Nematic phase in two-dimensional frustrated systems with power-law decaying interactions

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We address the problem of orientational order in frustrated interaction systems as a function of the relative range of the competing interactions. We study a spin model Hamiltonian with short-range ferromagnetic interaction competing with an antiferromagnetic component that decays as a power law of the distance between spins, $1/r^\alpha$. These systems may develop a nematic phase between the isotropic disordered and stripe phases. We evaluate the nematic order parameter using a self-consistent mean-field calculation. Our main result indicates that the nematic phase exists, at mean-field level, provided $0 < \alpha < 4$. We analytically compute the nematic critical temperature and show that it increases with the range of the interaction, reaching its maximum near $\alpha \sim 0.5$. We also compute a coarse-grained effective Hamiltonian for long wavelength fluctuations. For $0 < \alpha < 4$ the inverse susceptibility develops a set of continuous minima at wave vectors $|\vec{k}| = k_0(\alpha)$ which dictate the long-distance physics of the system. For $\alpha \to 4$, $k_0 \to 0$, making the competition between interactions ineffective for greater values of α.

I. INTRODUCTION

Local structures at different scales, that could break translational as well as rotational invariance, generally appear in systems with competing interactions. There is a variety of examples, ranging from solid state systems, like ultrathin ferromagnetic films [1–3] and strongly correlated electron systems [4–7], to soft matter systems like Langmuir monolayers [8], block copolymers [9,10], colloids and soft core systems [11–13]. Besides the intrinsic interest raised by the complexity of the phase behavior, their detailed knowledge could be relevant to understand basic phenomena such as high-temperature superconductivity, and also for technological applications like soft matter templates for nanoscale systems and future spintronic devices.

Competing interactions at different scales may give rise to complex phases and patterns, like stripes, lamellae, bubbles, and others [14]. In this intricate phase structure, the nematic phase [15], an homogeneous however nonisotropic state, plays an important role. It may appear as an intermediate phase between a fully disordered phase and a modulated phase. An interesting approach to study the phase transitions in systems with isotropic competing interactions was observed in Ref. [16]. Analyzing a generic Ginsburg-Landau model, whose main characteristic is the presence of a minimum in the spectrum of Gaussian fluctuations at a nonzero wave vector, it was shown that the model has a first order transition to a modulated phase. In recent works [15,17] we have shown that pure symmetry considerations, in the context of the renormalization group, lead to terms in the free energy that encode orientational order parameters. We were able to find a nematic phase at temperatures above the critical temperature for modulated phases. We have also found that the isotropic-nematic phase transition in the two-dimensional continuum system with isotropic competing interactions is in the Kosterlitz-Thouless universality class [17]. More recently, we developed a method to compute the nematic order parameter in a classical spin Hamiltonian system with competing

interactions [18]. We have applied the method to the Ising frustrated dipole ferromagnet [19] and to the short-ranged biaxial next-nearest-neighbor Ising model (BNNNI), or the $J1$-$J2$ model [20]. Interestingly, for the dipolar interaction model a nematic phase was found, while in the $J1$-$J2$ model this phase does not exist. This points to the relevance of the interaction range to develop a nematic phase.

Then, a natural question we address in this article is about the range of the frustrating interaction necessary to produce an intermediate nematic phase between the disordered isotropic and the striped phase. The necessity of long-ranged interactions is frequently invoked, but the actual influence of the relative range between the competing interactions is still an open problem. To answer this question we study a spin model Hamiltonian with short-range ferromagnetic interaction competing with an antiferromagnetic component that decays as a power law with the distance between spin sites $1/r^\alpha$ (where r is the distance between two spins, and α measure the range of the decay). For $\alpha = 3$, this is the known Ising frustrated dipolar model, while $\alpha = 1$ is equivalent to the frustrated Coulomb model [21]. We have evaluated the nematic order parameter using a self-consistent mean-field calculation. To do this, it is necessary to compute spin fluctuations, since the nematic order parameter is quadratic in the spin variable. Our main result indicates that the nematic phase exists, at mean-field level, provided $0 < \alpha < 4$. In other words, if the frustrating interaction decays faster than $1/r^4$, a pure orientational order is not possible. We have analytically computed the nematic critical temperature for $0 < \alpha < 4$ and have shown that the temperature window between the stripe and the nematic phase increases with the range of the interaction, reaching its maximum near $\alpha \sim 0.5$. We have also computed a coarse-grained effective Hamiltonian for long wavelength fluctuations. For $0 < \alpha < 4$ the inverse susceptibility develops a set of continuous minima at wave vectors $|\vec{k}| = k_0(\alpha)$ which rules the long-distance physics of the system. For $\alpha \to 4$, $k_0 \to 0$, making the competition ineffective for greater values of α. Also, the stiffness of

pattern formation is enhanced with the range of the competing interaction. For ranges shorter than the dipolar interaction $\alpha > 3$ it takes very small values, signaling a possible instability of mean-field order.

The paper is organized as follows: In Sec. II we present our model and compute the long wavelength effective field theory for any value of α. In Sec. III we compute the nematic order parameter and the critical temperature. Finally we discuss our results and conclusions in Sec. IV, leaving some technical details for the Appendix.

II. MODEL HAMILTONIAN AND EFFECTIVE FIELD THEORY

We consider a Hamiltonian written in terms of Ising variables $S_i = \pm 1$, with competition between short-range ferromagnetic and long-ranged antiferromagnetic interactions,

$$H = -\frac{J}{2} \sum_{\langle i,j \rangle} S_i S_j + \frac{g}{2} \sum_{(i,j)} \frac{S_i S_j}{r_{ij}^{\alpha}} + \sum_i B_i S_i. \quad (1)$$

The first sum runs over all pairs of nearest neighbor spins in a two-dimensional lattice, while the second one runs over all pairs of spins of the lattice; r_{ij} is the distance, measured in lattice units, between sites i and j. $J, g > 0$ measure the ferromagnetic exchange and the long-ranged frustrating antiferromagnetic interaction respectively. The range of the latter is controlled by the exponent α. The last term is the energy associated with an external magnetic field B_i. We are interested in the regime of small frustration $g < J$, since this is the relevant regime in some applications such as ferromagnetic thin films with perpendicular anisotropy in which the special case of $\alpha = 3$ (frustrated Ising-dipolar model) is usually considered to model the physical system.

Next, we analyze the effective long-wavelength behavior of this model as a function of α. As usual, the thermodynamic properties in the canonical ensemble are defined in terms of the partition function $Z(B) = \text{Tr} \exp(-\beta H)$. It is well known that long-ranged interactions with $\alpha < d$ ($d = 2$ is the dimensionality in this work), may lead to inequivalence between the canonical and the microcanonical ensembles [22]. The essential reason for this behavior is that the energy necessary to produce a homogeneous ground state is infinite in the thermodynamic limit. However, in competitive models like the ones we are considering, the phase transitions are dominated by the modulation scale [23]. In these cases, the energy is additive and in principle the canonical ensemble can be safely used.

It is convenient to rewrite the partition function in terms of real variables on the lattice ($-\infty < \Phi_i < \infty$). This can be done by means of a Hubbard-Stratonovich transformation [24]. Exactly summing up the Ising degrees of freedom S_i, we can rewrite the partition function in terms of an effective Hamiltonian written in the new variables [18],

$$\mathcal{H}[\{\Phi\}] = \frac{1}{4} \sum_{ij} \Phi_i J_{ij} \Phi_j - \frac{1}{2} \sum_i B_i \Phi_i$$

$$- \frac{1}{\beta} \sum_i \ln \cosh \left(\beta \sum_j J_{ij} \Phi_j \right). \quad (2)$$

In this expression, J_{ij} is the total interaction matrix. For nearest neighbors, it is essentially the constant J, while for all other components it is $-g/r_{ij}^{\alpha}$. The sums run over all pairs of sites in a two-dimensional square lattice, the inverse temperature $\beta = 1/T$, and B is an external magnetic field. It is not difficult to find a relation between the original discrete variables and the new continuous ones, by just differentiating the partition function with respect to the magnetic field in both representations. One immediately finds [24]:

$$\langle S_i \rangle = \frac{1}{2} \langle \Phi_i \rangle, \quad (3)$$

$$\langle S_i S_j \rangle = -\frac{1}{2\beta} J_{ij}^{-1} + \frac{1}{4} \langle \Phi_i \Phi_j \rangle. \quad (4)$$

It is instructive to write the Hamiltonian of Eq. (2) in the disordered high-temperature phase. In this regime, the entropic last term can be expanded in powers of $\beta\Phi$. Keeping the leading order term we find (for $B = 0$)

$$\mathcal{H}_d \sim \frac{1}{4} \sum_{ij} \Phi_i \{ \mathbf{J} \cdot (\mathbf{I} - 2\beta\mathbf{J}) \}_{ij} \Phi_j, \quad (5)$$

where $\mathbf{J}$ is the matrix whose components are J_{ij}, $\mathbf{I}$ is the identity matrix, and the "dot" indicates the usual matrix product. In reciprocal space, the quadratic Hamiltonian in two dimensions takes the simpler form,

$$\mathcal{H}_d \sim \frac{1}{4} \int_{\text{BZ}} \frac{d^2k}{(2\pi)^2} \Phi(k) \chi^{-1}(k) \Phi(-k), \quad (6)$$

in which BZ indicates the first Brillouin zone and $\chi^{-1}(k)$ is the Fourier transform of $\mathbf{J} \cdot (\mathbf{I} - 2\beta\mathbf{J})$.

For isotropic interactions and long-wave components ($ka \ll 1$, where a is the lattice constant), the inverse susceptibility $\chi^{-1}(k)$ in the high-temperature phase depends only on $k = |\vec{k}|$. For simple ferromagnets this function has a minimum at $k_0 = 0$, and then χ^{-1} can be expanded in Taylor series to arrive at the coarse-grained Hamiltonian

$$\mathcal{H}_d \sim \int_{|k| < 1/a} \frac{d^2k}{(2\pi)^2} \{ r + \rho k^2 \} |\Phi(k)|^2, \quad (7)$$

that correctly describes the ferromagnetic phase transition [25]. In Eq. (7),

$$r = \chi^{-1}(0), \quad (8)$$

$$\rho = \frac{1}{2!} \left. \frac{d^2 \chi^{-1}}{dk^2} \right|_{k=0}. \quad (9)$$

However, in frustrated systems $\chi^{-1}(k)$ may develop a minimum for a finite wave vector $\vec{k} = \vec{k}_0$. The effective long-distance Hamiltonian will then be dominated by this scale. Expanding the inverse susceptibility in Taylor series, the coarse-grained Hamiltonian is now given at leading order by

$$\mathcal{H}_d \sim \int_{|k-k_0| < \Lambda} \frac{d^2k}{(2\pi)^2} \{ r + \rho(k - k_0)^2 \} |\Phi(k)|^2, \quad (10)$$

where Λ is a cutoff and

$$r = \chi^{-1}(k_0), \quad (11)$$

$$\rho = \frac{1}{2!} \left. \frac{d^2 \chi^{-1}}{dk^2} \right|_{k=k_0}. \quad (12)$$

Therefore, the magnetic susceptibility $\chi(0)$ is always finite. On the other hand, $\chi(k_0) \sim 1/r$. If $r \to 0$, the susceptibility at the wave vector k_0 diverges, signaling a tendency of the magnetization to form modulated structures with wave vector k_0. For high temperatures $r > 0$, the correlation length $\xi \sim 1/\sqrt{r}$. The system tends to form stripe domains with wavelength $\lambda = 2\pi/k_0$, whose area is proportional to $1/r$. In this way, the magnetic susceptibility $\chi(k_0)$ is essentially a measure of the area of each stripe domain. These domains are, in principle, uncorrelated at high temperatures. The Hamiltonian of Eq. (10) was proposed a long time ago as an effective theory to study stripe phases [16]. More recently, a generalization of this model in the context of the renormalization group was studied [17]. It was shown that, in the continuum two-dimensional model, while the stripe long-ranged order cannot exist (at least for sufficiently short-ranged interactions), a pure orientational nematic order may develop indicating an orientational order of domain walls of local stripe order.

Let us now return to our "microscopic" model [Eq. (1)] and analyze the structure of the disordered susceptibility in terms of the range of the frustrating interaction. The interaction matrix can be cast in the form $J_{ij} = J J_{ij}^{\mathrm{f}} - g J_{ij}^{\mathrm{a}}$, where J_{ij}^{f} is a ferromagnetic short-ranged interaction and J_{ij}^{a} represents the antiferromagnetic long-ranged interaction. The Fourier transform of the first nearest neighbor ferromagnetic part in a square lattice is $J^{\mathrm{f}}(k) = \cos k_x + \cos k_y$, where for simplicity we consider the lattice spacing $a = 1$. For long wavelength with respect to the lattice spacing $k_x, k_y \ll 1$, this interaction turns out to be isotropic, $J^{\mathrm{f}}(k) \sim 2 - (1/2)(k_x^2 + k_y^2)$. On the other hand, the antiferromagnetic part, in the same isotropic approximation, takes the form [26] $J^{\mathrm{a}}(k) = 2^{1-\alpha}\Gamma(1 - \alpha/2)/\Gamma(\alpha/2)k^{\alpha-2}$ (for $\alpha \neq$ even). In this way, the interaction in reciprocal space can be written as (see the Appendix)

$$J(k) = 2J\left\{1 - \frac{1}{4}k^2 + \frac{1}{2}\left(\frac{g}{J}\right)\sigma(\alpha)\frac{k^{\alpha-2}}{\alpha-2}\right\}, \qquad (13)$$

where $\sigma(\alpha) = 2^{2-\alpha}\Gamma(2 - \alpha/2)/\Gamma(\alpha/2)$. Notice that, for $\alpha = 3$, the antiferromagnetic part reduces to gk, the well known long-distance behavior of the dipolar model. Equation (13) is not well defined for even values of α. A careful treatment of these cases leads to logarithmic corrections that are considered in detailed in the Appendix.

We see two special values, where the behavior of $J(k)$ changes qualitatively, $\alpha = 2$ and $\alpha = 4$. From Eq. (13) we see that, for $\alpha > 4$, the antiferromagnetic term is subleading in the long-distance limit ($k \ll 1$), in such a way that $J(k)$ has a maximum at the value $k_0 = 0$. Then, the long-distance effective Hamiltonian is that of Eq. (7). On the other hand, for $\alpha < 4$, we have two different situations. In the case $\alpha < 2$, $\lim_{k \to 0} J(k) = -\infty$, while if $\alpha > 2$, $J(0) = 2J$. We illustrate these two cases in Fig. 1. In Fig. 1(a) we depict two typical examples with $\alpha < 2$, while in Fig. 1(b) we show two cases with $\alpha > 2$. Despite the different behavior at the origin, we observe that for $\alpha < 4$, $J(k)$ develops a maximum at a finite scale $k_0 \neq 0$ given by

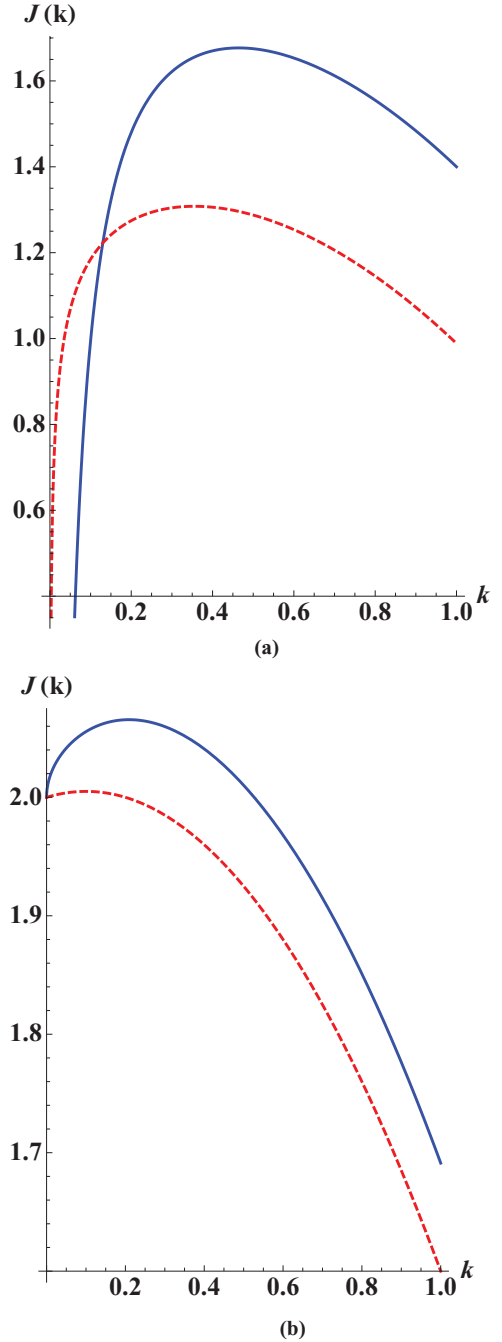$$k_0 = \left[\sigma(\alpha)\left(\frac{g}{J}\right)\right]^{1/(4-\alpha)}. \qquad (14)$$

FIG. 1. (Color online) $J(k)$ as a function of k in units of $1/a$. We have fixed $J = 1$ and $g = 0.1$. In (a), $\alpha < 2$. The bold line corresponds to $\alpha = 1$, while the dashed line corresponds to $\alpha = 1.8$. In (b), $\alpha > 2$. The bold line corresponds to $\alpha = 2.5$, while the dashed line corresponds to $\alpha = 3$.

In Fig. 2 we depict the values of k_0, given by Eq. (14), as a function of α for a fixed value of $g/J = 0.1$ We see that, in the regime of interest $g/J < 1$, a finite k_0 is developed all along the interval $0 < \alpha < 4$. After the dipolar value $\alpha > 3$, k_0 rapidly decays to zero, and for $\alpha > 4$ it is no more possible to have $k_0 \neq 0$. At the particular point $\alpha = 4$ Eq. (13) is not well defined, since $\sigma(\alpha)$ has a pole. We have studied in detail this special case in the Appendix and have shown that in this case k_0 is exponentially small $k_0 \sim \exp(-J/g)$. Then
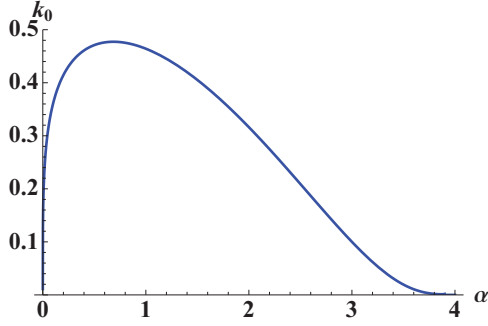
FIG. 2. (Color online) k_0 in units of the lattice spacing a, as a function of the parameter α for a fixed value of the competition parameter $g/J = 0.1$.

it can be considered zero for any practical purpose. Also, the limit of $\alpha \to 2$ is not well defined in Eq. (13). However, upon differentiation, the value of k_0 is perfectly well defined. This is also discussed in the Appendix.

Then, for $\alpha < 4$, we can expand Eq. (13) in powers of $k - k_0$ obtaining

$$J(k) = J[r_0 - \rho_0(k - k_0)^2] + O[(k - k_0)^3], \quad (15)$$

where $r_0 = J(k_0)/J \sim 2 + O(k_0^2)$ and $\rho_0 = (4 - \alpha)/2$. The inverse susceptibility in the same approximation reads

$$\frac{1}{J} \chi^{-1}(k) = J(k)[1 - 2\beta J(k)] \quad (16)$$

$$= r + \rho(k - k_0)^2 + \cdots . \quad (17)$$

Using Eq. (15) we obtain for the coefficients

$$r = r_0(1 - 2r_0\beta J), \quad (18)$$

$$\rho = \rho_0(4r_0\beta J - 1). \quad (19)$$

We see that, for a high-temperature regime, ρ could be negative, indicating an instability of the theory. This is a very well known limitation of the method, that has its origin in the use of the Hubbard-Stratonvich transformation for kernels which are not positive definite [25]. However, we are interested in the temperature regime in which the systems has a tendency to form patterns, i.e., $0 < r \ll 1$ in which $\rho \sim \rho_0$. In this regime the effective Hamiltonian Eq. (10), with the parameters k_0, r, ρ given by Eqs. (14), (18), and (19) respectively, is perfectly well defined.

In fact, the expansion in $k - k_0$, given by Eq. (17) is an excellent approximation of Eq. (16) near the the temperature $\beta_{st} \sim 1/2r_0 J$, where the susceptibility diverges at $k = k_0$, signaling the tendency to form striped patterns. In Fig. 3 we show two examples of the susceptibility computed from Eq. (16). The bold line corresponds to frustrated dipolar Ising model, $\alpha = 3$ at $\beta J = 1/4.012$, while the dashed line represents the susceptibility of the Coulomb model, $\alpha = 1$ at $\beta J = 1/3.3557$. We see that both curves are sharply picked at the corresponding value of k_0, given by Fig. 2, while the widths of the peaks are proportional to $1/\rho$. Moreover, the width of the curve $\alpha = 3$ is clearly larger than the one with $\alpha = 1$. This happens because the stiffness ρ grows with the range of the competing interaction (i.e., with decreasing α). In fact, from Eq. (19), we can observe that $\rho \sim \rho_0 \sim (4 - \alpha)$. Finally, the
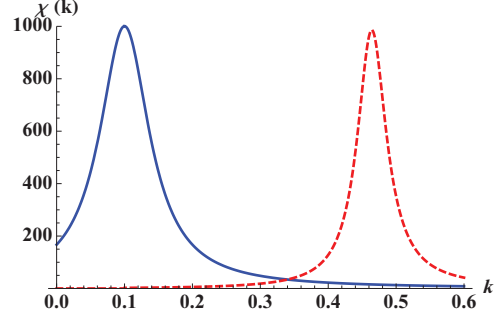

FIG. 3. (Color online) Magnetic susceptibility $\chi(k)$ as given by Eq. (16) for two typical values of α. The bold line corresponds to frustrated dipolar Ising model, $\alpha = 3$ with $\beta J = 1/4.012$, while the dashed line is the susceptibility of the frustrated Coulomb Ising model, $\alpha = 1$, with $\beta J = 1/3.3557$. In both cases we fixed $J = 1$ and $g = 0.1$.

small temperature difference between both examples are due to the quadratic corrections of $r_0 \sim 2 + O(k_0^2)$ that makes the stripe critical temperature β_{st}, α dependent.

III. MEAN-FIELD THEORY FOR THE NEMATIC ORDER PARAMETER

The orientational (nematic) order parameter is defined as [18]

$$Q = \sum_{ij} \langle S_i S_{i+\hat{x}} - S_i S_{i+\hat{y}} \rangle, \quad (20)$$

where $\hat{x}$ and $\hat{y}$ are unit vectors along the x and y axes of the square lattice. With this definition, the usual tensor nematic order parameter has only one component. If Q is positive, the director points along the x direction while if it is negative, the director mainly points in the y direction. These are the only two possible directions of the director. For this reason, if $Q \neq 0$, the resulting anisotropic phase is called Ising nematics, since it breaks the rotational point group of the lattice and it is invariant under rotations by π. On the other hand, in the continuum limit, the nematic order parameter acquires a phase, $Q = |Q|e^{i2\theta}$, since the direction in the plane is now arbitrary. In this case, if $Q \neq 0$, the director can point in any direction labeled by θ. The factor of 2 in the exponent guarantees the nematic symmetry $\theta \to \theta + \pi$. Throughout the article, for brevity, we generally use the term "nematic" to refer to any of these phases, however, whenever we deal with a square lattice model, "Ising nematic" should be understood.

Equation (20) can be written as

$$Q = \frac{1}{2} \sum_{ij} K_{ij} \langle S_i S_j \rangle = \frac{1}{8} \sum_{ij} K_{ij} \langle \Phi_i \Phi_j \rangle, \quad (21)$$

where

$$K_{ij} = \begin{cases} +1 & \text{if } j = i \pm \hat{x}, \\ -1 & \text{if } j = i \pm \hat{y}. \end{cases} \quad (22)$$

In the last equality we used Eq. (4) and the fact that $\text{Tr}(\mathbf{K} \cdot \mathbf{J}^{-1}) = 0$ by symmetry.

Therefore, to compute Q we need to evaluate the correlation function $\langle \Phi_i \Phi_j \rangle$ in the ordered ($Q \neq 0$) phase. We refer the

reader to Ref. [18] for a detailed mathematical formalism to compute this quantity, based on the Hamiltonian Eq. (2). Here, we sketch the main physical concepts behind this formalism. In the nematic ordered phase (if it exists), the system is homogeneous, however having a global anisotropy given by Q. The order parameter acts as a nematic mean field, in such a way that individual magnetic moments couple with Q, contributing to the mean-field Hamiltonian with an energy $JQ\Phi_i K_{ij}\Phi_j$. This is the simplest way that Q can be coupled to the magnetic moments satisfying rotational invariance. On the other hand, it is the leading order term in a homogeneous phase $\langle\Phi_i\rangle = 0$. Therefore, the mean-field Hamiltonian in the nematic phase has the form

$$\mathcal{H}_{\text{nematic}} \sim \frac{1}{4}\sum_{ij}\Phi_i\{\mathbf{J}\cdot(\mathbf{I}-2\beta\mathbf{J})-2JQ\mathbf{K}\}_{ij}\Phi_j. \quad (23)$$

The (anisotropic) correlation matrix is

$$\langle\Phi_i\Phi_j\rangle \sim [\mathbf{J}\cdot(\mathbf{I}-2\beta\mathbf{J})-2JQ\mathbf{K}]_{ij}^{-1}. \quad (24)$$

Substituting this expression into the definition of the order parameter [Eq. (21)] we find a self-consistent equation for Q given by [18]

$$Q = \frac{1}{16\beta}\text{Tr}\left\{\frac{\mathbf{K}}{\mathbf{J}-2\beta\mathbf{J}^2-2JQ\mathbf{K}}\right\}. \quad (25)$$

This equation is the analog of the Curie-Weiss approximation for the magnetization in the Ising model. If this equation has a nontrivial solution $Q \neq 0$, then the system exhibits an anisotropic but otherwise homogeneous phase with nematic symmetry. The presence or not of this phase depends on the detailed structure of the competing interactions, coded in the explicit form of the matrix $\mathbf{J}$. We can look for a critical point by expanding the right-hand side of Eq. (25) in powers of Q, to obtain

$$16\beta Q \approx 2JQ\text{Tr}(\chi\mathbf{K})^2 + 8J^3\ Q^3\text{Tr}(\chi\mathbf{K})^4, \quad (26)$$

where χ is the magnetic susceptibility matrix in the disordered isotropic phase. $Q = 0$ is always a solution of the self-consistent equation. If $Q \neq 0$ then for $Q \ll 1$

$$Q \approx \frac{1}{2}\left[\frac{8\beta J - \text{Tr}(\chi\mathbf{K})^2}{\text{Tr}(\chi\mathbf{K})^4}\right]^{1/2}. \quad (27)$$

This result implies a continuous, second order *isotropic-nematic* transition, at a critical temperature given by

$$\beta_c = \frac{1}{8}\text{Tr}[\chi(\beta_c)\mathbf{K}]^2. \quad (28)$$

In reciprocal space Eq. (28) reads

$$\beta_c = \frac{1}{8}\int_{\text{BZ}}\frac{d^2k}{(2\pi)^2}[\chi_c(\vec{k})K(\vec{k})]^2. \quad (29)$$

where $K(\vec{k}) = 2(\cos k_x - \cos k_y)$. In Ref. [18] we have numerically solved this equation in the case of the Ising-frustrated-dipolar model, i.e., for $\alpha = 3$, where the conditions for the existence of the nematic transition in terms of the microscopic parameters J and g was shown. Here, we show an approximate analytic solution for any value of α. To do this, we write a long-distance (continuous) approximation of Eq. (29) and we focus in the regime where the temperature

is very near the instability towards stripe formation $r \ll 1$. At long distances, $K(\vec{k}) \sim k_x^2 - k_y^2 = k^2\cos(2\theta)$. Using the results for the magnetic susceptibility found in Eq. (17) we find for the nematic critical temperature,

$$J\beta_c = \frac{1}{8}\int\frac{d^2k}{(2\pi)^2}\left[\frac{k^2\cos(2\theta)}{r_c+\rho_c(k-k_0)^2}\right]^2, \quad (30)$$

where r_c and ρ_c are given by Eqs. (18) and (19) at $\beta = \beta_c$. Performing the angular integration we finally find, in dimensionless quantities, the following self-consistent equation for the critical temperature:

$$8\pi J\beta_c = \int_0^1 dk\frac{k^5}{[r_c+\rho_c(k-k_0)^2]^2}. \quad (31)$$

Note that if $k_0 = 0$, the integral is completely regular, and almost temperature independent, dominated by the ultraviolet cutoff. However, for finite $k_0 \neq 0$, the integrand for $k \to k_0$ diverges as $r_c \to 0$. Then, the integral can be approximated at leading order in $r_c \ll 1$, and $\rho_c \sim \rho_0$,

$$\int_0^1 dk\frac{k^5}{[r_c+\rho_c(k-k_0)^2]^2} \sim \frac{8k_0^5}{3\rho_0^{1/2}}\frac{1}{r_c^{3/2}} + O(r_c^{-1/2}). \quad (32)$$

Substituting Eq. (32) into Eq. (31) and using Eqs. (18) and (19) we find

$$\beta_c = \frac{1}{2r_0J}(1-\delta), \quad (33)$$

where we have defined a small quantity $0 < \delta \ll 1$,

$$\delta = \frac{\beta_{\text{st}}-\beta_c}{\beta_{\text{st}}} = \left[\frac{2}{9\pi^2}\frac{k_0^{10}}{\rho_0}\right]^{1/3}. \quad (34)$$

Thus, for small frustration $g/J < 1$ and for long-ranged interactions $\alpha < 4$, the model of Eq. (1) presents a homogeneous but anisotropic phase with nematic symmetry. Note that, at the isotropic-nematic critical temperature $r_c = 2\delta \ll 1$ and the magnetic susceptibility at wave vector k_0, $\chi(k_0) \sim 1/\delta$ is finite, showing that the system is homogeneous. For interaction ranges longer than the dipolar model ($\alpha < 3$) the temperature window for the nematic phase grows. Conversely, for shorter-range interactions ($\alpha > 3$) the nematic critical temperature decreases. Furthermore, as $\alpha \to 4$ not only does the critical temperature decrease, but also the modulation wavelength grows rapidly, $\lambda = 2\pi/k_0 \to \infty$. In practice, there is an upper limit in the value of $\alpha \lesssim 4$, given by the finite size of the sample. In fact, for $k_0(\alpha)L \sim 1$ (where L is the linear dimension of the sample), the stripe wavelength $\lambda \sim L$. Near these values $\chi(k_0) \sim \chi(0)$ and the size of the domains are of the same order than the system size. Thus, the competition turns out to be ineffective and the system behaves, at long distances, as a uniform ferromagnet.

IV. SUMMARY AND CONCLUSIONS

We have addressed the role of the relative interaction ranges on the nematic phase in competing interaction models at different scales. We have studied a two-dimensional Ising model on a square lattice, with a short-ranged ferromagnetic interaction and a long-ranged antiferromagnetic one, whose range decays as a power law controlled by a parameter α.

We have mapped the microscopic model into a coarse-grained field theory that describes the long-distance behavior. We showed that, for small frustration $g/J < 1$ and long-ranged interactions $\alpha < 4$, the system develops a scale k_0 which dominates the low-energy physics. In this regime the effective field theory is formally equivalent to a Brazovskii model in two dimensions. On the other hand, for shorter-ranged interactions $\alpha > 4$, $k_0 = 0$ and the system behaves as a usual ferromagnet.

We focused on the isotropic-nematic transition, looking for a homogeneous phase with anisotropic correlations. To compute the nematic order parameter we used a self-consistent approach, previously applied [18] to the frustrated-dipolar-Ising interaction. We have shown that there is a temperature window above the stripe instability in which the nematic phase can be developed for ranges of $\alpha < 4$. This window grows with the range of the frustrating interaction up to very long ranges $\alpha < 1$. On the other hand, for ranges shorter than the dipolar interaction $\alpha > 3$ the critical temperature gets smaller and the stripe wavelength grows rapidly. For $\alpha \sim 4$ the domain size is very large and in the limit of $k_0 L \sim 1$ it is of the the same order of the system size. In this regime the competition turns out to be irrelevant. Although for $3 < \alpha < 4$ mean-field theory predicts a nematic phase, fluctuations may destroy nematic order since the "stiffness" ρ_0 is much weaker than in the region $\alpha \leqslant 3$. For $\alpha > 4$ there is neither stripe nor nematic solution even at mean field. A word of caution is in order here. The threshold $\alpha = 4$ for the existence of $k_0 \neq 0$ is valid in the asymptotic small frustration limit $g/J \ll 1$. If $g \sim J$, the system could develop a finite scale $\vec{k}_0 \neq 0$ very near the edge of the Brillouin zone. In this case, the isotropic approximation of the antiferromagnetic interaction is no longer valid. For strong frustration, it is necessary to take into account the short-ranged part of the competing interaction. This type of interactions, like for instance in the axial NNNI or BNNNI models, are necessarily anisotropic since they at least have the discrete symmetry of the lattice. These systems, although with short-range interactions, develop a finite scale k_0 which leads to the appearance of a stripe phase. However, the susceptibility $\chi(\vec{k})$ has a small number of isolated maxima [18], different from our model in which we have an infinitely degenerate set of maxima. This difference is at the heart of the existence of the nematic phase as we have studied it in the present work.

Interestingly enough, our result for the modulation period $k_0(\alpha)$ [Eq. (14) and Fig. 2] is in complete agreement with very general scaling properties studied in Ref. [27]. The scaling behavior of the modulation length with g/J resides on the homogeneity of the long-ranged interaction in Fourier space [Eq. (A5)]. Indeed, it was shown [27] that, for $T = 0$, the threshold exponent for the occurrence of modulated ground states is $\alpha = 3$ (for $d = 2$), while at high temperatures we have shown a very similar behavior at $\alpha = 4$. Thus, the scaling behavior of k_0 as a function of g/J is strongly temperature dependent. This could imply that, for models in the range $3 \leqslant \alpha < 4$, there could exist a critical temperature below which a disorder phase reappears from a high-temperature modulated phase. In fact, this reentrant behavior has been experimentally observed; see, for instance, Ref. [1]. Of course, in the context of the present work, we are not able to confirm this conjecture, since our calculation is restricted to high temperatures very near the critical point.

Summarizing, this paper is a contribution to understanding a sector of a complex phase diagram in general models with competing interactions at different scales. The results presented have to be considered as a qualitative guide to more precise calculations. The main difficulty to compute quantitative relevant results is that, in general, orientational order parameters in these kinds of systems are quadratic functions of fundamental degrees of freedom. Even at mean-field level the computation of the order parameter implies the evaluation of fluctuations in the original variables.

APPENDIX: FOURIER TRANSFORM OF LONG-RANGED INTERACTION KERNEL

The antiferromagnetic contribution to quadratic part of the effective Hamiltonian can be written as

$$H_2^a = \frac{g}{2} \sum_{ij} \Phi_i \frac{1}{|\vec{r}_i - \vec{r}_j|^\alpha} \Phi_j. \tag{A1}$$

In reciprocal space, it reads

$$H_2^a = \frac{g}{2} \int_{BZ} \frac{d^2k}{(2\pi)^2} f^a(\vec{k}) |\Phi(k)|^2. \tag{A2}$$

Assuming that the most relevant contribution to the phase transition comes from waves of lengths much longer than the lattice spacing, we can consider the continuum limit, in which

$$f^a(\vec{k}) = \int d^2r \frac{1}{|\vec{r}|^\alpha} e^{i\vec{k}\cdot\vec{r}}. \tag{A3}$$

This expression coincides with the exact form of $f^a(\vec{k})$ for small values of $|\vec{k}|a \ll 1$, where a is the lattice spacing.

The Fourier transform of $1/|\vec{r}|^\alpha$ in d dimensions and for α not even is given by [26]

$$\mathcal{F}(1/r^\alpha) = 2^{d-\alpha} \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} k^{\alpha-d}, \tag{A4}$$

where the coefficient is written in terms of usual Γ functions. Then, in two dimensions, and for $\alpha \neq 2, 4, \dots$,

$$f^a(k) = 2^{2-\alpha} \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} k^{\alpha-2}. \tag{A5}$$

Multiplying and dividing by $(2-\alpha)/2$ and using the Γ function property $z\Gamma(z) = \Gamma(z+1)$ we finally arrive at

$$f^a(k) = -\sigma(\alpha) \frac{k^{\alpha-2}}{\alpha-2}, \tag{A6}$$

where $\sigma(\alpha) = 2^{2-\alpha} \Gamma(\frac{4-\alpha}{2}) / \Gamma(\frac{\alpha}{2})$. Equation (A6) was used to build up the long-wavelength interaction in reciprocal space, Eq. (13).

As we have seen, for $\alpha > 4$, the exponent of k is greater than 2, then at small k the antiferromagnetic component is irrelevant

with respect to the ferromagnetic one. Conversely, for $\alpha \leqslant 4$ this term essentially changes the behavior of $J(k)$. However, note that the Γ function has poles at zero and negative integer values. Then, Eq. (A5) is not well defined for $\alpha = 2,4$.

The Fourier transform of negative integer exponents in two dimensions reads [26]

$$\mathcal{F}\left(\frac{2\pi}{r^{2m+2}}\right) = c_{-1}^{(2+2m)} k^{2m} \ln k + c_0^{(2+2m)} k^{2m}, \quad \text{(A7)}$$

where m is a positive integer and the coefficients c_0 and c_{-1} arise form the Laurent expansion of the Γ functions in Eq. (A5),

$$2^{2-\alpha} \pi \frac{\Gamma(\frac{2-\alpha}{2})}{\Gamma(\alpha/2)} = \frac{c_{-1}^{(2+2m)}}{2 - \alpha + 2m} + c_0^{(2+2m)}$$
$$+ c_1^{(2+2m)}(2 - \alpha + 2m) \ldots. \quad \text{(A8)}$$

The case $\alpha = 2$ corresponds to $m = 0$ in the preceding equation giving

$$\lim_{\alpha \to 2} f^a(k) = -\ln k + 1/20. \quad \text{(A9)}$$

Using this expression to compute $J(k)$, we find $k_0 = \sqrt{g/J}$ that coincides with Eq. (14) for $\alpha = 2$. Then, the curve $k_0(\alpha)$ depicted in Fig. 2 is continuous at $\alpha = 2$.

The other potentially problematic point is $\alpha = 4$. This corresponds to the value $m = 1$ in Eq. (A8). In this case,

$$\lim_{\alpha \to 4} f^a(k) = -k^2(1 - \ln k). \quad \text{(A10)}$$

Using this expression to build up $J(k)$, we find an exponentially small value of $k_0 \sim \exp(-\frac{J}{2g})$. The value of the stiffness in this case is also very small $\rho_0 \sim g/J$. Then, for all practical proposes, this limiting case can be safely ignored and $k_0(\alpha)$ is correctly represented in Fig. 2 for the entire range of α.

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