Universidade Federal do Rio Grande do Sul
Instituto de Matemática
Cadernos de Matemática e Estatística
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Centred Bimodules over Prime Rings: Closed submodules and applications to ring extensions

Miguel Ferrero

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Centred Bimodules over Prime Rings: Closed submodules and applications to ring extensions*

Miguel Ferrero†

Abstract. Let M be a bimodule over a prime ring R. In this paper we define and study a very useful class of sub-bimodules of M: the class of closed sub-bimodules. There is a canonical torsion-free extension of M to a Q-bimodule M^* which is always free over Q, where Q is the complete ring of right quotients of R. We prove that closed sub-bimodules of M are in one-to-one correspondence with closed sub-bimodules of M^*. The results are applied to study the torsion-free rank of a sub-bimodule of M and to study non-singular and strongly closed sub-bimodules. Also, the results are applied to study prime ideals in centred extensions and intermediate extensions. In particular, we complete and extend the results obtained in [5].

0. Introduction.

Prime ideals in ring extensions $R \subseteq S$ have extensively been studied in the last years. For example, when the extension is finite and generated by a set of R-centralizing elements, S is called a liberal extension [20,21]. A normalizing extension is again a finite extension which is generated by a set of R-normalizing generators [10,11,12,16]. Also, prime ideals in more general types of extensions (not necessarily finite) have been considered (e.g. [1,2,3,7,9,14,17,18,19]).

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In particular, the author in [4,5] studied prime ideals in polynomial rings and in free centred extensions. The method developed in these papers allows us to obtain precise information in infinite dimensional situations. Actually, a more general class of ideals called the closed ideals are studied and the results on prime ideals are obtained as applications of the general results. Closed ideals have also been used to study prime ideals in Ore extensions [3,7,14].

It turns out that the method developed in [5] can be extended to study submodules of (not necessarily free) centred bimodules over prime rings. The purpose of this paper is just to study closed submodules in this kind of bimodules. By this way we obtain results which can be applied to the theory of modules as well as to ring extensions. In particular, we extend all the results in [5].

Let R be a prime ring and let M be an R-bimodule. Following [20], we say that M is a centred bimodule over R if there exists a generating set of R-centralizing elements, i.e., there exists $X = (x_i)_{i\in\Omega} \subseteq M$ such that $M = \sum_{i\in\Omega} Rx_i$ and $ax_i = x_ia$, for every $a \in R$, $i \in \Omega$. Throughout this paper, submodule of M means sub-bimodule, unless otherwise stated.

In section 1, we define the closure $[N]_P$ of N in P, where $N \subseteq P$ are submodules of M. We say that N is closed in P if $[N]_P = P$. Then we study closed submodules and we obtain some useful characterization of this kind of submodules. This characterization is given via a free submodule L of M which is "dense" in M. Thus the description of closed submodules can always be reduced to the free case. The introduction of this "dense" free submodule of M and the characterization of $[N]_P$ are the main results of this section.

In Section 2 we study the extension of closed submodules from M to M^*, a canonical extension of M to a centred bimodule over Q, where Q is the maximal ring of right quotients of R. Corresponding to M^* we have also a C-vector space V, where C is the extended centroid of R. The main result here is a one-to-one correspondence between the closed submodules of M, M^*, and the subspaces of V. There is another interesting result in this section; the Q-module M^* is always free. This result says, loosely speaking, that a torsion-free bimodule M over a prime ring R is always free when considered as a Q-bimodule.

In Section 3 we obtain the first applications. We show that the torsion-free rank of a submodule N of M introduced in [20] reduces to the dimension of a C-vector space. As a consequence this notion becomes more tractable.

In Section 4 we study non-singular and strongly closed submodules of M. It follows that R is a prime non-singular ring if and only if every closed submodule N of P is non-singular in P (i.e., $Z(P/N) = 0$). A similar result is obtained concerning strongly prime rings and strongly closed submodules.

In Section 5 we study centred extensions of prime rings R and intermediate extensions, i.e., subrings W of S containing R. In this case we show that S^* is an extension of Q and the restriction V of S^* to a C-vector space is an algebra over C. Also, if W is an intermediate extension, W^* and $W^* \cap V$ are also rings and the one-to-one correspondence between closed submodules of them preserves closed ideals and closed prime ideals. Every R-disjoint prime ideal of S is closed, but we do not know whether the same is true for an intermediate extension.

In Section 6 we apply the former results to study strongly prime, non-singular prime, and primitive ideals. We prove that if R is a strongly prime (resp. non-singular prime, primitive) ring and W is an intermediate extension, then every ideal P of W which is maximal with respect to $P \cap R = 0$ is strongly prime (resp. non-singular, primitive). Also, under the same assumption we prove that every closed prime ideal P of W is strongly prime (resp. non-singular), provided that $W \subseteq V_S(X)$, where $V_S(X)$ is the centralizer of X in S.

In Section 7 we study prime ideals and radicals, under some finiteness conditions. In particular, we obtain that if W is an intermediate extension of finite rank, then every R-disjoint prime ideal P of W is closed and maximal with respect to the condition $P \cap R = 0$. Also, if W is torsion-free the prime radical of W is nilpotent and a finite intersection of minimal prime ideals. We have a similar result for the prime radical of a, so called, almost finite extension.

Finally, in Section 8 we apply the former results to an arbitrary centred extension of a (not necessarily prime) ring R.

The paper is reasonably self contained. It is clearly a natural sequel of [5], but except some facts which are based on that paper no heavy machinery is needed. Throughout, R is always a prime ring, except in Section 8, and M is a centred bimodule over R with $X = (x_i)_{i \in \Omega}$ as a set of R-centralizing generators. As we have already said, submodule means sub-bimodule. The notations $\subset$ and $\supset$ will mean strict inclusions.

§1. Closed submodules

Following [5], for submodules $N \subseteq P$ we define the closure of N in P by

$$[N]_P = [N] = \{x \in P : \text{there exists } 0 \neq H \lhd R \text{ such that } xH \subseteq N\}.$$

We will omit the subscript P when there is no possibility of misunderstanding.

It is clear that the closure $[N]_P$ of N in P is a submodule of M with $N \subseteq [N]_P \subseteq P$. A submodule N of P is said to be closed in P if $[N]_P = N$.

As in [5], the first thing we will do is to obtain a good characterization of $[N]$. We begin with the case of a free centred bimodule which is similar to the one developed in [5].

Assume that M is free over R with the centralizing basis $E = (e_i)_{i\in\Omega}$. Any $x \in M$ can be uniquely written as a finite sum $x = \sum_{i=1}^{n} a_i e_i$, where $a_i \in R$. The e-coefficient of x will be sometimes denoted by $x(e)$, i.e., for x given above $x(e_i) = a_i$, for $i = 1, 2, ..., n$. The support of x is defined as usual by $supp(x) = \{e \in E : x(e) \neq 0\}$.

Let N be a submodule of M. A non-zero element $x \in N$ is said to be of minimal support in N if for every $y \in N$ with $supp(y) \subset supp(x)$ we have $y = 0$. We denote by $m(N)$ the set of all the elements of minimal support in N. The minimality of N is defined by $Min(N) = \{supp(x) : x \in m(N)\}$. For $\Gamma \in Min(N)$ and $e \in \Gamma$ we denote by $\Theta_{\Gamma,e}(N)$ the ideal of R defined by $\Theta_{\Gamma,e}(N) = \{a \in R : \text{there exists } x \in N \text{ with } supp(x) = \Gamma \text{ and } x(e) = a\}$.

The following results can be proved in a similar way as in ([5], Sect. 1).

Lemma 1.1. Let M be a free centred bimodule over R and $N \subseteq P$ submodules of M. We have

i) $Min([N]_P) = Min(N)$.

ii) For any $x \in [N]_P$ there exists a non-zero ideal H of R such that for every $y \in [N]_P$ with $supp(y) \subseteq supp(x)$ we have $yH \subseteq N$.

Theorem 1.2. Let M be a free centred bimodule over R and $N \subseteq P$ submodules of M. Then $[N]_P$ is the largest submodule K of P which contains N and satisfies $Min(K) = Min(N)$. Also, $[N]$ is closed and, moreover, it is the smallest closed submodule of P which contains N. In particular, $[N]$ is the unique closed submodule of P which contains N and satisfies $Min([N]) = Min(N)$.

Now we return to the general case. An element $x \in M$ is said to be a torsion element if there exists a non-zero ideal H of R with $xH = 0$. Thus the torsion elements of M are the elements of the submodule $[0]_M$ of M. We will see soon that this definition agrees with the one given in ([20], Sect. 1).

The submodule N of M is said to be torsion-free (resp. torsion), if $[0]_N = 0$ (resp. $[0]_N = N$).

If every generator x_i of M, $i \in \Omega$, is a torsion element, then M is a torsion bimodule. Thus $[0]_P = P$, for every submodule P of M. It follows that P is the unique closed submodule of P. Consequently, it is natural to assume that there exist generators of M which are not torsion elements. It is easy to see that any such a generator is an element of M which is free over R.

Hereafter, we assume that M is not a torsion bimodule. Consequently, by the Zorn's Lemma there exists a subset $E = (x_j)_{j\in\Lambda}$ of X which is a maximal R-independent subset of X. Denote by L the (free) submodule of M which has E as a centralizing basis. There is a nice relation between M and L.

<u>Lemma</u> 1.3. Take any $y \in M$. Then there exists a non-zero ideal H of R such that $yH \subseteq L$ and $Hy \subseteq L$. Moreover, if we choose a representation of y as $\sum_{i=1}^{n} b_i x_i$, $b_i \in R$, we may choose the ideal H depending only on the set $\{x_1, x_2, ..., x_n\}$.

<u>Proof</u>. Suppose that $x \in X$ and $x \notin E$. By the maximality of E there exist $x_1, ..., x_t$ in E such that $\{x_1, ..., x_t, x\}$ is linearly dependent over R. Then there exist $a_1, ..., a_t, a$ in R with $a_1x_1 + ... + a_tx_t + ax = 0$, $a \neq 0$. Thus $xRaR \subseteq \sum_{i=1}^{t} x_iRa_iR \subseteq L$, where RaR is a non-zero ideal of R.

Now, take $y = \sum_{i=1}^{n} b_i x_i$, $b_i \in R$. We may assume $x_1, ..., x_s$ are in E and $x_{s+1}, ..., x_n$ are not in E. As above we find non-zero ideals H_j of R such that $x_jH_j \subseteq L$, $j = s+1, ..., n$. Hence the ideal $H = \cap_{j=s+1}^{n} H_j$ satisfies the required conditions.

Let $N \subseteq P$ be submodules of M. We say that N is a dense submodule of P if $[N]_P = P$. Equivalently, for every $x \in P$ there exists a non-zero ideal H of R with $xH \subseteq P$.

From the above it is clear that for every centred bimodule M over R there exists a dense submodule L which is free over R. We will refer to it as a free dense submodule of M.

In [20], an element $x \in M$ is said to be a torsion element if there are non-zero ideals A and B of R such that $AxB = 0$. As a first application of the existence of a free dense submodule we show a result which implies, in particular, that our definition is equivalent to this one. For liberal bimodules this equivalence was proved in ([20], Lemma 1.4). We have

Corollary 1.4. Let N be a submodule of M and $x \in M$. Then the following conditions are equivalent

i) There exists a non-zero ideal H of R such that $xH \subseteq N$.

ii) There exists a non-zero ideal F of R such that $Fx \subseteq N$.

iii) There are non-zero ideals A and B of R such that $AxB \subseteq N$.

Proof. If the factor bimodule M/N is a torsion bimodule, then the three conditions above are automatically satisfied. So we may assume there exists a free dense submodule L of M/N. Also, by factoring out N we may assume $N = 0$.

Let $AxB = 0$, where A and B are non-zero ideals of R. By Lemma 1.3 there exists $0 \neq H \lhd R$ with $xH \subseteq L$. Then $AxHB = 0$ and since L is free and R is prime we obtain $xH = 0$. Thus iii)$\to$i). The converse is clear and the proof of the equivalence ii)$\leftrightarrow$iii) is similar.

Remark 1.5. As in [5] we can define $[N]$ in a dual way. In fact, by Corollary 1.4 we have

$$[N]_P = \{x \in P : \text{there exists } 0 \neq H \lhd R \text{ such that } Hx \subseteq N\} =$$

$$= \{x \in P : \text{there are non-zero ideals } A \text{ and } B \text{ of } R \text{ with } AxB \subseteq N\}.$$

Lemma 1.6. Assume that $N \subseteq P$ are submodules of M such that N is dense in P. Then for any submodule K of P we have $[K]_P = [K \cap N]_P$.

Proof. It is clear that $[K \cap N]_P \subseteq [K]_P$. Take $x \in [K]_P$. Then $x \in P$ and $xH \subseteq K$ for a non-zero ideal H of R. Also, there exists a non-zero ideal F of R such that $xF \subseteq N$. Then $x(H \cap F) \subseteq (K \cap N)$ and so $x \in [K \cap N]_P$.

Now we can obtain the following interesting result.

Theorem 1.7. Assume that $N \subseteq P$ are submodules of M such that N is dense in P. Then there is a one-to-one correspondence between the set of all the closed submodules of P and the set of all the closed submodules of N. Moreover, this correspondence associates the closed submodule K of P with the closed submodule I of N if $K \cap N = I$ (equivalently $K = [I]_P$).

Proof. If K is a closed submodule of P, then $K \cap N$ is clearly a closed submodule of N and by the former lemma we have $[K \cap N]_P = [K]_P = K$. Conversely, assume that I is a closed submodule of N and put $K = [I]_P$. Then $K \cap N = I$ and $[K]_P = [K \cap N]_P = [I]_P = K$, i.e., K is closed in P.

The following is clear

Corollary 1.8. Assume that P is a submodule of M and L is a free dense submodule of M. Then

i) $[0]_M \cap L = 0$.

ii) There is a one-to-one correspondence via contraction between the set of all the closed submodules of P and the set of all the closed submodules of $P \cap L$.

iii) $P \not\subseteq [0]_M$ if and only if $P \cap L \neq 0$.

Now we can give a description of the closure $[N]_P$ of a submodule N of P, combining the results of Theorems 1.2 and 1.7. Choose a free dense submodule L of M with the basis E. For a submodule N of P we define the E-minimality of N as the minimality of the submodule $N \cap L$ of L. That is, $Min_E(N) = Min(N \cap L)$. We have $Min_E([N]_P) = Min_E([N]_P \cap L) = Min_E([N \cap L]_P \cap L) = Min([N \cap L]_{P \cap L}) = Min(N \cap L) = Min_E(N)$.
The following is now clear

Theorem 1.9. Assume that $N \subseteq P$ are submodules of M and let L be a free dense submodule of M with the centralizing basis E. Then $[N]_P$ is the largest submodule K of P which contains N and satisfies $Min_E(K) = Min_E(N)$. Also, $[N]_P$ is closed in P and, moreover, it is the smallest closed submodule of P which contains N. In particular, $[N]$ is the unique submodule of P containing N and satisfying $Min_E([N]) = Min_E(N)$.

In the proof of Theorem 1.2 we show that if K is a submodule of P such that $K \supseteq N$ and $Min(K) = Min(N)$ we have $K \subseteq [N]$. Actually, as it was pointed out in [5], we need only that K be a right (or left) submodule (see the remark preceding Corollary 1.6 in [5]). Thus we have

Corollary 1.10. Assume $K \subseteq N \subseteq P$ are submodules of M, I is a right (or left) submodule of P containing N and E is a basis of a free dense submodule of M. Then

i) $[K]_P \subseteq [N]_P$. In addition, if $Min_E(K) = Min_E(N)$, then $[K]_P = [N]_P$.

ii) If N is closed in P and $Min_E(I) = Min_E(N)$, then $I = N$.

The following corollary is very useful. First, suppose that a free dense submodule L of M has been chosen. For an element $x \in L$ we denote by $supp(x)$ the support of x with respect to the basis E. Let N be a submodule of M. An element $x \in L$ is said to be a remainder modulo N if for every $y \in N$ with $supp(y) \subseteq supp(x)$ we necessarily have $y = 0$.

Corollary 1.11. Let N be a submodule of M which is closed in P. If K is a right (or left) submodule of P such that $K \supset N$, then there exists $x \in m(L \cap K)$ which is a remainder modulo N.

Proof. Note that if $N = 0$, then $K \neq 0$ (so $K \cap L \neq 0$) and every element $0 \neq x \in K \cap L$ is a remainder modulo N. So we may assume $N \neq 0$.

By the way of contradiction, if for every $x \in m(L \cap K)$ there exists $0 \neq y \in N$ with $supp(y) \subseteq supp(x)$, it follows that $supp(y) = supp(x)$, because $N \subseteq K$, and hence $y \in m(N \cap L)$. Consequently, $Min_E(K) = Min_E(N)$ and we obtain $K = N$, by Corollary 1.10, (ii).

Now we compare our definition with the one given in ([8], p.18). Let C be a right R-module and let A be a submodule of C. We say that A is a closed submodule of C in the sense of [8] (G-closed, for short) if A has no proper essential extensions inside of C. We have

Corollary 1.12. Let $N \subseteq P$ be submodules of M.

i) If N is closed in P, then N is G-closed in P.

ii) If P is torsion-free and N is G-closed in P, then N is closed in P.

Proof. (i) Assume that N is a closed submodule of P but is not G-closed in P. Then there exists a right submodule K of P which is an essential extension of N. By Corollary 1.11, there exists $x \in m(K \cap L)$ which is a remainder modulo N, where L is a free dense submodule of M. Then $xR \neq 0$ and so $xR \cap N \neq 0$. Take $a \in R$ such that $0 \neq xa \in N$. Since $supp(xa) \subseteq supp(x)$ it follows that x is not a remainder modulo N, a contradiction.

(ii) If K is a non-zero right submodule of $[N]_P$, we choose any $0 \neq x \in K$. Then there exists a non-zero ideal H of R with $xH \subseteq K \cap N$. Since P is torsion-free we have $K \cap N \neq 0$. Thus $[N]_P$ is an essential extension of N and so $N = [N]_P$.

The proof of the following Lemma is straightforward.

Lemma 1.13. Let M and M' be two centred bimodules and $\varphi : M \rightarrow M'$ an epimorphism of R-bimodules. If $N' \subseteq P'$ are submodules of M', $N = \varphi^{-1}(N')$ and $P = \varphi^{-1}(P')$, we have $[N]_P = \varphi^{-1}([N']_{P'})$. In particular, N is closed in P if and only if N' is closed in P'.

Remark 1.14. The above Lemma allows us to make a reduction when we want to study the lattice of closed submodules of $P \subseteq M$. In fact, since $[0]_P \subseteq [N]_P$, for every submodule $N \subseteq P$, we may factor out $[0]_M$ and assume that M is torsion-free. With this reduction, the lattice of closed submodules of any submodule P of M is just the lattice of G-closed submodules.

Remark 1.15. Another consequence of the Lemma 1.13 is to give an alternative way for the description of the closure $[N]_P$ of a submodule of P. In fact, choose a free bimodule S over R with the basis $E = (e_i)_{i\in\Omega}$ and consider the canonical epimorphism $\varphi : S \rightarrow M$ given by $\varphi(e_i) = x_i$, $i \in \Omega$. Since there is a one-to-one correspondence between the set of all the closed submodules of $P \subseteq M$ and the set of all the closed submodules of $\varphi^{-1}(P) \subseteq S$ which contain $Ker\varphi$, the description of the closure in the free case gives a description for the general case. For example, the minimality of N may be defined as $Min(\varphi^{-1}(N))$, and so on.

2. Enlarging and Contracting Closed Submodules

Let Q be the maximal (complete) right quotient ring of R ([22], Chap IX, [13], Sect. 4.3). We say that T is a ring of right quotients of R if T is a subring of Q containing R. The extended centroid of R is the center of Q. We denote it here by C. For the basic properties we will use here the reader can see ([4], Lemmas 2.1 and 2.2).

Following ([5], Sect. 2), the purpose of this section is to extend the bimodule M to a Q-bimodule M^* and then to contract M^* to a vector space V over C. We will show that there is a one-to-one correspondence between the closed submodules of M, M^* and V.

We point out that we could reduce this study to the Martindale ring of right quotients of R, but we prefere to work with Q. We begin this section with the following particular case.

2.1. Free case. Let L be a free centred bimodule with the centralizing basis $E = (e_i)_{i\in\Omega}$. Denote by L^* the free Q-bimodule $\sum_{i\in\Omega} \oplus Qe_i$, where $E = (e_i)_{i\in\Omega}$ is a centralizing basis of L^*. For any subset S of Q, put $L^*_S = \sum_{i\in\Omega} Se_i$. In particular, L^*_C is a vector space over C with the basis $E = (e_i)_{i\in\Omega}$. We denote L^*_C by V. Also, if T is any ring of right quotients of R, L^*_T is a free T-bimodule with the same basis E and $L \subseteq L^*_T \subseteq L^*$. Finally, $L^*_R = L$.

The proofs of the following results are similar to the proofs given in ([5], Lemmas 2.1, 2.2, 2.3, Corollary 2.4 and Theorem 2.5). We will include only the proof of Theorem 2.5 because this case is more general than that one and also because something seems to be wrong in the proof of ([5], Theorem 2.5).

Lemma 2.1. Let N be a submodule of L, $\Gamma \in Min(N)$ and $e \in \Gamma$. Then there exists a unique element $m_{\Gamma,e} \in V$ such that for every $x \in N$ with $supp(x) = \Gamma$ we have $x = m_{\Gamma,e}x(e) = x(e)m_{\Gamma,e}$. Moreover, $supp(m_{\Gamma,e}) = \Gamma$ and $m_{\Gamma,e}(e) = 1$.

Given a submodule N of L we denote by $M_C(N)$ the set of all the elements

$m_{\Gamma,e}$ constructed in Lemma 2.1, where $\Gamma \in Min(N)$ and $e \in \Gamma$. So $M_C(N) \subseteq V$ and for every $m \in M_C(N)$ there exists a non-zero ideal H of R with $mH = Hm \subseteq N$. Also, for every $x \in m(N)$ we have $x = am$, for some $m \in M_C(N)$ and $a \in R$.

Let T be any ring of right quotients of R and let N be a T-submodule of L^*_T. Generalizing a former definition, we say that an element $0 \neq x \in L^*$ is a remainder modulo N if for every $y \in N$ with $supp(y) \subseteq supp(x)$ we have $y = 0$.

Now, consider $N \subseteq P$ submodules of L^*_T. We denote by $[N]_{T,P}$ the closure of N in P and we put $N_0 = N \cap L \subseteq P_0 = P \cap L$. If $N = 0$, then $[N]_{T,P} = 0$. Thus we assume $N \neq 0$. So N_0 is clearly a non-zero R-submodule of L and $Min(N_0) = Min(N)$. Also, $M_C(N_0)$ is defined as above. Under this notation we have

<u>Lemma</u> 2.2. Let $x \in L^*$. Then there exist elements $q_i \in Q$, $m_i \in M_C(N_0)$, $i = 1, 2, ..., n$, and $y \in L^*$ such that $x = \sum_{i=1}^{n} q_i m_i + y$, where either $y = 0$ or y is a remainder modulo N.

<u>Lemma</u> 2.3. Let N be a T-submodule of L^*_T and $x \in L^*$. Then the following conditions are equivalent

i) $x \in QM_C(N_0)$.

ii) There exists a dense right ideal J of R with $xJ \subseteq N_0$.

In addition, if $x \in L^*_T$ the above conditions are also equivalent to

iii) There exists a non-zero ideal H of T with $xH \subseteq N$.

<u>Corollary</u> 2.4. Assume $N \subseteq P$ are T-submodules of L^*_T, $N_0 = N \cap L$ and $P_0 = P \cap L$. Then we have $[N]_{T,P} = QM_C(N_0) \cap P$ and this submodule is also equal to the set of all the elements $x \in P$ such that there exists a dense right ideal J of R with $xJ \subseteq N_0$.

Note that if R is a simple ring, every submodule of M is closed in M. Then the set of all the C-subspaces of V coincides with the set of all the closed subspaces of V.

Now we can obtain one of the main results of this section. In the proof we will use freely the former results.

Theorem 2.5. Let T be a ring of right quotients of R, let L be a free centred bimodule over R and suppose that P is a submodule of L^*_T. Then there is a one-to-one correspondence between the following

i) The set of all the submodules of L which are closed submodules of $P \cap L$.

ii) The set of all the submodules of L^*_T which are closed submodules of P.

iii) The set of all the C-subspaces of $CM_C(P_0)$.

Moreover, this correspondence associates the closed submodule N of $P_0 = P \cap L$ with the closed submodule N^* of P and the subspace K of $CM_C(P_0)$ if $N^* \cap L = N$ and $N^* = QK \cap P$.

Proof. If $N \subseteq P$ is a submodule of L^*_T which is closed in P and $N_0 = N \cap L \subseteq P_0 = P \cap L$, we have $N = [N]_{T,P} = QM_C(N_0) \cap P$. Then $[N_0]_{R,P_0} = QM_C(N_0) \cap P_0 = N \cap P_0 = N \cap L = N_0$. So N_0 is closed in P_0. If N' is another closed submodule of P with $N' \cap L = N_0$ we have $N' = [N']_{T,P} = QM_C(N_0) \cap P = N$.

On the other hand, let I be a submodule of L which is closed in P_0. Then $N = QM_C(I) \cap P$ is a submodule of L^*_T which is contained in P and $N_0 = N \cap L = QM_C(I) \cap P_0 = [I]_{R,P_0} = I$. Thus $N = QM_C(N_0) \cap P$ and so N is a closed submodule of P with $N \cap L = I$. This establishes the correspondence between (i) and (ii).

To complete the proof it is enough to show the one-to-one correspondence between (ii) and (iii) for $T = Q$ and under the assumption that P is closed in L^*. In fact, we may assume P is closed in L^*_T by Theorem 1.7. Thus, by the first part already proved $P \cap L$ is a closed submodule of L and there exists a unique closed submodule P^* of L^* with $P^* \cap L = P \cap L$. It is clear that $P^* \cap L^*_T = P$. Applying again twice the first part we obtain a one-to-one correspodence between the set of all the closed submodules of P and the set of all the closed submodules of P^*.

So assume that $T = Q$ and P is closed in L^*. First we show that $P \cap V = CM_C(P_0)$. If $m \in M_C(P_0)$, there exists $0 \neq H \lhd R$ with $mH \subseteq P_0 \subseteq P$.

Thus $m \in P$ and we have $CM_C(P_0) \subseteq P \cap V$. Take a basis $\{v_j\}_{j\in\theta}$ of $CM_C(P_0)$ over C. If $P \cap V \supset CM_C(P_0)$ there exists $v \in P \cap V$ such that $A = \{v_j\}_{j\in\theta} \cup \{v\}$ is a linearly independent set over C. Since $L^* \simeq Q \otimes_C V$ we have that A is also Q-independent. However, $P = QM_C(P_0)$ and so $v = \sum_{i=1}^{n} q_i u_i$, for some elements $q_i \in Q$ and $u_i \in M_C(P_0)$. Writing each u_i as a linear combination of the elements v_j we obtain v as a linear combination of $\{v_j\}_{j\in\theta}$. This contradiction shows that $CM_C(P_0) = P \cap V$.

Now, let N be a submodule of P which is closed in P. Then N is closed in L^*, since P is closed. So by the same argument as above $CM_C(N_0) = N \cap V \subseteq P \cap V = CM_C(P_0)$. Then $N \cap V$ is a subspace of $CM_C(P_0)$ and $Q(N \cap V) = QM_C(N_0) = N$.

Conversely, let K be a subspace of $CM_C(P_0)$ and put $N = QK$. Then $N \subseteq QM_C(P_0) = P$. We show that N is closed in P and $K = N \cap V$.

First, assume $M_C(N_0) \not\subseteq K$. Take a basis $\{v_j\}_{j\in\theta}$ of K over C and an element $v \in M_C(N_0)$ such that $A = \{v_j\} \cup \{v\}$ is C-independent. Hence A is also a Q-independent subset of L^*. Since $vH \subseteq N_0 \subseteq N = QK$, for $0 \neq H \triangleleft R$, we obtain a contradiction arguing as above. Therefore $M_C(N_0) \subseteq K$ and so $QM_C(N_0) \subseteq N \subseteq [N]_Q = QM_C(N_0)$. Consequently, N is closed. Now, applying the former part we obtain $N \cap V = CM_C(N_0) \subseteq K \subseteq N \cap V$. Thus $N \cap V = K$ and the proof is complete.

The following consequence of Corollary 2.4 will be used next.

Corollary 2.6. Let M be a centred bimodule over R. Then M is torsion-free if and only if the following condition holds: if $x \in M$ and J is a dense right ideal of R such that $xJ = 0$, then $x = 0$.

Proof. If M is free over R, clearly the above condition holds. In general, assume that M is torsion-free and take a free centred bimodule L and an epimorphism $\varphi : L \to M$, as in Remark 1.15. Then $Ker\varphi$ is a closed submodule of L. Suppose $x \in M$ and $xJ = 0$, J a dense right ideal of R, and take $y \in L$ with $\varphi(y) = x$. Then $yJ \subseteq Ker\varphi$ an so $y \in Ker\varphi$, by Corollary 2.4. Hence $x = 0$. The converse is clear.

2.2. The canonical extension of M. Now we come back to the general case. Let M be a centred bimodule over R with $X = (x_i)_{i\in\Omega}$ a set of centralizing generators.

There always exists an extension of M to a Q-bimodule M^* ([22], Chap. IX). But we will present here a direct way to obtain M^*, independent of the results in the literature.

First, let P be a right R-module. We say that P is torsion-free if the following condition holds: $x \in P$ and $xJ = 0$, for a dense right ideal J of R, imply $x = 0$. By Corollary 2.6 this definition agrees with the one given in Section 1 for centred bimodules.

Definition 2.7. A pair (M^*, j) of a centred bimodule M^* over Q such that M^* is torsion-free as a right R-module and an R-bimodule homomorphism $j : M \to M^*$ is said to be a canonical torsion-free extension of M if for every right Q-module P which is torsion-free as right R-module and for every homomorphism of right R-modules $f : M \to P$, there exists a unique homomorphism of right Q-modules $f^* : M^* \to P$ such that $f^* \circ j = f$.

We have the following

Theorem 2.8. For every centred bimodule M over R there exists a canonical torsion-free extension (M^*, j).

Proof. Let L be a free R-bimodule with the basis $E = (e_i)_{i\in\Omega}$ and $\varphi : L \to M$ the R-bimodule homomorphism defined by $\varphi(e_i) = x_i$, $i \in \Omega$. Denote by L^* the extension of L to a free Q-bimodule with the same basis E and consider the canonical inclusion $L \to L^*$ as an identification. So assume $L \subseteq L^*$.

The submodule $I = \varphi^{-1}([0]_M)$ is a closed submodule of L and so there exists a closed submodule I^* of L^* such that $I^* \cap L = I$, by Theorem 2.5. Put $M^* = L^*/I^*$ and denote by $\pi : L^* \to M^*$ the canonical projection. Thus M^* is a centred Q-bimodule with $(\pi(e_i))_{i\in\Omega}$ as a generating set of centralizing elements and π is a Q-bimodule homomorphism. Also, since $\pi^{-1}(0) = I^*$ is closed we have that M^* is torsion-free as an R-module. We can easily see that $j(x_i) = \pi(e_i)$, $i \in \Omega$, induces a well-defined R-bimodule homomorphism $j : M \to M^*$.

Suppose that P is a right Q-module which is torsion-free as right R-module and $f : M \to P$ is an homomorphism of right R-modules. Every $x \in M^*$ can be written as $x = \sum_{i=1}^{n} \pi(e_i)q_i = \sum_{i=1}^{n} j(x_i)q_i$, for some $x_i \in X$, $q_i \in Q$, $i = 1, ..., n$. Assume $x = 0$. Then $\sum_{i=1}^{n} e_i q_i \in I^*$ and take a dense right ideal J of R with $q_i J \subseteq R$, $i = 1, ..., n$. We have $\sum_{i=1}^{n} e_i q_i J \subseteq I$ and so

$\sum_{i=1}^{n} x_i q_i J \subseteq [0]_M$. Hence, for every $a \in J$ there exists a non-zero ideal H_a of R with $\sum_{i=1}^{n} x_i q_i a H_a = 0$. It follows that $\sum_{i=1}^{n} f(x_i) q_i a H_a = 0$. Since P is torsion-free as right R-module we obtain $\sum_{i=1}^{n} f(x_i) q_i = 0$.

Consequently, the mapping $f^* : M^* \to P$ defined by $f^*(\sum_{i=1}^{n} j(x_i) q_i) = \sum_{i=1}^{n} f(x_i) q_i$ is a well-defined right Q-homomorphism such that $f^* \circ j = f$. The unicity of f^* is evident.

Remark 2.9. From the proof of Theorem 2.8 is clear that $(j(x_i))_{i \in \Omega}$ is a set of Q-centralizing generators of M^*. We can easily see that $Ker\, j = [0]_M$. So we may consider $M \subseteq M^*$ if and only if M is torsion-free. Finally, if M is free over R, then M^* is free over Q.

Lemma 2.10. Under the above notation, if P is a Q-bimodule and $f : M \to P$ is a homomorphism of R-bimodules, then $f^* : M^* \to P$ is a homomorphism of Q-bimodules.

Proof. It is enough to show that $q f(x_i) = f(x_i) q$, for every $q \in Q$, $i \in \Omega$. Let J be a dense right ideal of R with $qJ \subseteq R$ and take any $a \in J$. We have $(q f(x_i) - f(x_i) q) a = q f(x_i a) - f(x_i) q a = q f(x_i a) - f(x_i q a) = q a f(x_i) - q a f(x_i) = 0$. Since P is torsion-free the result follows.

Remark 2.11. From Lemma 2.10 we can easily see that the extension (M^*, j) is unique up to isomorphism, where isomorphism means isomorphism of pairs as usual.

Corollary 2.12. Let M and M' be two centred bimodules over R and $f : M \to M'$ an R-bimodule homomorphism. Then there exists a unique homomorphism of Q-bimodules $f^* : M^* \to M'^*$ such that $j' \circ f = f^* \circ j$, where j and j' are canonical.

In addition, if f is surjective (resp. injective) so is f^*.

Proof. The first part is straightforward. Also, it is not hard to show that f^* is onto when f is onto. Assume that f is injective and take $x = \sum_{i=1}^{n} j(x_i) q_i \in M^*$ such that $f^*(x) = \sum_{i=1}^{n} j' \circ f(x_i) q_i = 0$. Let J be a dense right ideal of R such that $q_i J \subseteq R$, for $i = 1, ..., n$. Thus $\sum_{i=1}^{n} j' f(x_i q_i a) = 0$, for every $a \in J$, and so $f(\sum_{i=1}^{n} x_i q_i a) \in [0]_{M'}$, by Remark 2.9. Let H be a non-zero ideal of R with $f(\sum_{i=1}^{n} x_i q_i a) H = 0$. It follows that $\sum_{i=1}^{n} x_i q_i a H = 0$, then $\sum_{i=1}^{n} x_i q_i a \in [0]_M$ and so $xJ = \sum_{i=1}^{n} j(x_i) q_i J = 0$. Therefore $x = 0$ because M^* is torsion-free.

Now we show the following main result.

Theorem 2.13. For every centred bimodule M over R, the canonical torsion-free extension M^* of M is free over Q. Moreover, if E is a basis of a free dense submodule L of M, then $(j(e))_{e \in E}$ is a centralizing basis of M^* over Q.

Proof. By Corollary 2.12, the canonical inclusion $L \subseteq M$ induces an inclusion $L^* \subseteq M^*$, where L^* is free over Q with the basis E. We show that $L^* = M^*$. It is enough to prove that for any $i \in \Omega$ we have $j(x_i) \in L^*$. If x_i is in E there is nothing to prove. So assume $x_i \notin E$ and denote it by x. Then, by the maximality of E there exist $x_1, ..., x_n$ in E such that $x_1, ..., x_n, x$ are linearly dependent over R. Thus there exist $a_1, ..., a_n$, $a \neq 0$, in R such that $y = a_1 x_1 + ... + a_n x_n + ax = 0$. For any $r \in R$ we have $ary - yra = 0$ so $ara_i = a_i ra$ and hence there are $c_i \in C$ with $c_i a = a_i$, for $i = 1, ..., n$. Then $(c_1 j(x_1) + ... + c_n j(x_n) + j(x))a = 0$, so $c_1 j(x_1) + ... + c_n j(x_n) + j(x))RaR = 0$ and we obtain $j(x) = -\sum_{i=1}^{n} c_i j(x_i) \in L^*$. The proof is complete.

As an immediate consequence of Corollary 2.12 and Theorem 2.13 we have

Corollary 2.14. Let M and M' be two centred bimodules over R and f, g two R-bimodule homomorphisms of M into M'. If $f/L = g/L$ for a free dense submodule of M, then $f = g$.

2.3. General case. Now we will show that the correspondence of Theorem 2.5 also holds when M is not necessarily free over R.

We know that M^* is a free Q-bimodule with the centralizing basis $E = (e_i)_{i \in \Lambda}$. As in the first part of the section we denote by M_T^* the free T-bimodule with the basis E, where T is any ring of right quotients of R, and $V = M_C^* = \sum_{i \in \Lambda} Ce_i$. It is clear that $M_R^* = L$ is a free dense submodule of M. We denote again here by $j : M \to M^*$ the canonical mapping.

Theorem 2.15. Let T be any ring of right quotients of R, M a centred bimodule over R and P a submodule of M_T^*. Then there is a one-to-one correspondence between the following

i) The set of all the submodules of M which are closed in $j^{-1}(P)$.

ii) The set of all the submodules of M_T^* which are closed in P.

iii) The set of all the C-subspaces of $CM_C(P \cap L)$.

Moreover, this correspondence associates the closed submodule N of $J^{-1}(P)$ with the closed submodule N^* of P and the subspace K of $CM_C(P \cap L)$ if $j^{-1}(N^*) = N$ and $N^* = QK \cap P$.

Proof. It follows easily from Theorems 1.7 and 2.5.

We finish this section with some additional remarks.

First, Theorem 2.15 gives, in particular, a one-to-one correspondence between the set of all the closed submodules of M, the set of all the closed submodules of M^* and the set of all the subspaces of V. On the other hand, if we factor out $[0]_M$ and, consequently, we assume that M is torsion-free, then the correspondence is given by intersection, i.e., N corresponds to N^* and K if $N^* \cap M = N$ and $N^* \cap V = K$.

Another interesting fact we point out is that for every $x \in M$ there exists a unique representation of $j(x)$ as $j(x) = \sum_{i=1}^{n} q_i e_i$, $0 \neq q_i \in Q$, $e_i \in E$, $i = 1, ..., n$. Then we can define the E-support of x as $supp(x) = \{e_1, ..., e_n\}$, and the E-minimality of a submodule N of M by the usual way and also give a description of $[N]_M$ using this concept.

Finally, the following is not hard to prove.

Corollary 2.16. Let P be a closed submodule of M and let L be a free dense submodule of M. Denote by P^* the closed submodule of M_T^* with $j^{-1}(P^*) = P$. Then we have $P^* = (P \cap L)^* = \{x \in M_T^*$: there exists a dense right ideal J of R with $xJ \subseteq P\} = \{x \in M_T^*$: there exists a dense right ideal J of R with $xJ \subseteq P \cap L\} = QM_C(P \cap L) \cap M_T^*$.

3. The torsion-free rank of a bimodule

Before considering ring extensions we give some applications.

Let N be a submodule of a centred bimodule M over a prime ring R. Following ([20], Definition 1.5), we define the rank of N as the length of the longest possible direct sum of non-zero torsion-free sub-bimodules of N, if such a bound exists, or ∞ in the contrary case. We denote the rank of N by rank (N).

In order to give an equivalent definition of rank (N) we begin with the following.

Lemma 3.1. Let N be any submodule of M. Then rank(N) =rank$([N])$ = rank$([N]/[0])$, where $[N]/[0]$ is a submodule of $M/[0]_M$.

Proof. Since $[[N]] = [N]$ it is enough to show that rank(N) = rank $([N]/[0])$.

Let $\sum_{i\in\Gamma} \oplus N_i \subseteq N$, where N_i is a torsion-free submodule of M, for all $i \in \Gamma$. Then we have $N_i \cap [0] = 0$ and so each N_i can be regarded as a torsion-free submodule of $M/[0]$ and $\sum_{i\in\Gamma} \oplus N_i \subseteq [N]/[0]$.

Conversely, assume that $(P_i)_{i\in\Gamma}$ is a family of submodules of M properly containing $[0]$ such that $\sum_{i\in\Gamma}(P_i/[0]) = \sum_{i\in\Gamma} \oplus(P_i/[0]) \subseteq [N]/[0]$ (note that every $P_i/[0]$ is automatically torsion-free). Let L be a dense submodule of M. Then $P_i \cap L$ is a non-zero torsion-free submodule of M with $\sum_{i\in\Gamma}(P_i \cap L) = \sum_{i\in\Gamma} \oplus(P_i \cap L) \subseteq [N]$. (Corollary 1.8). For every $i \in \Gamma$ take a non-zero element $y_i \in P \cap L$. So there exists a non-zero ideal H_i of R with $0 \neq y_i H_i \subseteq N$. Choose $a_i \in H_i$ such that $z_i = a_i y_i \neq 0$. We can easily see that $\sum_{i\in\Gamma} Rz_iR = \sum_{i\in\Gamma} \oplus Rz_iR \subseteq N$, where Rz_iR is a non-zero torsion-free submodule of M. The proof is complete.

The above Lemma shows that to compute rank(N) we may always assume that M is torsion-free and N is closed in M.

Now, denote by (M^*, j) the canonical torsion-free extension of M and by V the C-vector space M^*_C. We have

Theorem 3.2. Let N be a closed submodule of M, N^* the closed submodule of M^* with $j^{-1}(N^*) = N$ and $K = N^* \cap V$. Then rank(N)=$\dim_C(K)$, where $\dim_C(K)$ denotes the dimension of K as a C-vector space. In particular, rank(N) =rank(N^*).

Proof. We may assume M is torsion-free, i.e., $M \subseteq M^*$ and $N = N^* \cap M$.

Suppose $\sum_{i\in\Gamma} \oplus N_i \subseteq N$, where N_i is a non-zero (necessarily torsion-free) submodule of M, for all $i \in \Gamma$. Let N_i^* denote the extension of $[N_i]$ to a closed submodule of M^* and put $K_i = N_i^* \cap V$, $i \in \Gamma$. We easily see that $\sum_{i\in\Gamma}[N_i] = \sum_{i\in\Gamma} \oplus[N_i] \subseteq N$ and also $\sum_{i\in\Gamma} N_i^* = \sum_{i\in\Gamma} \oplus N_i^* \subseteq N^*$. Hence $\sum_{i\in\Gamma} K_i = \sum_{i\in\Gamma} \oplus K_i \subseteq K$, where K_i is non-zero. It follows that $\dim_C(K) \geq \mathrm{rank}(N)$.

Conversely, suppose that $\{v_i\}_{i\in\Gamma}$ is a linearly independent subset of K. Since $M^* \simeq Q \otimes_C V$ as Q-bimodules, we have that $\{v_i\}_{i\in\Gamma}$ is a set of Q-centralizing elements of N^* which are linearly independent over Q. Thus $\sum_{i\in\Gamma} Qv_i = \sum_{i\in\Gamma} \oplus Qv_i \subseteq N^*$. Consequently, $\sum_{i\in\Gamma}(Qv_i \cap M) = \sum_{i\in\Gamma} \oplus(Qv_i \cap M) \subseteq N$ and it follows that $\mathrm{rank}(N) \geq \dim_C(K)$.

Since $\mathrm{rank}(N)$ equals the dimension of a vector space, properties of rank will follow easily from well-known properties of vector spaces. In fact, many of the properties obtained in ([20], Sect. 1) can be reproved using this method. We give two examples.

Corollary 3.3. Let $N \subseteq P$ be closed submodules of M such that $\mathrm{rank}(N) = \mathrm{rank}(P) < \infty$. Then $N = P$.

Proof. If K and K' are the subspaces of V corresponding to N and P, respectively, we have $K \subseteq K'$ and $\dim_C(K) = \dim_C(K')$. Consequently, $K = K'$ and so $N = P$.

As a second example we will reprove Proposition 1.8 of [20]. First we need the following.

Lemma 3.4. Let M be a centred bimodule over R and let N be a submodule of M. Then M/N is a centred bimodule whose canonical torsion-free extension is $(M/N)^* \simeq M^*/N^*$, where M^* is the canonical torsion-free extension of M and N^* is the closed submodule of M^* which corresponds to $[N]$.

Proof. The canonical mapping $j : M \to M^*$ induces a homomorphism $g : M/N \to M^*/N^*$. If P is a right Q-module which is torsion-free as right R-module and $f : M/N \to P$ is a homomorphism of right R-modules, then there exists a Q-homomorphism $f' : M^* \to P$ such that $f \circ \pi = f' \circ j$, where $\pi : M \longrightarrow M/N$ is canonical. Since $N^* \subseteq Ker\, f'$ we obtain a Q-homomorphism $f^* : M^*/N^* \longrightarrow P$ with $f^* \circ g = f$. The result follows from the unicity of $(M/N)^*$.

Corollary 3.5 (c.f. [20], Proposition 1.8.). Assume that $N \subseteq P$ are submodules of a centred bimodule M. Then rank(P) =rank(N)+rank(P/N).

Proof. Let N^*, P^* be the closed submodules of M^* corresponding to N and P, respectively, and put $K = N^* \cap V$, $I = P^* \cap V$. By Lemma 3.1, rank(P/N) =rank$([P]/[N])$. Also, the closed submodule of $(M/[N])^* \cong M^*/N^*$ corresponding to $[P]/[N]$ is P^*/N^*. Therefore rank$(N) = \dim_C(K)$, rank$(P) = \dim_C(I)$ and rank$(P/N) = \dim_C(I/K)$. The result follows from the relation $\dim_C(I) = \dim_C(K) + \dim_C(I/K)$.

4. Non-singular and Strongly Closed Submodules

Recall that the singular submodule $Z(P)$ of a right R-module P is defined as the set of all the elements $x \in P$ such that the annihilator $r(x)$ of x in R is an essential right ideal of R. The module P is said to be non-singular if $Z(P) = 0$. We say that a submodule N of P is non-singular in P if $Z(P/N) = 0$ ([8], p.30-36).

When M is a bimodule over R and P is a submodule of M, we consider P as a right R-module. So $Z(P)$ is the right singular submodule of P an is, in fact, a sub-bimodule of M. We will say simply "singular submodule" and "non-singular", omitting "right". Also, $r(x)$ will denote the right annihilator of x in R.

For a ring R, the singular ideal of R is the ideal $Z(R)$, which is the singular submodule of R when considered as right R-module. We say that R is non-singular if $Z(R) = 0$.

Lemma 4.1. Let P be a submodule of a centred bimodule M and let N be a submodule of P. If N is non-singular in P, then N is closed in P.

Proof. Take $x \in [N]_P$. Then $x \in P$ and $xH \subseteq N$, for $0 \neq H \triangleleft R$. Hence $r(x+N) \supseteq H$, where $x+N \in P/N$. Since R is prime, $r(x+N)$ is an essential right ideal of R and so $x + N \in Z(P/N) = 0$. It follows that $x \in N$.

One of the purposes of this section is to study when the converse of Lemma 4.1 holds. We begin with the following

Lemma 4.2. Assume that R is a prime non-singular ring and $N \subseteq P$ are submodules of M. Then $Z(P/N) = [N]_P/N$.

Proof. By factoring out the submodule N we may assume $N = 0$.
If $x \in [0]_P$, then there exists a non-zero ideal H of R with $xH = 0$. Then $x \in Z(P)$ and so $[0]_P \subseteq Z(P)$. Assume that $Z(P) \supset [0]_P$ and let L be a free dense submodule of M. By Corollary 1.11 there exists $y = a_1e_1 + ... + a_ne_n \in m(Z(P) \cap L)$ which is a remainder modulo $[0]_P \cap L$, where $0 \neq a_i \in R$ and $e_i \in E$ (E is a basis of L), $i = 1, 2, ..., n$. We easily see that $r(y) = r(a_1)$ is an essential right ideal of R and so $a_1 \in Z(R) = 0$, a contradiction.

The following is clear

Corollary 4.3. Assume that R is a prime non-singular ring and $N \subseteq P$ are submodules of M. Then N is closed in P if and only if N is non-singular in P. In particular, P is torsion-free if and only if P is non-singular.

Combining Lemma 4.2 with ([4], Corollary 2.5) we have

Corollary 4.4 Let R be a prime non-singular ring and let I be an R-disjoint ideal of $R[X]$. Then there exists a unique monic polynomial $f_0 \in C[X]$ such that $Z(R[X]/I) = (f_0Q[X] \cap R[X])/I$, where $R[X]/I$ is considered a right R-module.

Now we obtain a converse of Corollary 4.3.

Lemma 4.5. Let N be a submodule of M which is not a torsion module. If $Z(N/[0]_M) = 0$, then R is non-singular.

Proof. By factoring out $[0]_M$ we may assume M is torsion-free, $Z(N) = 0$ and $N \neq 0$. Let L be a free dense submodule of M and take $0 \neq x \in N \cap L$, say $x = a_1e_1 + ... + a_ne_n$, $0 \neq a_i \in R$, $i = 1, ..., n$. If $a \in Z(R)$, then $r(a)$ is an essential right ideal of R. Then $r(xa)$ is also essential, thus $xa \in Z(N) = 0$. Therefore $a_1 Z(R) = 0$ and so $Z(R) = 0$.

We summarize the former results in the following

Theorem 4.6. Let M be a centred bimodule over the prime ring R and P a submodule of M which is not a torsion submodule. Then the following conditions are equivalent

i) R is non-singular.

ii) $Z(P/[0]_P) = 0$.

iii) Every closed submodule of P is non-singular in P.

iv) $Z(P/N) = [N]_P/N$, for every submodule N of P.

Corollary 4.7. A prime ring R is non-singular if and only if there exists a non-singular centred bimodule over R.

Now we turn for strongly prime rings. Recall that a ring R is said to be (right) strongly prime if every non-zero ideal I of R contains an insulator, i.e., there exists a finite set $F \subseteq I$ such that $Fa = 0$, $a \in R$, implies $a = 0$. An ideal P of R is said to be strongly prime if R/P is a strongly prime ring. For more details on strongly prime rings and ideals see [15].

By the results in ([5], Sect. 3), we may expect that there exists some result concerning strongly prime rings similar to Theorem 4.6. To obtain this result we give the following definition.

Let P be a submodule of M. A submodule N of P is said to be right strongly closed in P if for any submodule I of M with $N \subset I \subseteq P$ there exists a finite set $F \subseteq I$ such that $Fa \subseteq N$, $a \in R$, implies $a = 0$. Such a set F will be called an insulator. The submodule P is said to be strongly closed if the ideal (0) of P is strongly closed in P.

Every strongly closed submodule of P is closed in P. Moreover, we have

Lemma 4.8. Assume that $N \subseteq P$ are submodules of M and N is strongly closed in P. Then N is non-singular in P.

Proof. Assume $Z(P/N) = I/N$, where $I \supset N$ is a submodule of P. Then there exists a finite set $F \subseteq I$ which is an insulator. For every $x \in F$, $A_x = r(x + N)$ is an essential right ideal of R, where $x + N \in I/N$. Also, $F(\cap_{x\in F} A_x) \subseteq N$. Then $\cap_{x\in F} A_x = 0$, a contradiction.

It is easy to see that an intersection of strongly closed submodules of P is also a strongly closed submodule of P. Then the intersection of all the strongly closed submodules of P is the smallest strongly closed submodule of P. This submodule will be called the strongly closed radical of P and denoted by $s(P)$.

Lemma 4.9. Assume that R is a strongly prime ring. Then for submodules $N \subseteq P$ of M we have $s(P/N) = [N]_P/N$.

Proof. Assume that I is a submodule of M with $N \subset I \subseteq P$ and that I is strongly closed in P. If $[N]_P \not\subseteq I$, there exists a finite set $F \subseteq [N]_P + I$ such that $Fa \subseteq I$, $a \in R$, implies $a = 0$. Clearly we may assume $F \subseteq [N]_P$. Then for every $x \in F$ there exists a non-zero ideal H_x of R such that $xH_x \subseteq N$. Then $F(\cap_{x\in F} H_x) \subseteq N \subseteq I$ and so $\cap_{x\in F} H_x = 0$, a contradiction. Consequently $[N]_P \subseteq I$.

Now we show that $[N]_P$ is strongly closed in P, provided R is strongly prime. Assume that I is a submodule of P with $I \supset [N]_P$ and let L be a free dense submodule of M. Then there exists $y \in m(I \cap L)$ which is a remainder module $[N]_P \cap L$. Write $y = a_1e_1 + ... + a_ne_n$, $0 \neq a_i \in R$, for $i = 1, ..., n$. We define an ideal of R by $H = \{a \in R :$ there exists $y \in I \cap L$ with $supp(y) = \{e_1, ..., e_n\}$ and $y(e_1) = a\}$. Since R is strongly prime, there exists $F = \{b_1, ..., b_t\} \subseteq H$ such that $Fb = 0$, $b \in R$, implies $b = 0$. Now, for every $b_i \in F$ there exists $y_i \in I \cap L$ with $supp(y_i) = \{e_1, ..., e_n\}$ and $y_i(e_1) = b_i$, $i = 1, 2, ..., t$. Then we have that $\{y_1, ..., y_t\}$ is an insulator modulo $[N]_P$ which is contained in I. For if $b \in R$ and $y_ib \in [N]_P$, $i = 1, ..., t$, we have $y_ib = 0$, thus $Fb = 0$ and so $b = 0$. The proof is complete.

Corollary 4.10. Assume that R is a strongly prime ring and $N \subseteq P$ are submodules of M. Then N is strongly closed in P if and only if N is closed in P.

In particular, putting together Lemma 4.1, Lemma 4.8 and Corollary 4.10

we have

Corollary 4.11. Assume that R is a strongly prime ring and P is a submodule of M. Then the following conditions are equivalent

i) P is torsion-free

ii) P is non-singular

iii) P is strongly closed.

Now we prove the following converse of Corollary 4.10.

Lemma 4.11. Let N be a submodule of M which is not a torsion submodule. If $[0]_N$ is strongly closed in N, then R is a strongly prime ring.

Proof. By factoring out $[0]_M$ we may assume M is torsion-free and N is strongly closed. Let L be a free dense submodule of M and take any $0 \neq x \in N \cap L$, say $x = a_1e_1 + ... + a_ne_n$, $0 \neq a_i \in R$, $i = 1, ..., n$. Let H be a non-zero ideal of R and consider HxH, a non-zero submodule of N. By the assumption there exists an insulator $F \subseteq HxH$. Also, for every $x_i \in F$ we have $x_i = \sum_j c_{ij}xd_{ij}$, c_{ij}, $d_{ij} \in H$. Then we easily see that $\{d_{ij}\} \subseteq H$ is an insulator in R. Thus R is strongly prime.

As a direct consequence of the former results we have

Theorem 4.12. Let M be a centred bimodule over a prime ring R and P a submodule of M. Then the following conditions are equivalent.

i) R is strongly prime.

ii) $[0]_P$ is a strongly closed submodule of P.

iii) Every closed submodule of P is strongly closed in P.

iv) $s(P/N) = [N]_P/N$, for every submodule N of P.

Corollary 4.13. A ring R is strongly prime if and only if there exists a strongly closed centred bimodule over R.

5. Centred and intermediate extensions

Throughout this section R is again a prime ring and S is an extension of R. We say that S is a centred extension of R if S is a centred bimodule over R. That is, there exists a set of R-centralizing elements $X = (x_i)_{i\in\Omega}$ of S such that $S = \sum_{i\in\Omega} Rx_i$. Clearly we may assume that $1 \in X$. Closed and prime ideals in free centred extensions have been considered in [5].

Let S be a centred extension of R and let W denote a subring of S with $R \subseteq W$. Then we say that W is an intermediate extension of R.

An ideal I of an intermediate extension W of R is said to be R-disjoint if $I \cap R = 0$. The closure $[I]_W$ of an ideal I of W is defined as the closure of I as a submodule of W. Using Corollary 1.4 we easily see that the closure $[I]_W$ of an R-disjoint ideal I of W is also an R-disjoint ideal of W. Also, if $I \cap R \neq 0$, then $[I]_W = W$.

The ideal I of W is said to be closed in W if $[I]_W = I$. It is clear that a proper closed ideal is always R-disjoint. All the results we have proved in Section 1 applies to R-disjoint ideals and closed ideals.

To choose a free dense submodule L of S we consider a maximal R-independent subset $E = (e_i)_{i\in\Lambda}$ of X containing 1. Thus such a free dense submodule contains R, the canonical torsion-free extension S^* of S has a basis containing 1, and $Q \subseteq S^*$. As in the former sections we denote by V the C-vector space $S^*_C = \sum_{i\in\Lambda} Ce_i$.

Since S^* is free over Q with the basis E, the multiplication in S induces a multiplication in S^*. We can easily see that S^* is a ring and the canonical mapping $j : S \rightarrow S^*$ is a ring homomorphism. For every $i, k \in \Lambda$ we have $e_ie_k \in V_{S^*}(Q) = V$, where $V_{S^*}(Q)$ denotes the centralizer of S^* in Q. Consequently V is a C-algebra with the same basis E.

We have some problems to obtain a theorem of the type of Theorem 2.15 for closed and R-disjoint prime ideals of W. First, if T is any ring of right quotients of R, the T-submodule $S^*_T = \sum_{i\in\Lambda} Te_i$ of S^* need not be a subring, in general. So we have to restrict our attention to subrings T of Q with the following additional property: for every $e, e' \in E$ we have $ee' \in S^*_T$. We certainly can proceed with any ring of right quotients containing the central closure RC of R.

Hereafter we modify our notation for simplicity. We denote by Q any ring of right quotients of R containing RC and by S^* the ring $S^*_Q = \sum_{i\in\Lambda} Qe_i$. It is clear that $V \subseteq S^*$, $S^* \simeq Q \otimes_C V$, and S^* is free over Q with the basis E.

There is another problem concerning R-disjoint ideals of an intermediate extension W. If $W = S$ is easy to see that every R-disjoint prime ideal of W is closed. But we do not know if the same result holds for any intermediate extension. Moreover, we think the result is not true. We will consider the question afterwards. Meanwhile, an R-disjoint prime ideal which is closed will be called a closed prime ideal.

We begin the section with the following.

Lemma 5.1. Let $W \subseteq U$ be subrings of S such that W is dense in U. Then the correspondence of Theorem 1.7 preserves closed and closed prime ideals.

Proof. Let P be a closed submodule of U and put $P_0 = P \cap W$. If P_0 is an ideal of W, $x \in P$ and $y \in W$ we have non-zero ideals H and F of R with $xH \subseteq P_0$ and $Fy \subseteq W$. Then $FyxH \subseteq P_0 \subseteq P$ and so $yx \in P$, because P is closed. Similarly, $xy \in P$. Thus P is an ideal of U if and only if P_0 is an ideal of W.

Now, assume that P_0 is prime and let A, B be ideals of U with $AB \subseteq P$. If $x \in [A]_U$, $y \in [B]_U$ then $xH \subseteq A$ and $Fy \subseteq B$, for non-zero ideals H and F of R. Hence as above we obtain $yx \in P$. Therefore we may assume that A and B are closed. Thus $(A \cap W)(B \cap W) \subseteq P_0$ and so either $A \cap W \subseteq P_0$ or $B \cap W \subseteq P_0$. It follows that either $A = [A \cap W]_U \subseteq P$ or $B \subseteq P$.

Conversely, assume that P is prime and A and B are ideals of W with $AB \subseteq P_0$. As above we show that $[A]_U[B]_U \subseteq P$ and we have either $A \subseteq P \cap W = P_0$ or $B \subseteq P_0$.

Let W be an intermediate extension of R. Then it is easy to see that $[W]_S$ is also a subring of S containing R. By Lemma 5.1, in the proof of some results we may assume that W is closed. In this case we say that W is a closed intermediate extension.

We can do another simplification. Every closed ideal of W contains $[0]_W = [0]_S \cap W$. Thus, by factoring out the ideal $[0]_S$ we may assume that S is torsion-free. Henceforth we may consider $S \subseteq S^*$ and the correspondece of Theorem 2.15 is given by intersection.

Let W be an intermediate extension of R. Then there exists a closed submodule W^* of S^* with $W^* \cap S = [W]$ and put $W_0 = W^* \cap R$. We have

Lemma 5.2. If W is an intermediate extension of R, then $[W]$, W^* and W_0 are subrings of S, S^* and V, respectively.

Proof. We already know that $[W]$ is a ring. If $x, y \in W_0 \subseteq W^*$ there exist non-zero ideals H and F of R with $xH \subseteq W$ and $yF \subseteq W$. Then $xyHF = xHyF \subseteq W \subseteq W^*$, hence $xy \in W^* \cap V = W_0$. Thus W_0 is a subalgebra of V. The rest is clear because $W^* = QW_0$.

Now we can prove the following

Theorem 5.3. Let W be an intermediate extension of R. Then the correspondence of Theorem 2.15 is a one-to-one correspondence between the following

i) The set of all the closed (resp. closed prime) ideals of W.

ii) The set of all the closed (resp. closed prime) ideals of W^*.

iii) The set of all the (resp. prime) ideals of W_0.

Proof. We may assume W is closed. Let P denote a closed submodule of W, P^* the extension of P to W^* and $P_0 = P^* \cap W_0$. Using a similar argument to that in Lemma 5.2 we see that when one of the submodules P, P^* and P_0 is an ideal so are the others.

Assume that P is a closed prime ideal and A, B are ideals of W_0 with $AB \subseteq P_0$. Thus $(QA \cap W)(QB \cap W) \subseteq Q(AB) \cap W \subseteq P^* \cap W = P$. Then either $QA \cap W \subseteq P$ or $QB \cap W \subseteq P$ and it follows that either $A \subseteq P_0$ or $B \subseteq P_0$. Consequently P_0 is prime.

Now, assume that P_0 is prime and suppose that $AB \subseteq P^*$, where A and B are ideals of W^*. Then $(A \cap W_0)(B \cap W_0) \subseteq P_0$ and it follows easily that either $A \subseteq P^*$ or $B \subseteq P^*$.

Finally, assume that P^* is prime and $AB \subseteq P$, where A and B are ideals of W. Suppose that there exists $x \in A \setminus P$. As above we may assume that B is closed in W and let B^* denote the extension of B to W^*. For every $y \in B^*$ there exists a dense right ideal J of R with $yJ \subseteq B$. Then $xyJ \subseteq P \subseteq P^*$ and it follows that $xy \in P^*$. Therefore $xB^* \subseteq P^*$, where $x \in P^*$. Consequently $B = B^* \cap S \subseteq P^* \cap S = P$. The proof is complete.

Remark 5.4. i) The above Theorem generalizes ([5], Theorems 2.5 and 2.7).

ii) Changing B by A we see that the above correspondence is also a one-to-one correspondence between closed semiprime ideals.

iii) It is clear that the correspondence preserves intersections too.

Now we give some easy examples of R-disjoint prime ideals which are automatically closed.

Exemple 5.5. Let W be an intermediate extension and let P be an ideal of W which is maximal with respect to $P \cap R = 0$. Then it is easy to show that P is a closed prime ideal.

Example 5.6. Assume that the ring R satisfies the following condition: Every non-zero ideal of R contains a central element (for example, this condition holds if R is a PI ring). Then every R-disjoint prime ideal of W is closed. For, if $xH \subseteq P$ for $x \in W$, $0 \neq H \lhd R$, where P is an R-disjoint prime ideal of W, we obtain $xWcW \subseteq P$ for a central element $0 \neq c \in H$. Then $x \in P$.

Example 5.7. Let R be a subdirectly irreducible prime ring and let W be an intermediate extension. Then every R-disjoint prime ideal of W is closed. In fact, if $xH \subseteq P$, for $x \in W$, $0 \neq H \lhd R$, take any element $y \in W_0$ and let F be the minimal ideal of R. Then $yF \subseteq W$. Also, for every $z = yq$, $q \in Q$, we have $xzFHF = xyqFHF = xqFHyF \subseteq xHyF \subseteq P$, since $qF \subseteq R$. It follows easily that $xWFHF \subseteq P$ and consequently $x \in P$.

In Section 7 we will consider another case in which every R-disjoint prime ideal is closed.

6. Special types of prime ideals

The purpose of this section is to study strongly prime and non-singular prime ideals. At the end, we also include a Theorem concerning primitive ideals. These results are generalizations of the results in ([5], Sections 3 and 4).

Throughout this section W is an intermediate extension of a prime ring R. In [5], we proved that if R is a prime ring of some special type (e.g., strongly prime, non-singular, primitive), S is a free centred extension of R and P is an ideal of S which is maximal with respect to $P \cap R = 0$, then S/P is also a ring of the considered type ([5], Theorems 3.1, 3.2, 4.1). One of the purposes of this section is to extend this results. We have

Theorem 6.1. Let R be a strongly prime ring and let P be an ideal of W which is maximal with respect to $P \cap R = 0$. Then P is a strongly prime ideal of W.

Proof. Suppose that I is an ideal of W with $I \supset P$. Then $I \cap R \neq 0$ and so there exists a finite set $F \subseteq I \cap R$ such that $Fa = 0$, $a \in R$, implies $a = 0$. We show that F is an insulator in W/P. Put $K = \{y \in W : Fy \subseteq P\}$. Then K is a right ideal of W containing P. Assume, by contradiction, that $K \supset P$ and take a free dense submodule L of S with the basis $E = (e_i)_{i \in \Lambda}$. Then since P is closed, there exists an element $x \in m(K \cap L)$ which a remainder modulo $P \cap L$, by Corollary 1.11. Write $x = \sum_{i=1}^{n} a_i e_i$, $0 \neq a_i \in R$, $e_i \in E$, $i = 1, ..., n$. Since $Fx \subseteq P \cap L$ we have $Fx = 0$ and so $Fa_i = 0$, $i = 1, ..., n$. It follows that $x = 0$, a contradiction. Consequently $K = P$ and P is strongly prime.

Now we consider non-singular prime ideals. A prime ideal P of W is said to be non-singular if W/P is a (right) non-singular prime ring. Right annihilators of an element $\bar{x} = x + P \in W/P$, $x \in W$, will be denoted by $r_{W/P}(\bar{x})$.

Theorem 6.2. Let R be a non-singular prime ring and let P an ideal of W which is maximal with respect to $P \cap R = 0$. Then P is a non-singular prime ideal.

Proof. Assume, by contradiction, that $Z(W/P) = I/P \neq 0$, where I is an ideal of W. Then $I \cap R \neq 0$ and we may choose any $0 \neq a \in I \cap R$. We will reach a contradiction by showing that $a \in Z(R) = 0$.

Let J be a non-zero right ideal of R and let L be a free dense submodule of S with the basis $E = (e_i)_{i \in \Lambda}$. Since $(JW + P)/P$ is a non-zero right ideal of W/P there exists $x \in JW \setminus P$ such that $ax \in P$. Put $K = \{y \in JW : ay \in P\}$. Then K is a right ideal of W and $K \supset P$. Thus me may choose

$x \in m(K \cap L)$ which is a remainder modulo $P \cap L$, by Corollary 1.11. Also, $x = \sum_{i=1}^{n} a_i w_i$, for $a_i \in J$, $w_i \in W$, $i = 1, ..., n$. Take a non-zero ideal H of R with $w_i H \subseteq W \cap L$, for every i, and choose $b \in H$ such that $xb \neq 0$. It is easy to see that $xb = \sum_{j=1}^{m} b_j e_j$, for some $b_j \in J$, $e_j \in E$, $j = 1, ..., m$. Since $supp_E(xb) \subseteq supp_E(x)$, $xb \in K$ is also a remainder modulo $P \cap L$. Further, $axb \in P$ and so $axb = 0$. Consequently $ab_1 = 0$ and hence $r_R(a) \cap J \neq 0$. Therefore $a \in Z(R) = 0$, a contradiction.

Now we consider arbitrary closed prime ideals. In ([5], Sect. 3) we proved that similar results hold for every R-disjoint prime ideal of a free centred extension S of R, provided that the basis is a either a finite or a commuting set. We consider here any centred extension S of R with $X = (x_i)_{i \in \Omega}$ as a set of R-centralizing generators. Recall that $V_S(X)$ denotes the centralizer of X in S.

We can obtain the following generalization of the above mentioned result.

Theorem 6.3. Let R be a strongly prime ring and let W be an intermediate extension with $W \subseteq V_S(X)$. If P is closed prime ideal of W, then P is a strongly prime ideal of W.

Proof. By factoring out the ideal $[0]_S$ we may assume S is torsion-free. Suppose that I is an ideal of W with $I \supset P$. If $I \cap R \neq 0$, we obtain an insulator $F \subseteq I \cap R$ of W/P, by the same way as in Theorem 6.1. So we may assume $I \cap R = 0$.

Let L be a free dense submodule of S with the basis $E = (e_i)_{i \in \Lambda}$. Since P is closed there exists $x \in m(I \cap L)$ which is a remainder modulo $P \cap L$, say $x = \sum_{i=1}^{n} a_i e_i$, $0 \neq a_i \in R$, $i = 1, ..., n$. Then $H = \Theta_{\Gamma, e_1}(I \cap L)$ is a non-zero ideal of R and so there exists an insulator $F \subseteq H$. Put $F = \{b_1, b_2, ..., b_m\}$. For every $1 \leq i \leq m$ there exists $x_i = \sum_{j=1}^{n} b_{ij} e_j \in I \cap L$ with $b_{i1} = b_i$. Also, by Lemma 2.1, there exists $z \in M_C(I \cap L)$ such that $x_i = z b_i$, for $1 \leq i \leq m$, and z is a remainder modulo P^*, the extension of P to W^*. We show that $G = \{x_i : 1 \leq i \leq m\}$ is an insulator in W modulo P.

Put $K = \{y \in W : Gy \subseteq P\}$. If $K = P$ we are done. Assume that $K \supset P$ and take an element $y \in m(K \cap L)$ which is a remainder modulo $P \cap L$. Then $z b_i y \in P \subseteq P^*$, for $1 \leq i \leq m$. Since $W \subseteq V_S(X)$ it follows that $W^* \subseteq V_{S^*}(E)$. Hence $z W^* b_i y = W^* z b_i y \subseteq P^*$, $z \notin P^*$, and P^* is prime. Therefore $b_i y \in P^* \cap L = P \cap L$ and so $b_i y = 0$, $1 \leq i \leq m$. Since $y \in L$ and

F is an insulator we obtain $y = 0$, a contradiction. The proof is complete.

The corresponding result for non-singular prime ideals is the following.

Theorema 6.4. Let R be a non-singular prime ring and let W be an intermediate extension of R with $W \subseteq V_S(X)$. If P is a closed prime ideal of W, then P is a non-singular prime ideal.

Proof. By factoring out the ideal $[0]_S$ we may assume that S is torsion-free. Assume that $Z(W/P) = I/P \neq 0$, where I is an ideal of W. If $I \cap R \neq 0$ we arrive to a contradiction by the same way as in Theorem 6.2. So we consider the case $I \cap R = 0$.

Let L be a free dense submodule of S with the basis $E = (e_i)_{i \in \Lambda}$. Then there exists $x = \sum_{i=1}^{n} a_i e_i \in m(I \cap L)$ which is a remainder modulo $P \cap L$, where $0 \neq a_i \in R$, $e_i \in E$, $1 \leq i \leq n$. We show that $a_1 \in Z(R) = 0$, a contradiction.

Let J be a non-zero ideal of R and consider the non-zero ideal $(JW+P)/P$ of W/P. Then there exists $y \in JW \setminus P$ such that $xy \in P$. Thus $K = \{w \in JW : xw \in P\}$ is a right ideal of W and $K \supset P$. Therefore there exists $y \in m(K \cap L)$ which is remainder modulo $P \cap L$. Also, as in the proof of Theorem 6.2 we show that we may choose $y \in JE$. Finally, by Lemma 2.1 there exists $z \in M_C(I \cap L)$ with $za_1 = x$. We have $za_1y \in P \subseteq P^*$ and so $zW^*a_1y = W^*za_1y \subseteq P^*$, since $W^* \subseteq V_{S^*}(E)$. Consequently $a_1y \in P^* \cap L = P \cap L$, thus $a_1y = 0$ and so $r(a_1) \cap J \neq 0$. We obtain $a_1 \in Z(R)$, a contradiction. The proof is complete.

It is not surprising that in Theorem 6.3 we proved that an R-disjoint prime ideal is strongly prime only for closed ideals. In fact, we have

Proposition 6.5. Assume that P is a strongly prime ideal of an intermediate extension W of R such that $P \cap R = 0$. Then R is a strongly prime ring and P is closed.

Proof. If H is a non-zero ideal of R, then WHW is a non-zero ideal of W which is not contained in P. Then there exists a finite set $F \subseteq WHW$ such that $Fx \subseteq P$, $x \in W$, implies $x \in P$. Also, every $y_j \in F \subseteq WHW \subseteq S$ can be written as $y_j = \sum_i x_i a_{ij}$, for some elements $a_{ij} \in H$. Thus $\{a_{ij}\} \subseteq H$

is an insulator in R.

Now, assume that $[P] \supset P$. Then there exists a finite set $F \subset [P]$ such that $Fx \subseteq P$, $x \in W$, implies $x \in P$. However, since F is finite there exists $0 \neq H \lhd R$ with $FH \subseteq P$ and $H \not\subseteq P$. The contradiction shows that P is closed.

The corresponding of Proposition 6.5 for non-singular ideals is the following.

Proposition 6.6. Assume that there exists a closed non-singular prime ideal P of W. Then R is a non-singular prime ring.

Proof. Suppose that A and B are ideals of R with $AB = 0$. Then $AWBW \subseteq ABS = 0$, so either $AW \subseteq P$ or $BW \subseteq P$ and it follows that either $A = 0$ or $B = 0$. Thus R is prime.

Take $a \in Z(R)$ and let J be a right ideal of W with $J \supset P$. Choose an element $x = a_1e_1 + \ldots + a_ne_n \in m(J \cap L)$ which is a remainder modulo $P \cap L$, where L is a free dense submodule of S with the basis $E = (e_i)_{i \in \Lambda}$, $0 \neq a_i \in R$ for $1 \leq i \leq n$, and consider the right ideal $\Theta_{\Gamma,e_1}(J \cap L)$ of R. Then there exists $0 \neq c \in \Theta_{\Gamma,e_1}(J \cap L)$ such that $ac = 0$. Also, there exists $y = c_1e_1 + \ldots + c_ne_n \in J \cap L$ with $c_1 = c$. Assume that $ay = ac_se_s + \ldots + ac_ne_n$, where $ac_j \neq 0$ for $s \leq j \leq n$. Since $c_sR \neq 0$ there exists $b \in R$ such that $c_sb \neq 0$ and $ac_sb = 0$. Thus $0 \neq yb = ac_{s+1}be_{s+1} + \ldots + ac_nbe_n$. Repeating the argument we find an element $z \in J \cap L$ with $supp(z) = supp(y)$ and $az = 0$. Consequently, $r_{(W/P)}(a + P) \cap (J/P) = 0$ and we have $a \in P \cap R = 0$. Then $Z(R) = 0$ and the proof is complete.

As a direct consequence of the former results we have the following corollaries.

Corollary 6.7. Let W be an intermediate extension of R. Then R is strongly prime (resp. non-singular prime) if and only if every ideal P of W which is maximal with respect to $P \cap R = 0$ is a strongly prime (resp. non-singular prime) ideal.

Corollary 6.8. Let W be an intermediate extension of R with $W \subseteq V_S(X)$. An R-disjoint prime ideal P of W is strongly prime if and only if R is strongly prime and P is closed.

Corollary 6.9. Let W be an intermediate extension of R with $W \subseteq V_S(X)$. Then R is non-singular prime if and only if every closed prime ideal of W is non-singular.

Combining the above results with Examples 5.6 and 5.7 we have

Corollary 6.10. Let W be an intermediate extension of R with $W \subseteq V_S(X)$. Assume that one of the following condition is fulfilled.

i) $W = S$.

ii) Every non-zero ideal of R contains a central element.

iii) R is subdirectly irreducible.

Then R is strongly prime (resp.non-singular) if and only if every R-disjoint prime ideal of W is strongly prime (resp. non-singular).

Remark 6.11. i) An example given in ([6], Example 2.6) shows that Theorem 6.3 is not true if the condition $W \subseteq V_S(X)$ is not assumed. We could not find a similar example for Theorem 6.4. Thus the corresponding question for non-singular prime ideals is still open.

ii) We do not know if every non-singular R-disjoint prime ideal is always closed. This is true for strongly prime ideals (see Propositions 5.5 and 5.6).

iii) We did not succeed in proving that if $W \subseteq V_S(X)$, then every prime ideal P of W with $P \cap R = 0$ is closed.

To finish the section we prove the result corresponding to Theorems 6.1 and 6.2 on primitivity. This result is a partial extension of ([5], Theorem 4.1). Recall that an ideal P of W is said to be (right) primitive if there exists a maximal right ideal N of W such that $(N : W) = \{x \in W : Wx \subseteq N\} = P$, where $(N : W)$ is the largest ideal of W contained in N.

Theorem 6.12. Let R be a primitive ring and let P be an ideal of W which is maximal with respect to $P \cap R = 0$. Then P is a primitive ideal of W.

Proof. Let J be a maximal right ideal of R with $(J : R) = 0$. We show that $(JW + P) \cap R = J$. Assume, by contradiction, that $(JW + P) \cap R = R$. Then there exist $x \in JW$, $y \in P$ such that $x + y = 1$. Write x as a linear combination of the centralizing generators $(x_i)_{i \in \Omega}$ of S with coefficients in R. We easily see that we may put $x = \sum_{i=0}^{n} a_i x_i$, where $a_i \in J$ for $0 \leq i \leq n$ and x_0 denotes the identity of R. Consequently, there exists $y = \sum_{i=0}^{n} c_i x_i \in P$ such that $c_0 \notin J$ and $c_i \in J$ for $1 \leq i \leq n$. Let L be a free dense submodule of S with the basis $E = (e_i)_{i \in \Lambda}$ and let H be a non-zero ideal of R with $yH \subseteq L$. If $c_0 H \subseteq J$, using $c_0 R + J = R$ we easily obtain $H \subseteq J$. Therefore there exists $b \in H$ such that $c_0 b \notin J$. Hence, changing y by yb we see we may assume that $y = b_0 + b_1 e_1 + \ldots + b_t e_t \in P \cap L$, where $b_0 \notin J$ and $0 \neq b_i \in J$ for $i = 1, \ldots, t$. Also we may assume that t is minimal with respect to this conditions. Take $\Gamma \in Min_E(P \cap L)$ such that $\Gamma \subseteq \{e_0, e_1, \ldots, e_t\}$. It follows that there exists some i, say 1, such that $e_1 \in \Gamma$ and consider the ideal $\Theta_{\Gamma, e_1}(P \cap L)$. Since $(J : R) = 0$ we have $\Theta_{\Gamma, e_1}(P \cap L) \not\subseteq J$. So we can find an element $z \in P \cap L$ with $supp(z) = \Gamma$ and $z_1 = z(e_1) \notin J$.

On the other hand, $b_0 R + J = R$ and so $b_0 R z_1 \not\subseteq J$. Hence there exists $r \in R$ with $b_0 r z_1 \notin J$. Then the element $v = yrz_1 - b_1 rz \in P \cap L$ and it can be easily seen that $v_0 = v(e_0) \notin J$, $v_1 = v(e_1) = 0$ and $v(e_i) \in J$ for $2 \leq i \leq t$. This contradicts the minimality of t.

Therefore $(JW + P) \cap R = J$. Thus there exists a right ideal N of W which is maximal with respect to $N \supseteq (JW + P)$ and $N \cap R = J$. Clearly N is a maximal right ideal of W. Also $(N : W) \cap R \subseteq (J : R) = 0$ and so $(N : W) = P$. The proof is complete.

7. Some finiteness assumptions

In this section, we first consider intermediate extensions of finite rank. Note that if S is finitely generated over R (a liberal extension, according to [21]) and W is an intermediate extension, then rank $(W) < \infty$. The study of this situation is contained in [20].

Now, let S be an arbitrary centred extension of R and let W be an intermediate extension. We say that W is of finite rank if rank $(W) < \infty$.

As usual, in this section we denote by S^* the canonical torsion-free extension of S and by V the corresponding C-vector space. Also, W^* denotes the extension of the closed subring W of S and we put $W_0 = W^* \cap V$.

The prime (resp. strongly prime, Jacobson) radical of a ring B will be denoted by $P(B)$ (resp, $s(B)$, $J(B)$).

We begin this section with the following.

Theorem 7.1. Let W be an intermediate extension of finite rank of R with $[0]_W = 0$. Then the prime radical of W is nilpotent and is a finite intersection of minimal prime ideals. The minimal prime ideals of W are precisely those ideals of W which are maximal with respect to having zero intersection with R.

Proof. By factoring out from S the ideal $[0]_S$ we may assume S is torsion-free. First we assume that W is closed in S. By the assumption, $\dim_C(W_0) < \infty$. Then there exists a finite set of minimal prime ideals $\{K_1, ..., K_n\}$ of W_0 such that the prime radical $B = P(W_0)$ equals to $\cap_{i=1}^n K_i$ and $B^m = 0$, for some integer number $m \geq 1$. Also, $P_i = QK_i \cap W$ is a closed prime ideal of W, $i = 1, ..., n$, such that $A = QB \cap W = \cap_{i=1}^n P_i$ and $A^m = 0$. Hence we can easily see that A is the prime radical of W and $\{P_1, ..., P_n\}$ is the set of all the minimal prime ideals of W.

Suppose there exists an ideal I of W with $P_i \subseteq I$ and $I \cap R = 0$. We may assume that such an ideal I is maximal with respect to $I \cap R = 0$. Hence $I^* \cap V \supseteq P_i^* \cap V = K_i$ and therefore $I^* \cap V = K_i$. It follows that $I = P_i$.

In general, if W is any intermediate extension of finite rank we consider $[W]_S$. Applying Lemma 5.1 it is easy to complete the proof using similar arguments as above.

Corollary 7.2. Let W be an intermediate extension of finite rank of R. Then every R-disjoint prime ideal of W is closed.

Proof. Assume that P is any R-disjoint prime ideal of W. If $x \in W$ and $xH = 0$, for $0 \neq H \lhd R$, we have $x(HS \cap W) = 0$. Thus $x \in P$ since $HS \cap W$ is an ideal of W which is not contained is P. Consequently, $[0]_W \subseteq P$ and, by factoring out from S the ideal $[0]_S$ we may assume $[0]_W = 0$.

Using the same notation as in the proof of Theorem 7.1 we have that there exists $1 \leq i \leq n$ such that $P \supseteq P_i$. Hence $P = P_i$ is closed, by Example 5.5.

Since now we kwow that every R-disjoint prime ideal of W is closed, it is easy to repeat the arguments of the proof of Theorem 7.1 to obtain the following.

Corollary 7.3. Let W be an intermediate extension of finite rank of R and let I be a closed ideal of W. Then the prime radical of W/I is nilpotent and is a finite intersection of R-disjoint prime ideals of W/I.

As an immediate consequence of Theorems 6.1, 6.2 and 6.12 we have

Corollary 7.4. Let W be an intermediate extension of finite rank of a strongly prime (resp. non-singular prime, primitive) ring R. Then every R-disjoint prime ideal of W is strongly prime (resp. non-singular, primitive). In particular, in the first case $s(W) = P(W)$ and in the latest case $J(W) = P(W)$.

Now we consider another finiteness assumption. Let S be a centred extension of R with $X = (x_i)_{i\in\Omega}$ as a set of R-centralizing generators. We say that S is an almost finite centred extension of R if there exists a finite commuting subset $\{x_1, ..., x_n\}$ of X such that $S_0 = R[x_1, ..., x_n]$ is a dense subring of S, where $R[x_1, ..., x_n]$ denotes the submodule of S generated by all the elements of the type $x_1^{i_1}...x_n^{i_n}$, $i_j \geq 0$.

Theorem 7.5. Assume that S is an almost finite centred extension of R and let I be a closed ideal of S. Then the prime radical of S/I is nilpotent and is a finite intersection of minimal prime ideals of S/I all of which are R-disjoint.

Proof. By factoring out the ideal I we may assume $I = 0$ and S is torsion-free. Let L be a free dense submodule of $S_0 = R[x_1, ..., x_n]$ with a basis E contained in the set $\{x_1^{i_1}...x_n^{i_n}\}$ of generators of S_0. Then L is also a free dense submodule of S and we have $S^* = S_0^* = L^*$ is free over Q with the basis E. Then S^* is a homomorphic image $Q[x_1, ..., x_n]$ of a polynomial ring over Q in a finite number of indeterminates and also $V = C[x_1, ..., x_n]$. Hence V is a noetherian ring and so there exists a finite family $\{K_1, ..., K_t\}$ of minimal prime ideals of V such that the prime radical B of V equals $\cap_{i=1}^t K_i$ and $B^m = 0$, for some integer number $m \geq 1$. Then $P_i = QK_i \cap S$ is an R-disjoint prime ideal of S, $1 \leq i \leq t$, with $P(S) = \cap_{i=1}^t P_i$ and $P(S)^m = 0$. The proof can easily be completed as in Theorem 7.1.

Corollary 7.6. Assume that R is a strongly prime (resp. primitive) ring and S is an almost finite centred extension of R. If I is a closed ideal of S, then $s(S/I) = P(S/I)$ (resp. $J(S/I) = P(S/I)$).

Proof. Using the same notation as in Theorem 7.5 we have that $C[x_1, ..., x_n]$ is a commutative Jacobson ring. Then every ideal K_i is an intersection of maximal ideals of V. It follows that P_i is an intersection of ideals of S which are maximal with respect to having zero intersection with R. So it is enough to apply Theorem 6.1 (resp. Theorem 6.12).

Remark 7.7. Assume that S is an almost finite extension of R. If R is strongly prime (resp. non-singular prime), then every R-disjoint prime ideal of S is strongly prime (resp. non-singular). In fact, Theorem 6.3 (resp. Theorem 6.4) shows that this is true for the R-disjoint prime ideals of S_0. Now, it is not difficult to prove that if P is an R-disjoint prime ideal of S and $P \cap S_0$ is strongly prime (resp. non-singular), then P is strongly prime (resp. non-singular).

8. Some additional applications

The purpose of this section is to give some applications of the former results. Throughout, R is any ring (not necessarily prime) and S is a centred extension of R.

If P is a prime ideal of S, then $P \cap R$ is a prime ideal of R. Hence, to study S/P we may factor out from R and S the ideals $P \cap R$ and $(P \cap R)S$ (or even P), respectively. So we may assume that R is a prime ring and P is an ideal of S with $P \cap R = 0$. Then, as a direct application of Corollaries 6.8, 6.9 and 7.4 we obtain the following extension of ([5], Theorem 3.3).

Theorem 8.1. Let R be any ring and let S be a centred extension of R with X as a set of R-centralizing generators. Assume that one of the following conditions is fulfilled.

i) X is a commuting set

ii) rank $(S) < \infty$.

Then every prime ideal of R is strongly prime (resp. non-singular) if and only if the same is true of S.

We say that every prime ideal of R can be extended to S if for every prime ideal P of R there exists a prime ideal I of S with $I \cap R = P$. Note that this is the case when S is a free centred extension of R.

Another application is the following.

Proposition 7.2. Let S be a centred extension of R. Then $s(S) \cap R \supseteq s(R)$. In addition, if every prime ideal of R can be extended to S, then $s(S) \cap R = s(R)$.

Proof. If P is a strongly prime ideal of S, by factoring out from R the ideal $P \cap R$ and applying Proposition 6.5 we obtain $s(R) \subseteq P$. Thus $s(R) \subseteq s(S) \cap R$.

Now, if P is a strongly prime ideal of R and P can be extended to S, we take an ideal I of S which is maximal with respect to $I \cap R = P$. By Theorem 6.1, I is strongly prime. This completes the proof.

Remark 7.3. i) Applying similar arguments to those used in the proof of Proposition 6.6 we can prove that $Z(R) \subseteq Z(S) \cap R$. It is easy to see that $Z(R) = Z(S) \cap R$, provided that S is a free centred extension of R.

ii) As in Proposition 7.2 we can prove that if every prime ideal of R can be extended to S, then $J(S) \cap R \subseteq J(R)$.

iii) Similarly we obtain that $P(R) \subseteq P(S) \cap R$ and the equality holds provided that every prime ideal of R can be extended to S.

It seems to be very difficult to study prime ideals of intermediate extension W. To apply our results it should be necessary to reduce to the R-disjoint case. If P is a prime ideal of W, we cannot factor out convenient ideals of R, W and S so that in the new situation the image of P be R-disjoint. This is possible, for example, if S is a liberal extension of R ([20], Theorem 3.2).

To finish the paper we include here a case in which this is possible.

Theorem 7.4. Assume that R is a ring such that every prime factor of R is subdirectly irreducible, S is a centred extension of R and W is an intermediate extension. If P is a prime ideal of W, then $P \cap R$ is prime ideal of R and there exists a prime ideal I of S such that $I \cap R = P \cap R$ and $I \cap W \subseteq P$.

Proof. With minor modifications, the proof is the same as in ([20], Theorem 3.2). We use the same notation as in [20] to indicate the changes. We cannot use rank. Anyway we have $TX \subseteq P_t$ and this easily implies that $(P_t \oplus K) \cap T = P_t$. Also, $P_t \oplus K$ is a dense submodule of S and so $SY \subseteq P_t \oplus K$, where Y is the smallest ideal of R. The result follows.

From Corollary 6.10, the following is clear.

Corollary 7.5. Assume that R is a ring such that every prime factor of R is subdirectly irreducible, S is a centred extension of R with X as a centralizing generator set and W is an intermediate extension with $W \subseteq V_S(X)$. Then every prime ideal of R is strongly prime (resp. non-singular) if and only if the same is true of W.

References

[1] - W. Chin, Prime ideals in differential operator rings and crossed products of infinite groups, J. Algebra 106 (1987), 78-104.

[2] - W. Chin, Prime ideals in restricted operator rings, Israel J. Math. 60 (1987), 236-256.

[3] - E. Cisneros, M. Ferrero, and M. I. González, Prime ideals in skew polynomial rings and skew Laurent polynomial rings, Math. J. Okayama Univ. 32 (1990), 61-72.

[4] - M. Ferrero, Prime and principal closed ideals in polynomial rings, J. Algebra 134 (1990), 45-59.

[5] - M. Ferrero, Closed and prime ideals in free centred extensions, J. Algebra 148 (1992), 1-16.

[6] - M. Ferrero, E. Jespers, and E. R. Puczylowski, Prime ideals of graded rings and related matters, Comm. Algebra 18, n^0 11 (1990), 3819- 3834.

[7] - M. Ferrero and J. Matczuk, Prime ideals in skew polynomial rings of derivation type, Comm. Algebra 18, n^0 3 (1990), 689-710.

[8] - K. R. Goodearl, "Ring Theory: Non-singular rings and modules", Dekker, New York, 1976.

[9] - K. R. Goodearl, Prime ideals in skew polynomial rings and quantized Weyl algebras, J. Algebra 150 (1992), 324-377.

[10] - A. G. Heinicke and J. C. Robson, Normalizing extensions: Prime ideals and incomparability, J. Algebra 72 (1981), 237-268.

[11] - A. G. Heinicke and J. C. Robson, Normalizing extensions: nilpotency, J. Algebra 76 (1982), 459-470.

[12] - A. G. Heinicke and J. C. Robson, Intermediate normalizing extensions, Trans. Amer. Math. Soc. 282 (1984), 645-667.

[13] - J. Lambek, "Lectures on rings and modules", Chelsea, New York, 1976.

[14] - A. Leroy and J. Matczuk, Prime ideals of Ore extensions, Comm. Algebra 19, n^0 7 (1991), 1893-1907.

[15] - M. M. Parmenter, D, S. Passman, and P. N. Stewart, The strongly prime radical of crossed products, Comm. Algebra 12, n^0 9 (1984), 1099-1113.

[16] - D. S. Passman, Prime ideals in normalizing extensions, J. Algebra 73 (1981), 556-572.

[17] - D. S. Passman, Prime ideals in polycyclic crossed products, Trans. Amer. Math. Soc. 301 (1987), 737-759.

[18] - D. S. Passman, Prime ideals in enveloping rings, Trans. Amer. Math. Soc. 302(1987), 535-560.

[19] - D. S. Passman, Prime ideals in restricted enveloping rings, Comm. Algebra 16, n^0 7 (1988), 1411-1436.

[20] - J. C. Robson, Prime ideals in intermediate extensions, Proc. London Math. Soc. (3) 44 (1982), 372-384.

[21] - J. C. Robson and L. W. Small, Liberal extensions, Proc. London Math. Soc. (3) 42 (1981), 87-103.

[22] - B. Stenström, "Rings of quotients", Springer-Verlag, Berlim/Heidelberg, New York, 1975.

Instituto de Matemática

Universidade Federal do Rio Grande do Sul

91540-000 - Porto Alegre, Brasil.

e-mail: Ferrero@if1.ufrgs.br

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