else if $(0 > x_0)\ and\ (x_0 > -P_1(1, length(P_1)))$
$D = |(-P_1 - x_0)|;$
$[Y, I] = sort(D);$
$ffm = [x_0; y_0] + T(A[x_0; y_0] + b \times sign(y_0 + P_2(I(1))));$
else
$D = |(P_1 - x_0);$
$[Y, I] = sort(D);$
$ffm = [x_0; y_0] + T(A[x_0; y_0] + b\ sign(y_0 - P_2(I(1))));$
end;
end;
$x(i+1) = ffm(1); y(i+1) = ffm(2);$
$x_0 = ffm(1); y_0 = ffm(2);$
end;
$xx = P_1(1, length(P_1)) : T : 4;$
$yy = xx - xx;$
$P_1 = [P_1 xx];$
$P_2 = [P_2 yy];$

Fig. 4 shows the use of this switch curve on a nearby system when A, b taking their values at $\zeta = 0.4$, but the switch curve is the same as in Fig. 3. The system's trajectory still converges to almost its same maximum limit cycle (shown as dotted curve) showing some robustness of the disturbance switch curve. Any 2 × 2 matrix A, and any 2 × 1 matrix b can be entered in the above using the switch curve specified by ζ to explore other robustness properties.

For completeness, Fig. 5 shows the use of the switch curve (line $y = 0$) in selecting the most stressful disturbance for state response.

III. Conclusions

Time maximum disturbances that are bounded can be synthesized in feedback form using the obtained closed analytic form for the switch curve of stable second-order systems. Although this switch curve of closed analytic form was obtained for the maximum time severity disturbance index, it is easy using the above ideas to get the switch curve in closed analytic form for the usual time optimal task [3] for the damped harmonic oscillator and all other second-order systems.

References

[1] E. B. Lee and J. C. Luo, "Two methods for constructing aggrandized disturbances using delay tactics for stabilized control systems," in *Proc. ACC*, 1997, pp. 2189–2192.
[2] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.
[3] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*. New York: Wiley, 1967.
[4] T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, 2nd ed. Boston, MA: Birkhauser, 1995.
[5] E. B. Lee and J. C. Luo, "Disturbance stress for stabilized second order control systems," *Math. Models Methods Appl. Sci.*, vol. 8, pp. 1155–1182, 1998.
[6] M. M. Kogan, "Solution to the inverse problem of minimax control and worst case disturbance for linear continuous-time systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 670–674, May 1998.

Synthesis of Controllers for Continuous-Time Delay Systems with Saturating Controls via LMI's

Sophie Tarbouriech and João Manoel Gomes da Silva, Jr.

***Abstract*—The stabilization of linear continuous-time systems with time delay in the state and subject to saturating controls is addressed. Sufficient conditions obtained via a linear matrix inequality (LMI) formulation are stated to guarantee both the local stabilization and the satisfaction of some performance requirements. The method of synthesis consists in determining simultaneously a state feedback control law and an associated domain of safe admissible states for which the stability of the closed-loop system is guaranteed when control saturations effectively occur.**

***Index Terms*—Control saturation, linear matrix inequalities (LMI's), local stability, time-delay systems.**

I. Introduction

The problem of stabilizing linear systems with saturating controls has been widely studied these last years because of its practical interest: see, for example, [2] for a bibliographical overview. In this context, significant results have emerged in the scope of global [20] or semiglobal stabilization [18]. These studies require some stability properties for the open-loop system. Relaxing these assumptions, the local stabilization has been investigated [26], [5].

The stabilization of linear systems with a delayed state is also a problem of interest, because the existence of a delay may be a source of instability (as the occurence of the controls saturation) [10]. Different conditions for the stabilization of time-delay systems via memoryless control laws have been obtained. For an outline about the last results on the delay systems, consult, for example, [15], [11], and references therein, and the proceedings of the 13th World IFAC Congress (San Francisco, USA—July 1996) or the 35th IEEE-CDC (Kobe, Japan—December 1996).

Regarding linear systems with both delayed state and bounded controls, some results on local or global stabilization via memoryless control laws have been stated. We may cite [4], [7], [8], [16], [19], and [25]. The stability conditions presented in these papers are mainly based on the use of matrix measure, complex Lyapunov equations or, still, Razumikhin's approach.

This paper deals with the synthesis of stabilizing controllers for linear systems with state delay and saturating controls. The objective of the control design is twofold. It consists in determining both a memoryless state-feedback control law to ensure some performance requirements for the closed-loop system when the control is not saturated, and a set of safe initial conditions for which the asymptotic stability of the saturated closed-loop system is guaranteed. The performance requirements are treated in terms of closed-loop poles placement: the concept of β-stability is used (see [14], [15], and references therein). The method used is based on the Lyapunov–Krasovskii

Manuscript received June 30, 1998; revised July 13, 1998. Recommended by Associate Editor, A. Rantzer. This work was supported in part by FAPERGS, Brazil.

S. Tarbouriech is with the L.A.A.S.-C.N.R.S., 31077 Toulouse Cedex 4, France (e-mail: tarbour@laas.fr).

J. M. Gomes da Silva, Jr., is with the Department of Electrical Engineering, UFRGS, 90035-190 Porto Alegre-RS, Brazil (e-mail: jmgomes@iee.ufrgs.br).

Publisher Item Identifier S 0018-9286(00)00821-7.

approach [9]. The synthesis of both a suitable gain matrix and an associated set of initial conditions is carried out by solving linear matrix inequalities (LMI's) [3].

The paper is organized as follows. Section II presents the system with its properties and the problem to be solved. Section III deals with some preliminaries. The synthesis of the controller is presented in Section IV. Section V illustrates the results on a numerical example borrowed from the literature. Finally, in Section VI, concluding remarks end the paper.

Notations: $\Re^+$ is the set of nonnegative real numbers. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). For any real matrix A, A^T, and $A_{(i)}$ denote the transpose and the ith row of matrix A, respectively. I_n denotes the identity matrix in $\Re^{n\times n}$. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote, respectively, the maximal and minimal eigenvalue of matrix P. $co\{\cdot\}$ denotes a convex hull. $\mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \Re^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\Re^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix two-norm. $\|\phi\|_c = \sup_{-\tau\leq t\leq 0} \|\phi(t)\|$ stands for the norm of a function $\phi \in \mathcal{C}_\tau$. When the delay is finite, then "sup" can be replaced by "max." $\mathcal{C}_\tau^v$ is the set defined by $\mathcal{C}_\tau^v = \{\phi \in \mathcal{C}_\tau; \|\phi\|_c < v, v > 0\}$.

II. Problem Statement

Consider the linear continuous-time delay system subject to input saturation described by

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bsat(u(t)), \tag{1}$$

with the initial condition

$$x(t_0+\theta) = \phi(\theta), \quad \forall\theta \in [-\tau, 0], \quad (t_0, \phi) \in \Re^+ \times \mathcal{C}_\tau^v \tag{2}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the saturating control input, τ is the time-delay of the system, and A, A_d, and B are known real constant matrices of appropriate dimensions. Pair (A, B) is assumed to be stabilizable.

In the present paper, we consider a saturated state feedback, $sat(u(t)) = sat(Kx(t)), K \in \Re^{m\times n}$, with each component defined for $i = 1, \cdots, m$

$$sat(K_{(i)}x(t)) = \text{sign}(K_{(i)}x(t)) \min(|K_{(i)}x(t)|,\ u_{0(i)}), \tag{3}$$

where $u_{0(i)} > 0$, $\forall i = 1, \cdots, m$. Thus, we consider the following nonlinear system:

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bsat(Kx(t)). \tag{4}$$

In general, for a given stabilizing state feedback K, it is not possible to determine exactly the region of attraction of the origin with respect to system (4). Hence, a domain of initial conditions, for which the asymptotic stability of the system (4) is ensured, has to be determined. When the open-loop system $(u = 0)$ is stable, the global stabilization can be studied (see [8] and references therein). Throughout the paper, no assumption on the stability of the open-loop system is made. In this sense, the problem to be solved is a problem of *local stabilization*.

Remark II.1: When saturations do not occur, we get $sat(K_{(i)}x(t)) = K_{(i)}x(t), \forall i = 1, \cdots, m$. Thus, for all $x(t) \in \mathcal{S}(K, u_0)$ defined as

$$\mathcal{S}(K, u_0) = \{x \in \Re^n; -u_0 \leq Kx \leq u_0\}, \tag{5}$$

system (4) admits the linear model

$$\dot{x}(t) = (A + BK)x(t) + A_d x(t-\tau). \tag{6}$$

We cannot conclude, however, that any trajectory initiated in $\mathcal{S}(K, u_0)$ is a trajectory of system (6).

Let us now define the β-stability [14].

Definition II.1: Consider the solutions s of the equation $\det(F(s)) = 0$, $F(s) = sI_n - (A + BK) - e^{-s\tau}A_d$. The linear system (6) is said to be β-stable if $\text{Re}(s) + \beta < 0$. Or, equivalently, the system

$$\dot{y}(t) = (A + BK + \beta I_n)y(t) + e^{\beta\tau}A_d y(t-\tau) \tag{7}$$

is stable. In other words, system (6) is stable with the decay rate β.

The objective of the paper can be summarized as follows.

Problem II.1: Find both a state feedback K and a set of initial conditions $\mathcal{D}_0$ such as follows.

1) The asymptotic stability of the closed-loop system (4) is guaranteed in $\mathcal{D}_0$, that is, $\forall\phi(\theta) \in \mathcal{D}_0$, $\forall\theta \in [-\tau, 0]$, system (4) is asymptotically stable.
2) The linear closed-loop model (6) (that is, without saturations) is β-stable in the sense of Definition II.1; that is, the trajectories of (6) contained in the region $S(K, u_0)$ are stable with a decay rate β.

In [4], [16], and [25], conditions of stability for the closed-loop system (4) are given in a general form (by using condition of norm and matrix measure or Riccati equation) without explicitely defining the set in which the asymptotic stability is effectively ensured. At the converse, based on the Lyapunov–Krasovskii approach [6], our objective consists in quantifying a set of admissible initial conditions from which the asymptotic stability of the closed-loop saturated system is guaranteed.

Remark II.2:

- When no limits on the control vector are taken into account (saturation-free case), then the set $\mathcal{D}_0$ of the initial conditions that may be stabilized is the set $\mathcal{C}_\tau^v$ considered in (2), provided that the Krasovskii–Lyapunov theorem is satisfied [6].
- In both the delay-free $(\tau = 0)$ and the saturation-free case, the resolution of Problem II.1 simply consists in stabilizing a linear system $\dot{x}(t) = (A + A_d)x(t) + Bu(t)$ for which different solutions are available depending on the stabilizability property of pairs (A, B) or $((A + Ad), B)$.

III. Preliminaries

Let us write the saturation term as

$$sat(Kx(t)) = D(\alpha(x))Kx(t); \quad D(\alpha(x)) \in \Re^{m\times m}, \tag{8}$$

where $D(\alpha(x))$ is a diagonal matrix for which the diagonal elements $\alpha_{(i)}(x)$ are defined for $i = 1, \cdots, m$ as

$$\alpha_{(i)}(x) = \begin{cases} -\dfrac{u_{0(i)}}{K_{(i)}x}, & \text{if } K_{(i)}x < -u_{0(i)} \\ 1, & \text{if } -u_{0(i)} \leq K_{(i)}x \leq u_{0(i)} \\ \dfrac{u_{0(i)}}{K_{(i)}x}, & \text{if } K_{(i)}x > u_{0(i)} \end{cases} \tag{9}$$

and $0 < \alpha_{(i)}(x) \leq 1$. System (4) can then be written in the equivalent form

$$\dot{x}(t) = (A + BD(\alpha(x))K)x(t) + A_d x(t-\tau). \tag{10}$$

The coefficient $\alpha_{(i)}(x)$ can be viewed as an indicator of the degree of saturation of the ith entry of the control vector. In fact, the smaller $\alpha_{(i)}(x)$, the farther is the state vector from the region of linearity $\mathcal{S}(K, u_0)$. Notice that $\alpha_{(i)}(x)$ is a function of $x(t)$.

If we consider any compact set $S_0 \subset \Re^n$, it follows that for any $x(t)$ belonging to S_0, one may define a lower bound for $\alpha_{(i)}(x)$ as

$$\alpha_{\min(i)} = \min\{\alpha_{(i)}(x); x \in S_0\}. \tag{11}$$

Therefore, $\forall x(t) \in S_0$, the scalars $\alpha_{(i)}(x)$, $i = 1, \cdots, m$, satisfy $\alpha_{\min(i)} \leq \alpha_{(i)}(x) \leq 1$. Define now matrices A_j, $j = 1, \cdots, 2^m$, as follows [13]:

$$A_j = A + BD(\gamma_j)K \tag{12}$$

where $D(\gamma_j)$ is a diagonal matrix of positive scalars $\gamma_{j(i)}$, for $i = 1, \cdots, m$, which arbitrarily take the value one or $\alpha_{\min(i)}$. Thus, if $x(t) \in S_0$, then $\dot{x}(t)$ can be determined from the following polytopic model:

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_{j,t} A_j x(t) + A_d x(t - \tau) \tag{13}$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$, $\lambda_{j,t} \geq 0$. Note that the matrices A_j are the vertices of a convex polyhedron of matrices, and for $x \in S_0$, one gets $(A + BD(\alpha(x))K) \in co\{A_1, \cdots, A_{2^m}\}$. Note also that $\alpha_{\min(i)}$, $i = 1, \cdots, m$, define the polyhedral set

$$\mathcal{S}(K, u_0^\alpha) = \{x \in \Re^n;\ -u_0^\alpha \leq Kx \leq u_0^\alpha\}, \tag{14}$$

where every component of vector u_0^α is defined by $(u_{0(i)}/\alpha_{\min(i)})$, $i = 1, \cdots, m$. This set contains S_0 and corresponds to the maximal set in which model (13) can represent system (4) [or (10)].

IV. Synthesis of the Controller

Consider the following Lyapunov function:

$$\begin{aligned} V(x_t) &= x(t)^T P x(t) + \int_{t-\tau}^{t} x(\theta)^T S x(\theta)\, d\theta \\ P &= P^T > 0, \quad S = S^T > 0 \end{aligned} \tag{15}$$

where x_t, $\forall t \geq t_0$, denotes the restriction of x to the interval $[t - \tau, t]$ translated to $[-\tau, 0]$; that is, $x_t(\theta) = x(t + \theta), \forall \theta \in [-\tau, 0]$.

Assume now that the following data is given as follows.

- A positive scalar β that represents the desired decay rate in the zone of linear behavior.
- A vector $\alpha_{\min}$ such that each component $\alpha_{\min(i)}$, $i = 1, \cdots, m$, satisfies $0 < \alpha_{\min(i)} \leq 1$. This vector can be viewed as a specification on the saturation tolerance.

Then, by setting $\mathcal{J} = \{j \in [1, 2^m]; D(\gamma_j) \neq I_m\}$, which allows us to consider all of the matrices A_j described in (12), except the matrix $A + BK$, the following proposition can be stated to solve Problem II.1.

Proposition IV.1: If two symmetric positive-definite matrices $W \in \Re^{n \times n}$ and $R \in \Re^{n \times n}$ exist, a matrix $Y \in \Re^{m \times n}$ and a positive scalar κ solutions of the following LMI's:

$$\begin{bmatrix} WA^T + AW + Y^T B^T + BY + 2\beta W + R & e^{\beta\tau} A_d W \\ e^{\beta\tau} W A_d^T & -R \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} WA^T + AW + R + Y^T D(\gamma_j) B^T + BD(\gamma_j) Y & A_d W \\ W A_d^T & -R \end{bmatrix} < 0, \quad \forall j \in \mathcal{J} \tag{17}$$

$$\begin{bmatrix} W & \alpha_{\min(i)} Y_{(i)}^T \\ \alpha_{\min(i)} Y_{(i)} & \kappa u_{0(i)}^2 \end{bmatrix} \geq 0, \qquad i = 1, \cdots, m \tag{18}$$

then the state feedback matrix $K = YW^{-1}$ solves Problem II.1 with respect to system (4) for every initial condition in the ball

$$\begin{aligned} &\mathcal{B}(\delta) = \{\phi \in \mathcal{C}_\tau^v; \|\phi\|_c^2 \leq \delta\} \quad \text{with} \\ &\delta = \frac{\kappa^{-1}}{(\lambda_{\max}(W^{-1}) + \tau \lambda_{\max}(W^{-1} R W^{-1}))}. \end{aligned} \tag{19}$$

Proof: The existence of two symmetric positive-definite matrices $W \in \Re^{n \times n}$ and $R \in \Re^{n \times n}$, and a matrix $Y \in \Re^{m \times n}$ satisfying LMI (16), implies that the linear closed-loop system (6) is β-stable, according to Definition II.1 (see also [15]). Furthermore, by some algebraic manipulations, one can prove that the satisfaction of LMI (16) also implies that

$$\begin{bmatrix} WA^T + AW + Y^T B^T + BY + R & A_d W \\ W A_d^T & -R \end{bmatrix} < 0. \tag{20}$$

From (20) and provided that LMI's (17) are satisfied $\forall j \in \mathcal{J}$, by setting $K = YW^{-1}$, $P = W^{-1}$, and $S = W^{-1} R W^{-1}$, it follows by convexity:

$$\sum_{j=1}^{2^m} \lambda_{j,t} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} \mathcal{M}(j) & PA_d \\ A_d^T P & -S \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} < 0 \tag{21}$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$, $\lambda_{j,t} \geq 0$, and $\mathcal{M}(j) = (A + BD(\gamma_j)K)^T P + P(A + BD(\gamma_j)K) + S$. Moreover, the satisfaction of LMI's (18) means that the ellipsoid $\mathcal{S}(W^{-1}, \kappa^{-1}) = \mathcal{S}(P, \kappa^{-1})$ defined by

$$\mathcal{S}(W^{-1}, \kappa^{-1}) = \{x \in \Re^n; x^T W^{-1} x \leq \kappa^{-1}\} \tag{22}$$

is included in $\mathcal{S}(K, u_0^\alpha)$ [5]. Suppose now that $x(t) \in \mathcal{S}(K, u_0^\alpha)$. Hence, $\dot{x}(t)$ can be computed from the polytopic system (13), and it follows that the time-derivative of the Lyapunov functional defined in (15) along the trajectories of system (13) is given by

$$\begin{aligned} \dot{V}(x_t) = &\sum_{j=1}^{2^m} \lambda_{j,t} x(t)^T (A_j^T P + PA_j) x(t) \\ &+ 2x(t)^T PA_d x(t - \tau) \\ &+ x(t)^T S x(t) - x(t - \tau)^T S x(t - \tau). \end{aligned}$$

From (21) and by using the fact that $[x(t-\tau) - S^{-1}A_d^T Px(t)]^T S[x(t-\tau) - S^{-1}A_d^T Px(t)] \geq 0$ is equivalent to $-x(t-\tau)^T Sx(t-\tau) + 2x(t)^T PA_d x(t-\tau) \leq x(t)^T PA_d S^{-1} A_d^T Px(t)$, we get:

$$\dot{V}(x_t) \leq \sum_{j=1}^{2^m} \lambda_{j,t} x(t)^T (\mathcal{M}(j) + PA_d S^{-1} A_d^T P) x(t) \ < \ 0. \quad (23)$$

Thus, $\pi_3 \ > \ 0$ exists such that $\dot{V}(x_t) \leq -\pi_3 \|x(t)\|^2 \ < \ 0$ and therefore $V(x_t) \leq V(x_{t_0})$, provided that model (13) is valid, that is, provided that $x(t) \in \mathcal{S}(K, u_0^\alpha)$.

Furthermore, the Lyapunov functional defined in (15) satisfies

$$\pi_1 \|x(t)\|^2 \leq V(x_t) \leq \pi_2 \|x_t\|_c^2, \quad (24)$$

with $\pi_1 = \lambda_{\min}(P)$ and $\pi_2 = \lambda_{\max}(P) + \tau\lambda_{\max}(S)$.

From (23) and (24), if $x(t_0 + \theta) = \phi(\theta) \in \mathcal{B}(\delta)$, $\forall \theta \in [-\tau, 0]$, then it follows:

$$x(t)^T Px(t) \leq V(x_t) \leq V(x_{t_0}) \leq \kappa^{-1}, \quad \forall t \geq t_0.$$

Hence, for any initial condition in the ball $\mathcal{B}(\delta)$, we get $x(t) \in S(W^{-1}, \kappa^{-1})$, $\forall t \geq t_0$. Because LMI's (18) are satisfied, it follows that $x(t) \in \mathcal{S}(K, u_0^\alpha)$, $\forall t \geq t_0$. Thus, for any initial condition belonging to $\mathcal{B}(\delta)$, the model (13) represents the saturated system (4). Hence, from (23) and (24), we can conclude that system (4) verifies the conditions of the Krasovskii theorem [6] and $V(x_t)$ is a local strictly decreasing Lyapunov function. Therefore, provided that $\phi(\theta) \in \mathcal{B}(\delta)$, the asymptotic stability of system (4) is ensured. □

Remark IV.1:

- In the saturation-free case, only LMI (16) is of interest to prove the closed-loop stability under the initial condition hypothesis (2).
- In the delay-free case, the Lyapunov function is the quadratic one $V(x) = x^T Px$. Under the (A, B)-stabilizability hypothesis, the term $A_d x(t)$ can be considered as an uncertain term. For example, we can consider that this term is of the norm-bounded type [17] and therefore decompose A_d as $A_d = DFE$ with $F^T F \leq I_r$. In this case, LMI's (16)–(18) become the classical ones treating the problem of both saturation and uncertainty: see, for example, the book [22]. Furthermore, if the pair $((A + A_d), B)$ is stabilizable, then Proposition IV.1 can be modified to treat only the stabilization of saturated systems: see for example, Proposition 3 in [5], in which A is replaced by $A + A_d$.

Proposition IV.1 provides a condition allowing us to compute both a matrix K and a ball $\mathcal{B}(\delta)$ of initial condition such that Problem II.1 is solved. It is interesting to come up with a solution such that the ball of initial conditions is the largest as possible. In this sense, a first solution consists in minimizing κ. Another interesting solution consists in minimizing the term $[\lambda_{\max}(P) + \tau\lambda_{\max}(S)) = (\lambda_{\max}(W^{-1}) + \tau\lambda_{\max}(W^{-1}RW^{-1})]$. Nevertheless, the minimization of this term can be very hard, even impossible, to directly obtain. Then by considering some LMI's imposing conditions on the maximal eigenvalue of W^{-1} and R, some linear optimization criteria can be used. In particular, we suggest the following optimization problem:

$$\min \sigma_1 + \epsilon\kappa$$

subject to

a) $\sigma_1 > 0, \sigma_2 > 0, W > 0, R > 0, \kappa > 0$

b) $\begin{bmatrix} \sigma_1 I_n & I_n \\ I_n & W \end{bmatrix} \geq 0$

c) $\sigma_2 I_n - R \geq 0$

d) $\sigma_1 \geq \sigma_2$

e) LMI's (16)–(18) of Proposition IV.1. (25)

The satisfaction of LMI b) implies that $\lambda_{\max}(W^{-1}) = \lambda_{\max}(P) \leq \sigma_1$. In the same way, the satisfaction of LMI c) means that $\lambda_{\max}(R) \leq \sigma_2$, which corresponds to have $\lambda_{\max}(W^{-1}RW^{-1}) = \lambda_{\max}(S) \leq \sigma_2\sigma_1^2$. Hence, we get $\lambda_{\max}(P) + \tau\lambda_{\max}(S) \leq \sigma_1 + \tau\sigma_2\sigma_1^2$ and from LMI d) it follows $\lambda_{\max}(P) + \tau\lambda_{\max}(S) \leq \sigma_1 + \tau\sigma_2\sigma_1^2 \leq \sigma_1 + \tau\sigma_1^3$. Therefore, because $\delta = 1/(\kappa(\lambda_{\max}(P) + \tau\lambda_{\max}(S)))$, we get $\delta \geq 1/(\kappa(\sigma_1 + \tau\sigma_1^3))$. If we minimize the criterion as defined in (25), then we maximize the bound $1/(\kappa(\sigma_1 + \tau\sigma_1^3))$ and thus greater δ tends to be. In other words, by using the optimal problem (25), we orient the solutions of LMI's (16)–(18) in the sense to obtain the ball $\mathcal{B}(\delta)$, associated with the gain $K = YW^{-1}$, as large as possible.

With respect to the optimization problem (25), we can formulate the following comments.

1) In problem (25), we consider a criterion with multiple objectives in which ϵ is a positive scalar that can be used to assign relative weight to σ_1 or κ. Thus, the positive scalar ϵ can be considered as a parameter of synthesis. For a given pair $(\alpha_{\min}, \beta)$ [such that LMI's (a)–(e) are feasible], one can iterate on ϵ in view to obtain the best associated δ.

2) In fact, the solution of (25) is a tradeoff between the performance requirements in terms of decay rate β around the origin, the tolerance of saturation $\alpha_{\min}$, and the size of the resulting ball $\mathcal{B}(\delta)$. This tradeoff will be shown in the illustrative example. Moreover, if the designer imposes a certain level of requirements, the problem (25) may be not solvable; i.e., the resulting LMI's (a)–(e) may be not feasible.

3) In practice, the given data are often constituted by both the decay rate and a minimal ball of initial states $\mathcal{B}(\delta_{\min})$. Given the pair $(\delta_{\min}, \beta)$, the objective consists then, if possible, in fixing the parameters $\alpha_{\min(i)}$, $i = 1, \cdots, m$ and ϵ to obtain W, Y and κ giving $K = YW^{-1}$ and satisfying $\delta_{\min} \leq 1/(\kappa(\lambda_{\max}(P) + \tau\lambda_{\max}(S)))$. In this case, we have to add in the optimization problem an upper bound for σ_1 depending on $\delta_{\min}$.

4) One difficulty regarding the application of Proposition IV.1 or problem (25) resides in the necessary *a priori* choice of the vector $\alpha_{\min}$, that is, of the different matrices $D(\gamma_j)$. We conjecture that the smaller the components of vector $\alpha_{\min}$, both the larger the domain of admissible initial states and the more stringent the performance specification for which it is possible to find a solution, verifying the conditions given in Proposition IV.1 (and therefore solving Problem II.1). This can be justified in part by the fact that, for a given stabilizing matrix gain K, if we consider vector $\alpha_{\min}$ with smaller components, larger is the region $S(K, u_0^\alpha)$, where the ball $\mathcal{B}(\delta)$ can be contained. Moreover, we claim that more stringent performances are associated, in general, with larger gains and, in consequence, smaller regions of linearity. These facts are illustrated through the proposed numerical example. We are not able, however, to prove these conjectures at this time. Nevertheless, in order to avoid the *a priori* choice of $\alpha_{\min}$, we can solve problem (25) (for a given β) iteratively in two steps as follows.

   a) Fix $\alpha_{\min}$ and solve (25) for W, Y, R, σ_1, σ_2, κ.
   b) Fix Y and solve (25) for W, $\alpha_{\min}$, R, σ_1, σ_2, κ, in adding the constraints $0 < \alpha_{\min(i)} \leq 1, i = 1, \cdots, m$.

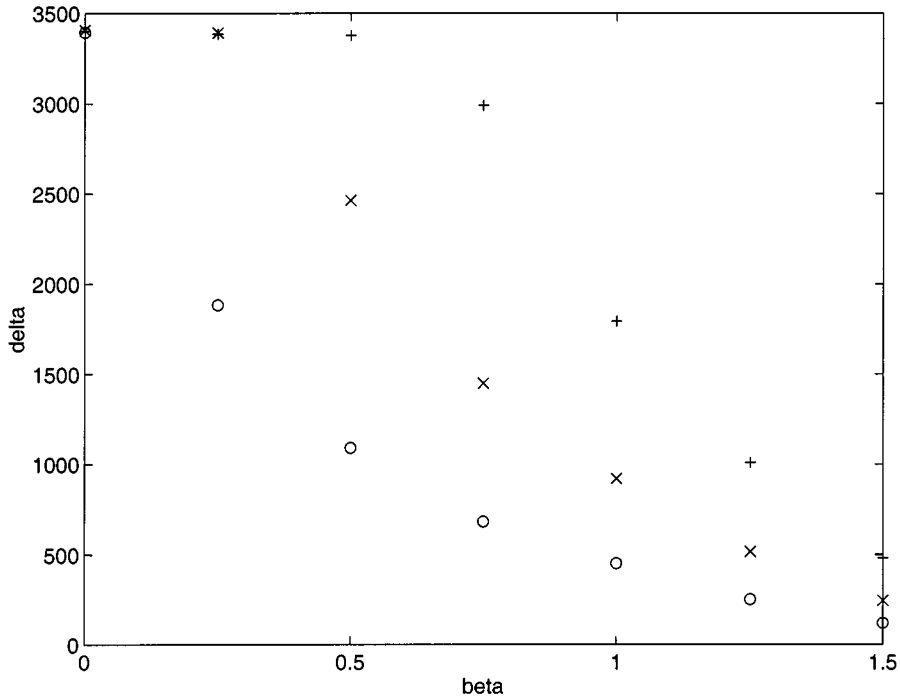

Fig. 1. "o" $-\alpha_{\min} = 1$; "×" $-\alpha_{\min} = 0.7$; and "+" $-\alpha_{\min} = 0.5$.

The objective is to obtain the largest ball $\mathcal{B}(\delta)$ as possible, the stopping criterion depends on the values of σ_1 and κ. Note that, for a given β, our method may be not able to give a solution to problem (25).

5) To avoid some numerical problems, we can add some condition number on R and W or consider a security margin in the verification of LMI's. For example, we can consider LMI (16) by replacing the right term of the inequality by $-\omega I_{2n}$, with $\omega > 0$ chosen small enough.

6) In the nonsymmetrical saturation case, we consider

$$sat(K_{(i)}x) = \begin{cases} u_{\max(i)}, & \text{if } K_{(i)}x > u_{\max(i)} \\ K_{(i)}x, & \text{if } -u_{\min(i)} \leq K_{(i)}x \leq u_{\max(i)} \\ -u_{\min(i)}, & \text{if } K_{(i)}x < -u_{\min(i)} \end{cases}$$

with $u_{\max(i)},\ u_{\min(i)} > 0,\ \forall i = 1, \cdots, m$. With this definition, results of Section III apply. Thus, Proposition IV.1 and problem (25) apply by considering in (18) $\min(u_{\min(i)}, u_{\max(i)})^2$ instead of $u_{0(i)}^2$.

V. Illustrative Example

Consider the example borrowed from [24]. System (1) is described by the following data:

$$A = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 10 \\ 1 \end{bmatrix}, \qquad \tau = 1.$$

We want to compute a saturated control law as defined in (3) with $u_0 = 15$. In this sense, we apply optimization problem (25) with $\epsilon = 7000$. We choose the coefficient of tolerance of saturation $\alpha_{\min} = 0.5$ and the desired decay rate $\beta = 1$. By using the LMI tool in MATLAB (with accuracy = 0.01), we obtain the following matrices:

$$W = \begin{bmatrix} 0.6901 & -0.0062 \\ -0.0062 & 0.6629 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0258 & -0.0239 \\ -0.0239 & 1.500 \end{bmatrix}$$
$$K = [-0.3592 \quad -0.1421].$$

The ball of admissible initial conditions $\mathcal{B}(\delta)$ is defined by $\delta = 1.7919 * 10^3$. Fig. 1 depicts the evolution of δ in function of the chosen decay rate β for different given coefficients of tolerance $\alpha_{\min}$ (with $\epsilon = 7000$). Thus, it shows the tradeoff among the size of the ball of the admissible initial conditions, the desired decay rate, and the tolerance of saturation. Note that if we decrease $\alpha_{\min}$, that is, we allow a greater level of saturation, then we obtain larger domains $\mathcal{B}(\delta)$. Furthermore, the larger is β, that is, the more stringent is the exigence in terms of linear performance, then the smaller is the domain $\mathcal{B}(\delta)$ for which we guarantee the asymptotic stability of the saturated system.

By numerical simulation, we show in Fig. 2 the trajectories of the saturated closed-loop system and the different sets of interest, namely, $\mathcal{B}(\delta)$, $S(W^{-1}, \kappa^{-1})$ and $S(K, u_0^{\alpha})$. Of course, the real region of attraction of the origin is nonconvex and greater than the set $S(W^{-1}, \kappa^{-1})$, but such a region can be obtained in general only by simulation and for very simple example.

We compare our approach with some published results.

- In [21], we have studied the same example via an approach based on Riccati equations (ARE-based approach). The obtained ball of admissible initial conditions is smaller because we obtain $\delta = 21.0751$. Hence, for this example, we have a gain of 8502.4% relative to the size of $\mathcal{B}(\delta)$.
- Consider the results proposed in [4] and, more especially, Corollary 1, p. 873. Our notation $\alpha_{\min}$ corresponds to w in [4]. By applying relation (2.27) in [4] to our system, it follows $-(\|Ad\|/\mu_2(A+((1+\alpha_{\min})/2)BK)) = \tilde{r}_0$, where the matrix measure $\mu_2(M) = \lambda_{\max}(M + M^T)/2$, whereas condition

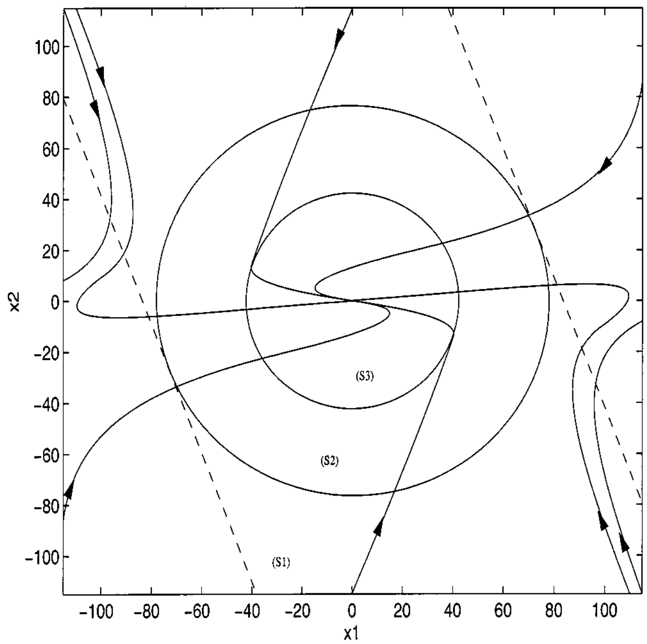

Fig. 2. State trajectories of the closed-loop saturated system with $\phi(\theta) = x(0), \forall\theta \in [-1, 0]$. (S1): $\mathcal{S}(K, u_0^\alpha)$; (S2): $\mathcal{S}(W^{-1}, \kappa^{-1})$; (S3): $\mathcal{B}(\delta)$.

(2.28) reads: $\tilde{r}_0 < (1 + ((1 - \alpha_{\min})\|B\| \; \|K\|/2\mu_2(A + ((1 + \alpha_{\min})/2)BK)))$. Hence, by using our numerical data, we get $\alpha_{\min} = w = 0.5$ and $\tilde{r}_0 = 0.6292$, which implies that condition (2.28) in [4] is not satisfied. Therefore, at the converse of our results, from [4], it is not possible to conclude to the stability of the closed-loop system (4). In this sense, our results are less conservative than those of [4]. Moreover, by choosing $\alpha_{\min} = w = 0.8$, we get $\tilde{r}_0 = 0.4835$, which satisfies condition (2.28) in [4] because $\tilde{r}_0 < 0.8123$. Nevertheless, we can observe that for

$$\phi(\theta) = \begin{bmatrix} 115 \\ -115 \end{bmatrix}, \quad \forall\theta \in [-1, 0]$$

the closed-loop system (4) is unstable, which contradicts the result given in [4] (see Fig. 2). Thus, the necessity to explicitely define a set of admissible stabilizable initial conditions clearly appears.

VI. Concluding Remarks

- The local stabilization of linear continuous-time systems with saturating controls and time delay in the state was addressed through the use of a Lyapunov–Krasovskii technique. The control law and a domain of safe initial conditions, for which the stability asymptotic of the saturated closed-loop system is guaranteed, were determined from a convex optimization problem with LMI constraints. An optimization problem was presented to maximize the size of the ball of admissible initial values. The conservativeness of the results proposed in this paper is partly because of the representation chosen for the saturated system. Indeed, only the trajectories of system (4) contained in $\mathcal{S}(K, u_0^\alpha)$ can be represented by those of system (13). Hence, all of the conditions obtained from this representation are only sufficient. Furthermore, the considered optimization problem can lead to a conservative solution. Some other LMI relaxation schemes and optimization problems could be investigated. Given the complexity of the problem caused by both the time delay and the saturation occurence, however, the proposed method provides an interesting systematic procedure for computing an admissible solution.
- In the time-varying delay case, that is, in the case in which the delay satisfies: $0 \le \tau(t) \le \tau_{\max}$ and $\dot{\tau} \le \zeta < 1$. In [12], the authors study the quadratic stabilization of continuous-time systems with time-varying delay and norm-bounded time-varying uncertainties, but without control constraints. Our results apply by considering in the Lyapunov function defined in (15) $(1/(1-\zeta))S$ instead of S. Thus, the ball of initial condition will be defined by $\delta = \kappa^{-1}/(\lambda_{\max}(P) + (\tau_{\max}/(1-\zeta))\lambda_{\max}(R))$.
- In this paper, the considered control was based on Krasovskii approach. Nevertheless, the Razumikhin approach [6], [15], [16] could be considered. In the context of Problem II.1, we have shown in [21] how the method based on Krasovskii approach leads to less restrictive results than those obtained with the Razumikhin approach.
- In this paper, the considered control law was of the memoryless type. Nevertheless, the desired control law may be expressed under the form: $u(t) = sat(Kx(t) + K_1x(t-\tau))$. Then, we can consider LMI's (17), $\forall j = 1 = 1, \cdots, 2^m$ by replacing the term A_dW by $A_dW + BD(\gamma_j)Y_1$, to compute $K = YW^{-1}$ and $K_1 = Y_1W^{-1}$. In this last case, the knowledge about the delay could be directly used in the control law. Such a study should be the subject of a forthcoming publication.
- When $\mathcal{B}(\delta)$ is an arbitrarily large known bounded set, the problem to be treated is related for systems without delay $(\tau = 0)$ to the semiglobal stabilization: see, for example, [1], [18]. Recall that, in this case, such a study requires some stability properties for the open-loop system: all of the eigenvalues of $A + A_d$ must have nonpositive real part. In the delay case $(\tau \neq 0)$, some studies on the necessary open-loop stability, that is, on the system $\dot{x}(t) = Ax(t) + A_dx(t-\tau)$, are needed. Such a study constitutes an open problem.
- Proposition IV.1 could be extended to uncertain systems provided some modifications of relations (16) and (17) Such an extension should be studied in the future. A first answer for uncertain systems with polytopic uncertainty is proposed in [23].

References

[1] J. Alvarez-Ramirez, R. Suarez, and J. Alvarez, "Semiglobal stabilization of multi-input linear system with saturated linear state feedback," *Syst. Contr. Lett.*, vol. 23, pp. 247–254, 1994.
[2] D. S. Bernstein and A. N. Michel, "A chronological bibliography on saturating actuators," *Int. J. Robust Nonlinear Contr.*, vol. 5, pp. 375–380, 1995.
[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, "Linear matrix inequalities in system and control theory," in *SIAM Studies Appl. Math.* Philadelphia, PA, 1994.
[4] B. S. Chen, S. S. Wang, and H. C. Lu, "Stabilization of time-delay systems containing saturating actuators," *Int. J. Contr.*, vol. 47, pp. 867–881, 1988.
[5] J. M. Gomes da Silva Jr., A. Fischman, S. Tarbouriech, J. M. Dion, and L. Dugard, "Synthesis of state feedback for linear systems subject to control saturation by an LMI-based approach," in *Proc. ROCOND*, Budapest, Hungary, June 1997, pp. 219–234.
[6] J. K. Hale, *Theory of Functional Differential Equations*. New York: Springer-Verlag, 1977.
[7] J. C. Hennet and S. Tarbouriech, "Stability and stabilization of delay differential systems," *Automatica*, vol. 33, pp. 347–354, 1997.
[8] M. Klaï, S. Tarbouriech, and C. Burgat, "Some independent-time-delay stabilization of linear systems with saturating controls," in *Proc. IEE Contr.'94*, Coventry, U.K., Mar. 1994, pp. 1358–1363.
[9] N. N. Krasovskii, *Stability of Motion*. Stanford, CA: Stanford Univ. Press, 1963.

[10] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*. New York: Academic, 1986.
[11] J. F. Lafay and G. Conte, "Analysis and design methods for delay systems," in *Proc. 34th IEEE-CDC*, New Orleans, Dec. 1995, pp. 2035–2069.
[12] M. S. Mahmoud and N. F. Al-Muthairi, "Quadratic stabilization of continuous-time systems with state-delay and norm-bounded time-varying uncertainties," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2135–2139, Oct. 1994.
[13] A. P. Molchanov and E. S. Pyatnitskii, "Criteria of asymptotic stability of differential and difference inclusions encountered in control theory," *Syst. Contr. Lett.*, vol. 13, pp. 59–64, 1989.
[14] T. Mori, N. Fukuma, and M. Kuwahara, "On an estimate of the decay rate for stable linear delay systems," *Int. J. Contr.*, vol. 36, pp. 95–97, 1982.
[15] S.-I. Niculescu, "H^∞ memoryless control with an α-stability constraint for time-delays systems: An LMI approach," in *Proc. 34th IEEE-CDC*, New Orleans, Dec. 1995, pp. 1507–1512.
[16] S.-I. Niculescu, J.-M. Dion, and L. Dugard, "Robust stabilization for uncertain time-delay systems containing saturating actuators," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 742–747, May 1996.
[17] I. R. Petersen, "Stabilization algorithm for a class of uncertain linear systems," *Syst. Contr. Lett.*, vol. 8, pp. 181–188, 1987.
[18] A. Saberi, Z. Lin, and A. R. Teel, "Control of linear systems with saturating actuators," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 368–378, Mar. 1996.
[19] T. J. Su, P. L. Liu, and J. T. Tsay, "Stabilization of delay-dependence for saturating actuator systems," in *Proc. 30th IEEE-CDC*, Brighton, U.K., Dec. 1991, pp. 2891–2892.
[20] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Trans. Automat. Contr.*, vol. 39, pp. ?–?, 1994.
[21] S. Tarbouriech, "Local stabilization of continuous-time systems with bounded inputs," in *Stability and Control of Time-Delay Systems*. New York: Springer-Verlag, 1997, ch. 14.
[22] S. Tarbouriech and G. Garcia, *Control of Uncertain Systems with Bounded Inputs*, S. Tarbouriech and G. Garcia, Eds. New York: Springer-Verlag, 1997.
[23] ——, "Robust stability of uncertain linear delay systems with saturating inputs: An LMI's approach," in *Proc. IFAC Conf. Syst. Structure Contr.*, Nantes, France, July 1998, pp. 379–384.
[24] A. Towsen, "Stable sampled-data feedback control of dynamic systems with time-delays," *Int. J. Systems Sci.*, vol. 13, no. 12, pp. 1379–1384, 1982.
[25] E. Tissir and A. Hmamed, "Further results on the stabilization of time delay systems containing sturating actuators," *Int. J. Systems Sci.*, vol. 23, pp. 615–622, 1992.
[26] G. F. Wredenhagen and P. R. Blanger, "Piecewise-linear LQ control for systems with inputs constraints," *Automatica*, vol. 30, pp. 403–416, 1994.

A Learning Approach to Tracking in Mechanical Systems with Friction

Seong-Il Cho and In-Joong Ha

***Abstract*—This note describes a novel learning control scheme for tracking periodic trajectories in mechanical systems with friction. It is based on the fact that the solution of the closed-loop system tends to be periodic in steady state. When the closed-loop system reaches the steady state, the proposed learning control scheme updates the control input. By doing this iteratively, the proposed learning control scheme eventually can drive the tracking error to zero. Neither the information of the system mass nor the parametric model for friction is required for successful tracking. In particular, the proposed learning control scheme can be implemented at cheap cost on a commercially available microprocessor. Furthermore, its generality is well supported through rigorous convergence analysis.**

***Index Terms*—Friction, iterative update, learning, periodic trajectory tracking, steady-state oscillation.**

I. Introduction

In motion control systems, friction is a primary source of disturbance, which can degrade control performance significantly. In this context, various friction compensation methods have been considered by many authors [2]. One of such friction compensation methods is the model-based friction compensation approach taken in [3]–[9]. It seems well suited for the case that an accurate parametric model for friction is available. In reality, however, it is often hard to obtain an accurate parametric model for friction. For this reason, a design method without using a parametric model for friction was taken in [10], in which state-dependent parasitic effects such as friction were approximated by Gaussian neural networks. It requires a large computational load, however, for the identification and compensation of friction.

On the other hand, it is well known that a learning approach is quite effective in the case in which *a priori* knowledge of the system model is limited. In fact, learning control schemes have been popularly used in robotic manipulators, gas-metal arc welding process, and CNC machines [11]–[16]. Furthermore, the feasibility of a learning control scheme for friction compensation has been well demonstrated in [15] through experiments using a repetitive control scheme. The learning control schemes are usually computationally simple and, hence, can be implemented at cheap cost. Instead, iterative operations along with some memory are required.

In this correspondence, we also take a learning approach to friction compensation. Our learning control scheme extends basically the work in [16] to the problem of periodic trajectory tracking in mechanical systems with friction. In particular, our learning control scheme differs fundamentally from the repetitive control schemes [14], [15] and the conventional iterative learning schemes [11]–[13]. Roughly speaking, we apply the periodic signal and then wait for the closed-loop system to settle until making the update.

Compared with other friction compensation methods, the proposed learning control scheme has the following merits. First, it requires a small amount of computational load for the compensation of friction. It does need some memory to store feedforward input for one period,

Manuscript received February 26, 1998; revised September 14, 1998 and April 15, 1999. Recommended by Associate Editor, J. A. Farrell.

The authors are with the School of Electrical Engineering, Seoul National University, Kwanak Ku, Seoul 151-742, Korea (e-mail: ijha@asri.snu.ac.kr).

Publisher Item Identifier S 0018-9286(00)00822-9.