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Some comments about a static-localized solution for the massive Thirring model

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We derive the stability equation for a static-localized solution of the two-dimensional massive Thirring model. The corresponding eigenvalue spectrum is determined and compared with the eigenvalue spectrum of the stability equation for the soliton solution of the sine-Gordon theory.

Stationary-localized solutions for the two-dimensional massive Thirring model[1] have been found by Lee *et al.*[2] and, independently, by Chang *et al.*[3] By definition, the just-mentioned solutions are of the form

$$\psi(t,x)=e^{-iEt}\Phi(x)\,. \tag{1}$$

Here, E is the frequency of oscillation of the stationary wave (a real number) and $\Phi(x)$ is a two-component spinor which satisfies the boundary conditions

$$\Phi(x)\xrightarrow[x\to\pm\infty]{}0\,, \tag{2}$$

as required by confinement.

It has been shown[4,5] that only for those values of E in the interval

$$0\leq E<m$$

the solutions (1) are localized and yield a positive energy density for all points in space. Therefore, setting $E=0$ in Eq. (1) we obtain a physically meaningful static-localized solution for the two-dimensional massive Thirring model. It is precisely the eigenvalue spectrum of the stability equation associated with this static solution that we attempt to discuss in this work.

Our starting point is the Lagrangian density[1]

$$\mathcal{L}(x)=\tfrac{1}{2}i\{\bar\psi(x)\gamma^{\mu}[\partial_{\mu}\psi(x)]-[\partial_{\mu}\bar\psi(x)]\gamma^{\mu}\psi(x)\}-m\bar\psi(x)\psi(x)+\mathcal{L}_I(\bar\psi\psi)\,, \tag{3}$$

where $\mathcal{L}_I$ contains the Fermi interaction of vector-type,

$$\mathcal{L}_I=\tfrac{1}{2}g(\bar\psi\gamma^{\mu}\psi)(\bar\psi\gamma_{\mu}\psi)\,.$$

Since we are working in a two-dimensional space-time, $\psi(x)$ is a two-component spinor $[\bar\psi(x)=\psi^{\dagger}(x)\gamma_0]$, m is the fermion mass, and g is a positive coupling constant. As usual, repeated Greek indices imply summation. Our metric is $g_{00}=-g_{11}=+1$, $g_{01}=g_{10}=0$.

We would like to point out that we interpret all classical fermion fields as commuting c-number functions and not as the classical limit of quantum field operators. In this connection it is interesting to see the discussion in Appendix B of Ref. 3.

The covariant relativistic notation of Eq. (3) will not be used again in this paper and it will be replaced by $x^0=t$, $x^1=x$, as we already did when writing Eqs. (1) and (2).

We complete the specification of the model (3) assuming, throughout, the γ-matrices representation

$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\,,\quad\gamma_1=-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\,.$$

Using Euler-Lagrange equations one can derive from (3) the equations of motion for ψ and $\bar\psi$ from which one is immediately led to the time-independent equations to be satisfied by Φ and $\bar\Phi$. By exploiting this last set of equations and the boundary conditions (2) the authors of Ref. 5 have shown that one does not lose generality by assuming that Φ is a real spinor, and we shall do so from now on.

It is easy to verify that if $\Phi_0(x)$ is a static-localized solution for the two-dimensional massive Thirring model, its two real components

$$\Phi_0(x)=\begin{bmatrix}u_0(x)\\v_0(x)\end{bmatrix} \tag{4}$$

are to be determined by solving the two-equation system

$$\frac{dv_0}{dx}+mu_0-g(u_0^2+v_0^2)u_0=0\,,\qquad \frac{du_0}{dx}+mv_0+g(u_0^2+v_0^2)v_0=0 \tag{5}$$

together with the boundary conditions

$$u_0(\pm\infty)=v_0(\pm\infty)=0\,. \tag{6}$$

The static-localized solution in whose analysis we are interested is easily obtained from Ref. 2 and reads

$$u_0=(2m/g)^{1/2}e^{mx}(e^{2mx}+1)(e^{4mx}+1)^{-1}\,, \tag{7a}$$

$$v_0=(2m/g)^{1/2}e^{mx}(e^{2mx}-1)(e^{4mx}+1)^{-1}\,. \tag{7b}$$

Let us now consider a real two-component time-independent spinor $\Phi(x)$

$$\Phi(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}. \tag{8}$$

The spinor $\Phi(x)$ is not supposed to be a solution of the equations of motion (5), although it is supposed to vanish fast enough as $x \to \pm\infty$ in order to make the total energy integral H

$$H = H(u, v) = \int_{-\infty}^{+\infty} dx \left[u\frac{dv}{dx} - v\frac{du}{dx} + m(u^2 - v^2) - \frac{g}{2}(u^2 + v^2)^2 \right] \tag{9}$$

finite.[6] One can see that the functions $u(x)$ and $v(x)$ which make of H an extremum are necessarily solutions of Eqs. (5). Therefore,

$$\left.\frac{\delta H(u,v)}{\delta u}\right|_{\substack{u=u_0\\ v=v_0}} = \left.\frac{\delta H(u,v)}{\delta v}\right|_{\substack{u=u_0\\ v=v_0}} = 0. \tag{10}$$

We now make a functional Taylor expansion of $H(u,v)$ around the solution u_0, v_0. We shall call Δ the sum of all terms involving second-order partial functional derivatives in the just-mentioned expansion. After some algebra we arrive at the following expression for Δ:

$$\Delta = \sum \omega^{(n)} |C_n|^2, \tag{11}$$

where the $\omega^{(n)}$'s are to be found by solving the linear eigenvalue problem

$$H_D \chi(x) = \omega\chi(x). \tag{12}$$

Here, H_D is the Hermitian operator

$$H_D = i\alpha_1 \frac{d}{dx} + \beta m + V \tag{13}$$

which acts in a two-dimensional space. χ is the two-component eigenfunction corresponding to the eigenvalue ω. The 2×2 matrices α_1 and β are given by

$$\alpha_1 = \gamma_0\gamma_1, \quad \beta = \gamma_0. \tag{14}$$

Finally, the elements of the 2×2 Hermitian matrix V,

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{15}$$

turn out to be

$$\begin{aligned} V_{11} &= -2m(2\cosh 2mx + 1)\cosh^{-2} 2mx, \\ V_{12} = V_{21} &= -2m \sinh 2mx \cosh^{-2} 2mx, \\ V_{22} &= -2m(2\cosh 2mx - 1)\cosh^{-2} 2mx. \end{aligned} \tag{16}$$

It is perhaps obvious to remark that Eq. (12) is the stability equation associated with the solution (7). Our main purpose in this note is to determine and discuss its eigenvalue spectrum.

From Eqs. (13)–(16) it clearly follows that the eigenvalue problem (12) is mathematically equivalent to the eigenvalue problem for the one-dimensional Dirac equation when the external potential is, precisely, V. Then, there will be nonvanishing eigenfunctions which are solutions of (12) for negative values of ω and, as a consequence, the classical confined solution (7) is not stable against small disturbances. This lack of stability can also be seen from the fact that the energy (9) associated with the static solution is not bounded from below, and consequently, it can not be absolutely stable.

Nevertheless, we shall not at this stage discard (7) as a solution around which the theory can be quantized because we think that any consistent fermion-field quantization scheme one could design to work in the one-soliton sector should solve the inconsistency of having negative-energy states, much as it is solved in the vacuum-sector quantization; i.e., all negative-energy states are eliminated from the theory through an appropriate mechanism contained in the quantization rules. If this is so, it is still a physically meaningful task to find the eigenvalues corresponding to the bound states of (12).

It is obvious that for bound states we must have

$$\chi(x) \xrightarrow[x\to\pm\infty]{} 0. \tag{17}$$

By using Eqs. (12) and (17) we arrive at

$$\chi^\dagger(x)\alpha_1\chi(x) = \overline{\chi}(x)\gamma_1\chi(x) = 0. \tag{18}$$

This last condition only holds for bound states. In fact, for scattering states it must be replaced by $\overline{\chi}\gamma_1\chi = \text{constant} \neq 0$.

From Eqs. (12) and (18) it follows that all bound-state eigenfunctions are real. Then we can write

$$\chi(x) = \rho(x) \begin{pmatrix} \cos\alpha(x) \\ \sin\alpha(x) \end{pmatrix}, \tag{19}$$

where $\rho(x)$ and $\alpha(x)$ are real functions of the variable x.

Since the soliton solution (7) is also real it can be written as[2]

$$\begin{aligned} u_0 &= R(x)\cos\theta(x), \\ v_0 &= R(x)\sin\theta(x), \end{aligned} \tag{20}$$

where $R(x)$ and $\theta(x)$ are real functions of the variable x.

By going back with (19) and (20) into (12), we can recast the eigenvalue problem for the bound states to read

$$\frac{d\phi}{dx} - m\cos 2\phi = \omega + 2m\cos 2\theta, \tag{21a}$$

$$\frac{1}{\rho}\frac{d\rho}{dx}=m\sin 2\phi\ , \tag{21b}$$

where

$$\phi=\alpha-2\theta\,. \tag{22}$$

In the language of Eq. (19) the boundary condition (17) translates into

$$\rho(x)\xrightarrow[x\to\pm\infty]{}0\,. \tag{23}$$

Furthermore, from the structure of the linear operator H_D, given in Eq. (13), and what is known about the Dirac equation it follows that all bound-state eigenvalues (ω) must satisfy

$$|\omega|<m\,. \tag{24}$$

Also note that the variable ρ decouples entirely from the differential equation determining ϕ.

It can be easily checked that the spinor

$$\chi(x;\omega=0)=\begin{bmatrix}\dfrac{du_0}{dx}\\ \\ \dfrac{dv_0}{dx}\end{bmatrix} \tag{25}$$

is the solution of the two-equation system (21) for $\omega=0$. It is obvious from Eq. (7) that $\chi(x)$ given in (25) verifies the boundary condition $\chi(\pm\infty)=0$, as it is required for the bound states. One arrives at (25) either by solving Eqs. (21) explicitly or by realizing that, after d/dx is taken in both sides of Eqs. (5), one reproduces Eq. (12) with χ given by (25) and $\omega=0$.

Thus, the zero-frequency mode required by spatial translation invariance is present.

When $\omega\neq 0$ our analysis goes as follows. From Eq. (21b) we obtain

$$\rho(x)=\rho(a)\exp\left[m\int_a^x \sin 2\phi(y)\,dy\right]\,. \tag{26}$$

On the other hand, from Eqs. (7) and (20) we find that

$$\cos 2\theta=\cosh^{-1}2mx\underset{x\to\pm\infty}{\sim}2e^{-2m|x|}\,.$$

Therefore, in the asymptotic region ($|x|\to\infty$) Eq. (21a) can be approximated by

$$\frac{d\phi}{dx}-m\cos 2\phi=\omega\ ,$$

whose solution is well known to be[2]

$$\phi(x;\omega)=\tan^{-1}\{\alpha\tanh[\beta_0(x+C)]\}\,. \tag{27}$$

Here,

$$\alpha=+\left(\frac{m+\omega}{m-\omega}\right)^{1/2},\quad \beta_0=+(m^2-\omega^2)^{1/2}\ ,$$

and C is an arbitrary constant. We now choose both limits of the integral

$$\int_a^x \sin 2\phi(y)\,dy$$

to be in the asymptotic region. This allows us to find the asymptotic behavior of $\rho(x)$. By inserting (27) into (26) we obtain

$$\rho(x)\underset{|x|\to\infty}{\sim}C_1\left\{\frac{2m}{m-\omega}\cosh^2[(m^2-\omega^2)^{1/2}(x+C)]-m-\omega\right\}^{1/2}, \tag{28}$$

where C_1 is a constant. From this last expression it follows that

$$\rho(x)\xrightarrow[|x|\to\infty]{}\infty\,. \tag{29}$$

Then, we conclude that there are no bound states with $\omega\neq 0$ or, what amounts to the same thing, that the soliton (7) does not exhibit excited states.

Note that when $\omega=0$ the term $\cos 2\theta$ in (21a) cannot be neglected, not even in the asymptotic region. In this case a bound-state solution of (21) exists and it is given in (25).

It has been shown by Coleman[7] that the massive Thirring model is equivalent to the sine-Gordon (SG) theory in the charge-zero sector. Also, the authors of Ref. 3 have noted that the relative phase θ between the upper and lower components of the spinor ϕ_0 obeys the SG equation. Then, it seems to us interesting to close this note with some conclusions arising after comparing the eigenvalue spectrum obtained by us with the eigenvalue spectrum of the stability equation corresponding to the soliton solution of the SG theory.[8]

If one admits that the negative-energy states of our Eq. (12) can be eliminated through a set of adequate quantization rules, one concludes that both spectra are qualitatively equal; i.e., in both cases one observes a bound state at $\omega=0$ and a continuum of positive-energy scattering states. Furthermore, for $\omega=0$ the solution to Eqs. (21) is, as we already said, the spinor (25). When this spinor is written as indicated in Eq. (19) and the static-confined solution (7) is written as in (20) it turns out that $\alpha=3\theta+\pi/2$, which in turns implies that [see Eq. (22)] for $\omega=0$

$$\phi(x)=\theta(x)+\pi/2\,. \tag{30}$$

On the other hand, from Eqs. (21a) and (21.6) one easily arrives at the following differential equation for ρ^2:

$$\left(-\frac{d^2}{dx^2}+\mu^2+2\mu^2\cos 2\phi\cos 2\theta\right)\rho^2=0\ , \tag{31}$$

where $\mu=2m$. After taking into account Eq. (30)

and the fact that the static-confined solution for the massive Thirring model (7) can, as far as the phase $\theta(x)$ is concerned, also be written[2]

$$\theta(x) = \tan^{-1}(\tanh mx)\ ,$$

we can recast Eq. (31) to read

$$\left(-\frac{d^2}{dx^2} + \mu^2 - \frac{2\mu^2}{\cosh^2 \mu x}\right)\rho^2 = 0\ . \qquad (32)$$

Thus, we conclude that for $\omega = 0$ the differential equation satisfied by ρ^2 is equivalent to the stability equation for the soliton solution of the SG theory,[8] also for $\omega = 0$. This tells us that the presence of a zero-frequency mode in our eigenvalue problem (12) could have been predicted from the presence of a similar mode in the SG stability equation. We have not pursued the investigation of this equivalence further.

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