Rotating Skyrmion in 2+1 dimensions

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The collective rotation of the Skyrmion in two-dimensional space is considered. In contradistinction to the three-dimensional case, inertial effects do not spoil the hedgehog form and can, therefore, be investigated consistently without great computational difficulty. The energy, the moment of inertia, and the mean radius of the rotating soliton are calculated for a wide range of model parameters. It is found that the ''frozen hedgehog'' treatment—commonly assumed adequate in the Skyrme model on the basis of large N_C (number of colors) arguments—is invalid in a sizable portion of parameter space. The phase shifts associated with radial fluctuations of the rotating soliton are also investigated and are found to be significantly affected by the rotation. [S0556-2821(96)04613-9]

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I. INTRODUCTION

The Skyrme model [1], in its simplest version, is a nonlinear theory of pion fields endowed with an SU(2)⊗ SU(2) chiral symmetry, weakly broken by the pion mass. Large N_C (number of colors) arguments have revived it as a candidate for an effective theory of hadron physics at low and intermediate energies [2].

The description of baryon structure in this model is based on a static soliton solution of the field equations with unit topological charge, in which the pion-field triplet takes on the so-called hedgehog form, pointing out radially at each point. The field strength is given by a radial profile function, which is obtained by minimizing the energy of the static soliton. Baryon spin and isospin states are constructed by quantizing the collective rotations of the hedgehog. For the hedgehog, an isospin rotation is equivalent to a coordinate-space rotation and consequently, the isospin is equal to the angular momentum.

In the simplest—and most usual—description, the profile function of the rotating soliton is taken to be equal to that of the static soliton. This approximation ignores the reaction of the soliton to the rotation and results in the quantum mechanics of a spherical top with constant moment of inertia. We shall frequently use the phrase ''frozen hedgehog'' approximation to refer to this picture. The rationale for it is found in the assumption that the Skyrme model is a realization of large N_C quantum chromodynamics (QCD), which allows the specification of the N_C dependence of the model parameters. On this basis, it is argued that the rotational contribution to the energy of the soliton is of order N_C^{-2} compared to the static mass; therefore, rotational effects are small and can be treated to lowest order in $1/N_C$. One should, however, note that N_C does not appear explicitly in the Skyrme model Lagrangian. Once the parameters of the model are fixed phenomenologically, the validity of the ''frozen hedgehog'' approximation becomes a question to be settled within the model itself, without relying upon its supposed relation to QCD.

One way to take inertial effects into account is to minimize the energy of the rotating hedgehog with respect to variations of the profile function, for fixed angular momentum [3,4]. However, this procedure is rather unsatisfactory since the field equations do not admit rotating hedgehog solutions in 3+1 dimensions. This simple kinematical fact is independent of the dynamics of the model and can be noticed by considering the application of the d'Alembertian operator to the rotating hedgehog. One verifies easily that the resulting field vector will not in general be radial, although it will be contained in the meridian plane defined with respect to the rotation axis. Therefore, the reduction of the isovector-field equations to a single equation for the profile function will not be possible. In physical terms, a rotating nonrigid object is expected to acquire an oblate shape, losing the spherical symmetry of the hedgehog. Thus, in 3+1 dimensions, rotating solitons only possess cylindrical symmetry and their construction requires the solution of coupled partial-differential equations involving two field functions. Attempts in this direction have been made in the context of the nonlinear σ model [5], but no complete solution has been presented.

The relevance of rotating soliton solutions has recently been emphasized by an analysis of collective quantization in the framework of the phase-space path integral [6,7]. It has been shown that the saddle-point condition on the effective action for nonvanishing angular momentum (or isospin) leads to a field configuration which minimizes, not the static energy, but the energy augmented by a rotational contribution. In fact, the resulting equation is identical to that satisfied by the intrinsic field of a classical soliton rotating with constant angular velocity. The distortion of the rotating soliton away from the hedgehog shape in 3+1 dimensions has also been stressed in [7].

The present work is motivated by the elementary observation that, in two-dimensional space, the meridian plane collapses to the radial direction and it can therefore be expected that the rotating hedgehog ansatz reduces the field equations to a single differential equation for the profile function, from which exact classical solutions corresponding to rotating solitons can be constructed. Thus, there is at least no *a priori* inconsistency in studying inertial effects in the framework of the collective rotation of the hedgehog in two-dimensional space. Preliminary results for rotating Skyrmion properties in 2+1 dimensions have already been presented in [8]. Similar calculations have also been performed by Piette *et al.* [9]. The first part of the present work complements these analyses through a more complete mapping of the parameter space of the model and a discussion of the limitations of the standard ''frozen hedgehog'' or ''large N_C'' approximation.

Inertial effects caused by the soliton rotation are also expected to be present in the calculation of meson-soliton scattering phase shifts. As demonstrated in [6,7], these should be extracted from the quantum field fluctuations about the rotation-distorted soliton instead of the static hedgehog, as has been assumed in all extensive studies of meson-baryon scattering in the Skyrme model [10,11]. Although the complete formulation of this problem is beyond the scope of this work, we shall perform an analysis of the phase shifts associated with radial quantum fluctuations about the rotating Skyrmion in 2+1 dimensions. Preliminary results have been reported in [12]. A similar study has already been presented in 3+1 dimensions [3] but, as mentioned above, the use of the hedgehog ansatz is questionable in that case.

Of course, by considering a two-dimensional toy model, we give up the possibility of making direct contact with the real world. In compensation, by working with a model that admits rotating hedgehog solutions, we should be able to draw some reliable conclusions, from which one may hope to gain at least qualitative insight on the importance of rotational inertial effects for the calculation of baryon properties and meson-baryon interactions in the physical three-dimensional Skyrme model.

This paper is organized as follows. In Sec. II, we summarize the relevant features of the Skyrme model in 2+1 dimensions. The differential equation satisfied by the profile function of a classical rotating hedgehog is derived, together with the expressions of the associated physical quantities.

In Sec. III, we perform a partial quantization of the rotating hedgehog, treating the rotation angle and the profile function as dynamical variables. The rotating mean-field approximation then yields a quantum-mechanical interpretation of the classical solutions introduced in Sec. II. We perform a detailed analysis of the properties of the first few rotational collective states, over a large range of model parameters. By comparing our results to those obtained ignoring inertial effects, we assess the limitations of standard ''large N_C'' arguments, which are usually invoked to neglect such effects.

In Sec. IV, we analyze the quantum fluctuations of the profile function of a rotating hedgehog, in a linearized approximation. We extract the phase shifts associated to the scattering of a field quantum off a rotating soliton. Again, the results are compared to those obtained by taking the static hedgehog as background field, in the usual fashion.

Section V contains our conclusions and final comments.

II. SKYRME MODEL IN 2+1 DIMENSIONS

The Skyrme model in 3+1 dimensions is a nonlinear realization of SU(2)⊗ SU(2) chiral symmetry in terms of pion fields. The chiral symmetry is weakly broken by the pion mass. The Lagrangian density for this model is written in terms of an SU(2) field matrix U as

$$\mathcal{L}=-\frac{1}{4}f_\pi^2\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)+\frac{1}{8}\epsilon^2\mathrm{Tr}[\partial_\mu UU^\dagger,\partial_\nu UU^\dagger]^2 + \frac{1}{2}m_\pi^2 f_\pi^2(\mathrm{Tr}U-2), \tag{1}$$

where f_π is the pion decay constant, ϵ is a phenomenological coupling constant and m_π is the pion mass. The constants f_π and ϵ have dimensions $[f_\pi]=L^{-1}$ and $[\epsilon]=L^0$, respectively.

As SU(2) ⊗ SU(2) ∼O(4), the Lagrangian density can also be written as an invariant function of a unitary four-vector in field space, $\mathbf{\Phi}\equiv\{\Phi_0,\vec{\Phi}\}$, with

$$\mathbf{\Phi}\cdot\mathbf{\Phi}=1. \tag{2}$$

This is implemented by inserting in the Lagrangian density (1) the following parametrization of U:

$$U=\Phi_0+i\vec{\tau}\cdot\vec{\Phi}, \tag{3}$$

where $\vec{\tau}$ is a three-vector whose components are the Pauli matrices. This leads to

$$\mathcal{L}=\frac{1}{2}f_\pi^2\partial_\mu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi}+\frac{1}{2}\epsilon^2[\partial_\mu\mathbf{\Phi}\cdot\partial_\nu\mathbf{\Phi}\partial^\mu\mathbf{\Phi}\cdot\partial^\nu\mathbf{\Phi} -(\partial_\mu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi})^2]+m_\pi^2 f_\pi^2(\Phi_0-1). \tag{4}$$

Since, for localized configurations, the field must tend to its vacuum value at large distance, $\{\Phi_0,\vec{\Phi}\}\to\{1,\vec{0}\}$, solitons correspond to mappings of the compactified geometrical space S_S^3 onto the field manifold S_F^3.

An analogous model in 2+1 dimensions is obtained by considering the Lagrangian density as an O(3)-invariant function of a unitary three-vector in field space, $\mathbf{\Phi}\equiv\{\Phi_0,\vec{\Phi}\}$, where $\vec{\Phi}$ is now a two-component vector and $\mathbf{\Phi}$ again satisfies the constraint (2). Solitons are now $S_S^2\to S_F^2$ mappings. In this two-dimensional model, the constants f_π and ϵ are easily seen to have dimensions $[f_\pi]=L^{-1/2}$, $[\epsilon]=L^{1/2}$. Although we are now dealing with a toy model, we shall continue to refer to $\mathbf{\Phi}$ as the pion field and to m_π as the pion mass.

It is convenient to make the space-time variables adimensional by rescaling them by f_π/ϵ. Rescaling the pion Compton wavelength in the same way, we introduce the adimensional parameter $\mu=\epsilon m_\pi/f_\pi$. The Lagrangian density is correspondingly rescaled by a factor ϵ^2/f_π^4 and then takes the form

$$\mathcal{L}_{\mathrm{adim}}=\frac{1}{2}\partial_\mu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi}+\frac{1}{2}[\partial_\mu\mathbf{\Phi}\cdot\partial_\nu\mathbf{\Phi}\partial^\mu\mathbf{\Phi}\cdot\partial^\nu\mathbf{\Phi} -(\partial_\mu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi})^2]+\mu^2(\Phi_0-1). \tag{5}$$

From here on, we shall use adimensional variables and work with the Lagrangian density (5).

The field equations are

$$\partial_\mu\partial^\mu\mathbf{\Phi}+2(\partial_\nu\mathbf{\Phi}\cdot\partial_\mu\partial^\mu\mathbf{\Phi}\partial^\nu\mathbf{\Phi}+\partial_\nu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi}\partial_\mu\partial^\nu\mathbf{\Phi}$$
$$-\partial_\nu\mathbf{\Phi}\cdot\partial^\nu\mathbf{\Phi}\partial_\mu\partial^\mu\mathbf{\Phi}-\partial_\nu\mathbf{\Phi}\cdot\partial_\mu\partial^\nu\mathbf{\Phi}\partial^\mu\mathbf{\Phi})$$
$$+\lambda\mathbf{\Phi}-\mu^2\mathbf{e}_0=0, \tag{6}$$

where $\mathbf{e}_0$ is a unit three-vector in field space: $\mathbf{e}_0\equiv\{1,\vec{0}\}$. The Lagrange multiplier λ, introduced to enforce the constraint (2), is given by

$$\lambda=\partial_\mu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi}+2[\partial_\nu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi}\partial^\nu\mathbf{\Phi}\cdot\partial_\mu\mathbf{\Phi}$$
$$-(\partial_\mu\mathbf{\Phi}\cdot\partial^\mu\mathbf{\Phi})^2]+\mu^2\Phi_0. \tag{7}$$

The energy-momentum tensor is easily constructed. From it, one derives the expressions for the energy,

$$E=\frac{1}{2}\int d^2x[(\dot{\mathbf{\Phi}}\cdot\dot{\mathbf{\Phi}}+\partial_i\mathbf{\Phi}\cdot\partial_i\mathbf{\Phi})(1+\partial_j\mathbf{\Phi}\cdot\partial_j\mathbf{\Phi})$$
$$-2\dot{\mathbf{\Phi}}\cdot\partial_i\mathbf{\Phi}\dot{\mathbf{\Phi}}\cdot\partial_i\mathbf{\Phi}-\partial_i\mathbf{\Phi}\cdot\partial_j\mathbf{\Phi}\partial_i\mathbf{\Phi}\cdot\partial_j\mathbf{\Phi}+2\mu^2(1\text{-}\Phi_0)], \tag{8}$$

and the angular momentum,

$$J=\mathcal{E}_{ij}\int d^2xx^i[\partial_j\mathbf{\Phi}\cdot\dot{\mathbf{\Phi}}(1+2\partial_k\mathbf{\Phi}\cdot\partial_k\mathbf{\Phi})$$
$$-2\partial_j\mathbf{\Phi}\cdot\partial_k\mathbf{\Phi}\dot{\mathbf{\Phi}}\cdot\partial_k\mathbf{\Phi}]. \tag{9}$$

Since the vacuum configuration is $\mathbf{\Phi}_{vac}\equiv\{1,\vec{0}\}$, the original O(3) symmetry of Eq. (5) is broken to the U(1) symmetry corresponding to rotations of the $\vec{\Phi}$ field. The associated charge, i.e., the isospin, is

$$T=\int d^2x[(1+2\partial_i\mathbf{\Phi}\cdot\partial_i\mathbf{\Phi})\dot{\vec{\Phi}}\times\vec{\Phi}-2\dot{\mathbf{\Phi}}\cdot\partial_i\mathbf{\Phi}\partial_i\vec{\Phi}\times\vec{\Phi}]. \tag{10}$$

The mappings $S_S^2\to S_F^2$ are characterized by the topological charge

$$B=\int d^2xb^0=\frac{1}{8\pi}\mathcal{E}_{ij}\int d^2x\mathbf{\Phi}\cdot(\partial_i\mathbf{\Phi}\times\partial_j\mathbf{\Phi}). \tag{11}$$

We shall use as a measure of the soliton size the rms radius R of the topological charge distribution, defined as

$$R^2=\int d^2xr^2b^0. \tag{12}$$

On the basis of the arguments presented in the Introduction, the two-dimensional model, in contrast with the three-dimensional one, is expected to possess exact time dependent classical solutions corresponding to the rigid rotation of a hedgehog, i.e.,

$$\Phi_i(\vec{x},t)=R_{ij}[\varphi(t)]\frac{x^j}{r}\sin f(r),\quad \Phi_0(\vec{x},t)=\cos f(r), \tag{13}$$

where $R_{ij}[\varphi(t)]$ denotes the 2×2 orthogonal matrix which operates a rotation by the angle $\varphi(t)$. Indeed, the substitution of this ansatz in the field equations (6) reduces these consistently to a single differential equation for the profile function $f(r)$, if the angular velocity $\omega=\dot{\varphi}$ is assumed to be constant. This equation reads

$$r^2f''+rf'+(1-\omega^2r^2)[\sin2f(f'^2-\tfrac{1}{2})+2\sin^2ff'']-2(1$$
$$+\omega^2r^2)\sin^2f\frac{f'}{r}-\mu^2r^2\sin f=0\,. \tag{14}$$

For a soliton of unit topological charge (11), the boundary conditions

$$f(0)=\pi,\quad f(\infty)=0 \tag{15}$$

must be satisfied.

The solutions of Eqs. (14) and (15) have been obtained for one value of the parameter μ in [9]. A more complete study will be presented in the next section.

The physical quantities associated with the classical rotating hedgehog can easily be obtained by substituting Eq. (13) in the expressions (8)–(12). For the angular momentum, one gets, from Eqs. (13) and (9),

$$J=\omega I[f], \tag{16}$$

where the moment of inertia $I[f]$ is

$$I[f]=2\pi\int_0^\infty drr\sin^2f(1+2f'^2). \tag{17}$$

As expected for the hedgehog, an identical expression is obtained for the isospin, by substituting Eq. (13) into Eq. (10).

The energy of the rotating hedgehog, obtained by inserting Eq. (13) in Eq. (8), is

$$E=M[f]+\frac{1}{2}\omega^2I[f], \tag{18}$$

with

$$M[f]=\pi\int_0^\infty drr[f'^2+r^{-2}\sin^2f(1+2f'^2)$$
$$+2\mu^2(1-\cos f)], \tag{19}$$

a quantity that we shall refer to as the intrinsic mass of the rotating soliton.

Finally, for the hedgehog (13), the rms topological charge radius (12) is given by

$$R^2=-\frac{1}{2}\int_0^\infty drr^2\sin ff'. \tag{20}$$

For completeness and posterior use, we list below the expressions of the dimensional physical quantities in terms of the adimensional ones. In addition to already-defined quantities, we record the scaling of the action, denoted by S. The dimensional angular velocity is Ω. In the case of quantities denoted by roman upper-case letters, the corresponding script letter is used to represent the dimensional quantity:

$$\mathcal{S}=\epsilon f_\pi S, \quad \mathcal{M}=f_\pi^2 M, \quad \mathcal{E}=f_\pi^2 E,$$

$$\Omega=f_\pi \omega/\epsilon, \quad \mathcal{J}=\epsilon f_\pi J, \quad \mathcal{I}=\epsilon^2 I,$$

$$m_\pi=f_\pi \mu/\epsilon, \quad \mathcal{R}=\epsilon R/f_\pi. \tag{21}$$

We remark that, since we started out with natural units such that $\hbar=1$, and rescaled the action by the factor $(\epsilon f_\pi)^{-1}\equiv\alpha$, the quantum of action in our new units is α, a number which depends on the parameters of the original dimensional model. Correspondingly, the mass of the pion is no longer equal to its inverse Compton wavelength. The adimensional inverse Compton wavelength is μ and the adimensional mass is $\alpha\mu$.

III. ROTATING HEDGEHOG IN THE MEAN-FIELD APPROXIMATION

The relevance, for the quantum field theory, of the classical solutions introduced in the previous section can be established by performing a partial quantization of the hedgehog, in which the rotation angle and the profile function are treated as dynamical variables.

Introducing in Eq. (5) the ansatz (13), with the rotation angle $\varphi(t)$ and the profile function $f(r,t)$ considered as functions of time, and integrating over space, we get the Lagrangian

$$L=\int_0^\infty dr\rho(f)\frac{\dot{f}^2}{2}+I[f]\frac{\dot{\varphi}^2}{2}-M[f], \tag{22}$$

where

$$\rho(f)=2\pi(r+2r^{-1}\sin^2 f). \tag{23}$$

The conjugate momenta are

$$J=\frac{\partial L}{\partial\dot{\varphi}}=I[f]\dot{\varphi} \tag{24}$$

and

$$p=\frac{\delta L}{\delta\dot{f}}=\rho(f)\dot{f}. \tag{25}$$

The corresponding Hamiltonian is, therefore,

$$H=\int_0^\infty dr p\dot{f}+J\dot{\varphi}-L=\int_0^\infty dr\frac{p^2}{2\rho(f)}+\frac{J^2}{2I[f]}+M[f]. \tag{26}$$

When we go over to quantum mechanics, ordering problems appear in Eq. (26), but it can be easily verified that they may be ignored at the levels of approximation used in this work.

Since the angle variable φ does not appear in the Hamiltonian (26), the angular momentum J is conserved. The energy eigenstates may therefore be classified according to the eigenvalues of J, which, following the standard quantization of angular momentum in two dimensions, are given by $n\hbar$, with n integer. As already mentioned, the quantum of action is $\hbar=(\epsilon f_\pi)^{-1}\equiv\alpha$ in our rescaled units.

The dynamics of the profile function may then be studied within a sector of given J. In keeping with the usual picture of solitons as semiclassical objects, it is natural to replace, in first approximation, the quantum field $f(r,t)$ by its expectation value $\overline{f(r)}$, assumed to be time independent, so that the corresponding mean-field momentum $\overline{p(r)}$ vanishes. The equation of motion for $p(r,t)$ then reduces to the condition that the mean-field energy,

$$\overline{E}=M[\overline{f}]+\frac{J^2}{2I[\overline{f}]}, \tag{27}$$

be stationary with respect to variations of $\overline{f}$. This leads to the differential equation (14) for $\overline{f}$, with

$$\omega=J/I[\overline{f}]. \tag{28}$$

For a soliton of unit topological charge, the boundary conditions (15) should be satisfied. We shall denote by $\overline{f_\omega}$ (or also by $\overline{f_J}$) the corresponding solution, indicating that it depends parametrically on ω (or equivalently on J).

It is easily verified that, at large distance, the solution of Eq. (14) tends to a modified Bessel function:

$$\overline{f_\omega}(r)\underset{r\to\infty}{\sim}CK_1(\sqrt{\mu^2-\omega^2}r)\sim\frac{C'}{\sqrt{r}}\exp(-\sqrt{\mu^2-\omega^2}r), \tag{29}$$

so that localized solutions exist only for $\omega<\mu$. Thus, the pion-mass term is necessary to ensure the stability of the rotating soliton [13,14]. We note that, even though the nonlinear O(3) σ model, which corresponds to the first term of Eq. (5) only, possesses static localized solutions in 2+1 dimensions, the Skyrme term becomes necessary to prevent the collapse of the soliton, once the pion-mass term is introduced. Indeed, from Eq. (5) one sees easily that, under the scale change $\vec{x}\to\lambda\vec{x}$, the contribution to the soliton mass from the first term is invariant, while the contribution from the pion-mass term scales as λ^2. Therefore, the soliton collapses to $\lambda=0$. This situation is remedied by the contribution from the Skyrme term, which scales as λ^{-2}.

The calculation of the properties of a rotating soliton of angular momentum $J=n\hbar$ requires the solution of Eq. (14), with ω determined self-consistently using Eq. (28). Once the profile function $\overline{f_J}$ is determined, the intrinsic mass $\overline{M_J}$, the moment of inertia $\overline{I_J}$, the energy (or total mass) $\overline{E_J}$, and the radius $\overline{R_J}$ can be computed from Eqs. (19), (17), (27), and (20), respectively (here and below, we use the notation $\overline{M[f]}\equiv\overline{M}$, etc.). In addition to the value of the angular-momentum quantum number n, two adimensional parameters are input to such calculations: the pion inverse Compton wavelength μ and the combination $\alpha=(\epsilon f_\pi)^{-1}$ which fixes the value of $\hbar$.

We shall present and discuss below results for $n=1$ and $n=2$, mapping a fairly wide domain of values of μ and α.

Our main motivation is to assess the validity of the standard treatment of rotational excitations, which uses $f_{J=0}$ to calculate $\bar{M}$, $\bar{I}$, and $\bar{R}$.

Although we are dealing with a two-dimensional model, our choice of parameter ranges will be guided by the requirement that the size and mass of the static soliton crudely match the physics of nonstrange hadrons. We note first that, according to Eq. (21), $m_\pi \mathcal{R} = \mu R$. It is, therefore, reasonable to restrict μ to values such that the product μR_0 be roughly equal to the ratio of the nucleon size to the pion Compton wavelength, i.e., $\simeq 0.5$. In fact, somewhat smaller values for μR_0 might be more adequate since the nucleon radius will be larger than R_0, because of the swelling due to the rotation. We find that the value of μR_0 varies slowly from 0.1 to 0.5 as μ varies from 0.001 to 0.1. We, therefore, choose to restrict our numerical study to this range of values of μ. Similarly, in order to limit the range of values of α, we note that from Eq. (21), we have $\mathcal{M}/m_\pi = f_\pi \epsilon M/\mu$. This suggests choosing α such that $\alpha^{-1} M_0/\mu$ be roughly equal to the ratio of the nucleon mass to the pion mass, i.e., $\simeq 7$. For $\mu \simeq 0.1$ this gives $\alpha \simeq 20$. Somewhat smaller values of $\alpha^{-1} M_0/\mu$, and therefore larger values of α, could be more appropriate since M_0 is smaller than the physical nucleon mass, which includes a rotational contribution. Smaller values of μ also lead to larger values of α. We shall present results for values of α running up to 50.

Since we are primarily interested in evaluating the relative importance of inertial effects, we shall present our results in the form of ratios. We begin by comparing the rotational energy to the static soliton mass. In order for the standard ''large N_C'' arguments to hold, this should be small (of order N_C^{-2}). A contour plot of the quantity $(E_J - M_0)/M_0$ is shown in Fig. 1. It is seen that, over the parameter domain under consideration, this ratio does not exceed 20% for the first rotational excitation and 40% for the second one.

It is often argued that, since the rotational contribution to the energy is small, the dynamics of the soliton is little affected by the rotation and, in particular, the rotational energy itself can be computed using the moment of inertia of the static soliton. This is the standard ''frozen hedgehog'' picture, in which the total energy of the rotating soliton is taken as $E_J^F = M_0 + J^2/(2I_0)$. The accuracy of this approximation can be gauged by comparing the rotational energy derived from it to that obtained from the exact solution of the field equations. The relative difference between these rotational energies is given by the quantity $(E_J^F - E_J)/(E_J^F - M_0)$, for which a contour plot is given in Fig. 2. One sees that the ''frozen'' treatment is good to 40% or less for the first rotational state, but already can lead to an error of more than 60% for the second one.

In fact, inertial effects are even more drastic than these results suggest. This is because, in the total energy E_J, the intrinsic mass part M_J and the rotational kinetic part $J^2/(2I_J)$ are affected in opposite directions, since both M_J and I_J are increased. This can be seen in Figs. 3 and 4, which show contour plots of the quantities $(M_J - M_0)/M_0$ and $(I_J - I_0)/I_0$, respectively. The inertial effect on M is rather small, no more than 5% for $n=1$ and up to 20% for $n=2$. This is not surprising since this quantity reaches a minimum

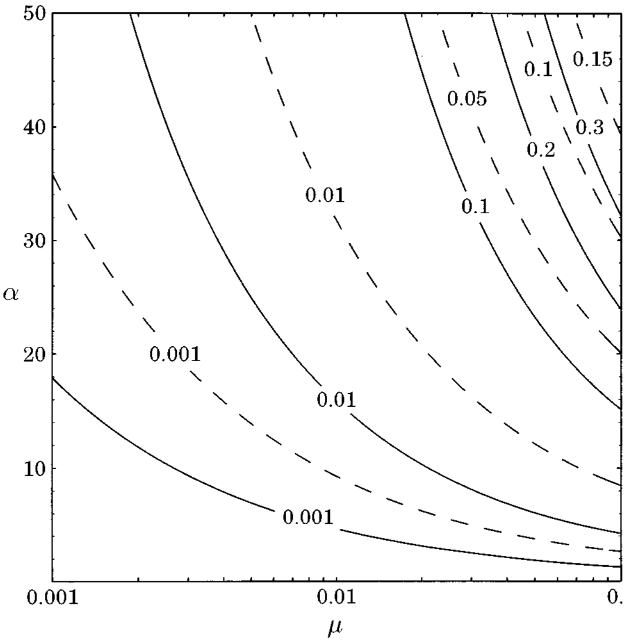

FIG. 1. Contour plot of the relative contribution of the rotation to the soliton energy, $(E_J - M_0)/M_0$, as a function of the model parameters, for the first two rotational states. Dashed curves for $n=1$; solid curves for $n=2$.

for the static profile function. Note, however, that the corresponding effect on the rotational contribution to the energy is significant, because the mass is much larger than the rotational kinetic energy. The moment of inertia is, on the contrary, quite strongly affected by the rotation, the inertial modification already reaching over 100% for $n=1$.

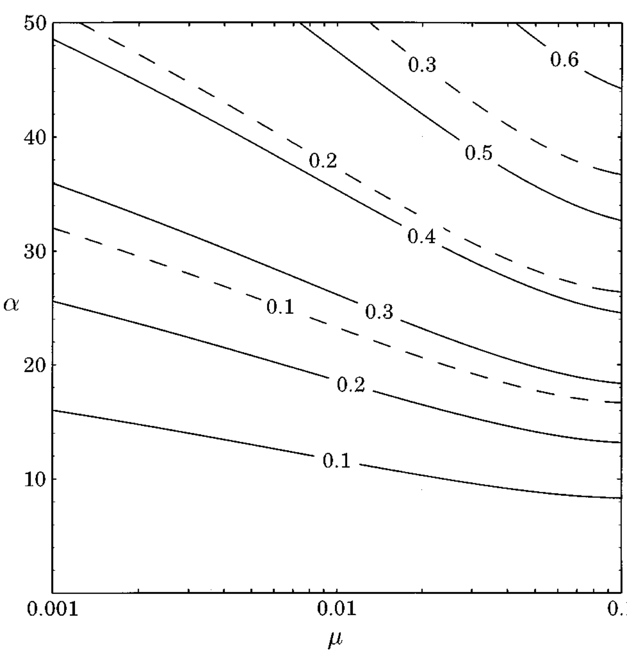

FIG. 2. Contour plot of the relative error made in the ''frozen hedgehog'' approximation to the rotational energy, $(E_J^F - E_J)/(E_J^F - M_0)$, as a function of the model parameters, for the first two rotational states. Dashed curves for $n=1$; solid curves for $n=2$.

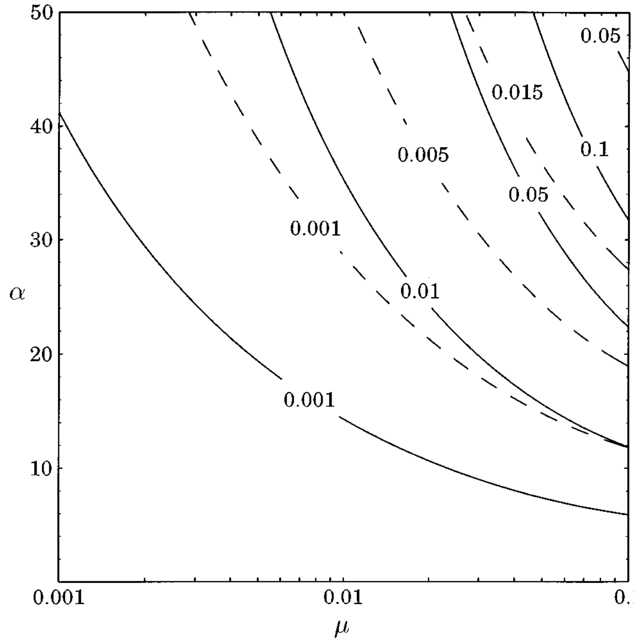

FIG. 3. Contour plot of the relative difference between the intrinsic masses of rotating and static solitons, $(\overline{M}_J-\overline{M}_0)/\overline{M}_0$, as a function of the model parameters, for the first two rotational states. Dashed curves for $n=1$; solid curves for $n=2$.

The severe increase in the moment of inertia caused by the rotation is indicative of a significant swelling of the soliton. This is confirmed by calculations of the rms radius (20). The contour plot of Fig. 5, for the quantity $(R_J-R_0)/R_0$, shows that the radii of the first and second rotational states can be, respectively, up to 50% and 100% larger than that of the static ground state.

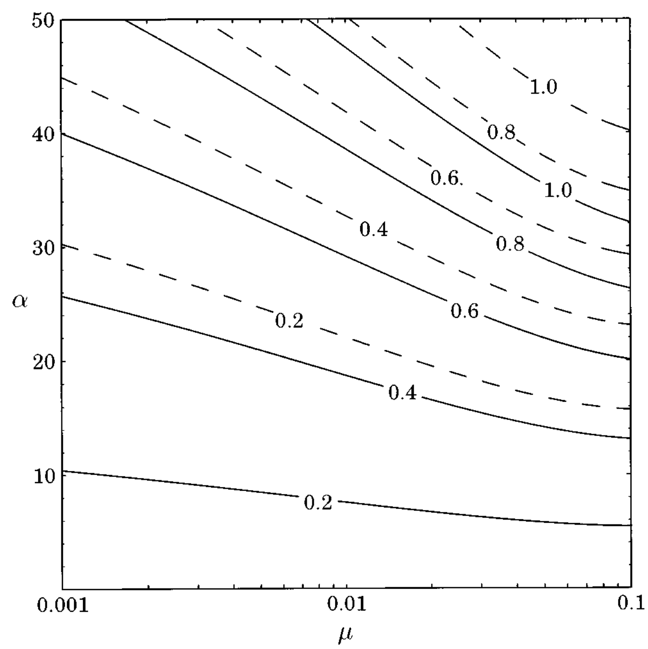

FIG. 4. Contour plot of the relative difference between the moments of inertia of rotating and static solitons, $(I_J-I_0)/I_0$, as a function of the model parameters, for the first two rotational states. Dashed curves for $n=1$; solid curves for $n=2$.

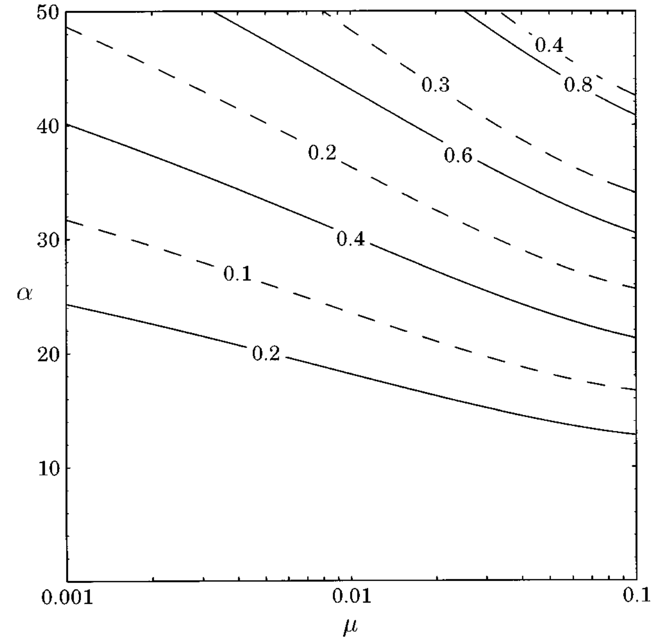

FIG. 5. Contour plot of the relative difference between the rms topological charge radii of rotating and static solitons, $(R_J-R_0)/R_0$, as a function of the model parameters, for the first two rotational states. Dashed curves for $n=1$; solid curves for $n=2$.

To close this section, we remark that, although the angular velocity ω of a stable rotating soliton is limited to values less than the pion inverse Compton wavelength μ, there is no limit to the angular momentum such a soliton can carry. This is because the moment of inertia becomes arbitrarily large as ω tends to μ. It has been shown [9] that in this limit the soliton energy approaches a linearly increasing function of the angular momentum.

IV. RADIAL FLUCTUATIONS OF THE PROFILE FUNCTION

Meson-baryon scattering phase shifts in the Skyrme model can be extracted from the quantum fluctuations about the mean field. As has been emphasized recently [6,7], the rotation-modified mean field should, in principle, be used as zeroth-order approximation, while all extensive studies so far performed have relied on the ''frozen'' approximation.

Although we do not intend to perform a full calculation of meson-soliton scattering in the two-dimensional model, we shall investigate the effect of the rotation on the phase shifts associated with the radial field modes. This can be done in the framework of the Hamiltonian (26), by quantizing the small-amplitude oscillations of the profile function, within a sector of given angular momentum J. We let

$$f(r,t)=\overline{f}(r)+\eta(r,t) \tag{30}$$

and expand the Hamiltonian (26) to second order in the fluctuation η. We get

$$H=\overline{E}+\frac{1}{2}\int dr\frac{p^2}{\overline{\rho}}+\frac{1}{2}\int dr\int dr'\left[\frac{\delta^2\overline{M}}{\delta\overline{f(r)}\delta\overline{f(r')}}-\frac{J^2}{2\overline{I}^2}\frac{\delta^2\overline{I}}{\delta\overline{f(r)}\delta\overline{f(r')}}+\frac{J^2}{\overline{I}^3}\frac{\delta\overline{I}}{\delta\overline{f(r)}}\frac{\delta\overline{I}}{\delta\overline{f(r')}}\right]\times\eta(r,t)\eta(r',t), \tag{31}$$

where $\overline{\rho}=\rho(\overline{f})$. We note that this procedure differs from the one used in available studies of radial fluctuations in the three-dimensional model [15,16] by the fact that we are expanding about the profile function $\overline{f_J}$ of the rotating soliton, instead of about the profile function f_0 of the static soliton. Carrying out the functional derivatives, using the expressions of M and I given by Eqs. (19) and (17) with $f\equiv\overline{f}$, we obtain

$$H=\overline{E}+\frac{1}{2}\int_0^\infty dr\left[\frac{p^2}{\overline{\rho}}+\overline{U}\eta'^2+\overline{V}\eta^2\right]+\frac{1}{2}\left[\int_0^\infty dr\overline{W}\eta\right]^2, \tag{32}$$

where

$$\overline{U}=2\pi\left[r+\frac{2}{r}(1-\omega^2r^2)\sin^2\overline{f}\right], \tag{33}$$

$$\overline{V}=2\pi\left[\frac{1}{r}(1-\omega^2r^2)(1-2\overline{f'^2})\cos2\overline{f}+2(1+\omega^2r^2)\frac{\overline{f'}}{r^2}\sin2\overline{f}-2(1-\omega^2r^2)\frac{\overline{f''}}{r}\sin2\overline{f}+\mu^2r\cos\overline{f}\right], \tag{34}$$

$$\overline{W}=2\pi\frac{\omega r}{\sqrt{\overline{I}}}\left[4\left(\frac{\overline{f'}}{r}+\overline{f''}\right)\sin^2\overline{f}+(2\overline{f'^2}-1)\sin2\overline{f}\right]. \tag{35}$$

We stress that these functions of r depend parametrically on ω — or more properly on J — both through the explicit appearance of ω in their expressions and through the parametric dependence of the profile function $\overline{f}$ on ω.

The Hamiltonian (32) implies the following equation for the fluctuation field $\eta(r,t)$:

$$\overline{\rho}\ddot{\eta}-\overline{U'}\eta'-\overline{U}\eta''+\overline{V}\eta+\overline{W}\int_0^\infty dr'\overline{W}\eta=0. \tag{36}$$

The somewhat unusual integral term in this equation originates from the last term in the brackets of Eq. (31).

The field $\eta(r,t)$ may be expanded in terms of creation and annihilation operators, in the form

$$\eta(r,t)=\sum_\kappa\frac{1}{\sqrt{2E_\kappa}}[\psi_\kappa(r)a_\kappa e^{-iE_\kappa t/\alpha}+\psi_\kappa^*(r)a_\kappa^\dagger e^{iE_\kappa t/\alpha}], \tag{37}$$

where the denominators in the time-dependent phases appear because, in our adimensional units, the Planck constant is $(\epsilon f_\pi)^{-1}\equiv\alpha$. Substituting this expansion in the field equation (36), one obtains the following integro-differential eigenvalue equation to be satisfied by the modes $\psi_\kappa(r)$:

$$-\frac{d}{dr}\left[\overline{U(r)}\frac{d}{dr}\psi_\kappa(r)\right]+\overline{V(r)}\psi_\kappa(r)+\overline{W(r)}\int_0^\infty dr'\overline{W(r')}\psi_\kappa(r')=\frac{E_\kappa^2}{\alpha^2}\overline{\rho(r)}\psi_\kappa(r). \tag{38}$$

Since the boundary conditions (15) are satisfied by $f(r,t)$ as well as by $\overline{f(r)}$, the functions $\psi_\kappa(r)$ must vanish at the origin and at infinity.

Given the asymptotic behavior (29) of the profile function $\overline{f}$, the functions appearing in Eq. (38) are easily seen from Eq. (23) and Eqs. (33)–(35) to approach exponentially the following forms as r increases:

$$\overline{\rho(r)}\sim\overline{U(r)}\sim2\pi r,\quad \overline{V(r)}\sim2\pi\left[\frac{1}{r}+(\mu^2-\omega^2)r\right],\quad \overline{W(r)}\sim0. \tag{39}$$

Then Eq. (38) reduces to

$$r^2\frac{d^2}{dr^2}\psi_\kappa+r\frac{d}{dr}\psi_\kappa+\left[r^2\left(\frac{E_\kappa^2}{\alpha^2}-\mu^2+\omega^2\right)-1\right]\psi_\kappa=0. \tag{40}$$

The solutions of this equation are the Bessel functions of the first kind $J_1(\kappa r)$ and $Y_1(\kappa r)$, with the (adimensional) wave number related to the energy E_κ by

$$\kappa=\sqrt{\frac{E_\kappa^2}{\alpha^2}-\mu^2+\omega^2}. \tag{41}$$

Therefore, at large distance ($\sqrt{\mu^2-\omega^2}r\gg1$), the regular solution of Eq. (38) takes the form

$$\psi_\kappa(r)\sim\cos\delta(\kappa)J_1(\kappa r)+\sin\delta(\kappa)Y_1(\kappa r), \tag{42}$$

where $\delta(\kappa)$ is the phase shift, which can be extracted in standard fashion by integrating Eq. (38) numerically, with the boundary condition $\psi_\kappa(0)=0$, out to a sufficiently large radius and matching the solution to the form (42).

The unusual dispersion relation (41) might lead one to conclude that the pion mass is renormalized by the rotation [17], in such a way that $\alpha\sqrt{\mu^2-\omega^2}$ would be the physical pion mass in the presence of a soliton rotating with angular velocity ω. In order to settle this issue, we study the pion field at large distance in the fixed frame, for a mode of wave number κ. Since the mean profile function vanishes exponentially and the fluctuation is small by assumption, it follows from Eqs. (13), (30), and (37) that

$$\Phi_i(\vec{x},t)\sim R_{ij}(t)\frac{x^j}{r}\psi_\kappa(r)a_\kappa e^{-iE_\kappa t/\alpha}+\text{c.c.}, \tag{43}$$

where an irrelevant normalization factor has been dropped. The d'Alembertian of this expression is easily calculated; the piece involving time derivatives is

$$\partial_t^2\Phi_i\sim-\omega^2\Phi_i-\frac{E_\kappa^2}{\alpha^2}\Phi_i+\cdots, \tag{44}$$

where only those terms that will contribute to the mass have been displayed. The Laplacian of the field can be seen, with the help of Eq. (40), to take the form

$$\vec{\nabla}^2\Phi_i\sim\left(\mu^2-\omega^2-\frac{E_\kappa^2}{\alpha^2}\right)\Phi_i. \tag{45}$$

Collecting these results, one gets

$$\partial_\nu\partial^\nu\Phi_i\sim-\mu^2\Phi_i+\cdots. \tag{46}$$

Thus, the correct pion-mass term appears in the Klein-Gordon equation for the asymptotic field, without renormalization. The above calculation shows that this result follows from the cancellation of the ω^2 term in the dispersion relation (41), valid in the rotating frame, by the term arising from the second derivative of the rotation matrix, so that the pion energy in the fixed frame $E_{\kappa,f}$ is given by the dispersion relation

$$E_{\kappa,f}=\alpha\sqrt{\mu^2+\kappa^2}. \tag{47}$$

We note that the dimensional wave number k is related to the adimensional one κ through $\kappa=\epsilon k/f_\pi$, so that, since $\alpha=(\epsilon f_\pi)^{-1}$, the above relation implies $E_{\kappa,f}=\sqrt{m_\pi^2+k^2}/f_\pi^2=\mathcal{E}_{k,f}/f_\pi^2$, where $\mathcal{E}_{k,f}$ is the dimensional pion energy. Comparing this with Eq. (21), one checks that meson and soliton energies scale in the same way, as they should.

Before turning to the discussion of numerical results, a brief comment about the term represented by the ellipse in Eq. (44) is in order. This term arises from applying one of the time derivatives to the rotation matrix and the other one to the intrinsic field; it is oriented tangentially in the rotating frame. The presence of this undesirable term indicates that, even in two dimensions, the rotating hedgehog ansatz no longer corresponds to a solution of the field equations once the profile function depends on time. In order to achieve full consistency with the field equations, the restriction to radial fluctuations should be lifted when quantizing the meson field.

We present in Fig. 6 the results of phase-shift calculations for three values of the pion inverse Compton wavelength μ and two values of the parameter α, which sets the value of Planck's constant in rescaled units. These values were chosen so as to provide a fair sampling of the parameter space considered in Sec. III. For each set of parameters, the phases associated with radial scattering of a meson off a soliton in the first rotational state (analogous to pion-nucleon scattering) and in the second rotational state (analogous to pion-Δ scattering) are compared to those produced by the static soliton. The phase shifts are plotted as functions of κ/μ, which is equal to the corresponding ratio of dimensional quantities k/m_π.

The influence of the rotation on the phase shifts is significant in all cases, and of course, considerably larger for $n=2$ than that for $n=1$. The rotational effect depends quite strongly on the value of the parameter α, being small to mild for $\alpha=10$, but reaching already dramatic proportions for

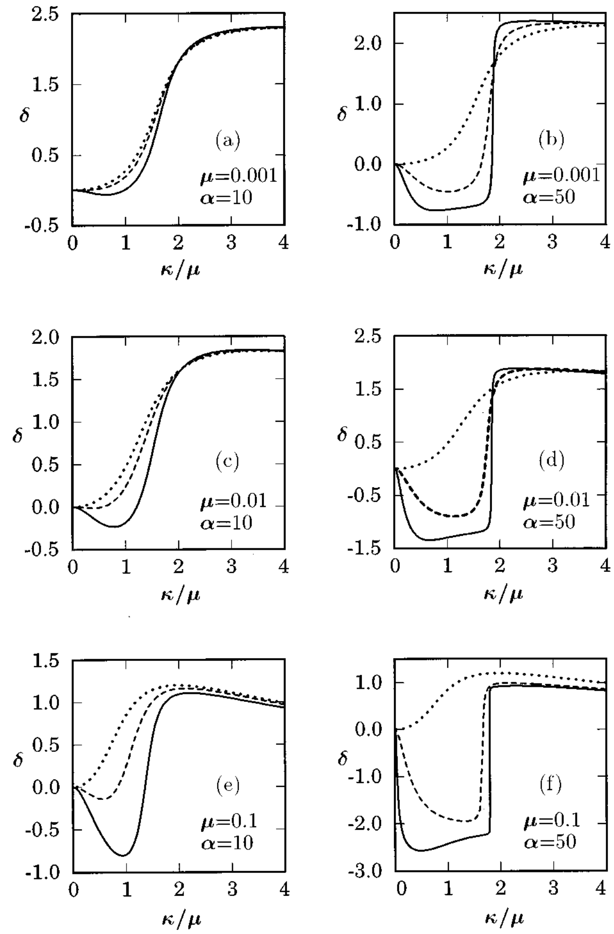

FIG. 6. Meson-soliton phase shifts from radial fluctuations, as functions of the meson momentum-to-mass ratio, for six sets of model parameters and three values of the angular momentum. Dotted curves for $n=0$ (static soliton); dashed curves for $n=1$; solid curves for $n=2$.

$\alpha=50$. The size of the effect also increases—though less spectacularly—with μ. By referring to the figures of Sec. III and the accompanying discussion, one verifies that the rotational effect on the phase shifts generally increases as one moves in parameter space from regions in which the mean-field results conform well to the ''large N_C'' approximation towards regions where they differ from it considerably.

It is also worth remarking that, for $\kappa/\mu\gtrsim3$, the phase shifts are almost unaffected by the rotation. This reflects the long-range character of the rotational effect, which mostly stretches the tail of the profile function. Finally, one may notice that, except for the largest value of μ, the phase shifts go through $\pi/2$ in the region close to $\kappa/\mu\sim2$. It is curious that the position of this resonance is little affected by the rotation.

V. CONCLUSIONS

The Skyrme model in 2+1 dimensions has been used to investigate the influence of the collective rotation of a topological soliton on its intrinsic structure.

The use of such a toy model was motivated by the fact that, in 3+1 dimensions, a rotating soliton possesses cylindrical symmetry only and is, therefore, much more difficult to construct than the spherically symmetric static soliton. Studies that propose to describe the rotating three-dimensional Skyrmion as a spherically symmetric object, namely, a hedgehog, are flawed from the start and may lead to misleading results.

On the other hand, in two-dimensional space, a soliton with circular symmetry can maintain that symmetry when set in rotation, and more specifically, the hedgehog ansatz for a rotating soliton is compatible with the field equations. Thus, in that case, one may hope to draw some theoretically sound—though not necessarily phenomenologically relevant—conclusions while avoiding excessive computational complexity.

The rotating hedgehog ansatz yields exact solutions of the classical field equations if the angular velocity is constant and the profile function describing the intrinsic structure of the soliton is independent of time. These solutions can be interpreted in the quantum field theory as describing angular-momentum eigenstates in the mean-field approximation. Even though the angular velocity of a stable rotating soliton cannot exceed the meson mass, there is no corresponding restriction on the angular-momentum quantum number.

The physical properties of the first two rotational states have been investigated over a fairly broad range of values of the parameters of the model, chosen to crudely mock up essential features of the realistic hadronic physics. The main purpose of this study has been to assess the validity of standard ''large N_C'' arguments, which are usually invoked to treat the rotating soliton as an undeformable object whose intrinsic structure is described by the static solution. It has been found that, even though the rotational contribution to the energy is generally rather small, the rotation may strongly affect the radius and the moment of inertia—and, therefore, also the splitting between rotational levels—of the soliton.

The influence of the rotation upon the soliton intrinsic mean field implies corresponding modifications of the phase shifts associated with quantum fluctuations about this mean field, which describe meson-baryon scattering. Although we have not performed a complete consistent quantization, we have presented an exploratory study limited to quantum fluctuations of the radial profile function. Not surprisingly, we have found that, in those regions of parameter space where the effect of the rotation on the mean field is significant, its effect on the phase shifts is also considerable, at least for meson momenta less than a few times the meson mass.

Translated into realistic hadron physics, these results suggest that the reliance upon ''large N_C'' arguments in Skyrme model calculations may lead to crude or even incorrect results for physical quantities such as the N-Δ splitting, baryon radii, and meson-baryon scattering cross sections. They also do not lend support to the hope that a ''band cutoff'' mechanism—such as has been demonstrated in a hybrid quark-meson model [18]— could make the unobserved high spin baryon excitations unstable already at the semiclassical level.

Of course, models defined in different numbers of spatial dimensions may possess qualitatively different features. That this is sometimes the case for Skyrme-type models is already clear from the fact that the massless nonlinear σ model admits static hedgehog solutions in two but not in three dimensions. Thus, one should obviously refrain from abusively extrapolating from the toy model considered in this work to the realistic model. In spite of this caveat, it is our opinion that the results presented in this work strongly suggest that the influence of the collective rotation on the intrinsic structure of the Skyrmion deserves more thorough study.

As has been demonstrated recently [6,7], and as was assumed in the present work, the correct starting point for a study of a quantum soliton of nonvanishing angular momentum is the intrinsic field configuration corresponding to a time dependent solution of the classical field equations describing a soliton rotating with constant angular velocity. In order to construct such solutions in the three-dimensional Skyrme model, it is necessary to generalize the hedgehog ansatz to a field oriented in an *a priori* arbitrary direction in the meridian plane defined with respect to the rotation axis. The components of the field in this plane are then functions of two variables (e.g., the distance to the rotation axis and the height above the equatorial plane). The solution of the corresponding field equations is a somewhat arduous but not impossible computational task.

Only when such solutions are obtained will it be possible to assess by comparison the adequacy of the standard ''large N_C'' approximation for the realistic Skyrme model.

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[1] T. H. S. Skyrme, Proc. R. Soc. London **A260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).

[2] G. S. Adkins, C. P. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983); G. S. Adkins and C. P. Nappi, *ibid.* **B233**, 109 (1984).

[3] K.-F. Liu, J.-S. Zhang, and G. R. E. Black, Phys. Rev. D **30**, 2015 (1984).

[4] R. Rajaraman, H. M. Sommermann, J. Wambach, and H. W. Wyld, Phys. Rev. D **33**, 287 (1986).

[5] H. Otsu and T. Sato, Nucl. Phys. **B334**, 489 (1990).

[6] N. Dorey, J. Hughes, and M. P. Mattis, Phys. Rev. D **49**, 3598 (1994).

[7] N. Dorey, J. Hughes, and M. P. Mattis, Phys. Rev. D **50**, 5816 (1994).

[8] M. Betz, H. B. Rodrigues, and T. Kodama, in *Proceedings of the 3rd Workshop on Relativistic Aspects of Nuclear Physics*, Rio de Janeiro, Brazil, 1993, edited by K. C. Chung *et al.* (World Scientific, Singapore, 1995), p. 384.

[9] B. M. A. G. Piette, B. J. Schroers, and W. J. Zakrzewski, Nucl. Phys. **B439**, 205 (1995).

[10] B. Schwesinger, H. Weigel, G. Holzwarth, and A. Hayashi, Phys. Rep. **173**, 173 (1989).
[11] M. P. Mattis and M. Karliner, Phys. Rev. D **31**, 2833 (1985); M. P. Mattis and M. Peskin, *ibid.* **32**, 58 (1985); M. Karliner and M. P. Mattis, *ibid.* **34**, 1991 (1986).
[12] M. Betz and H. B. Rodrigues, in *''Hadron Physics 94,''* proceedings of the 4th Workshop on Hadron Physics, Gramado, Brazil, 1994, edited by V. E. Herscovitz, C. A. Z. Vasconcellos, and E. Ferreira (World Scientific, Singapore, 1995), p. 261.
[13] M. Bander and F. Hayot, Phys. Rev. D **30**, 1837 (1984).
[14] E. Braaten and J. P. Ralston, Phys. Rev. D **31**, 598 (1985).
[15] J. D. Breit and C. R. Nappi, Phys. Rev. Lett. **53**, 889 (1984).
[16] E. L. M. Koopman, A. Lande, and O. Scholten, Nucl. Phys. **A562**, 659 (1993).
[17] T. Okazaki, K. Fujii, and N. Ogawa, Int. J. Mod. Phys. A **7**, 6763 (1992).
[18] J. P. Blaizot and G. Ripka, Phys. Rev. D **38**, 1556 (1988).