Low energy limit of the Chern-Simons theory coupled to fermions

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We study the nonrelativistic limit of the theory of a quantum Chern-Simons field minimally coupled to Dirac fermions. To get the nonrelativistic effective Lagrangian one has to incorporate vacuum polarization and anomalous magnetic moment effects. In addition to that, an unsuspected quartic fermionic interaction may also be induced. As a by-product, the method we use to calculate loop diagrams, separating low and high loop momenta contributions, allows us to identify how a quantum nonrelativistic theory nests in a relativistic one. [S0556-2821(97)01418-5]

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I. INTRODUCTION

Nonrelativistic quantum field theories are important to the description and clarification of conceptual aspects of the physics of systems in the low energy regime. One example of such a situation is provided by the treatment of the Aharonov-Bohm effect [1] by means of the model of a scalar field minimally coupled to a Chern-Simons (CS) field. In order to achieve accordance between the exact and perturbative one-loop calculations, it was necessary to include in the perturbative approach a quartic scalar self-interaction with a coupling tuned to eliminate divergences and to restore the conformal invariance of the tree approximation [2]. It was later also shown that this quadrilinear interaction automatically arises in the low energy limit of the corresponding full-fledged relativistic quantum theory [3].

In this paper we will extend the above considerations to the fermionic case. Specifically, we study the low energy limit of the (2+1)-dimensional theory of a CS field minimally coupled to fermions, as specified by the Lagrangian density [4]

$$\mathcal{L} = \frac{\theta}{4} \epsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha + \frac{i}{2} \overline{\psi} \gamma^\mu \partial_\mu \psi - \frac{i}{2} (\partial_\mu \overline{\psi}) \gamma^\mu \psi - m \overline{\psi} \psi$$
$$+ e \overline{\psi} \gamma^\mu \psi A_\mu, \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and ψ is a two-component Dirac field representing a fermion and an antifermion of the same spin [5]. In particular, we investigate up to which extent the Pauli-Schrödinger (PS) nonrelativistic Lagrangian

$$\mathcal{L} = \phi^\dagger \left(i \frac{d}{dt} - eA^0 \right) \phi - \frac{1}{2m} (\vec{\nabla}\phi - ie\vec{A}\phi)^\dagger \cdot (\vec{\nabla}\phi - ie\vec{A}\phi)$$
$$+ \frac{e}{2m} B \phi^\dagger \phi + \frac{\theta}{4} \epsilon_{\mu\nu\rho} A^\mu F^{\nu\rho}, \tag{1.2}$$

where ϕ is a one-component anticommuting Pauli field, correctly describes the low energy limit of Eq. (1.1). Here, $B = -F_{12}$ is the magnetic field.

By rescaling the A^μ field we could formulate the models above with only one dimensionless interaction strength ($e/\sqrt{\theta}$). Actually, this is the small parameter which appears in our perturbative expansions. Nevertheless, Eqs. (1.1) and (1.2) are conventional in the literature, A_μ having the same canonical dimension as a Maxwell field in 2+1 dimensions.

To fix ideas, we choose once and for all to work in the Coulomb gauge, where the A_μ free propagator is known to be

$$\Delta_{\mu\nu}(k) = \frac{1}{\theta} \epsilon_{\mu\nu\rho} \frac{\overline{k}^\rho}{\overline{k}^2}, \tag{1.3}$$

where $\overline{k}^\alpha \equiv (0, \vec{k})$. The k^0 independence of this propagator is the basic reason why the use of the Coulomb gauge is convenient for nonrelativistic calculations.

Let us begin by pointing out that Eq. (1.2) cannot fully reproduce the two and three point vertex functions arising from Eq. (1.1). This is due to the absence of one-loop radiative corrections in Eq. (1.2), which follows from the antisymmetry of Eq. (1.3) and the fact that nonrelativistic fermions only propagate forward in time. On the other hand, the radiative corrections arising from Eq. (1.1) do not vanish. The contributions to the two and three point functions are, actually, superficially divergent and need to be subtracted.

We begin our study by investigating the nonrelativistic model (1.2). In Sec. II we verify the absence of one-loop radiative corrections to the two and three point vertex functions. We study also the fermion-fermion scattering amplitude and confirm the assertion made in [2] that the Pauli's interaction term regularizes the theory and gives a contribution essential to reproduce the correct Aharonov-Bohm amplitude up to order e^4.

In Sec. III we examine the relativistic model (1.1). After summarizing the renormalization program [6], we prove that radiative corrections induce both an "anomalous" magnetic term and a Maxwell term, absent in Eq. (1.2). To calculate the one-loop corrections to the fermion-fermion scattering amplitudes we will employ a scheme which separates the contributions of the low and high momenta intermediary

FIG. 1. Feynman rules for the PS theory.

states [7]. This allows a direct simplification of the integrands and it is closely related to the methods of effective field theories [8]. We will show that the low momenta intermediary part coincides with the result for the same process calculated from the Lagrangian (1.2). On the other hand, the contributions from the high intermediary momenta can be thought as coming from new interactions in the effective nonrelativistic theory. We finish this section presenting a discussion of our results.

II. NONRELATIVISTIC THEORY

Our purpose here is to use Eq. (1.2) to study the interaction of the CS field with fermions up to the one-loop approximation. Afterwards, we shall compare these results with those for Dirac fermions, calculated from Eq. (1.1) in the low energy regime.

As said in the Introduction, we shall work in the radiation gauge where the free gauge field propagator is given in Eq. (1.3). From that expression we can get the free propagator

$$\Delta_B(x) = \langle TB(x)A^0(0)\rangle = -\frac{i}{\theta}\delta^3(x), \qquad (2.1)$$

which is also necessary to construct Feynman amplitudes.

As the second quantized free fermionic field is

$$\phi(\vec{x},t) = \int \frac{d^2k}{2\pi} b(\vec{k})\exp[-i(\vec{k}^2/2m\,t - \vec{k}\cdot\vec{x})], \quad (2.2)$$
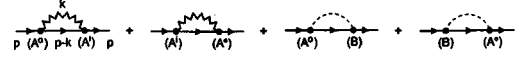
where the annihilation operator $b(\vec{k})$ satisfies

FIG. 2. One-loop contributions to the PS fermion self-energy.

$$\{b(\vec{k}), b^\dagger(\vec{k}')\} = \delta^2(\vec{k}-\vec{k}'), \qquad (2.3)$$

the free ϕ propagator turns out to be given by

$$G(x-y) \equiv \langle T\phi(x)\phi^\dagger(y)\rangle = \theta(x^0-y^0)\langle\phi(x)\phi^\dagger(y)\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3}\exp[-ik^0(x^0-y^0-\eta)+i\vec{k}\cdot(\vec{x}-\vec{y})]$$

$$\times \frac{i}{k^0-\vec{k}^2/2m+i\epsilon}, \qquad (2.4)$$

where η and ϵ are positive parameters to be set to zero at the end of the calculations. In the above definition a time-ordering prescription was chosen so that the propagator is zero for $x^0=y^0$. This implies that any closed fermion loop automatically vanishes.

Our graphical notation is shown in Fig. 1. Using these rules, one can demonstrate that there are no one-loop corrections to the propagators and vertices. The would-be one-loop corrections to the fermion two point function are represented in Fig. 2. The first two of those diagrams cancel each other due to the antisymmetry of the A_μ propagator and the third and fourth are in fact closed fermionic loops [see Eq. (2.1)] and, therefore, give no contribution as remarked following Eq. (2.4). Similar arguments allow one to extend these conclusions to all remaining one-loop vertex and propagators graphs.

We will next study the fermion-fermion elastic scattering. To consider the possibility of the scattering of nonidentical fermions we will not antisymmetrize our amplitudes. The incoming and outgoing fermions are assumed to have momenta $p_1 = (\vec{p}_1^2/2m, \vec{p}_1)$, $p_2 = (\vec{p}_2^2/2m, \vec{p}_2)$ and $p_1' = (\vec{p}_1'^2/2m, \vec{p}_1')$, $p_2' = (\vec{p}_2'^2/2m, \vec{p}_2')$, respectively. We shall work in the center-of-mass frame where $\vec{p}_1 = -\vec{p}_2 = \vec{p}$, $\vec{p}_1' = -\vec{p}_2' = \vec{p}'$, and $|\vec{p}| = |\vec{p}'|$.

Up to one loop the nonvanishing contributions come from the diagrams in Figs. 3 and 4. The amplitude corresponding to the diagrams in Fig. 3 is

$$A^{(0)} = \frac{e^2}{m\theta}\left[1 + i\frac{\vec{s}\times\vec{q}}{\vec{q}^2}\right], \qquad (2.5)$$

where $\vec{s} = \vec{p} + \vec{p}'$, $\vec{q} = \vec{p}' - \vec{p}$, and $\vec{s}\times\vec{q}$ stands for $\epsilon^{ij}s_iq_j$. The $\vec{q}$-dependent term result from the graph containing the propagator $\langle TA^0A^i\rangle$ and the $\vec{q}$-independent one comes from the contact Pauli interaction mediated by $\langle TBA^0\rangle$.

Let us now examine the one-loop diagrams. We will first do the k^0 integration and then regularize the remaining $\vec{k}$ integration by a cutoff Λ. The box diagrams [Fig. 4(a)] yield

$$A_{\text{box}}^{(1)} = -\frac{e^4 i}{m^2 \theta^2} \int_0^\Lambda \frac{d^3 k}{(2\pi)^3} \frac{1}{k^0 + (\vec{p}^2 - \vec{k}^2)/2m + i\epsilon} \frac{1}{-k^0 + (\vec{p}^2 - \vec{k}^2)/2m + i\epsilon} \epsilon_{ij}\epsilon_{ln} \frac{(k-p')^j}{|\vec{k}-\vec{p}\,'|^2} \frac{(k-p)^n}{|\vec{k}-\vec{p}|^2} (-p'-k)^i (p'+k)^l$$

$$= -\frac{4e^4}{m\theta^2} \int_0^\Lambda \frac{d^2 k}{(2\pi)^2} \frac{1}{\vec{k}^2 - \vec{p}^2 - i\epsilon} \frac{\vec{p}' \times \vec{k}}{|\vec{k}-\vec{p}\,'|^2} \frac{\vec{p} \times \vec{k}}{|\vec{k}-\vec{p}|^2} = \frac{e^4}{4\pi m \theta^2} \left\{ \ln\left[\frac{\vec{q}^2}{\vec{p}^2}\right] + i\pi \right\} + O(\vec{p}^2/\Lambda^2), \tag{2.6}$$

while the ''contact'' diagram [Fig. 4(b)] gives

$$A_{\text{cont}}^{(1)} = i\frac{e^4}{m^2\theta^2} \int_0^\Lambda \frac{d^3 k}{(2\pi)^3} \frac{1}{k^0 + (\vec{p}^2 - \vec{k}^2)/2m + i\epsilon} \frac{1}{-k^0 + (\vec{p}^2 - \vec{k}^2)/2m + i\epsilon}$$

$$= -\frac{e^4}{m\theta^2} \int_0^\Lambda \frac{d^2 k}{(2\pi)^2} \frac{1}{\vec{k}^2 - \vec{p}^2 - i\epsilon} = -\frac{e^4}{4\pi m \theta^2} \left\{ \ln\frac{\Lambda^2}{\vec{p}^2} + i\pi \right\} + O(\vec{p}^2/\Lambda^2). \tag{2.7}$$

Finally, the triangle graphs [Figs. 4(c) and 4(d)] give

$$A_{\text{tri}}^{(1)} = -\frac{2e^4 i}{m\theta^2} \int_0^\Lambda \frac{d^3 k}{(2\pi)^3} \frac{1}{-k^0 + \vec{p}^2/2m - (|\vec{p}+\vec{k}|^2)/2m + i\epsilon} \epsilon^{il}\epsilon^{in} \frac{k^l}{|\vec{k}|^2} \frac{(p'-p-k)^n}{|\vec{p}'-\vec{p}-\vec{k}|^2} = -\frac{e^4}{m\theta^2} \int_0^\Lambda \frac{d^2 k}{(2\pi)^2} \frac{\vec{k}\cdot(\vec{p}'-\vec{p}-\vec{k})}{|\vec{k}|^2 |\vec{p}'-\vec{p}-\vec{k}|^2},$$
$$\tag{2.8}$$

where the k_o integration was done symmetrically. The final result is then

$$A_{\text{tri}}^{(1)} = -\frac{e^4}{4\pi m \theta^2} \left\{ \ln\frac{\vec{q}^2}{\Lambda^2} \right\} + O(\vec{p}^2/\Lambda^2). \tag{2.9}$$

By summing up the above contributions we get

$$A = A^{(0)} + A_{\text{cont}}^{(1)} + A_{\text{box}}^{(1)} + A_{\text{tri}}^{(1)} = \frac{e^2}{m\theta} \left[1 + i\frac{\vec{s}\times\vec{q}}{\vec{q}^2} \right] + O(\vec{p}^2/\Lambda^2). \tag{2.10}$$

This result shows that, up to one loop, there is no radiative correction to the nonrelativistic scattering. This holds for all values of the coupling constant e. In the model of a nonrelativistic boson coupled to a CS field, a similar result was first obtained in [2]. There, the role of the contact Pauli interaction was played by a $\lambda(\phi^\dagger \phi)^2$ interaction with λ chosen to restore the scale invariance [9] present in the tree approximation. Observe also that in terms of the angle χ between $\vec{p}$ and $\vec{p}\,'$, Eq. (2.10) becomes

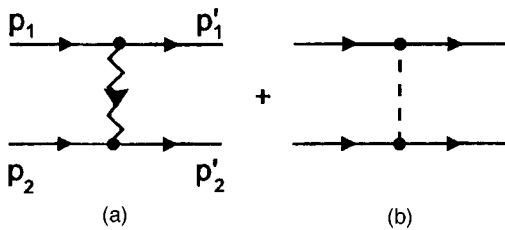$$i\frac{e^2}{m\theta} \frac{e^{-i\chi/2}}{\sin(\chi/2)}, \tag{2.11}$$

FIG. 3. Graphs for the PS fermion-fermion scattering in the tree approximation.

which is the expansion up to order e^4 of the Aharonov-Bohm amplitude for fermions [10].

III. RELATIVISTIC THEORY

We now consider the relativistic theory defined by Eq. (1.1). The corresponding Feynman rules are depicted in Fig. 5. By power counting the model is renormalizable, the degree of divergence of a graph γ being $\delta(\gamma) = 3 - F - B$, where F and B are the number of external fermion and boson lines, respectively. Thus, the only divergences are those associated with the fermion two point function, the CS two point function, and with the vertex. The renormalization of the model in the Coulomb gauge, up to one loop, was studied in [6], using dimensional regularization. Here, for completeness, we just stress the main points of that calculation.

The ambiguities in the finite parts are eliminated by adding to Eq. (1.1) the counterterm Lagrangian density

$$\mathcal{L}_c = \delta Z \left(\frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{i}{2} (\partial_\mu \bar{\psi}) \gamma^\mu \psi \right) - \delta m \bar{\psi}\psi$$

$$+ \frac{\delta\theta}{4} \epsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha + \delta e \, \bar{\psi} \gamma^\mu A_\mu \psi, \tag{3.1}$$

where the coefficients δZ, δm, $\delta\theta$, and δe are fixed by the normalization conditions specifying the ψ field intensity, the values of the physical mass, CS parameter, and charge, respectively.
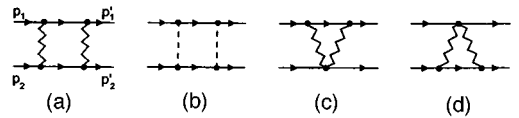

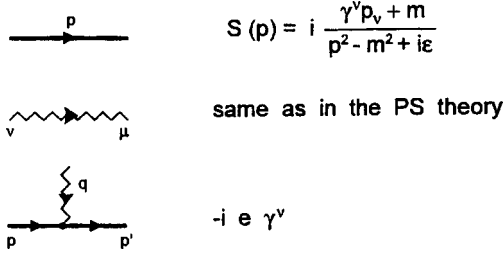FIG. 4. Nonvanishing contributions to the PS fermion-fermion scattering in one-loop approximation.

$$S(p) = i\, \frac{\gamma^\nu p_\nu + m}{p^2 - m^2 + i\varepsilon}$$

same as in the PS theory

$$-i\, e\, \gamma^\nu$$

FIG. 5. Feynman rules for the relativistic theory of a Dirac fermion coupled to a CS field.

First, consider the CS self-energy,

$$\prod{}^{\alpha\beta}(k) = -e^2 \int \frac{d^3q}{(2\pi)^3} \frac{\mathrm{Tr}[\gamma^\alpha(\slashed{q}+\slashed{k}+m)\gamma^\beta(\slashed{q}+m)]}{[(k+q)^2-m^2](q^2-m^2)}$$
$$-2\delta\theta\epsilon^{\alpha\beta\rho}k_\rho$$
$$= -i[k^2 g^{\alpha\beta}-k^\alpha k^\beta]A(k^2) + \epsilon^{\alpha\beta\rho}k_\rho[B(k^2)$$
$$-2\delta\theta], \tag{3.2}$$

where

$$A(k^2) = \left(\frac{m}{k^2}+\frac{1}{4m}\right)B(k^2)-\frac{e^2 m}{4\pi k^2}, \tag{3.3}$$

and

$$B(k^2) = \frac{e^2 m}{4\pi}\int_0^1 dx\, \frac{1}{\sqrt{k^2 x(x-1)+m^2-i\epsilon}}. \tag{3.4}$$

By choosing $\delta\theta = B(k=0)/2 = (e^2/8\pi)\,\varepsilon(m)$, where ε denotes the sign function, we fix θ to be the renormalized CS parameter (this renormalization could, equivalently, be interpreted as a wave function renormalization for A_μ). For low momentum, $\Pi^{\alpha\beta}$ approaches the expression

$$\prod{}^{\alpha\beta}(k) = -i\,\frac{e^2}{12\pi|m|}(k^2 g^{\alpha\beta}-k^\alpha k^\beta), \tag{3.5}$$

showing the well-known phenomena of induction of a Maxwell term in the effective Lagrangian of the model [11].

The fermion self-energy is

$$\sum(p) = -\frac{e^2 i}{\theta}\int \frac{d^3 k}{(2\pi)^3} \frac{\gamma^\mu(\slashed{k}+\slashed{p}+m)\gamma^\nu}{(k+p)^2-m^2}\frac{\epsilon_{\mu\nu\rho}\overline{k}^\rho}{(\vec{k})^2}$$
$$+i\delta Z\slashed{p}-i\delta m$$
$$= -\frac{ie^2}{2\pi\theta}\left[\frac{(\vec{p}^2-m\vec{p}\cdot\vec{\gamma})}{m+w_p}-m\right]+i\delta Z\slashed{p}-i\delta m. \tag{3.6}$$

Choosing $\delta m = me^2/2\pi\theta$, we guarantee that the pole of the fermion propagator up to this order is at $p^2 = m^2$. In addition to that, taking $\delta Z = 0$ the form of the propagator in the fermion rest frame is the same as for the free case [12].

Finally, the radiative correction to the vertex is given by

$$\Gamma^\rho(p,p') = \frac{ie^3}{\theta}\int \frac{d^3 k}{(2\pi)^3}\frac{\gamma^\mu(\slashed{p}'-\slashed{k}+m)\gamma^\rho(\slashed{p}-\slashed{k}+m)\gamma^\alpha\epsilon_{\alpha\mu\nu}\overline{k}^\nu}{[(p'-k)^2-m^2+i\epsilon][(p-k)^2-m^2+i\epsilon](-\vec{k}^2)} + i(e+\delta e)\gamma^\rho. \tag{3.7}$$

Thus, choosing $\delta e = 0$, we get in the low momentum regime

$$\bar{u}(p')\Gamma^0 u(p) = ie, \tag{3.8}$$

$$\bar{u}(p')\Gamma^i u(p) = ie\left(1-\frac{e^2}{4\pi\theta}\right)\frac{(p+p'^i)}{2m}$$
$$+e\left(1+\frac{e^2}{4\pi\theta}\right)\epsilon^{ij}\frac{(p'-p)^j}{2m}. \tag{3.9}$$

In a covariant gauge the magnetic moment of the fermion could be read as the coefficient of $\epsilon^{ij}(p'-p)^j$ in this last expression. This happens because only the first of the three diagrams of Fig. 6, which appear in the calculation of the scattering of the fermion by an external field $\mathcal{A}^\rho$, is nonvanishing on shell. In the Coulomb gauge this is not so and only after taking into account the contribution of all three diagrams we get $e^3/(4\pi m\theta)$ for the anomalous magnetic moment of the fermion. This result is in accord with calculations in covariant gauges where only graph [Fig. 6(a)]

contributes [13]. If we take into account that the spin also changes [4,13,14] we conclude that relation $\mu = e/mS$ between the magnetic moment μ and the spin S of the particle holds, keeping $g=2$.

It is now clear that, up to one loop, instead of Eq. (1.2), these radiative corrections induce the effective nonrelativistic Lagrangian,

$$\mathcal{L}_{\mathrm{eff}} = \phi^\dagger\left(i\frac{d}{dt}-eA^0\right)\phi - \frac{1}{2m}(\vec{\nabla}\phi-ie\vec{A}\phi)^\dagger\cdot(\vec{\nabla}\phi-ie\vec{A}\phi)$$
$$+\frac{e}{2m}gB\phi^\dagger\phi+\frac{\theta}{4}\epsilon_{\mu\nu\rho}A^\mu F^{\nu\rho}-\frac{1}{4}\left(\frac{e^2}{12\pi m}\right)F^{\mu\nu}F_{\mu\nu}, \tag{3.10}$$

where $g\equiv 1+e^2/2\pi\theta$.

We shall next look for the appearance of a $(\phi^\dagger\phi)^2$ vertex in the effective nonrelativistic Lagrangian. We so focus on the elastic fermion-fermion scattering amplitude [15]. In the center-of-mass frame, the incoming and outgoing fermions
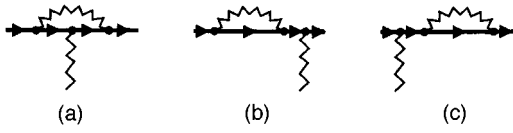
FIG. 6. One-loop contributions to the fermion anomalous magnetic moment.

are assumed to have momenta $p_1=(w_p,\vec{p})$, $p_2=(w_p,-\vec{p})$, and $p_1'=(w_p,\vec{p}')$, $p_2'=(w_p,-\vec{p}')$, where $|\vec{p}|=|\vec{p}'|$ and $w_p=\sqrt{m^2+p^2}$.

The various contributions, up to one loop, are shown in Figs. 7 and 8. The tree approximation (Fig. 7) is given by

$$T^{(0)}=-ie^2\,\overline{u}(\vec{p}')\gamma^\mu u(\vec{p})\Delta_{\mu\nu}(\vec{p}'-\vec{p})\overline{u}(-\vec{p}')\gamma^\nu u(-\vec{p}).$$
$$(3.11)$$

Its low energy approximation is gotten by expanding $w_p=(m^2+p^2)^{1/2}$ in powers of $(|\vec{p}|/m)\,(\ll 1)$. To leading order, we have

$$T^{(0)}=\frac{e^2}{\theta m}\left(1+i\frac{\vec{s}\wedge\vec{q}}{\vec{q}^2}\right).$$
$$(3.12)$$

Observe that Eq. (3.12) is the same as the e^2 amplitude (2.5) in the PS theory, due to exchange of one photon, including the contribution of the Pauli interaction.

Self-energy and vertex radiative corrections to the tree approximation (Fig. 7), in leading $1/m$ order, give

$$T_R=\frac{e^4}{12\pi m\theta^2}+\frac{e^4}{2\pi m\theta^2}=\frac{7e^4}{12\pi m\theta^2},$$
$$(3.13)$$

where the first and second terms in the first equality come, respectively, from the vacuum polarization and vertex insertions. T_R must not be considered for the induction of a term $(\phi^\dagger\phi)^2$ since self-energy and vertex corrections have already been incorporated in Eq. (3.10) through the fermion anomalous magnetic moment and the Maxwell terms.

It remains to calculate the graphs in Figs. 8(a) and 8(b). They are, respectively, given by (the subscripts B and X stand for box and crisscross two photons exchange amplitudes)

$$T_B=ie^4\int\frac{d^3k}{(2\pi)^3}[\overline{u}(\vec{p}')\gamma^\mu S_F(r)\gamma^\nu u(\vec{p})]$$
$$\times[\overline{u}(-\vec{p}')\gamma^\alpha S_F(r')\gamma^\beta u(-\vec{p})]$$
$$\times\Delta_{\nu\beta}(\vec{k}-\vec{p})\Delta_{\alpha\mu}(\vec{k}-\vec{p}')\qquad(3.14)$$

and

$$T_X=ie^4\int\frac{d^3k}{(2\pi)^3}[\overline{u}(\vec{p}')\gamma^\mu S_F(r)\gamma^\nu u(\vec{p})]$$
$$\times[\overline{u}(-\vec{p}')\gamma^\alpha S_F(t)\gamma^\beta u(-\vec{p})]$$
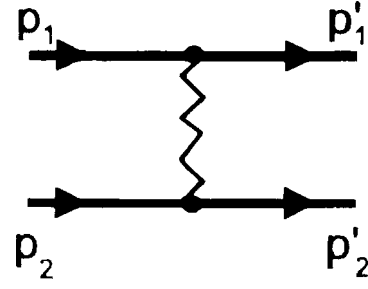$$\times\Delta_{\nu\alpha}(\vec{k}-\vec{p})\Delta_{\beta\mu}(\vec{k}-\vec{p}'),\qquad(3.15)$$

FIG. 7. Graph for the relativistic fermion-fermion scattering in the tree approximation.

where $r\equiv(w_p+k^0,\vec{k})$, $r'\equiv(w_p-k^0,-\vec{k})$, and $t\equiv(w_p+k^0,\vec{k}-\vec{p}-\vec{p}')$.

In what follows, the free fermion propagator will be written in terms of fermion and antifermion wave functions [5]:

$$S_F(p)=i\frac{u(\vec{p})\overline{u}(\vec{p})}{p^0-w_p+i\epsilon}+i\frac{v(-\vec{p})\overline{v}(-\vec{p})}{p^0+w_p-i\epsilon}.$$
$$(3.16)$$

This device greatly simplifies the calculation of the k^0 integrals. As a by-product, we can trace the contribution of fermions or antifermions in intermediary states.

Replacing Eq. (3.16) in the expressions above, we get

$$T_B=-ie^4\int\frac{d^3k}{(2\pi)^3}\Delta_{\nu\beta}(\vec{k}-\vec{p})\Delta_{\alpha\mu}(\vec{k}-\vec{p}')$$
$$\times\overline{u}(\vec{p}')\gamma^\mu\left[\frac{u(\vec{k})\overline{u}(\vec{k})}{k^0+w_p-w_k+i\epsilon}\right.$$
$$\left.+\frac{v(-\vec{k})\overline{v}(-\vec{k})}{k^0+w_p+w_k-i\epsilon}\right]\gamma^\nu u(\vec{p})\overline{u}(-\vec{p}')\gamma^\alpha$$
$$\times\left[\frac{u(-\vec{k})\overline{u}(-\vec{k})}{w_p-k^0-w_k+i\epsilon}+\frac{v(\vec{k})\overline{v}(\vec{k})}{w_p-k^0+w_k-i\epsilon}\right]\gamma^\beta u(-\vec{p})$$
$$(3.17)$$

and

$$T_X=-ie^4\int\frac{d^3k}{(2\pi)^3}\Delta_{\nu\alpha}(\vec{k}-\vec{p})\Delta_{\beta\mu}(\vec{k}-\vec{p}')$$
$$\times\overline{u}(\vec{p}')\gamma^\mu\left[\frac{u(\vec{k})\overline{u}(\vec{k})}{k^0+w_p-w_k+i\epsilon}\right.$$
$$\left.+\frac{v(-\vec{k})\overline{v}(-\vec{k})}{k^0+w_p+w_k-i\epsilon}\right]\gamma^\nu u(\vec{p})$$
$$\times\overline{u}(-\vec{p}')\gamma^\alpha\left[\frac{u(\vec{k}-\vec{s})\overline{u}(\vec{k}-s)}{k^0+w_p-w_{k-s}+i\epsilon}\right.$$
$$\left.+\frac{v(\vec{s}-\vec{k})\overline{v}(\vec{s}-\vec{k})}{k^0+w_p+w_{k-s}-i\epsilon}\right]\gamma^\beta u(-\vec{p}).\qquad(3.18)$$
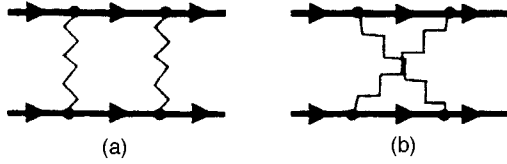
FIG. 8. Graphs contributing to the relativistic fermion-fermion scattering in one-loop approximation.

The integration in k^0 can be done by closing the contour in the upper half k^0 complex plane. After some simplifications, we get

$$T_B = T_B^{\text{el,el}} + T_B^{\text{pos,pos}}, \qquad (3.19)$$

where

$$T_B^{\text{el,el}} = -\frac{e^4}{2} \int \frac{d^2k}{(2\pi)^2} \frac{w_k + w_p}{\vec{k}^2 - \vec{p}^2 + i\epsilon} F^*(\vec{k}, \vec{p}\,') F(\vec{k}, \vec{p}) \qquad (3.20)$$

and

$$T_B^{\text{pos,pos}} = -\frac{e^4}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{w_k + w_p} H(\vec{p}\,', \vec{k}) H^*(\vec{p}, \vec{k}). \qquad (3.21)$$

For T_X it results

$$T_X = \frac{e^4}{2} \int \frac{d^2k}{(2\pi)^2} \frac{1}{w_k + w_{k-s}} [G(\vec{k} - \vec{s}, -\vec{p}, \vec{p}\,')$$
$$\times G^*(\vec{k} - \vec{s}, -\vec{p}\,', \vec{p}) + G(\vec{k}, \vec{p}, -\vec{p}\,') G^*(\vec{k}, \vec{p}\,', -\vec{p})], \qquad (3.22)$$

where, as before, $\vec{s} = \vec{p} + \vec{p}\,'$, and

$$F(\vec{k}, \vec{p}) = [\bar{u}(\vec{k}) \gamma^\mu u(\vec{p})] \Delta_{\mu\nu}(\vec{k} - \vec{p})[\bar{u}(-\vec{k}) \gamma^\nu u(-\vec{p})],$$

$$H(\vec{p}, \vec{k}) = [\bar{u}(\vec{p}) \gamma^\mu v(-\vec{k})] \Delta_{\mu\nu}(\vec{p} - \vec{k})[\bar{u}(-\vec{p}) \gamma^\nu v(\vec{k})], \qquad (3.23)$$

$$G(\vec{a}, \vec{b}, \vec{c}) = [\bar{u}(\vec{a}) \gamma^\mu u(\vec{b})] \Delta_{\mu\nu}(\vec{a} - \vec{b})$$
$$\times [\bar{u}(\vec{c}) \gamma^\nu v(\vec{b} - \vec{a} - \vec{c})].$$

The terms $T_B^{\text{el,el}}$ and $T_B^{\text{pos,pos}}$ are, respectively, the contributions of the u and v fermion wave functions (electron and positron) to the two internal fermion lines in Fig. 8(a). Mixed contributions, in which u "runs" in one line and v in the other, cancel in this model. On the other hand, each of the two terms in T_X corresponds to a graph in which one of the internal fermion line is a u and the other a v.

We will break the integration region $|\vec{k}| = (0, \infty)$ in two ranges

(1) Contributions of the nonrelativistic intermediate states, corresponding to the loop momentum in the range $|\vec{k}| = (0, \Lambda)$ where Λ is a parameter satisfying $|\vec{p}| \ll \Lambda \ll m$.

(2) The relativistic energy intermediate states contributions, corresponding to the range $|\vec{k}| = (\Lambda, \infty)$ for the loop momentum.

In the region $(0, \Lambda)$, the integrands can be expanded in powers of $1/m$ up to the desired order of approximation $(w_k = m + \vec{k}^2/2m + \vec{k}^4/8m^3 + \cdots)$. We will limit ourselves to the leading $(1/m)$ order which suffices for comparison with the nonrelativistic PS theory. For the region (Λ, ∞), we will expand w_p around $\vec{p} = 0$, but keep w_k exact. So, to extract the leading $(1/m)$ approximation of this part of the integral, an extra expansion in $1/m$ must be made after the integral is computed. With these mentioned approximations, we can write Eqs. (3.14) and (3.15) in leading order as

$$T_B^{\text{el,el}} = -\frac{e^4}{\theta^2 m} \int_0^\Lambda \frac{d^2k}{(2\pi)^2} \frac{1}{\vec{k}^2 - \vec{p}^2 + i\epsilon}$$
$$\times \left[1 + 4 \frac{\vec{k} \wedge \vec{p}}{(\vec{k} - \vec{p})^2} \frac{\vec{k} \wedge \vec{p}\,'}{(\vec{k} - \vec{p}\,')^2} \right]$$
$$- \frac{e^4}{2\theta^2} \int_\Lambda^\infty \frac{d^2k}{(2\pi)^2} \frac{w_k + m}{\vec{k}^2 w_k^2}, \qquad (3.24)$$

$$T_B^{\text{pos,pos}} = -\frac{e^4}{2\theta^2} \int_\Lambda^\infty \frac{d^2k}{(2\pi)^2} \frac{1}{(w_k + m) w_k^2}, \qquad (3.25)$$

$$T_X = \frac{e^4}{m\theta^2} \int_0^\Lambda \frac{d^2k}{(2\pi)^2} \frac{\vec{k}}{\vec{k}^2} \cdot \frac{(\vec{k} - \vec{q})}{(\vec{k} - \vec{q})^2} + \frac{e^4 m^2}{\theta^2} \int_\Lambda^\infty \frac{d^2k}{(2\pi)^2} \frac{1}{\vec{k}^2 w_k^3}. \qquad (3.26)$$

In the integration region $(0, \Lambda)$ of $T_B^{\text{el,el}}$, the contributions of graphs in which one of the photon propagators is $\langle TA^0 A^i \rangle$ and the other $\langle TA^0 B \rangle$, vanish after integrating (they have not been written above); moreover, in $T_B^{\text{pos,pos}}$, the integrand does not have a $1/m$ order contribution. Actually, its leading contribution starts at $(1/m)^3$ which lies beyond the approximation we want to keep.

The low energy parts of T_B and T_X can be identified with amplitudes in the PS theory: The first term in the low energy part of $T_B^{\text{el,el}}$ corresponds to the Fig. 4(b) and the second term to the diagram Fig. 4(a) of the PS theory; the low energy part of T_X exactly corresponds to the PS result (2.8) coming from the Figs. 4(c) and 4(d).

After performing the integrations, one obtains

$$T_B^{\text{el,el}} = \left[-\frac{e^4}{4\pi m \theta^2} \ln \frac{\Lambda^2}{\vec{q}^2} \right]_{\text{low}} + \left[-\frac{e^4}{4\pi m \theta^2} \ln \frac{2m^2}{\Lambda^2} \right]_{\text{high}}, \qquad (3.27)$$

$$T_B^{\text{pos,pos}} = \left[-\frac{e^4}{4\pi m \theta^2} \ln 2 \right]_{\text{high}}, \qquad (3.28)$$

$$T_X^{\text{pos,el}} = \left[-\frac{e^4}{4\pi m \theta^2} \ln\frac{\vec{q}^2}{\Lambda^2} \right]_{\text{low}}$$

$$+ \left[-\frac{e^4}{4\pi m \theta^2} \left(2 + \ln\frac{\Lambda^2}{4m^2} \right) \right]_{\text{high}}, \qquad (3.29)$$

where low and high refer to the integration intervals $|\vec{k}| < \Lambda$ and $|\vec{k}| > \Lambda$, respectively, of the loop momentum $\vec{k}$. Observe that for each graph the sum of the high and low parts is actually Λ independent, as they should be.

If Λ is thought as an ultraviolet cutoff ($\Lambda \to \infty$), each graph of the nonrelativistic theory (*low* part of the relativistic theory) diverges. On the other hand, the corresponding amplitudes in the relativistic Dirac theory are finite. It is interesting to see that their high energy parts exactly provide the counterterms to render the nonrelativistic PS theory finite.

Separately adding the low and the high energy parts of the above amplitudes, we obtain

$$T_B + T_X = \left[-\frac{e^4}{2\pi m \theta^2} \right]_{\text{high}}. \qquad (3.30)$$

The cancellation of the sum of all low energy parts is connected with the absence of scale anomalies in the PS theory. As already observed at the end of Sec. II, in the scalar nonrelativistic theory it was first noticed in [2].

The high energy result (3.30), which is of the same order in $1/m$ as the tree approximation (3.12), is new and could not be suspected from the PS theory. If we are restricted to the model (1.1) with fermions of just one flavor and spin, it in fact gives no contribution after antisymmetrization of the amplitude. Let us so enlarge our model (1.1) by assuming that ψ is an N flavor fermion field. If, analogously, ϕ now is also an N flavor PS fermion, the theory equivalent to the enlarged Eq. (1.1) model will be

$$\mathcal{L}_{\text{eff}} = \phi^\dagger \left(i \frac{d}{dt} - e A^0 \right) \phi - \frac{1}{2m} (\vec{\nabla}\phi - ie\vec{A}\phi)^\dagger \cdot (\vec{\nabla}\phi - ie\vec{A}\phi)$$

$$+ \frac{e}{2m}(1 + e^2/2\pi\theta)B\phi^\dagger\phi + \frac{\theta}{4}\epsilon_{\mu\nu\rho}A^\mu F^{\nu\rho}$$

$$- \frac{1}{4}\left(\frac{Ne^2}{12\pi m} \right)F^{\mu\nu}F_{\mu\nu} + \frac{e^4}{4\pi m \theta^2}(\phi^\dagger\phi)^2. \qquad (3.31)$$

Using this new Lagrangian, the total fermion-fermion scattering amplitude, up to one loop, before antisymmetrization, is

$$T = i\frac{e^2}{m\theta}\frac{e^{-i\chi/2}}{\sin(\chi/2)} + \frac{Ne^4}{12\pi m \theta^2}. \qquad (3.32)$$

For nonidentical fermions the last term survives and provides a correction to the PS result.

Our study has been restricted to the investigation of the induction of terms in the effective Lagrangian in leading order of $1/m$. Of course, a whole series of new terms will be induced in higher orders.

The above Lagrangian summarizes our main results. The low energy limit of the theory of a CS field minimally coupled to Dirac fermions differs from the PS theory by an anomalous magnetic moment, a Maxwell term, and a quartic fermionic term, all of the same $1/m$ order. They are purely quantum field theoretical effects. These results show that taking the nonrelativistic limit of a classical relativistic Lagrangian and then quantizing, leads to a different theory than first quantizing and then taking the nonrelativistic limit.

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$$u(\vec{p}) = \left(\frac{w_p + m}{2w_p} \right)^{1/2} \begin{bmatrix} 1 \\ \frac{p^2 - ip^1}{w_p + m} \end{bmatrix}, \quad v(\vec{p}) = \left(\frac{w_p + m}{2w_p} \right)^{1/2} \begin{bmatrix} \frac{p^2 + ip^1}{w_p + m} \\ 1 \end{bmatrix},$$

where $w_p = (m^2 + \vec{p}^2)^{1/2}$ and the normalizations were chosen so that $\bar{u}u = -\bar{v}v = m/w_p$.

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