

# Gallai-Edmonds decomposition of unicyclic graphs from null space

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## Abstract

In this paper, we compute the Gallai-Edmonds decomposition of a unicyclic graph  $G$  using linear algebraic tools. More precisely, the Gallai-Edmonds decomposition of  $G$  is obtained from the null space associated with adjacency matrices of its subtrees.

## 1 Introduction

The aim of spectral graph theory is to obtain structural properties of a graph using the eigenvalues and eigenvectors of the matrices associated with the graph. In particular, relating classical parameters with spectral parameters is quite useful and this is the general goal of this paper. Here, we use the null space of a unicyclic graph  $G$ , which is the null space of its adjacency matrix, to compute the Gallai-Edmonds decomposition of  $G$ . More precisely, we obtain the Gallai-Edmonds decomposition (see definition below) of unicyclic graphs from the null decomposition of its subtrees.

Null decomposition of a graph (see definition in Section 2), is defined from subsets of vertices of a graph that satisfy certain properties related to the null space. We remark that the null decomposition provides information on structure of a graph, as for example, matching and independence number [2, 3, 14, 15].

In order to explain our results, we need a few definitions here. For an undirected graph  $G = (V, E)$ , a matching  $M$  in  $G$  is a set of pairwise non-adjacent edges. A maximum matching is a matching of largest cardinality in  $G$  and the matching number of  $G$ , denoted by  $\nu(G)$ , is the size of a set of any maximum matching.  $\mathcal{M}(G)$  denotes the set of all maximum matchings of  $G$ . A vertex is saturated by  $M$ , if it is an endpoint of one of the edges in the matching  $M$ . Otherwise the vertex is said to be non-saturated. The neighborhood of a vertex  $v \in V$  in  $G$  is defined to be  $N(v) = \{u \in V : \{u, v\} \in E\}$ . The Gallai-Edmonds

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decomposition of  $G$  [8, 11] is a partition of  $V$  into three sets with certain properties related to maximum matchings of  $G$ . More precisely,  $V$  is partitioned into three sets  $EG(G)$ ,  $R(G)$  and  $S(G)$  where:

- $EG(G) = \{v \in V : \exists M \in \mathcal{M}(G) \text{ such that } M \text{ does not saturate } v\}$ .
- $R(G) = N(EG(G)) - EG(G)$ , where  $N(EG(G)) = \bigcup_{v \in EG(G)} N(v)$ .
- $S(G) = V - (EG(G) \cup R(G))$ .

To give a glimpse of our results, we will compute the Gallai-Edmonds decomposition of the unicyclic graph  $G$  in Figure 1 (left) using our technique which is based on linear algebra. Consider  $C$  the cycle of  $G$ . First, using the null space we compute the support (the subset of vertices for which at least one of its corresponding coordinates of the eigenvectors of the null space of the adjacency matrix is nonzero), core (neighborhood of the support) and  $N$ -vertices (the remaining vertices) of the graph  $G - C$ . Note that  $Supp(G - C) = \{f, g\}$ ,  $Core(G - C) = \{e\}$  and  $V(\mathcal{G}_N(G - C)) = \{a, b, j, \ell\}$ . Since  $|V(C)|$  is even, by Theorem 4.6, the Gallai-Edmonds decomposition of the unicyclic graph  $G$  is given by  $EG(G) = Supp(G - C) = \{f, g\}$ ,  $R(G) = Core(G - C) = \{e\}$  and  $S(G) = V(\mathcal{G}_N(G - C)) \cup V(C) = \{a, b, c, d, i, h, j, \ell\}$  as depicted in Figure 1 (right).

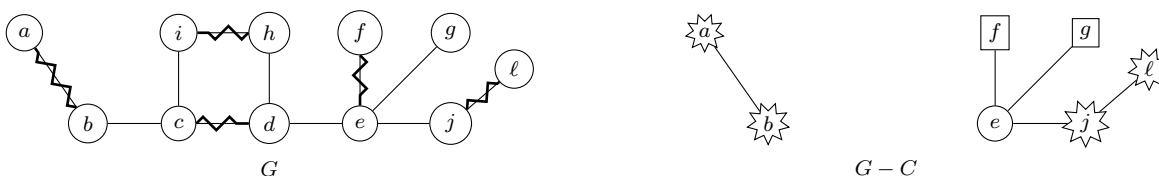


Figure 1: Unicyclic graph  $G$  and its subtrees  $G - C$ .

Gallai-Edmonds decomposition was independently introduced by Gallai [10, 11] and Edmonds [8]. This classical decomposition has several applications [7, 16] and it has been studied by many mathematicians [1, 4, 6, 7, 9, 17, 20] over the years. Many tools for matching theory have been developed from this decomposition. For instance, the well-known Gallai-Edmonds structure theorem ([18], Chapter 3) which states the following:

- The subgraph induced by  $S(G)$  has a perfect matching.
- $\nu(G) = \frac{1}{2} (|V| - c(EG(G)) + |R(G)|)$ , where  $c(EG(G))$  denotes the number of components of the graph induced by  $EG(G)$ .

Cymer [7] shows that the Gallai-Edmonds decomposition can be applied to the pruning methods of constraint programming. The authors of [19, 21] used, respectively, the notions of path-matching and matching cover to obtain more general versions of the Gallai-Edmonds structure theorem for graphs and hypergraphs. For any graph, the Gallai-Edmonds decomposition can be obtained in polynomial time via the Edmonds Matching Algorithm (see [18], Chapters

3 and 9). Unicyclic graphs are important because there are many molecules that can be modeled as a unicyclic graph [12, 22, 24] and, using the spectral graph theory, it is possible to discuss and predict some behaviors of these molecules [5]. Moreover, a chemical application of null space of a molecular graph  $G$  is related to its dimension, because if nullity of  $G$  is greater than 0, then the corresponding chemical compound is highly reactive and unstable, or nonexistent [13].

In this paper, we compute the Gallai-Edmonds decomposition of a unicyclic graph  $G$  through linear algebra. More precisely, we use the null decomposition of its subtrees to obtain  $EG(G)$ ,  $R(G)$  and  $S(G)$ .

The outline of the paper is as follows. In Section 2, we recall some basic definitions and preliminary results. In Sections 3 and 4, we obtain our main results. More precisely, we give a relationship between the Gallai-Edmonds decomposition of a unicyclic graph and the null decomposition of its subtrees. This relationship gives a nice way to compute the Gallai-Edmonds decomposition of unicyclic graphs from the null space of its subtrees, that is, from linear algebra.

## 2 Basic definitions and preliminary results

The support of a graph  $G$  is given by

$$Supp(G) = \{v \in V(G) : \exists x \in \mathcal{N}(G) \text{ such that } x_v \neq 0\},$$

where  $\mathcal{N}(G)$  denotes the null space of  $G$ , that is, the null space of its adjacency matrix. Support gives important information about structure of trees. The next two lemmas show the relationship between the support of trees and their independent sets and matchings.

**Lemma 2.1.** [14] *Let  $T$  be a tree. Then  $Supp(T)$  is an independent set of  $T$ .*

**Lemma 2.2.** [2] *Let  $T$  be a tree. Then  $EG(T) = Supp(T)$ .*

The core of  $G$ , denoted by  $Core(G)$ , is defined by:

$$Core(G) = \bigcup_{v \in Supp(G)} N(v).$$

The set of  $N$ -vertices of  $G$ , denoted by  $V(\mathcal{G}_N(G))$ , is given by:

$$V(\mathcal{G}_N(G)) = V(G) - (Supp(G) \cup Core(G)).$$

For a tree  $T$ , we have that  $Core(T) = R(T)$  and  $V(\mathcal{G}_N(T)) = S(T)$ , see [2, 14]. Null decomposition of  $G$  is a pair of induced subgraphs of  $G$ . The first subgraph is induced by  $Supp(G)$  and  $Core(G)$  and the second one is induced by  $V(\mathcal{G}_N(G))$ . Lemma 2.3 gives a nice way to compute the independence and matching numbers of trees using its null decompositions.

**Lemma 2.3.** [14] *Let  $T$  be a tree. Then*

$$\begin{aligned}\nu(T) &= |Core(T)| + \frac{|V(\mathcal{G}_N(T))|}{2} \\ \alpha(T) &= |Supp(T)| + \frac{|V(\mathcal{G}_N(T))|}{2}.\end{aligned}$$

Let  $G$  be a unicyclic graph and let  $C$  be the unique cycle of  $G$ . For each vertex  $v \in V(C)$ , we denote by  $G\{v\}$  the induced connected subgraph of  $G$  with maximum number of vertices, which contains the vertex  $v$  and no other vertex of  $C$ .  $G\{v\}$  is called the pendant tree of  $G$  at  $v$  (see Figure 3). The unicyclic graph  $G$  is said to be of Type *I* if there exists at least one pendant tree  $G\{v\}$  such that  $v \notin Supp(G\{v\})$ , otherwise,  $G$  is said to be of Type *II* (for more details see [2]). Notice that the set of unicyclic graphs can be divided into two classes, the set of unicyclic graphs of Type *I* and the set of unicyclic graphs of Type *II*.

Lemmas 2.4 and 2.5 give a nice way to compute the matching number of unicyclic graphs using the null decomposition of its subtrees.

**Lemma 2.4.** [2] *If  $G$  is a unicyclic graph of Type *I* and  $G\{v\}$  a pendant tree such that  $v \notin Supp(G\{v\})$ , then*

$$\begin{aligned}\nu(G) &= |Core(G\{v\})| + |Core(G - G\{v\})| + \frac{|V(\mathcal{G}_N(G\{v\}))| + |V(\mathcal{G}_N(G - G\{v\}))|}{2} \\ &= \nu(G\{v\}) + \nu(G - G\{v\}).\end{aligned}$$

**Lemma 2.5.** [2] *Let  $G$  be a unicyclic graph and  $C$  its cycle. Let  $G - C = \bigcup_{i=1}^k T_i$ , where  $T_i$  is a connected component of  $G - C$ . If  $G$  is a unicyclic graph of Type *II*, then*

$$\begin{aligned}\nu(G) &= \left\lfloor \frac{|V(C)|}{2} \right\rfloor + \sum_{i=1}^k |Core(T_i)| + \frac{|V(\mathcal{G}_N(T_i))|}{2} \\ &= \nu(C) + \sum_{i=1}^k \nu(T_i).\end{aligned}$$

In the next sections we will study the Gallai-Edmonds decomposition of unicyclic graphs of Type *I* and *II*, respectively.

### 3 Gallai-Edmonds decomposition of unicyclic graphs of Type I

In this section, we obtain a relationship between the Gallai-Edmonds decomposition of a unicyclic graph  $G$  of Type *I* and the null decompositions of  $G\{v\}$  and  $G - G\{v\}$ , where  $v \notin Supp(G\{v\})$ .

The next technical lemmas and remarks will be used to prove our main result in this section (Theorem 3.9).

**Remark 3.1.** Let  $G$  be a unicyclic graph and  $G\{v\}$  a pendant tree. Let  $u, w \in N(v) \cap V(G - G\{v\})$ . Notice that  $E(G\{v\})$ ,  $E(G - G\{v\})$  and  $\{\{u, v\}, \{w, v\}\}$  form a partition of  $E(G)$ , thus,  $E(G) = E(G\{v\}) \cup E(G - G\{v\}) \cup \{\{u, v\}, \{w, v\}\}$ . Hence, given  $M$  a matching in  $G$  we have that

$$M = (M \cap E(G\{v\})) \cup (M \cap E(G - G\{v\})) \cup (M \cap \{\{u, v\}, \{w, v\}\}).$$

**Remark 3.2.** Note that for all graph  $G$  and  $v \in V(G)$  we have that

$$\nu(G) - 1 \leq \nu(G - v) \leq \nu(G).$$

**Lemma 3.3.** Let  $T$  be a tree and  $v \in V(T)$ . Then  $v \in \text{Supp}(T)$  if and only if  $\nu(T - v) = \nu(T)$ .

*Proof.* Suppose that  $v \in \text{Supp}(T)$ , then by Lemma 2.2 there is  $M \in \mathcal{M}(T)$  such that  $M$  does not saturate  $v$ . Thus,  $M$  is a matching in  $T - v$ . Then  $\nu(T) = |M| \leq \nu(T - v)$ . Given  $M \in \mathcal{M}(T - v)$ . Note that  $M$  is a matching in  $T$ , because  $T - v \subseteq T$ . Thus,  $\nu(T - v) = |M| \leq \nu(T)$ . Therefore,  $\nu(T) = \nu(T - v)$ .

Conversely, suppose that  $\nu(T) = \nu(T - v)$ . Given  $M \in \mathcal{M}(T - v)$ . Notice that  $M$  is a matching in  $T$ , because  $T - v \subseteq T$ . Since  $\nu(T) = \nu(T - v)$ , we have that  $M \in \mathcal{M}(T)$ . Moreover,  $M$  does not saturate  $v$ . Hence, by Lemma 2.2 we conclude that  $v \in \text{Supp}(T)$ .  $\square$

**Lemma 3.4.** Let  $G$  be a unicyclic graph of Type I and  $G\{v\}$  a pendant tree such that  $v \notin \text{Supp}(G\{v\})$ . If  $M \in \mathcal{M}(G)$ , then  $M \cap E(G - G\{v\}) \in \mathcal{M}(G - G\{v\})$ .

*Proof.* Suppose there is  $M \in \mathcal{M}(G)$  such that  $M \cap E(G - G\{v\}) \notin \mathcal{M}(G - G\{v\})$ . Thus, we have that

$$|M \cap E(G - G\{v\})| \leq \nu(G - G\{v\}) - 1.$$

Let  $u, w \in N(v) \cap V(G - G\{v\})$ .

**Case 1:**  $\{u, v\} \notin M$  and  $\{w, v\} \notin M$

Since  $|M \cap E(G - G\{v\})| \leq \nu(G - G\{v\}) - 1$ , we have that

$$\begin{aligned} \nu(G) &= |M| \\ &= |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| \\ &\leq \nu(G\{v\}) + \nu(G - G\{v\}) - 1 \\ &< \nu(G\{v\}) + \nu(G - G\{v\}), \end{aligned}$$

which is a contradiction, because by Lemma 2.4, we have that  $\nu(G) = \nu(G\{v\}) + \nu(G - G\{v\})$ .

**Case 2:**  $\{u, v\} \in M$  or  $\{w, v\} \in M$

Note that in this case  $M \cap E(G\{v\})$  does not saturate  $v$ , because  $v$  is saturate by  $\{u, v\}$  or by  $\{w, v\}$ . Since  $v \notin \text{Supp}(G\{v\})$ , by Lemma 2.2 we conclude that  $M \cap E(G\{v\}) \notin \mathcal{M}(G\{v\})$ . Thus,  $|M \cap E(G\{v\})| \leq \nu(G\{v\}) - 1$ .

Hence,

$$\begin{aligned}
 \nu(G) &= |M| \\
 &= |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| + |M \cap \{\{u, v\}, \{w, v\}\}| \\
 &= |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| + 1 \\
 &\leq \nu(G\{v\}) - 1 + \nu(G - G\{v\}) - 1 + 1 \\
 &\leq \nu(G\{v\}) + \nu(G - G\{v\}) - 1 \\
 &< \nu(G\{v\}) + \nu(G - G\{v\}),
 \end{aligned}$$

which is a contradiction, because by Lemma 2.4, we have that  $\nu(G) = \nu(G\{v\}) + \nu(G - G\{v\})$ .  $\square$

**Lemma 3.5.** *Let  $G$  be a unicyclic graph of Type I and  $G\{v\}$  a pendant tree such that  $v \notin \text{Supp}(G\{v\})$ . Let  $u, w \in N(v) \cap V(G - G\{v\})$  and  $M \in \mathcal{M}(G)$ . If  $\{u, v\} \notin M$  and  $\{w, v\} \notin M$ , then  $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$ , otherwise,  $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\} - v)$ .*

*Proof. Case 1:*  $\{u, v\} \notin M$  and  $\{w, v\} \notin M$

Using Remark 3.1 and Lemma 2.4, we have that

$$\nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G) = |M| = |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})|.$$

Since, by Lemma 3.4,

$$|M \cap E(G - G\{v\})| = \nu(G - G\{v\}),$$

we have that

$$\nu(G\{v\}) = |M \cap E(G\{v\})|.$$

Hence,  $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$ .

**Case 2:**  $\{w, v\} \in M$  or  $\{u, v\} \in M$

Note that  $\{w, v\}$  and  $\{u, v\}$  can not belong simultaneously to  $M$ , because both are incident in  $v$ . Moreover,  $M \cap E(G\{v\})$  does not saturate  $v$ , because  $\{w, v\}$  or  $\{u, v\}$  is incident in  $v$ . Thus,  $M \cap E(G\{v\})$  is a matching in  $G\{v\} - v$ . Using Remark 3.1 and Lemma 2.4, we have that

$$\nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G) = |M| = |M \cap E(G\{v\})| + |M \cap E(G - G\{v\})| + 1.$$

Since, by Lemma 3.4,

$$|M \cap E(G - G\{v\})| = \nu(G - G\{v\}),$$

we have that

$$|M \cap E(G\{v\})| = \nu(G\{v\}) - 1.$$

Therefore, using Lemma 3.3 and Remark 3.2 we conclude that  $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\} - v)$ .  $\square$

**Lemma 3.6.** *Let  $G$  be a unicyclic graph of Type I,  $G\{v\}$  a pendant tree such that  $v \notin \text{Supp}(G\{v\})$ ,  $u \in N(v) \cap V(G - G\{v\})$  and  $M \in \mathcal{M}(G)$ . If  $u \notin \text{Supp}(G - G\{v\})$ , then  $\{u, v\} \notin M$ .*

*Proof.* Suppose there is  $M \in \mathcal{M}(G)$  such that  $\{u, v\} \in M$ . Note that  $M \cap E(G - G\{v\})$  is matching in  $G - G\{v\}$ . Moreover,  $M \cap E(G - G\{v\})$  does not saturate  $u$ . Since  $u \notin \text{Supp}(G - G\{v\})$ , by Lemma 2.2 we conclude that  $M \cap E(G - G\{v\}) \notin \mathcal{M}(G - G\{v\})$ . Thus, using Lemma 3.4 we conclude that  $M \notin \mathcal{M}(G)$ , which is a contradiction.  $\square$

**Proposition 3.7.** *Let  $G$  be a unicyclic graph of Type I and  $G\{v\}$  a pendant tree such that  $v \notin \text{Supp}(G\{v\})$ . Let  $u, w \in N(v) \cap V(G - G\{v\})$  such that  $u, w \notin \text{Supp}(G - G\{v\})$ . Then  $M \in \mathcal{M}(G)$  if and only if  $M = M_1 \cup M_2$ , where  $M_1 \in \mathcal{M}(G\{v\})$  and  $M_2 \in \mathcal{M}(G - G\{v\})$ .*

*Proof.* Given  $M \in \mathcal{M}(G)$ . Using Lemma 3.6 we conclude that  $\{u, v\} \notin M$  and  $\{w, v\} \notin M$ . Moreover, using Remark 3.1, we have that  $M = (M \cap E(G\{v\})) \cup (M \cap E(G - G\{v\}))$ . By Lemmas 3.5 and 3.4, we have that  $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$  and  $M \cap E(G - G\{v\}) \in \mathcal{M}(G - G\{v\})$ .

Conversely, given  $M = M_1 \cup M_2$ , where  $M_1 \in \mathcal{M}(G\{v\})$  and  $M_2 \in \mathcal{M}(G - G\{v\})$ . Thus, we have that  $|M| = |M_1| + |M_2| = \nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G)$ . Hence,  $M \in \mathcal{M}(G)$ .  $\square$

**Lemma 3.8.** [3] *Let  $G$  be a unicyclic graph of Type I and  $G\{v\}$  a pendant tree such that  $v \notin \text{Supp}(G\{v\})$ . Then  $\text{Supp}(G\{v\}) \subseteq \text{Supp}(G\{v\} - v)$ .*

Now, we are able to present the relationship between the Gallai-Edmonds decomposition of a unicyclic graph of Type I and the null decomposition of its subtrees. Theorem 3.9 gives a way to obtain the Gallai-Edmonds decomposition of a unicyclic graph from the null decompositions of  $G\{v\}$ ,  $G - G\{v\}$  and  $G\{v\} - v$ , where  $v \notin \text{Supp}(G\{v\})$ .

**Theorem 3.9.** *Let  $G$  be a unicyclic graph of Type I,  $G\{v\}$  a pendant tree such that  $v \notin \text{Supp}(G\{v\})$  and  $u, w \in N(v) \cap V(G - G\{v\})$ . If  $u, w \notin \text{Supp}(G - G\{v\})$ , then*

$$(i) \quad EG(G) = \text{Supp}(G\{v\}) \cup \text{Supp}(G - G\{v\})$$

$$(ii) \quad R(G) = \text{Core}(G\{v\}) \cup \text{Core}(G - G\{v\})$$

$$(iii) \quad S(G) = V(\mathcal{G}_N(G\{v\})) \cup V(\mathcal{G}_N(G - G\{v\}))$$

otherwise,

$$(i) \quad EG(G) = \text{Supp}(G\{v\} - v) \cup \text{Supp}(G - G\{v\})$$

$$(ii) \quad R(G) = \{v\} \cup \text{Core}(G\{v\} - v) \cup \text{Core}(G - G\{v\})$$

$$(iii) \quad S(G) = V(\mathcal{G}_N(G\{v\} - v)) \cup V(\mathcal{G}_N(G - G\{v\})).$$

*Proof. Case 1:*  $u, w \notin \text{Supp}(G - G\{v\})$

- (i) Given  $x \in EG(G)$ . Thus, there is  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . By Proposition 3.7 we conclude that  $M = M_1 \cup M_2$ , where  $M_1 \in \mathcal{M}(G\{v\})$  and  $M_2 \in \mathcal{M}(G - G\{v\})$ . Note that  $M_1$  and  $M_2$  do not saturate  $x$ . Therefore, if  $x \in V(G\{v\})$ , then by Lemma 2.2 we conclude that  $x \in \text{Supp}(G\{v\})$ . Similarly, if  $x \in V(G - G\{v\})$ , then we conclude that  $x \in \text{Supp}(G - G\{v\})$ .

Now, given  $x \in \text{Supp}(G\{v\}) \cup \text{Supp}(G - G\{v\})$ . We will obtain  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . If  $x \in \text{Supp}(G\{v\})$ , then by Lemma 2.2 there is  $M_1 \in \mathcal{M}(G\{v\})$  such that  $M_1$  does not saturate  $x$ . Consider  $M_2 \in \mathcal{M}(G - G\{v\})$ . Define  $M = M_1 \cup M_2$ . We have that  $|M| = |M_1| + |M_2| = \nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G)$ . Hence,  $M \in \mathcal{M}(G)$  and  $M$  does not saturate  $x$ , that is,  $x \in EG(G)$ . Similarly, if  $x \in \text{Supp}(G - G\{v\})$ , then we define  $M = M_1 \cup M_2$ , where  $M_2 \in \mathcal{M}(G - G\{v\})$  such that  $M_2$  does not saturate  $x$  and  $M_1 \in \mathcal{M}(G\{v\})$ . We conclude that  $M \in \mathcal{M}(G)$  and  $M$  does not saturate  $x$ , thus,  $x \in EG(G)$ .

- (ii) Notice that

$$V(G - G\{v\}) \cap N(V(G\{v\})) = \{u, w\} \text{ and } V(G\{v\}) \cap N(V(G - G\{v\})) = \{v\}.$$

Since  $u, w \notin \text{Supp}(G - G\{v\})$ ,  $N(\text{Supp}(G\{v\})) \cap V(G - G\{v\}) = \emptyset$  and  $N(\text{Supp}(G - G\{v\})) \cap V(G\{v\}) = \emptyset$ . Moreover, note that

$$\begin{aligned} N(EG(G)) &= N(\text{Supp}(G\{v\}) \cup \text{Supp}(G - G\{v\})) \\ &= \text{Core}(G\{v\}) \cup \text{Core}(G - G\{v\}). \end{aligned}$$

By Lemma 2.1, we have that  $\text{Supp}(G\{v\}) \cup \text{Supp}(G - G\{v\})$  is an independent set, thus,

$$\begin{aligned} (\text{Supp}(G\{v\}) \cup \text{Supp}(G - G\{v\})) \cap (\text{Core}(G\{v\}) \cup \text{Core}(G - G\{v\})) &= \emptyset \\ EG(G) \cap N(EG(G)) &= \emptyset. \end{aligned}$$

Therefore,  $N(EG(G)) - EG(G) = R(G) = \text{Core}(G\{v\}) \cup \text{Core}(G - G\{v\})$ .

- (iii) Just use the items (i) and (ii).

**Case 2:**  $u \in \text{Supp}(G - G\{v\})$  or  $w \in \text{Supp}(G - G\{v\})$

- (i) Given  $x \in EG(G)$ . Thus, there is  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . Note that in this case we can have  $\{u, v\} \in M$  or  $\{w, v\} \in M$ . If  $\{u, v\} \in M$  or  $\{w, v\} \in M$ , then by Lemma 3.5  $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\} - v)$ . Moreover, by Lemma 3.4, we have that  $M \cap E(G - G\{v\}) \in \mathcal{M}(G - G\{v\})$ . Notice that  $M \cap E(G\{v\})$  and  $M \cap E(G - G\{v\})$  do not saturate  $x$ . Thus, if  $x \in V(G\{v\} - v)$ , then using Lemma 2.2 we conclude that  $x \in \text{Supp}(G\{v\} - v)$ . Similarly, if  $x \in V(G - G\{v\})$ , then we conclude that  $x \in \text{Supp}(G - G\{v\})$ .



If  $\{u, v\} \notin M$  and  $\{w, v\} \notin M$ , then by same argument as in item (i) of the Case 1 we have that  $x \in \text{Supp}(G\{v\}) \cup \text{Supp}(G - G\{v\})$ . Thus, using Lemma 3.8, we have that  $x \in \text{Supp}(G\{v\} - v) \cup \text{Supp}(G - G\{v\})$ .

Now, given  $x \in \text{Supp}(G\{v\} - v) \cup \text{Supp}(G - G\{v\})$ . We will obtain  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . If  $x \in \text{Supp}(G\{v\} - v)$  and  $u \in \text{Supp}(G - G\{v\})$ , then by Lemma 2.2 there are  $M_1 \in \mathcal{M}(G\{v\} - v)$  and  $M_2 \in \mathcal{M}(G - G\{v\})$  such that  $M_1$  and  $M_2$  do not saturate  $x$  and  $u$ , respectively. We define  $M = M_1 \cup M_2 \cup \{u, v\}$ . We have that  $|M| = |M_1| + |M_2| + 1 = \nu(G\{v\} - v) + \nu(G - G\{v\}) + 1 = \nu(G\{v\}) + \nu(G - G\{v\}) = \nu(G)$ . Therefore,  $M \in \mathcal{M}(G)$  and  $M$  does not saturate  $x$ . Similarly, if  $x \in \text{Supp}(G\{v\} - v)$  and  $w \in \text{Supp}(G - G\{v\})$ , then we define  $M = M_1 \cup M_2 \cup \{w, v\}$ , where  $M_1 \in \mathcal{M}(G\{v\} - v)$  and  $M_2 \in \mathcal{M}(G - G\{v\})$  and  $M_1$  and  $M_2$  do not saturate  $x$  and  $w$ , respectively. Hence,  $M \in \mathcal{M}(G)$  and  $M$  does not saturate  $x$ . If  $x \in \text{Supp}(G - G\{v\})$ , then we define  $M = M_1 \cup M_2$ , where  $M_1 \in \mathcal{M}(G\{v\})$  and  $M_2 \in \mathcal{M}(G - G\{v\})$  and  $M_2$  does not saturate  $x$ . Therefore,  $M \in \mathcal{M}(G)$  and  $M$  does not saturate  $x$ .

(ii) Note that

$$\begin{aligned} N(EG(G)) &= N(\text{Supp}(G\{v\} - v) \cup \text{Supp}(G - G\{v\})) \\ &= \{v\} \cup \text{Core}(G\{v\} - v) \cup \text{Core}(G - G\{v\}). \end{aligned}$$

Moreover  $N(EG(G)) \cap EG(G) = \emptyset$ . Therefore,  $N(EG(G)) - EG(G) = R(G) = \{v\} \cup \text{Core}(G\{v\} - v) \cup \text{Core}(G - G\{v\})$ .

(iii) Just use the items (i) and (ii).

□

In the following example, we use the Theorem 3.9 to obtain the Gallai-Edmonds decomposition of the unicyclic graph  $G$  in Figure 2. Analyzing the entries of the vectors of the basis of the  $\mathcal{N}(G\{v\})$  and  $\mathcal{N}(G - G\{v\})$  we obtain that  $\text{Supp}(G\{v\}) = \{z, x, k\}$ ,  $\text{Core}(G\{v\}) = \{v\}$ ,  $V(\mathcal{G}_N(G\{v\})) = \{c, y\}$ ,  $\text{Supp}(G - G\{v\}) = \{e, d, i, g, h, m, n, o, p, s, t\}$ ,  $\text{Core}(G - G\{v\}) = \{a, j, l, q, r\}$  and  $V(\mathcal{G}_N(G - G\{v\})) = \{u, f, w, b\}$ .

Note that  $G$  is a unicyclic graph of Type I, because  $v \notin \text{Supp}(G\{v\})$ . Moreover,  $u, w \notin \text{Supp}(G - G\{v\})$ . Therefore, by Theorem 3.9 the Gallai-Edmonds decomposition of  $G$  is given by:

$$\begin{aligned} EG(G) &= \text{Supp}(G\{v\}) \cup \text{Supp}(G - G\{v\}) = \{z, x, k, e, d, i, g, h, m, n, o, p, s, t\}, \\ R(G) &= \text{Core}(G\{v\}) \cup \text{Core}(G - G\{v\}) = \{a, j, l, q, r, v\}, \\ S(G) &= V(\mathcal{G}_N(G\{v\})) \cup V(\mathcal{G}_N(G - G\{v\})) = \{u, f, w, b, c, y\}. \end{aligned}$$

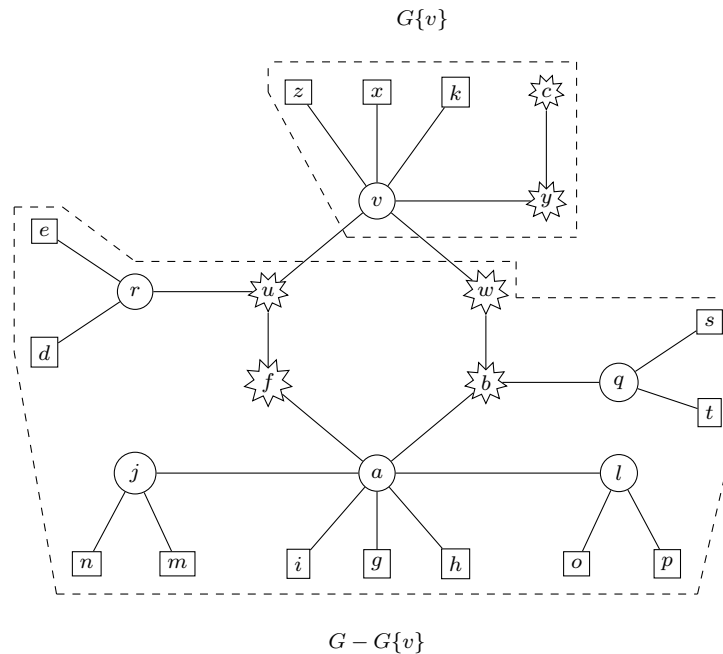


Figure 2: Unicyclic graph of Type I and its subtrees  $G\{v\}$  and  $G - G\{v\}$ .

## 4 Gallai-Edmonds decomposition of unicyclic graphs of Type II

In this section, we obtain a relationship between the Gallai-Edmonds decomposition of a unicyclic graph  $G$  of Type II and the null decompositions of  $G - C$  or  $G\{v\}$ , where  $C$  is the unique cycle of  $G$  and  $v \in V(C)$ .

Next, we will define the set of intermediate edges. We study under what circumstance these edges are in a maximum matching of  $G$  (see Lemma 4.4).

**Definition 4.1.** Let  $G$  be a unicyclic graph and  $C$  its cycle. The set of intermediate edges, denoted by  $\mathcal{IE}(G)$ , is defined as  $\mathcal{IE}(G) = E(G) - (E(C) \cup E(G - C))$ .

The following lemmas and remarks will be crucial to prove our main results in this section (Theorems 4.6 and 4.8).

**Remark 4.2.** Let  $G$  be a unicyclic graph and  $C$  its cycle. Notice that  $E(C)$ ,  $E(G - C)$  and  $\mathcal{IE}(G)$  form a partition of  $E(G)$ , thus,  $E(G) = E(C) \cup E(G - C) \cup \mathcal{IE}(G)$ . Therefore, given a matching  $M$  in  $G$  we have that

$$\begin{aligned} M &= (M \cap E(C)) \cup (M \cap E(G - C)) \cup (M \cap \mathcal{IE}(G)) \\ &= (M \cap E(C)) \cup \bigcup_{v \in V(C)} M \cap E(G\{v\}). \end{aligned}$$

**Lemma 4.3.** Let  $G$  be a unicyclic graph of Type II and  $C$  its cycle. If  $M \in \mathcal{M}(G)$ , then  $M \cap E(C) \in \mathcal{M}(C)$  and  $M \cap E(G\{v\}) \in \mathcal{M}(G\{v\})$  for all  $v \in V(C)$ .

*Proof.* Suppose there is  $M \in \mathcal{M}(G)$  such that  $M \cap E(C) \notin \mathcal{M}(C)$  or  $M \cap E(G\{w\}) \notin \mathcal{M}(G\{w\})$  for some  $w \in V(C)$ . That is,  $|M \cap E(C)| \leq \nu(C) - 1$  or  $|M \cap E(G\{w\})| \leq \nu(G\{w\}) - 1$ . Note that given  $u \in V(C)$  we have that  $u \in \text{Supp}(G\{u\})$ . Then by Lemma 3.3,  $\nu(G\{u\}) = \nu(G\{u\} - u)$ . Consider  $G - C = \bigcup_{i=1}^k T_i$ , where  $T_i$  is a connected component of  $G - C$ . Thus, we have that

$$\begin{aligned} \nu(G) &= |M| \\ &= |M \cap E(C)| + \sum_{u \in V(C)} |M \cap E(G\{u\})| \quad (\text{Remark 4.2}) \\ &= |M \cap E(C)| + |M \cap E(G\{w\})| + \sum_{u \in V(C) - \{w\}} |M \cap E(G\{u\})| \\ &\leq +\nu(C) + \nu(G\{w\}) - 1 + \sum_{u \in V(C) - \{w\}} \nu(G\{u\}) \\ &= -1 + \nu(C) + \sum_{u \in V(C)} \nu(G\{u\}) \\ &= -1 + \nu(C) + \sum_{u \in V(C)} \nu(G\{u\} - u) = -1 + \nu(C) + \sum_{i=1}^k \nu(T_i) \\ &< \nu(C) + \sum_{i=1}^k \nu(T_i), \end{aligned}$$

which is a contradiction, because by Lemma 2.5  $\nu(G) = \nu(C) + \sum_{i=1}^k \nu(T_i)$ . □

**Lemma 4.4.** *Let  $G$  be a unicyclic graph of Type II,  $C$  its cycle and  $M \in \mathcal{M}(G)$ . Let  $e \in \mathcal{IE}(G)$ . If  $|V(C)|$  is even, then  $e \notin M$ .*

*Proof.* Let  $e \in \mathcal{IE}(G)$ . Suppose there is  $M \in \mathcal{M}(G)$  such that  $e \in M$ . Consider  $G - C = \bigcup_{i=1}^k T_i$ , where  $T_i$  is a connected component of  $G - C$ . Define  $e = \{u, v\}$  such that  $u \in V(T_j)$  for some  $j \in \{1, 2, \dots, k\}$  and  $v \in V(C)$ . Note that  $M \cap E(C)$  is a matching in  $C$ . Moreover,  $M \cap E(C)$  do not saturate  $v$ , because  $v$  is saturate by  $e$ . As  $|V(C)|$  is even and  $\nu(C) = \frac{|V(C)|}{2}$ , all maximum matchings in  $C$  are perfect matching. Therefore,  $M \cap E(C) \notin \mathcal{M}(C)$ . Hence, using Lemma 4.3, we have that  $M \notin \mathcal{M}(G)$ , which is a contradiction. □

**Lemma 4.5.** [2] *Let  $G$  be a unicyclic graph and  $C$  its cycle. Let  $G\{v\}$  be a pendant tree such that  $v \in \text{Supp}(G\{v\})$ . If  $u \in N(v) \cap V(G\{v\})$ , then  $u \notin \text{Supp}(G - C)$ .*

We are now ready to prove our main results in this section. Theorems 4.6 and 4.8 provide a way to obtain the Gallai-Edmonds decomposition of a unicyclic graph of Type II from the null decompositions of  $G - C$  and  $G\{v\}$ , respectively.

**Theorem 4.6.** *Let  $G$  be a unicyclic graph of Type II and  $C$  its cycle. If  $|V(C)|$  is even, then*

(i)  $EG(G) = Supp(G - C)$

(ii)  $R(G) = Core(G - C)$

(iii)  $S(G) = V(\mathcal{G}_N(G - C)) \cup V(C)$ .

*Proof.* (i) Consider  $G - C = \bigcup_{i=1}^k T_i$ , where  $T_i$  is a connected component of  $G - C$ . Given  $x \in EG(G)$ . Thus, there is  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . Since  $|V(C)|$  is even, using Remark 4.2 and Lemma 4.4 we conclude that

$$M = (M \cap E(C)) \cup \left( \bigcup_{i=1}^k (M \cap E(T_i)) \right).$$

Note that by Lemma 4.3, we have that  $M \cap E(C) \in \mathcal{M}(C)$ . Moreover, we have that  $M \cap E(T_i) \in \mathcal{M}(T_i)$ , otherwise

$$\nu(G) = |M| < \nu(C) + \sum_{i=1}^k \nu(T_i),$$

which is a contradiction by Lemma 2.5. As  $|V(C)|$  is even and  $\nu(C) = \frac{|V(C)|}{2}$ , all maximum matchings in  $C$  are perfect matching. Hence,  $x \notin V(C)$ , otherwise would be saturated by  $M \cap E(C)$  and consequently saturated by  $M$ . That is,  $x \in V(T_s)$  for some  $s \in \{1, \dots, k\}$ . Since  $M \cap E(T_s) \in \mathcal{M}(T_s)$ , by Lemma 2.2  $x \in Supp(T_s)$ , that is,  $x \in Supp(G - C) = \bigcup_{i=1}^k Supp(T_i)$ .

Now, given  $x \in Supp(G - C)$ , we will obtain  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . Consider  $G - C = \bigcup_{i=1}^k T_i$ , where  $T_i$  is a connected component of  $G - C$ . Note that

$$Supp(G - C) = \bigcup_{i=1}^k Supp(T_i). \text{ Thus, there is } s \in \{1, \dots, k\} \text{ such that } x \in Supp(T_s).$$

Since  $x \in Supp(T_s)$ , by Lemma 2.2 there is  $M_s \in \mathcal{M}(T_s)$  such that  $M_s$  does not saturate  $x$ . Let  $M_i \in \mathcal{M}(T_i)$  with  $i \in \{1, \dots, k\} - \{s\}$  and  $M_c \in \mathcal{M}(C)$ . Define  $M = M_c \cup \left( \bigcup_{i=1}^k M_i \right)$ . Note that  $M$  is a matching in  $G$  and  $M$  does not saturate  $x$ . Moreover, by Lemma 2.5 we conclude that  $M \in \mathcal{M}(G)$ .

(ii) Using Lemma 4.5 we conclude that  $N(Supp(G - C)) \cap V(C) = \emptyset$ . Thus,  $N(EG(G)) = N(Supp(G - C)) = Core(G - C)$ . Moreover, by Lemma 2.1, we have that  $N(EG(G)) \cap EG(G) = N(Supp(G - C)) \cap Supp(G - C) = \emptyset$ . Therefore,

$$\begin{aligned}
 R(G) &= N(EG(G)) - EG(G) \\
 &= N(\text{Supp}(G - C)) - \text{Supp}(G - C) \\
 &= \text{Core}(G - C).
 \end{aligned}$$

(iii) Just use the items (i) and (ii). □

**Lemma 4.7.** *Let  $G$  be a unicyclic graph of Type II and  $G\{v\}$  a pendant tree. Let  $u \in V(G)$  and  $M \in \mathcal{M}(G)$ . If  $u \in \text{Core}(G\{v\}) \cup V(\mathcal{G}_N(G\{v\}))$ , then  $M$  saturates  $u$ .*

*Proof.* Let  $C$  be the cycle of  $G$ . Suppose there is  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $u$ . Note that  $M \cap E(G\{v\})$  is a matching of  $G\{v\}$  and does not saturate  $u$ , thus, by Lemma 2.2 we conclude that  $M \cap E(G\{v\}) \notin \mathcal{M}(G\{v\})$ . Hence, using Lemma 4.3 we conclude that  $M \notin \mathcal{M}(G)$ , which is a contradiction. □

**Theorem 4.8.** *Let  $G$  be a unicyclic graph of Type II and  $C$  its cycle. If  $|V(C)|$  is odd, then*

$$(i) \quad EG(G) = \bigcup_{v \in V(C)} \text{Supp}(G\{v\})$$

$$(ii) \quad R(G) = \bigcup_{v \in V(C)} \text{Core}(G\{v\})$$

$$(iii) \quad S(G) = \bigcup_{v \in V(C)} V(\mathcal{G}_N(G\{v\})).$$

*Proof.* (i) Given  $x \in EG(G)$ . Thus, there is  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . Then by Lemma 4.7 we conclude that  $x \in \text{Supp}(G\{v\})$  for some  $v \in V(C)$ .

Given  $x \in \text{Supp}(G\{v\})$  for some  $v \in V(C)$ . We will obtain  $M \in \mathcal{M}(G)$  such that  $M$  does not saturate  $x$ . Suppose first  $x = v$ . Since  $|V(C)|$  is odd, there is  $M_c \in \mathcal{M}(C)$  such that  $M_c$  does not saturate  $v$ . Let  $M_w \in \mathcal{M}(G\{w\})$  such that  $M_w$  does not saturate  $w$  for all  $w \in V(C)$  (notice that  $M_w$  exists for Lemma 2.2). Define  $M = M_c \cup \left( \bigcup_{w \in V(C)} M_w \right)$ .

Consider  $G - C = \bigcup_{i=1}^k T_i$ , where  $T_i$  is a connected component of  $G - C$ . Note that  $M$  is a matching in  $G$  does not saturate  $v$  and for all  $w \in V(C)$  we have that  $\nu(G\{w\}) = \nu(G\{w\} - w)$  (see Lemma 3.3). Moreover, we have that

$$\begin{aligned}
 |M| &= |M_c| + \sum_{w \in V(C)} |M_w| \\
 &= \nu(C) + \sum_{w \in V(C)} \nu(G\{w\}) \\
 &= \nu(C) + \sum_{w \in V(C)} \nu(G\{w\} - w) \\
 &= \nu(C) + \sum_{i=1}^k \nu(T_i) = \nu(G).
 \end{aligned}$$

Then  $M \in \mathcal{M}(G)$ . Therefore,  $x \in EG(G)$ .

Now, suppose  $x \neq v$ . Let  $M_v \in \mathcal{M}(G\{v\})$  and  $M_c \in \mathcal{M}(C)$  such that  $M_v$  does not saturate  $x$  and  $M_c$  does not saturate  $v$  (notice that  $M_v$  by Lemma 2.2). Consider  $M_u \in \mathcal{M}(G\{u\})$  such that  $M$  does not saturate  $u$  for all  $u \in V(C) - \{v\}$ . Define  $M = M_c \cup \left( \bigcup_{u \in V(C)} M_u \right)$ . Similarly we conclude that  $M \in \mathcal{M}(G)$  and does not saturate  $x$ . Hence,  $x \in EG(G)$ .

(ii) Note that  $N(EG(G)) = V(C) \cup \left( \bigcup_{v \in V(C)} Core(G\{v\}) \right)$  and  $N(EG(G)) \cap EG(G) = V(C)$ . Therefore,

$$\begin{aligned}
 R(G) &= N(EG(G)) - EG(G) \\
 &= N \left( \bigcup_{v \in V(C)} Supp(G\{v\}) \right) - \bigcup_{v \in V(C)} Supp(G\{v\}) \\
 &= \bigcup_{v \in V(C)} Core(G\{v\}).
 \end{aligned}$$

(iii) Just use the items (i) and (ii). □

In this example, we use Theorem 4.8 to obtain the Gallai-Edmonds decomposition of the unicyclic graph  $G$  in Figure 3. Consider  $C$  the cycle of  $G$ . Analyzing the entries of the vectors of the basis of the  $\mathcal{N}(G\{a\})$ ,  $\mathcal{N}(G\{b\})$ ,  $\mathcal{N}(G\{c\})$ ,  $\mathcal{N}(G\{d\})$  and  $\mathcal{N}(G\{e\})$  we conclude that support, core and  $N$ -vertices set of the pendant trees of  $G$  are given in Table 1.

Support	Core	$N$ -vertices
$Supp(G\{a\}) = \{g, h, a, l, t\}$	$Core(G\{a\}) = \{f, m\}$	$V(\mathcal{G}_N(G\{a\})) = \{i, j\}$
$Supp(G\{b\}) = \{b\}$	$Core(G\{b\}) = \emptyset$	$V(\mathcal{G}_N(G\{b\})) = \emptyset$
$Supp(G\{c\}) = \{c, n, p, q\}$	$Core(G\{c\}) = \{o\}$	$V(\mathcal{G}_N(G\{c\})) = \{v, r, s, u\}$
$Supp(G\{d\}) = \{d\}$	$Core(G\{d\}) = \emptyset$	$V(\mathcal{G}_N(G\{d\})) = \emptyset$
$Supp(G\{e\}) = \{e\}$	$Core(G\{e\}) = \emptyset$	$V(\mathcal{G}_N(G\{e\})) = \emptyset$

Table 1: Support, Core and  $N$ -vertices of the pendant trees of  $G$ .

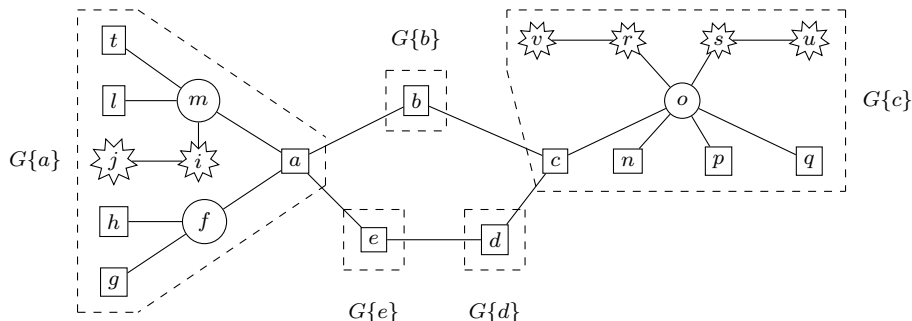


Figure 3: Unicyclic graph of Type II and its pendant trees.

Note that  $G$  is a unicyclic graph of Type II, because  $a \in Supp(G\{a\})$ ,  $b \in Supp(G\{b\})$ ,  $c \in Supp(G\{c\})$ ,  $d \in Supp(G\{d\})$  and  $e \in Supp(G\{e\})$ . Moreover,  $|V(C)|$  is odd. Therefore, by Theorem 4.8 the Gallai-Edmonds decomposition of  $G$  is given by:

$$\begin{aligned}
 EG(G) &= \bigcup_{v \in V(C)} Supp(G\{v\}) = \{a, b, c, d, e, g, h, l, t, n, p, q\}, \\
 R(G) &= \bigcup_{v \in V(C)} Core(G\{v\}) = \{f, m, o\}, \\
 S(G) &= \bigcup_{v \in V(C)} V(\mathcal{G}_N(G\{v\})) = \{v, r, s, u, i, j\}.
 \end{aligned}$$

## 5 Concluding remark

In this paper, we have provided a way to obtain the Gallai-Edmonds decomposition of unicyclic graphs from the null space of its adjacency matrix.

Similarly to the Gallai-Edmonds decomposition, the Zito decomposition [23] is also a partition of the vertex set. The difference is that Zito decomposition is defined through certain properties of the maximum independent sets. In trees and  $C_{4k}$ -free bipartite graphs all three decompositions coincide, that is, form the same partition of the vertex set [2, 14, 15]. For graphs in general, this is not the case (consider, for example, the pan graph of order 6, that is, the graph obtained by joining the cycle graph  $C_6$  to the complete graph  $K_1$  with an

edge).

In view of this fact, the following question naturally arises: Is there a relationship between Zito decomposition of unicyclic graphs and null decomposition of its subtrees?

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