UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
PROGRAMA DE PÓS-GRADUAÇÃO EM ENGENHARIA ELÉTRICA

UNIVERSITÉ PARIS-SACLAY
ECOLE DOCTORALE SCIENCES ET TECHNOLOGIES DE L'INFORMATION ET DE LA COMMUNICATION

LEONARDO BROERING GROFF

EVENT-TRIGGERED CONTROL FOR PIECEWISE AFFINE DISCRETE-TIME SYSTEMS

Porto Alegre
2020

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Thesis presented to Programa de Pós-Graduação em Engenharia Elétrica of Universidade Federal do Rio Grande do Sul and to Ecole Doctorale Sciences et Technologies de l'Information et de la Communication of Université Paris-Saclay in partial fulfillment of the requirements for the double degree of Doctor in Engenharia Elétrica (UFRGS) and Automatique (Université Paris-Saclay).
Area: Controle e Automação

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Porto Alegre
2020

CIP - Catalogação na Publicação

Broering Groff, Leonardo
EVENT-TRIGGERED CONTROL FOR PIECEWISE AFFINE DISCRETE-TIME SYSTEMS / Leonardo Broering Groff. -- 2020.
129 f.
Orientador: João Manoel Gomes da Silva Jr..

Coorientador: Giorgio Valmorbida.

Tese (Doutorado) -- Universidade Federal do Rio Grande do Sul, Escola de Engenharia, Programa de Pós-Graduação em Engenharia Elétrica, Porto Alegre, BR-RS, 2020.

1. Controle baseado em eventos. 2. Controle de sistemas em rede. 3. Sistemas em tempo discreto. I. Gomes da Silva Jr., João Manoel, orient. II. Valmorbida, Giorgio, coorient. III. Título.

Elaborada pelo Sistema de Geração Automática de Ficha Catalográfica da UFRGS com os dados fornecidos pelo(a) autor(a).

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Porto Alegre, September 2020.

DEDICATION

To my late grandparents, Guilherme, Izary, Dorothéia and Ivo, who unfortunately were not be able to see this thesis finished.

Aos meus falecidos avós, Guilherme, Izary, Dorothéia e Ivo, que infelizmente não puderam ver este trabalho concluído.

ACKNOWLEDGMENTS

I would like to thank:

my parents, Marcos and Cecilia, for the patience and support provided, and for understanding the distance and absence during the production of this work;

my good friends Gregory, Marcos, Duda, Dudinha, Luis, Gordo and Bonatto for the good laughs and the important moments of joy whenever I was available;

the friends I made during the doctorate, especially Luciano and Charles;

my advisors, João and Giorgio, for the insurmountable dedication and patience, invaluable insights and careful revisions, without which this work would not have been possible;

and finally, to CAPES for the financial support provided by a PhD Scholarship and the Programa de Doutorado Sanduíche no Exterior.

ABSTRACT

In the present work, we study the problems of stability analysis of piecewise-affine (PWA) discrete-time systems, and trigger-function design for discrete-time event-triggered control systems. We propose a representation for piecewise-affine systems in terms of ramp functions, and we rely on Lyapunov theory for the stability analysis.

The proposed implicit piecewise-affine representation prevents the shortcomings of the existing stability analysis approaches of PWA systems. Namely, the need to enumerate regions and allowed transitions of the explicit representations. In this context, we can emphasize two benefits of the proposed approach: first, it makes possible the analysis of uncertainty in the partition and, thus, the transitions. Secondly, it enables the analysis of event-triggered control systems for the class of PWA systems since, for ETC, the transitions cannot be determined as a function of the state variables. The proposed representation, on the other hand, implicitly encodes the partition and the transitions.

The stability analysis is performed with Lyapunov theory techniques. We then present conditions for exponential stability. Thanks to the implicit representation, the use of piecewise quadratic Lyapunov functions candidates becomes simple. These conditions can be solved numerically using a linear matrix inequality formulation. The numerical analysis exploits quadratic expressions that describe ramp functions to verify the positiveness of extended quadratic forms.

For ETC, a piecewise quadratic trigger function defines the event generator. We find suitable parameters for the trigger function with an optimization procedure. As a result, this function uses the information on the partition to reduce the number of events, achieving better results than the standard quadratic trigger functions found in the literature.

We provide numerical examples to illustrate the application of the proposed representation and methods.

Keywords: Event-triggered control, networked control systems, discrete-time systems.

RÉSUMÉ

Ce manuscrit présente des résultats sur l'analyse de stabilité des systèmes affines par morceaux en temps discret et sur le projet de fonctions de déclenchement pour des stratégies de commande par événements. Nous proposons une représentation pour des systèmes affines par morceaux et l'on utilise la théorie de stabilité de Lyapunov pour effectuer l'analyse de stabilité globale de l'origine.

La nouvelle représentation implicite que nous proposons rend plus simple l'analyse de stabilité car elle évite l'énumération des régions et des transitions entre régions tel que c'est fait dans le cas des représentations explicites. Dans ce contexte nous pouvons souligner deux avantages principaux, à savoir I) la possibilité de traiter des incertitudes dans la partition qui définit le système et, par conséquent des incertitudes dans les transitions, II) l'analyse des stratégies de commande par événements pour des systèmes affines par morceaux. En effet, dans ces stratégies les transitions ne peuvent pas être définies comme des fonctions des variables d'état.

La théorie de stabilité de Lyapunov est utilisée pour établir des conditions pour la stabilité exponentielle de l'origine. Grâce à la représentation implicite des partitions nous utilisons des fonctions de Lyapunov quadratique par morceaux. Ces conditions sont données par des inégalités dont la solution numérique est possible avec une formulation par des inégalités matricielles linéaires. Ces formulations numériques se basent sur des expressions quadratiques décrivant des fonctions rampe.

Pour des stratégies par événement, une fonctions quadratique par morceaux est utilisée pour le générateur d'événements. Nous calculons les paramètres de ces fonctions de déclenchement a partir de solutions de problèmes d'optimisation. Cette fonction de déclenchement quadratique par morceaux permet de réduire le nombre de d'événements en comparaison avec les fonctions quadratiques utilisées dans la littérature.

Nous utilisons des exemples numériques pour illustrer les méthodes proposées.

Keywords: Comande déclenché par événements, systèmes de contrôle en réseau, systèmes à temps discret.

RESUMO

No presente trabalho, são estudados os problemas de análise de estabilidade de sistemas afins por partes e o projeto da função de disparo para sistemas de controle baseado em eventos em tempo discreto. É proposta uma representação para sistemas afins por partes em termos de funções rampa, e é utilizada a teoria de Lyapunov para a análise de estabilidade.

A representação afim por partes implícita proposta evita algumas das deficiências das abordagens de análise de estabilidade de sistemas afins por partes existentes. Em particular, a necessidade de anumerar regiões e transições admissíveis das representações explícitas. Neste contexto, dois benefícios da abordagem proposta podem ser enfatizados: primeiro, ela torna possível a análise de incertezas na partição, e, assim, nas transições. Segundo, ela permite a análise de sistemas de controle baseado em eventos para a classe de sistemas afins por partes, já que, para o controle baseado em eventos, as transições não podem ser determinadas como uma função das variáveis de estado. A representação proposta, por outro lado, codifica implicitamente a partição e as transições.

A análise de estabilidade é realizada com técnicas da teoria de Lyapunov. Condições para a estabilidade exponencial são então apresentadas. Graças à representação implícita, o uso de funções candidatas de Lyapunov se torna simples. Essas condições podem ser resolvidas numéricamente usando uma formulação de desigualdades matriciais lineares. A análise numérica explora expressões quadráticas que descrevem funções de rampa para verificar a postividade de formas quadráticas extendidas.

Para o controle baseado em eventos, uma função de disparo quadrática por partes define o gerador de eventos. Parâmetros adequados para a função de disparo são encontrados com um procedimento de otimização. Como resultado, esta função usa informação da partição para reduzir o número de eventos, obtendo resultados melhores do que as funções de disparo quadráticas encontradas na literatura.

Exemplos numéricos são fornecidos para ilustrar a aplicação da representação e métodos propostos.

Palavras-chave: Controle baseado em eventos, controle de sistemas em rede, sistemas em tempo discreto.

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LIST OF ACRONYMS

ETC	Event Triggered Control
HH	Hinging Hyperplane
LFR	Linear Fractional Representation
LMI	Linear Matrix Inequality
LQ	Linear Quadratic
LR	Lattice Representation
MMPS	Max-min-plus-scaling
MPC	Model Predictive Control
NCS	Networked Control System
OP	Optimization Problem
PWA	Piecewise Affine
PWL	Piecewise Linear
PWQ	Piecewise Quadratic
RHOC	Receding Horizon Optimal Control

LIST OF SYMBOLS

$\sum$	Sum
$0_{n\times m}$	Null matrix with n rows and m columns
0	Null matrix of appropriate dimensions
I_n	Identity matrix of dimension n
$M^\top$	Transpose of matrix M
$M > (\geq)\ 0$	Matrix M is positive (semi-)definite
$M < (\leq)\ 0$	Matrix M is negative (semi-)definite
$M \succcurlyeq (\preccurlyeq) 0$	Matrix M has no elements larger (smaller) than 0
$\mathcal{E}(M)$	Set $\mathcal{E} = \{x : x'Mx \leq 1\}$
$\mathbb{R}^n$	Euclidean space of dimension n
$\mathbb{R}^{n\times m}$	Set of real matrices of dimension $n \times m$
$\mathbb{S}^n$	Set of symmetric matrices of dimension $n \times n$
$\mathbb{D}^n$	Set of diagonal matrices of dimension $n \times n$
$\mathbb{P}^{n\times m}$	Set of matrices of dimension $n \times m$ with non-negative elements
diag$(M_1, ..., M_n)$	Block-diagonal matrix with blocks M_1 to M_n
trace(M)	Trace of matrix M

CONTENTS

1 INTRODUCTION

The popularization of the networked control systems (NCSs) in the last few decades, driven by the developments in the technology of communication networks, as highlighted in the surveys (ZHANG; BRANICKY; PHILLIPS, 2001; TIPSUWAN; CHOW, 2003; YANG, 2006), made visible some of the problems related to the implementation of classical discrete-time feedback loops operating in networks. Firstly, shared networks, generally implemented with a single communication bus, have severe bandwidth limitations (LIAN; MOYNE; TILBURY, 1999; SUZUKI et al., 2011), limiting the sampling rate and therefore the performance of control systems based on periodic control updates. Secondly, remote control systems over a wireless network require a high amount of energy to transmit data (AKYILDIZ et al., 2001), and usually have limited amount of energy available, often stored in batteries. In this sense, the sensor measurements update rate is a parameter that may improve the performance if increased, but, on the other hand, transmitting the information from sensors may increase the energy consumption, thus decreasing the battery charge duration (SADI; ERGEN, 2017).

In this context, aiming at a better use of network resources, aperiodic control strategies have been studied, where a control task, that is, the update of the controller and the transmission of data between the sensor, the controller and the actuator, is executed only when some condition is met. Among these strategies, the event-triggered control (ETC) (TABUADA, 2007) and the self-triggered control (VELASCO; FUERTES; MARTI, 2003) paradigms stand out. In the first case, an event generator monitors the state or the output of the system and an event happens when a criterion based on the monitored variables is verified. Differently, on the self-triggered control, whenever an event happens, the event generator determines the next event instant based on the current state or output information. The main advantage of the self-triggered control consists on the fact that it does not require a constant monitoring of plant signals, and therefore the event generator and the sensor do not have to be co-located. On the other hand, this strategy comes with some drawbacks, such as a higher computational complexity. Also, this class of control systems operates without any sort of feedback between two events, posing a problem in the presence of unmodeled dynamics or unforeseen uncertainties. In turn, the ETC strategy requires the constant monitoring of the states or outputs of the plant, but have a lower computational complexity and is less prone to perturbations. Moreover, ETC has shown a more efficient utilization of the network resources, as observed in the comparative study (MAZO; TABUADA, 2008), and in other recent studies implementing recent event-triggered and self-triggered techniques (YI et al., 2018; BRUNNER; HEEMELS; ALLGÖWER, 2019). For the above reasons, this work considers the ETC strategy. Furthermore, since in general NCSs are implemented over digital platforms, the present work focuses on discrete-time (also called periodic) ETC, that is, it is considered that the event generator only monitors the variables of interest in periodic time instants.

Even though the ETC strategies gained popularity recently, the state of the art still presents some gaps, especially for nonlinear systems. Indeed, the literature regarding methods of ETC for piecewise affine (PWA) systems is very scarce, with (MA; WU; CUI, 2018; JIANG et al., 2020) being exceptions. Piecewise affine systems are a class of nonlinear systems in which the state-space is partitioned in convex polyhedron regions, and the behavior of the system in each of these regions is defined by an affine dynamic. Several authors have studied this class of systems since it can model a wide array of nonlinear dynamical systems, such as systems with actuator saturation, nonlinear circuits and closed-loop systems under a model predictive control (MPC). Also, PWA functions have been shown to have universal approximation properties (LIN; UNBEHAUEN, 1992; XU; XIE, 2014), meaning they can approximate smooth nonlinear functions arbitrarily well in bounded intervals. Hence, more complex dynamics may be modeled by uncertain PWA systems.

Regarding stability analysis, the most used methods are based on Lyapunov techniques. Although several representations have been proposed to described PWA functions, most methods used for the stability analysis of this class of systems rely on a standard explicit representation that enumerates the regions of the state space partition and associates affine dynamics to each of these regions. Also, the stability conditions for discrete-time systems require the computation of admissible transitions and transition sets (BISWAS et al., 2005). These facts entail some issues. First, enumerating the regions and computing all the possible transitions for higher order systems demand a large amount of computational power. Second, for any control law design and, in particular for an ETC strategy, it is not possible, in general, to know the transition sets before the control law (or the ETC strategy) parameters are defined. Thus the systematic design of an event generator that ensures the stability of a PWA system is still an open problem.

Based on the above considerations, the following general objective is defined for this work:

— Propose an ETC design method for discrete-time PWA systems.

From this general objective, the following specific objectives are considered:

— Propose a representation for PWA systems that is suited to the stability analysis and does not require the enumeration of the regions and admissible transitions of the partition.
— Propose conditions for the stability analysis of discrete-time PWA systems in the proposed representation and cast these conditions as linear matrix inequalities.
— Propose a systematic method allowing the emulation based synthesis of the event generator parameters preserving the stability and reducing the trigger activity through convex optimization problems.

1.1 Thesis Outline

This thesis is organized as follows:

In Chapter 2, a review of the state of the art regarding ETC and PWA systems is presented. The main characteristics of the class of PWA systems are shown, as well as the characteristics of the ETC paradigm, including some challenges in applying an ETC strategy to a PWA system.

In Chapter 3, a new representation for continuous PWA functions is presented. Some properties of this representation are explored and the relation and advantages of this new representation to previous results in the literature of PWA systems are discussed.

In Chapter 4, conditions for the stability analysis of PWA systems described using the representation proposed in Chapter 3 are derived. These conditions rely on a a characterization of the ramp function. Considering a piecewise quadratic (PWQ) Lyapunov candidate function, these conditions are cast in terms of LMIs and extended to deal with polytopic uncertainties. Some numerical examples are used to illustrate the effectiveness of the results.

In Chapter 5, the problem of designing an ETC strategy for a discrete-time PWA system is addressed, considering the representation proposed in Chapter 3 and the stability conditions derived in Chapter 4. A new PWQ trigger function is proposed, and conditions for the design of such a function using convex optimization problems are presented. The simulation of numerical examples show the potential of the proposed ETC design method in reducing the number of control updates.

Lastly, Chapter 6 contains the conclusion of the thesis, including the final remarks and future work perspectives.

2 STATE OF THE ART

This chapter is dedicated to a bibliographical review of event-triggered control and piecewise affine systems. Some of the main results regarding these two areas are presented, and some of their shortcomings are shown. Also, important concepts to the understanding of the present work are introduced.

2.1 Event-Triggered Control

Most NCSs are implemented over digital platforms. Because of this, these systems are usually based on a periodic sampling of the states or outputs of the plant, and thus, the application of classic discrete time control techniques is possible. However, the application of these techniques to networked systems causes an inefficient utilization of network resources, since the sensors information must be sent to the actuators at every periodic sampling time. In this sense, the event-triggered control paradigm has been proposed as a way to use these resources more efficiently, through an aperiodic transmission strategy, as shown in (KOPETZ, 1993; ÅSTRÖM; BERNHARDSSON, 1999).

The event-triggered control differs from the classic discrete-time control by not updating the controller and sending the signal to the actuator at each period, but only when an event is generated, which happens whenever a given trigger function violates a threshold (HESPANHA; NAGHSHTABRIZI; XU, 2007). In this context, the update of the controller and the sending of the signal from the controller to the actuator is called a control task. This way, an event-triggered control strategy can be divided in two distinct parts: a stabilizing control law, which provides the control signal to the actuator based on the information sent by the sensors, and the event generator, which evaluates the trigger function and determines when the control must be updated. Thus, the trigger function, which measures the monitored state degradation, is compared with a determined constant, and when the trigger condition is violated (i.e. when the trigger function value goes beyond a given threshold), an event is generated, and the control signal applied to the plant must be updated.

2.1.1 Discrete-time Event-Triggered Control

A large portion of the literature about ETC considers that the system operates in continuous time and that the variables can be continually monitored, and also that the control tasks can be executed at any time instant, for instance as in (TABUADA, 2007; VELASCO; MARTÍ; BINI, 2009; SBARBARO; TARBOURIECH; GOMES DA SILVA JR, 2014; MOREIRA et al., 2016; ABDELRAHIM et al., 2018). This assumption, however, is not generally true, since the implementation of NCSs is usually done over digital platforms, so that the control operation happens in discrete time, i.e. the evaluation of the sensor states can only happen in determined sampling intervals, and so does the update of the control signal. So, after the design is done in continuous time, a discretization must be performed, so that the implemented system is merely an approximation of the one designed (HEEMELS; DONKERS; TEEL, 2011). Moreover, in the continuous time ETC there is the possibility of Zeno behavior, which does not happen in the discrete-time ETC. Broadly speaking, Zeno behavior is the occurrence of infinitely many events at the same instant, or the occurrence of inter-event times that tend to 0 either as the time goes to infinity or as the time goes to some finite instant. When designing control strategies for continuous time systems, one must rule out such behavior, see for instance (TABUADA, 2007; MOREIRA et al., 2017; TARBOURIECH et al., 2017).

The discrete-time ETC, also called periodic ETC, on the other hand, considers that the monitoring of the trigger function and the execution of control tasks can only be performed in given time instants. This approach can be found, for instance, in (EQTAMI; DIMAROGONAS; KYRIAKOPOULOS, 2010; HEEMELS; DONKERS; TEEL, 2013; GROFF et al., 2016; LINSENMAYER; DIMAROGONAS; ALLGÖWER, 2019). When dealing with discrete-time systems, there is already a minimum innate inter-event time defined by the nominal sampling period of the system (HEEMELS;

DONKERS; TEEL, 2011), so it suffices to guarantee that that the trigger function is smaller than the threshold at the instants in which an event occurs to ensure that the Zeno behavior is avoided.

2.1.2 Network Topologies

According to (HESPANHA; NAGHSHTABRIZI; XU, 2007), of the many possible network topologies the single channel topology is of particular interest for the study of NCSs. This architecture is characterized by having the sensor and the actuator nodes separated by a single communication channel. Although simple, this topology captures several important characteristics of NCSs, such as bandwidth limitations. This topology can also be used to represent different network configurations, depending on the positioning of the controller in the actuator node or the sensor node.

In the first case, when the controller and the actuator are co-located, the information of the plant states are only available to the controller at the instants when a control tasks is executed. Thus, between two events, the controller dynamics happen based on the information received through the network in the last control task. On the other hand, the controller information is available to the actuator at every periodic instant. This kind of architecture is studied in (LEHMANN; JOHANSSON, 2012; GOLABI et al., 2016).

On the second case, when the sensor and the controller are co-located, the information of the plant states are available to the controller at every periodic instant, but this information is only sent to the actuator when a control task is executed. This configuration is of particular interest to the study of observer-based control, since the observer will be periodically updated, leading to a faster convergence of the observed states. The results found in (TALLAPRAGADA; CHOPRA, 2012; HEEMELS; DONKERS; TEEL, 2013; SBARBARO; TARBOURIECH; GOMES DA SILVA JR, 2014; GROFF et al., 2016) can be highlighted for approaching this configuration.

A third configuration is also possible, where the controller is located in a distinct node from the sensor and the actuator. In this case, both the information the controller receives from the sensors and the update of the signal applied to the actuator happen only when a control task is executed. This network arrangement is studied in (TRIPATHY; KAR; PAUL, 2017).

It is also important to point that in the case of a static feedback, such as in (WU; REIMANN; LIU, 2014; GROFF; MOREIRA; GOMES DA SILVA JR, 2016; MAH-

MOUD, 2017), the three architectures are equivalent.

2.1.3 Event Generator

The event generator is a vital part of an ETC system, as it is responsible for dictating when a control task should be executed, that is, when there should be a sampling of the states and an update of the control signal. This is done by monitoring a trigger function, and an event happens whenever this function violates a given threshold. A generic trigger strategy can be described by

Algorithm 1 Event-triggered control algorithm

if $f(x(k),\delta(k)) > \epsilon$ **then**

  Generate an event;

  $n_{i+1} = k$;

end if

where $f(x(k),\delta(k))$ is the trigger function, ϵ is the threshold, x is the system state, δ is the *state degradation* defined as

$$\delta(k) \triangleq x(n_i) - x(k), \tag{1}$$

and k and n_i are the current time instant and the time instant of the i-th (last) event, respectively. Thus, one of the main challenges of this control paradigm consists in choosing an appropriate function so that the closed-loop system is stable, has the desired performance, and at the same time efficiently uses the network resources. In this sense, the quadratic trigger functions, that is, functions that can be written in the form

$$f(x,\delta) = \begin{bmatrix} x \\ \delta \end{bmatrix}^\top Q \begin{bmatrix} x \\ \delta \end{bmatrix}, \tag{2}$$

with a matrix Q of appropriate dimensions, constitute an important class of trigger functions, with ample applications in the literature. However, the design of generic quadratic functions, with a free matrix Q, is generally a complex task (HEEMELS; DONKERS; TEEL, 2011). Thus, some particular structure is usually applied to Q in order to obtain a tractable design or analysis problem. Three of the most prevalent trigger strategies, namely the absolute error, the relative error and the weighted relative error are detailed below. Also, other results considering quadratic functions can be found in (MARCHAND et al., 2013) where a quadratic Lyapunov function is used for the trigger or in (SEURET et al., 2013) where a function based on a linear-quadratic cost is used.

2.1.3.1 Absolute Error Threshold

This triggering strategy consists of generating an event when the norm of the difference between the current and the last sampled state reaches a given threshold as summarized by the following algorithm

Algorithm 2 Absolute error threshold trigger

if $||x(n_i) - x(k)|| \geq \epsilon_{abs}$ **then**
  Generate an event;
  $n_{i+1} = k$;
end if

The term ϵ_{abs} is the threshold and is the only design parameter available in this case. This strategy can also be extended by applying different weights to the error of each state variable (as well as their cross-products), yielding a strategy that can be described by the algorithm

Algorithm 3 Weighted absolute error threshold trigger

if $\delta(t)^\top Q_\delta \delta(t) > \epsilon_{abs}$ **then**
  Generate an event;
  $n_{i+1} = k$;
end if

With this, the matrix Q_δ is also a design parameter, allowing more degrees of freedom than the threshold based simply on the Euclidean norm of the state. The main problem with this strategy is that, since there is no normalization of the degradation norm, as the state gets closer to the origin, the degradation norm becomes smaller, up to the point where the event generator will no longer trigger. After this point, if the system is stable, it will operate in open-loop indefinitely and converge to a state close to the origin according to the open-loop dynamics. On the other hand, if it is open-loop unstable, the degradation norm will increase again until the threshold is reached, and eventually reach either a limit cycle or develop a chaotic behavior. Because of these properties, asymptotic stability cannot be achieved with this strategy.

2.1.3.2 Relative Error Threshold

One alternative to the absolute error is to normalize the state degradation with respect to the current state, leading to the relative error threshold, described by the algorithm

Algorithm 4 Relative error threshold trigger

if $\frac{||\delta(t)||}{||x(t)||} \geq \sigma_0$ **then**
  Generate an event;
  $n_{i+1} = k$;
end if

This condition was proposed in (TABUADA, 2007), where the possibility of asymptotic stability was demonstrated for a large class of systems. The design parameter in this case is the variable σ_0, and the greater its value, the larger the degradation of the state can be before an event occurs, so that less events tend to occur for a larger σ_0.

2.1.3.3 Weighted Relative Error Threshold

The previous trigger condition can be extended by considering weights for the different variables, leading to a weighted relative error condition, that can be summarized by the algorithm

Algorithm 5 Weighted relative error threshold trigger

if $\delta(t)^\top Q_\delta \delta(t) - x^\top(t) Q_x x(t) > 0$ **then**
  Generate an event;
  $n_{i+1} = k$;
end if

The design parameters in this case are the matrices Q_δ and Q_x, which are symmetric positive-definite matrices that act as weights in the relative errors. The relation between these has the same role as σ_0 in the relative error strategy, so a “larger” Q_x and a "smaller" Q_δ" will allow a larger deviation of the current state from tha last sampled one before an event is generated. In this sense, "larger" and "smaller" refer to the eigenvalues of the matrices, since they are involved in quadratic forms of x and δ. Also, note that taking $Q_\delta = I$ and $Q_x = \sigma_0 I$, one retrives the relative error trigger conditions, so this strategy is indeed a generalization of the previous one.

Methods for applying this strategy were introduced as part of the author's research and can be found in (GROFF et al., 2016; GROFF; MOREIRA; GOMES DA SILVA JR, 2016).

2.2 Continuous Piecewise Affine Systems

The class of PWA systems is comprised of systems with dynamics that can be described by

$$x^+ = f(x), \tag{3}$$

where $f(x)$ is a PWA function, and x^+ denotes the state vector at the next time instant, that is, $x^+ = x(k+1)$. This is class of systems has been used to represent a large number of nonlinear systems, such as nonlinear circuits (KAHLERT; CHUA, 1992; JULIAN; DESAGES; AGAMENNONI, 1999; POGGI; COMASCHI; STORACE, 2010; PASOLLI; RUDERMAN, 2019), where even simple piecewise affine nonlinearities can lead to complex behavior, and other engineered systems such as power converters (MOLLA-AHMADIAN et al., 2014) and pneumatic systems (ANDRIKOPOULOS et al., 2013), or in other control systems problems, such as biosystems control (AZUMA; YANAGISAWA; IMURA, 2008). Moreover, systems presenting some hard nonlinearities, such as saturation or deadzone, can also be modelled by this class of systems, since these functions are indeed piecewise affine (GOMES DA SILVA JR.; TARBOURIECH, 1999; LATHUILIÈRE; VALMORBIDA; PANTELEY, 2018).

The practical interest on PWA continuous functions in discrete-time systems also appears in the context of Receding Horizon Optimal Control (RHOC) (BEMPORAD et al., 2002), where multi-parametric linear or quadratic programs can be solved offline to obtain PWA control laws associated to a specific partition on the state space, termed Explicit Model Predictive Control (EMPC).

2.2.1 Models for PWA Functions

Several representations for PWA functions have been proposed in the literature, each with their own advantages and disadvantages, being applied in different areas of interest. Among the most important ones, are the standard PWA representation, the canonical PWA representation, the Hinging Hyperplanes (HH) representation and the lattice representation, which are detailed below.

2.2.1.1 Standard PWA Representation

The standard representation of PWA systems traces back to (SONTAG, 1981), where the following explicit representation for a PWA function was proposed

$$f(x) = A_i x + b_i,\ \forall x \in \Gamma_i \subset \mathbb{R}^n,\ i \in \mathcal{L} \tag{4}$$

where Γ_i is a convex polyhedron defined as

$$\Gamma_i = \{x \in \mathbb{R}^n | E_i x + e_i \geq 0\}, \tag{5}$$

and the index set $\mathcal{L} = \{1,...,L\}$, L being the number of regions in the partition.

2.2.1.2 Canonical PWA Representation

Originally proposed in (CHUA; KANG, 1977), the canonical representation of a PWA funciton is given by the following expression

$$f(x) = a_0 + a_1^\top x + \sum_{i=1}^{L} b_i |\alpha_i^\top x - \beta_i| \tag{6}$$

This form is capable of representing every scalar, continuous, PWA function of one variable, but not every continuous mutivariate PWA function can be represented. In particular, a function must have the consistent variation property in order to be represented by a canonical PWL function (see (CHUA; KANG, 1977), Theorem 1).

In order to extend the Canonical PWA Representation to functions that do not exhibit the consistent variation property, generalizations based on nested absolute values were proposed in (KAHLERT; CHUA, 1990), (GUZELIS; GOKNAR, 1991). A generalized formulation allowing to represent any multivariate continuous PWA function was presented in (LIN; XU; UNBEHAUEN, 1994), with the following definition.

Definition 1. *The canonical formulation* (6) *is called the first-level canonical PWA function. The K-th level canonical PWA function, for $K > 1$, takes the form*

$$f(x) = f_0(x) + C|g(x)|, \tag{7}$$

where $C \in \mathbb{R}^{M \times I}$, with a finite integer I, and f_0 and g are canonical PWA functions of at most $K - 1$ level.

This was an important advance in the theory of PWA systems, since it allowed the use of the framework developed for canonical PWA functions, mainly used in the field of electronic circuits, to be applied to general continuous PWA functions.

2.2.1.3 Hinging Hyperplanes Representation

The HH (Hinging Hyperplanes) representation was proposed in (BREIMAN, 1993) for function approximation and classification, as an alternative to the use of neural networks. It is based on a sum of hinge functions which consist of two hyperplanes joined together at a hinge. Thus, a function in this representation is given by

$$f(x) = \sum_{i=1}^{L} \max(A_{i1}x + b_{i1}, A_{i2}x + b_{i2}). \tag{8}$$

As it was shown in (WANG; SUN, 2005), the HH model has a close relation to the canonical representation, and also suffers from the same shortcomings, that is, it can only model function that have consistent variation. Therefore, in the aforementioned paper, the following generalization that can represent any continuous PWA function was proposed

$$f(x) = \sum_{i=1}^{L} \max(A_{i1}x + b_{i1}, \ldots, A_{in}x + b_{in}). \tag{9}$$

2.2.1.4 Lattice PWA Representation

It was first proposed in (WILKINSON, 1963) that any continuous PWA function can be expressed by a max-min composition of affine components, that is

$$f(x) = \max_{i=1,\ldots,L} (\min_{j \in I_i}(\ell_j)) \tag{10}$$

where I_i is an index set and ℓ_j is an affine function, i.e., $\ell_j = A_j x + b_j$. Later, it was formally demonstrated in (TARELA; MARTINEZ, 1999) that any continuous PWA function can indeed be described by (10), then called the lattice PWA representation.

2.2.2 Stability of PWA Systems

In this section, results regarding the stability of PWA systems are reviewed. In the study of PWA systems, the Lyapunov theory has been proven an important tool for the characterization of the stability of the system's equilibrium points (BISWAS et al., 2005). The Lyapunov characterization of stability is presented in the appendix.

It should be pointed out that the vast majority of the results for stability analysis in the literature are carried out using the standard PWA representation (4) (HEEMELS; DE SCHUTTER; BEMPORAD, 2001). Thus, through this section, it is assumed that the system is modeled by the standard representation (4) unless otherwise noted.

2.2.2.1 Quadratic Lyapunov Functions

Quadratic functions as

$$V(x) = x^\top P x, \tag{11}$$

where P is a symmetric positive definite matrix, are perhaps the most popular class of differentiable functions used as Lyapunov function candidates (CHEN, 1999). For linear stable systems, the existence of a quadratic function is necessary and sufficient to show the stability of the origin. For PWA systems, quadratic functions are used to formulate sufficient stability conditions for equilibria. PWA systems admitting a single function as (11) as the LF are called quadratically stable. (MIGNONE; FERRARI-TRECATE; MORARI, 2000). However, this is only a sufficient condition, and stable systems may not satisfy it. Actually, if there exists a set of positive-definite matrices $R_i, i = 1, \ldots, L$, satisfying

$$\sum_{i=1}^{L} (A_i^\top R_i A_i - R_i) > 0, \tag{12}$$

then the system does not admit a common quadratic Lyapunov function (FENG, 2002).

Note that this approach is rather conservative, since no information about the partition is included in the stability analysis.

2.2.2.2 *Piecewise Quadratic Lyapunov Functions*

Piecewise quadratic functions have been proposed for PWA discrete-time systems by (MIGNONE; FERRARI-TRECATE; MORARI, 2000; FENG, 2002; JOHANSSON, 2003). The idea consists in using Lyapunov functions of the class

$$V(x) = V_i(x),\ \forall x \in \Gamma_i,\ \ i \in \mathcal{L}, \tag{13}$$

where $V_i(x) = x^\top P_i x$, and $P_i, i \in \mathcal{L}$ are symmetric positive definite matrices. The origin of the system can be shown to be stable if there exists a set of matrices P_j verifying

$$\Delta V_{ij}(x) < 0,\ \forall\{i,j\} \in \Omega, \tag{14}$$

where $\Delta V_{ij}(x) = V_j(x^+) - V_i(x)$ and Ω is the set of all possible transitions from one region into another. In order to further reduce the conservatism, this condition can be modified to account for the partition, requiring $\Delta V_{ij}(x) < 0$ to be satisfied only when $x \in \Gamma_i$. This can be done by defining

$$G_i(x) = E_i x + e_i. \tag{15}$$

Since, by definition, $G_i(x) \geq 0$ whenever $x \in \Gamma_i$, if

$$\Delta V_{ij}(x) + G_i(x)^\top U_{ij} G_i(x) < 0,\ \forall\{i,j\} \in \Omega, \tag{16}$$

with $U \succcurlyeq 0$, that is, all elements of U are non-negative, then the origin of the system is stable.

Several improvements to the method have been proposed since it was first proposed, aiming at reducing the conservatism by applying relaxations and exploiting specific characteristics of PWA systems, such as in (HOVD; OLARU, 2013; IERVOLINO; TANGREDI; VASCA, 2017; ZHU et al., 2018).

However, a major disadvantage of this method is that it requires the computation of the set Ω, that is, the computation of all possible transitions between regions of the system.

2.2.2.3 *Piecewise Affine Lyapunov Functions*

A piecewise affine Lyapunov function is a function of the form

$$V(x) = V_i(x) = x^\top \ell_i + c_i,\ \forall x \in \Gamma_i,\ i \in \mathcal{L}, \tag{17}$$

where $\ell_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ are such that

$$V_i(x) > 0,\ \forall x \in \Gamma_i,\ i \in \mathcal{L}. \tag{18}$$

This class of functions has been originally proposed for continuous systems in (JOHANSSON, 2003), and extended to discrete-time systems in (BISWAS et al., 2005). Similarly to the case of PWQ functions, if

$$\Delta V_{ij} \leq 0,\ \forall \{i,j\} \in \Omega, \tag{19}$$

then the origin of the system is stable. One of the advantages of this class of functions is that the stability conditions can be checked with a linear program.

This approach has been extended in (RUBAGOTTI et al., 2011) and improved in (RUBAGOTTI; ZACCARIAN; BEMPORAD, 2016).

Similar to the PWQ approach, this method relies on the computation of the set Ω to cast the stability conditions as a linear program. Moreover, it also requires the computation of the transition sets, that is, the set of states that lead to each particular transition between regions.

2.2.2.4 *Other Approaches*

Other approaches to the stability of PWA systems have been presented. In particular, (BIANCHINI; PAOLETTI; VICINO, 2008) presents an $\mathcal{L}_2$ stability test for systems in the hinging hyperplanes representation. The method consists in converting the system to a linear fractional representation (LFR), and using integral quadratic constraints to check the stability of the system. One of the main motivations behind the development of the method was to avoid the enumeration of transitions between regions, because, in the context of the work, a large number of regions and transitions would lead to a large computation time to check the stability of the system. Though promising, the technique only applies to hyperplanes defined as (8), that is, only hyperplanes without nesting, so it cannot be used to study the stability of functions without the consistent variation property, which require nested hyperplanes. Also, the transformation into an LFR equivalent required the linear part of the system to be Schur-stable, but no guarantees could be provided that this assumption would hold, even for stable systems.

Another approach was presented in (BEMPORAD et al., 2010, 2011), where the function is described based on a simplicial partition with a canonical PWA basis function. The stability test then relies on the computation of a PWA Lyapunov function. The main problem of the approach is that the number of regions on the simplex grows very quickly, specially for higher dimension systems.

2.2.3 Reachability Analysis

As seen on the previous section, the stability analysis of PWA systems in the standard representation relies on the set of possible transitions Ω. This happens because it is not possible to know *a priori* in what region Γ_i the state will be at time instant k nor in what region Γ_j the state will be at time instant $k+1$, thus, it is required that the Lyapunov function is decreasing for all possible combinations of regions Γ_i and Γ_j.

One way to compute the set Ω, as described in (BISWAS et al., 2005), is as follows. First, a matrix T is defined with elements $t_{i,j}$ as

$$t_{i,j} = \begin{cases} 1, & \text{if } \exists x \in \Gamma_i, \text{ such that } f(x) \in \Gamma_j \\ 0, & \text{otherwise} \end{cases}. \tag{20}$$

Next, the matrix T is used to construct the set $\Omega \triangleq \{i,j \in \mathbb{R} \mid t_{i,j} = 1\}$. As for determining the values of $t_{i,j}$, consider the following definition

Definition 2. *For system* (3)*, the set* $\mathcal{G}(x,\mathcal{H}) \triangleq \{\chi \in \mathbb{R}^n \mid \chi = f(x) \wedge x \in \mathcal{H}\}$ *is the one-step reachable set from* $\mathcal{H}$*.*

Then, if $\mathcal{G}(x,\Gamma_i) \cap \Gamma_j = \{\}$, then $t_{i,j} = 0$. On the other hand, if the intersection is not empty, there exists some $x \in \Gamma_i$ such that $f(x) \in \Gamma_j$, and thus $t_{i,j} = 1$. For more information on the computation of reachable sets for PWA systems, see (BEMPORAD, 2003), for instance.

2.3 Event-Triggered Control of Piecewise Affine Systems

The literature on event-triggered controlled piecewise affine systems is currently very scarce, with the few exceptions in (MA; WU; CUI, 2018; JIANG et al., 2020).

Firstly, interest in event-triggered control for nonlinear systems has only recently started to get attention. Secondly, the methods for certifying the stability of PWA systems rely on computing reachable sets. In the case of event-triggered controlled systems, the error due to the control strategy can be modeled as a perturbation, so that the system is a perturbed system. Computing reachable sets for perturbed systems is not a difficult task, see (NAM; PATHIRANA; TRINH, 2015), for example, when the perturbation is bounded. However, for event-triggered systems, the modeled perturbation depends on the current state and the state in the last event, and since no *a priori* information is known about the evolution of the system between the events, finding a bound to this perturbation can be difficult, especially when a triggering function is yet to be designed. The reachability analysis of linear ETC systems is approached in (FU; MAZO, 2018), but an upper bound to the number of time instants between two events has to be imposed, and the application to PWA systems is not straightforward. The cited papers have not addressed this problem, and it is assumed that they consider all transitions between states as possible, which results in a great conservatism.

2.4 Conclusion

In this chapter, a bibliographical review regarding event-triggered control and PWA systems was presented. The main features of the event-triggered control paradigm and results from the literature were presented. Also, the tools for the representation and stability analysis of piecewise affine systems were summarized, as well as the reachability analysis problem. Analyzing the literature that deals with the intersection of these themes, that is, the event-triggered control of piecewise affine systems, it becomes evident that this is still an open topic.

In particular, a methodology that enables the stability analysis of PWA systems without relying on enumerating transitions would help to bridge the gap between event-triggered control and PWA systems. In the next chapters, methods satisfying these criteria will be presented, providing a new tools for stability analysis of PWA systems.

These tools will then be applied to event-triggered control.

3 IMPLICIT PIECEWISE AFFINE REPRESENTATION

As discussed in the state of the art review, there exist several representations for piecewise affine systems. Models such as hinging hyperplanes(BREIMAN, 1993) and lattice (TARELA; MARTINEZ, 1999) were shown to be suitable for system identification methods. Also, models for nonlinear active devices, such as diodes, lead to the canonical PWA representation (CHUA; KANG, 1977), which is a natural representation for nonlinear circuits. However, thanks to an intuitive mathematical description, the explicit representation has been widely used for stability analysis (FENG, 2002; RUBAGOTTI et al., 2011; HOVD; OLARU, 2013). The stability analysis methods of PWA systems using the explicit representation (4) often require the enumeration of the partition as well as the enumeration of possible transitions between partitions. In some cases, the enumeration of these transitions may be difficult or impossible, such as when the transition is driven by an external signal, or whenever there is an uncertainty in the partition.

In this chapter, we propose a representation for continuous PWA functions that does not require the regions and partitions to be enumerated to perform stability analysis with PWA Lyapunov functions. Some of the results in this chapter were introduced by the author in (GROFF; VALMORBIDA; GOMES DA SILVA JR, 2019).

3.1 Implicit Continuous PWA Functions

Consider the mapping $f : \mathbb{R}^n \to \mathbb{R}^{n_f}$ defined by

$$f(x) = F_1 x + F_2 \phi(y(x)) \tag{21a}$$

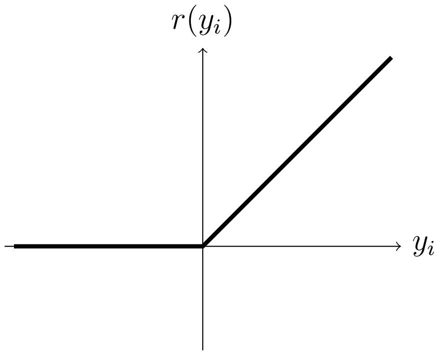
$$y(x) = F_3 x + F_4 \phi(y(x)) + f_5 \tag{21b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n_y}$, $F_1 \in \mathbb{R}^{n_f \times n}$, $F_2 \in \mathbb{R}^{n_f \times n_y}$, $F_3 \in \mathbb{R}^{n_y \times n}$, $F_4 \in \mathbb{R}^{n_y \times n_y}$, $f_5 \in \mathbb{R}^{n_y}$ and the vector function $\phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ is defined element-wise by the ramp function as

$$\phi_i(y) = r(y_i) := \begin{cases} 0 & \text{if} \quad y_i < 0 \\ y_i & \text{if} \quad y_i \geq 0 \end{cases}, i = 1, \ldots, n_y \tag{22}$$

as depicted in Figure 1.

Figure 1 – Nonlinear function $r(y_i)$.



Source: The author

From the definition of the ramp function (22), we obtain

$$\phi(y) = \Delta(y) y \tag{23}$$

where $\Delta : \mathbb{R}^m \to \mathbb{D}_{\{0,1\}}$. We have

$$\Delta_{ii} := \begin{cases} 0 & \text{if} \quad y_i < 0 \\ 1 & \text{if} \quad y_i \geq 0 \end{cases}, i = 1, \ldots, n_y. \tag{24}$$

The equation (21b) is, in general, an implicit equation for variable y (in Section 3.1.1 will state a result allowing to verify the well-posedeness of this equation). That is, to obtain y for a given x, we need to solve the equation

$$y - F_4\phi(y) = F_3x + f_5. \tag{25}$$

We will assume this equation is well-posed, namely, that for each x it corresponds a unique y, thus defining a map $y : \mathbb{R}^n \to \mathbb{R}^m$. Using (23)-(25), we then have that $y(x)$ satisfy

$$y(x) = (I - F_4\Delta(y(x)))^{-1}F_3x + (I - F_4\Delta(y(x)))^{-1}f_5. \tag{26}$$

The matrix $(I - F_4\Delta(y(x)))$ has an inverse since the implicit equation is well-posed. From the above expression we also obtain, using (21a), (23) and (26).

$$f(x) = \left(F_1 + F_2\Delta(y(x))(I - F_4\Delta(y(x)))^{-1}F_3\right)x + F_2\Delta(y(x))(I - F_4\Delta(y(x)))^{-1}f_5 \tag{27}$$

that is, (26)-(27) are expressions that can be written as

$$\begin{aligned} f(x) &= A(x)x + b(x) \\ y(x) &= C(x)x + d(x) \end{aligned} \tag{28}$$

with

$$\begin{aligned} A(x) &= (F_1 + F_2\Delta(y(x))(I - F_4\Delta(y(x)))^{-1}F_3) \\ b(x) &= F_2\Delta(y(x))(I - F_4\Delta(y(x)))^{-1}f_5 \\ C(x) &= (I - F_4\Delta(y(x)))^{-1}F_3 \\ d(x) &= (I - F_4\Delta(y(x)))^{-1}f_5. \end{aligned}$$

$A(x)$, $b(x)$ $C(x)$, $d(x)$ are thus determined by $\Delta(y(x))$ as in (27) which, in turn, depend on the sign of the solution y to (25). Since diagonal elements of $\Delta(y(x))$ belong to $\{0,1\}$, it has at most 2^m possible values. Therefore, $A(x)$, $b(x)$ $C(x)$, $d(x)$ can have at most 2^m values. We can then use (28) to obtain an explicit representation as

$$f(x) = A_ix + b_i \;\; x \in \Gamma_i$$

$i \in \{1, \ldots, 2^m\}, \Gamma_i = \left\{x \in \mathbb{R}^n \mid y_j \geq 0, \forall j \in \mathcal{J}_i, y_j < 0, \forall j \notin \mathcal{J}_i, i = 1 + \sum_{j \in \mathcal{J}_i} 2^{(j-1)}\right\}$, where $\mathcal{J}_i$ is a set containing the indexes of the elements of y that are non-negative in the region Γ_i.

That is, from (21)-(22), it is the vector function $\phi(y(x))$ and the regions where its arguments are not negative that define the PWA partition of $\mathbb{R}^n$. Since these partitions and the value is defined implicitly we refer to (21)-(22) as an *implicit representation* of PWA systems.

Note that when $F_4 = 0$, or for some particular structures of F_4, explicit solutions to (21b) can be obtained (see section 3.1.1). We should also observe that, thanks to the well-posedness and the continuity of ϕ, we have that $f(x)$ is continuous.

Next, we illustrate the representation (21) with two examples, showing their relation to the explicit representation.

Example 1: Consider (21) with $n = 2$, $n_y = 3$, $n_f = 1$ and

$$\begin{aligned} F_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad & F_2 &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ F_3 &= \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad & F_4 &= \begin{bmatrix} 0 & -2/3 \\ -1 & 0 \end{bmatrix}, \quad f_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{29}$$

As $y \in \mathbb{R}^2$, the partition of the state space corresponding to an explicit representation can be obtained by considering all the 2^{n_y} possibilities involving $y_i \geq 0$ and $y_i < 0$, that is

- $\Gamma_1 = \{x \in \mathbb{R}^2 | y_1(x) < 0, y_2(x) < 0\}$.
  In this case, it follows that $\phi_1 = 0$ and $\phi_2 = 0$, and we have

$$\begin{aligned} y_1 &= -x_1 - x_2 < 0 \Leftrightarrow -x_1 < x_2 \\ y_2 &= x_1 - x_2 < 0 \Leftrightarrow x_1 < x_2. \end{aligned}$$

- $\Gamma_2 = \{x \in \mathbb{R}^2 | y_1(x) \geq 0, y_2(x) < 0\}$.
  In this case, it follows that $\phi_1 = y_1$ and $\phi_2 = 0$, and we have

$$\begin{aligned} y_1 &= -x_1 - x_2 \geq 0 \Leftrightarrow -x_1 \geq x_2 \\ y_2 &= x_1 - x_2 - (-x_1 - x_2) < 0 \Leftrightarrow x_1 < 0. \end{aligned}$$

- $\Gamma_3 = \{x \in \mathbb{R}^2 | y_1(x) < 0, y_2(x) \geq 0\}$.

In this case, it follows that $\phi_1 = 0$ and $\phi_2 = y_2$, and we have

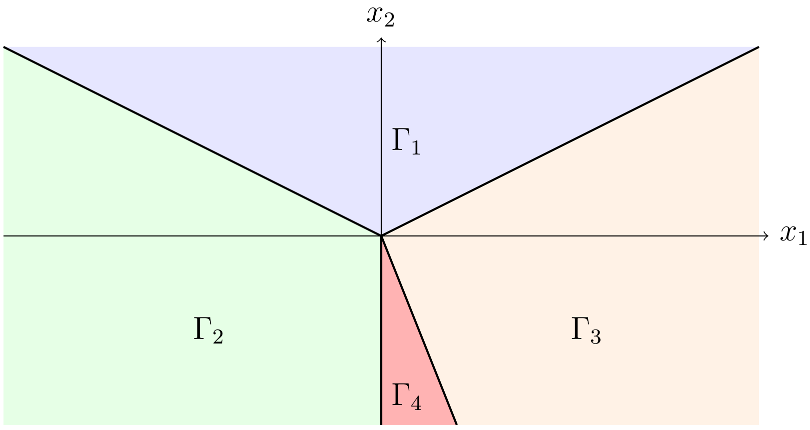
$$y_1 = -x_1 - x_2 - \tfrac{2}{3}(x_1 - x_2) < 0 \Leftrightarrow -5x_1 < x_2$$
$$y_2 = x_1 - x_2 \geq 0 \Leftrightarrow x_1 \geq x_2.$$

- $\Gamma_4 = \{x \in \mathbb{R}^2 | y_1(x)) \geq 0, y_2(x) \geq 0\}$.
  In this case, it follows that $\phi_1 = 0$ and $\phi_2 = y_2$, and we have

$$y_1 = -x_1 - x_2 - \tfrac{2}{3}(x_1 - x_2) \geq 0 \Leftrightarrow -5x_1 \geq x_2$$
$$y_2 = x_1 - x_2 - (-x_1 - x_2) \geq 0 \Leftrightarrow x_1 \geq 0.$$

These regions are depicted in Figure 2. An explicit representation for $f(x)$ as in (4) is

Figure 2 – Partition of $\mathbb{R}^2$ for $f(x)$ defined by (21), (29).



Source: The author

given by

$$f(x) = \begin{cases} x_2, & x \in \Gamma_1 = \{x \in \mathbb{R}^2 | -x_1 < x_2; x_1 < x_2\}, \\ -x_1, & x \in \Gamma_2 = \{x \in \mathbb{R}^2 | x_1 < 0; x_2 \leq -x_1\}, \\ x_1, & x \in \Gamma_3 = \{x \in \mathbb{R}^2 | x_2 \leq x_1; x_2 > -5x_1\}, \\ x_1, & x \in \Gamma_4 = \{x \in \mathbb{R}^2 | 0 \leq x_1; x_2 \leq -5x_1\}. \end{cases}$$

Note that $f(x)$ is the same in the regions Γ_3 and Γ_4. Since, additionally $\Gamma_3 \cup \Gamma_4$ is a convex polyhedron, the explicit representation can be done considering only three regions:

$$f(x)=\begin{cases} x_2, & x\in\Gamma_1=\{x\in\mathbb{R}^2|-x_1<x_2;x_1<x_2\},\\ -x_1, & x\in\Gamma_2=\{x\in\mathbb{R}^2|x_1<0;x_2\le -x_1\},\\ x_1, & x\in\Gamma_3\cup\Gamma_4=\{x\in\mathbb{R}^2|x_2\le x_1;0\le x_1;\}.\end{cases}$$

Example 2: Given a matrix $K\in\mathbb{R}^{n_f\times n}$, and vectors $\overline{\mu}\in\mathbb{R}^{n_f}$ and $\underline{\mu}\in\mathbb{R}^{n_f}$ consider the asymmetric saturation function $f(x)=\text{sat}_{[\underline{\mu},\overline{\mu}]}(Kx):\mathbb{R}^n\to\mathbb{R}^{n_f}$, defined elementwise as follows

$$f_i(x)=\begin{cases}\underline{\mu}_i & x\in D_{1,i}=\{x\in\mathbb{R}^n|(Kx)_i\le\underline{\mu}_i\}\\ K_ix & x\in D_{2,i}=\{x\in\mathbb{R}^n|\underline{\mu}_i<(Kx)_i<\overline{\mu}_i\}\\ \overline{\mu}_i & x\in D_{3,i}=\{x\in\mathbb{R}^n|(Kx)_i\ge\overline{\mu}_i\},\end{cases}$$

$i=1,\ldots,n_f$. To obtain an implicit representation, consider that

$$\begin{aligned} y_j &= -K_jx+\underline{\mu}_j, j=1,...,n_f\\ y_j &= K_jx-\overline{\mu}_{j-n_f}, j=n_f+1,...,2n_f.\end{aligned}$$

Hence:

- If $x\in D_{1,i}$, it follows that $y_i\ge 0$ and $y_{i+n_f}<0$, hence

$$\begin{aligned}\phi_i &= -K_ix+\underline{\mu}_i\\ \phi_{i+n_f} &= 0,\end{aligned}$$

  and $f_i=K_ix+\phi_i-\phi_{i+n_f}=\underline{\mu}_i$.
- If $x\in D_{2,i}$, it follows that $y_i<0$ and $y_{i+n_f}<0$, hence

$$\begin{aligned}\phi_i &= 0\\ \phi_{i+n_f} &= 0,\end{aligned}$$

  and $f_i=K_ix+\phi_i-\phi_{i+n_f}=K_ix$.

- If $x \in D_{3,i}$, it follows that $y_i < 0$ and $y_{i+n_f} \geq 0$, hence

$$\phi_i = 0$$
$$\phi_{i+n_f} = K_1 x - \overline{\mu}_i,$$

and $f_i = K_i x + \phi_i - \phi_{i+n_f} = \overline{\mu}_i$.

This function can be described as in (21) by considering

$$\begin{aligned} F_1 &= K, & F_2 &= \begin{bmatrix} I_{n_f} & -I_{n_f} \end{bmatrix} \\ F_3 &= \begin{bmatrix} -K \\ K \end{bmatrix}, & F_4 &= 0_{n_y \times n_y} \quad f_5 = \begin{bmatrix} \underline{\mu} \\ -\overline{\mu} \end{bmatrix} \end{aligned} \tag{30}$$

with $n_y = 2n_f$. This corresponds to a partition with 3^{n_f} regions (TARBOURIECH et al., 2011; GOMES DA SILVA JR.; TARBOURIECH, 1999), For instance, with $n = 2$, $n_f = 1$, $K = \begin{bmatrix} -1 & 1 \end{bmatrix}$, $\underline{\mu} = -1$ and $\overline{\mu} = 2$ we obtain the following partition of $\mathbb{R}^2$ in terms of y

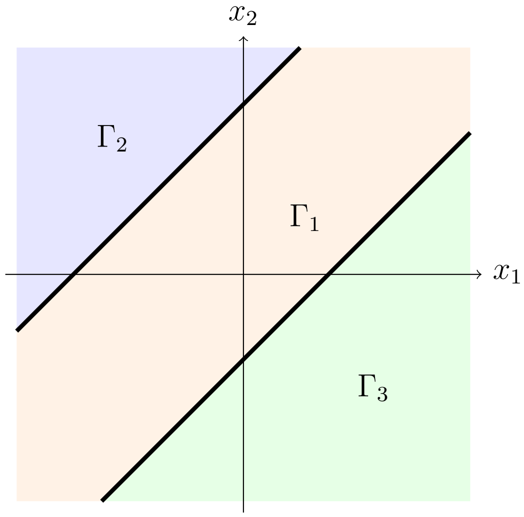
$$\begin{aligned} \Gamma_3 &= \{x \in \mathbb{R}^2 | y_1(x) < 0, y_2(x) \geq 0\} \\ \Gamma_1 &= \{x \in \mathbb{R}^2 | y_1(x) < 0, y_2(x) < 0\} \\ \Gamma_2 &= \{x \in \mathbb{R}^2 | y_1(x) \geq 0, y_2(x) < 0\}, \end{aligned}$$

which is depicted in Figure 3. It is worth to note that the region where $y_1(x) > 0$ and $y_2(x) > 0$ is empty, since the signal cannot be simultaneously above the upper saturation limit and below the lower one. An explicit representation for $f(x)$ as in (4) in this case is given by

$$f(x) = \begin{cases} -1, & x \in \Gamma_3 = \{x \in \mathbb{R}^2 | Kx < -1\} \\ Kx, & x \in \Gamma_1 = \{x \in \mathbb{R}^2 | -1 \leq Kx \leq 2\} \\ 2, & x \in \Gamma_2 = \{x \in \mathbb{R}^2 | 2 < Kx\}. \end{cases}$$

In the following we use (21) as a model for discrete-time PWA systems. The main feature of (21) that will be exploited in the formulation of stability conditions for discrete-time systems is the characterization of the ramp function in terms of identities and inequalities, presented in Section 4.1. These identities and inequalities will be key

Figure 3 – Partition of $\mathbb{R}^2$ for $f(x)$ defined in (30).



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to obtain numerically tractable conditions for the verification of Lyapunov inequalities as carried out in the stability analysis of systems with sector bounded nonlinearities using sector inequalities. The implicit representation also simplifies the stability analysis since the partitions and possible transitions between sets of the partition do not have to be explicitly accounted for in the piecewise quadratic Lyapunov inequalities.

The representation (21) will also be useful for uncertainty representation. Besides considering uncertainties in the dynamics, the representation can also be used to represent uncertainties in the partition by considering uncertain matrices F_3, F_4 and f_5, which cannot be done with an explicit representation. These uncertainties can be described by matrix sets such as polytopic or norm-bounded ones (BOYD et al., 1994).

3.1.1 Conditions for well-posedness

Note that in Example 2 above, the solution to equation (21b) is explicit since $F_4 = 0$, giving $y = F_3x + f_5$. It is then straightforward to compute $f(x)$ using the value of y. Explicit solutions can also be obtained in case matrix F_4 is structured, for instance for a strictly lower or upper triangular structure. As an example, take $m = 4$ and consider the structured algebraic loop, with an upper-triangular matrix F_4

$$\begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} F_{3a} \\ F_{3b} \end{bmatrix} x + \begin{bmatrix} 0_{2\times 2} & \tilde{F}_4 \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} \phi\left(\begin{bmatrix} y_a \\ y_b \end{bmatrix}\right) + \begin{bmatrix} f_{5a} \\ f_{5b} \end{bmatrix}$$

$y_a, y_b \in \mathbb{R}^2$ which can be rewritten as

$$\begin{aligned} y_b &= F_{3b}x + f_{5b} \\ y_a &= F_{3a}x + \tilde{F}_4\phi(F_{3b}x + f_{5b}) + f_{5a} \end{aligned}$$

In general, with $F_4 \neq 0$ (21b) is an implicit equation and the existence of a unique solution y for all $x \in \Re^n$ must be ensured. With this aim, below a condition for the well-posedness of (21b) is provided, that is, the existence and uniqueness of solutions to

$$\tilde{f}(y(x)) = y(x) - F_4\phi(y(x)) = F_3x + f_5 \ \ \forall x \in \mathbb{R}^n. \tag{31}$$

In (ZACCARIAN; TEEL, 2002, Proposition 2) it is shown that for a locally Lipschitz function $\tilde{f}(y(x))$ such that the Jacobian satisfies $J_y\tilde{f}(y) \in \mathcal{M} \subset \mathbb{R}^{n_y \times n_y}$ for almost all $y \in \mathbb{R}^{n_y}$, where $\mathcal{M}$ is a compact, convex set, with each of its elements being non-singular, there exists a unique globally Lipschitz function $y(\xi)$ satisfying $\tilde{f}(y) = \xi$. Such a result is used in (ZACCARIAN; TEEL, 2002) to obtain a condition for the well-posedness of an algebraic loop involving saturation and deadzone functions.

Using the definition of the ramp function in (22), it is true that the Jacobian with respect to y of $\tilde{f}(y)$ in (31) is given by $J_y\tilde{f}(y) = (I - F_4\Delta)$ with $\Delta \in \mathcal{D} = \{\Delta \in \mathbb{D}^n | \Delta_{(i,i)} \in [0, 1]\}$, which is a compact and convex set of matrices. Thus, following (ZACCARIAN; TEEL, 2002, Proposition 2) a unique solution to (31) exists if $(I - F_4\Delta)$ is non-singular for all $\Delta \in \mathcal{D}$. A condition for the well-posedness is then cast as an LMI constraint (see (VALMORBIDA; DRUMMOND; DUNCAN, 2018; ZACCARIAN; TEEL, 2002)) as in the proposition below.

Proposition 1 ((ZACCARIAN; TEEL, 2002, Proposition 1))**.** *If there exist a matrix* $W \in \mathbb{D}^{n_y}$ *such that* $-2W + WF_4 + F_4^\top W < 0$ *then* $(I - F_4\Delta)^{-1}$ *exists* $\forall \Delta \in \mathcal{D}$.

In the following, it will be assumed that the condition for well-posedness of (31) of Proposition 1 holds. Whenever well posed holds, we might be interested to compute the value of the function as for instance, for the implementation of state-feedback PWA control laws. In this case, for a given x ,the PWA function has to be computed to generate a control input. Different methods can be used to solve the algebraic loop yielding the solution to (31). One systematic approach is obtained by casting the equation as a Linear Complementarity Problem (COTTLE; PANG; STONE, 1992) and to obtain its solution, see Remark 4.3 below.

3.1.2 Relation to other implicit representations

Different models for PWA functions have been studied in the literature in the context of nonlinear circuits and control systems. A comparison among different modelling methods is discussed in (HEEMELS; DE SCHUTTER; BEMPORAD, 2001). This section relates the proposed model to other models for PWA systems that do not explicitly define the partition as (4). To this end, two representations from the literature are introduced. These representations yield (31) with a structured matrix F_4 which, as mentioned in section 3.1.1, lead to an explicit solution of the algebraic loop.

3.1.2.1 MMPS functions

This section relates representation (21) to one of the models discussed in (HEEMELS; DE SCHUTTER; BEMPORAD, 2001), namely max-min-plus-scaling (MMPS). Using the results in (HEEMELS; DE SCHUTTER; BEMPORAD, 2001), where the equivalence of MMPS and other PWA models is discussed, we can then relate (21) and the other models studied therein.

An MMPS function is a mapping $f : \mathbb{R}^n \to \mathbb{R}$ which is recursively defined by a grammar (DE SCHUTTER; VAN DEN BOOM et al., 2000)

$$g(x,g_k,g_\ell) := (x_i|\alpha|\max(g_k,g_\ell)|\min(g_k,g_\ell)|g_k) + (g_\ell|\beta g_k) \tag{32}$$

where g_k and g_ℓ are themselves MMPS expressions and $i \in 1,\dots,n$. The symbol “|" in expression (32) denotes an “or" operator (see details in (DE SCHUTTER; VAN DEN BOOM et al., 2000)).

To obtain an MMPS model from (21), we write the ramp function as the MMPS function

$$r(y_i(x)) = \max(0,y_i(x)) \tag{33}$$

which is an expression (32) with $g_k = 0$ and $g_\ell = y_i$. Using (33) we can write $f(x)$ in (21) as an MMPS expression. Let us illustrate these steps with an example.

Example 3: For Example 2 above, using (30) with $n_f = 1$, $n = 2$, we have

$$\begin{cases} f(x) = K_{11}x_1 + K_{12}x_2 - r(y_1) + r(y_2) \\ y_1 = K_{11}x_1 + K_{12}x_2 - \overline{\mu} \\ y_2 = -K_{11}x_1 - K_{12}x_2 + \underline{\mu}. \end{cases}$$

Using (33) it gives

$$f(x) = K_{11}x_1 + K_{12}x_2 - \max(0, K_{11}x_1 + K_{12}x_2 - \overline{\mu}) + \max(0, -K_{11}x_1 - K_{12}x_2 + \underline{\mu}).$$

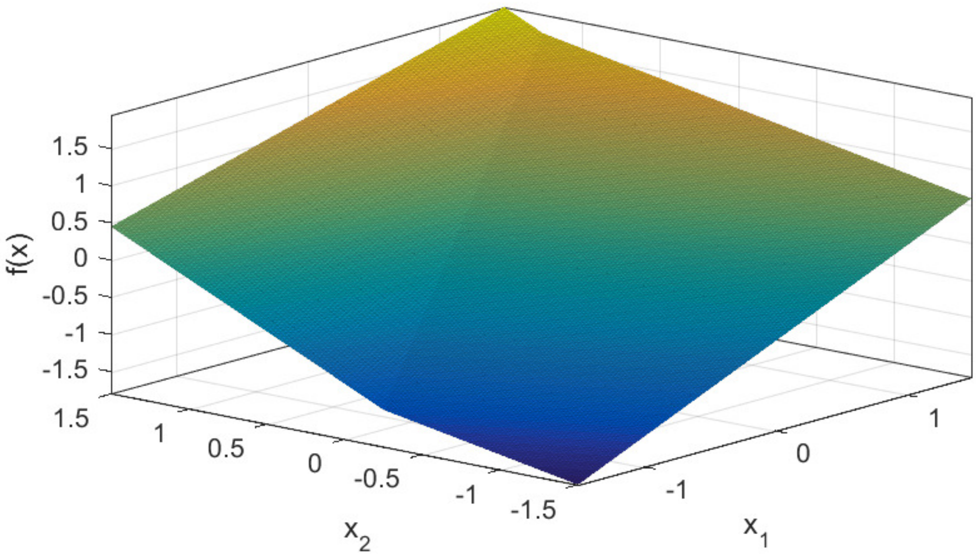
Combining the terms of this expression we can obtain (32), that is an MMPS with

$$\begin{aligned} &g_1 = x_1, \\ &g_2 = x_2, \\ &g_3 = K_{11}g_1, \\ &g_4 = K_{12}g_2, \\ &g_5 = g_3 + g_4, \\ &g_6 = -\overline{\mu}, \\ &g_7 = \underline{\mu}, \\ &g_8 = (-1)g_5, \\ &g_9 = g_5 + g_6, \\ &g_{10} = g_7 + g_8, \\ &g_{11} = \max(0, g_9), \\ &g_{12} = \max(0, g_{10}), \\ &g_{13} = (-1)g_{11}, \\ &g_{14} = g_{12} + g_{13}, \\ &f(x) = g_5 + g_{14}. \end{aligned}$$

Conversely, to obtain (21) from an MMPS expression it suffices to consider the identities $\max(g_k, g_\ell) = -\min(-g_k, -g_\ell)$ and $\max(g_k, g_\ell) = g_k + r(g_\ell - g_k)$ and perform the composition of terms using the expressions g_i.

Example 4: Consider the following function, adapted from (BEMPORAD; ROLL;

Figure 4 – Values of $f(x)$ with the parameters from (34)



Source: The author

LJUNG, 2001):

$$f(x) = 0.8x_1 + 0.4x_2 + \max(0, -0.3x_1 + 0.6x_2 - 0.3)$$

using the identity relating $\max$ and the ramp function, gives

$$\begin{aligned} f(x) &= 0.8x_1 + 0.4x_2 + (0) + r((-0.3x_1 + 0.6x_2 - 0.3) - (0)) \\ &= \begin{bmatrix} 0.8 & 0.4 \end{bmatrix} x + r(\begin{bmatrix} -0.3 & 0.6 \end{bmatrix} x - 0.3)) \end{aligned}$$

that is, defining $y_1 = -0.3x_1 + 0.6x_2 - 0.3$, we can write $f(x)$ as in (21) with

$$\begin{aligned} F_1 &= \begin{bmatrix} 0.8 & 0.4 \end{bmatrix}, & F_2 &= \begin{bmatrix} 1 \end{bmatrix}, \\ F_3 &= \begin{bmatrix} -0.3 & 0.6 \end{bmatrix}, & F_4 &= \begin{bmatrix} 0 \end{bmatrix}, \; f_5 = \begin{bmatrix} -0.3 \end{bmatrix}. \end{aligned} \tag{34}$$

Figure 4 illustrates the function $f(x)$ of the above example.

Other PWA function representations are defined as MMPS functions, such as the HH representation (BREIMAN, 1993; WANG; SUN, 2005) and the LR (TARELA; ALONSO; MARTINEZ, 1990; WEN; MA; YDSTIE, 2009), presented in Section 2.2.1.

The results presented in this section thus apply to obtain an equivalent representation in the form of (21) for both HH and LR. The possibility of readily obtaining an implicit PWA representation for systems in the HH form is particularly interesting, since there are several methods of system identification using the HH representation (WEN et al., 2007; XU; HUANG; WANG, 2009; XU et al., 2020), but stability analysis methods are scarce. Thus, it is possible to identify the systems using HH methods and transform the result into an implicit PWA model to perform the stability analysis.

3.1.2.2 PWA Canonical Representation

This section discusses the relation between (21) and a representation introduced in the context of non-linear circuit analysis, the so-called canonical representation for PWA functions (KAHLERT; CHUA, 1992), (JULIAN, 2003). In particular, it is shown that from the canonical representation, it is always possible to obtain (21) with a particular lower triangular block structure for matrix F_4. This structure leads to explicit solutions for equation (21b).

The main definitions required to obtain the canonical representation as presented in (JULIAN, 2003) are briefly recalled. The basic element for the generation of the canonical form are the N_h hyperplanes generating the partition of the state space. Each of these hyperplanes can be described by a PWA function $q : \mathbb{R}^n \to \mathbb{R}$, with $q = q_i$, $i = 1, \ldots, N_h$, where

$$q_i(x) = \alpha_i^\top x + \beta_i, \tag{35}$$

$\alpha_i \in \mathbb{R}^n$, $\beta_i \in \mathbb{R}$. Moreover, it relies on the generating function $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$\gamma(v_1,v_2) = \frac{1}{4}\left(||-v_1| + v_2| - ||v_2| - v_1| + |-v_1| + |v_2| - |-v_1 + v_2|\right) \tag{36}$$

from which a family of nested functions γ^k can be obtained as follows: $\gamma^0(v_1) = v_1$, $\gamma^1(v_1) = \gamma(v_1,v_1)$, $\gamma^2(v_1,v_2) = \gamma(v_1,v_2)$, ...

$$\gamma^k(v_1,v_2,\ldots,v_k) = \gamma(v_1,\gamma^{k-1}(v_2,\ldots,v_k)). \tag{37}$$

As stated in (JULIAN, 2003, Theorem 1), any continuous PWA function $f : \mathbb{R}^n \to$

$\mathbb{R}$ can be expressed by a canonical form of level k, $k \geq 1$, which takes the form

$$f(x) = a^\top x + b + \sum_{j=1}^{k} \sum_{\ell=1}^{N_k(j)} c_{j,\ell} \times \gamma^j (d_1^{(j),\ell,m} \tilde{q}(x)^{(j),\ell,1}, \ldots, d_j^{(j),\ell,m} \tilde{q}(x)^{(j),\ell,j}) \tag{38}$$

$a \in \mathbb{R}^n$, $b \in \mathbb{R}$, $c_{j,\ell} \in \mathbb{R}$, $d_k^{(j),\ell,m} \in \mathbb{R}$, where each $\tilde{q}(x)^{(j),\ell,s}$, $s = 1, \ldots, j$, corresponds to some function $q_i(x)$ as in (35), j corresponds to the order of a degenerate intersection from which the arguments of function γ^j are computed, $N_k(j)$ denotes the number of degenerate intersections of order j in the partition, and m is the index associated to one of the degenerated intersections of level $j-1$ generating the intersection of level j with index ℓ. More details about how to obtain (38), namely the functions $\tilde{q}(x)^{(j),\ell,s}$ and the corresponding coefficients $d_s^{(j),\ell,m}$ can be found in (JULIAN, 2003).

Since

$$|v_1| = |-v_1| = 2r(v_1) - v_1 \tag{39}$$

and from the recursive definition of γ^k in (37), the expression (38) can be re-written using ramp functions. Thus, generalizing (37) for a vector function $f : \mathbb{R}^n \to \mathbb{R}^{n_f}$, it is true that for each level j, new vectors $y^{(j)} \in \mathbb{R}^{n_{y(j)}}$ and corresponding functions $\phi(y^{(j)})$ have to be defined. In order to do so, first consider the following lemma.

Lemma 1. *The following identity is verified*

$$\gamma(v_1,v_2) = r(v_1 - r(v_1 - v_2)) \tag{40}$$

Proof. From (39), it follows that

$$r(v_1 - r(v_1 - v_2)) = \frac{1}{4}\left(v_1 + v_2 + |v_1 + v_2 - |v_1 - v_2|| - |v_1 - v_2|\right) \tag{41}$$

Now consider the following relation

$$4\gamma(v_1,v_2) + |-v_1 + v_2| = 4r(v_1 - r(v_1 - v_2)) + |v_1 - v_2|,$$

which leads to

$$||-v_1| + v_2| - ||v_2| - v_1| + |-v_1| + |v_2| = v_1 + v_2 + |v_1 + v_2 - |v_1 - v_2||, \qquad (42)$$

and can be demonstrated by analyzing the following cases:

Case 1: $v_1 < 0$, $v_2 > 0$. Then

$$\begin{aligned} \overbrace{|-v_1+v_2| - |v_2 - v_1|}^{0} -v_1 + v_2 &= v_1 + v_2 + |v_1 + v_2 - |v_1 - v_2|| \\ -v_1 + v_2 &= v_1 + v_2 + |v_1 + v_2 - v_2 + v_1| \\ -v_1 + v_2 &= v_1 + v_2 + |2v_1|. \end{aligned}$$

Case 2: $v_1 > 0$, $v_2 < 0$. Then

$$\begin{aligned} \overbrace{|v_1+v_2| - |-v_2 - v_1|}^{0} +v_1 - v_2 &= v_1 + v_2 + |v_1 + v_2 - |v_1 - v_2|| \\ v_1 - v_2 &= v_1 + v_2 + |v_1 + v_2 - v_1 + v_2| \\ v_1 - v_2 &= v_1 + v_2 + |2v_2|. \end{aligned}$$

Case 3: $v_1 > 0$, $v_2 > 0$, $v_1 > v_2$. Then

$$\begin{aligned} |v_1 + v_2| - |v_2 - v_1| + v_1 + v_2 &= v_1 + v_2 + |v_1 + v_2 - |v_1 - v_2|| \\ v_1 + v_2 - v_1 + v_2 + v_1 + v_2 &= v_1 + v_2 + |v_1 + v_2 - v_1 + v_2| \\ v_1 + 3v_2 &= v_1 + v_2 + |2v_2|. \end{aligned}$$

Case 4: $v_1 > 0$, $v_2 > 0$, $v_1 < v_2$. Then

$$\begin{aligned} |v_1 + v_2| - |v_2 - v_1| + v_1 + v_2 &= v_1 + v_2 + |v_1 + v_2 - |v_1 - v_2|| \\ v_1 + v_2 + v_1 - v_2 + v_1 + v_2 &= v_1 + v_2 + |v_1 + v_2 + v_1 - v_2| \\ 3v_1 + v_2 &= v_1 + v_2 + |2v_1|. \end{aligned}$$

Case 5: $v_1 < 0$, $v_2 < 0$, $v_1 > v_2$. Then

$$\begin{aligned} |-v_1 + v_2| - |-v_2 - v_1| - v_1 - v_2 &= v_1 + v_2 + |v_1 + v_2 - |v_1 - v_2|| \\ v_1 - v_2 + v_2 + v_1 - v_1 - v_2 &= v_1 + v_2 + |v_1 + v_2 - v_1 + v_2| \\ v_1 - v_2 &= v_1 + v_2 + |2v_2|. \end{aligned}$$

Case 6: $v_1 < 0$, $v_2 < 0$, $v_1 < v_2$. Then

$$\begin{aligned} |-v_1+v_2| - |-v_2-v_1| - v_1 - v_2 &= v_1 + v_2 + |v_1+v_2-|v_1-v_2|| \\ -v_1+v_2+v_2+v_1-v_1-v_2 &= v_1+v_2+|v_1+v_2+v_1-v_2| \\ -v_1+v_2 &= v_1+v_2+|2v_1|. \end{aligned}$$

Thus, from the definition of γ in (36), (39) and (41), it follows that

$$\begin{aligned} 4\gamma(v_1,v_2) + |-v_1+v_2| &= 4r(v_1 - r(v_1-v_2)) + |v_1-v_2| \\ 4\gamma(v_1,v_2) + |v_1-v_2| &= 4r(v_1 - r(v_1-v_2)) + |v_1-v_2| \end{aligned} \tag{43}$$

holds, hence the identity (40) is true. □

Now, consider the k-th level function

$$\gamma^k = \gamma(d_1^{(j),\ell,m}\tilde{q}(x)^{(j),\ell,1},\gamma^{k-1}), \tag{44}$$

with $\tilde{q}(x)^{(j),\ell,1} = \alpha^{(j),\ell,1}x + \beta^{(j),\ell,1}$. From (40), it follows that

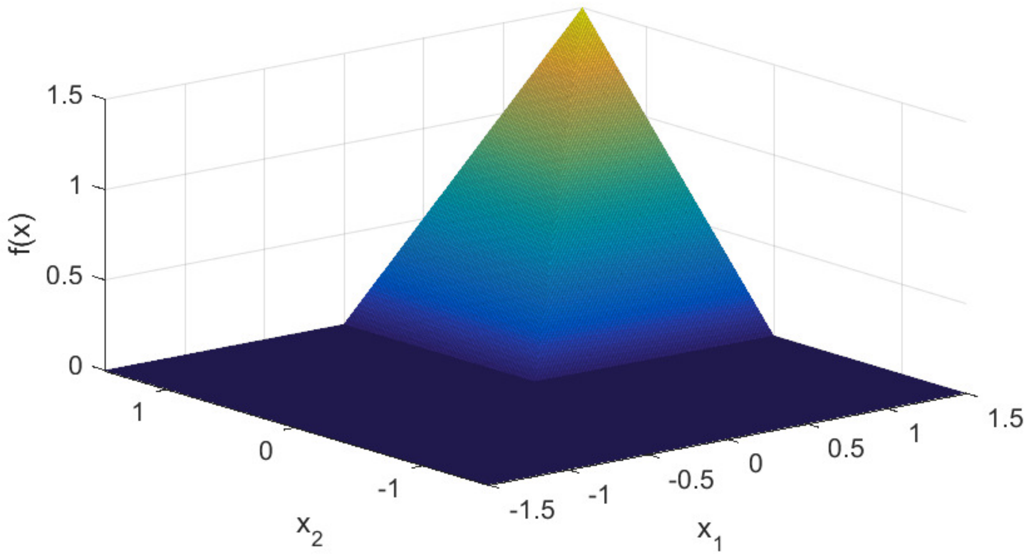
$$\gamma^k = r(\alpha^{(j),\ell,1}x + \beta^{(j),\ell,1} - r(\alpha^{(j),\ell,1}x + \beta^{(j),\ell,1} - \gamma^{k-1})). \tag{45}$$

Applying this relation recursively, it leads to a representation (21) with the following structure

$$\begin{gathered} F_1 = A_k,\ F_2 = \begin{bmatrix} F_2^{(1)} & F_2^{(2)} & \cdots & F_2^{(k)} \end{bmatrix} \\ F_3 = \begin{bmatrix} F_3^{(1)} \\ F_3^{(2)} \\ F_3^{(3)} \\ \vdots \\ F_3^{(k)} \end{bmatrix}, F_4 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ F_4^{(2),1} & 0 & 0 & \cdots & 0 \\ F_4^{(3),1} & F_4^{(3),2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_4^{(k),1} & F_4^{(k),2} & \cdots & F_4^{(k),k-1} & 0 \end{bmatrix} \\ f_5 = \begin{bmatrix} f_5^{(1)\top} & f_5^{(2)\top} & \cdots & f_5^{(k)\top} \end{bmatrix}^\top. \end{gathered} \tag{46}$$

To illustrate, consider the function $f(x) = \gamma(x_1,x_2)$, used as a basis function in (JULIAN; DESAGES; AGAMENNONI, 1999). Using the identity (40), this function

Figure 5 – Values of $f(x)$ with the parameters from (47)



Source: The author

can be written as:

$$f(x) = r(x_1 - r(x_1 - x_2)).$$

which, defining $y_1 = x_1 - x_2$ and $y_2 = x_1 - \phi(y_1)$, can be put in the form (21) with

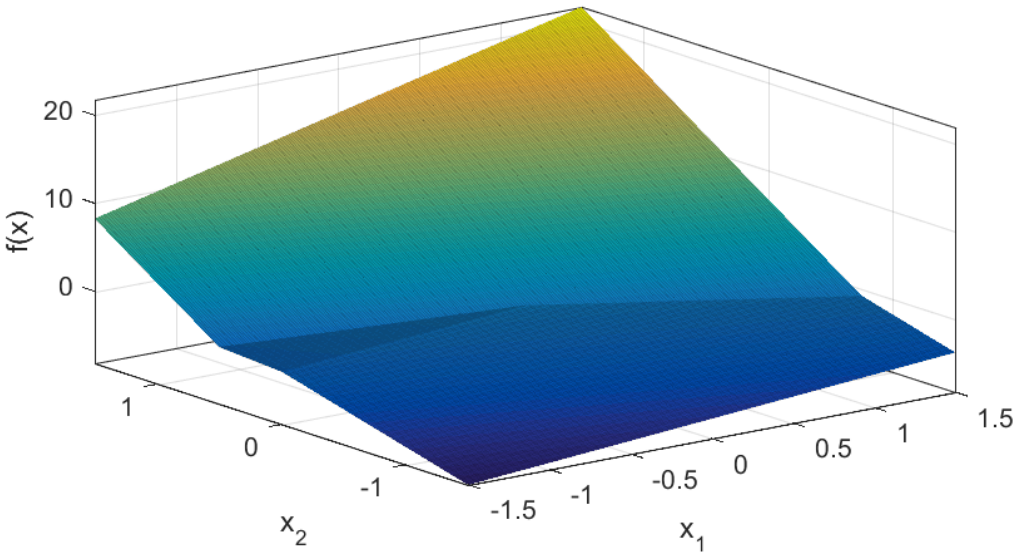
$$\begin{aligned} F_1 &= \begin{bmatrix} 0 & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \\ F_3 &= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \; F_4 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, f_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{47}$$

The plot of the function $f(x)$ can be seen in Figure 5.

The relation (39) can also be used to find an equivalent representation for functions in the canonical form (6). For instance, consider the following example, taken from (GUZELIS; GOKNAR, 1991)

$$f(x) = 1.5x_1 + 2.5x_2 - 1.5|x_2| + 1.5|x_1 + x_2 - |x_2||$$

Figure 6 – Values of $f(x)$ with the parameters from (48)



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Using the identity (39), it follows that

$$-1.5|x_2| = 1.5x_2 - 3r(x_2)$$

and that

$$\begin{aligned} 1.5|x_1 + x_2 - |x_2|| &= 1.5|x_1 + 2x_2 - 2r(x_2)| \\ &= -1.5x_1 - 2x_2 + 2r(x_2) + 3r(x_1 + 2x_2 - 2r(x_2)) \end{aligned}$$

so that this function can be written as

$$f(x) = 2x_2 - r(x_2) + 3r(x_1 + 2x_2 - 2r(x_2)),$$

which, defining $y_1 = x_2$ and $y_2 = x_1 + 2x_2 - 2\phi(y_1)$ can be put in the form (21) with

$$\begin{aligned} F_1 &= \begin{bmatrix} 0 & 2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -1 & 3 \end{bmatrix}, \\ F_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}, \quad f_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{48}$$

The Figure 6 illustrates the surface of the function $f(x)$.

3.1.3 Existence of an Equivalent Representation with F_4 Strictly Triangular

Depending on the structure of the matrix F_4 of the representation (21), an implicit PWA function representation might require the solution of an algebraic loop. However, an explicit solution can always be computed if the matrix F_4 is strictly triangular.

The next proposition states that given a continuous PWA function, there always exists an implicit representation (21) with F_4 strictly triangular.

Proposition 2. *Any continuous PWA function $f : \mathbb{R}^n \to \mathbb{R}$ can be expressed as in* (21) *with a strictly triangular matrix F_4.*

Proof. Since any continuous PWA function $f : \mathbb{R}^n \to \mathbb{R}$ can be expressed as (38), and from this representation it is always possible to find an equivalent representation (21) with a strictly triangular matrix F_4 as discussed in the previous section, then the proposition is true. □

In Example 1 above, the implicit function is described with an F_4 matrix that is not strictly triangular, and thus requires the solution of an algebraic loop. However, from Proposition 2, there must exist an implicit PWA function that does not require the solution of an algebraic loop for that particular $f(x)$. Indeed, that same function can be written in the form (21) with

$$\begin{aligned} F_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad & F_2 &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \\ F_3 &= \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad & F_4 &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; f_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{49}$$

which admits an explicit solution, since y_1 can be computed directly from the state x, and y_2 can then be obtained from x and y_1.

3.1.4 Existence of a Representation with Non-Positive f_5

Finding a representation with a non-positive vector f_5 will be useful in the study of the stability of PWA systems, as will be seen in Chapter 4. In this sense, we here demonstrate that such a representation can always be found, provided $f(0) = 0$.

Proposition 3. *Any continuous PWA function* $f : \mathbb{R}^n \to \mathbb{R}$ *with* $f(0) = 0$ *can be expressed as in* (21)*, with* $f_5 \preccurlyeq 0$.

Proof. First, let us note that the ramp function r satisfies the identity

$$r(y_i) = y_i + r(-y_i). \tag{50}$$

Indeed, if $y_i \geq 0$, $r(y_i) = y_i$ and $r(-y_i) = 0$. On the other hand, if $y_i < 0$, $r(y_i) = 0$ and $r(-y_i) = -y_i$. Now, suppose that F_4 is strictly triangular, which is always possible for a continuous PWA function, as stated in Proposition 2. Also suppose that the i-th element $f_{5,i}$ of the vector f_5 is positive, and that no element $f_{5,j}$, with $j < i$, is positive. We now want to represent the implicit PWA function in terms of a new algebraic vector $\tilde{y}$, which satisfies $\tilde{y}_j = y_j, \forall j \neq i$ and $\tilde{y}_i = -y_i$, hence the constant term associated to the i-th element of $\tilde{y}$ is negative. Define $\mathcal{Q}_i = \text{diag}(0_{i-1}, -1, 0_{n_y - i})$, $\mathcal{Q}_{iF} = (I_{n_y} + 2\mathcal{Q}_i)$, then the following equivalent representation can be found

$$f(x) = F_1 x + F_2(\phi(\tilde{y}) - \mathcal{Q}_i(F_3 x + F_4 \phi(\tilde{y}(x)) + f_5)) \tag{51a}$$

$$\tilde{y}(x) = \mathcal{Q}_{iF}(F_3 x + F_4 \phi(\tilde{y}(x)) + f_5 - F_4 \mathcal{Q}_i(F_3 x + f_5)) \tag{51b}$$

To demonstrate this, first note that from the definition of $\tilde{y}$, $(I + \mathcal{Q}_i)\tilde{y} = (I + \mathcal{Q}_i)y$, and $\mathcal{Q}_i \tilde{y} = -\mathcal{Q}_i y$, hence,

$$\tilde{y} = (I + 2\mathcal{Q}_i)y = \mathcal{Q}_{iF}(F_3 x + F_4 \phi(y) + f_5). \tag{52}$$

Similarly, from the definition of $\tilde{y}$, it follows that $(I + \mathcal{Q}_i)\phi(\tilde{y}) = (I + \mathcal{Q}_i)\phi(y)$ and $\mathcal{Q}_i \phi(\tilde{y}) = \mathcal{Q}_i \phi(-y)$. Hence,

$$\phi(\tilde{y}) = \phi(y) + \mathcal{Q}_i \phi(y) - \mathcal{Q}_i \phi(-y).$$

From (50), this is equivalent to

$$\phi(\tilde{y}) = \phi(y) + \mathcal{Q}_i y,$$

and thus

$$\phi(y) = \phi(\tilde{y}) - \mathcal{Q}_i y = \phi(\tilde{y}) - \mathcal{Q}_i(F_3 x + F_4 \phi(y) + f_5)$$

Since F_4 is strictly diagonal, if follows that $\mathcal{Q}_i F_4 \phi(y) = \mathcal{Q}_i F_4 \phi(\tilde{y})$, thus

$$\phi(y) = \phi(\tilde{y}) - \mathcal{Q}_i(F_3 x + F_4 \phi(\tilde{y}) + f_5)$$

Applying this relation to (52), and taking into account that $F_4 \mathcal{Q}_i F_4 = 0$, yields (51b). Also, applying this relation to (21a) yields (51a).

Also note that, since F_4 is strictly lower diagonal, the first i elements of $(\mathcal{Q}_{iF} + F_4 \mathcal{Q}_i) f_5$ are non-positive. The iterative application of this procedure so that all elements of f_5 is non-positive yields

$$\begin{aligned} f(x) &= \bar{F}_1 x + \bar{F}_2 \phi(\bar{y}(x)) + \sigma \\ \bar{y}(x) &= \bar{F}_3 x + \bar{F}_4 \phi(\bar{y}(x)) + \bar{f}_5, \end{aligned} \tag{53}$$

where $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$, $\bar{F}_4$, $\bar{f}_5$ and σ are found by iterating (51). Since all elements of $\bar{f}_5$ are non-positive and the system is in the form (46), it is clear that $\phi(\tilde{y}(0)) = 0$. Thus, since $f(0) = 0$, it necessarily follows that $\sigma = 0$, and (53) is an equivalent representation with $\bar{f}_5 \preccurlyeq 0$. □

To illustrate this procedure, consider the following representation of the symmetric saturation function, with $n = 1$ and $n_f = 1$.

$$\begin{aligned} f(x) &= \overbrace{-K}^{F_1} x + \overbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}^{F_2} \phi(y) \\ y &= \underbrace{\begin{bmatrix} -K \\ K \end{bmatrix}}_{F_3} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{F_4} \phi(y) + \underbrace{\begin{bmatrix} \mu \\ \mu \end{bmatrix}}_{f_5}. \end{aligned}$$

The first iteration of (51) yields:

$$\begin{aligned} f(x) &= 0x + \begin{bmatrix} -1 & 1 \end{bmatrix} \phi(y) - \mu \\ y &= \begin{bmatrix} K \\ K \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \phi(\bar{y}) + \begin{bmatrix} -\mu \\ \mu \end{bmatrix}. \end{aligned}$$

The second iteration yields:

$$
\begin{aligned}
f(x) &= \overbrace{K}^{\bar{F}_1} x + \overbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}^{\bar{F}_2} \phi(\bar{y}) \\
\bar{y} &= \underbrace{\begin{bmatrix} K \\ -K \end{bmatrix}}_{\bar{F}_3} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\bar{F}_4} \phi(y) + \underbrace{\begin{bmatrix} -\mu \\ -\mu \end{bmatrix}}_{\bar{f}_5},
\end{aligned}
$$

which is an equivalent representation, and $\bar{f}_5$ has only non-positive elements.

3.2 Final Remarks

In this chapter, a representation for PWA system was presented. This representation is based on a ramp function and implicitly encodes the system partition, hence not requiring that the regions and transitions are enumerated. The representation is composed of two equations, one of which can present an algebraic loop. In this case, conditions for the well-posedness, that is, the existence and uniqueness, of functions in the proposed form were presented. Furthermore, it was shown that any continuous PWA function can be represented in the proposed form, and a representation with an explicit solution of the algebraic variables always exist. Finally, the relation of the proposed representation with other representation for PWA functions was illustrated.

This representation will be used in the next Chapters to study the stability of PWA systems.

4 STABILITY OF IMPLICIT PIECEWISE AFFINE SYSTEMS

We proposed a new representation for PWA functions in the previous chapter. The proposed representation in the study of PWA systems will be suitable for stability analysis of discrete-time systems and, in particular, for systems usisng event triggered control strategies. In this chapter, conditions for the stability analysis of systems in the *implicit* PWA representation are presented. We formulate conditions for the exponential stability of the origin of PWA systems using piecewise quadratic Lyapunov candidate functions. To verify the stability conditions, expressed in terms of generalized quadratic forms conditions depending on x and the ramp functions describing PWA functions $\phi(y(x))$, we propose linear matrix inequalities that can be solved using semi-definite programming (SDP).

4.1 Positivity of Generalized Quadratic Forms

In this section, we describe ramp functions as a set of inequalities and identities. Using these identities we then propose conditions to verify the positivity (non-negativity) of generalized quadratic forms. These conditions will be instrumental for the stability analysis of PWA systems.

4.1.1 Preliminary Results

The properties of sector bounded nonlinearities have been used to study stability of nonlinear systems since the results from Lur'e. In this context, several results to verify the positivity of generalized quadratic forms involving sector nonlinearities rely on sector inequalities that hold either globally or locally (HU; TEEL; ZACCARIAN, 2006; TARBOURIECH et al., 2011). The main drawback in this case is that these standard sector inequalities cover a broad class of nonlinearities lying in the considered sector. In consequence, the conditions implicitly certifies the stability not only for the particular system of interest, but for a large class of systems, leading to an inherent conservatism.

In particular, the ramp function is a nonlinearity that lies in the sector $[0,\ \ 1]$. An immediate approach would be to consider a sector bound relation to formulate stability conditions for a system presenting such nonlinearity. However, in case it is the only nonlinearity in the system description, such as PWA systems using the representation presented in the previous chapter, we might be interested in a description that is more precise than the sector inequality. It is therefore suitable, if possible, to consider more information about the specific nonlinearity to derive the stability conditions.

In the following we show how to obtain a finer characterization of the ramp function (22), by using identities and inequalities. These relations will be instrumental to formulate stability conditions for PWA systems, namely dynamical systems defined by PWA functions. The key properties of the ramp function can be summarized in the following lemma.

Lemma 2. *The ramp function $r(\theta)$ satisfies the relations*

$$(r(\theta) - \theta)\, r(\theta) = 0 \tag{54a}$$

$$r(\theta) \geq 0 \tag{54b}$$

$$(r(\theta) - \theta) \geq 0. \tag{54c}$$

Proof. Since for $\theta < 0$ we have $r(\theta) = 0$ and $(r(\theta) - \theta) = (0 - \theta) > 0$, and for $\theta \geq 0$ we have that $(r(\theta) - \theta) = 0$, the relations (54a) and (54c) hold for all $\theta \in \mathbb{R}$. Inequality (54b) comes directly from the definition of the ramp function. □

From Lemma 2, the following generalizations regarding the function ϕ can be stated.

Lemma 3. *For any matrix $T \in \mathbb{D}^{n_y}$ the function ϕ in* (22) *satisfies the identity*

$$s_1(T, y) := \phi^\top(y) T(\phi(y) - y) = 0,\ \forall y \in \mathbb{R}^{n_y}. \tag{55}$$

Proof. Since the elements of ϕ are ramp functions we have

$$s_1(T,y) = \sum_{i=1}^{n_y} T_{i,i} r(y_i)(r(y_i) - y_i)$$

which, using (54a) gives (55). □

Lemma 4. *For any matrix $M \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$ the vector function ϕ in* (22) *satisfies the inequality*

$$s_2(M,y) := \begin{bmatrix} 1 \\ \phi(y) \\ \phi(y) - y \end{bmatrix}^\top M \begin{bmatrix} 1 \\ \phi(y) \\ \phi(y) - y \end{bmatrix} \geq 0, \forall y \in \mathbb{R}^{n_y} \tag{56}$$

Proof. We have

$$\begin{aligned} s_2(M,y) = {} & M_{11} + 2\textstyle\sum_{i=1}^{n_y} M_{1(1+i)} r(y_i) + M_{1(1+n_y+i)}(r(y_i) - y_i) \\ & + \textstyle\sum_{i=1}^{n_y}\sum_{j=1}^{n_y+1} \big(M_{(1+i)(1+j)} r(y_i) r(y_j) \\ & + 2M_{(1+n_y+i)(1+j)} r(y_i)(r(y_j) - y_j) \\ & + M_{(1+n_y+i)(1+n_y+j)}(r(y_i) - y_i)(r(y_j) - y_j)\big) . \end{aligned}$$

Since each element of matrix M is non-negative and the ramp function verifies (54b) and (54c) it is concluded that each term in the above expression is not negative, therefore (56) holds. This condition, however, might introduce some conservatism. □

Remark 4.1. *In (PRIMBS; GIANNELLI, 2001), relations describing the saturation nonlinearity were obtained using the Karush-Kuhn-Tucker (KKT) optimality conditions. The relations in Lemma 2 can be obtained using the same approach. A ramp function can be expressed as the solution to an optimization problem parameterized in θ as follows*

$$\underset{r}{\text{minimize}} \ \frac{1}{2}(r-\theta)^2 \quad \textit{subject to} \quad -r \leq 0. \tag{57}$$

Note that if $\theta \geq 0$, the optimal value is given by $r = \theta$. On the other hand, if $\theta < 0$, the optimal value is $r = 0$. With the Lagrangian associated to the optimization problem, $\mathcal{L}(r,\lambda) = \frac{1}{2}(r-\theta)^2 - \lambda r$, we obtain the KKT conditions

$$(r-\theta) - \lambda = 0;\ \lambda \geq 0;\ r \geq 0;\ \lambda r = 0 \tag{58}$$

which are necessary optimality conditions (note that $r \geq 0$ comes from the optimization problem). These relations offer a characterization in terms of linear and quadratic identities and inequalities in three variables (θ,r,λ). To obtain a description in the variables (θ,r) one can use $\lambda = (r - \theta)$ above to obtain

$$(r - \theta) \geq 0 \tag{59a}$$

$$r \geq 0 \tag{59b}$$

$$(r - \theta)r = 0. \tag{59c}$$

These relations correspond to (54a), (54b) *and* (54c)*, therefore they are also necessary to describe the ramp function.*

4.1.2 Conditions for Positivity of Generalized Quadratic Forms

In this section, the above lemmas are used to set conditions to verify the positivity of generalized quadratic forms of the type

$$h(x) = \begin{bmatrix} 1 \\ x \\ \phi(y(x)) \\ \phi(y(x)) - y(x) \end{bmatrix}^\top H \begin{bmatrix} 1 \\ x \\ \phi(y(x)) \\ \phi(y(x)) - y(x) \end{bmatrix} = \chi(x)^\top H \chi(x), \tag{60}$$

where the dependence of y on x follows from the implicit equation described in (21).

Proposition 4. *Given a generalized quadratic form $h(x)$ as in* (60)*, if there exist matrices $T \in \mathbb{D}^{n_y}$, and $M \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$ such that*

$$h(x) + 2s_1(T, y(x)) - s_2(M, y(x)) \geq 0 \tag{61}$$

then

$$h(x) \geq 0 \quad \forall x \in \mathbb{R}^n. \tag{62}$$

Proof. From Lemma 3, which hold for all $y(x)$, if (61) is satisfied it follows that

$$h(x) \geq s_2(M, y(x)), \forall x \in \mathbb{R}^n.$$

Then, using Lemma 4, it is concluded that $h(x) \geq 0, \forall x \in \mathbb{R}^n$. □

Remark 4.2. *The verification of the non-negativity of a generalized quadratic form as* (60) *from the solution to the inequality* (61) *makes possible to verify Lyapunov inequalities for PWA systems. Moreover, if matrix H is affine on unknown variables, the inequality* (61) *can be cast as an LMI, therefore yielding constraints of a semi-definite program which can be solved with freely available optimization software (LOFBERG, 2004).*

Remark 4.3. *The relations given in* (54) *can be used to obtain a solution to the algebraic loop* (21b)*. From* (54) *and* (22)*, it follows that*

$$(\phi_i - y_i) \geq 0 \tag{63a}$$

$$\phi_i \geq 0 \tag{63b}$$

$$(\phi_i - y_i)\phi_i = 0, \tag{63c}$$

$i = 1, \ldots, n_y$*. Set* $\xi = F_3 x + f_5$ *in equation* (21b)*, and use* $y_i = (F_4\phi + \xi)_i$ *in the above expressions to obtain respectively*

$$((I - F_4)\phi - \xi)_i \geq 0 \tag{64a}$$

$$\phi_i \geq 0 \tag{64b}$$

$$((I - F_4)\phi - \xi)_i \phi_i = 0. \tag{64c}$$

$i = 1, \ldots, n_y$.

The problem of solving for ϕ the above inequalities (64a), (64b) *affine in ϕ, and equations* (64c)*, quadratic in ϕ, is called a mixed Linear Complementarity Problem (LCP). For a given ξ, the solution ϕ to* (64) *thus provides a solution to the implicit equation $y = F_4\phi(y) + \xi$. Please refer to the Lemke algorithm presented in (ADEGBEGE; HEATH, 2017, Section 5.1) for a strategy to solve LCPs yielding solutions to algebraic loops. Also, as one should expect, the condition for the well posedness of LCPs in (ADEGBEGE; HEATH, 2017, Proposition 7.1) applied to* (64) *holds if the condition in Proposition 1 is satisfied.*

4.2 Global Stability Analysis

In this section the results for the verification of non-negativity of generalized quadratic forms presented in the previous section are applied to study the global stability of discrete-time systems defined by the implicit PWA function (21). In other words, we are interested in the stability analysis of the following system:

$$x^+ = f(x) = F_1 x + F_2 \phi(y) \tag{65a}$$

$$y = F_3 x + F_4 \phi(y) + f_5, \tag{65b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n_y}$, $F_1 \in \mathbb{R}^{n\times n}$, $F_2 \in \mathbb{R}^{n\times n_y}$, $F_3 \in \mathbb{R}^{n_y\times n}$, $F_4 \in \mathbb{R}^{n_y\times n_y}$, $f_5 \in \mathbb{R}^{n_y}$, the vector function $\phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ is defined elementwise by the ramp function given in (22). For notational simplicity we dropped the dependence on the discrete time k, $k \in \mathbb{N}$, i.e. we consider that $x = x(k)$ and $x^+ = x(k+1)$. From (65b) we have

$$y^+ = F_3 x^+ + F_4 \phi(y^+) + f_5. \tag{66}$$

It is assumed that $x = 0$ is an equilibrium point, i.e $f(0) = 0$. It is also assumed that $f_5 \preccurlyeq 0$, which, according to Proposition 3, is always possible to obtain a representation (21) satisfying this. This condition will be required to guarantee an upper bound for PWQ Lyapunov function candidates

The following theorem provides conditions for the global exponential stability of the origin of (65a)-(65b), considering a generic Lyapunov function candidate and the results from Lemmas 3 and 4.

Theorem 4.1. *If there exist a function $V : \mathbb{R}^n \to \mathbb{R}$, matrices $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $M_1 \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$, $M_2 \in \mathbb{P}^{(1+4n_y)\times(1+4n_y)}$ and positive scalars $\eta < 1$, ϵ_1 and ϵ_2 such that the following inequalities are verified along the trajectories of* (65)*, for all $x \in \mathbb{R}^n$*

$$V(x) \leq \epsilon_2 x^\top x \tag{67}$$

$$(V(x) - \epsilon_1 x^\top x) + s_1(T_1, y(x)) - s_2(M_1, y(x)) \geq 0 \tag{68}$$

$$-(V(x^+) - (1-\eta)V(x)) + s_1(T_2, \tilde{y}(x)) - s_2(M_2, \tilde{y}(x)) \geq 0 \tag{69}$$

where $\tilde{y} = \begin{bmatrix} y^\top & y^{+\top} \end{bmatrix}^\top$*, with* y^+ *given by* (66)*, then the origin is globally exponentially stable.*

Proof. From Lemmas 3 and 4 , if (67), (68) and (69) hold it follows that

$$\epsilon_1 \|x\|^2 \leq V(x) \leq \epsilon_2 \|x\|^2 \tag{70a}$$

$$V(x^+) \leq (1-\eta)V(x). \tag{70b}$$

Note that (70a) implies that $V(x)$ is radially unbounded. Thus, (70) allows to conclude that $\Delta V(x) = V(x^+) - V(x) \leq -\eta V(x)$, which ensures that $\lim_{k\to\infty} x(k) = 0$, $\forall x(0) \in \mathbb{R}^n$. Moreover, it follows from (70) that $V(x(k)) \leq (1-\eta)^k V(0)$, and $\epsilon_1 \|x(k)\| \leq \epsilon_2 (1-\eta)^k \|x(0)\|$, hence $\|x(k)\| \leq \gamma \alpha^k \|x(0)\|$ with $\gamma = (\frac{\epsilon_2}{\epsilon_1})^{\frac{1}{2}}$, $\alpha = (1-\eta)^{\frac{1}{2}}$, $\forall x(0) \in \mathbb{R}^n$. which implies the global exponential convergence to the origin. □

As seen in Chapter 2, several results in the literature have studied the class of PWA systems using the explicit representation (4) or alternatives as detailed in (HEEMELS; DE SCHUTTER; BEMPORAD, 2001). Regarding stability analysis, piecewise quadratic Lyapunov functions have been considered and the formulation of stability conditions often requires a first evaluation of the possible transitions between sets of the partition (FENG, 2002; IERVOLINO; VASCA; IANNELLI, 2015; HOVD; OLARU, 2013). Moreover, in general, the possible transitions between sets in the partition do not correspond to the neighbouring sets (as in continuous-time systems) and it might be difficult to obtain all the transitions from a description of each set in the partition. A conservative approach is then to assume that all sets in the partition are reachable in one step. An advantage of Theorem 4.1, with respect to these conditions proposed in the literature, is that the enumeration and the evaluation of the possible transitions is not required. This is implicitly taken into account by the implicit system description.

The idea now is the formulation of testable conditions in LMI form that satisfy conditions in Theorem 4.1 for a suitable Lyapunov function V. With this aim, we consider a continuous piecewise quadratic Lyapunov function, given by a generalized quadratic form on x and the function $\phi(y(x))$. Hence, differently from previous approaches, the definition of an explicit quadratic form on x for each set of the partition is not required. More specifically, the Lyapunov candidate functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq} 0$, $V(0) = 0$ considered are given by

$$V(x) = \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}^\top P \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}. \tag{71}$$

with $P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix}$, $P_1 \in \mathbb{S}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n_y}$, $P_3 \in \mathbb{S}^{n_y\times n_y}$, and $y(x)$ is given by (65b). No *a priori* assumptions are made about the sign definiteness of the matrices P_1, P_2 and P_3. Note that $V(x^+)$ depends on y^+, which is defined in (66). Also, note that this class of PWQ functions includes the class of quadratic functions and contains information about the partition.

To consider V above as a candidate Lyapunov function we need to check the inequalities of Theorem 4.1. Next, we show that it satisfies (67) with an appropriate scalar ϵ_2. For this, defining $\overline{y} := y - f_5$ and substituting it in (65b) yields

$$\overline{y} = F_3 x + F_4\phi(\overline{y} + f_5).$$

Since we assume that $f(0) = 0$ and $f_{5i} \leq 0$, one has that $\overline{y} + f_5 \geq 0$ implies that $\overline{y} \geq -f_5 \geq 0$, and then $\phi(\overline{y}+f_5) = \Delta\overline{y}$ is obtained with $\Delta \in \mathcal{D} = \{\Delta \in \mathbb{D}^n | \Delta_{(i,i)} \in [0,1]\}$. From the well-posedness assumption, $(I - F_4\Delta)$ is invertible for all $\Delta \in \mathcal{D}$, thus

$$\overline{y} = (I - F_4\Delta)^{-1} F_3 x$$

and

$$\phi(y) = \phi(\overline{y} + f_5) = \Delta\overline{y} = \Delta(I - F_4\Delta)^{-1}F_3 x,$$

yielding

$$\|\phi(y(x))\| \leq \sigma \|x\|,$$

with $\sigma = \max_{\Delta\in\mathcal{D}} \|\Delta(I - F_4\Delta)^{-1}F_3\|$. From (71), it follows that

$$\begin{aligned} V(x) &\leq \|P_1\| \|x\|^2 + 2\|P_2\| \|x\| \|\phi\| + \|P_3\| \|\phi\|^2 \\ &\leq (\|P_1\| + 2\sigma\|P_2\| + \sigma^2\|P_3\|) \|x\|^2, \end{aligned} \tag{72}$$

i.e., $\epsilon_2 = \|P_1\| + 2\sigma\|P_2\| + \sigma^2\|P_3\|$.

The relations (68) and (69) can be written in the generic quadratic form given by (60)-(61), where the corresponding matrices H present an affine dependence on the elements of matrix P. Hence, from Remark 4.2, conditions in LMI form can be obtained to ensure (68) and (69).

This is formally stated in the following theorem.

Theorem 4.2. *If there exist matrices* $P_1 \in \mathbb{S}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n_y}$, $P_3 \in \mathbb{S}^{n_y\times n_y}$, $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{2n_y}$, $M_1 \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$, $M_2 \in \mathbb{P}^{(1+4n_y)\times(1+4n_y)}$ *and positive scalars* $\eta < 1$ *and* ϵ_1 *such that the following LMIs are verified*

$$H + He\left(\mathcal{X}_T^\top T_1 \mathcal{I}_T\right) - \mathcal{X}_M^\top M_1 \mathcal{X}_M \geq 0 \tag{73}$$

$$-\tilde{H} + He\left(\tilde{\mathcal{X}}_T^\top T_2 \tilde{\mathcal{I}}_T\right) - \tilde{\mathcal{X}}_M^\top M_2 \tilde{\mathcal{X}}_M \geq 0 \tag{74}$$

with

$$\mathcal{X}_T = \begin{bmatrix} -f_5 & -F_3 & I - F_4 \end{bmatrix}, \quad \mathcal{X}_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_{n_y} \\ -f_5 & -F_3 & I - F_4 \end{bmatrix}$$

$$\mathcal{I}_T = \begin{bmatrix} 0 & 0 & I_{n_y} \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & P_1 - \epsilon_1 I_n & P_2 \\ 0 & P_2^\top & P_3 \end{bmatrix},$$

$$\tilde{\mathcal{X}}_T = \begin{bmatrix} -f_5 & -F_3 & I - F_4 & 0 \\ -f_5 & -F_3F_1 & -F_3F_2 & I - F_4 \end{bmatrix}, \quad \tilde{\mathcal{X}}_M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & I_{n_y} & 0 \\ 0 & 0 & 0 & I_{n_y} \\ -f_5 & -F_3 & -F_4 & 0 \\ -f_5 & -F_3F_1 & -F_3F_2 & I - F_4 \end{bmatrix}$$

$$\tilde{\mathcal{I}}_T = \begin{bmatrix} 0 & 0 & I_{n_y} & 0 \\ 0 & 0 & 0 & I_{n_y} \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & F_1^\top P_1 F_1 - \bar{\eta} P_1 & F_1^\top P_1 F_2 - \bar{\eta} P_2 & F_1^\top P_2 \\ 0 & F_2^\top P_1 F_1 - \bar{\eta} P_2^\top & F_2^\top P_1 F_2 - \bar{\eta} P_3 & F_2^\top P_2 \\ 0 & P_2^\top F_1 & P_2^\top F_2 & P_3 \end{bmatrix},$$

$$\bar{\eta} = 1 - \eta$$

then the origin of (65) *is globally exponentially stable.*

Proof. Consider $V(x)$ defined as in (71) with

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix}$$

Note that this function $V(x)$ satisfies condition (67), as shown in (72).

Consider the vector $\chi = \begin{bmatrix} 1 & x^\top & \phi(y)^\top \end{bmatrix}^\top$. Note that $\mathcal{X}_T \chi = \phi(y) - y$ and $\mathcal{X}_M \chi = \begin{bmatrix} 1 & \phi(y)^\top & (\phi(y) - y)^\top \end{bmatrix}^\top$. Left and right multiplying the matrix in (73) by $\chi^\top$ and χ, respectively, it follows that condition (68) is verified, since $\chi^\top H \chi = V(x)$, $\chi^\top \mathcal{X}_T^\top T_1 \mathcal{I}_T \chi = s_1(T_1, y(x))$ and $\chi^\top \mathcal{X}_M^\top M_1 \mathcal{X}_M \chi = s_2(M_1, y(x))$.

Recalling that $\tilde{y} = [y(x) \;\; y(x^+)]$, consider the vector

$$\tilde{\chi} = \begin{bmatrix} 1 \\ x \\ \phi(\tilde{y}) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ \phi(y) \\ \phi(y^+) \end{bmatrix}.$$

Note that $\tilde{\mathcal{X}}_T \tilde{\chi} = \phi(\tilde{y}) - \tilde{y}$ and $\tilde{\mathcal{X}}_M \tilde{\chi} = \begin{bmatrix} 1 & \phi(\tilde{y})^\top & (\phi(\tilde{y}) - \tilde{y})^\top \end{bmatrix}^\top$. Taking into account that

$$\bar{\eta} V(x) = \tilde{\chi}^\top \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{\eta} P_1 & \bar{\eta} P_2 & 0 \\ 0 & \bar{\eta} P_2^\top & \bar{\eta} P_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tilde{\chi},$$

$$V(x^+) = \begin{bmatrix} x^+ \\ \phi(y^+) \end{bmatrix}^\top P \begin{bmatrix} x^+ \\ \phi(y^+) \end{bmatrix},$$

and

$$\begin{bmatrix} x^+ \\ \phi(y^+) \end{bmatrix} = \begin{bmatrix} 0 & F_1 & F_2 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tilde{\chi},$$

hence

$$V(x^+) = \tilde{\chi}^\top \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & F_1^\top P_1 F_1 & F_1^\top P_1 F_2 & F_1^\top P_2 \\ 0 & F_2^\top P_1 F_1 & F_2^\top P_1 F_2 & F_2^\top P_2 \\ 0 & P_2^\top F_1 & P_2^\top F_2 & P_3 \end{bmatrix} \tilde{\chi},$$

left and right multiplying the matrix in (74), by $\tilde{\chi}^\top$ and $\tilde{\chi}$, respectively, it follows that inequality (69) is verified, since $\tilde{\chi}^\top \tilde{H} \tilde{\chi} = V(x^+) - \bar{\eta} V(x)$, $\tilde{\chi}^\top \tilde{\mathcal{X}}_T^\top T_2 \tilde{\mathcal{I}}_T \tilde{\chi} = s_1(T_2, \tilde{y}(x))$ and $\tilde{\chi}^\top \tilde{\mathcal{X}}_M^\top M_2 \tilde{\mathcal{X}}_M \tilde{\chi} = s_2(M_2, \tilde{y}(x))$.

We conclude that if (73)-(74) are verified, the inequalities of Theorem 4.1 are fulfilled along the trajectories of system (65) and the global exponential stability of the origin follows. □

From Theorem 4.2, we conclude that the stability assessment of the piecewise affine system (65) can be numerically performed through an LMI feasibility problem.

4.3 Robust Stablity Analisys

A usual way to study to stability of uncertain PWA systems is to consider the explicit formulation (4), which for uncertain systems is given by:

$$x^+ = A_i(\lambda)x + b_i(\lambda), \forall x \in \Gamma_i. \tag{75}$$

where $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^{n_\lambda}$ is a vector of uncertain parameters. $\{\Gamma_i\}_{i \in \mathbb{I}} = \mathbb{R}^n$, where $\mathbb{I}$ is an index set, denotes a partition of the state space into a number of closed polyhedral sets.

This explicit formulation is for instance considered in (TRIMBOLI; RUBAGOTTI; BEMPORAD, 2011) where a PWA Lyapunov candidate function is used to demonstrate the stability of the system, and in (HOVD; OLARU, 2018), where a parameter dependant PWQ Lyapunov candidate function is used. While these approaches are effective when the system dynamics in each subset Γ_i is not precisely known, it assumes a partition given *a priori*, so it is difficult to treat the cases where the uncertain parameter affects the partition, even more so when the number of subsets in the partition changes. In particular, (NGUYEN et al., 2016) provides conditions to analyze the stability of linear systems under PWA control laws when the partition is also uncertain, but the method is based on enumerating the vertices of the partition, so it cannot be applied when the number of vertices can change due to the uncertainty.

To elucidate the problem, consider a two-dimensional uncertain piecewise linear system described by (65) with

$$F_2 = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$F_3 = \begin{bmatrix} -0.15 & 0.15 \\ 0.15 & -0.15 \\ d_1 & d_2 \end{bmatrix}, \quad F_4 = 0_{3\times 3}, \quad f_5 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \tag{76}$$

where d_1 and d_2 are uncertain parameters in the following ranges

$$-0.1 \leq d_1 \leq 0.06, \quad -0.03 \leq d_2 \leq 0.05,$$

and F_1 is some matrix defining the behavior of the system, which does not impact the partition. A graphical illustration of the partition dependence on the uncertain parameters regarding the above system is presented in Figure 7 considering two scenarios: $d_1 = -d_2$, on the left and $d_1 = d_2$, on the right. Note that in this case, the explicit approach is difficult to use, since the partition $\{\Gamma_i\}_{i\in\mathbb{I}} \subset \mathbb{R}^2$ and even the index set $\mathbb{I}$ change depending on the parameters d_1 and d_2.

On the other hand, it is simpler to handle the uncertainties in the partition using the implicit representation (77) since the uncertainty is expressed as a parametric uncertainty in matrices F_3, F_4 and f_5.

In this section, we focus on the problem of robust stability analysis of piecewise affine systems. Considering polytopic uncertainties on both the dynamics (i.e. on matrices F_1 and F_2) and the partition (i.e. matrices F_3, F_4 and vector f_5), we formulate

LMIs that can be solved to assess the global exponential stability of the origin of uncertain PWA systems.

4.3.1 The uncertain model

Consider the following class of uncertain PWA discrete-time systems:

$$x^+ = F_1(\lambda)x + F_2(\lambda)\phi(y) \tag{77a}$$

$$y = F_3(\lambda)x + F_4(\lambda)\phi(y) + f_5(\lambda), \tag{77b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n_y}$,$\phi : \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ is defined element-wise by the ramp function given in (22). Matrices $F_1(\lambda) \in \mathbb{R}^{n \times n}$, $F_2(\lambda) \in \mathbb{R}^{n \times n_y}$, $F_3(\lambda) \in \mathbb{R}^{n_y \times n}$, $F_4(\lambda) \in \mathbb{R}^{n_y \times n_y}$, $f_5(\lambda) \in \mathbb{R}^{n_y}$ depend on a (possibly time-varying) vector $\lambda(k) \in \mathbb{R}^N$. Note that, in this case

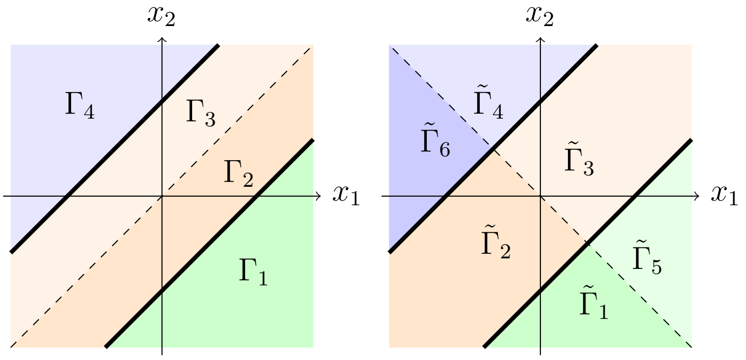
$$y^+ = F_3(\lambda^+)x^+ + F_4(\lambda^+)\phi(y^+) + f_5(\lambda^+), \tag{78}$$

where $\lambda^+ = \lambda(k+1)$.

We assume that these matrices belong to a convex and bounded polytopic set of matrices $\mathcal{F}$ as follows

$$\mathbf{F}(\lambda) = \begin{bmatrix} F_1(\lambda) & F_2(\lambda) & 0 \\ F_3(\lambda) & F_4(\lambda) & f_5(\lambda), \end{bmatrix}$$

Figure 7 – Partitions of $\mathbb{R}^2$ for $f(x)$ for different values of d_1 and d_2. On the left, $d_1 = -d_2$. On the right, $d_1 = d_2$. The dashed line represents $y_3 = 0$.



Source: The author

and it is supposed that:

$$\mathbf{F}(\lambda) \in \mathcal{F} = \left\{ \mathbf{F}(\lambda) \ : \ \mathbf{F}(\lambda) = \sum_{j=1}^{N} \lambda_j \mathbf{F}_j, \ \lambda \in \Lambda \right\} \tag{79}$$

$$\Lambda = \left\{ \lambda \ : \ \sum_{j=1}^{N} \lambda_j = 1, \lambda_j \geq 0, \ \forall j \right\} \tag{80}$$

where

$$\mathbf{F}_j = \begin{bmatrix} F_{1j} & F_{2j} & 0 \\ F_{3j} & F_{4j} & f_{5j} \end{bmatrix}$$

It follows that $\mathcal{F}$ corresponds to the convex hull of matrices $\mathbf{F}_j$

$$\mathcal{F} = \text{Co} \left\{ \mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N \right\}. \tag{81}$$

Remark 4.4. *Note that the well-posedness of the uncertain system can be tested by verifying the conditions from Proposition 1 on the vertices of $F_4(\lambda)$. This is a sufficient condition, and henceforth it will be assumed it is verified.*

4.3.2 Stability Conditions - Parameter Independent Lyapunov Function

In this section new LMI conditions for the stability of the origin of the uncertain PWA discrete-time system (77) are proposed, considering Lyapunov candidate functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as defined in (71).

Also, similarly to before, the upper bound for $V(x)$ which will be useful the prove the exponential stability of the origin, given by

$$\begin{aligned} V(x) &\leq \|P_1\| \|x\|^2 + 2 \|P_2\| \|x\| \|\phi\| + \|P_3\| \|\phi\|^2 \\ &\leq \left(\|P_1\| + 2\sigma \|P_2\| + \sigma^2 \|P_3\| \right) \|x\|^2 && (82) \\ &= \varepsilon_2 \|x\|^2 , && (83) \end{aligned}$$

still holds, with

$$\sigma = \max_{\Delta \in \mathcal{D}, \lambda \in \Lambda} \left\| \Delta (I - F_4(\lambda)\Delta)^{-1} F_3(\lambda) \right\| . \tag{84}$$

Note that in the case where the matrices F_1, F_2, F_3, F_4 or the vector f_5 are uncertain, the convexity of Theorem 4.2 is lost. This comes basically by the cross products between involving this matrices that appear in $\tilde{H}$ and also in $\tilde{\mathcal{X}}_T$ and $\tilde{\mathcal{X}}_M$ appearing in (73) and (74).

To overcome this problem, we propose now a slightly different approach inspired by the Finsler Lemma. Instead of directly replacing x^+ by the expression given in (77a), we use the dynamics as an algebraic constraint involving vectors x^+, x and ϕ. This is formalized in the following theorem.

Theorem 4.3. *If there exist $P \in \mathbb{S}^{(n+n_y)\times(n+n_y)}$, matrices $T_{1j} \in \mathbb{D}^{n_y}$, $T_{2jl} \in \mathbb{D}^{n_y}$, $T_{3jl} \in \mathbb{D}^{n_y}$, $M_{1j} \in \mathbb{P}^{1+2n_y}$, $M_{2jl} \in \mathbb{P}^{1+4n_y}$, $L_1 \in \mathbb{R}^{(1+n+2n_y)\times n_y}$, $L_2 \in \mathbb{R}^{(1+2n+4n_y)\times 2n_y}$, $L_3 \in \mathbb{R}^{(1+2n+4n_y)\times n}$ and positive scalars $\eta < 1$ and ϵ_1 such that*

$$\Pi_1 + \mathit{He}(\Pi_{2j} + L_1G_{1j}) - \Pi_{3j} > 0, \quad \forall j \in \{1,...,N\} \tag{85}$$

$$-\Pi_4 + \mathit{He}(\Pi_{5jl} + L_2G_{2jl} + L_3G_{3j}) - \Pi_{6jl} > 0, \quad \forall j,l \in \{1,...,N\} \tag{86}$$

with

$$\Pi_1 = \begin{bmatrix} P_1 - \epsilon_1 I_n & P_2 & 0 & 0 \\ P_2^\top & P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \Pi_{2j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & T_{1j} & -T_{1j} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_{3j} = \Theta_1^\top M_{1j}\Theta_1, \ \Theta_1 = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & I & -I & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Pi_4 = \begin{bmatrix} -(1-\eta)P_1 & 0 & -(1-\eta)P_2 & 0 & 0 & 0 & 0 \\ 0 & P_1 & 0 & P_2 & 0 & 0 & 0 \\ -(1-\eta)P_2^\top & 0 & -(1-\eta)P_3 & 0 & 0 & 0 & 0 \\ 0 & P_2^\top & 0 & P_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_{6jl} = \Theta_2^\top M_{2jl}\Theta_2., \ \Theta_2 = \begin{bmatrix} 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Pi_{5jl} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_{2jl} & 0 & -T_{2jl} & 0 & 0 \\ 0 & 0 & 0 & T_{3jl} & 0 & -T_{3jl} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{1j} = \begin{bmatrix} -F_{3j} & -F_{4j} & I & -f_{5j} \end{bmatrix},$$

$$G_{2jl} = \begin{bmatrix} -F_{3j} & 0 & -F_{4j} & 0 & I & 0 & -f_{5j} \\ 0 & -F_{3l} & 0 & -F_{4l} & 0 & I & -f_{5l} \end{bmatrix},$$

$$G_{3j} = \begin{bmatrix} -F_{1j} & I & -F_{2j} & 0 & 0 & 0 & 0 \end{bmatrix},$$

then the origin of the PWA system (77) *is globally exponentially stable.*

Proof. As F_{1j}, F_{2j}, F_3j, F_{4j}, f_{5j}, T_{1j}, T_{2jl}, T_{3jl}, M_{1j} and M_{2jl} appear affinely in (85) and (86), from convexity arguments, it follows that

$$\Pi_1 + \text{He}(\Pi_2(\lambda) + L_1 G_1(\lambda)) - \Pi_3(\lambda) \geq 0 \tag{87}$$

$$-\Pi_4 + \text{He}(\Pi_5(\lambda,\lambda^+) + L_2 G_2(\lambda,\lambda^+) + L_3 G_3(\lambda,\lambda^+)) - \Pi_6(\lambda,\lambda^+) \geq 0, \tag{88}$$

with $\Pi_2(\lambda) = \sum_{j=1}^N \lambda_j \Pi_{2j}$, $G_1(\lambda) = \sum_{j=1}^N \lambda_j G_{1j}$, $\Pi_3(\lambda) = \sum_{j=1}^N \lambda_j \Pi_{3j}$, $\Pi_5(\lambda,\lambda^+) = \sum_{l=1}^N \lambda_l^+ \sum_{j=1}^N \lambda_j \Pi_{5jl}$, $G_2(\lambda,\lambda^+) = \sum_{l=1}^N \lambda_l^+ \sum_{j=1}^N \lambda_j G_{2jl}$, $G_3(\lambda) = \sum_{j=1}^N \lambda_j G_{3j}$, $\Pi_6(\lambda,\lambda^+) = \sum_{l=1}^N \lambda_l^+ \sum_{j=1}^N \lambda_j \Pi_{6jl}$.

Define the vectors

$$\xi_1 = \begin{bmatrix} x \\ \phi(y) \\ y \\ 1 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} x \\ x^+ \\ \phi(y) \\ \phi(y^+) \\ y \\ y^+ \\ 1 \end{bmatrix}, \tag{89}$$

then we have that

$$\xi_1^\top \left(\Pi_1 + \text{He}(\Pi_2(\lambda) + L_1 G_1(\lambda)) - \Pi_3\right) \xi_1 \geq 0 \tag{90}$$

$$\xi_2^\top(-\Pi_4 + \text{He}(\Pi_5(\lambda,\lambda^+) + L_2 G_2(\lambda,\lambda^+) + L_3 G_3(\lambda)) - \Pi_6(\lambda,\lambda^+))\xi_2 \geq 0, \tag{91}$$

From the definition of $G_1(\lambda)$, $G_1(\lambda)\xi_1 = 0$ on the trajectories of system (77), and note that $\xi_1^\top \Pi_2(\lambda)\xi_1 = s_1(T_1(\lambda),y) = 0$, where $T_1(\lambda) = \sum_{j=1}^N \lambda_j T_{1j}$. Also, since all entries of $M_1(\lambda)$ are non-negative, it follows that $\xi_1^\top \Pi_3(\lambda)\xi_1 = s_2(M_1,y) \geq 0$. This means that, since $\xi_1^\top \Pi_1 \xi_1 = V(x)$, if (90) is verified, then condition (68) from Theorem 4.1 is also verified for the uncertain system (77).

Similarly, note that from the definition of $G_2(\lambda,\lambda^+)$ and $G_3(\lambda)$, and taking into account that y^+ is given by (78), we also have that $G_2(\lambda,\lambda^+)\xi_2 = 0$ and $G_3(\lambda)\xi_2 = 0$ on the trajectories of system (77). Also, note that $\xi_2^\top \Pi_5(\lambda, \lambda^+)\xi_2 = s_1(T_2(\lambda, \lambda^+), y) + s_1(T_3(\lambda, \lambda^+), y^+) = 0$, where $T_2(\lambda, \lambda^+) = \sum_{l=1}^N \lambda_l^+ \sum_{j=1}^N \lambda_j T_{2jl}$ and $T_3(\lambda, \lambda^+) = \sum_{l=1}^N \lambda_l^+ \sum_{j=1}^N \lambda_j T_{3jl}$. Furthermore, since all entries of $M_2(\lambda, \lambda^+)$ are non-negative, it follows that $\xi_2^\top \Pi_6(\lambda, \lambda^+)\xi_2 = s_2(M_2(\lambda, \lambda^+),\tilde{y}) \geq 0$. Hence, since $\xi_2^\top \Pi_4 \xi_2 = V(x^+) - (1-\eta)V(x)$, if (91) is verified, condition (69) from Theorem 4.1 is also verified for the uncertain system (77).

Now, recall that the condition (67) is guaranteed for the PWQ Lyapunov candidate function, with an upper bound given by (83). Hence, if (85) and (86) hold, (90) and (91) also hold, so all conditions form Theorem 4.1 are verified, and we conclude that the origin of (77) is exponentially stable. □

Remark 4.5. *Theorem 4.3 covers the generic case of a possibly time-varying parameter. If the parameter is uncertain, but constant, a particular formulation of Theorem 4.3 can* *be adopted. As in this case $\lambda^+ = \lambda$, it suffices to consider only one convex sum, that is $l = j$, and thus, the LMIs* (85) *and* (86) *become:*

$$\Pi_1 + He(\Pi_{2j} + L_1 G_{1j}) - \Pi_{3j} > 0, \quad \forall j \in \{1,...,N\}$$
$$-\Pi_4 + He(\Pi_{5j} + L_2 G_{2j} + L_3 G_{3j}) - \Pi_{6j} > 0, \quad \forall j \in \{1,...,N\}.$$

4.3.3 Stability Conditions - Parameter Dependent Lyapunov Function

In the last section, conditions to analyze the stability of uncertain PWA systems was proposed, using a piecewise quadratic Lyapunov function candidate. Now, conditions based on a parameter dependent piecewise quadratic Lyapunov function candidate are presented.

Consider the parameter dependent Lyapunov function candidate

$$V(x,\lambda) = \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}^\top P(\lambda) \begin{bmatrix} x \\ \phi(y(x)) \end{bmatrix}. \tag{92}$$

with $P(\lambda) = \sum_{j=1}^{N} \lambda_j P_j$.

To find an upper bound on V, define:

$$\rho_1 \triangleq \max_{\lambda \in \Lambda} \|P_1(\lambda)\|, \quad \rho_2 \triangleq \max_{\lambda \in \Lambda} \|P_2(\lambda)\|, \quad \rho_3 \triangleq \max_{\lambda \in \Lambda} \|P_3(\lambda)\|, \tag{93}$$

then

$$V(x) \leq (\rho_1 + 2\sigma\rho_2 + \sigma^2\rho_3) \|x\|^2 = \epsilon_2 \|x\|^2, \tag{94}$$

with σ as defined in (84).

With this, the following theorem formalizes the stability conditions with a parameter dependent Lyapunov function.

Theorem 4.4. *If there exist $P_j \in \mathbb{S}^{(n+n_y)\times(n+n_y)}$, matrices $T_{1j} \in \mathbb{D}^{n_y}$, $T_{2j} \in \mathbb{D}^{n_y}$, $T_{3j} \in \mathbb{D}^{n_y}$, $M_{1j} \in \mathbb{P}^{1+2n_y}$, $M_{2j} \in \mathbb{P}^{1+4n_y}$, $L_1 \in \mathbb{R}^{(1+n+2n_y)\times n_y}$, $L_2 \in \mathbb{R}^{(1+2n+4n_y)\times 2n_y}$, $L_3 \in \mathbb{R}^{(1+2n+4n_y)\times n}$ and positive scalars $\eta < 1$ and ϵ_1 such that*

$$\Pi_{1j} + He(\Pi_{2j} + L_1 G_{1j}) - \Pi_{3j} > 0, \quad \forall j \in \{1,...,N\} \tag{95}$$

$$-\Pi_{4jl} + He(\Pi_{5jl} + L_2 G_{2jl} + L_3 G_{3j}) - \Pi_{6jl} > 0, \quad \forall j,l \in \{1,...,N\} \tag{96}$$

with

$$\Pi_{1j} = \begin{bmatrix} P_{1j} - \epsilon_1 I_n & P_{2j} & 0 & 0 \\ P_{2j}^\top & P_{3j} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Pi_{4jl} = \begin{bmatrix} -(1-\eta)P_{1j} & 0 & -(1-\eta)P_{2j} & 0 & 0 & 0 & 0 \\ 0 & P_{1l} & 0 & P_{2l} & 0 & 0 & 0 \\ -(1-\eta)P_{2j}^\top & 0 & -(1-\eta)P_{3j} & 0 & 0 & 0 & 0 \\ 0 & P_{2l}^\top & 0 & P_{3l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and Π_{2j}*,* Π_{3j}*,* Π_{5jl} *and* Π_{6jl} *as defined in theorem 4.3, then the origin of the PWA system* (77) *is globally exponentially stable.*

Proof. Following the same reasoning presented in Theorem 4.3, it follows that (95) ensure that

$$\begin{bmatrix} x^\top \\ \phi(y)^\top \end{bmatrix}^\top P_j \begin{bmatrix} x \\ \phi(y) \end{bmatrix} - \epsilon_1 \|x\|^2 + s_1(T_{1j},y) - s_2(M_{1j},y) \geq 0, \quad \forall j \in \{1,...,N\}, \tag{97}$$

which, by convexity, implies that

$$V(x,\lambda) - \epsilon_1 \|x\|^2 + s_1(T_1(\lambda),y) - s_2(M_1(\lambda),y) \geq 0, \tag{98}$$

and, hence, condition (68) from Theorem 4.1 is verified.

On the other hand, since

$$V(x^+,\lambda^+) = \begin{bmatrix} x^{+\top} \\ \phi(y^+)^\top \end{bmatrix}^\top P(\lambda^+) \begin{bmatrix} x^+ \\ \phi(y^+) \end{bmatrix},$$

with $P(\lambda^+) = \sum_{l=1}^{N} \lambda_l^+ P_l$, it follows that (96) ensures that

$$-\left(\begin{bmatrix} x^{+\top} \\ \phi(y^+)^\top \end{bmatrix}^\top P_l \begin{bmatrix} x^+ \\ \phi(y^+) \end{bmatrix} - \begin{bmatrix} x^\top \\ \phi(y)^\top \end{bmatrix}^\top (1-\eta) P_j \begin{bmatrix} x \\ \phi(y) \end{bmatrix} \right) + \\ + s_1(T_{2jl}, y) + s_1(T_{3jl}, y^+) - s_2(M_{2jl}, \tilde{y}) \geq 0, \quad \forall j,l \in \{1,...,N\}, \tag{99}$$

which, by convexity, guarantees that

$$-\big(V(x^+,\lambda^+) - (1-\eta)V(x,\lambda)\big) + \\ + s_1(T_2(\lambda,\lambda^+), y) + s_1(T_3(\lambda,\lambda^+), y^+) - s_2(M_2(\lambda,\lambda^+), \tilde{y}) \geq 0, \tag{100}$$

which, in turn, ensures condition (69) from Theorem 4.1 is satisfied, even for a time-varying uncertainty.

Also, the condition (67) is guaranteed for the parameter dependent PWQ Lyapunov function by the upper bound (94). Thus, all conditions from Theorem (4.1) are verified, and we conclude that origin of the system (77) is exponentially stable. □

4.4 Numerical Examples

In this section, the results from Theorem 4.1 are illustrated with some numerical examples. In the first, the global stability of a generic piecewise linear system is demonstrated. In the second one, the global stability of a linear system subject to actuator saturation is analyzed. A third example treats a benchmark example borrowed from the explicit MPC literature. The fourth example illustrates the stability analysis of a generic PWA uncertain system. Finally, an uncertain spring-mass-damper system driven by a saturating actuator is studied. In all examples, the main objective is just to show that the origin is stable, hence the variable η was chosen to be very small (just above the floating-point accuracy, $\eta = 2.2204.10^{-16}$).

Example I. Consider a piecewise linear system given by (65) with

$$F_1 = \begin{bmatrix} 0.5 & 0.1+\kappa \\ -1 & 0.5 \end{bmatrix} \quad F_2 = \kappa \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and F_3, F_4 and f_5 as in (29), where κ is a design parameter.

Applying the conditions of Theorem 4.2 we can show that the system is globally stable for $\kappa = 0.63$, and (71) is a Lyapunov function for the system with

$$P = \begin{bmatrix} 2.2172 & -0.0151 & -0.4494 & 0.0094 \\ -0.0151 & 1.6462 & 0.0094 & 0.3570 \\ -0.4494 & 0.0094 & -1.2060 & -0.8242 \\ 0.0094 & 0.3570 & -0.8242 & -0.4758 \end{bmatrix}.$$

Note that the matrix P is not positive definite. Indeed the positive definiteness of matrix P is not imposed by the conditions in Theorem 4.2. However, since (68) holds we have that the Lyapunov function is guaranteed to be positive definite. Some trajectories of the system are shown in Figure 8, along with the level sets of the decreasing Lyapunov function. For comparison, the dual problem presented in (FENG, 2002, Section II) demonstrate that there does not exist a quadratic Lyapunov function, that is $V(x) = x^\top P_1 x$, with $P_1 \in \mathbb{R}^{n\times n}$, that certifies the stability for $\kappa \geq 0.357$, and through simulation, we find that the origin of the system is stable for $-0.35 < \kappa < 0.7$. It should also be pointed out that with the method proposed in (FENG, 2002), using a piecewise quadratic Lyapunov function, is not possible to certify the stability of the system for $\kappa \geq 0.51$ (considering known all the admissible transitions between regions), which shows that our conditions lead to less conservative results.

Example II. Consider the system taken from (DRUMMOND; VALMORBIDA; DUNCAN, 2017) and discretized with a sampling period of 100ms and subject to asymmetric actuator saturation

$$x^+ = Ax + B\text{sat}_{[-1,15]}(Kx)$$

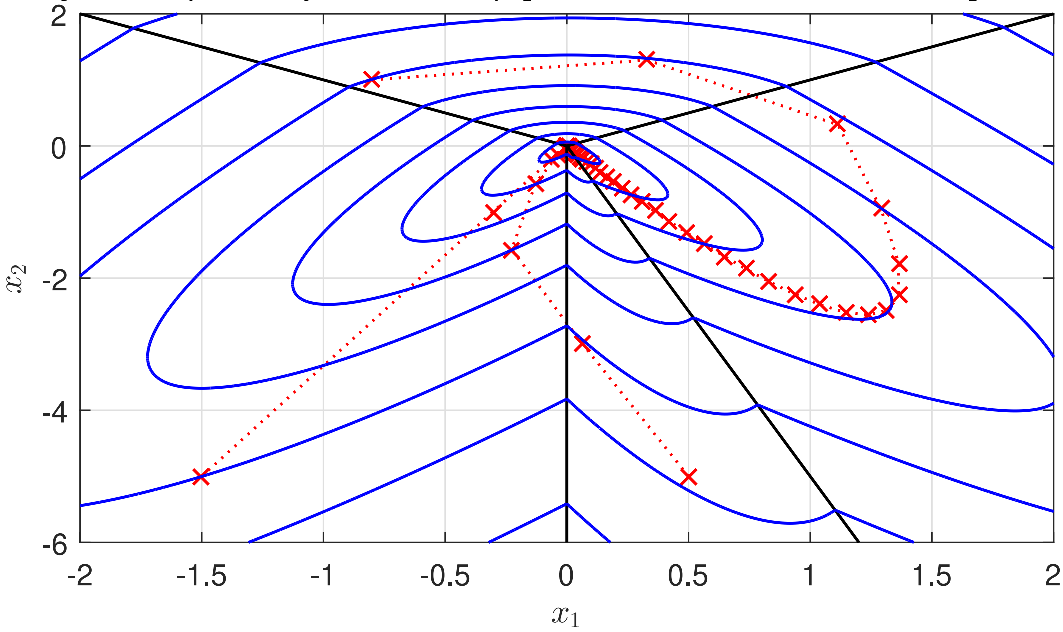
with

$$A = \begin{bmatrix} 0.9464 & 0.0957 \\ -0.9568 & 0.9033 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0049 \\ 0.0959 \end{bmatrix},$$
$$K = \begin{bmatrix} 9.9000 & 0.4950 \end{bmatrix}.$$

Using (30) we have that the right hand side of the above system is written as (65) with $f(x)$ defined by

$$F_1 = A + BK, \quad F_2 = \begin{bmatrix} -B & B \end{bmatrix}$$

Figure 8 – System trajectories and Lyapunov function level sets for Example I.



Source: The author

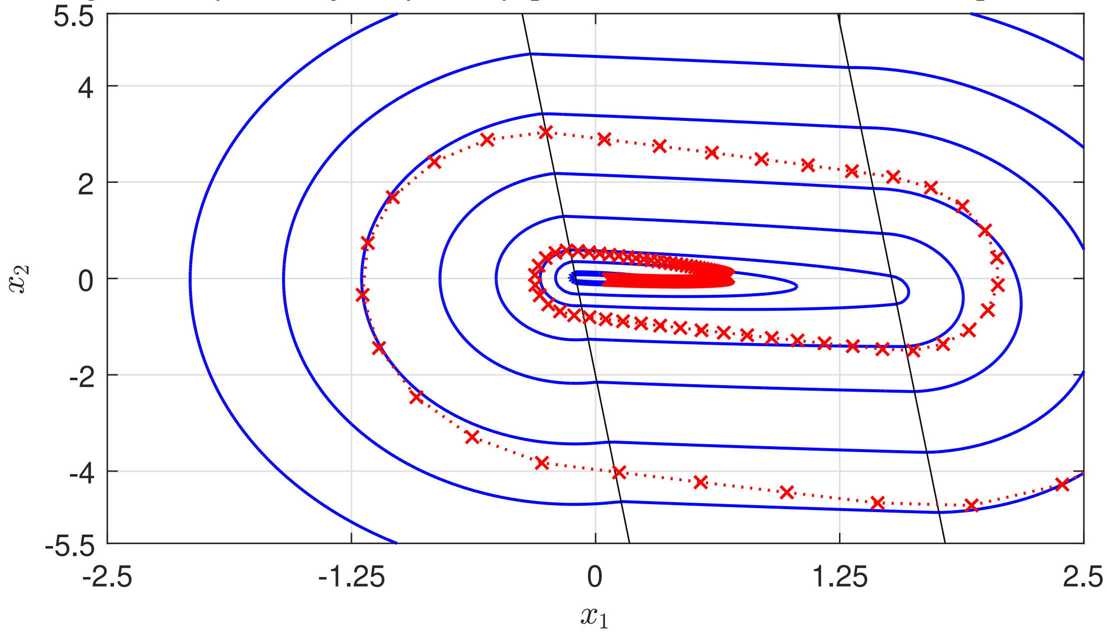
and F_3, F_4 and f_5 as in (30).

It can be shown (see (FENG, 2002, Section II)) that there does not exist a common quadratic Lyapunov function for the linear systems defined by A and $(A + BK)$ Since the quadratic global stability of a linear system subject to a saturating linear state feedback imposes that the Lyapunov function be common for the open-loop and the closed-loop without saturation, we conclude that there is no quadratic function to assess the stability of the system. However, considering a piecewise quadratic Lyapunov function as in (71) and applying Theorem 4.2, we can certify that the system is globally stable with

$$P = \begin{bmatrix} 0.1372 & 0.1684 & -0.0030 & -0.0241 \\ 0.1684 & 1.0349 & -0.0241 & 0.0668 \\ -0.0030 & -0.0241 & 0.1042 & -0.0073 \\ -0.0241 & 0.0668 & -0.0073 & 0.0934 \end{bmatrix}.$$

Note that though in this case, the computed P is a positive-definite matrix, this is not imposed *a priori*.

Figure 9 – System trajectory and Lyapunov function level sets for Example II.



Source: The author

In Figure 9, a trajectory of the system and the level sets of the decreasing Lyapunov function are depicted.

Example III. Consider the following closed-loop system

$$x^+ = Ax + Bu,$$
$$A = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix},$$

with u being given by the explicit MPC law computed in (BEMPORAD et al., 2002) leading to the explicit PWA representation given in Table 1. From this explicit representation, we obtain the following compact representation of the closed-loop system (65),

Table 1 – Explicit MPC law: regions of the partition and associated piecewise linear control laws

Region	$u(k)$
$\begin{bmatrix} -5.9220 & -6.8883 \\ 5.9229 & 6.8883 \\ -1.5379 & 6.8296 \\ 1.5379 & -6.8296 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} -5.9220 & -6.8883 \end{bmatrix} x$
$\begin{bmatrix} -6.4159 & -4.6953 \\ -0.0275 & 0.1220 \\ 6.4159 & 4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x$ $+0.6423$
$\begin{bmatrix} 6.4159 & 4.6953 \\ 0.0275 & -0.1220 \\ -6.4159 & -4.6953 \end{bmatrix} x \leq \begin{bmatrix} 1.3577 \\ -0.0357 \\ 2.6423 \end{bmatrix}$	$\begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix} x$ -0.6423
$\begin{bmatrix} -3.4155 & 4.6452 \\ 0.1044 & 0.1215 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	2
$\begin{bmatrix} 0.0679 & -0.0924 \\ 0.1259 & 0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	2
$\begin{bmatrix} -0.0679 & 0.0924 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} -0.0524 \\ -0.0519 \end{bmatrix}$	-2
$\begin{bmatrix} 3.4155 & -4.6452 \\ -0.1044 & -0.1215 \\ -0.1259 & -0.0922 \end{bmatrix} x \leq \begin{bmatrix} 2.6341 \\ -0.0353 \\ -0.0267 \end{bmatrix}$	-2

Source: (BEMPORAD et al., 2002)

with $f(x)$ described in the implicit proposed representation (21) with

$$F_1 = A + BK_1,\ F_2 = B \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \phi(y)$$

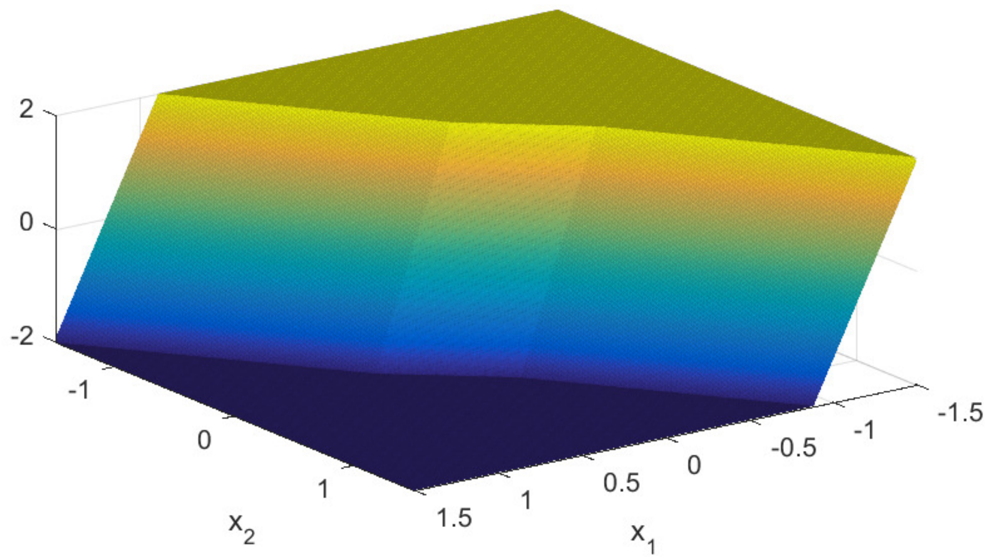
$$F_3 = \begin{bmatrix} K_2 - K_1 \\ K_1 - K_2 \\ -K_1 \\ K_1 \end{bmatrix},\ F_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix},$$

$$f_5^T = \begin{bmatrix} -0.6423 & -0.6423 & -2 & -2 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} -5.9220 & -6.8883 \end{bmatrix},\ K_2 = \begin{bmatrix} -6.4159 & -4.6953 \end{bmatrix}.$$

The value of the control signal u is depicted in Figure 10.

Figure 10 – Value of u as a function of x for Example III



Source: The author

Applying Theorem 4.2, we can find a quadratic Lyapunov function that certifies the stability of the system. In this case $V(x) = x^\top P_1 x$ (a quadratic function is a particular case of the generic form (71), in which we consider $P_2 = 0$ and $P_3 = 0$) with

$$P_1 = \begin{bmatrix} 0.9262 & 0.4674 \\ 0.4674 & 1.0815 \end{bmatrix}.$$

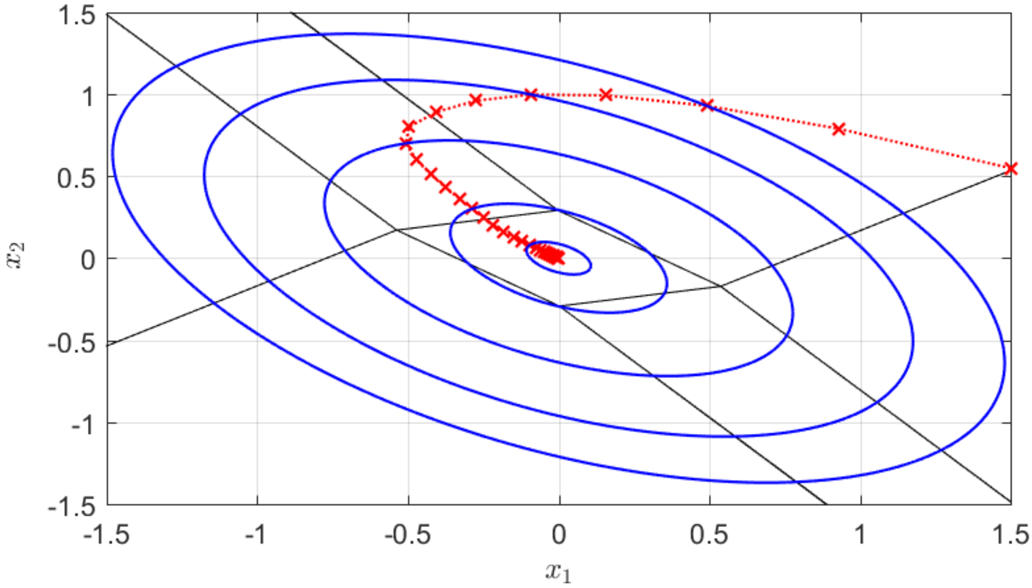
A trajectory and the level sets of the obtained Lyapunov function are shown in Figure 11.

Example IV. Consider an uncertain piecewise linear system given by (65) with

$$F_1 = \begin{bmatrix} 0.85 & 0.25 \\ -0.8 & 0.8 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} -0.15 & 0.15 \\ 0.15 & -0.15 \\ d_1 & d_2 \end{bmatrix}, \quad F_4 = 0_{3\times 3}, \quad f_5 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix},$$

Figure 11 – System trajectory and Lyapunov function level sets for Example III.



Source: The author

where d_1 and d_2 are uncertain parameters such that:

$$-0.1 \leq d_1 \leq 0.06, \quad -0.03 \leq d_2 \leq 0.05.$$

The partition of $\mathbb{R}^2$ for this system is the same partition discussed in section 4.3. Note that in this example, even enumerating transitions between sets of the partition is not possible, since the number of regions can vary depending on the values of the parameters d_1 and d_2, as can be seen in Figure 7. However, using the Lyapunov function as (71) we certify the global stability of the uncertain system with

$$P = \begin{bmatrix} 5.9646 & -0.7936 & -4.8724 & 3.7880 & 1.3481 \\ -0.7936 & 1.6585 & -1.7570 & 2.1793 & 3.2269 \\ -4.8724 & -1.7570 & 2.5663 & -1.6874 & -0.0837 \\ 3.7880 & 2.1793 & -1.6874 & 4.0464 & 2.3755 \\ 1.3481 & 3.2269 & -0.0837 & 2.3755 & 3.7548 \end{bmatrix}.$$

This shows the flexibility of the proposed method, where the regions and transitions are implicitly encoded in the representation.

Example V. Consider an uncertain piecewise linear system given by (65) with:

$$F_1 = \begin{bmatrix} 0 & 1 \\ -kd & -cd \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ -d & d \end{bmatrix},$$
$$F_3 = \begin{bmatrix} K_c \\ -K_c \end{bmatrix}, \quad F_4 = 0_{2\times 2}, \quad f_5 = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

which represents a spring-mass-damper system with a saturating actuator, where k is the spring constant, c is damping constant, $d = 1/m$, where m is the mass and K_c represents the feedback gains of the controller. Assume that the values for these constants are $k = 8$, $c = 2$, $K_c = [-5 \ \ 1.5]$ and the mass can vary with the operation of the system. Using the results from Theorem 4.3, we can guarantee the stability of the system for $d \in [0.5 \ \ 2]$, with a Lyapunov with an associated matrix

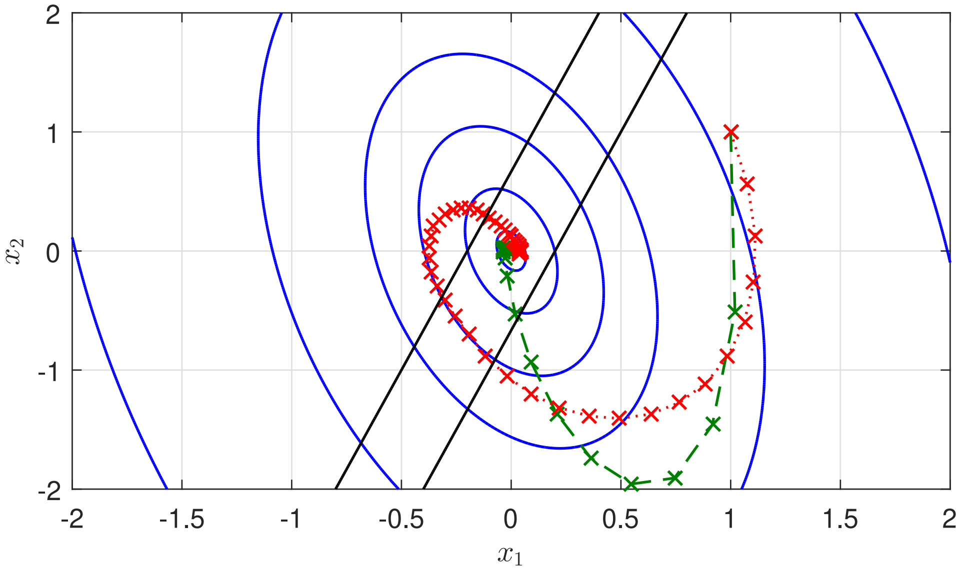
$$P = \begin{bmatrix} 4.2670 & 0.7879 & -0.1270 & 0.0291 \\ 0.7879 & 0.6912 & 0.0291 & 0.0002 \\ -0.1270 & 0.0291 & 0.0665 & -0.0227 \\ 0.0291 & 0.0002 & -0.0227 & 0.0277 \end{bmatrix}.$$

The Lyapunov function level sets can be seen in Figure 12 along with two trajectories that start at $x = [1 \ \ 1]'$, with different values of m.

On the other hand, using the results from Theorem 4.4, we can guarantee the stability of the system for $d \in [0.46 \ \ 2.5]$, with a parameter dependent Lyapunov function with associated matrices

$$P_1 = \begin{bmatrix} 1.4190 & 0.4241 & 0.0116 & 0.0029 \\ 0.4241 & 0.2522 & 0.0029 & -0.0070 \\ 0.0116 & 0.0029 & 0.0114 & -0.0023 \\ 0.0029 & -0.0070 & -0.0023 & 0.0176 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.0265 & 0.2404 & -0.0332 & 0.0048 \\ 0.2404 & 0.1900 & 0.0048 & 0.0039 \\ -0.0332 & 0.0048 & 0.0244 & -0.0058 \\ 0.0048 & 0.0039 & -0.0058 & 0.0137 \end{bmatrix}.$$

Figure 12 – Level sets of the Lyapunov function and two trajectories, with $d = 2$ (dashed green) and $d = 0.5$ (dashed red)



Source: The author

4.5 Final Remarks

In this chapter a new framework for the stability analysis of discrete-time PWA systems was presented. For this, a novel implicit representation of PWA functions was introduced, based on the use of ramp functions. In particular, we present the connections of this representation with results in the literature.

Exploiting a characterization of ramp functions in terms of a set of equalities and inequalities, it was shown how to use them to guarantee the verification of Lyapunov inequalities related to piecewise quadratic functions candidates. This is done by casting these inequalities in a generic quadratic form depending on the ramp functions, which leads to conditions to assess stability in the form of LMIs.

The following aspects of the proposed approach represent advantages with respect to existing results in the literature and can be highlighted.

— There is no need of defining the quadratic function associated to each set of the partition: this is implicitly done in a generalized quadratic form.

— There is no need of enumerating possible transitions between regions to reduce the conservativeness of the stability analysis.

— The conditions for stability are cast in two LMIs and can be efficiently tested by standard optimization packages.
— The use of properties associated to ramp functions applies only this class of function and therefore are less conservative than generic sector bounded conditions.
— From the novel proposed representation, the matrices describing the dynamics of the system appears affinely on the stability LMI conditions, which allows to directly consider uncertainties both in the system dynamics as well as the partition.

The potential of the proposed approach has been illustrated in several examples, including a generic PWA system, a linear system with saturating inputs and the explicit solution of an MPC problem.

5 EVENT-TRIGGERED CONTROL FOR PIECEWISE AFFINE SYSTEMS

In this chapter, the problem of designing an event-triggered control strategy applied to a piecewise affine system is addressed. For this, the representation proposed in Chapter 3 and the stability conditions presented in Chapter 4 are used. Similarly to Chapter 4, conditions to verify the stability of the origin with a generic Lyapunov function are presented. Finally, LMI conditions are formulated for the computation of the parameters of a trigger function that ensures the asymptotic stability of the origin of the closed-loop system.

In particular, a new PWQ trigger function is proposed to employ information from the state partition in the event generator. Based on the derived conditions, a convex optimization problem is formulated to design the trigger function aiming at reducing the number of events, i.e. the number of control updates.

It is important to recall that these conditions do not require the set of transitions between regions to be computed, which is a great advantage to the application of the method for the stability analysis of ETC systems.

5.1 Event-Triggered Controlled Piecewise Affine Systems

The system considered throughout this chapter is a discrete-time closed-loop plant, represented by the following equation

$$x(k+1) = f_1(x(k)) + u(k), \tag{101}$$

with

$$f_1(x(k)) = F_{11}x(k) + F_{21}\phi(y_p(x(k)))$$
$$y_p(x(k)) = \tilde{F}_3x(k) + \tilde{F}_4\phi(y_p(x(k))) + \tilde{f}_5,$$

with $F_{11} \in \mathbb{R}^{n\times n}$, $F_{21} \in \mathbb{R}^{n\times n_{y_p}}$, $\tilde{F}_3 \in \mathbb{R}^{n_{y_p}\times n}$, $\tilde{F}_4 \in \mathbb{R}^{n_{y_p}\times n_{y_p}}$ and $\tilde{f}_5 \in \mathbb{R}^{n_{y_p}}$, and u being the control input. To stabilize the system, a state-feedback PWA control law given by

$$u(k) = f_2(x(k)) \tag{103}$$

is considered, with

$$f_2(x(k)) = F_{12}x(k) + F_{22}\phi(y_p(x(k))) \tag{104}$$

where $F_{12} \in \mathbb{R}^{n\times n}$, $F_{22} \in \mathbb{R}^{n\times n_{y_p}}$.

Remark 5.1. *Note that system* (101) *with* $u(k)$ *given by* (103)-(104) *encompasses the case where the control signal is applied to the system through an input matrix, that is, the control law is defined as*

$$u(k) = Bv(k) = B\bar{F}_{12}x(k) + B\bar{F}_{22}\phi(y_p(x(k))). \tag{105}$$

In this case, it suffices to consider

$$F_{12} = B\bar{F}_{12}, F_{22} = B\bar{F}_{22}. \tag{106}$$

With a given stabilizing control law (103) that updates the control periodically, that is at each time instant k, a control input $u(k)$ is computed and updated based on the value of the state $x(k)$, we would like to obtain an event-triggered strategy. Recall that in the ETC paradigm, an event generator monitors the state of the system, and the control signal is only updated when a conditions is violated, reducing the number of control updates required for the stability of the system, and thus reducing the use of network resources. In this case, the control is updated only when an event occurs, namely whenever a function f_t, to be computed, exceeds a triggering value.

Considering $k = n_i, i \in \mathbb{N}$ as the time instants in which an event is triggered, the control signal that is generated by the triggering law and that is effectively applied to the system is given as follows

$$u(k) = u(n_i) = f_2(x(n_i)), \quad \forall k \in [n_i,\ n_{i+1}), \tag{107}$$

i.e. between two events, the control signal is kept constant with the value computed in the last trigger instant. For simplicity of notation, the time dependence of $x(k)$ is omitted, hence $x(k)$ is henceforth denoted by x. Also, $x(k+1)$ is denoted by x^+ and $x(n_i)$ is denoted by x_n. Then, the event-triggered implementation of the control law can be written as

$$f_2(x_n) = F_{12}x_n + F_{22}\phi(y_p(x_n)) \tag{108a}$$

$$y_p(x_n) = \tilde{F}_3 x_n + \tilde{F}_4\phi(y_p(x_n)) + \tilde{f}_5, \tag{108b}$$

Defining $\delta = x - x_n$, that is, δ is a measure of the difference between the state on the last control update and the current state, which we call the *state degradation*. Thus, from (108a) and considering that $x_n = x - \delta$, the closed-loop dynamics resulting from the application of the event-triggered control (107) can be expressed as

$$\begin{aligned} x^+ &= (F_{11}+F_{12})x + F_{21}\phi(y_p(x)) + F_{22}\phi(y_p(x-\delta)) - F_{12}\delta \\ y_p(x) &= \tilde{F}_3 x + \tilde{F}_4\phi(y_p(x)) + \tilde{f}_5 \\ y_p(x-\delta) &= \tilde{F}_3 x + \tilde{F}_4\phi(y_p(x-\delta)) + \tilde{f}_5 - \tilde{F}_3\delta \end{aligned} \tag{109}$$

Defining

$$y(x,\delta) = \begin{bmatrix} y_p(x) \\ y_p(x-\delta) \end{bmatrix},\ F_1 = F_{11} + F_{12},\ F_2 = \begin{bmatrix} F_{21} & F_{22} \end{bmatrix},$$
$$F_3 = \begin{bmatrix} \tilde{F}_3 \\ \tilde{F}_3 \end{bmatrix},\ F_4 = \begin{bmatrix} \tilde{F}_4 \\ \tilde{F}_4 \end{bmatrix},\ f_5 = \begin{bmatrix} \tilde{f}_5 \\ \tilde{f}_5 \end{bmatrix},\ F_\delta = \begin{bmatrix} 0 \\ \tilde{F}_3 \end{bmatrix},$$

the closed-loop dynamics of (101) can be written as

$$\begin{aligned} x^+ &= F_1 x + F_2\phi(y(x,\delta)) - F_{12}\delta \\ y(x,\delta) &= F_3 x + F_4\phi(y(x,\delta)) + f_5 - F_\delta\delta \end{aligned} \tag{110}$$

Note that this representation is similar to (21), with the addition of the state degradation δ, due to the event-triggered implementation. The well-posedness of the closed-loop ETC system can be tested as stated in Proposition 1, since this test relies only on the matrix F_4.

As discussed in section 2.1, the ETC strategy consists in evaluating a trigger function $f_t(x,\delta)$ at each instant k to determine whether the control must be updated or not. This is summarized by the following algorithm

Algorithm 6 ETC algorithm

if $f(x,\delta) > 0$ **then**

Generate an event;

$n_{i+1} = k$;

end if

5.2 Stability Analysis for Event-Triggered Controlled Piecewise Affine Systems

In this section, the stability conditions presented in the previous chapter are applied to the global stability analysis of closed-loop piecewise affine systems with the event-triggered strategy described in Section 5.1. In other words, the stability conditions are extended to the stability analysis of the system (110).

First, conditions for the global exponential stability of the origin of (110), considering a generic Lyapunov function candidate are presented in the following theorem.

Theorem 5.1. *If there exist a function $V : \mathbb{R}^n \to \mathbb{R}$, a function $f_t : \mathbb{R}^{n+n_\delta} \to \mathbb{R}$, matrices matrices $T_1 \in \mathbb{D}^{n_y}$, $T_2 \in \mathbb{D}^{n_y}$, $M_1 \in \mathbb{P}^{(1+2n_y)\times(1+2n_y)}$, $M_2 \in \mathbb{P}^{(1+4n_y)\times(1+4n_y)}$ and positive scalars $\eta < 1$, ϵ_1 and ϵ_2 such that the following inequalities are verified along the trajectories of* (110)*, for all $x \in \mathbb{R}^n$,*

$$f_t(x,\delta) \leq 0, \text{ for } \delta = 0 \tag{111}$$

$$V(x) \leq \epsilon_2 x^\top x \tag{112}$$

$$(V(x) - \epsilon_1 x^\top x) + s_1(T_1, y) - s_2(M_1, y) \geq 0 \tag{113}$$

$$-(V(x^+) - (1-\eta)V(x)) + s_1(T_2, \tilde{y}) - s_2(M_2, \tilde{y}) + f_t(x,\delta) \geq 0 \tag{114}$$

with $\tilde{y} = [y^\top \; y^{+\top}]^\top$, then the origin of (110) *is globally exponentially stable.*

Proof. From Lemmas 2 and 3, using the arguments presented in Proposition 4, if (112) and (113) are satisfied, it follows that $\epsilon_1 \|x\|^2 \leq V(x) \leq \epsilon_2 \|x\|^2$. It also follows from (114), using the same arguments, that

$$V(x^+) - (1-\eta)V(x) \leq f_t(x,\delta). \tag{115}$$

The remaining of the proof is carried out considering the time intervals $k = n_i$ and $k \in]n_i \;\; n_{i+1}[$.

First, considering $k = n_i$, it means that an event occurs at the time instant k. With this, from Algorithm 6, it follows that $x_n = x$ and, consequently, $\delta = 0$. From (111), $f_t(x,0) \leq 0$, which means, from (115), that $\Delta V(x) \leq -\eta V(x)$ whenever $k = n_i$.

Considering $k \in]n_i \;\; n_{i+1}[$, it means that an event does not occur at the time instant k, which, in turn, implies from Algorithm 6 that $f_t(x,\delta) \leq 0$, because otherwise an event would have occurred and the control state would have been updated, leading to the situation analyzed for $k = n_i$. Then, from (115), this leads to $\Delta V(x) \leq -\eta V(x)$ whenever $k \in]n_i \;\; n_{i+1}[$.

With both cases, we can conclude that $\Delta V(x) \leq -\eta V(x)$ whenever (114) is verified. This, in turn, implies the global exponential convergence of the origin, as shown in the proof of Theorem 4.1. □

Remark 5.2. *Note that this theorem applies to an event generator with any trigger function, provided it follows Algorithm 6 and the trigger function satisfies* (111).

5.3 Event Generator

It is assumed that the event generator has access to the state of the system, and an event generator based on a PWQ trigger function is presented, leading to a piecewise weighted relative error threshold trigger. This is an extension of the weighted relative error threshold trigger based on a quadratic trigger function, and it uses the fact that the system behaviour is piecewise defined to try reduce the conservatism by using $\phi(y_p(x))$, that encodes the partition of the system. The PWQ trigger function considered is then

$$f_t(x,\delta) = x^\top Q_x x + \phi(y_p(x))^\top Q_\phi \phi(y_p(x)) + \delta^\top Q_\delta \delta \tag{116}$$

where Q_δ is a symmetric positive-definite matrix and Q_x and Q_ϕ are symmetric matrices such that

$$x^\top Q_x x + \phi(y_p(x))^\top Q_\phi \phi(y_p(x)) \leq 0. \tag{117}$$

5.4 Constructive LMI Stability Conditions

Theorem 5.1 provides conditions for the stability analysis of the origin of closed-loop PWA discrete-time systems based on an event-triggered control strategy. In this section, testable LMI conditions based on the trigger function described in Section 5.3 and a particular class of Lyapunov function candidates, namely, continuous PWQ functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ of the form

$$V(x) = \begin{bmatrix} x \\ \phi(y(x,\delta)) \end{bmatrix}^\top P \begin{bmatrix} x \\ \phi(y(x,\delta)) \end{bmatrix}, \tag{118}$$

with $P = \left[\begin{smallmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{smallmatrix}\right]$, $P_1 \in \mathbb{S}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n_y}, P_2 = \begin{bmatrix} \tilde{P}_2 & 0_{n\times n_{y_p}} \end{bmatrix}$, $P_3 \in \mathbb{S}^{n_y \times n_y}$, $P_3 = \text{diag}(\tilde{P}_3, 0_{n_{y_p}})$, $\tilde{P}_2 \in \mathbb{R}^{n\times n_{y_p}}$ and $\tilde{P}_3 \in \mathbb{S}^{n_{y_p}\times n_{y_p}}$, are presented. Note that this particular structure means that $V(x)$ is given by

$$V(x) = \begin{bmatrix} x \\ \phi(y_p(x)) \end{bmatrix}^\top \tilde{P} \begin{bmatrix} x \\ \phi(y_p(x)) \end{bmatrix}, \tag{119}$$

with $\tilde{P} = \left[\begin{smallmatrix} P_1 & \tilde{P}_2 \\ \tilde{P}_2^\top & \tilde{P}_3 \end{smallmatrix}\right]$, hence it is indeed a PWQ function of x.

When considering candidate functions of the class above, the inequality (114) becomes dependent of the vector δ^+. Thus, properties of the dynamics of δ and its connection to x and x^+ should be exploited. As such, consider the following lemma.

Lemma 5. *The function δ satisfies the identity*

$$\delta^{+\top} N(\delta^+ - \delta + x - x^+) = 0. \tag{120}$$

for any matrix $N \in \mathbb{R}^{n\times n}$.

Proof. Suppose there is an event at instant $k+1$. Then, $\delta^+ = 0$ and the identity (120) is satisfied. On the other hand, suppose that there is no event at instant $k+1$. In that case, $\delta^{+} - \delta = x^{+} - x$, and the identity is also satisfied. □

Note that the upper bound for $V(x)$ derived in chapter 4, given by

$$V(x) \leq \left(\|P_1\| + 2\sigma \|P_2\| + \sigma^2 \|P_3\|\right) \|x\|^2, \tag{121}$$

with $\sigma = \max_{\Delta \in \mathcal{D}} \|\Delta(I - F_4\Delta)^{-1}F_3\|$, still apply, and can be used in the proof of the exponential stability of the origin.

With this, the following Theorem can be stated.

Theorem 5.2. *If there exist matrices $P \in \mathbb{S}^{(n+n_{\tilde{y}})\times(n+n_{\tilde{y}})}$, $T_1 \in \mathbb{D}^{n_y}$, $\tilde{T} \in \mathbb{D}^{n_y}$ $T_2 \in \mathbb{D}^{n_y}$, $T_3 \in \mathbb{D}^{n_y}$, $M_1 \in \mathbb{P}^{1+2n_y}$, $\tilde{M} \in \mathbb{P}^{1+2n_y}$, $M_2 \in \mathbb{P}^{1+4n_y}$, $L_1 \in \mathbb{R}^{(1+2n+2n_y)\times n_y}$, $\tilde{L} \in \mathbb{R}^{(1+2n+2n_y)\times n_y}$, $L_2 \in \mathbb{R}^{(1+4n+4n_y)\times 2n_y}$, $L_3 \in \mathbb{R}^{(1+4n+4n_y)\times n}$, $Q_\delta \in \mathbb{S}^{n\times n}$, $Q_\sigma \in \mathbb{R}^{n\times n}$ and $Q_\phi \in \mathbb{R}^{n_y\times n_y}$, and positive scalars $\eta < 1$ and ϵ_1 such that*

$$\Pi_1 + \textit{He}(\Pi_2 + L_1G_1) - \Pi_3 \geq 0 \tag{122}$$

$$-\Pi_4 + \textit{He}(\Pi_5 + L_2G_2 + L_3G_3 + \mathcal{N}) - \Pi_6 + \mathcal{Q}_t \geq 0 \tag{123}$$

$$\tilde{\Pi} + \textit{He}(\tilde{\Pi}_2 + \tilde{L}_1G_1) - \tilde{\Pi}_3 \geq 0, \tag{124}$$

with

$$\Pi_1 = \begin{bmatrix} P_1 - \epsilon_1 I & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ P_2^\top & 0 & P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_1 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & T_1 & -T_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Pi_3 = \Theta_1^\top M_1 \Theta_1,$$

$$\Theta_1 = \begin{bmatrix} 0 & 0 & I_{n_y} & 0 & 0 \\ 0 & 0 & I_{n_y} & -I_{n_y} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Pi_4 = \begin{bmatrix} -(1-\eta)P_1 & 0 & 0 & 0 & -(1-\eta)P_2 & 0 & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 & 0 & P_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1-\eta)P_2^\top & 0 & 0 & 0 & -(1-\eta)P_3 & 0 & 0 & 0 & 0 \\ 0 & P_2^\top & 0 & 0 & 0 & P_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{N} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ N & -N & -N & N & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{Q}_t = \mathit{diag}(Q_x, 0_n, Q_\delta, 0_n, \tilde{Q}_\phi, 0_{n_y}, 0_{n_y}, 0_{n_y}, 0),\ \tilde{Q}_\phi = \mathit{diag}(Q_\phi, 0_{n_{y_p}})$$

$$\Pi_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & T_2 & 0 & -T_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T_3 & 0 & -T_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; \Pi_6 = \Theta_2^\top M_2 \Theta_2,$$

$$\Theta_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & I_{n_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_y} & 0 & -I_{n_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_y} & 0 & -I_{n_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -F_3 & F_\delta & -F_4 & I_{n_y} & -f_5 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -F_3 & 0 & F_\delta & 0 & -F_4 & 0 & I_{n_y} & 0 & -f_5 \\ 0 & -F_3 & 0 & F_\delta & 0 & -F_4 & 0 & I_{n_y} & -f_5 \end{bmatrix},$$

$$G_3 = \begin{bmatrix} -F_1 & I_n & F_{12} & 0 & -F_2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\Pi} = \begin{bmatrix} -Q_\sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Q_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{\Pi}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{T} & -\tilde{T} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; \tilde{\Pi}_3 = \Theta_1^\top \tilde{M} \Theta_1,$$

then the origin of the closed-loop PWA system (110) *is globally exponentially stable under the proposed ETC strategy.*

Proof. Define the vectors

$$\xi_1 = \begin{bmatrix} x \\ \delta \\ \phi(y) \\ y \\ 1 \end{bmatrix}, \ \xi_2 = \begin{bmatrix} x \\ x^+ \\ \delta \\ \delta^+ \\ \phi(y) \\ \phi(y^+) \\ y \\ y^+ \\ 1 \end{bmatrix}.$$

Assuming (122) and (123) are verified, it follows that

$$\xi_1^\top \left(\Pi_1 + \text{He}(\Pi_2 + L_1 G_1) - \Pi_3\right) \xi_1 \geq 0 \tag{125}$$

$$\xi_2^\top (-\Pi_4 + \text{He}(\Pi_5 + L_2 G_2 + L_3 G_3 + \mathcal{N}) - \Pi_6 + \mathcal{Q}_t)\xi_2 \geq 0. \tag{126}$$

Note that from the definition of G_1, $G_1\xi_1 = 0$ on the trajectories of the system (110). Furthermore, note that $\xi_1^\top \Pi_2 \xi_1 = s_1(T_1,y) = 0$. Also, since all entries of M_1 are non-negative, recall that $\xi_1^\top \Pi_3 \xi_1 = s_2(M_1,y) \geq 0$. Hence, as $\xi_1^\top \Pi_1 \xi_1 = V(x) - \epsilon_1 x^\top x$ we conclude that condition (113) from theorem 5.1 is verified on the trajectories of the system (110).

Following the same steps, (124) ensures that (117) is verified, since

$$\xi_1^\top \tilde{\Pi} \xi_1 = -x^\top Q_x x - \phi(y_p(x))^\top Q_\phi \phi(y_p(x)).$$

On the other hand, note that from the definition of G_2 and G_3, we also have that $G_2\xi_2 = 0$ and $G_3\xi_2 = 0$ on the trajectories of system (110). Also, note that $\xi_2^\top \Pi_5 \xi_2 = s_1(T_2,y) + s_1(T_3,y^+) = 0$ and since all entries of M_2 are non-negative, it follows that $\xi_2^\top \Pi_6 \xi_2 = s_2(M_2,\tilde{y}) \geq 0$. Moreover, from (120), $\xi_2^\top \mathcal{N} \xi_2 = 0$. Furthermore, $\xi_2^\top \mathcal{Q}_t \xi_2 = f_t(x,\phi,\delta)$, and from the event-triggered strategy, $f_t(x,\phi,\delta) \leq 0$, since an event would have happened if the trigger function was positive, and, in that case, δ is reset to zero and the function would again become non-positive, due to (117), as demonstrated in the proof of Theorem 5.1. Hence, as $\xi_2^\top \Pi_4 \xi_2 = V(x^+) - (1-\eta)V(x)$, we can conclude that the condition (114) from theorem 5.1 is verified on the trajectories of the system (110).

Finally, since the condition (111) is guaranteed by (117) and the condition (112) is guaranteed by (121), all conditions from Theorem 5.1 are verified, and we conclude that the origin of system (110) is globally exponentially stable. □

Remark 5.3. *The inequalities from theorem 5.2 are cast in a similar form to those from theorems 4.3 and 4.4. Thanks to this fact, the results can be readily applied to the stability analysis of ETC PWA systems with polytopic uncertainties.*

5.5 Optimization Problem

In Section 5.4, we provided conditions to test the stability of PWA systems under an event-triggered control strategy. Furthermore, the matrices Q_δ, Q_x and Q_ϕ can be considered as free variables in the LMIs (122) and (123). In this case, we propose a convex optimization problem that aims to find parameters of the trigger function that lead to a reduction in the trigger activity, i.e., reduce the number of control updates.

In (MOREIRA; GROFF; GOMES DA SILVA JR, 2016), it is proposed that the optimization of a quadratic trigger function $f_q(x,\delta) = \delta^\top Q_\delta \delta - x^\top Q_\sigma x$ can be performed by finding suitable $Q_\delta \geq 0$ and $Q_\sigma \geq 0$. Theses matrices are obtained with the choice of the objective function trace$(Q_\delta + Q_\sigma^{-1})$. Following a similar idea, with the trigger function (116), we introduce an auxiliary variable Q_σ such that the following LMI is verified

$$\tilde{\Pi}_a + \text{He}(\tilde{\Pi}_2 + \tilde{L}G_1) - \tilde{\Pi}_3 \geq 0, \tag{127}$$

with

$$\tilde{\Pi}_a = \begin{bmatrix} -Q_\sigma - Q_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Q_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\tilde{\Pi}_2$, $\tilde{L}_1$ and $\tilde{\Pi}_3$ constructed as Π_2, L_1 and Π_3 in (122). This inequality is analogous to (124) and, if verified, implies

$$x^\top Q_x x + \phi^\top Q_\phi \phi \leq -x^\top Q_\sigma x. \tag{128}$$

Thus, we can implicitly optimize f_t by optimizing f_q. With this, we propose the following optimization problem.

$$\begin{aligned} \text{minimize} \quad & \text{trace}(Q_\delta - Q_\sigma) \\ \text{subject to:} \quad & (122),\ (123),\ (127). \end{aligned} \tag{129}$$

In this case, matrices Q_x and Q_ϕ are symmetric, but no assumptions on their sign definiteness are made, thus, as long as (128) is verified, Q_x and Q_ϕ can be indefinite.

Note also that, at any given time instant, the value of the piecewise quadratic trigger function obtained by this optimization problem will be equal to or smaller than the value of the quadratic function $f_q(x,\delta) = \delta^\top Q_\delta \delta - x^\top Q_\sigma x$, as, from (128), it follows that

$$x^\top Q_x x + \phi^\top Q_\phi \phi + \delta^\top Q_\delta \delta \leq -x^\top Q_x x + \delta^\top Q_\delta \delta.$$

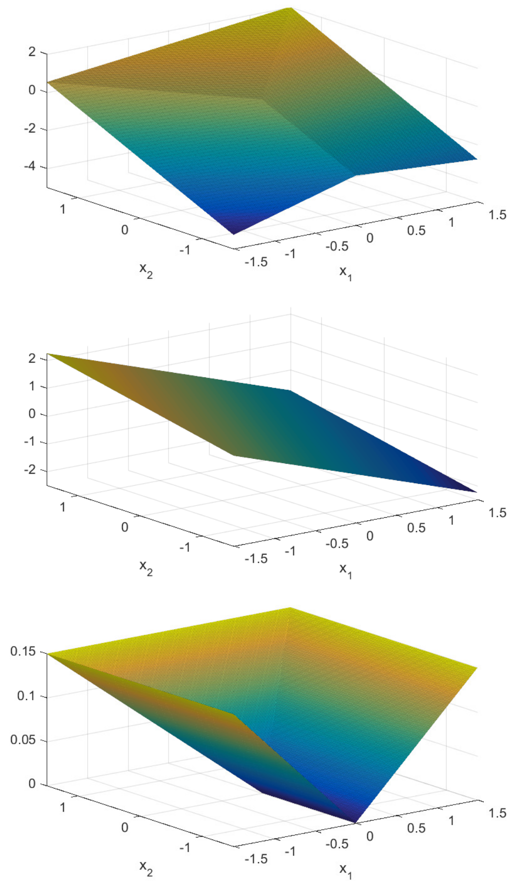
Hence, given an event, the number of time instants until the next event using the obtained piecewise quadratic function $f_t(x,\delta)$ will be greater than or equal to the number of time instants until the next event using the quadratic function $f_q(x,\delta)$. In this sense, the piecewise quadratic trigger function is a generalization of the quadratic trigger function and can lead to less conservative results, as will be shown in the numerical examples.

5.6 Numerical Examples

Example I: Consider the following piecewise-linear system, given by with

$$F_{11} = \begin{bmatrix} 0.5 & 0.85 \\ -1 & 0.5 \end{bmatrix}, \quad F_{21} = \begin{bmatrix} 0.75 & 0.75 \\ 0 & 0 \end{bmatrix},$$
$$\tilde{F}_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \tilde{F}_4 \begin{bmatrix} 0 & -1/3 \\ -1 & 0 \end{bmatrix}, \quad \tilde{f}_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

Figure 13 – First and second components of $F_{11}x + F_{21}\phi(y)$ (top and middle, respectively) and first component of $F_{12}x + F_{22}\phi(y)$ (bottom).



Source: The author

and an event-triggered controller given by, with

$$F_{12} = \begin{bmatrix} 0 & \kappa \\ 0 & 0 \end{bmatrix}, \quad F_{22} = \begin{bmatrix} \kappa & \kappa \\ 0 & 0 \end{bmatrix},$$

where κ is a design parameter of the controller.

We plot $F_{11}x + F_{21}\phi(y(x))$ in Figure 13. Also, in the same figure, the values of the first component of $F_{12}x + F_{22}\phi(y)$ are shown. Note that the second component of this function is equal to zero.

Considering $K = -0.1$, and using the LMI formulation from Theorem 5.2, with the optimization problem (129), the following parameters for the triggering function were computed

$$Q_x = \begin{bmatrix} -2.2359 & -1.0205 \\ -1.0205 & -0.5772 \end{bmatrix}, \quad Q_\phi = \begin{bmatrix} -0.0350 & 0.0636 \\ 0.0636 & 0.0491 \end{bmatrix},$$
$$Q_\delta = \begin{bmatrix} 2.7701 & 0.3361 \\ 0.3361 & 1.3508 \end{bmatrix}.$$

A Lyapunov function with the following matrix $\tilde{P} = \left[\begin{smallmatrix} P_1 & \tilde{P}_2 \\ \tilde{P}_2^\top & \tilde{P}_3 \end{smallmatrix}\right]$ was obtained

$$\tilde{P} = \begin{bmatrix} 0.0169 & -0.0067 & -2.3250 & -2.3127 \\ -0.0067 & 0.0162 & -2.3127 & 2.3163 \\ -2.3250 & -2.3127 & -4.6369 & 1.5363 \\ -2.3127 & 2.3163 & 1.5363 & 4.6183 \end{bmatrix}.$$

Note that from (119), $P = \left[\begin{smallmatrix} \tilde{P} & 0_{n \times n_{yp}} \\ 0_{n_{yp} \times n} & 0_{n_{yp} \times n_{yp}} \end{smallmatrix}\right]$.

To assess the efficiency of the event-triggered strategy in reducing the control updates, the simulation of 100 initial conditions, evenly distributed in a unitary circle around the origin, was made, with $k = [0 \ 50]$. The 50 time instants were enough for the states to converge to the origin. The average number of events over the simulation period was $n_{\text{avg}} = 27.83$, meaning that the control updates were reduced almost by half. Simulation results for $x_0 = [-0.9989 \ \ 0.0476]$ and $x_0 = [-0.0317 \ \ -0.9995]$ are depicted in figures 14 and 15, respectively, where the bars represent the instants when an event happened with a value of 1, and the instants when an event didn't happen with a value of 0.

Whenever we impose $Q_\phi = 0$ it results that Theorem 5.2 yields a quadratic trigger function. In this case, the following parameters were obtained

$$Q_x = \begin{bmatrix} -0.4268 & -0.2126 \\ -0.2126 & -0.1063 \end{bmatrix}, \quad Q_\delta = \begin{bmatrix} 1.0534 & 0.0030 \\ 0.0030 & 1.0141 \end{bmatrix}.$$

all the simulations had 51 events, meaning an event happened at every single time instant and therefore the ETC strategy had no impact in reducing the control updates. This example shows that the term $\phi(y)^\top Q_\phi \phi(y)$, which includes information of the partition in the trigger function can help reduce the events.

For the sake of comparison, using the results from (MA; WU; CUI, 2018) and fixing the values of the controller, the stability can not be shown for any trigger function. This shows that the stability test proposed in this chapter is less conservative than the one used in the aforementioned work.

Example II: Consider the MPC control problem presented in the Example 3, in Chapter 4. Now, suppose the control law u is implemented with an event-triggered control strategy. In this case, the system can be described by:

$$F_{11} = A,\ F_{21} = 0,$$

$$\tilde{F}_3 = \begin{bmatrix} K_2 - K_1 \\ K_1 - K_2 \\ -K_1 \\ K_1 \end{bmatrix},\ \tilde{F}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix},$$

$$\tilde{f}_5^T = \begin{bmatrix} -0.6423 & -0.6423 & -2 & -2 \end{bmatrix},$$

$$F_{12} = BK_1,\ F_{22} = B\begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}\phi(y).$$

Using the proposed optimization problem (129), one can find the following parameters for the triggering function

$$Q_x = \begin{bmatrix} -1.8916 & -1.1546 \\ -1.1546 & -1.1703 \end{bmatrix},\quad Q_\phi = \begin{bmatrix} 0.0705 & -0.0135 & -0.0054 & 0.0031 \\ -0.0135 & 0.0680 & -0.0337 & -0.0053 \\ -0.0054 & -0.0337 & 0.0432 & -0.0024 \\ 0.0031 & -0.0053 & -0.0024 & 0.0436 \end{bmatrix},$$

$$Q_\delta = \begin{bmatrix} 16.4579 & 18.0110 \\ 18.0110 & 21.9858 \end{bmatrix}.$$

Also, a Lyapunov function that certifies the stability of the system can be found, with the following matrix $\tilde{P} = \begin{bmatrix} P_1 & \tilde{P}_2 \\ \tilde{P}_2^\top & \tilde{P}_3 \end{bmatrix}$

$$\tilde{P} = \begin{bmatrix} 235.7184 & 162.9185 & -13.4113 & 17.7559 & -13.7545 & -7.3717 \\ 162.9185 & 322.5319 & -34.2679 & 16.8953 & -16.4932 & -4.1465 \\ -13.4113 & -34.2679 & 3.2818 & -0.5158 & 2.4523 & 0.3444 \\ 17.7559 & 16.8953 & -0.5158 & -10.4631 & -3.7240 & 0.5601 \\ -13.7545 & -16.4932 & 2.4523 & -3.7240 & 2.8074 & 0.7273 \\ -7.3717 & -4.1465 & 0.3444 & 0.5601 & 0.7273 & -4.0712 \end{bmatrix}.$$

The efficiency of the event-triggered strategy was assessed by simulating 1000 initial conditions generated with uniformly distributed random x_1 and x_2 in the interval $[-1.5 \;\; 1.5]$, for $k \in [0 \;\; 25]$. The uniform distribution was chosen because, unlike the previous PWL example, this system is not homogeneous due to the (nonlinear) affine terms in y. The average number of events was $n_{\text{avg}} = 13.50$, which represents a reduction of 48% in the number of control updates. Figure 18 presents the trajectories for the initial conditions $x_0 = [-1.1804 \;\; 0.4036]$ and $x_0 = [0.7791 \;\; -0.7600]$. The simulation results of the of the states and the trigger instants are shown in Figures 16 and 17, where the bars represent the instants when an event happened with a value of 1, and the instants when an event didn't happen with a value of 0. A significant reduction of the control updates can be observed, showing the effectiveness of the proposed ETC strategy.

5.7 Conclusions

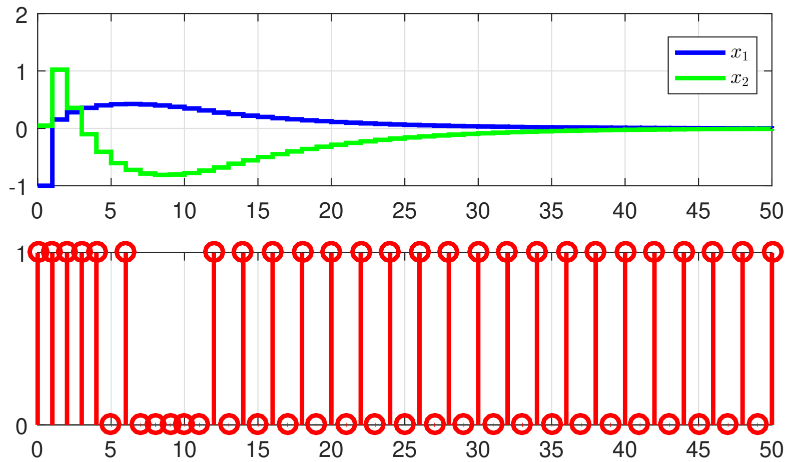
In this chapter, a new methodology for the emulation based design of an ETC strategy for a PWA system was proposed. First, it was shown how to represent an event-triggered PWA system with the implicit representation proposed in Chapter 3. Then, stability conditions based on the results presented in Chapter 4 were derived for this class of system, and an optimization problem was applied to design a PWQ trigger function. Finally, some numerical examples were used to illustrate the results.

The following advantages of the proposed method can be pointed out.

— The use of the implicit representation means that the stability analysis can be carried out without needing a reachability analysis, since the transitions are implicitly encoded and do not have to be enumerated.

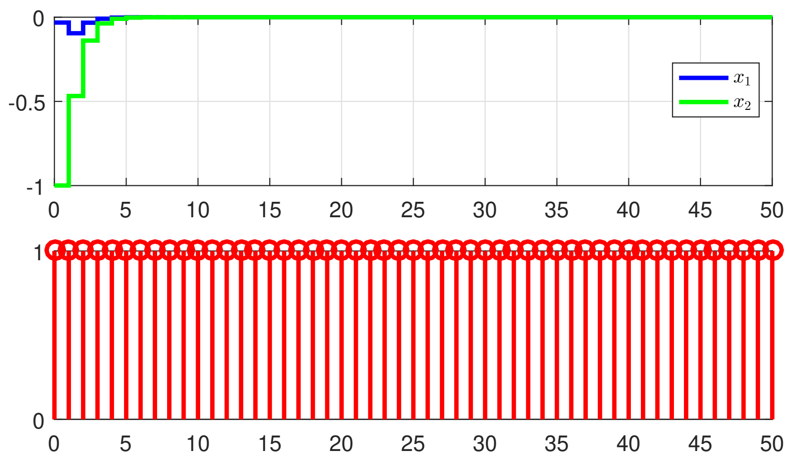
- The numeric examples have shown that the conditions presented are less conservative than previous conditions in the literature in assessing the stability of the origin of the system.
- The proposed PWQ trigger function yielded better results than a quadratic trigger function, this is illustrated by the Example I.

Figure 14 – Simulation with $x_0 = [-0.9989 \;\; 0.0476]$: 26 events occurred.



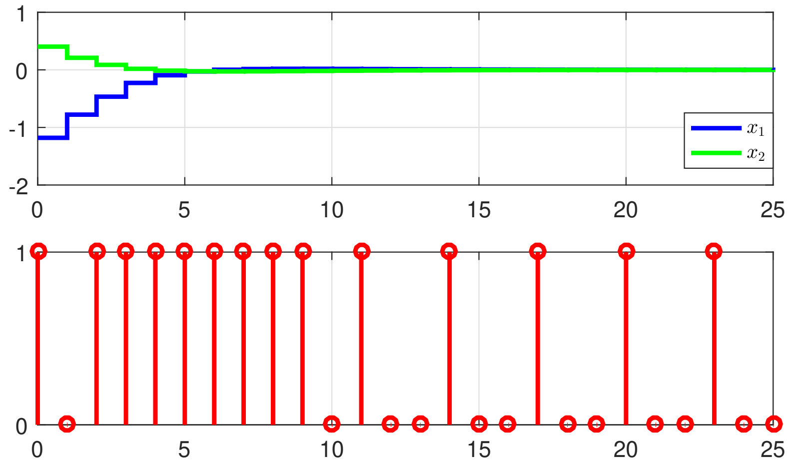
Source: The author

Figure 15 – Simulation with $x_0 = [-0.0317 \;\; -0.9995]$: 51 events occurred.



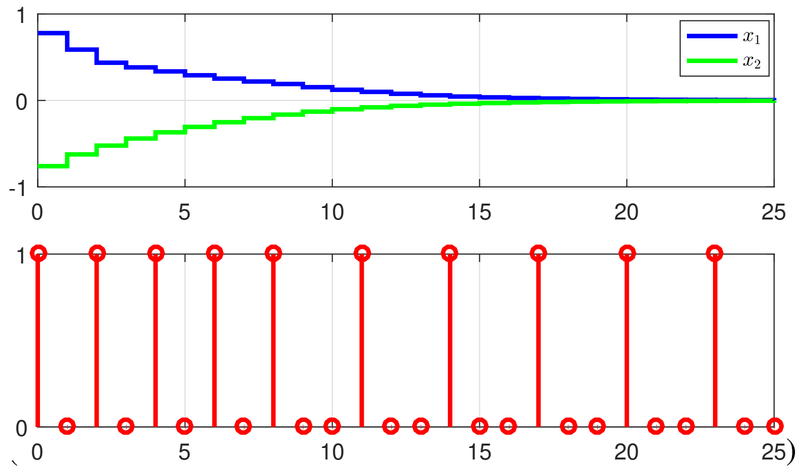
Source: The author

Figure 16 – Simulation with $x_0 = [-1.1804 \;\; 0.4036]$: 14 events occurred.



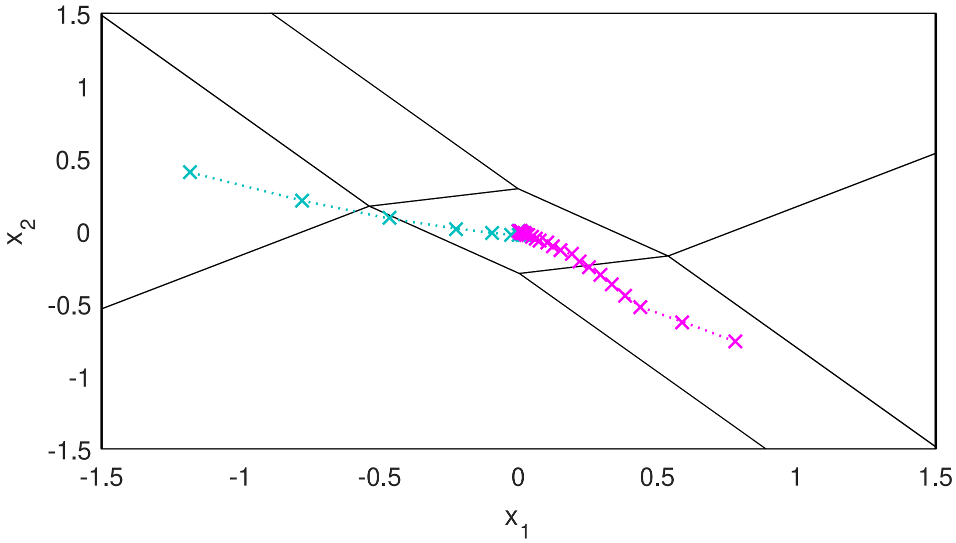
Source: The author

Figure 17 – Simulation with $x_0 = [0.7791 \quad -0.7600]$: 10 events occurred.



Source: The author

Figure 18 – Trajectories of the initial conditions. In cyan, $x_0 = [-1.1804 \;\; 0.4036]$, in magenta, $x_0 = [0.7791 \;\; -0.7600]$



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6 CONCLUSION

In this thesis, the problems of stability analysis and event generator design for PWA systems under an event-triggered strategy was studied. For such, a new implicit representation for continuous PWA functions was introduced and some properties of this representation were studied. Then, stability conditions based on the Lyapunov theory were derived for systems described by the new implicit representation, and considering PWQ Lyapunov candidate functions, testable LMI conditions were presented. Finally, a method to design event generators based on a PWQ trigger function guaranteeing the preservation of the stability properties of PWA systems under an ETC strategy was proposed.

Given this summary, the following contributions of the thesis can be discussed in more detail.

- Introduction of a new implicit representation for continuous PWA functions

  In Chapter 3, a new representation for continuous PWA functions was proposed, based on ramp functions. Conditions for the well-posedness of functions in the proposed form were presented and the relation of this representation with other representations in the literature were illustrated. Moreover, the existence of a representation with an explicit solution for any continuous PWA function was demonstrated. Among the advantages of the proposed representation, it can be pointed that:

  — The proposed implicit representation does not require the regions of the state-space partition to be explicitly described and enumerated;

— The representation is based on ramp functions, which have properties that can be exploited to demonstrate the positivity of quadratic forms, and thus is well suited for the stability analysis, as shown in Chapter 4.

- Stability conditions for PWA systems in the implicit representation

  In Chapter 4, stability conditions for PWA systems described by the proposed implicit representation were derived using the Lyapunov theory, exploiting some propertied of ramp functions. Then, considering PWQ Lyapunov candidate functions, these conditions were cast as LMIs, which can be numerically tested. Some advantages of the proposed method in comparison to existing methods are:

  — The conditions are based on a generalized implicit piecewise quadratic form, meaning there is no need to define individual quadratic functions to each region;

  — Since the regions of the partition are implicitly encoded by the ramp functions, the stability analysis can be carried out with no need to map the set of possible transitions;

  — The implicit encoding the the regions also enables the analysis of systems with polytopic uncertainties both in the dynamics and in the partition, which was not possible with the other approaches in the literature;

  — Since the conditions are cast in LMIs, they can be efficiently tested with standard optimization packages;

  — The properties of the ramp functions used resulted in less conservatism than previous methods in the literature, which include sector bounded inequalities. In fact, the used properties apply only to ramp functions and allow their exact characterization, i.e. these properties are verified if and only if the function is a ramp.

- Design of event generators for PWA systems under an ETC strategy

  In Chapter 5, the problem of designing an event generator for the ETC of PWA systems was addressed considering a PWQ trigger function, and an optimization problem was proposed to find suitable parameters for this trigger function.

— A new condition relating the dynamics of the state to the state degradation was proposed.
— The use of the implicit representation for PWA systems overcame the problem of conducting a reachability analysis before the stability conditions are tested, which poses problems in the case of ETC.
— The PWQ trigger function includes information about the partition and results in a reduced number of events when compared to quadratic trigger functions found in the literature.
— The LMI-based conditions can be readily extended to deal with systems containing polytopic uncertainties.

6.1 Future Work

Here, some ongoing research as well as problems that remain open are summarized.

- Address the local stability analysis of PWA systems in the implicit representation. It is not always possible to guarantee the global stability of nonlinear systems, and PWA systems are no exception. In this sense, tools to analyze the local stability, the existence of multiple equilibria and limit cycles are very important, and this topic is currently being researched.

- Consider discontinuous PWA functions. The proposed representation based on ramp functions was shown to model any continuous PWA function, but that is only a subclass of PWA functions. Modifications, possibly involving the use of step functions, are being investigated to deal with discontinuities.

- Consider the co-design of the event-generator and a PWA control law. The results from this work focused on the emulation based design, that is, given a control law that stabilizes the system with a classical discrete-time update scheme, find an event generator that preserves the stability under an event-triggered update strategy. However, the simultaneous design of the control law and the event-generator might improve the performance of the overall ETC strategy.

- Consider systems with transport delay. Communication delay is a common problem in the study of NCSs. In the present work, it was considered that the communication channel can transmit information quickly enough that this problem can be disregarded, however that is not always the case, and tools to consider the effect of this delay on the stability analysis of systems in the implicit representation is an important open topic.

- In the robust stability analysis, consider a polynomial dependency of the Lyapunov candidate function on the uncertainties, which can lead to less conservative results.

- In the MPC problem, find an implicit representation directly from the solution of the KKT conditions. In this case, an explicit PWA model, enumerating the regions of the partition, and the conversion of this model to the implicit representation, would not be required for the stability analysis.

- In the ETC problem, consider a trigger function based on the system output, to deal with systems where the complete state is not available.

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APPENDIX A STABILITY OF DYNAMIC SYSTEMS

One of the most important characteristics of a given system is the stability, since closed-loop unstable systems are, in general, of little use and often dangerous (SLOTINE; LI et al., 1991). The Lyapunov theory is the main tool for the stability analysis of nonlinear systems (KHALIL, 1996), and has been proven an efficient mean to the characterization of internal stability of event-triggered systems (HEEMELS; JOHANSSON; TABUADA, 2012), as well as for PWA systems (BISWAS et al., 2005), which are addressed in this work. Thus, in the appendix, the concept of stability in the sense of Lyapunov, as well as the second method of Lyapunov (also know as the indirect method) are presented, and so are its applications to discrete-time systems.

A.1 Stability in the Sense of Lyapunov

The study of the stability of a system based on Lyapunov theory is related to the stability of its equilibrium points. Thus, considering the following discret-time dynamical system:

$$x(k+1) = g(x(k)), \tag{130}$$

the following definitions can be made

Definition 3. *A state x^* is considered an equilibrium point if, once $x(k) = x^*$ in $k = k_1$, $x(k)$ remains equal to x^* for any $k \geq k_1$, that is, we have that $g(x^*) = x^* \ \ \forall k \geq k_1$.*

For the stability analysis in the sense of Lyapunov, it is convenient that the equilibrium point being analyzed is at the origin, that is, $x^* = 0$. When this is not the case, the point can be transfered to the origin by a change of variables (SLOTINE; LI et al., 1991).

Definition 4. *Let* $x = 0$ *be an equilibrium point of system* (130)*. It is considered stable if for all* $\overline{r} > 0$ *there exists* $\underline{r} > 0$ *such that:*

$$\|x(0)\| < \underline{r} \implies \|x(k)\| < \overline{r},\ \forall k \geq 0. \tag{131}$$

This point is considered asymptotically stable if, besides that,

$$\lim_{k \to \infty} \|x(k)\| = 0. \tag{132}$$

This stability definition, called stability in the sense of Lyapunov, means that, if an equilibrium point it stable, the trajectories of the system can be kept arbitrarily close to the equilibrium point if they start sufficiently close to it. If this is not possible, the equilibrium point is considered unstable (SLOTINE; LI et al., 1991).

One way of verifying the internal stability of the system is the application of the second method of Lyapunov, which consists in using a scalar function of the system states. If this function is positive and decreasing in time in a region around the equilibrium point, the trajectories will get closer to it, and thus it is asymptotically stable. The following Theorem formalizes this method.

Theorem A.1 (Lyapunov Theorem for Discrete Time Systems, (ÅSTRÖM; WITTENMARK, 1997))**.** *Let* $x = 0$ *be an equilibrium point of* (130) *and* $\mathcal{D} \subset \mathbb{R}^n$*, it will be considered stable if there exists a scalar function* $V : \mathcal{D} \mapsto \mathbb{R}$ *continuous in x, such that* $V(0) = 0$ *and:*

$$V(x) > 0,\ \forall x \in \mathcal{D} - \{0\}, \tag{133}$$

$$\Delta V(x) = V(g(x)) - V(x) \leq 0,\ \forall x \in \mathcal{D}. \tag{134}$$

Furthermore, if $\Delta V(x) < 0, \forall x \neq 0$*, it will be considered locally asymptotically stable.*

An equilibrium point can also be considered globally stable, if $\mathcal{D} = \mathbb{R}^n$ and $V(x)$ is radially unbounded, that is, $\lim_{\|x\| \to \infty} V(x) = \infty$.

In general, many Lyapunov functions can exist for the same system. Since the conditions from the Lyapunov Theorem are only sufficient for the stability of the origin, in case a candidate function verifies the conditions of the Lyapunov Theorem, then the equilibrium point in question is stable. However, if it does not satisfy these conditions, nothing can be asserted, except that some other candidate function might be able to demonstrate the stability. Hence, the main challenge in using the theory of Lyapunov resides in finding an adequate function.

A commonly used Lyapunov candidate function is the quadratic function, given by

$$V(x) = x^\top P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \tag{135}$$

where P is a symmetric positive definite matrix. When (130) is a linear system, that is, $g(x) = Ax(k)$, the variation of $V(x)$ is given by:

$$\Delta V(x) = x(k)^\top (A^\top P A - P) x(k), \tag{136}$$

so that the set of LMIs

$$P > 0, \quad A^\top P A - P < -Q, \tag{137}$$

with $Q > 0$, provides a sufficient and necessary condition for the stability of the system in question (ÅSTRÖM; WITTENMARK, 1997).

APPENDIX B LINEAR MATRIX INEQUALITIES

The use of LMIs is very important in the analysis and synthesis of control systems, because a wide array of control problems can be reduced to convex optimization problems, which can be efficiently numerically solved (BOYD et al., 1994). The challenge consists, thus, in formulating conditions for solving the problem in terms of LMIs, since oftentimes nonlinear terms are presente in the resulting formulation. In this appendix, the definition of an LMI and some techniques employed in deal with them are presented.

A strict LMI is an inequality of the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_x > 0, \tag{138}$$

where $x \in \mathbb{R}^m$ is a vector of variables and the symmetric matrices $F_i = F_i^\top \in \mathbb{R}^{n\times n}$ are known. In case $F(x) \geq 0$, the inequality is said non-strict. One of the fundamental properties of LMIs is that the restriction (138) is convex in x, that is, the set of solutions $\mathcal{X} = \{x : F(x) > 0\}$ is convex, so that a problem of the form

$$\begin{aligned} \text{minimize: } & c^\top x \\ \text{subject to: } & F(x) > 0, \end{aligned} \tag{139}$$

is a convex optimization problem (BOYD et al., 1994).

B.1 Schur Complement

The Schur Complement is a very useful tool for converting nonlinear matrix inequalities in equivalent LMIs. It can be enunciated as follows

Lema B.1 (Schur Complement (BOYD et al., 1994)). *Let $Q = Q'$, $R = R'$ and S be real matrices of appropriate dimensions. Then:*

$$i)R > 0,\ Q - SR^{-1}S^{\top} > 0, \tag{140}$$

$$ii) \begin{bmatrix} Q & S \\ S^{\top} & R \end{bmatrix} > 0 \tag{141}$$

are equivalent.

In this way, the quadratic inequality (140) can be transformed into the LMI (141), which can be efficiently treated by computational methods.

B.2 S-Procedure

Commonly there are cases where it is desired to guarantee the difinition in sign of a quadratic function whenever another quadratic function is defined in sign. The S-procedure is a tool that allows the approximation of this kind of restriction through an LMI. The S-procedure for strict inequalities is presented next.

Lema B.2 (S-procedure (BOYD et al., 1994)). *Let$T_0, ..., T_p \in \mathbb{R}^{n\times n}$ be symetric matrices. If there exist scalars $\tau_1, ..., \tau_p$ such that:*

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0, \tag{142}$$

then

$$x^{\top}T_0 x > 0,\ \text{for all } x \neq 0 \text{ such that } x^{\top}T_i x \geq 0,\ i = 1,...,p \tag{143}$$

is verified.

B.3 Finsler's Lemma

Through the use of Finsler's Lemma, it is possible to obtain equivalent conditions for the test of LMIs. Moreover, Finsler's Lemma can be directly employed in the elimination of variables in some LMIs (BOYD et al., 1994).

Lema B.3 (Finsler's Lemma (DE OLIVEIRA; SKELTON, 2001))**.** *Let* $x \in \mathbb{R}^m$, $Q \in \mathbb{R}^{m\times m}$ *and* $N \in \mathbb{R}^{n\times m}$*, such that rank*$(\mathcal{B}) < n$*, then the following statements are equivalent:*

$$i)\;\; x^\top Q x < 0, \quad \forall \mathcal{B}x = 0,\; x \neq 0 \tag{144}$$

$$ii)\;\; \mathcal{B}_0^\top Q \mathcal{B}_0 < 0, \quad \forall \mathcal{B}\mathcal{B}_0 = 0 \tag{145}$$

$$iii)\;\; Q - \alpha \mathcal{B}^\top \mathcal{B} < 0, \quad \forall \alpha \in \mathbb{R} \tag{146}$$

$$iv)\;\; Q + N\mathcal{B} + \mathcal{B}^\top N^\top < 0, \quad \forall N. \tag{147}$$