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A Simple Method for Obtaining PBW-Basis for Some Small Quantum Algebras

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Abstract

In this paper, using a much simpler method than the previous existing ones, we explicitly describe the PBW-generators of the multiparameter quantum groups $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra of small dimension, while the main parameter of quantization q is not a root of the unity.

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1 Introduction

Since the Poincaré-Birkhoff-Witt Theorem proven around 1900, PBW-bases have an important role in Lie Theory. One of the reasons is that in many cases they provide a finite basis even for an infinite dimensional universal enveloping Lie algebra. The same thing happens when we discuss quantum enveloping Lie algebras. Most of these quantum algebras are infinite dimensional, so it is very useful to have a finite way to represent then.

The definition of a quantum enveloping algebra (also called quantum group) is somehow sophisticated. The PBW-bases of these algebras are profoundly connected to the root systems related to the ground Lie algebra, and this theory is also complex. In this paper we present a method to find the PBW-bases of some examples of quantum groups using only combinatoric arguments. These PBW-bases are already known (see, for example, [1]), however the previous methods for finding these bases are really complex, so it is interesting to find a much simpler proof. We also have two more special reasons for the interest in this development. First, we do not simply calculate a PBW-basis (which is not unique), but we obtain a special PBW-basis, the only one formed by hard supper-letters (see [3]). Second, in this proof there are no restrictions in the base field **k** (in [1] **k** needs to be algebraically closed and have characteristic zero). It is important to notice that, although the studied quantum enveloping algebras are Hopf algebras, we use only their algebra structure (and not the coalgebra) to calculate their PBW-generators.

In the second and third sections, following [3] and [5], we introduce main concepts and general results about hard super-letters and PBW-generators. In sections 4, 5 and 6 we present the considered quantum algebras $U_q^+(\mathfrak{g})$ and explicitly calculate their hard super-letters. Finally, on the last section, our main theorem exhibits the PBW-generators for these algebras.

2 Preliminaries

Along this work, A represents an algebra over a field **k** and G is a multiplicative group.

Definition 2.1. *Let A be an algebra over a field $\boldsymbol{k}$ and B its subalgebra with a fixed basis $\{b_j | j \in J\}$. A linearly ordered subset $W \subseteq A$ is said to be a set of PBW-generators of A over B if there exists a function $h : W \to \mathbb{Z}^+ \cup \infty$, called the height function, such that the set of all products*

$$b_j w_1^{n_1} w_2^{n_2} \dots w_k^{n_k}, \tag{2.1}$$

where $j \in J$, $w_1 < w_2 < \dots < w_k \in W$, $n_i < h(w_i)$, $1 \leq i \leq k$ is a basis of A. The value $h(w)$ is referred to as the height of w in W. If $B = \boldsymbol{k}$ is the ground field, then we shall call W simply as a set of PBW-generators of A.

Definition 2.2. *Let W be a set of PBW-generators of A over a subalgebra B. Suppose that the set of all words in W as a free monoid has its own order $\prec$ (that is, $a \prec b$ implies $cad \prec cbd$ for all words $a, b, c, d \in W$). A leading word of $a \in A$ is the maximal word $m = w_1^{n_1} w_2^{n_2} \dots w_k^{n_k}$ that appears in the decomposition of a in the basis (2.1). A leading term of a is the sum ba of all terms $\alpha_i b_i m$, $\alpha_i \in \boldsymbol{k}$, that appear in the decomposition of a in the basis (2.1), where m is the leading word of a.*

Definition 2.3. *Let $X = \{x_i | i \in I\}$ be a set of indeterminates and G a group. A constitution of a word u in $G \cup X$ is a family of non-negative integers $\{m_x, x \in X\}$ such that u has m_x occurrences of x. Certainly almost all m_x in the constitution are zero.*

Let us fix an arbitrary complete order $<$ on the set X, and let Γ^+ be the free additive (commutative) monoid generated by X. The monoid Γ^+ is a completely ordered monoid with respect to the following order:

$$m_1x_{i_1} + m_2x_{i_2} + \ldots + m_kx_{i_k} > m'_1x_{i_1} + m'_2x_{i_2} + \ldots + m'_kx_{i_k} \tag{2.2}$$

if the first from the left nonzero number in $(m_1 - m'_1, m_2 - m'_2, \ldots, m_k - m'_k)$ is positive, where $x_{i_1} > x_{i_2} > \ldots > x_{i_k}$ in X. We associate a formal degree $D(u) = \sum_{x\in X} m_x x \in \Gamma^+$ to a word u in $G \cup X$, where $\{m_x | x \in X\}$ is the constitution of u. Respectively, if $f = \sum \alpha_i u_i \in G\langle X\rangle$, $0 \neq \alpha_i \in \mathbf{k}$ then

$$D(f) = max_i\{D(u_i)\}. \tag{2.3}$$

On the set of all words in X we fix the lexicographical order with the priority from the left to the right, where a proper beginning of a word is considered to be greater than the word itself.

Definition 2.4. *A non-empty word u is called a standard word (or Lyndon word, or Lyndon-Shirshov word) if $vw > wv$ for each decomposition $u = vw$ with non-empty v, w.*

Definition 2.5. *A non-associative word is a word where brackets $[,]$ are somehow arranged to show how multiplication applies.*

If $[u]$ denotes a non-associative word, then by u we denote an associative word obtained from $[u]$ by removing the brackets. Of course, $[u]$ is not uniquely defined by u in general.

Definition 2.6. *The set of standard non-associative words is the biggest set SL that contains all variables x_i and satisfies the following properties:*

1. *If $[u] = [[v],[w]] \in SL$ then $[v],[w] \in SL$, and $v > w$ are standard.*
2. *If $[u] = [[[v_1],[v_2]],[w]] \in SL$ then $v_2 \leq w$.*

By the Shirshov's Theorem, every standard associative word has only one alignment of brackets such that the defined non-associative word is standard. In order to find this alignment we use the following procedure: the factors v, w of the non-associative decomposition $[u] = [[v],[w]]$ are standard words such that $u = vw$ and v has the minimal length (see [6]).

Definition 2.7. *A super-letter is a polynomial that equals a non-associative standard word. A super-word is a word in super-letters.*

We will denote by $[u]$ the super-letter obtained from u using Shirshov's Theorem. The order on the super-letters is defined in the natural way: $[u] > [v] \Leftrightarrow u > v$.

Definition 2.8. *A super-letter $[u]$ is called hard in H if its value in H is not a linear combination of super-words of the same degree* (2.3) *in super-letters smaller than $[u]$.*

Proposition 2.9. ([3, Corollary 2]) *A super-letter $[u]$ is hard in H if and only if the value in H of the standard word u is not a linear combination of values of smaller words of the same degree (2.3).*

Proposition 2.10. ([4, Lemma 4.8]) *Let B be a set of super-letters containing $x_1, \ldots, x_n$. If each pair $[u], [v] \in B$, $u > v$, satisfies one of the following conditions*

1) $[[u], [v]]$ *is not a standard non-associative word;*

2) *the super letter $[[u], [v]]$ is not hard in H;*

3) $[[u], [v]] \in B$*;*

then the set B includes all hard in H super-letters.

3 Skew-commutator and quantum algebras

We define a bilinear skew-commutator on homogeneous linear combinations of words by the formula

$$[x_i, x_j] = x_i x_j - p_{ij} x_j x_i, \tag{3.1}$$

where $p_{ij} \in \mathbf{k}$. These brackets are related to the product by the following identities

$$[u \cdot v, w] = p_{vw}[u, w] \cdot v + u \cdot [v, w], \quad [u, v \cdot w] = [u, v] \cdot w + p_{uv} v \cdot [u, w].$$

Thus, for example, we deduce that

$$[x_i, x_j \cdot x_k] = x_i x_j x_k - p_{ij} p_{ik} x_j x_k x_i,$$

$$[x_i \cdot x_j, x_k] = x_i x_j x_k - p_{ik} p_{jk} x_k x_i x_j,$$

$$[x_i, [x_j, x_k]] = x_i [x_j, x_k] - p_{ij} p_{ik} [x_j, x_k] x_i. \tag{3.2}$$

Definition 3.1. *(see, for example, [4, section 2]) Let $C = \| a_{ij} \|$ be a generalized Cartan matrix symmetrizable by $D = diag(d_1, \ldots, d_n)$, $d_i a_{ij} = d_j a_{ji}$. Denote by $\mathfrak{g}$ a Kac-Moody algebra defined by C (see [2]). Suppose that the quantification parameters $p_{ij} = p(x_i, x_j) = \chi^i(g_j)$ are related by*

$$p_{ii} = q^{d_i}, \quad p_{ij} p_{ji} = q^{d_i a_{ij}}, \quad 1 \le i, j \le n. \tag{3.3}$$

The multiparameter quantization $U_q^+(\mathfrak{g})$ of the Borel subalgebra $\mathfrak{g}^+$ is a character Hopf algebra generated by $x_1, \dots, x_n, g_1, \dots, g_n$ and defined by Serre relations with the skew brackets (3.1) in place of the Lie operation:

$$[[\dots[[x_i, x_j], x_j], \dots], x_j] = 0, \quad 1 \leq i \neq j \leq n, \tag{3.4}$$

where x_j appears $1 - a_{ji}$ times.

In this paper we are going to consider the quantum algebras $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra of type A_2, A_3, B_2, B_3 and C_3. The cases C_2, D_2 and D_3 are not considered since they are isomorphic as Lie algebras to B_2, A_2 and A_3, respectively.

Definition 3.2. *We say that the height of a hard in H super-letter $[u]$ equals $h = h([u])$ if h is the smallest number such that*

1. *p_{uu} is a primitive t-th root of 1 and either $h = t$ or $h = tl^r$, where $l = char(\boldsymbol{k})$,*
2. *the value of $[u]^h$ in H is a linear combination of super-words of the same degree (2.3) in super-letters smaller than $[u]$.*

If there exists no such number then the height equals infinity.

Theorem 3.3. ([3, Theorem 2]) *The values of all hard in H super-letters with the above defined height function form a set of PBW-generators for H over* $\mathbf{k}[\mathrm{G}]$.

4 Quantizations of type A

The algebra $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra of type A_2, is defined by two generators x_1, x_2 and two relations

$$[x_1, [x_1, x_2]] = 0 = [[x_1, x_2], x_2] \tag{4.1}$$

where the brackets mean the skew commutator $[x_i, x_j] = x_i x_j - p_{ij} x_j x_i$ (3.1) and relations (3.3) take up the form $p_{11} = p_{22} = q$, $p_{12}p_{21} = q^{-1}$. We notice that relation $[x_1, [x_1, x_2]] = 0$ is equivalent to $[[x_2, x_1], x_1] = 0$, which appears in (3.4), and the same occurs in all following cases.

Proposition 4.1. *Let $\mathfrak{g}$ be the simple Lie algebra of type A_2. The set of all hard in $U_q^+(\mathfrak{g})$ super-letters is given by $\mathcal{A}_2 = \{x_1, [x_1, x_2], x_2\}$.*

Proof. This follows directly from Proposition 2.10 and the defining relations (4.1). Since $x_1 > [x_1, x_2] > x_2$ all possibilities to be considered are $[x_1, [x_1, x_2]], [x_1, x_2]$ and $[[x_1, x_2], x_2]$, where $[x_1, x_2] \in \mathcal{A}_2$ and $[x_1, [x_1, x_2]] = 0 = [[x_1, x_2], x_2]$, so they are not hard super-letters. $\square$

Remark 4.2. *If we consider p_{uu} as the the element obtained in the relation $[u,u] = u^2 - p_{uu}u^2$, using relation (3.2) we can see that $p_{uu} = p_{[u][u]}$ and we can easily calculate $p_{AA} = p_{BB} = p_{CC} = q$.*

Now we consider the algebra $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra of type A_3. This algebra is defined by three generators x_1, x_2, x_3 and five relations

$$[x_1,[x_1,x_2]] = [[x_1,x_2],x_2] = [x_2,[x_2,x_3]] = [[x_2,x_3],x_3] = [x_1,x_3] = 0 \quad (4.2)$$

where the brackets mean the skew commutator and we have $p_{11} = p_{22} = p_{33} = q$, $p_{12}p_{21} = q^{-1} = p_{23}p_{32}$ and $p_{13}p_{31} = 1$.

Proposition 4.3. *Let $\mathfrak{g}$ be the simple Lie algebra of type A_3. The set of all hard in $U_q^+(\mathfrak{g})$ super-letters is given by $\mathcal{A}_3 = \{x_1, [x_1,x_2], [x_1,[x_2,x_3]], x_2, [x_2,x_3], x_3\}$.*

Proof. For simplicity we call $x_1 = [A]$, $[x_1,x_2] = [B]$, $[x_1,[x_2,x_3]] = [C]$, $x_2 = [D]$, $[x_2,x_3] = [E]$, $x_3 = [F]$ and notice that $[A] > [B] > [C] > [D] > [E] > [F]$. From proposition 2.10 we have 15 possible cases to consider. Using item 3) of the proposition, we exclude the cases $[[A],[D]] = [B]$, $[[A],[E]] = [C]$ and $[[D],[F]] = [E]$. From the defining relations we see that $[[A],[B]] = [[A],[F]] = [[B],[D]] = [[D],[E]] = [[E],[F]] = 0$, so these five super-letters are not hard in $U_q^+(\mathfrak{g})$. Also, the super-letters $[[B],[E]] = [[x_1,x_2],[x_2,x_3]]$, $[[B],[F]] = [[x_1,x_2],x_3]$ and $[[C],[F]] = [[x_1,[x_2,x_3]],x_3]$ are not standard since they can be written as $[[[u],[v]],[w]]$ with $v > w$.

Now, for the remaining cases $[[A],[C]]$, $[[B],[C]]$, $[[C],[D]]$ and $[[C],[E]]$ we use the defining relations of the algebra to describe their associated standard words as a linear combination of smaller words of the same degree, proving that they are not hard in $U_q^+(\mathfrak{g})$ by Proposition 2.9.

Expanding the defining relations (4.2) of the algebra we obtain the following equations:

$$x_1x_3 = p_{13}x_3x_1, \quad (4.3)$$

$$x_1x_2^2 = p_{12}(1+p_{22})x_2x_1x_2 - p_{12}^2p_{22}x_2^2x_1, \quad (4.4)$$

$$x_1^2x_2 = p_{12}(1+p_{11})x_1x_2x_1 - p_{11}p_{12}^2x_2x_1^2, \quad (4.5)$$

$$x_2x_3^2 = p_{23}(1+p_{33})x_3x_2x_3 - p_{23}^2p_{33}x_3^2x_2, \quad (4.6)$$

$$x_2^2x_3 = p_{23}(1+p_{22})x_2x_3x_2 - p_{22}p_{23}^2x_3x_2^2. \quad (4.7)$$

The super-letter $[[A],[C]]$ is not hard in $U_q^+(\mathfrak{g})$ since from relation (4.5) the standard word $AC = x_1^2x_2x_3 = p_{12}(1+p_{11})x_1x_2x_1x_3 - p_{11}p_{12}^2x_2x_1^2x_3$ is a linear combination of smaller standard words. For the super-letter $[[C],[D]]$ we see that $CD = x_1x_2x_3x_2$, which is the leading term of $x_1 \cdot (4.7) - (4.4) \cdot x_3$. In the case $[[C],[E]]$, we obtain CE as the greatest term in $x_1x_2 \cdot (4.6) - (4.4) \cdot x_3^2$. Finally, the super-letter $[[B],[C]]$ is not hard since calculating $(4.5) \cdot x_2x_3 - x_1^2 \cdot (4.7)$ we have $p_{12}(1+p_{11})BC = p_{11}p_{12}^2x_2x_1^2x_2x_3 + p_{22}p_{23}^2x_1^2x_3x_2^2 -$

$p_{23}(1+p_{22})x_1^2x_2x_3x_2$ and now using (4.3) and (4.5) we obtain $p_{12}(1+p_{11})BC = p_{11}p_{12}^2x_2x_1^2x_2x_3 + p_{22}p_{23}^2p_{13}^2x_3x_1^2x_2^2 - p_{23}p_{12}(1+p_{22})^2x_1x_2x_1x_3x_2 + p_{23}p_{12}^2p_{11}(1+p_{22})x_2x_1^2x_2x_3$, writing BC as a linear combination of smaller words and proving the proposition.

□

Remark 4.4. *In this case we obtain* $p_{AA} = p_{BB} = p_{CC} = p_{DD} = p_{EE} = p_{FF} = q$.

5 Quantizations of type B

Let us consider the algebra $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra of type B_2. This algebra is defined by two generators x_1, x_2 and two relations

$$[x_1,[x_1,x_2]] = [[[x_1,x_2],x_2],x_2] = 0. \tag{5.1}$$

In this case, $p_{11} = p_{22}^2 = q^2$ and $p_{21}p_{12} = q^{-2}$.

Proposition 5.1. *Let $\mathfrak{g}$ be the simple Lie algebra of type B_2. The set of all hard in $U_q^+(\mathfrak{g})$ super-letters is given by $\mathcal{B}_2 = \{x_1, [x_1,x_2], [[x_1,x_2],x_2], x_2\}$.*

Proof. We call $x_1 = [A], [x_1,x_2] = [B], [[x_1,x_2],x_2] = [C], x_2 = [D]$ where $[A] > [B] > [C] > [D]$. From Proposition 2.10 we have 6 possibilities. Using item 3) we exclude the cases $[[A],[D]] = [B]$ and $[[B],[D]] = [C]$. From the defining relations we see that $[[A],[B]] = [[C],[D]] = 0$, so these two super-letters are not hard in $U_q^+(\mathfrak{g})$.

The remaining cases are $[[A],[C]]$ and $[[B],[C]]$, so we use the defining relations of the algebra to describe their associated standard words as linear combinations of smaller words of the same degree and prove that they are not hard in $U_q^+(\mathfrak{g})$.

Expanding the defining relations of the algebra we obtain the equations:

$$x_1^2x_2 = p_{12}(1+p_{11})x_1x_2x_1 - p_{11}p_{12}^2x_2x_1^2 \tag{5.2}$$

$$x_1x_2^3 = p_{12}(1+p_{22}+p_{22}^2)x_2x_1x_2^2 - p_{22}p_{12}^2(1+p_{22}+p_{22}^2)x_2^2x_1x_2 + p_{22}^2p_{12}^3x_2^3x_1 \tag{5.3}$$

If we multiply the equation (5.2) from the right by x_2^2, we have AC as a linear combination of smaller words. Thus the super-letter is not hard. Now, let us multiply the equation (5.2) from the right by x_2, while relation (5.3) from the left by x_1. The leading term of the difference equals:

$$(-p_{12}(1+p_{22}+p_{22}^2) + p_{12}(1+p_{11}))BC$$

Therefore BC is also a linear combination of smaller words and the super-letter is not hard by Proposition 2.9. □

Remark 5.2. *In this case we compute* $p_{AA} = p_{CC} = q^2$ *and* $p_{BB} = p_{DD} = q$.

Now we consider the algebra $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra of type B_3. This algebra is defined by three generators x_1, x_2, x_3 and five relations

$$[[x_1, x_2], x_2] = [x_1, [x_1, x_2]] = [[[x_2, x_3], x_3], x_3] = [x_2, [x_2, x_3]] = [x_1, x_3] = 0. \quad (5.4)$$

Relations (3.3) take up the form $p_{11} = p_{22} = q^2$, $p_{33} = q$, $p_{12}p_{21} = q^{-2} = p_{23}p_{32}$ and $p_{13}p_{31} = 1$.

Proposition 5.3. *Let* $\mathfrak{g}$ *be the simple Lie algebra of type* B_3. *Then the set of all hard in* $U_q^+(\mathfrak{g})$ *super-letters is constituted by* $\mathcal{B}_3 = \{x_1, [x_1, x_2], [x_1, [x_2, x_3]], [x_1, [[x_2, x_3], x_3]], [[x_1, [[x_2, x_3], x_3]], x_2], x_2, [x_2, x_3], [[x_2, x_3], x_3], x_3\}$.

Proof. As done before, we define $[A] = x_1, [B] = [x_1, x_2], [C] = [x_1, [x_2, x_3]], [D] = [x_1, [[x_2, x_3], x_3]], [E] = [[x_1, [[x_2, x_3], x_3]], x_2], [F] = x_2, [G] = [x_2, x_3], [H] = [[x_2, x_3], x_3]$ and $[I] = x_3$. From propositiom 2.10 we have 36 possibilities. Using item 3) we exclude the cases $[[A], [F]] = [B], [[A], [G]] = [C], [[A], [H]] = [D], [[D], [F]] = [E], [[F], [I]] = [G]$ and $[[G], [I]] = [H]$. From the defining relations we see that $[[A], [B]] = [[A], [I]] = [[B], [F]] = [[F], [G]] = [[H], [I]] = 0$, so these five super-letters are not hard in $U_q^+(\mathfrak{g})$. The super-letters $[[B],[G]], [[B],[H]], [[B],[I]], [[C],[H]], [[C], [I]], [[D], [I]], [[E], [G]], [[E], [H]]$ and $[[E], [I]]$ are not standard since they can be written as $[[[u], [v]], [w]]$ with $v > w$.

For the 16 remaining cases we use the defining relations of the algebra to describe their associated standard words as a linear combination of smaller words of the same degree, proving that they are not hard in $U_q^+(\mathfrak{g})$ by Proposition 2.9.

Expanding the defining relations of the algebra we obtain the equations:

$$x_1x_3 = p_{13}x_3x_1 \quad (5.5)$$

$$x_1^2x_2 = p_{12}(1 + p_{11})x_1x_2x_1 - p_{11}p_{12}^2x_2x_1^2 \quad (5.6)$$

$$x_1x_2^2 = p_{12}(1 + p_{22})x_2x_1x_2 - p_{12}^2p_{22}x_2^2x_1 \quad (5.7)$$

$$x_2^2x_3 = p_{23}(1 + p_{22})x_2x_3x_2 - p_{22}p_{23}^2x_3x_2^2 \quad (5.8)$$

$$x_2x_3^3 = p_{23}(1+p_{33}+p_{33}^2)x_3x_2x_3^2 - p_{33}p_{23}^2(1+p_{33}+p_{33}^2)x_3^2x_2x_3 + p_{33}^3p_{23}^3x_3^3x_2 \quad (5.9)$$

The super-letters $[[A], [C]], [[A], [D]]$ and $[[A], [E]]$ are not hard in $U_q^+(\mathfrak{g})$ since from relation (5.6) we write the standard word $AC = x_1^2x_2x_3 = p_{12}(1 + p_{11})x_1x_2x_1x_3 - p_{11}p_{12}^2x_2x_1^2x_3$ as a linear combination of smaller standard words, and also $AD = AC \cdot x_3$, $AE = AC \cdot x_3x_2$. Analogously, the super-letter $[[F], [H]]$ is not hard in $U_q^+(\mathfrak{g})$ since from relation (5.8) the standard word $FH = x_2^2x_3^2 = p_{23}(1 + p_{22})x_2x_3x_2x_3 - p_{22}p_{23}^2x_3x_2^2x_3$ is a linear combinations of smaller standard words.

The super-letter $[[B],[C]]$ is not hard since calculating $(5.6)\cdot x_2x_3 - x_1^2\cdot(5.8)$ we have $p_{12}(1+p_{11})BC = p_{11}p_{12}^2x_2x_1^2x_2x_3 + p_{22}p_{23}^2x_1^2x_3x_2^2 - p_{23}(1+p_{22})x_1^2x_2x_3x_2$ and using (5.6) and (5.5) we obtain $p_{12}(1+p_{11})BC = p_{11}p_{12}^2x_2x_1^2x_2x_3 + p_{22}p_{23}^2p_{13}^2x_3x_1^2x_2^2 - p_{23}p_{12}(1+p_{22})(1+p_{11})x_1x_2x_1x_3x_2 + p_{23}p_{11}p_{12}^2(1+p_{22})x_2x_1^2x_3x_2$. The cases $[[B],[D]]$ and $[[B],[E]]$ follow from the previous case since $BD = BC\cdot x_3$ and $BE = BC\cdot x_3x_2$.

For the super-letter $[[C],[D]]$ we compute $x_1x_2x_1\cdot(5.9)$ and obtain that $p_{13}CD = p_{23}^{-1}(1+p_{33}+p_{33}^2)^{-1}(x_1x_2x_1x_2x_3^3 + p_{23}^2p_{33}(1+p_{33}+p_{33}^2)x_1x_2x_1x_3^2x_2x_3 - p_{23}^3p_{33}^3x_1x_2x_1x_3^3x_2)$. Notice that we have $x_1x_2x_1x_2x_3^3 = BD\cdot x_3 = p_{12}^{-1}(1+p_{11})^{-1}(p_{11}p_{12}^2x_2x_1^2x_2x_3^3 - p_{23}p_{12}(1+p_{11})(1+p_{22})x_1x_2x_1x_3x_2x_3^2 + p_{11}p_{23}p_{12}^2(1+p_{22})x_2x_1^2x_3x_2x_3^2 + p_{22}p_{23}^2p_{13}^2x_3x_1^2x_2^2x_3^2)$. Now, we replace that word in the first relation obtaining $(p_{13}+p_{12}p_{23}(1+p_{11})(1+p_{22}))CD = p_{23}^{-1}(1+p_{33}+p_{33}^2)^{-1}(p_{12}^{-1}(1+p_{11})^{-1}(p_{11}p_{12}^2x_2x_1^2x_2x_3^3 + p_{11}p_{23}p_{12}^2(1+p_{22})x_2x_1^2x_3x_2x_3^2 + p_{22}p_{23}^2p_{13}^2x_3x_1^2x_2^2x_3^2 + p_{23}^2p_{33}(1+p_{33}+p_{33}^2)x_1x_2x_1x_3^2x_2x_3 - p_{23}^3p_{33}^3x_1x_2x_1x_3^3x_2)$. Thus CD is a linear combination of smaller words. In the case $[[C],[E]]$ we just notice that $CE = CD\cdot x_2$.

In the case $[[C],[F]]$ we see that $CF = x_1x_2x_3x_2$, which is the leading term of $x_1\cdot(5.8) - (5.7)\cdot x_3$. Thus the super-letter is not hard. As a consequence, the super-letter $[[C],[G]]$ is not hard since $CG = CF\cdot x_3$.

For the super-letter $[[D],[G]]$ we compute $x_1x_2\cdot(5.9) - (5.7)\cdot x_3^3$ and obtain $p_{23}^2p_{33}(1+p_{33}+p_{33}^2)DG = p_{23}^2p_{33}(1+p_{33}+p_{33}^2)x_1x_2x_3^2x_2x_3 = p_{23}^3p_{33}^3x_3^3x_2 - p_{12}(1+p_{22})x_2x_1x_2x_3^3 + p_{12}^2p_{22}x_2^2x_1x_3 + p_{23}(1+p_{33}+p_{33}^2)x_1x_2x_3x_2x_3^2$. Note that the word DG is smaller than the word $x_1x_2x_3x_2x_3^2$. Now we calculate $x_1\cdot(5.8)\cdot x_3^2$ and find $x_1x_2x_3x_2x_3^2 = (p_{23}(1+p_{22}))^{-1}(x_1x_2^2x_3^3 + p_{22}p_{23}^2x_1x_3x_2^2x_3^2)$, so using (5.8) we obtain $x_1x_2x_3x_2x_3^2 = (p_{23}(1+p_{22}))^{-1}(p_{12}(1+p_{22})x_2x_1x_2x_3^3 - p_{12}^2p_{22}x_2^2x_1x_3^3 + p_{22}p_{23}^2x_1x_3x_2^2x_3^2)$. Now we can replace the word $x_1x_2x_3x_2x_3^2$ and write DG as a linear combination of smaller words. For the super-letter $[[D],[H]]$ we see that $DH = DG\cdot x_3$.

The super-letter $[[D],[E]]$ is not hard since computing $x_1x_2x_1x_3\cdot(5.9)\cdot x_2$ we have $p_{23}(1+p_{33}+p_{33}^2)DE = p_{23}p_{31}^2(1+p_{33}+p_{33}^2)x_1x_2x_1x_3^2x_2x_3^2x_2 = -x_1x_2x_1x_3x_2x_3^3x_2 + p_{33}p_{23}^2(1+p_{33}+p_{33}^2)x_1x_2x_1x_3^3x_2x_3x_2 - p_{33}^3p_{23}^3x_1x_2x_1x_3^4x_2^2$. We notice that the word DE is smaller than $x_1x_2x_1x_3x_2x_3^3x_2$, but $x_1x_2x_1x_3x_2x_3^3x_2 = CD\cdot x_3$, thus DE can be written as a linear combination of smaller words.

In the case $[[E],[F]]$, calculating $x_1\cdot(5.8)\cdot x_2x_3$ we have $x_1x_2^2x_3x_2x_3 = p_{23}(1+p_{22})x_1x_2x_3x_2^2x_3 - p_{22}p_{23}^2x_1x_3x_2^3x_3$ and using (5.8) we obtain $p_{23}^3p_{22}(1+p_{22})EF = p_{23}^3p_{22}(1+p_{22})x_1x_2x_3^2x_2^2 = -x_1x_2^2x_3x_2x_3 + p_{23}^2(1+p_{22})^2x_1x_2x_3x_2x_3x_2 - p_{22}p_{23}^2x_1x_3x_2^3x_3$. Note that the words $x_1x_2^2x_3x_2x_3$ and $x_1x_2x_3x_2x_3x_2$ are bigger than EF, but $x_1x_2^2x_3x_2x_3 = (5.7)\cdot x_3x_2x_3 = p_{12}(1+p_{22})x_2x_1x_2x_3x_2x_3 - p_{22}p_{12}^2x_2^2x_1x_3x_2x_3$ and $x_1x_2x_3x_2x_3x_2 = CG\cdot x_2 = p_{22}p_{23}(1+p_{22})^{-1}x_1x_3x_2^2x_3x_2 - p_{12}p_{23}^{-1}x_2x_1x_2x_3^2 + p_{12}^2p_{22}p_{23}^{-1}(1+p_{22})^{-1}x_2^2x_1x_3^2$, as calculated in the previous cases. By replacing these words we write EF as a linear combination of smaller words.

Finally, the supper-letter $[[G],[H]]$ is not hard since computing $x_2\cdot(5.9)$

we have $p_{23}(1+p_{33}+p_{33}^2)GH = p_{23}(1+p_{33}+p_{33}^2)x_2x_3x_2x_3^2 = x_2^2x_3^3 + p_{23}^2p_{33}(1+p_{33}+p_{33}^2)x_2x_3^2x_2x_3 - p_{23}^3p_{33}^3x_2x_3^3x_2$. Note that the word GH is smaller than $x_2^2x_3^3$, however $x_2^2x_3^3 = FH\cdot x_3 = p_{23}(1+p_{22})GH - p_{22}p_{23}^2x_3x_2^2x_3^2$. Replacing the word $x_2^2x_3^3$ we obtain $(p_{23}(1+p_{33}+p_{33}^2) - p_{23}(1+p_{22}))GH = p_{23}^2p_{33}(1+p_{33}+p_{33}^2)x_2x_3^2x_2x_3 - p_{22}p_{23}^2x_3x_2^2x_3^2 - p_{23}^3p_{33}^3x_2x_3^3x_2$. □

Remark 5.4. *In this case we calculate* $p_{AA} = p_{BB} = p_{EE} = p_{FF} = q^2$, $p_{CC} = p_{GG} = p_{II} = q$ *and* $p_{DD} = p_{HH} = q^4$.

6 Quantizations of type C

In this section we are going to explicit a set of PBW-generators for $U_q^+(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra of type C_3. This algebra is defined by three generators x_1, x_2, x_3 and five relations

$$[x_1,[x_1,x_2]] = [[x_1,x_2],x_2] = [x_2,[x_2,[x_2,x_3]]] = [[x_2,x_3],x_3] = [x_1,x_3] = 0 \tag{6.1}$$

where the brackets mean the skew commutator and we have $p_{11} = p_{22} = q$, $p_{33} = q^2$, $p_{12}p_{21} = q^{-1}$, $p_{23}p_{32} = q^{-2}$ and $p_{13}p_{31} = 1$.

Proposition 6.1. *Let* $\mathfrak{g}$ *be the simple Lie algebra of type* C_3*. The set of all hard in* $U_q^+(\mathfrak{g})$ *super-letters is given by* $\mathcal{C}_3 = \{x_1, [x_1,x_2], [[x_1,x_2],[x_1,[x_2,x_3]]], [x_1,[x_2,x_3]], [[x_1,[x_2,x_3]],x_2], x_2, [x_2,[x_2,x_3]], [x_2,x_3], x_3\}$.

For simplicity we call $x_1 = [A], [x_1,x_2] = [B], [[x_1,x_2],[x_1,[x_2,x_3]]] = [C], [x_1,[x_2,x_3]] = [D], [[x_1,[x_2,x_3]],x_2] = [E], x_2 = [F], [x_2,[x_2,x_3]] = [G], [x_2,x_3] = [H], x_3 = [I]$ where $[A] > [B] > [C] > [D] > [E] > [F] > [G] > [H] > [I]$. From proposition 2.10 we have 36 possible cases to consider. Using item 3) of the proposition, we exclude the cases $[[A],[F]] = [B]$, $[[A],[H]] = [D]$, $[[B],[D]] = [C]$, $[[D],[F]] = [E]$, $[[F],[H]] = [G]$ and $[[F],[I]] = [H]$. From the defining relations we see that $[[A],[B]] = [[A],[I]] = [[B],[F]] = [[F],[G]] = [[H],[I]] = 0$, so these five super-letters are not hard in $U_q^+(\mathfrak{g})$. The nine super-letters $[[B],[I]] = [[x_1,x_2],x_3]$, $[[C],[E]] = [[[x_1,x_2],[x_1,[x_2,x_3]]],[[x_1,[x_2,x_3]],x_2]]$, $[[C],[G]] = [[[x_1,x_2],[x_1,[x_2,x_3]]],[x_2,[x_2,x_3]]]$, $[[C],[H]] = [[[x_1,x_2],[x_1,[x_2,x_3]]],[x_2,x_3]]$, $[[C],[I]] = [[[x_1,x_2],[x_1,[x_2,x_3]]],x_3]$, $[[D],[I]] = [[x_1,[x_2,x_3]],x_3]$, $[[E],[G]] = [[[x_1,[x_2,x_3]],x_2],[x_2,[x_2,x_3]]]$, $[[E],[H]] = [[[x_1,[x_2,x_3]],x_2],[x_2,x_3]]$ and $[[E],[I]] = [[[x_1,[x_2,x_3]],x_2],x_3]$ are not standard since they can be written as $[[[u],[v]],[w]]$ with $v > w$.

Now, for the remaining 16 cases we use the defining relations of the algebra to describe their associated standard words as a linear combination of smaller words of the same degree, proving that they are not hard in $U_q^+(\mathfrak{g})$ by Proposition 2.9.

Expanding the defining relations (6.1) of the algebra we obtain the following equations:

$$x_1x_3 = p_{13}x_3x_1, \tag{6.2}$$

$$x_1^2x_2 = p_{12}(1+p_{11})x_1x_2x_1 - p_{11}p_{12}^2x_2x_1^2, \tag{6.3}$$

$$x_1x_2^2 = p_{12}(1+p_{22})x_2x_1x_2 - p_{12}^2p_{22}x_2^2x_1, \tag{6.4}$$

$$x_2x_3^2 = p_{23}(1+p_{33})x_3x_2x_3 - p_{23}^2p_{33}x_3^2x_2, \tag{6.5}$$

$$x_2^3x_3 = p_{23}(1+p_{22}+p_{22}^2)x_2^2x_3x_2 - p_{22}p_{23}^3(1+p_{22}+p_{22}^2)x_2x_3x_2^2 + p_{22}^3p_{23}^3x_3x_2^3. \tag{6.6}$$

Multiplying equation (6.3) on the right by $x_1x_2x_3$, x_3 or x_3x_2 we obtain linear combinations whose leader term is AC, AD and AE, respectively. Calculating (6.4) $\cdot\, x_3$ we have $AG = BH$ as a combination of smaller words, and to obtain BG we compute (6.4) $\cdot\, x_2x_3$. In the case GI, we multiply $x_2 \cdot$ (6.5) and obtain the needed combination.

In order to write DH, GH and DG as linear combinations of smaller words, we compute the leading term of equations $x_1x_2\cdot(6.5)-(6.4)\cdot x_3^2$, $(6.6)\cdot x_3 - x_2^2\cdot(6.5)$ and $x_1\cdot(6.6)\cdot x_3 - (6.4)\cdot x_2x_3^2 - p_{23}(1+p_{22}+p_{22}^2)(6.4)\cdot x_3x_2x_3$, respectively.

For the super-letter $[[E],[F]]$ we subtract $x_1 \cdot (6.6) - (6.4)\cdot x_2x_3$ and obtain $EF = p_{23}^{-1}p_{22}^{-1}x_1x_2^2x_3x_2 + p_{23}p_{22}^2(1+p_{22}+p_{22}^2)^{-1}x_1x_3x_2^2 - p_{23}^{-2}p_{22}^{-1}p_{12}(1+p_{22})(1+p_{22}+p_{22}^2)^{-1}x_2x_1x_2^2x_3 + p_{23}^{-2}p_{12}^2(1+p_{22}+p_{22}^2)^{-1}x_2^2x_1x_2x_3 = p_{23}^{-1}p_{22}^{-1}p_{12}(1+p_{22})x_2x_1x_2x_3x_2 - p_{23}^{-1}p_{12}^2x_2^2x_1x_3x_2 + p_{23}p_{22}^2(1+p_{22}+p_{22}^2)^{-1}x_1x_3x_2^2 - p_{23}^{-2}p_{22}^{-1}p_{12}(1+p_{22})(1+p_{22}+p_{22}^2)^{-1}x_2x_1x_2^2x_3 + p_{23}^{-2}p_{12}^2(1+p_{22}+p_{22}^2)^{-1}x_2^2x_1x_2x_3$ where the second equality follows from using (6.4) $\cdot\, x_2x_3$.

In the case of the super-letters $[[B],[E]]$ and $[[C],[F]]$ we notice that $BE = x_1x_2x_1x_2x_3x_2 = CF$. Multiplying (6.3) $\cdot\, x_2x_3x_2$ we obtain $x_1^2x_2^2x_3x_2 = p_{12}(1+p_{11})BE - p_{11}p_{12}^2x_2x_1^2x_2x_3x_2$. Note that the word BE is smaller than $x_1^2x_2^2x_3x_2$. Now, we multiply $x_1^2 \cdot$ (6.6) and have $x_1^2x_2^3x_3 = p_{23}(1+p_{22}+p_{22}^2)x_1^2x_2^2x_3x_2 - p_{23}^2p_{22}(1+p_{22}+p_{22}^2)x_1^2x_2x_3x_2^2 + p_{22}^3p_{33}^3x_1^2x_3x_2^3 = p_{23}(1+p_{22}+p_{22}^2)x_1^2x_2^2x_3x_2 - p_{23}^2p_{22}(1+p_{22}+p_{22}^2)x_1^2x_2x_3x_2^2 + p_{22}^3p_{33}^3p_{13}^2x_3x_1^2x_2^3$. We still have that the word BE smaller than the words $x_1^2x_2^3x_3$ and $x_1^2x_2x_3x_2^2$. We multiply (6.3) $\cdot\, x_2^2x_3$ and (6.3) $\cdot\, x_3x_2^2$ and we obtain $x_1^2x_2^3x_3 = p_{12}(1+p_{11})x_1x_2x_1x_2^2x_3 - p_{11}p_{12}^2x_2x_1^2x_2^2x_3$ and $x_1^2x_2x_3x_2^2 = p_{12}(1+p_{11})x_1x_2x_1x_3x_2^2 - p_{11}p_{12}^2x_2x_1^2x_3x_2^2$. The word $x_1x_2x_1x_2^2x_3$ is bigger than BE, but we multiply $x_1x_2 \cdot (6.4)\cdot x_3$ and we have $x_1x_2x_1x_2^2x_3 = p_{23}(1+p_{33})x_1x_2^2x_1x_2x_3 - p_{12}^2p_{22}x_1x_2^3x_1x_3$. Thus we have all words smaller than BE. Substituting all of these words we have BE as a linear combination of smaller words.

For the super-letter $[[D],[E]]$ we have $DE = p_{31}x_1x_2x_1x_3x_2x_3x_2$ and using (6.3), we get $DE = \alpha x_1^2x_2x_3x_2x_3x_2 + \beta x_2x_1^2x_3x_2x_3x_2$ where $\alpha = p_{31}p_{12}^{-1}(1+p_{11})^{-1}$ and $\beta = p_{31}p_{12}p_{11}(1+p_{11})^{-1}$. However the word DE is smaller than $x_1^2x_2x_3x_2x_3x_2$. Using (6.5) we obtain that $DE = \gamma x_1^2x_2^2x_3^2x_2 + \delta x_1^2x_2x_3^2x_2^2 + \beta x_2x_1^2x_3x_2x_3x_2$, where $\gamma = \alpha p_{23}^{-1}(1+p_{33})^{-1}$ and $\delta = \alpha p_{23}p_{33}(1+p_{33})^{-1}$. Note that $x_1^2x_2x_3^2x_2^2 = AD \cdot x_3x_2^2$ which is a linear combination of words smaller than DE. Calculating (6.6) $\cdot x_3 - x_2^2 \cdot$ (6.5) and using (6.5) twice, we obtain $x_1^2x_2^2x_3^2x_2 = \lambda x_1^2x_2x_3x_2^2x_3 + X'$ where $\lambda = p_{22}^{-1}(1+p_{22}+p_{22}^2)(1+p_{22}(1+p_{33})^{-1})^{-1}$. Now, we use (6.3) and we obtain $DE = \rho x_1x_2x_3x_1x_2^2x_3 + \delta AD\cdot x_3x_2^2 + X$ where $\rho = \gamma\lambda p_{13}p_{12}(1+p_{11})$ and X and X' are linear combinations of words beginning with x_2.

For the super-letter $[[B],[C]]$ we have $BC = x_1x_2x_1x_2x_1x_2x_3$. We compute $x_1 \cdot$ (6.3) and using again (6.3), we obtain $x_1^3x_2 = \varepsilon x_1x_2x_1^2 + X_1$ where $\varepsilon = p_{12}^2(1+p_{11}+p_{11}^2)$ and X_i is a linear combination of words beginning with x_2 for every $i \in \{1,2,3,4,5,6,7\}$. Now, we multiply $x_1^3x_2 \cdot x_2 = \varepsilon x_1x_2x_1^2x_2 + X_1x_2$ and from (6.3), we obtain $x_1^3x_2^2 = \mu x_1x_2x_1x_2x_1 + X_2$ where $\mu = \varepsilon(p_{12}(1+p_{11}))$. We are calculating (6.3) $\cdot x_2^2$ and using (6.4) we obtain $x_1^2x_2^3 = X_3$. Then we calculate (6.4) $\cdot x_2 + p_{12}(1+p_{22})x_2 \cdot$ (6.4), and we get $x_1x_2^3 = \eta x_2^2x_1x_2 + \sigma x_2^3x_1$ where $\eta = p_{12}^2((1+p_{22})^2 - p_{22})$ and $\sigma = -p_{12}^3p_{22}(1+p_{22})$. Using (6.3) we obtain $x_1x_2x_1x_2x_1x_2 = \xi x_1^2x_2^2x_1x_2 + X_4$. Now we use (6.4) and then we use (6.3) in the previous equation to get $x_1x_2x_1x_2x_1x_2 = \tau x_1^3x_2^3 + X_5$. We multiply $x_1x_2x_1x_2x_1x_2$ on the right by x_3 and using (6.6) we have $BC = x_1x_2x_1x_2x_1x_2x_3 = \eta x_1^3x_2x_3x_2^2 + \sigma x_1^3x_2^2x_3x_2 + \phi x_1^3x_3x_2^3 + X_6$, where $\phi = p_{22}^3p_{23}^3$. Now we use the relations obtained previously, and write $BC = \sigma\mu x_1x_2x_1x_2x_1x_3x_2 + p_{13}^2\eta\varepsilon x_1x_2x_3x_1^2x_2^2 + p_{13}^2\phi x_3x_1^2x_2^2 + X_6$. Thus BC is a linear combination of smaller words.

Finally, for the case $[[C],[D]]$ we will use the previous case. First we have $x_1^3x_2^3x_3^2 = \mu x_1x_2x_1x_2x_1x_2x_3^2 + X_7$ and using (6.6) we obtain $p_{23}(1 + p_{22} + p_{22}^2)x_1^3x_2^2x_3x_2x_3 - p_{23}^2p_{22}(1 + p_{22} + p_{22}^2)x_1^3x_2x_3x_2^2x_3 + p_{22}^3p_{23}^3x_1^3x_3x_2^3x_3 = \mu x_1x_2x_1x_2x_1x_2x_3^2 + X_7$. Now using the relations obtained for $x_1^3x_2$ and $x_1^3x_2^2$ and (6.5), we get the equation $\omega p_{31}CD = \omega x_1x_2x_1x_2x_1x_3x_2x_3 = p_{23}^2p_{22}(1+p_{22}+p_{22}^2)p_{13}^2\varepsilon x_1x_2x_3x_1^2x_2^2x_3 - p_{22}^3p_{23}^3p_{13}^3x_3x_1^3x_2^3x_3 - \mu p_{23}^2p_{33}x_1x_2x_1x_2x_1x_3^2x_2 + X_7$ where $\omega = p_{23}(1+p_{22}+p_{22}^2)(1-p_{12}^3(1+p_{11})(1+p_{33}))$. Thus CD is a linear combination of smaller words.

Remark 6.2. *In this case we calculate* $p_{AA} = p_{BB} = p_{DD} = p_{EE} = p_{FF} = p_{HH} = q$ *and* $p_{CC} = p_{GG} = p_{II} = q^2$.

7 Explicit PBW-Generators for Quantizations

In this section we finally prove that the previous sets are actually the PBW-bases of the considered algebras.

Theorem 7.1. *If q is not a root of 1, then the sets $\mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_2, \mathcal{B}_3, \mathcal{C}_3$ constitute sets of PBW-generators for $U_q^+(\mathfrak{g})$ over $\mathbf{k}[G]$, where $\mathfrak{g}$ is a simple Lie algebra of type A_2, A_3, B_2, B_3 and C_3, respectively. Also, each super-letter from each set has infinite height.*

Proof. This Theorem is a consequence of Propositions 4.1, 4.3, 5.1, 5.3 and 6.1, since they prove that all the hard in $U_q^+(\mathfrak{g})$ super-letters are contained in the previous sets. If they are all hard and not zero, from Theorem 3.3, they form a set of PBW-generators for $U_q^+(\mathfrak{g})$ over $\mathbf{k}[G]$. Now we only have to see that all heights are infinite.

From Remarks 4.2, 4.4, 5.2, 5.4 and 6.2 we know that for every hard super-letter $[u]$, either $p_{uu} = q$, $p_{uu} = q^2$ or $p_{uu} = q^4$. But we are supposing that q is not a root of 1, so $p(u, u)$ is not a primitive t-th root of 1 for any t, and from Definition 3.2 we have that $h([u])$ is infinite. □

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