^*. We mimic the definition from $\mathbb{R}^n$.

Definition 4.2.2. *1. The level sets of u are $\Omega_t = \{x \in \Omega \mid |u(x)| > t\}$.*

2. The distribution function associated to u is

$$\mu(t) = \text{vol}(\Omega_t).$$

3. The decreasing rearrangement of u into $[0, \infty]$ is

$$u^{\#}(s) = \inf\{t \geq 0; \mu(t) < s\}.$$

4. The spherically symmetric rearrangement of u is

$$u^*(V(s)) = u^{\#}(s).$$

The rearrangement of u in M^* has the same properties as the rearrangement in $\mathbb{R}^n$.

Theorem 4.2.3. *The functions u, $u^\#$ and u^* are equimeasurable, i. e., they have the same distribution function. The distribution function associated to u, μ, is differentiable almost everywhere and its derivative is given by*

$$\mu'(t) = -\int_{\{u=s\}} \frac{1}{|\operatorname{grad} u|} dH^{n-1}(\{u=t\}) \tag{4.5}$$

at the points where it is differentiable. Besides $u^\#(\mu(t)) = t$ and $(\Omega_t)^* = (\Omega^*)_t$.

The proof of this theorem follows, using the co-area formula, the same steps as its version in the Euclidean space. Therefore we won't write it here.

Using observation (4.4), we obtain the following corollary.

Corollary 4.2.4. $\operatorname{vol}(\partial\Omega_t) \geq \operatorname{vol}(\partial\Omega_t^*)$ *for all* $t \in [0, \operatorname{vol}(M))$.

Lemma 4.2.5. *Some properties hold for integrals of u. They are*

1. $\int_\Omega u^p = \int_{\Omega^*} (u^*)^p \; \forall\, p > 0$.

2. $\int_\Omega |\operatorname{grad} u|^2 \geq \int_{\Omega^*} |\operatorname{grad} u^*|^2 \; \forall\, u \in H^1(M)$.

3. $\int_{\Omega_t} u = \int_0^{\mu(t)} u^\#(s) ds$.

Proof. 1. The first item follows easily from the co-area formula.

$$\begin{aligned} \int_\Omega u^p &= \int_0^{\sup u} \int_{\{u=t\}} \frac{u^p(x)}{|\operatorname{grad} u|} dH^{n-1}(\{u=t\}) dt \\ &= \int_0^{\sup u} t^p(-\mu_u'(t)) dt = \int_{\Omega^*} (u^*)^p, \end{aligned}$$

since $\mu_u = \mu_{u^*}$.

2. From now on we will denote $dH^{n-1}(\{u=t\})$ by da_t. In order to demonstrate 2., we apply the co-area formula taking $f = u$ and $g = |\operatorname{grad} u|^2$. By Sards Theorem, the set $A_u = \{t \in \mathbb{R} \mid \exists\, x \in \{u = t\}$ with $\operatorname{grad} f(x) = 0\}$ has measure zero. Hence,

$$\int_\Omega |\operatorname{grad} u|^2 dx = \int_0^{\sup u} \left(\int_{\{u=t\}} |\operatorname{grad} u| da_t \right) dt,$$

From Cauchy-Schwarz inequality,

$$\left(\int_{\{u=t\}} da_t\right)^2 \leq \left(\int_{\{u=t\}} |\operatorname{grad} u| da_t\right)\left(\int_{\{u=t\}} |\operatorname{grad} u|^{-1} da_t\right) \quad (4.6)$$

and replacing u by u^* the equality holds since $|\operatorname{grad} u^*|$ is constant in $\{u^* = t\}$. Hence, denoting $\{u = t\}$ by $G(t)$, μ_u by μ and μ_{u^*} by μ^*,

$$\begin{aligned}
\int_{G(t)} |\operatorname{grad} u| da_t &\geq \frac{|G(t)|^2}{\left(\int_{G(t)} |\operatorname{grad} u|^{-1} da_t\right)} \\
&= \frac{|G(t)|^2}{-\mu'(t)} \geq \frac{|G^*(t)|^2}{-\mu^{*\prime}(t)} \\
&= \int_{\{u^*=t\}} |\operatorname{grad} u^*| da_t.
\end{aligned}$$

Thus, observing that the symmetrization process makes any function smoother, $u^* \in H^1(\Omega^*)$ and

$$\begin{aligned}
\int_\Omega |\operatorname{grad} u|^2 dx &= \int_0^{\sup u} \left(\int_{G(t)} |\operatorname{grad} u| da_t\right) dt \\
&\geq \int_0^{\sup u = \sup u^*} \left(\int_{G^*(t)} |\operatorname{grad} u^*| da_t\right) dt \\
&= \int_\Omega |\operatorname{grad} u^*|^2 dx.
\end{aligned}$$

3. The third item follows from observing that the set $\Omega_t^\# = \{u^\# > t\}$ is the interval of extremal points 0 and $\mu(t)$, then from the co-area formula,

$$\int_{\Omega_t} u = \int_{\Omega_t^\#} u^\# = \int_0^{\mu(t)} u^\#(s) ds.$$

□

A Version of Faber Krahn's Theorem

Theorem 4.2.6. *If we consider the symmetrization above and $\lambda(\Omega)$ denotes the first Dirichlet eigenvalue of Ω, then $\lambda(\Omega) \geq \lambda(\Omega^*)$.*

Proof. In order to demonstrate the theorem, we use the Raylaigh characterization of the first eigenvalue, which states that

$$\lambda(\Omega) = \inf\{\mathcal{R}(u) \mid u \in H_0^1(\Omega), u \neq 0\}, \text{ where } \mathcal{R}(u) = \frac{\int_\Omega |\operatorname{grad} u|^2}{\int_\Omega u^2}.$$

From Lemma 4.2.5, $\mathcal{R}(u) \geq \mathcal{R}(u^*)$ for all $u \in H_0^1(\Omega)$, concluding the proof. □

A version of Chiti's Theorem

Theorem 4.2.7. *Consider $u : \Omega \to \mathbb{R}$ the first eigenfunction of a domain $\Omega \subseteq M$, a Riemannian manifold with isoperimetric function H. Let $z(r)$ be the the first eigenfunction in S_1, the ball in M^* centered at the origin (with radius R) with the same eigenvalue as Ω. Assume that u and z are normalized such that*

$$\int_\Omega u^2 = \int_{S_1} z^2. \tag{4.7}$$

Then there is $r_0 \in (0, R)$, such that $z(r) \geq u^*(r)\ \forall\ r \in (0, r_0)$ and $z(r) \leq u^*(r)\ \forall\ r \in (r_0, R)$

Remark 4.2.8. *The existence of S_1 is guaranteed by the fact that $\lambda(\Omega^*)$ is less than or equal to $\lambda(\Omega)$ and the eigenvalues increase up to infinity when the ball radius decreases to zero.*

Proof. Applying the divergence theorem and observing that for almost every t, the inner normal vector of Ω_t is $\frac{\operatorname{grad} u}{|\operatorname{grad} u|}$, we obtain

$$\begin{aligned} -\Delta u = \lambda u \Rightarrow \lambda \int_{\Omega_t} u &= -\int_{\Omega_t} \operatorname{div}(\operatorname{grad u}) \\ &= \int_{\partial\Omega_t} \operatorname{grad} u \frac{\operatorname{grad} u}{|\operatorname{grad} u|} dH_{n-1} \\ &= \int_{\partial\Omega_t} |\operatorname{grad} u| dH_{n-1}. \end{aligned}$$

Now we use the expression to the derivative of μ given by (4.5) and (4.6) remembering that replacing u by u^* equality holds.

Putting all together,

$$|\partial\Omega_t|^2 \leq -\mu'(t)\lambda \int_{\Omega_t} u.$$

From Corollary 4.2.4, $|\partial\Omega_t| \geq |\partial\Omega_t^*|$ and, of course, if $\Omega_t = \Omega_t^*$, equality holds.

$$|\partial\Omega_t| \geq |\partial\Omega_t^*| = \mathrm{vol}(\mathbb{S}^{n-1})a^{n-1}(r(t)) = V'(V^{-1}(|\Omega_t^*|)).$$

Hence,

$$\lambda \int_{\Omega_t} u \geq \frac{1}{-\mu'(t)} V'(V^{-1}(\mu(t)))^2.$$

From Theorem 4.2.3,

$$\int_{\Omega_t} u = \int_0^{\mu(t)} u^{\#}(s)ds.$$

If we take $s = \mu(t)$,

$$\lambda \int_0^s u^{\#} \geq -u^{\#\prime}(s)V'(V^{-1}(s))^2. \tag{4.8}$$

If we replace u by z, we always have equalities, so

$$\lambda \int_0^s z^{\#} = -z^{\#\prime}(s)V'(V^{-1}(s))^2. \tag{4.9}$$

We claim that either $u^{\#}$ and $z^{\#}$ are identical or they cross exactly once in $[0, |S_1|]$. Since u and z are normalized by (4.7), they must be identical or cross at least once. Let's assume that they cross twice or more times. Then there are two points s_1 and s_2 in $[0, |S_1|]$, such that

1. $0 \leq s_1 < s_2 < |S_1|$;
2. $u^{\#}(s) > z^{\#}(s)\ \forall\ s \in (s_1, s_2)$;
3. $u^{\#}(s_2) = z^{\#}(s_2)$;
4. Either $u^{\#}(s_1) = z^{\#}(s_1)$ or $s_1 = 0$.

We set

$$v(s) = \begin{cases} u^{\#}(s) \text{ in } (0, s_1) \text{ if } \int_0^{s_1} u^{\#} > \int_0^{s_1} z^{\#}, \\ z^{\#}(s) \text{ in } (0, s_1) \text{ if } \int_0^{s_1} u^{\#} \leq \int_0^{s_1} z^{\#}, \\ u^{\#}(s) \text{ in } (s_1, s_2), \\ z^{\#}(s) \text{ in } [s_2, |S_1|]. \end{cases}$$

Then, because of (4.8) and (4.9),

$$-\frac{dv}{ds}(s) \leq \lambda V'(V^{-1}(s))^{-2} \int_0^s v(t)dt \tag{4.10}$$

$\forall\ s \in [0, |S_1|]$. Let us now define the test function $\Psi(r, \theta) = v(V(r))$, which is the way we obtain u back from $u^{\#}$. Define $r(S_1)$ the radius of S_1 in M^*.

By the Rayleigh quotient, if $u^{\#}$ and $z^{\#}$ are not identical,

$$\lambda \int_{S_1} \Psi^2 dV < \int_{S_1} |\operatorname{grad} \Psi|^2 dV = \int_0^{r(S_1)} \left(v'(V(r))V'(r)\right)^2 V'(r)dr$$

Observe that using (4.10),

$$r = V^{-1}(s) \Rightarrow -v'(V(r)) \leq \lambda V'(r)^{-2} \int_0^{V(r)} v(t)dt$$

Hence,

$$\begin{aligned} \lambda \int_{S_1} \Psi^2 dV &\leq \int_0^{r(S_1)} -v'(V(r))V'(r)^3 \left(\lambda V'(r)^{-2} \int_0^{V(r)} v(t)dt \right) dr \\ &= \lambda \int_0^{r(S_1)} -v'(V(r))V'(r) \left(\int_0^{V(r)} v(t)dt \right) dr \\ &= \lambda \int_0^{|S_1|} v^2(s)ds \\ &= \lambda \int_{S_1} \Psi^2 dV. \end{aligned}$$

The second equality comes from an integration by parts:

$$\int_0^{r(S_1)} -v'(V(r))V'(r)\left(\int_0^{V(r)} v(t)dt\right)dr$$

$$= \int_0^{r(S_1)} -\left(v(V(r))\right)'\left(\int_0^{V(r)} v(t)dt\right)dr$$

$$= -\left[v(V(r)\int_0^{V(r)} v(t)dt)\right]_{r=0}^{r=r(S_1)} + \int_0^{r(S_1)} (v(V(r))\, v(V(r))V'(r)dr$$

$$= \int_0^{|S_1|} v^2(s)ds.$$

□

Corollary 4.2.9. *Among all domains of first eigenvalue λ in all manifolds in which H is an isoperimetric function, the ball in M^* maximizes the quotient*

$$\frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{L^2(\Omega)}},$$

for u the eigenfunction associated to the eigenvalue λ.

Application

In [21], the following theorem was proved.

Theorem 4.2.10. *(Kleiner) Let M^3 be a complete, simply connected 3 dimensional Riemannian manifold with sectional curvature $K_M \leq k \leq O$, and let $\mathbb{Q}^3(k)$ be the model space with constant sectional curvature k. If $\Omega \subset \mathbb{Q}^3(k)$ is a compact domain with smooth boundary $\partial\Omega$ and Ω^* is a geodesic ball in $\mathbb{Q}^3(k)$ with the same volume as Ω, then*

$$\mathrm{vol}(\partial\Omega) \geq \mathrm{vol}(\partial\Omega^*).$$

As a consequence of this theorem the isoperimetric profile of $\mathbb{Q}^3(k)$ is an isoperimetric function in M, if $K_M \leq k$. Let us now work with the particular case of $K_M \leq 0$. Then the isoperimetric profile of $\mathbb{R}^3$ is an isoperimetric function in M. Using the fact that balls are ioperimetric domains in the Euclidean space, one easily calculates

$$H(v) = 3\omega_3^{1/3} v^{2/3}$$

is the isoperimetric profile of $\mathbb{R}^3$.

Applying Lemma 4.2.1, we obtain that $f(t) = t$ is the function that determines M^* implying that M^* is $\mathbb{R}^3$. Since $u(x) = \frac{\sin(|x|)}{|x|}$ is the first eigenfunction of $B_\pi(0)$ associated to eigenvalue 1, we have the following result.

Theorem 4.2.11. *If M is a complete, simply connected $3-$dimensional Riemannian manifold with sectional curvature $K_M \leq 0$ and $\Omega \subset M$ is a compact domain of eigenvalue 1, then the eigenfunction associated to 1 in Ω satisfies*

$$\frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{L^2(\Omega)}} \leq \frac{\|\frac{\sin(|x|)}{|x|}\|_{L^\infty(B_\pi(0))}}{\|\frac{\sin(|x|)}{|x|}\|_{L^2(B_\pi(0))}} \approx 0.839728.$$

4.3 An estimate obtained directly from the isoperimetric function

The ideas of this section follow [10] and [28].

Let M be a $n-$dimensional complete Riemannian manifold in with isoperimetric function H.

Theorem 4.3.1. *Let u be a solution of $-\Delta u = c$ in Ω, $u = 0$ on $\partial\Omega$. Then*

$$\sup u \leq cf\left(|\Omega|\right),$$

where

$$f(t) = \int_0^t \frac{s}{H(s)^2} ds.$$

Remark 4.3.2. *We assume that H is nice enough at the origin in order that the integrand is well defined.*

Proof. Let u be a solution of $-\Delta u = c$ in Ω, $u = 0$ on $\partial\Omega$. Consider μ the distribution function of u (Definition 4.2.2). For all $t > 0$, the set $\{|u(x)| > t\}$ is at a positive distance from the boundary $\partial\Omega$. It is compactly contained in Ω, has boundary $\{u = t\}$ with inner normal vector $\frac{\operatorname{grad} u}{|\operatorname{grad} u|}$, well defined for almost all $t > 0$ by Sard's Theorem. If the inner normal vector is well defined for t, we apply the Divergent Theorem on the differential equation, obtaining

$$\int_{\{u>t\}} c dx = \int_{\{u=t\}} |\operatorname{grad} u| da_t. \tag{4.11}$$

From the expression for the derivative of μ a. e. (4.5) and the Cauchy-Schwarz inequality (4.6), we obtain

$$|\{u = t\}| \leq (-\mu'(t))^{1/2} \left(\int_{\{u=t\}} |\operatorname{grad} u| H_{n-1}(dx) \right)^{1/2}.$$

On the other hand,

$$|\{u = t\}| \geq H(\mu(t)),$$

by the isoperimetric inequality applied for Ω_t.

Hence, joining all the expressions,

$$-\mu'(t)c\mu(t) = (-\mu'(t)) \int_{\{u=t\}} |\operatorname{grad} u| da_t \geq |\{u = t\}|^2 \geq H^2(\mu(t)),$$

for almost all $t > 0$, which implies that

$$1 \leq \frac{c\mu(t)(-\mu'(t))}{H^2(\mu(t))}.$$

Note that

$$-c\frac{d}{dt} f(\mu(t))$$

is the right hand side of the above inequality, so that one can integrate it in $[0, t]$, obtaining

$$t \leq -c \left(f(\mu(t)) - f(|\Omega|) \right)$$

Taking $t = \sup u$, $\mu(t) = 0$ and the proof is complete. □

Remark 4.3.3. *In the hyperbolic plane* $\varphi(s) = 4\pi s + s^2$, *which implies that* $f(s) = \ln(\frac{4\pi+s}{4\pi})$ *and applying the theorem* $\sup u \leq c \ln(1 + \frac{|\Omega|}{4\pi})$.

In order to continue the process we have to observe that f is an invertible function, which happens by its definition. Besides, since the derivative of f is positive, f^{-1} is also an increasing function.

Theorem 4.3.4. *Consider $\Omega \subset M$ a bounded domain and let w be a solution of*

$$\begin{cases} -\Delta w = \lambda w \ \text{in} \ \Omega \\ w = 0 \ \text{on} \ \partial\Omega \end{cases}$$

for λ any eigenvalue of Ω. Then $\frac{1}{2\lambda} \leq f\left(\left[\frac{2\|w\|_r}{\|w\|_\infty}\right]^r\right)$ where f was defined in theorem above.

Proof. Denote by $K = \|w\|_\infty$ and let us assume that $\max w = \max |w|$. Fix $\rho \geq 1$ and $\widetilde{\Omega} = \{x \in \Omega | |w(x)| > K/\rho\}$. Then,

$$\|w\|_r^r = \int_\Omega |w|^r dx \geq \int_\Omega |w|^r dx \geq \left(\frac{K}{\rho}\right)^r |\widetilde{\Omega}| \tag{4.12}$$

On the other hand,

$$-\Delta w = \lambda w \leq \lambda K.$$

By the comparison principle, $w \leq u$ in $\widetilde{\Omega}$ where u is solution of

$$\begin{cases} -\Delta v = \lambda K \text{ in } \widetilde{\Omega} \\ v = K/\rho \text{ on } \partial\widetilde{\Omega} \end{cases}$$

From Theorem 4.3.1,

$$K = \sup u \leq \frac{K}{\rho} + \lambda K f\left(|\widetilde{\Omega}|\right).$$

Hence,

$$|\widetilde{\Omega}| \geq f^{-1}\left(\frac{\rho K - K}{\lambda K \rho}\right).$$

Taking $\rho = 2$ and remembering equation (4.12), we obtain

$$\|w\|_r^r \geq \left(\frac{K}{2}\right)^r f^{-1}\left(\frac{1}{2\lambda}\right).$$

Applying f in the inequality, the proof ends. □

In the particular case of $H(v) = Dv^{(n-1)/n}$, Theorem 4.3.1 states that if u is a solution of $-\Delta u = c$ in Ω, $u = 0$ on $\partial\Omega$, then

$$\sup u \leq \frac{nc|\Omega|^{2/n}}{2D^2}.$$

Theorem 4.3.4 states that a solution u of

$$\begin{cases} -\Delta w = \lambda w \text{ in } \Omega \\ w = 0 \text{ on } \partial\Omega \end{cases}$$

for λ any eigenvalue of Ω satisfies

$$\max u \leq 2(n\lambda)^{\frac{n}{2r}} D^{-n/r} \|u\|_r$$

for any $r > 0$. Furthermore,

$$|\Omega_t| \geq \left(\frac{2(\|u\|_\infty - t)}{n\lambda \|u\|_\infty} \right)^{n/2} D^n.$$

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