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The radiation of water waves by heaving horizontal disk at a shallow submergence depth is considered. The problem is formulated as a Fredholm integral equation. The resonant frequencies originating from the shallow submergence are examined and their locations are computed by a modified Newton's method. The hydrodynamic coefficients in a neighborhood of a resonance pole are found to be represented by circles in the complex plane. An asymptotic approximation is obtained for small values of the submergence depth. The relationship between the small-submergence first order problem and the dock problem is explored and found to be clearly represented by the added mass and damping coefficients.
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I. INTRODUCTION

We consider the radiation of water waves with a thin rigid circular disk in three dimensions. The disk is assumed to be submerged below the free surface. This problem is associated with the *dock problem*, a situation in which the disk is at the free surface. The circular dock problem has been studied by several authors.[1–7] Dock problems can be reduced to the solution of a boundary integral equation for the velocity potential ϕ; this equation is a Fredholm integral equation of the second kind.

The position of the thin circular disk underneath the water surface poses additional and interesting mathematical difficulties. There are some previous works on the subject which we now briefly review. Yu and Chwang[8] used matched eigenfucntions expansions for studying the scattering by a horizontal disk in water of finite depth. Martin and Farina[9] described a rigorous method for axisymmetric motions of a horizontal disk in deep water. They transformed the governing hypersingular integral equation for $[\phi]$ into a one-dimensional Fredholm integral equation of the second kind for a new unknown function; the new equation is a generalization of Love's integral equation, common in the theory of electrostatics of a circular-plate capacitor.[10] Numerical results of the heaving added mass and damping were obtained.

Farina and Martin[11] considered three-dimensional scattering by a thin disk, in deep water. The governing hypersingular integral equation is solved numerically using a expansion-collocation method. Similarly to the radiation problem, they found that the scattering problem presents a strong dependence on the frequency, especially when the plate is close to the free surface. The relationships between the scattering cross section and the peaks in the added mass have been explored.

Yu[12] uses analytical, numerical, and semiempirical methods and summarizes the functional performance of a submerged and essentially horizontal plate for offshore wave control. Emphasis is put on the hydrodynamic force and on the reflection and transmission coefficients. In particular, effects of the porosity of the plate and fluid viscosity are discussed and reviewed.

Roy and Ghosh[13] solve Laplace's equation using the method of separation of variables in order to calculate forces on a circular thin disk vertically submerged in shallow water. Morison's equation is used for the determination of wave force. Interesting related problems of interaction of water waves with a flexible disk have been studied (see, for instance, Refs. 14–16).

The occurrence of peaks in the hydrodynamic force is observed in problems of interaction of water waves with bodies. See for instance Refs. 17–20. Time-dependent problems were considered in Refs. 21 and 22.

In this work, we investigate the connections between the peaks present in the added mass and damping coefficients for a heaving circular disk in deep water and the so-called resonant frequencies. Emphasis will be given to the case where the body is very near the free surface. Extending the work by Martin and Farina,[9] we approach the problem using the generalized Love's integral equation as the governing equation. The reason for choosing this approach is based on the simplification present in Love's equation and on its efficient numerical solution. The resonant frequencies are linked to the existence of a pole in the complex K-plane, which contains all the information about the resonance. In order to locate the poles we search for values of complex wavenumber which make the matrix $\mathcal{M}$, related to the discretized generalized Love's integral equation, singular. Thus, we use a modified Newton's method to find the zeros of det $\mathcal{M}$. In addition we conclude that the resonance poles are of order 1, i.e., simple poles. In particular, we associate the hydrodynamic coefficients with circles in the complex plane, when K is near a resonance pole.

In a second part of this paper, we derive an asymptotic approximation for small values of the submergence depth. The result is a governing Fredholm integral equation of the first kind. This equation is solved numerically. We observe that the graph of the first order added-mass approximation is the perfect underlying curve for the added mass of the shallow submerged disk without the spikes. For small values of the wavenumber K, the added mass behaves like $-\pi/Ka$, where a is the radius of the disk. Also, the damping coefficient for the dock problem is recovered by the first order damping approximation. In addition, we find that the added mass of the dock problem equals the first order added mass approximation added to π/Ka, a result previously found by Martin and Farina[9] using the full solution of the generalized Love's integral equation. We found that, physically, the first order added mass corresponds to the potential of the fluid above the disk, in a simple state of uniform vertical oscillation in phase with the disk.

II. FORMULATION

Consider a thin, with zero thickness, rigid plate, S, completely submerged beneath the free surface in water of infinite depth. We assume that S is represented by a smooth open surface with a smooth edge ∂S. We take Cartesian coordinates (x,y,z) with the origin in the mean free surface; the water occupies the region $z<0$. Linear water-wave theory is employed. Thus, under the usual conditions, the time-harmonic velocity potential is

$$\mathrm{Re}\{\phi(x,y,z)e^{-i\omega t}\},$$

where ϕ satisfies Laplace's equation in the water,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{1}$$

the linearized free surface condition

$$\frac{\partial\phi}{\partial z}-K\phi=0, \quad \text{on } z=0, \tag{2}$$

and a boundary condition on the plate,

$$\frac{\partial\phi}{\partial n}=V, \tag{3}$$

where V is prescribed and $K=\omega^2/g$ is the wavenumber. We also require that ϕ vanishes as $z\to-\infty$ and a radiation condition at infinity, given by

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial\phi}{\partial r}-iK\phi\right)=0, \tag{4}$$

where $r=(x^2+y^2)^{1/2}$.

Let us now introduce Green's function G, given by

$$G(P;Q)\equiv G(x,y,z;\xi,\eta,\zeta)$$
$$=[R^2+(z-\zeta)^2]^{-1/2}+⨍_0^\infty \frac{k+K}{k-K}e^{k(z+\zeta)}J_0(kR)dk, \tag{5}$$

where P and Q are arbitrary points in the water, $R=[(x-\xi)^2+(y-\eta)^2]^{1/2}$ and J_0 is a Bessel function. This fundamental solution to our problem satisfies Eq. (1), except at $P=Q$ where it has a singularity. G also satisfies Eqs. (2) and (4). By using Green's theorem it is possible to represent ϕ as

$$\phi(P)=\frac{1}{4\pi}\int_S[\phi(q)]\frac{\partial}{\partial n_q}G(P,q)dS_q, \tag{6}$$

where

$$[\phi(q)]=\phi(q^+)-\phi(q^-)$$

is the discontinuity in ϕ across the plate, where $q\in S$, q^+ and q^- are corresponding points on S^+ and S^-, respectively, $S^\pm$ are the two sides of the plate, and $\partial/\partial n_q$ denotes normal differentiation at q in the direction from S^+ into the water. Applying the boundary condition on S^+ gives

$$\frac{1}{4\pi}\frac{\partial}{\partial n_p}\int_S[\phi(q)]\frac{\partial}{\partial n_q}G(p,q)dS_q=V(p^+), \quad p\in S. \tag{7}$$

The same equation is obtained by applying the boundary condition on S^-; $V(p^-)=-V(p^+)$ as the plate is rigid. The integrodifferential Eq. (7) is to be solved subject to the edge condition

$$[\phi]=0 \quad \text{on} \quad \partial S; \tag{8}$$

ϕ is discontinuous across the plate only.

Interchanging the order of integration and normal differentiation in Eq. (7) produces a hypersingular integral. Such a procedure is proper as long as the resulting integral is then interpreted as a finite-part integral. We obtain

$$\frac{1}{4\pi}\int\!\!\!\!\!\times_S[\phi]\frac{\partial^2G(p,q)}{\partial n_p\,\partial n_q}dS_q=V(p), \quad p\in S, \tag{9}$$

which is to be solved subject to Eq. (8). The cross indicates that the integral is a finite-part integral.

III. AXISYMMETRIC PROBLEM AND THE GENERALIZED LOVE'S INTEGRAL EQUATION

Consider now that S is a horizontal circular disk. Introduce cylindrical polar coordinates (r,θ,z), so that $x=r\cos\theta$ and $y=r\sin\theta$. Then, the disk is given by

$$S=\{(r,\theta,z):0\le r<a,\ -\pi\le\theta<\pi,\ z=-b/2\}. \tag{10}$$

It has radius a and is submerged at a distance $b/2$ below the free surface; we can take $a=1$ without loss of generality. The factor of 1/2 in the submergence depth is included in order to simplify expressions that appear later on in the text.

The radiation of waves by heaving (vertical) oscillations of the disk can be formulated by choosing

$$V(r,\theta)=1.$$

In this case the solution is axisymmetric. Martin and Farina[9] have shown that the solution can be written as

$$[\phi]=-\frac{4}{\pi}\int_r^1\frac{\psi(t)}{\sqrt{t^2-r^2}}dt,$$

where ψ is an auxiliary function (see Ref. 9) satisfying

$$\psi(t) - \frac{1}{\pi}\int_{-1}^{1} \frac{b\psi(s)}{(t-s)^2+b^2}ds - \frac{2K}{\pi}\int_{-1}^{1} \psi(s)\Phi_0(t-s,b)ds = t, \quad -1 \le t \le 1, \tag{11}$$

where Φ_0 is a two-dimensional wave-source potential given by

$$\Phi_0(X,Y) = \oint_0^\infty e^{-kY}\cos kX \frac{dk}{k-K}, \tag{12}$$

which can be computed conveniently using an expansion derived by Yu and Ursell:[23]

$$\Phi_0(X,Y) = -e^{-KY}\{(\log KS - i\pi + \gamma)\cos KX + \beta \sin KX\} + \sum_{m=1}^{\infty} \frac{(-KS)^m}{m!}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right)\cos m\beta, \tag{13}$$

where S and β are defined by $X=S\sin\beta$ and $Y=S\cos\beta$, and $\gamma=0.5772\ldots$ is Euler's constant. Equation (11) generalizes the well-known *Love's integral equation*, as we shall see in Sec. III A. The solution depends on the wavenumber K and the submergence b. To emphasize this dependence we will sometimes write $\psi(x)=\psi(x;K,b)$.

The hydrodynamic force can be decomposed in terms of the added mass $\mathcal{A}$ and damping $\mathcal{B}$ coefficients.[24] For the submerged heaving horizontal disk they are given by

$$\begin{aligned}\mathcal{A}+i\mathcal{B} &= -\int_S [\phi]dS = -2\pi\int_0^1 [\phi]rdr \\ &= 2\pi\int_0^1\int_r^1 \frac{4}{\pi}\frac{\psi(t)}{\sqrt{t^2-r^2}}dtrdr \\ &= 8\int_0^1 \psi(t)tdt.\end{aligned} \tag{14}$$

A. Previous results

In the deep submergence limit, i.e., when $b\to\infty$, it is shown by Martin and Farina[9] that the solution of Eq. (11) is $\psi(x,K,\infty)=x$. From Eq. (14), we recover the known result, given in[25]

$$\mathcal{A} = \tfrac{8}{3} \quad \text{and} \quad \mathcal{B}=0 \quad (b\to\infty),$$

which corresponds to a single disk oscillating in an unbounded fluid.

When $K=0$, Eq. (11) reduces to

$$\psi(x) - \frac{b}{\pi}\int_{-1}^{1}\frac{\psi(y)}{b^2+(x-y)^2}dy = x, \quad -1\le x\le 1.$$

This equation with 1 instead of x on the right-hand side is the original Love's equation that arises in the electrostatic problem of a circular plate condenser. In our case, we can see that setting $K=0$ in the free-surface condition (2) is equivalent to having a mirror disk, in other words, two coaxial disks separated by the distance b. No closed-form solution of Love's equation is known.

As shown by Martin and Farina,[9] when $K\to\infty$, Eq. (11) has also a very similar form to Love's equation:

$$\psi(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{\psi(y)}{b^2+(x-y)^2}dy = x, \quad -1\le x\le 1.$$

The general case, without further assumptions on K and b, was treated numerically by Martin and Farina.[9] It was found that the added mass has negative values and strong peaks at the resonant frequencies, when b becomes small. Apart from the spikes in the damping, the underlying trend of the damping coefficient is very close to the damping of the heaving dock problem ($b=0$). The same is not true for the added mass. In fact, the behavior of the potential in the layer between the submerged disk and the dock is one of the key for understanding what is occurring. To clarify this phenomenon, we construct an asymptotic approximation for small values of b in Sec. V.

IV. RESONANT FREQUENCIES

The occurrence of resonant frequencies in the interaction of water waves with submerged bodies, when this is close to the free surface has been previously observed. Our objective now is to analyze in more detail the resonant frequencies for the problem we are treating. The resonances are presented in both scattering and radiation problems and, although we exhibit certain resonant frequencies which appear in both problems, the treatment in this work will deal mainly with the radiation problem.

The peaks in the hydrodynamic coefficients occur at or near the resonant frequencies. They correspond to singularities in the matrix of the linear system obtained from the discretization of Eq. (11) as a function of complex K, for fixed b. We denote the complex wavenumber by $\mathcal{K}$.

Resonant frequencies are linked to the existence of a pole in the complex $\mathcal{K}$-plane. The pole contains a great deal of information about the resonance. The real part of $\mathcal{K}$ corresponds to the resonant frequency observed, and the imaginary part of $\mathcal{K}$ is proportional to the width of the peak in the hydrodynamic coefficients, and inversely proportional to the height of the peak, as seen in Sec. IV A.

A. The resonance poles

We now outline the mathematical theory which explains the origin of the resonance poles. Introduce

$$T(\mathcal{K})\psi(t) = \psi(t) - \frac{1}{\pi}\int_{-1}^{1}\frac{b\psi(s)}{(t-s)^2+b^2}ds - \frac{2\mathcal{K}}{\pi}\int_{-1}^{1}\psi(s)\Phi_0(t-s,b;\mathcal{K})ds, \quad -1\le t\le 1.$$

Thus, solving the heaving disk problem becomes equivalent to solving

TABLE I. Resonance poles computed using the modified Newton's method.

b	Resonance poles by Newton's method		
0.2	(0.386 386, -5.0068×10^{-02})	(2.151 037, -0.376 180)	(5.069 478, -0.829 091)
0.12	(0.261 035, -1.6390×10^{-02})	(1.431 881, -0.154 277)	(3.554 856, -0.382 447)
0.08	(0.186 130, -6.2313×10^{-03})	(1.025 591, -7.2383×10^{-02})	(2.548 165, -0.189 149)

$$T(\mathcal{K})\psi(t)=t.$$

It can be shown by using Steinberg (Ref. 26, theorem 1) (see also Alves and Ha Duong)[27] that the operator valued function $T^{-1}(\mathcal{K})$, inverse of $T(\mathcal{K})$, is meromorphic on $\mathbb{C}$ with a discrete set of poles in $\mathbb{C}^-$.

Now suppose a pole $\mathcal{K}_r$ lies sufficiently near to the real K-axis, such that the Laurent expansion

$$T^{-1}(\mathcal{K})=\sum_{j=-m}^{\infty}(\mathcal{K}-\mathcal{K}_r)^jT_j,\quad \text{for some } m\geq 1,$$

where T_j are bounded operators, is valid in a neighborhood (α,β) of $\text{Re}(\mathcal{K}_r)$. Thus it is suggested from

$$\psi=T^{-1}(\mathcal{K})t=\sum_{j=-m}^{\infty}(\mathcal{K}-\mathcal{K}_r)^jT_jt, \tag{15}$$

that ψ and therefore the hydrodynamic coefficients, as functions of $\mathcal{K}$, present a maxima or peaks in (α,β).

B. Finding the resonances

Similarly to the work in Ref. 9, we can solve the generalized Love's integral equation by a Guass–Legendre quadrature method. Then, an approximation $\tilde{\psi}$ of the solution ψ satisfies

$$M\tilde{\psi}=\tilde{t}, \tag{16}$$

where the matrix M is given by

$$M=\left(I-\frac{1}{\pi}A-\frac{2K}{\pi}B\right), \tag{17}$$

here

$$A=\{A_{ij}\}=w_j\frac{b}{(t_i-t_j)^2+b^2},$$

$$B=\{B_{ij}\}=w_j\Phi_0(t_i-t_j,b),$$

$$\tilde{\psi}=[\psi(t_1),\ldots,\psi(t_N)],$$

$$\tilde{t}=(t_1,\ldots,t_N),$$

and $\{t_i\}_0^N$ are the Gauss–Legendre quadrature nodes and $\{w_j\}_0^N$ are the corresponding weights. The value $N=120$ was chosen, based on the observed numerical convergence of the method.

To find the poles, we search for the values of $\mathcal{K}$ such that

$$\det\mathcal{M}=0,$$

where $\mathcal{M}$ is the matrix M evaluated at complex K. Before we proceed by describing any method to do this task, it is opportune to show how we can evaluate $\mathcal{M}$. From Eq. (17), it is seen that the part of M which includes values of K is

$$\frac{2K}{\pi}B,$$

where

$$B=\{B_{ij}\}=w_j\Phi_0(t_i-t_j,b).$$

Thus our task is to find the wave-source potential $\Phi_0(X,Y)$ for complex K. Thus,

$$\Phi_0(X,Y)=\oint_0^{\infty}e^{-kY}\cos kX\frac{dk}{k-K}$$
$$=2\pi ie^{-KY}\cos KX+\int_0^{\infty}e^{-kY}\cos kX\frac{dk}{k-K},$$

where the second integral runs above the pole K. Then, for $\text{Im}(\mathcal{K})<0$, Φ_0 is given by

$$\Phi_0(X,b)=2\pi ie^{-b\mathcal{K}}\cos\mathcal{K}X+\int_0^{\infty}e^{-kb}\cos kX\frac{dk}{k-\mathcal{K}}, \tag{18}$$

where the integral is nonsingular.

We applied a modified Newton's method to locate the zeros of $\det\mathcal{M}$. See Appendix A for the details of this method. Table I shows the resonance poles, for a number of submergences, b, obtained by employing the modified Newton's method.

In Figs. 1 and 2 the relationship between the resonance poles and the peaks in the added mass for a heaving disk is clearly seen. Note that the higher the peaks, the narrower they are.

For values of the submergence depth, $b\leq 0.04$, the modified Newton's method poses convergence difficulties. Finding estimates for these frequencies, for small values of b, based on perturbation methods might be an option to overcome this problem. Anther approach seems to be the one proposed by Meylan and Gross[28] where an alternative search algorithm is used to find the poles.

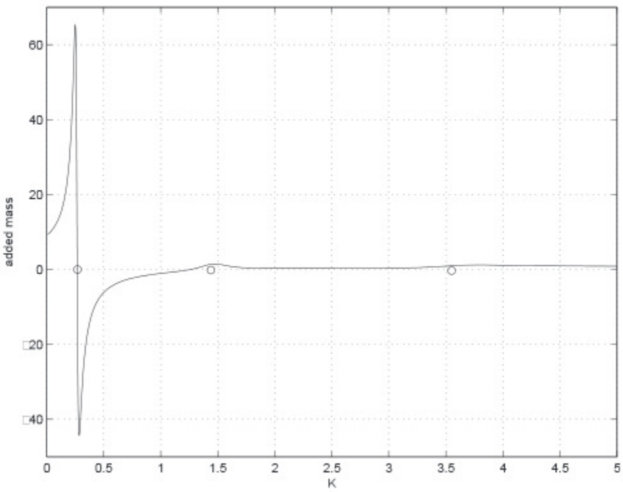

FIG. 1. The added mass (solid line) and the resonance poles (circles) computed by Newton's method. The submergence is given by $b=0.12$.

C. Classification of the poles

In this section we aim to classify the resonance poles according to their order. From Eq. (15) it follows that the added mass and damping, as functions of K and near the real part of the resonance pole $\mathcal{K}_r$, are given by

$$\mathcal{A}(K) + i\mathcal{B}(K) = 8\int_0^1 \psi t\, dt = 8\sum_{j=-N}^{\infty} (K-\mathcal{K}_r)^j \int_0^1 t\, T_j\, t\, dt. \tag{19}$$

Now suppose that the order of the pole K_r is 1. Then we have

$$\mathcal{A}(K) + i\mathcal{B}(K) \simeq \frac{\alpha_1}{K-\mathcal{K}_r} + \alpha_0, \tag{20}$$

where α_1 and α_0 are complex constants.

The region of the complex plane given by the right hand side of Eq. (20) is a circle. Therefore $\mathcal{K}_r$ being a simple pole implies that $\mathcal{A}(K)+i\mathcal{B}(K)$ determine a circular curve near $\mathcal{K}_r$.

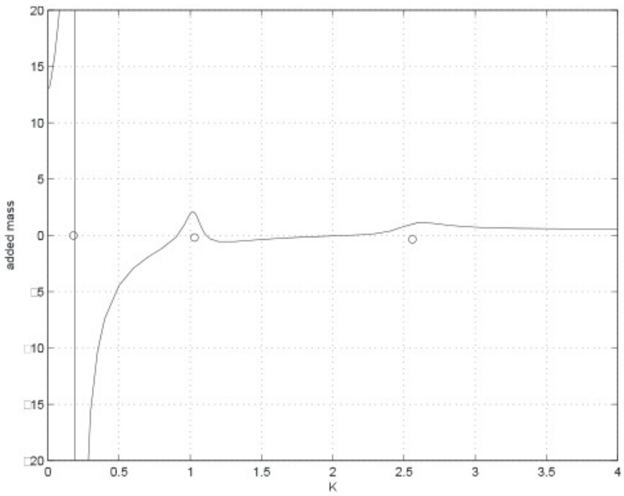

FIG. 2. The added mass (solid line) and the resonance poles (circles) computed by Newton's method. The submergence is given by $b=0.08$.

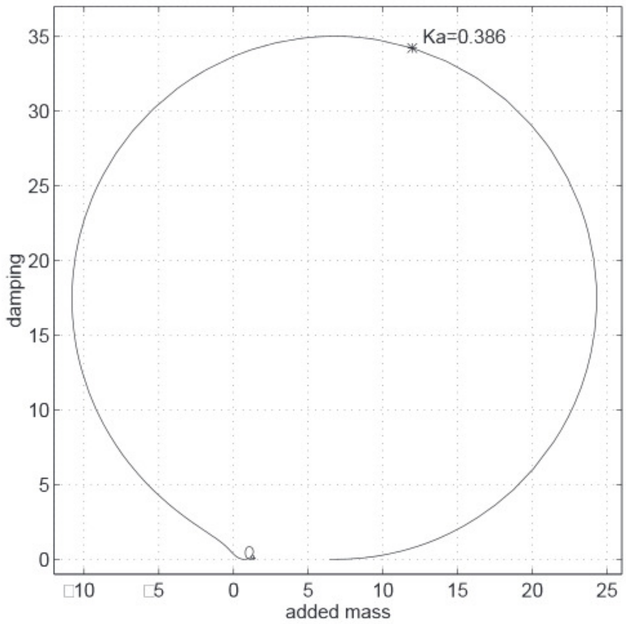

FIG. 3. The function $\mathcal{A}(Ka)+i\mathcal{B}(Ka)$, for $0.01 \leq Ka \leq 12$. The depth of submergence is given by $b=0.2$.

Moreover if $\mathcal{K}_r$ has an order greater than 1, the dominant terms in the Laurent series (15) do not represent a circle in $\mathbb{C}$.

We found convincing numerical evidence that the complex function $\mathcal{A}(K)+i\mathcal{B}(K)$ takes a neighborhood of a resonance pole into a circle. This fact was observed in all cases of b considered and examples are shown in Figs. 3–6, where the proportion in diameter between the circles can also be noted.

The association of resonances with circles in the com-

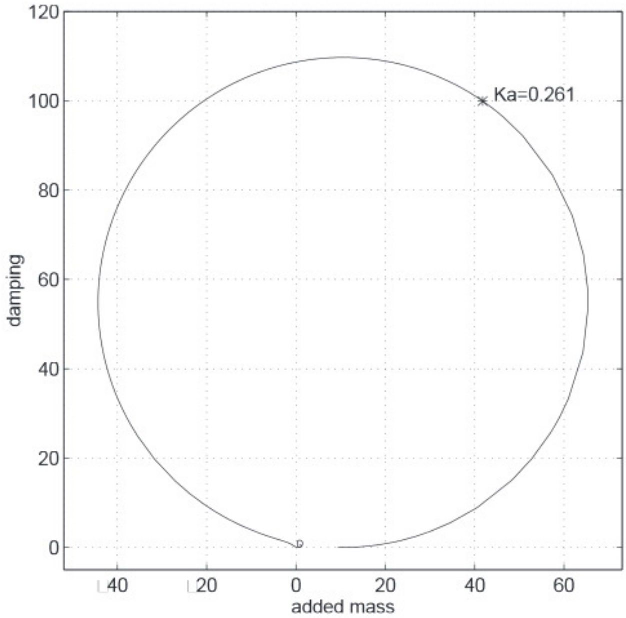

FIG. 4. The function $\mathcal{A}(Ka)+i\mathcal{B}(Ka)$, for $0.01 \leq Ka \leq 5$. The depth of submergence is given by $b=0.12$.

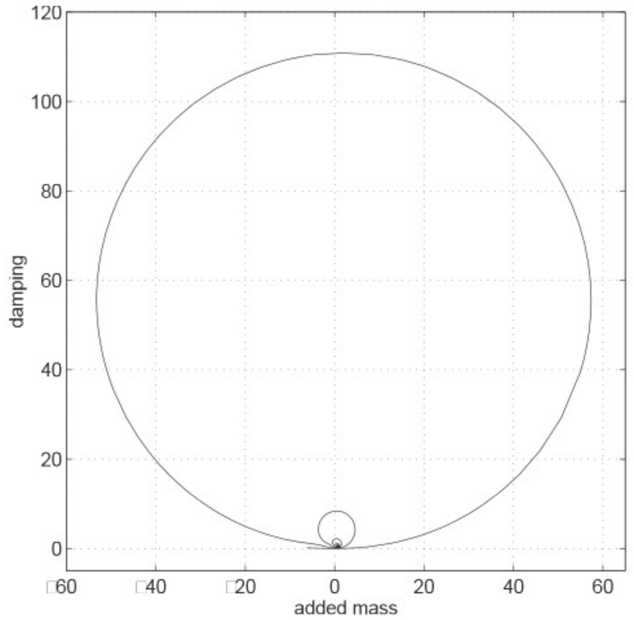

FIG. 5. The function $\mathcal{A}(Ka)+i\mathcal{B}(Ka)$, for $0.2 \le Ka \le 12.3$. The large circle is due to the second resonance. The depth of submergence is given by $b=0.02$.

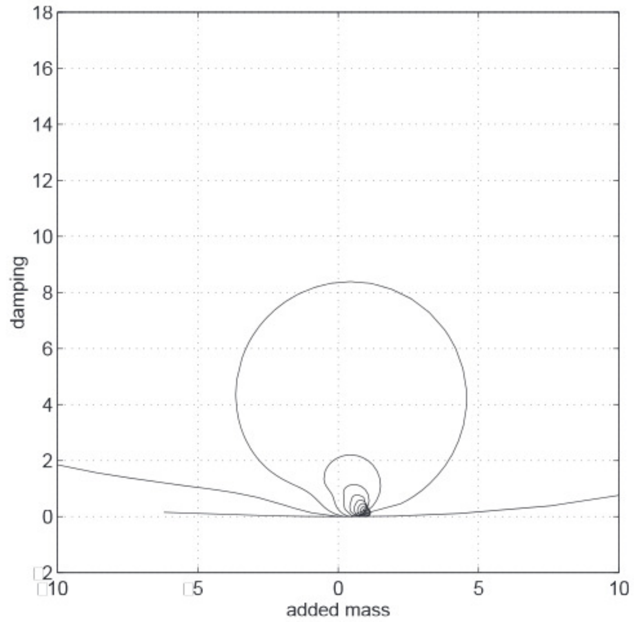

FIG. 6. The function $\mathcal{A}(Ka)+i\mathcal{B}(Ka)$, for $0.2 \le Ka \le 12.3$. The 3rd up to the 11th resonance are completed represented. The depth of submergence is given by $b=0.02$.

plex plane was mentioned in a paper by Linton and Evans[18] and the more general idea of plotting the added mass against damping is suggested by Jefferys.[29]

V. SMALL SUBMERGENCE $(b \searrow 0)$

In this section we look for an approximation of the solution when the disk submergence is very small. The intention is to study by a perturbation method how the submerged disk problem approaches the dock problem.

Let us show that for small b, relatively to radius of the disk (and t bounded away from ± 1),

$$\frac{b}{\pi}\int_{-1}^{1}\frac{f(s)}{(t-s)^2+b^2}ds \simeq f(t)+\frac{b}{\pi}⨎_{-1}^{1}\frac{f(s)}{(t-s)^2}ds. \tag{21}$$

For completeness we present the derivation of that result here.

The integral on the left-hand side of Eq. (21) is defined for $b>0$ and $-1<t<1$. We approximate it for small values of b, with t bounded away from the end-points, ± 1. We start by splitting the range of integration into three, so that

$$\frac{b}{\pi}\int_{-1}^{1}\frac{f(s)}{(t-s)^2+b^2}ds \simeq I_1+I_2,$$

where

$$I_1=\frac{b}{\pi}\int_{t-\varepsilon}^{t+\varepsilon}\frac{f(s)}{(t-s)^2+b^2}ds=\frac{b}{\pi}\int_{-\varepsilon}^{\varepsilon}\frac{f(t+u)}{b^2+u^2}du,$$

$$I_2=\frac{b}{\pi}\int_{-1}^{t-\varepsilon}\frac{f(s)}{b^2+(t-s)^2}ds+\frac{b}{\pi}\int_{t+\varepsilon}^{1}\frac{f(s)}{b^2+(t-s)^2}ds,$$

if $\varepsilon \ll 1$. In I_2, t is not in the range of integration, so we can readily approximate for small b:

$$I_2=\frac{b}{\pi}\int_{-1}^{t-\varepsilon}\frac{f(s)}{(t-s)^2}\{1+O(b^2)\}ds +\frac{b}{\pi}\int_{t+\varepsilon}^{1}\frac{f(s)}{(t-s)^2}\{1+O(b^2)\}ds,$$

as $b\to 0$. In I_1, the range of integration is small, so we expand $f(t+u)$ about $u=0$; as the range of integration is symmetric about $u=0$, we obtain, using [Ref. 30, formula (4.4.42)]

$$\begin{aligned}I_1&=\frac{b}{\pi}f(t)\int_{-\varepsilon}^{\varepsilon}\frac{du}{b^2+u^2}+\frac{b}{2\pi}f''(t)\int_{-\varepsilon}^{\varepsilon}\frac{u^2du}{b^2+u^2}+R\\&=\frac{2}{\pi}f(t)\tan^{-1}\frac{\varepsilon}{b}+\frac{b}{\pi}f''(t)\left\{\varepsilon-b\tan^{-1}\frac{\varepsilon}{b}\right\}+R\\&=f(t)-\frac{2b}{\pi}f(t)\left\{\frac{1}{\varepsilon}+O(b^3)\right\}\\&\quad+\frac{b}{\pi}f''(t)\left\{\varepsilon-\pi\frac{b}{2}+O(b^3)\right\}+R\end{aligned}$$

as $b\to 0$, where R is due to the remainder in Taylor's theorem; using Lagrange's form of this remainder, we obtain

$$|R|\le\frac{2b}{\pi}\frac{L}{4!}\int_{0}^{\varepsilon}\frac{u^4}{b^2+u^2}du\le\frac{bL\varepsilon^3}{12\pi},$$

where L is a bound on $f^{(iv)}$. Hence, neglecting the error terms, and letting $\varepsilon\to 0$, we obtain

$$\frac{b}{\pi}\int_{-1}^{1}\frac{f(s)}{(t-s)^2+b^2}ds$$
$$\simeq f(t)+\frac{b}{\pi}⨎_{-1}^{1}\frac{f(s)}{(t-s)^2}ds-\frac{1}{2}b^2 f''(t), \qquad (22)$$

after making use of definition of one-dimensional finite-part integral:

$$⨎_{a}^{b}\frac{f(t)}{(x-t)^2}dt=\lim_{\varepsilon\to 0}\left\{\int_{a}^{x-\varepsilon}\frac{f(t)}{(x-t)^2}dt+\int_{x+\varepsilon}^{b}\frac{f(t)}{(x-t)^2}dt-\frac{2f(x)}{\varepsilon}\right\}, \qquad (23)$$

which is valid whenever f'' is a Hölder-continuous function, $f\in C^{1,\beta}$.

Let us now consider the limit $b\to 0$ preserving the wave effects, i.e., K is finite and positive. Supposing $X\neq 0$ and using

$$\Phi_0(X,b)\simeq\Phi_0(X,0)+\left. b\frac{\partial}{\partial z}\Phi_0(X,z)\right|_{z=0}$$

in conjunction with the free-surface condition which is satisfied by Φ_0, we get

$$\Phi_0(X,b)=(1-Kb)\Phi_0(X,0)+O(b^2) \quad \text{as } b\to 0. \qquad (24)$$

Substituting Eq. (24) in Eq. (11), and using Eq. (21), we have as $b\to 0$ that

$$(1-Kb)S\psi+\frac{b}{2}H\psi=g(t), \quad -1\le t\le 1, \qquad (25)$$

where

$$g(t)=-\frac{1}{2}\pi t,$$

$$(S\psi)(t)=K\int_{-1}^{1}\psi(s)\Phi_0(t-s,0)ds,$$

$$(H\psi)(t)=⨎_{-1}^{1}\frac{\psi(s)}{(t-s)^2}ds.$$

Let us assume that ψ has a regular expansion in powers of b. Then from Eq. (25),

$$\psi=(1+Kb)\psi_0-\frac{b}{2}\psi_1+O(b^2) \quad \text{as } b\to 0,$$

where

$$S\psi_0=g,$$
$$S\psi_1=H\psi_0. \qquad (26)$$

We look at basic Eq. (26), in what follows. Thus, the governing equation for our small-submergence approximation, $S\psi=g$, can be written as

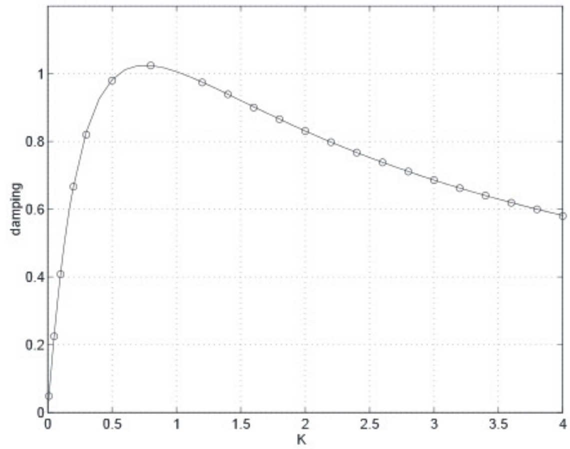

FIG. 7. The damping as a function of K. The solid line is computed with the method in this section. The circles are the damping of the dock problem.

$$\int_{-1}^{1}\psi(s)\Phi_0(t-s,0)ds=\frac{-\pi t}{2K}, \quad -1\le t\le 1. \qquad (27)$$

This is a Fredholm integral equation of the first kind for ψ. From Eq. (13), the kernel in Eq. (27) can be written as

$$\Phi_0(t-s,0)$$
$$=\cos K|t-s|(i\pi-\log K|t-s|-\gamma)-\frac{\pi}{2}\sin K|t-s|$$
$$+\sum_{m=1}^{\infty}\frac{(-K|t-s|)^m}{m!}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right)\cos m\frac{\pi}{2}. \qquad (28)$$

In order to solve Eq. (27), we employ the numerical method described in Appendix B.

A. Results

We found that the solution of Eq. (27) produces interesting results. The damping coefficient of the dock problem is recovered as Fig. 7 shows. In this figure and also in Fig. 10, the dock problem results are obtained solving a two-dimensional Fredholm integral equation of the second kind by boundary element method (see Ref. 31 for details).

Moreover the first order added mass approximation has the behavior (see Fig. 8)

$$\mathcal{A}_{ap}\sim-\frac{\pi}{K}, \quad K\to 0, \qquad (29)$$

as predicted in Ref. 9. Further, this added mass is identified as the lower underlying curve sought for the added mass of the slightly submerged disk, as can be seen from Fig. 9. The dashed line in this figure represents the added mass for the disk submerged with $b=0.02$ and was computed solving the generalized Love's integral equation by the method described in Ref. 9.

The major question is why only the damping is recovered. An interesting result is obtained if the function $h(K)=\pi/K$ is added to our first order added mass approxi-

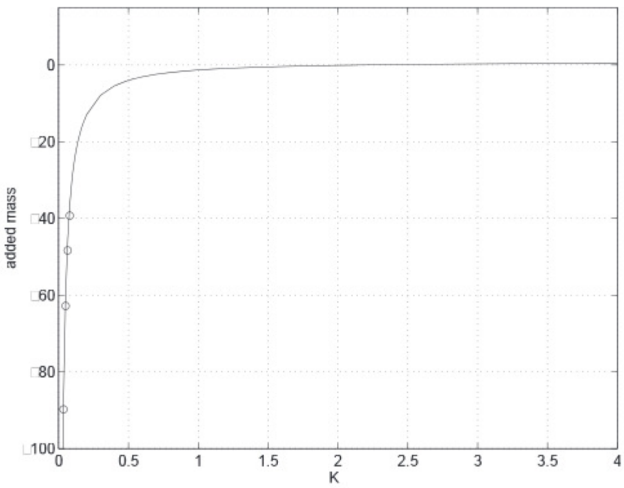

FIG. 8. The first order added mass approximation as a function of K. The circles are values of the function $-\pi/K$.

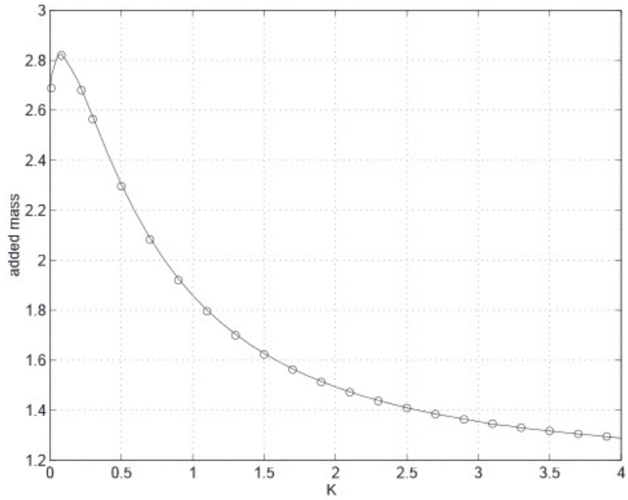

FIG. 10. The dock problem added mass, $\overline{\mathcal{A}}$ (solid line). The circles represent the first order added mass approximation added to $h(K)$.

mation. Denote the added mass and the damping of the dock problem by $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, respectively. We find that

$$\mathcal{A}_{ap} + h(K) = \overline{\mathcal{A}}. \tag{30}$$

Figure 10 graphically shows Eq. (30).

The essence of relation (30) and a theoretical explanation of why this is obtained is given by Martin and Farina.[9] For small values of b, the potential beneath the submerged disk and the dock is the same. However, the potential in the thin layer above the submerged disk is significant; there the fluid is in a simple state of uniform vertical oscillation in phase with the disk, with potential $z-1/K$ (see Ref. 17). Thus, $[\phi] \approx \phi_d$, where ϕ_d is the potential on the dock. Hence, $\mathcal{A}_{ap}+i\mathcal{B}_{ap} \approx \overline{\mathcal{A}}+i\overline{\mathcal{B}}-\pi/K$, in agreement with the numerical results. In that work, however, the reasoning proposed did not considered the first order approximation, solution of Eq. (27), however; only the full solution of the generalized Love's integral Eq. (11) was solved.

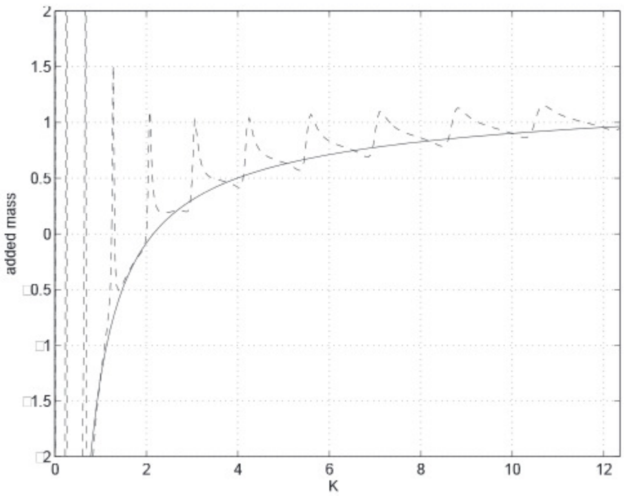

FIG. 9. The first order added mass approximation as a function of K and the added mass for the submerged disk (b=0.02) (– –) as a function of K.

The asymptotics make the solution of Eq. (27) represent a first order approximation of the small submergence problem. Although this solution does not include the resonances, it shows mathematically the connection between the slightly submerged disk with the dock problem. Furthermore, physically, we can see from the results and discussion above that the first order added mass approximation is due to the potential of the fluid above the submerged disk, for lower frequency waves.

VI. DISCUSSION

The radiation of water waves by a submerged circular disk has been examined. Emphasis was put on the case where the submergence is very small. In this critical regime, the hydrodynamic force presents a strong dependence on the frequency. In particular the peaks in the added mass and damping coefficients have been studied and located by a modified Newton's method. We also showed definitive numerical evidence that the added mass and damping coefficients are mapped into circles in the complex plane, near the resonant frequencies.

In order to understand the connections between this problem with the dock problem, we derive an asymptotic approximation for small values the submergence. We found that the first order approximation possesses most the of features appearing in the dock added mass and damping. Additionally, we found that, physically, the first order added mass corresponds to the potential of the fluid above the disk, in a simple state of uniform vertical oscillation in phase with the disk. The associated problem of interaction of water waves with nonplanar perturbations of a disk is being investigated and will be described elsewhere.

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TABLE II. Convergence of Newton's method to the pole $1.025\,591 - i0.072\,382$ with the initial guess $1.2 - i0.4$ and submergence depth $b = 0.08$.

n	$\mathcal{K}_n$	$\lvert\det \mathcal{M}\rvert$
1	$(1.2,\ -0.4)$	4.2825×10^{-04}
2	$(1.166\,841,\ -0.257\,773)$	1.5311×10^{-04}
3	$(1.111\,380,\ -0.136\,350)$	4.7031×10^{-05}
4	$(1.049\,537, -7.8199\times 10^{-02})$	8.7012×10^{-06}
5	$(1.027\,035, -7.1837\times 10^{-02})$	5.1648×10^{-07}
6	$(1.025\,591, -7.2376\times 10^{-02})$	2.2279×10^{-09}
7	$(1.025\,591, -7.2382\times 10^{-02})$	4.8755×10^{-14}
8	$(1.025\,591, -7.2382\times 10^{-02})$	3.7023×10^{-19}
9	$(1.025\,591, -7.2382\times 10^{-02})$	7.7350×10^{-20}
10	$(1.025\,591, -7.2382\times 10^{-02})$	2.1982×10^{-20}

APPENDIX A: THE MODIFIED NEWTON'S METHOD

The Newton's method to find the zeros of a complex valued function $f(w)$ is the iteration procedure

$$w_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)}. \qquad \text{(A1)}$$

Since we do not possess an analytical expression for the derivative of $\det \mathcal{M}$, we actually use the approximation (with $\epsilon \ll 1$)

$$\frac{\det \mathcal{M}(\mathcal{K}) - \det \mathcal{M}(\mathcal{K}+\epsilon)}{\epsilon}, \qquad \text{(A2)}$$

for the derivative

$$\frac{d}{d\mathcal{K}} \det \mathcal{M}(\mathcal{K}).$$

The parameter ϵ was taken, in our experiments, as $10^{-7} + i10^{-7}$. Using Eqs. (A2) in Eq. (A1), we get the so called modified Newton's method:

$$\mathcal{K}_{n+1} = \mathcal{K}_n - \frac{\epsilon \det \mathcal{M}(\mathcal{K}_n)}{\det \mathcal{M}(\mathcal{K}_n) - \det \mathcal{M}(\mathcal{K}_n + \epsilon)}. \qquad \text{(A3)}$$

This scheme achieved convergence very fast. For $b \geq 0.08$, ten iterations were usually sufficient to determine a zero with good accuracy (see Table II).

The resonance poles have small and negative imaginary parts. This suggests that the initial guess, $\mathcal{K}_0$, in Eq. (A3) is chosen accordingly. However determining the region of convergence,

$$\Omega = \{\mathcal{K}_0 : \text{Newton's method Eq. (A3) converges}\},$$

is a delicate problem. It is very likely that the boundary of Ω is a fractal. See Ref. 32 for more information about Newton's method and fractals. We remark that a second method could also be used to find the zeros of $\mathcal{M}(\mathcal{K})$.

APPENDIX B: THE BOUNDARY ELEMENT METHOD

The logarithmic singularity present in the kernel of Eq. (27) prevents the method of solution from having the same collocation and quadrature points, so that a simple quadrature method cannot be used. Instead we use a boundary element method to solve this equation.

We start by subdividing the interval $[-1, 1]$ into N' subintervals, S_j, such that

$$S_j = [s_j^1, s_j^2], \quad j = 1, \ldots, N',$$

where

$$s_1^1 \to -1 \quad \text{and} \quad s_{N'}^2 \to 1 \quad \text{as } N \to \infty.$$

The collocation points, $\{s_j\}_1^N$ lie on the midpoint of the subintervals. Thus

$$s_j = \frac{s_j^2 + s_j^1}{2}.$$

Equation (27) can then be approximated as

$$\sum_{j=1}^{N} \int_{S_j} \psi(s)\Phi_0(t-s,0)ds = \frac{-\pi t}{2K}. \qquad \text{(B1)}$$

This approximation is exact if the union of the subintervals is $[-1, 1]$. Assuming constant elements, we have

$$\sum_{j=1}^{N} \psi(s_j) \int_{S_j} \Phi_0(t-s,0)ds = \frac{-\pi t}{2K}. \qquad \text{(B2)}$$

Evaluating Eq. (B2) at the collocation points we get

$$\sum_{j=1}^{N} \psi(s_j) \int_{S_j} \Phi_0(s_i - s,0)ds = \frac{-\pi s_i}{2K}. \qquad \text{(B3)}$$

Denoting the vectors $\tilde{\psi}$ and $\tilde{g}$ as

$$\tilde{\psi} = \{\psi(s_1, \ldots, \psi(s_{N'})\}, \qquad \text{(B4)}$$

$$\tilde{g} = (s_1, \ldots, s_{N'}), \qquad \text{(B5)}$$

we write Eq. (B3) as the matrix equation:

$$A\tilde{\psi} = \tilde{g}, \qquad \text{(B6)}$$

where

$$A = \{A_{ij}\} = \int_{S_j} \Phi_0(s_i - s,0)ds.$$

The solution of Eq. (B6) gives a discrete solution $\tilde{\psi}$. This approximation to ψ is constant on each subinterval S_j. The value of $N'=76$ was used to assure numerical convergence. From Eq. (14), the first order added mass and damping coefficients can be written as

$$\mathcal{A}_{ap} + i\mathcal{B}_{ap} \approx 4\sum_{j=1}^{N} \psi(s_j)[(s_j^2)^2 - (s_j^1)^2].$$

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