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Potts model in a random field

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The q-state Potts model in a random field with a discrete distribution of statistically independent fields ordered along any of the q states is studied in mean-field theory. Detailed phase diagrams are obtained in a two-component order-parameter theory for $q=3$ and a one-component theory for general q. Lines of critical *and* tricritical points are found in the first case and lines of critical points in the second one, in the presence of a sufficiently large, constant uniform field.

I. INTRODUCTION

The order of the phase transition of the q-state Potts model[1] has already been a subject of great interest for some time.[2-5] A Landau expansion of the free energy in mean-field theory, which is exact in the limit of dimension $d \rightarrow \infty$, yields a first-order phase transition for $q>2$ in all d. In contrast, a strong d dependence of the order of the phase transition follows from exact work in two dimensions,[3] a $1/q$ expansion in three dimensions,[4] and various renormalization-group calculations.[5] A continuous transition, for short-range "ferromagnetic" interactions, follows if $q \leq q_c^0(d)$, while the transition is of first order otherwise. The critical value $q_c^0(d)$ varies in the range $2 \leq q_c^0(d) \leq \infty$ between the upper critical dimension $d_u=4$ and $d \rightarrow 1^+$, in what seems to be a monotonic decreasing behavior with d.

The q-state Potts model in a quenched random field that couples linearly to the order parameter has been studied in recent works.[6,7] Explicit calculations on the mean-field free energy in terms of a one-component order parameter yield a first-order phase transition between a ferromagnetic and a paramagnetic phase for all q. An attempt to obtain a changeover to a second-order phase transition at a tricritical point, in a Landau expansion for the free energy, failed so far, independently of the form of the distribution function for the random field.[7]

Thus the situation for the general q-state Potts model in a random field seems to be different from that in the Ising model.[8,9] Indeed, depending on some general features of the distribution function for the random field, the second-order phase transition for the Ising model may become of first order at a tricritical point.

It is not known whether or not a random field increases the tendency towards a first-order phase transition in the $q(>2)$-state Potts model. The mean-field theory results referred to above,[6,7] in the presence of a random field, may be mainly a manifestation of the first-order transition already present without a random field.

An argument based on dimensional reduction[10] has been used to suggest that fluctuations may turn the first-order phase transition for the three- and four-state Potts models in a random field into continuous transitions at a tricritical point.[7] However, a rigorous proof on the two-dimensional Ising model[11] in a random field shows that dimensional reduction in the usual sense (with the *same* shift everywhere between the upper and lower critical dimensions) does *not* hold for the Ising model. The same conclusion is expected to apply to other discrete symmetry models as the Potts model.

The purpose of the present paper is to investigate, within mean-field theory, the tendency towards ordering via a first-order or a continuous phase transition for the q-state Potts model in a random field. We do this by introducing a constant uniform field and consider either a one-component order-parameter theory for general q or a two-component theory for $q=3$.

Mean-field theory with a two-component order parameter for the three-state Potts model is known to yield a changeover from a first-order to a continuous phase transition at a tricritical point, in the absence of a random field.[12] All that is needed is a sufficiently large uniform field, as shown schematically in Fig. 1(a).

If a random field with an appropriate distribution favors a first-order transition in a system with discrete symmetry, as the available results on the Ising model in a random field seem to indicate, with the appearance of a tricritical point, then one may ask (a) what occurs with the tricritical point already present in the Potts model, and (b) can there be a second tricritical point at which the continuous phase transition changes back to a first-order transition, for sufficiently large random field, as shown in Fig. 1(b)? A phase diagram with two tricritical points, where a first-order transition becomes continuous and then returns to first order was discussed in the context of fluctuation-induced transitions not long ago.[13]

The restriction to a single order-parameter mean-field theory for general q is only for simplicity. Although there is no tricritical point in this case, there is a critical point in finite uniform field, as will be shown here.

In Sec. II we present the model and in Sec. III we derive the two versions of mean-field theory: that with a

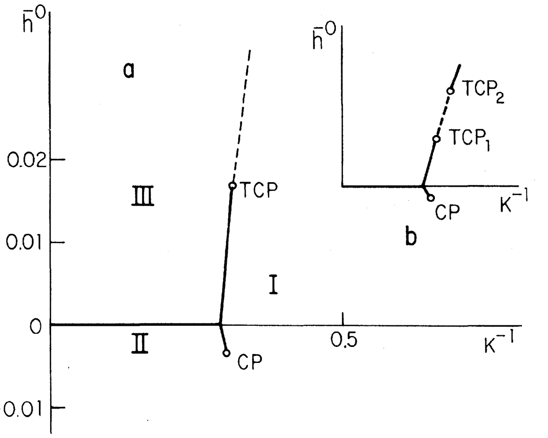

FIG. 1. Mean-field phase diagram, described in Sec. IV, for the three-state Potts model in a constant uniform field h^0, with $K=\beta J$ and $\bar{h}^0=-h^0/\sqrt{6}J$ (Sec. II). Heavy lines indicate first-order transitions and dashed lines continuous transitions. Critical and tricritical points (CP and TCP) appear (a) and one may speculate on a second TCP_2 (b) in a large random field.

one-component order parameter for all q and with a two-component order parameter for $q=3$. The phase diagrams that yield lines of critical and tricritical points for the three-state model are established in Sec. IV, while further results for general q are studied in Sec. V, in particular the limit $q\to\infty$. We end with concluding remarks in Sec. VI.

II. THE MODEL IN A RANDOM FIELD

The q-state Potts model in a uniform and a random field, $\mathbf{h}^0$ and $\mathbf{h}_i$, respectively, on the sites $1\le i\le N$ of a d-dimensional lattice is defined here in the standard representation by the Hamiltonian

$$\mathcal{H}=-\tfrac{1}{2}\sum_{i,j}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j-\mathbf{h}^0\cdot\sum_i\mathbf{S}_i-\sum_i\mathbf{h}_i\cdot\mathbf{S}_i\ , \qquad (2.1)$$

with the "spins" S_i that can be in q states which are the vectors $\mathbf{a}_i^\alpha=\{a_k^\alpha(i)\}$, $1<\alpha<q$ and $1\le k\le q-1$, to the vertices of a hypertetrahedron in $q-1$ dimensions. Thus we write

$$\mathcal{H}=-\tfrac{1}{2}\sum_{i,j}J_{ij}\mathbf{a}_i^\mu\cdot\mathbf{a}_j^\nu-\mathbf{h}^0\cdot\sum_i\mathbf{a}_i^\mu-\sum_i\mathbf{h}_i^\lambda\cdot\mathbf{a}_i^\mu\ , \qquad (2.2)$$

with both fields taken along the Potts vectors $\mathbf{a}^\alpha$ to preserve the permutational symmetry of the model. Specifically, we take

$$\mathbf{h}^0=h^0\mathbf{a}^1\ ,\qquad \mathbf{h}_i^\lambda=h\tau_i^\lambda,\quad 1\le\lambda\le q\ , \qquad (2.3)$$

where h^0 is assumed to remain constant, and τ_i^λ is quenched to one of the Potts vectors at each site.

Taking further[14]

$$a_k^\alpha=\begin{cases}0 & \text{if } \alpha<k\\ \left[\dfrac{q-k}{q-k+1}\right]^{1/2} & \text{if } \alpha=k\\ \left[\dfrac{q-k}{q-k+1}\right]^{1/2}\left[\dfrac{-1}{q-k}\right] & \text{if } \alpha>k\ ,\end{cases} \qquad (2.4)$$

where the site dependence has been suppressed for simplicity, and which satisfy the relationships

$$\sum_{\alpha=1}^{q}a_k^\alpha a_l^\alpha=\delta_{kl}\ ,\qquad \sum_{k=1}^{q-1}a_k^\alpha a_k^\beta=\delta_{\alpha\beta}-1/q\ ,\qquad \sum_{\alpha=1}^{q}a_k^\alpha=0\ , \qquad (2.5)$$

we have for the three-state model

$$a_1^1=2/\sqrt{6},\quad a_2^1=0\ ,\qquad a_1^2=-1/\sqrt{6},\quad a_2^2=1/\sqrt{2}\ ,\qquad a_1^3=-1/\sqrt{6},\quad a_2^3=1/\sqrt{2}\ . \qquad (2.6)$$

Although the representation used here in Eq. (2.2) differs from the more usual one[2,6,7] in which

$$\mathcal{H}=-\tfrac{1}{2}\sum_{i,j}J_{ij}(q\delta_{\lambda_i,\lambda_j}-1)-h^0\sum_i(q\delta_{\lambda_i,1}-1)-\sum_i\sum_\alpha h_i^\alpha(q\delta_{\alpha,\sigma_i}-1)\ , \qquad (2.7)$$

where λ_i is a spinlike variable taking q values, which may be chosen as the q roots of unity, the results that will be presented in the following sections turn out to be the same. Also, infinite range interactions $J_{ij}=J/N$, for all i and j, will be taken in this work so that mean-field theory becomes exact.

We consider a statistically independent distribution of random fields on each site, neglecting correlations between different sites, so that $P\{\mathbf{h}^\lambda(i)\}=\prod_i p\{\mathbf{h}^\lambda(i)\}$. Keeping the magnitude h of the random field fixed and assigning the same probability $1/q$ to a field along each of the q vectors we have

$$p\{\mathbf{h}^\lambda(i)\}=\frac{1}{q}\sum_{\mu=1}^{q}\delta(\mathbf{h}^\lambda(i)-\mathbf{h}^\mu(i))\ . \qquad (2.8)$$

For the three-state model Eq. (2.6) yields

$$p\{\mathbf{h}^\lambda\}=\frac{1}{3}\left[\delta\left[h_1^1-\frac{2}{\sqrt{6}}h\right]\delta(h_2^1)+\delta\left[h_1^2+\frac{1}{\sqrt{6}}h\right]\delta\left[h_2^2-\frac{1}{\sqrt{2}}h\right]+\delta\left[h_1^3+\frac{1}{\sqrt{6}}h\right]\delta\left[h_2^3+\frac{1}{\sqrt{2}}h\right]\right]\ , \qquad (2.9)$$

where, for simplicity, the site dependence has been suppressed.

In contrast to the discrete distribution used here, which averages over orientations of the random field, a Gaussian distribution that averages over the magnitude has also been considered in previous works.[6,7] Although more complicated to deal with, it reproduces basically the results of the discrete distribution. We comment more on this point below.

The average free energy per spin f,

$$\beta f = \lim_{N\to\infty} (\beta F/N) \tag{2.10}$$

in which $\beta = 1/k_B T$, follows from

$$-\beta F = (\ln Z)_{\rm av} \equiv \sum_{\{\mathbf{h}_i^\lambda\}} P\{\mathbf{h}_i^\lambda\} \ln Z\{\mathbf{h}_i^\lambda\} \tag{2.11}$$

as the random-field average of the logarithm of the partition function

$$Z\{\mathbf{h}_i^\lambda\} = \mathop{\rm Tr}_{\{\mathbf{a}_i^\mu\}} \exp(-\beta \mathcal{H}\{\mathbf{a}_i^\mu, \mathbf{h}_i^\lambda\}) , \tag{2.12}$$

in accordance with the standard procedure for quenched random fields. The trace is taken over the vectors $\mathbf{a}^\mu$, keeping the τ_i^λ in Eq. (2.3) fixed.

In the next section we proceed with the calculation of F in mean-field theory.

III. MEAN-FIELD THEORY

The partition function, Eq. (2.12), for the Hamiltonian given by Eq. (2.2) may be written by means of a Gaussian integration as

$$Z\{\mathbf{H}_i^\lambda\} = \mathop{\rm Tr}_{\{\mathbf{a}_i^\mu\}} \int \prod_k \frac{dm^k}{\sqrt{2\pi/NK}} \times \exp\left[-\frac{NK\mathbf{m}^2}{2} + K\sum_i \mathbf{m}\cdot\mathbf{a}_i^\mu + \mathbf{H}^0\cdot\sum_i \mathbf{a}_i^\mu + \sum_i \mathbf{H}_i^\lambda\cdot\mathbf{a}_i^\mu \right] \tag{3.1}$$

in which $\mathbf{m}$ is a $(q-1)$-dimensional vector of components m^k, while $K=\beta J$, $\mathbf{H}^0 = \beta\mathbf{h}^0$, and $\mathbf{H}_i^\lambda = \beta \mathbf{h}_i^\lambda$. Recognizing that the last three terms in the exponential involve only single-site terms, we have

$$Z\{\mathbf{H}_i^\lambda\} = \int \prod_k \frac{dm^k}{\sqrt{2\pi/NK}} \exp\left[-\frac{NK\mathbf{m}^2}{2} + N \ln \mathop{\rm Tr}_{\{\mathbf{a}^\mu\}} \exp[(K\mathbf{m} + \mathbf{H}^0 + \mathbf{H}^\lambda)\cdot\mathbf{a}^\mu] \right] . \tag{3.2}$$

For large N, the integral may be done by steepest descent. Keeping only the contribution at the saddle point we obtain, up to a trivial additive constant,

$$-\frac{1}{N}\ln Z\{\mathbf{H}^\lambda\} = \frac{K\mathbf{m}^2}{2} - \ln \mathop{\rm Tr}_{\{\mathbf{a}^\mu\}} \exp[(K\mathbf{m} + \mathbf{H}^0 + \mathbf{H}^\lambda)\cdot\mathbf{a}^\mu] . \tag{3.3}$$

The mean-field free energy that follows from Eqs. (2.10) and (2.11) is now given by

$$\beta f = \frac{K\mathbf{m}^2}{2} - \left[\ln \mathop{\rm Tr}_{\{\mathbf{a}^\mu\}} \exp[(K\mathbf{m} + \mathbf{H}^0 + \mathbf{H}^\lambda)\cdot\mathbf{a}^\mu] \right]_{\rm av} , \tag{3.4}$$

where the order parameter $\mathbf{m}$ is determined by the equations $\partial f/\partial m^k = 0$ which yield

$$\mathbf{m} = (\langle \mathbf{a}^\mu \rangle)_{\rm av} . \tag{3.5}$$

The thermal average is given here by

$$\langle \mathbf{a}^\mu \rangle = \mathop{\rm Tr}_{\{\mathbf{a}^\nu\}} [\mathbf{a}^\mu \exp(-\beta\mathcal{H}_{\rm eff})] / \mathop{\rm Tr}_{\{\mathbf{a}^\nu\}} \exp(-\beta\mathcal{H}_{\rm eff}) \tag{3.6}$$

in terms of the effective Hamiltonian

$$\mathcal{H}_{\rm eff} = -(J\mathbf{m} + \mathbf{h}^0 + \mathbf{h}^\lambda)\cdot\mathbf{a}^\mu . \tag{3.7}$$

So far, $\mathbf{m}$ is a $(q-1)$-dimensional order parameter. The study of the phase diagram for general q can be considerably simplified by the replacement with a one-component order parameter, as done in recent work.[6,7] Here we deal with both, a one-component theory for general q and a two-component theory for $q=3$.

A. One-component theory

We consider the order-parameter variable

$$\mathbf{m} = m\,\mathbf{a}^1 \tag{3.8}$$

along the external field $\mathbf{h}^0$ [cf. Eq. (2.3)]. This is a one-component order parameter, as can be seen from Eq. (2.4), with m determined by $\partial f/\partial m = 0$.

The random-field-averaged free energy, Eq. (3.4), becomes

$$\beta f = \frac{K\mathbf{m}^2}{2} - G(m, H^0, H) , \tag{3.9}$$

where $H^0 = \beta h^0$, $H = \beta h$, and

$$G(m,H^0,H) = (1/q)\ln\{\exp[(q-1)(A+B)] + (q-1)\exp[-(A+B)]\} + [(q-1)/q]\ln\{\exp[(q-1)A - B] + \exp[-A + (q-1)B] + (q-2)\exp[-(A+B)]\} , \tag{3.10}$$

in which

$$A \equiv (Km + H^0)/q , \quad B \equiv H/q . \tag{3.11}$$

For the order parameter we find

$$m = \frac{e^{qA}}{e^{qA} + e^{qB} + q - 2} - \frac{1}{\exp[q(A+B)] + q - 1} . \tag{3.12}$$

Equation (3.10) may be expanded in powers of m to obtain the Landau expansion for the free energy. We do not pursue this here since such a form in a one-component theory has not proven to be very useful. Instead, the equations presented here are used below to obtain analytical and numerical results for the phase diagrams with general q.

B. Two-component theory for the three-state model

For $q=3$, $\mathbf{m}=(m_1,m_2)$ has two components. Noting that $\mathbf{H}^0\cdot\mathbf{a}^\mu=H^0a^\mu$, with h^0 given by Eq. (2.3), it is convenient to replace m_1 in Eq. (3.3) by m_1+H^0/K. Calculating the trace, with the explicit representation for $\mathbf{a}^\mu$ given by Eq. (2.6), and performing the random-field average, Eq. (2.11), with the discrete distribution of Eq. (2.9) yields

$$\beta f=\frac{K}{2}[(m_1-H^0/K)+m_2^2]-\tfrac{1}{3}\ln\prod_{\mu=1}^{3}T_\mu(\mathbf{m},H)\ , \tag{3.13}$$

where

$$\begin{aligned}T_1&=\exp[2K(s_1+\bar h/3)]+\exp[-K(s_1-s_2+\bar h/3)]\\&\quad+\exp[-K(s_1+s_2+\bar h/3)]\ ,\\T_2&=\exp[K(2s_1-\bar h/3)]+\exp[-K(s_1-s_2-2\bar h/3)]\\&\quad+\exp[-K(s_1+s_2+\bar h/3)]\ ,\\T_3&=\exp[K(2s_1-\bar h/3)]+\exp[-K(s_1-s_2+\bar h/3)]\\&\quad+\exp[-K(s_1+s_2-2\bar h/3)]\ ,\end{aligned} \tag{3.14}$$

in which

$$s_1=m_1/\sqrt{6},\quad s_2=m_2/\sqrt{2},\quad \bar h=h/J\ . \tag{3.15}$$

The two components, m_1 and m_2, are determined by the equations $\partial f/\partial m_1=\partial f/\partial m_2=0$.

The Landau expansion for the free energy in the present, two-component theory, truncated at fourth order, becomes

$$\begin{aligned}\beta f&=F_0+\tfrac{1}{2}r(m_1^2+m_2^2)-w(m_1^3-3m_1m_2^2)\\&\quad+u(m_1^2+m_2^2)^2-H^0m_1\ ,\end{aligned} \tag{3.16}$$

with the coefficients given by

$$F_0=-\ln(e^{2H/3}+2e^{-H/3})+(H^0)^2/2K\ , \tag{3.17}$$

$$r=K[1-K(1-\tau^2)/3]\ , \tag{3.18}$$

$$w=\frac{K^3}{18\sqrt{6}}(1-\tau)^2(1+2\tau)\ , \tag{3.19}$$

$$u=\frac{K^4}{144}(1-\tau)(1+\tau-2\tau^2-6\tau^3)\ , \tag{3.20}$$

in which

$$\tau=(e^{2H/3}-e^{-H/3})/(e^{2H/3}+2e^{-H/3}) \tag{3.21}$$

is a monotonically increasing function of H, so that $0\le\tau\le1$ for $0\le H\le\infty$.

As in Ref. 7, u is a decreasing function of τ which becomes negative when $\tau\gtrsim\tau^*=0.542$. In that case at least the sixth-order terms in the free energy will be needed for stability. In order to avoid this problem, and mainly because the Landau expansion is rather inaccurate even for small random fields. As will be pointed out below, we keep in the following the exact mean-field theory given by Eqs. (3.13)–(3.15).

IV. PHASE DIAGRAMS FOR THE THREE-STATE MODEL

The most interesting features of the phase diagram for the three-state Potts model follow from the two-component theory. As shown in Fig. 1(a), in the absence of a random field, there is a disordered phase I in which $m_1\neq0$ and $m_2=0$ but $m_1\to0$ as $H^0\to0$; an ordered phase II where $m_1\neq0$ and $m_2=0$ but $m_1\not\to0$ as $H^0\to0$, and an ordered phase III where $m_1\neq0$ and $m_2\neq0$. The "order parameter" along the first-order line between phases I and II is the discontinuity of m_1 which vanishes at the Ising critical point CP, with a nonzero magnetization m_{1c}. The discontinuities in both m_1 and m_2 along the first-order I-III phase boundary vanish at a "normal" tricritical point TCP and along the second-order I-III phase boundary.[15]

The reason for the Ising critical behavior on the first-order I-II phase boundary is that in a negative uniform field the q-state Potts model has the symmetry of the $(q-1)$-state Potts model.

We consider first phases I and II, where $m_2=0$. Defining

$$\bar h^0=-h^0/\sqrt{6}J \tag{4.1}$$

and with $\bar h$ given in Eq. (3.15), we find the surface of first-order I-II transitions. The projection on the plane $\bar h^0=0$, shown in Fig. 2, is also the surface of first-order II-III transitions, the two phases lying on opposite sides of the plane of the figure. The curve shown is the I-II phase boundary also found by Nishimori.[6]

Consider next the phase diagram for nonzero $\bar h^0$. Defining

$$x=e^{3Ks_1},\quad a=e^{K\bar h}\ , \tag{4.2}$$

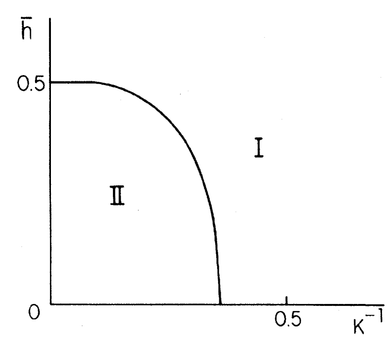

FIG. 2. Projection on the plane $\bar h^0=0$ of the phase diagram for the three-state Potts model in a random field h, where $\bar h=h/J$, in mean-field theory.

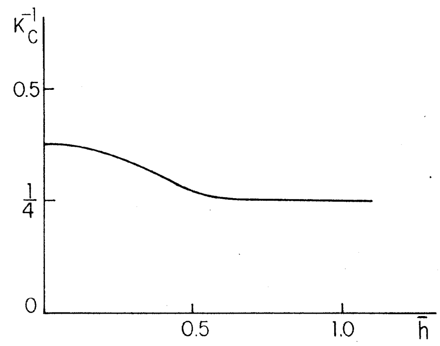

FIG. 3. Projection on the plane $\bar{h}^0=0$ of the line of critical points for the three-state Potts model in a random field $\bar{h}=h/J$, in mean-field theory.

where s_1 is given in Eq. (3.15) and $K\bar{h}=H$, we find the line of critical points shown in Figs. 3 and 4, ending the surface of first-order I-II transitions, and given by the equations

$$\phi(x,a)=0, \quad d\phi(x,a)/dx=0 , \tag{4.3}$$

in which

$$\phi(x,a)=1/Kx-(a+1)/(a+1+x)^2-a/(ax+2)^2 . \tag{4.4}$$

These are to be combined with the order parameter determined by Eq. (3.12).

Note that, except in the region of small fields, a larger constant uniform field is needed to reach a critical point the larger the size of the random field. This is what one would expect, if the role of the random field is to push the system deeper into the first-order transition region already present in the three-state Potts model.

It is also reasonable to expect that a small random field would first slightly smooth out the discontinuity in the magnetization at the boundary of first-order transitions, with a consequent *decrease* of the constant uniform field needed to reach a critical point, as shown in the inset of Fig. 4.

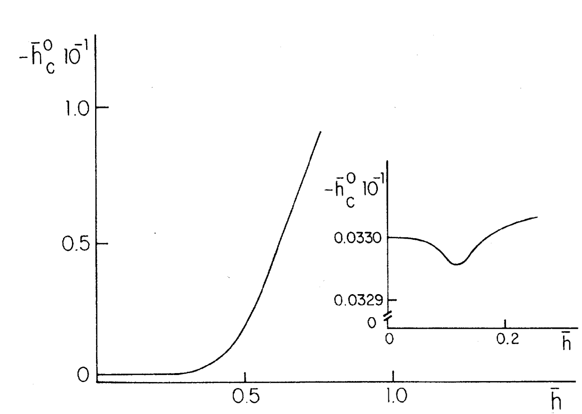

FIG. 4. Projection on the plane $K^{-1}=0$ of the critical line for the three-state Potts model in a random and constant uniform field, $\bar{h}$ and $\bar{h}^0$.

Note also, incidentally, that the critical point in zero random field, given here by $K_c^{-1}=\frac{3}{8}$ and $\bar{h}^0\cong-3.3\times10^{-3}$ in the *full* mean-field theory differs considerably from the result $K_c^{-1}=\frac{7}{18}$ and $\bar{h}^0\cong-6.2\times10^{-3}$ that follows from the Landau expansion to fourth order. Even larger discrepancies start to appear when $\bar{h}\lesssim 0.5$.

Before presenting the results for the tricritical line, it is interesting to discuss the zero-temperature behavior of all three phase boundaries, shown in Fig. 5. In phase I the free energy f, Eq. (3.13), and the order parameter are given by

$$f=-\tfrac{1}{2}(h^0)^2J^{-1}-\tfrac{2}{3}h , \qquad m_1=h^0/J, \quad m_2=0 . \tag{4.5}$$

In phase II,

$$f=-\tfrac{1}{2}(h^0)^2J^{-1}-\frac{2}{\sqrt{6}}h^0-\tfrac{1}{3}J , \qquad m_1=h^0/J+\frac{2}{\sqrt{6}}, \quad m_2=0 , \tag{4.6}$$

and the I-II phase boundary is given by

$$\bar{h}=-\bar{h}/3+\tfrac{1}{6} . \tag{4.7}$$

There are two solutions in phase III involving the two components of the order parameter m_1 and m_2. In region A, to the right of $\bar{h}=\frac{2}{3}$, the stable free energy and the corresponding order parameter are given by

$$f=-\tfrac{1}{2}(h^0)^2J^{-1}+\frac{1}{\sqrt{6}}h^0-\tfrac{1}{3}h-\tfrac{1}{9}J , \qquad m_1=h^0/J-\frac{1}{\sqrt{6}}, \quad m_2=\frac{1}{3\sqrt{2}} . \tag{4.8}$$

In region B, to the left of $\bar{h}=\frac{2}{3}$, the more stable free energy and the order parameter are, instead,

$$f=-\tfrac{1}{2}(h^0)^2J^{-1}+\frac{1}{\sqrt{6}}h^0-\tfrac{1}{3}J , \qquad m_1=h^0/J-\frac{1}{\sqrt{6}}, \quad m_2=\frac{1}{\sqrt{2}} , \tag{4.9}$$

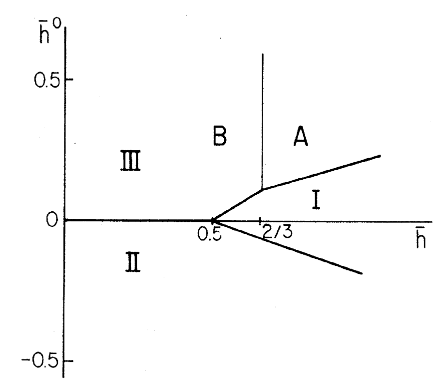

FIG. 5. Zero-temperature behavior of the phase boundaries of first-order transitions for the three-state Potts model in a random and a constant uniform field, $\bar{h}$ and $\bar{h}^0$, discussed in Sec. IV.

with the same free energy on the line $\bar{h}=\frac{2}{3}$. As the temperature increases we expect the discontinuity in m_2 within phase III to disappear.

Accordingly, the zero-temperature I-III phase boundary has two parts. An upper one is given by

$$\bar{h}^0=\bar{h}/3-\tfrac{1}{9} \tag{4.10}$$

and a lower one by

$$\bar{h}^0=2\bar{h}/3-\tfrac{1}{3} . \tag{4.11}$$

This one ends on the phase boundary $\bar{h}^0=0$, $\bar{h}\le\frac{1}{2}$, between phases II and III.

We consider next the tricritical line that ends the first-order I-III phase boundary. If the break in the slope of the phase boundary has a persistence to higher temperature, one should expect a crossover in the rate of growth of $\bar{h}^0$ with $\bar{h}$ on the tricritical line, and that is precisely what follows from our mean-field calculations that we outline next, as shown in Fig. 6.

Since s_2, Eq. (3.15), vanishes at the tricritical point and on the part of continuous transitions on the I-III phase boundary, a more transparent but still exact mean-field free energy, asymptotically close to the continuous transition, may be obtained expanding in powers of s_2 to yield

$$f=\frac{1}{3K^2}\ln^2 x+\frac{1}{3K}(1+6\bar{h}^0)\ln x+\frac{1}{3K}\ln a+B_0(a,x)+B_2(a,x)s_2^2+B_4(a,x)s_2^4 , \tag{4.12}$$

in terms of x and a defined in Eq. (4.2), and where

$$B_0(a,x)=-\frac{1}{3K}\ln[(ax+2)(a+x+1)^2] , \tag{4.13}$$

$$B_2(a,x)=1-\frac{K}{3}\left[\frac{1}{ax+2}+\frac{(a+1)x+4a}{(a+x+1)^2}\right] , \tag{4.14}$$

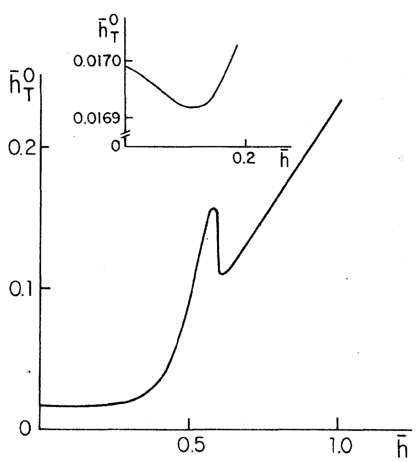

FIG. 6. Projection on the plane $K^{-1}=0$ of the tricritical line for the three-state Potts model in a random, constant, uniform field, $\bar{h}$ and $\bar{h}^0$. Note that small initial dip in the inset and the crossover to the high-temperature extension of the phase boundary I-A in Fig. 5, for larger $\bar{h}$.

$$B_4(a,x)=-\frac{K^3}{36}\left[\frac{ax-4}{(ax+2)^2}-\frac{A(a,x)}{(a+x+1)^4}\right] , \tag{4.15}$$

$$A(a,x)=(1+a)x^3-4(a-1)^2x^2+[1+a^3-13a(1+a)]x+16a(a^2-4a+1) .$$

The surface of continuous transitions on the I-III phase boundary is given by

$$B_2(a,x)=0 , \tag{4.16}$$

$$\frac{2}{K}\ln x+1+6\bar{h}^0+3KxB_0'(a,x)=0 , \tag{4.17}$$

where $B_0'\equiv dB_0/dx$. The tricritical line follows from Eqs. (4.16) and (4.17) together with

$$\frac{2}{3K^2x^2}(1-\ln x)-\frac{1}{3Kx^2}(1+6\bar{h}_0)+B_0''(a,x)-\frac{[B_2'(a,x)]^2}{2B_4(a,x)}=0 , \tag{4.18}$$

in which $B_0''\equiv d^2B_0/dx^2$ and $B_2'\equiv dB_2/dx$. For the tricritical point in zero random field we find

$$K_T^{-1}=\tfrac{7}{18},\quad \bar{h}^0_T=\tfrac{1}{18}(\tfrac{7}{3}\ln\tfrac{7}{4}-1) , \tag{4.19}$$

which again differs considerably from the tricritical point $K_T^{-1}=\frac{39}{96}$, $\bar{h}^0_T=\frac{1}{32}$ obtained in the Landau expansion for the free energy, in both s_2 *and* s_1.

The small initial drop in the constant uniform field on the line of tricritical points and the further increase with a larger random field again follows the expected behavior discussed above, as shown in Fig. 6.

To explore next if there is a second tricritical point for sufficiently large $\bar{h}$, at which the phase transition on the I-III boundary changes back from a continuous to a first-order transition, consider the limit $\bar{h}\to\infty$. In this case there are two possibilities: (i) x remains finite, which means from Eqs. (4.16) and (4.17) that the I-III boundary of continuous transitions, if present at all, must shrink to a point at $T=0$. However, the zero-temperature analysis that yields only first-order transitions eliminates this possibility. Otherwise, (ii) $x\to 0$ such that ax remains finite, and we consider this next. If so, $K_T^{-1}=\frac{7}{54}$ is the limiting *non*zero value of K_T^{-1} with which the curve in Fig. 7 has

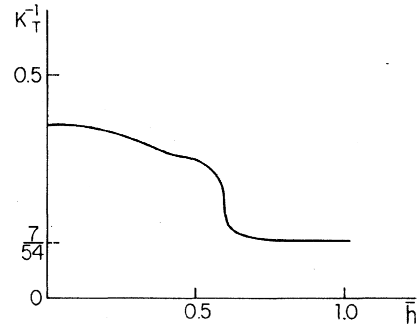

FIG. 7. Projection on the plane $\bar{h}^0=0$ of the line of tricritical points for the three-state Potts model in a random field $\bar{h}=h/J$, in mean-field theory.

been plotted. One also finds that $\bar{h}_T^0 \cong \bar{h}/3$, as $\bar{h} \to \infty$. Combined with the monotonic behavior of $\bar{h}_T^0(\bar{h})$, at the larger values of $\bar{h}$, shown in Fig. 6, together with the fact that *both* the uniform and the random field couple *linearly* to the Potts vectors in the Hamiltonian of the model, we infer that $\bar{h}_T^0(\bar{h})$ is never a two-valued function of $\bar{h}$, excluding therefore a second tricritical point.

V. RESULTS FOR GENERAL q

We extend here the work of Nishimori,[6] in a one-component mean-field theory, to include the effects of a constant uniform field.[16]

The phase transition in only a random field is known to be of first order for all $q > 2$. The addition of a uniform field yields a line of critical points for the three-state Potts model, if the uniform field is larger than a threshold value which depends on the random field as shown in Fig. 4. Due to the small dip in the curve (which may be expected to become smaller for larger q) the value of $|\bar{h}_c^0|$ at $\bar{h} = 0$ is a good approximation to the minimum threshold. This is used to find out if there is a critical point for general q when there is a finite minimum threshold for the constant uniform field.

Figure 8 shows that there is always a finite $|h_c^0|$, for any finite q, but that for very large q the minimum threshold becomes asymptotically large. This is necessary to invert the strong tendency towards a first-order transition for large q, already present in the model in the absence of a random field.

We are interested next in the behavior of the infinite state model, also considered by Nishimori.[6] It is convenient to take

$$f_\infty \equiv \lim_{q \to \infty} f/\ln q \tag{5.1}$$

and to define

$$\bar{K} = K/\ln q, \quad \tilde{h}^0 = h^0/J , \tag{5.2}$$

in which $K = \beta J$, together with $\bar{h} = h/J$ introduced before. Equations (3.9) and (3.10) yield then

$$f_\infty = \tfrac{1}{2}\bar{K}m^2 - \lim_{q \to \infty} \ln(q^{\bar{K}(m+\tilde{h}^0)} + q^{\bar{K}\bar{h}} + q)/\ln q . \tag{5.3}$$

Depending on which of the terms in the first natural log argument dominates, one finds one of three phases. A "ferromagnetic" phase exists when $m + \tilde{h}^0 > \sup(\bar{h}, \bar{K}^{-1})$, where

$$f_\infty = -\tfrac{1}{2}\bar{K} - \bar{K}\tilde{h}^0, \quad m = 1 . \tag{5.4}$$

A paramagnetic phase (para. I, in Nishimori's notation) exists when $\bar{h} > \sup(m + \tilde{h}^0, \bar{K}^{-1})$, with

$$f_\infty = -\bar{K}\bar{h}, \quad m = 0 , \tag{5.5}$$

and a second paramagnetic phase (para. II) for $\bar{K}^{-1} > \sup(m + \tilde{h}^0, \bar{h})$ with

$$f_\infty = -1, \quad m = 0 . \tag{5.6}$$

It follows from Eqs. (5.4)–(5.6) that, as in the absence of a nonrandom uniform field, there are three phase boundaries of first-order transitions, as shown in Fig. 9. For a given strength of the random field, the ferromagnetic phase occurs only for a sufficiently large additional nonrandom field.

Although there is no true spin-glass order in a random field,[6] the paramagnetic phases may be distinguished by a spin-glass order parameter defined as

$$Q = (\langle \mathbf{a}^\mu \rangle \cdot \langle \mathbf{a}^\nu \rangle) . \tag{5.7}$$

Explicit calculation in the limit $q \to \infty$ yields $Q = 1$ when $\bar{K}\bar{h} > 1$ (the para. I phase) while $Q = 0$ if $\bar{K}\bar{h} < 1$ (the para. II phase), regardless of the size of the uniform field $\tilde{h}^0$. We also found that $Q = 1$ in the ferromagnetic phase. These results agree with those of Nishimori[6] when $\tilde{h}^0 = 0$.

VI. SUMMARY AND CONCLUDING REMARKS

We have obtained a number of new results about the phase diagrams for the q-state Potts model in a random field, within mean-field theory, with an additional nonrandom uniform field. Lines of critical and tricritical points are obtained for the three-state model in a two-component order-parameter theory, while only lines of critical points follow for general q (other than two) in a one-component order-parameter theory.

We find that there is a clear *increase* in the tendency to

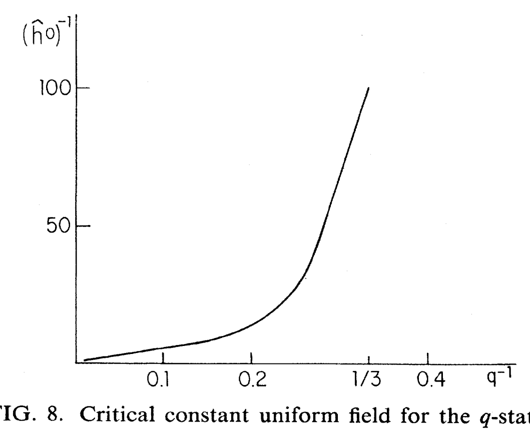

FIG. 8. Critical constant uniform field for the q-state Potts model as a function of q, in mean-field theory.

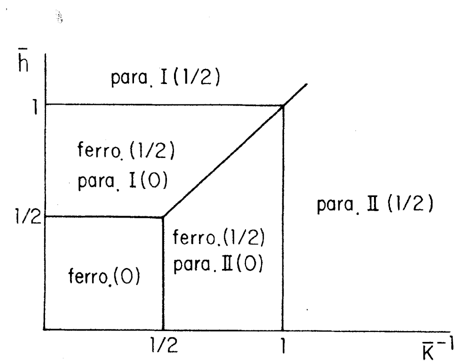

FIG. 9. Mean-field phase diagrams for the infinite-state Potts model in a random field for zero or finite normalized uniform field $\tilde{h}^0$, the values of which are in parentheses and where $\bar{K} = K/\ln q$.

order via a first-order phase transition for the $q(>2)$-state Potts model, due to the presence of the random field *beyond* small values, and that the first-order transition ends either at a line of critical and tricritical points in the presence of a sufficiently large nonrandom field. It is also interesting to note, as we find here, that a small random field reduces the magnitude of the necessary critical or tricritical nonrandom field. The initial effect of the random field is thus to reduce the discontinuity in the magnetization at the boundary of first-order phase transitions.

There are two comments concerning the random-field distribution. The first one is that we only considered a statistically independent distribution of random fields, in which there is no correlation between the random fields at different sites. Although this is a limitation, it is a consistent one with the usual mean-field theory employed in this work in which the same saddle point is taken for all sites. Second, we considered a discrete distribution of random fields which preserves the permutational symmetry of the model and we averaged over the orientations of the random field. Such a distribution is expected to enhance most clearly a first-order transition, guided by what one knows on the Ising model.[8]

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