Duality symmetry in the Schwarz-Sen model

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The continuous extension of the discrete duality symmetry of the Schwarz-Sen model is studied. The corresponding infinitesimal generator Q turns out to be local, gauge invariant, and metric independent. Furthermore, Q commutes with all the conformal group generators. We also show that Q is equivalent to the nonlocal duality transformation generator found in the Hamiltonian formulation of Maxwell theory. We next consider the Batalin-Fradkin-Vilkovisky formalism for the Maxwell theory and demonstrate that requiring a local duality transformation leads us to the Schwarz-Sen formulation. The partition functions are shown to be the same, which implies the quantum equivalence of the two approaches. [S0556-2821(97)06522-3]

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It is nowadays accepted that the known string theories are different perturbative versions of an underlying M theory [1]. This idea was originated by the several duality symmetries present in the string theories. It is therefore important to study in detail duality symmetries in field theory and understand their implications. In this respect the old electric-magnetic duality present in Maxwell's equations has again been the subject of intensive study [2–7]. These studies show that there is a conflict between electric-magnetic duality symmetry and manifest Lorentz covariance when we attempt to implement duality at the action level. If manifest Lorentz covariance is maintained, then the action is either nonpolynomial [2] or requires an infinite set of fields [3]. If we give up manifest Lorentz covariance, then duality symmetry can be implemented in the Hamiltonian formulation of Maxwell theory in a nonlocal way [4]. However, a recent proposal made by Schwarz and Sen implements duality in a local way at the expenses of introducing one more potential [5]. Although the Schwarz-Sen formulation is not manifestly covariant, it is Poincaré covariant both at the classical [5] and quantum [7] levels. In this paper we shall investigate some consequences of such a proposal. In particular, we will construct the generator of duality transformations and discuss its meaning and relation with the corresponding nonlocal generator found in the first order Hamiltonian formulation described in [4]. We will also verify that the Schwarz-Sen proposal can be understood as a formulation where the nonlocal duality transformation of [4] is turned into a local one. This is shown in the Batalin-Fradkin-Vilkovisky path integral formalism. As a consequence, this also implies a quantum equivalence of the two approaches. Although we consider only the free field case, our work exemplifies the use of a method that may be helpful in more general situations.

The Schwarz-Sen action [5] involves two gauge potentials $A^{\mu a}$ ($1\leqslant a\leqslant 2$ and $0\leqslant\mu\leqslant 3$) and is given by

$$S=-\frac{1}{2}\int d^4x(B^{a,i}\epsilon^{ab}E^{b,i}+B^{a,i}B^{a,i}), \tag{1}$$

where

$$E^{a,i}=-F^{a,0i}=-(\partial^0A^{a,i}-\partial^iA^{a,0}), \tag{2}$$

$$B^{a,i}=-\frac{1}{2}\epsilon^{ijk}F^a_{jk}=-\epsilon^{ijk}\partial_jA^a_k, \tag{3}$$

and ϵ is the Levi-Civitá symbol ($\epsilon^{12}=1$, $\epsilon^{123}=1$) and $1\leqslant i,j,k\leqslant 3$. This action is separately invariant under the local gauge transformations

$$A^{a,0}\to A^{a,0}+\psi^a, \tag{4}$$

$$A^{a,i}\to A^{a,i}-\partial^i\Lambda^a \tag{5}$$

and the global SO(2) rotations

$$A'^{\mu a}=A^{\mu a}\cos\theta+\epsilon^{ab}A^{\mu b}\sin\theta, \tag{6}$$

which reduces to the usual discrete duality transformation for $\theta=\pi/2$.

The Noether's charge associated with this SO(2) symmetry is

$$Q=-\frac{1}{2}\int d^3x\epsilon^{jik}(\partial_jA^a_i)A^a_k=\frac{1}{2}\int d^3xB^{ak}A^a_k. \tag{7}$$

Notice that Q is a SO(2)-invariant Chern-Simons term. Hence, up to surface terms, it is gauge invariant. It is also metric independent and so its algebraic form also holds for curved spaces.

Using the Coulomb gauge equal time commutators [7]

$$[A^{a,i}(\vec{x}),A^{b,j}(\vec{y})]=-i\epsilon_{ab}\epsilon^{ijk}\frac{\partial^x_k}{\nabla^2}\delta(\vec{x}-\vec{y}), \tag{8}$$

it is straightforward to verify that Q indeed generates infinitesimal SO(2) rotations:

$$[Q,A^b_j(y)]=-i\epsilon^{ba}A^a_j(y). \tag{9}$$

The Fock space of states is constructed through the action of creation and annihilation operators $a^\dagger_\lambda$ and a_λ introduced via the Fourier decomposition of $A^{a,i}$:

$$A^{a,i}(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3p}{\sqrt{2|\vec{p}|}}\sum_{\lambda=1}^{2}[e^{-ipx}\epsilon_\lambda^{ai}a_\lambda(p)+e^{ipx}\epsilon_\lambda^{ai}a_\lambda^\dagger(p)],\tag{10}$$

where $px\equiv|\vec{p}|x^0-\vec{p}\cdot\vec{x}$ and $\epsilon_\lambda^{a,i}(p)$, $\lambda=1,2$, are unit norm polarization vectors, orthogonal to $\vec{p}$ and satisfying

$$(g_{ij}\delta_{ab}p_0-\epsilon_{ab}\epsilon_{ilj}p^l)\epsilon_\lambda^{bj}(p)=0.\tag{11}$$

This means that $(\hat{p},\epsilon_1^a(\vec{p}),\epsilon_2^a(\vec{p}))$, $a=1,2$, are two orthonormal bases rotated by $\pi/2$ in the direction of $\hat{p}$. The operators a_λ and $a_\lambda^\dagger$ satisfy the usual algebra

$$[a_\lambda(\vec{p}),a_{\lambda'}^\dagger(\vec{p}\,')]=\delta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}\,').\tag{12}$$

In terms of these operators the charge Q can be rewritten as

$$Q=i\int d^3k(a_1^\dagger a_2-a_2^\dagger a_1)\tag{13}$$

and becomes diagonal,

$$Q=\int d^3k(a_L^\dagger a_L-a_R^\dagger a_R),\tag{14}$$

in the base of circularly polarized operators, defined by

$$a_R^\dagger=\frac{a_1^\dagger+ia_2^\dagger}{\sqrt{2}},\tag{15}$$

$$a_L^\dagger=\frac{a_1^\dagger-ia_2^\dagger}{\sqrt{2}}.\tag{16}$$

From Eq. (14) one sees that, in a generic state, Q counts the number of left minus right polarized photons.

It is easily checked that Q commutes with the components $\theta^{0\mu}$ of the energy-momentum tensor. Hence, it commutes with all the generators of the conformal group as should be expected from an internal symmetry.

Identifying the operators a_λ with the corresponding ones in Maxwell's theory one could work backward and find that the charge Q has the nonlocal expression

$$Q=\frac{1}{2}\int d^3x(-\vec{A}\cdot\nabla\times\vec{A}+\vec{E}\cdot\nabla^{-2}\nabla\times\vec{E}),\tag{17}$$

where $\vec{E}$ is the electric field and $\vec{A}$ the vector potential in the Coulomb gauge. As described in [4], Eq. (17) arises from a first order Hamiltonian formulation of the Maxwell action.

The expression (17) for the charge Q can also be arrived at through formal manipulations using the equations of motion. In fact, using the gauge freedom (4) Schwarz and Sen have shown that

$$\vec{B}^2=\vec{E}^1.\tag{18}$$

Thus, taking the curl of this equation and using the Coulomb gauge condition one has formally

$$\vec{A}^2=-\nabla^{-2}\nabla\times\vec{E}^1.\tag{19}$$

Equation (17) follows from the replacement of Eqs. (18) and (19) into Eq. (7).

Our discussion indicates that the Deser-Teitelboim and Schwarz-Sen implementations of duality are equivalent and both formulations have the same physical content. We now show how this equivalence can be understood in the path integral framework. To see that let us use the Batalin-Fradkin-Vilkovisky formalism [8] for constrained systems. The generating functional for the Maxwell theory is

$$Z=\int\mathcal{D}A_\mu\mathcal{D}\pi_\mu\mathcal{D}c\mathcal{D}\overline{\mathcal{P}}\mathcal{D}\overline{c}\mathcal{D}\mathcal{P}e^{iS_{\rm eff}},\tag{20}$$

with the effective action given by

$$S_{\rm eff}=\int d^4x(\pi_i\dot{A}^i+\pi_0\dot{A}^0+\dot{\mathcal{P}}\overline{c}+\dot{c}\overline{\mathcal{P}}-\mathcal{H}_0-\{\Psi,Q_B\}).\tag{21}$$

As usual, π_μ is the conjugate momentum of A_μ, c and $\overline{c}$ are ghosts, and $\mathcal{P}$ and $\overline{\mathcal{P}}$ their conjugate momenta. At equal times they satisfy

$$\{\overline{\mathcal{P}}(\vec{x}),c(\vec{y})\}=-\delta(\vec{x}-\vec{y}),\quad\{\mathcal{P}(\vec{x}),\overline{c}(\vec{y})\}=-\delta(\vec{x}-\vec{y}).\tag{22}$$

The Becchi-Rouet-Stora (BRS) charge and Hamiltonian densities are given by

$$Q_B=\int d^3x(\partial_i\pi^ic-i\mathcal{P}\pi_0),\tag{23}$$

$$\mathcal{H}_0=-\frac{1}{2}(\pi^i\pi_i+B^iB_i).\tag{24}$$

It is convenient to choose the gauge-fixing function Ψ as

$$\Psi=i\overline{c}\chi+\overline{\mathcal{P}}A_0\tag{25}$$

and fix χ in such way that the Coulomb condition holds. This can be achieved if $\chi=(1/\epsilon)\partial_iA^i$, we redefine the fields $\pi_0\to\epsilon\pi_0$, $\overline{c}\to\epsilon\overline{c}$, and let ϵ go to zero. Notice that this scaling transformation produces a trivial Jacobian and is compatible with the BRS transformation,

$$\delta_{\rm BRS}A_i=\partial_ic,\quad\delta_{\rm BRS}\pi_i=0,\quad\delta_{\rm BRS}A_0=i\mathcal{P},\quad\delta_{\rm BRS}\pi_0=0,\tag{26}$$

$$\delta_{\rm BRS}c=0,\quad\delta_{\rm BRS}\overline{c}=i\pi_0,\quad\delta_{\rm BRS}\overline{\mathcal{P}}=-\partial_i\pi^i,\quad\delta_{\rm BRS}\mathcal{P}=0,\tag{27}$$

which leaves invariant the generating functional Z. After taking the limit $\epsilon\to0$ we get

$$S_{\rm eff}=\int d^4x(\pi_i\dot{A}^i+\dot{c}\overline{\mathcal{P}}-\mathcal{H}_0+A_0\partial_i\pi^i+\pi_0\partial_iA^i+i\overline{\mathcal{P}}\mathcal{P}-i\overline{c}\nabla^2c).\tag{28}$$

It is interesting to observe that already at this level there is a duality transformation of the fields which leaves Z invariant. The infinitesimal form of the transformation is given by

$$\delta A_i = \theta \nabla^{-2} \epsilon_{ijk} \partial^j \pi^k, \quad \delta \pi_i = \theta \epsilon_{ijk} \partial^j A^k, \tag{29}$$

the variations of the remaining fields being zero. Notice that the variation of A_i and π_i is the nonlocal duality transformation introduced by Deser and Teitelboim. δ commutes with the BRS transformation, as can be checked.

Integrating Eq. (20) in π_0 and A^0 produces the usual δ functions which characterize the Coulomb gauge. Performing also the trivial integrations in the ghosts $\overline{\mathcal{P}}$ and $\mathcal{P}$ we finally have

$$Z = \int \mathcal{D}A_i \mathcal{D}\pi_i \mathcal{D}c \mathcal{D}\overline{c}\, e^{iS_{\mathrm{eff}}} \delta(\partial_i A^i)\, \delta(\partial_i \pi^i), \tag{30}$$

where now S_{eff} is the action considered by Deser and Teitelboim:

$$S_{\mathrm{eff}} = \int d^4x (\pi_i \dot{A}^i - \mathcal{H}_0 - i\, \overline{c} \nabla^2 c). \tag{31}$$

This action is, of course, invariant under the Deser-Teitelboim duality transformation, given by Eqs. (29) and $\delta c = \delta \overline{c} = 0$.

We will now show that the Schwarz-Sen formulation can be obtained by requiring that the duality transformation be local. To this end we introduce two auxiliary fields C_i and P_i so that

$$\delta A_i = \theta C_{iT}, \quad \delta \pi_i = \theta P_{iT}, \tag{32}$$

where the index T indicates the transversal part of the field, as required by the Coulomb gauge condition on A_i and π_i. These fields are fixed through the equations

$$\nabla^2 C_{iT} = + \epsilon_{ijk} \partial^j \pi^k, \quad P_{iT} = \epsilon_{ijk} \partial^j A^k, \tag{33}$$

and the transversality conditions $\partial^i C_i = - \partial^i P_i = 0$.

From Eqs. (33) and (29) we have

$$\nabla^2 \delta C_{iT} = + \theta \epsilon_{ijk} \epsilon^{klm} \partial^j \partial_l A_m = - \theta \nabla^2 A_i. \tag{34}$$

Hence, the transversality condition implies that $\delta C_i = - \theta A_i$. Furthermore, proceeding in an analogous way one can prove that $\delta P_i = - \theta \pi_i$.

Comparing these equations with the infinitesimal form of Eq. (6) for the Schwarz-Sen approach allows us to identify their fields as follows:

$$A_i^{(1)} = A_i, \quad A_i^{(2)} = C_i. \tag{35}$$

The conjugate momenta to these fields are

$$\pi_i^{(1)} = \frac{1}{2} \pi_i, \quad \pi_i^{(2)} = \frac{1}{2} P_i. \tag{36}$$

To construct the Schwarz-Sen action first notice that

$$P^i \dot{C}_i = - \epsilon^{ijk} \partial_j A_k \epsilon_{ilm} \nabla^{-2} \partial^l \dot{\pi}^m = - A^i \dot{\pi}_i \tag{37}$$

and

$$\epsilon^{ijk} \nabla^2 \partial_j C_{kT} = - \nabla^2 \pi^i, \tag{38}$$

from which $\pi_i = - \epsilon_{ijk} \partial^j C_T^k$. We can therefore write Eq. (31) as

$$S_{\mathrm{eff}} = \int d^4x \left(\frac{1}{2} \pi_i \dot{A}^i + \frac{1}{2} P_i \dot{C}^i - \mathcal{H}_0 - i\, \overline{c} \nabla^2 c \right), \tag{39}$$

where now $\mathcal{H}_0 = (1/2)[(\nabla \times \vec{C})^2 + (\nabla \times \vec{A})^2]$.

Up to the ghost term, Eq. (39) is the Schwarz-Sen action in Hamiltonian form. Although our discussion already demonstrates the equivalence of the two approaches, we will prove now that the generating functional Z given by Eq. (30) is equal to the corresponding functional obtained using the Batalin-Fradkin-Vilkovisky formalism for the Schwarz-Sen action. After the introduction of the auxiliary fields C_i and P_i, Z is obviously equal to

$$Z = \int \mathcal{D}A_i \mathcal{D}\pi_i \mathcal{D}C_i \mathcal{D}P_i \mathcal{D}c \mathcal{D}\overline{c}\, \delta(C_{iT} - \nabla^{-2} \epsilon_{ijk} \partial^j \pi^k)\, \delta(P_{iT} - \epsilon_{ijk} \partial^j A^k)\, \delta(\partial_i A^i)\, \delta(\partial_i \pi^i)\, \delta(\partial_i C^i)\, \delta(\partial_i P^i) e^{iS_{\mathrm{eff}}}. \tag{40}$$

Now, we can write

$$\delta(C_{iT} - \nabla^{-2} \epsilon_{ijk} \partial^j \pi^k) = \det{}^{-1}(\nabla^{-2} \epsilon_{ijk} \partial^j)\, \delta(\pi_i + \epsilon_{ijk} \partial^j C^k). \tag{41}$$

But because of the transversality property guaranteed by the δ functions in Eq. (40), $\det^{-1}(\nabla^{-2} \epsilon_{ijk} \partial^j) = \det(\epsilon_{ijk} \partial^j)$. So

$$Z = \int \mathcal{D}A_i \mathcal{D}\pi_i \mathcal{D}C_i \mathcal{D}P_i \mathcal{D}c \mathcal{D}\overline{c}\, \delta(\partial_i A^i)\, \delta(\partial_i \pi^i) \times \det(\epsilon_{ijk} \partial^j)\, \delta(\pi_i + \epsilon_{ijk} \partial^j C^k)\, \delta(P_{iT} - \epsilon_{ijk} \partial^j A^k)\, \delta(\partial_i C^i)\, \delta(\partial_i P^i) e^{iS_{\mathrm{eff}}}. \tag{42}$$

As shown in [7] the Schwarz-Sen action exhibits both first and second class constraints. They are, respectively,

$$\Omega_0^a \equiv \pi_0^a \approx 0, \quad \Omega^a \equiv \partial^i \pi_i^a \approx 0, \tag{43}$$

and

$$\Omega_{iT}^a \equiv \pi_{iT}^a + \frac{1}{2} \epsilon_{ab} \epsilon_{ijk} \partial^j A_T^{b,k} \approx 0. \tag{44}$$

At equal times, the first class constraints satisfy an Abelian algebra whereas for the second class constraints we have

$$\{\Omega_{iT}^a(\vec{x}), \Omega_{jT}^b(\vec{y})\} = - \epsilon_{ab} \epsilon_{ijk} \partial_x^k \delta(\vec{x} - \vec{y}). \tag{45}$$

Therefore, in the gauge $A_0^a = \partial^i A_i^a = 0$, the generating functional for the Schwarz-Sen approach is given by

$$\widetilde{Z} = \int \mathcal{D}A_i^a \mathcal{D}\pi_i^a\, \delta(\partial^i A_i^a)\, \delta(\partial^i \pi_i^a) \det(\nabla^2) \times \delta\left(\pi_T^{ia} + \frac{1}{2} \epsilon^{ab} \epsilon^{ijk} \partial_j A_{kT}^b \right) \det{}^{1/2}(\epsilon^{ab} \epsilon^{ijk} \partial_k) e^{i\widetilde{S}_{\mathrm{eff}}}, \tag{46}$$

where

$$\widetilde{S}_{\text{eff}}=\int d^4x\left(\pi_i^a\dot{A}_a^i-\frac{1}{2}(\nabla\times\vec{A}^a)^2\right) \tag{47}$$

and the trivial sector A_0^a, π_0^a has already been integrated out. We see that the $\det(\nabla^2)$ factor in Eq. (46) arises also in Eq. (42) due to the ghost contribution. Moreover, as

$$\det(\epsilon_{ab}\epsilon_{ijk}\partial^k)=\det{}^2(\epsilon_{ijk}\partial^k), \tag{48}$$

we see from Eqs. (35) that the two generating functionals Z and $\widetilde{Z}$ are actually identical.

We have thus shown the quantum equivalence of Schwarz-Sen and Maxwell theories. This was expected as both are free field theories that are classically equivalent. Notice that the field equations were not used. Notice also that, although the off-shell descriptions look different, the physical contents of both formulations are the same since their generating functionals are equal. The important point is that the Schwarz-Sen approach can be understood as a way to implement duality as a local symmetry in Maxwell theory. This is possible thanks to the introduction of the auxiliary fields C_i and P_i. There are, of course, other possibilities to turn the duality transformation local and presumably this may be related to the already known formulations where duality is realized locally. We expect that this method may be also used for the interacting case since in essence it replaces nonlocal terms by auxiliary fields in a BRS-invariant way.

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